Weighted L^p theory for vector potential operators in three-dimensional exterior domains

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Abstract

In the present paper we study the vector potential problem in exterior domains of $\mathbb{R}^3$. Our approach is based on the use of weighted spaces in order to describe the behaviour of functions at infinity. As a first step of the investigation, we prove important results on the Laplace equation in exterior domains with Dirichlet or Neumann boundary conditions. As a consequence of the obtained results on the vector potential problem, we establish usefull results on weighted Sobolev inequalities and Helmholtz decompositions of weighted spaces.

Keywords: Vector potential; Sobolev inequalities; Helmholtz decomposition; weighted spaces; unbounded domains

1 Introduction

Let Ω' be a simply-connected bounded domain of $\mathbb{R}^3$ assumed to have at least a Lipschitz-continuous boundary $\partial\Omega$ and let Ω denote the complement of $\overline{\Omega'}$, in other words, the exterior of Ω'. For a given divergence-free vector field $\boldsymbol{u}$, the vector potential problem in Ω consists in looking for a vector field $\boldsymbol{\psi}$ that satisfies

$$\boldsymbol{u} = \textbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \text{div}\, \boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\boldsymbol{u}$ and $\boldsymbol{\psi}$ satisfy appropriate boundary conditions.

This problem has many applications in fluid mechanics, particularly in fluid flow past an obstacle described by the Navier-Stokes equations and related linearized problems. We refer for instance to [7, 14, 15] and references therein, for the direct use of (1.1) in order to solve the Stokes equations in exterior domains.

Problem (1.1) also allows to establish important results on Helmholtz decompositions which are powerful tools in the study of the Navier-Stokes equations (see for instance [11, 12] and references therein). We also refer to [9] for discussions and applications of (1.1) in fluid mechanics and other fields.

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In this work, since the domain Ω is unbounded, we propose to investigate (1.1) by setting the problem in a class of weighted Sobolev spaces. These spaces enable to describe the growth or the decay of functions at infinity. They were introduced and studied by Hanouzet in [18]. They were used by Girault to study (1.1) in the whole space $\mathbb{R}^3$ [13] and in exterior domains [14]. Let us notice that in these works, the study was done in a Hilbertian framework. Since we are interesting in applications of (1.1) to nonlinear exterior problems, we choose to use an L^p-theory where $1 < p < \infty$. Thus, this work is an extension of [13] and [14].

Let us mention now that the potential vector problem has been studied in other types of domains. We can refer for example to [3, 8, 19] for bounded domains and [10] for the half-space. Finally, for exterior domains we can refer to [21] where the asymptotic behaviour of regular vector potentials is studied and to [20] where the problem is set in Sobolev-Besov or Besov spaces.

This paper is organized as follow. In the next section we introduce the class of weighted Sobolev spaces and we recall some of their basic properties. In Section 3 we solve two important auxiliary problems needed for the investigation of (1.1). The first problem is the divergence equation. The second problem is the Laplace equation in exterior domains with Dirichlet or Neumann boundary conditions. The results established here extend the ones proved in [6] in the sense that we are interested in the resolution of these problems for various decay properties at infinity. In Section 4 we solve the vector potential problem. We start with the case of the whole space and then extend the obtained results to exterior domains. Finally in Section 5, we give two important applications of the resolution of the vector potential problem. We first derive weighted Sobolev inequalities that extend important imbedding results well known in bounded domains (see for instance [3] and [8]). The inequalities we proved are extensions of the ones proved in [14] and [23]. Then we end by proving results on Helmholtz decompositions.

We now give the Notation we use throughout the paper. In what follows, p is a real number in the interval $]1,\infty[$. The dual exponent of p denoted p' is defined by the relation $1/p + 1/p' = 1$. We will use bold characters for vector or matrix fields. A point in $\mathbb{R}^3$ is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$ and its distance to the origin by

$$r = |\boldsymbol{x}| = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}.$$

Let $\mathbb{N}$ denote the set of nonnegative integers, $\mathbb{Z}$ the set of all integers and $\mathbb{Z}^-$ the set of nonpositive integers. For any multi-index $\boldsymbol{\lambda} \in \mathbb{N}^3$, we denote by $\partial^{\boldsymbol{\lambda}}$ the differential operator of order $\boldsymbol{\lambda}$,

$$\partial^{\boldsymbol{\lambda}} = \frac{\partial^{|\boldsymbol{\lambda}|}}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2} \partial x_3^{\lambda_3}}, \quad |\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \lambda_2.$$

We denote by $[s]$ the integer part of s. For any $k \in \mathbb{Z}$, $\mathscr{P}_k$ stands for the space of polynomials of degree less than or equal to k and $\mathscr{P}_k^{\Delta}$ the harmonic polynomials of $\mathscr{P}_k$. If k is a negative integer, we set by convention

$\mathscr{P}_k = \{0\}$. We denote by $\mathscr{D}(\Omega)$ the space of $\mathscr{C}^\infty$ functions with compact support in Ω. We recall that $\mathscr{D}'(\Omega)$ is the well-known space of distributions defined on Ω and $\mathscr{S}'(\mathbb{R}^3)$ is the space of tempered distributions. We recall that $L^p(\Omega)$ is the well-known Lebesgue real space and for $m \geq 1$, we recall that $W^{m,p}(\Omega)$ is the well-known classical Sobolev space and the space $\mathring{W}^{m,p}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in $W^{m,p}(\Omega)$. We shall write $u \in W^{m,p}_{loc}(\Omega)$ to mean that $u \in W^{m,p}(\mathscr{O})$, for any bounded domain $\mathscr{O}$, with $\overline{\mathscr{O}} \subset \Omega$. For $R > 0$, we denote B_R the open ball of radius R centered at $\mathbf{0}$. We set $\Omega_R = \Omega \cap B_R$ and $C_R = \mathbb{R}^3 \setminus \overline{B}_R$. The notation $\langle ., . \rangle$ will denote adequate duality pairing and will be specified when needed. If not specified, $\langle ., . \rangle_{\partial\Omega}$ will denote the duality pairing between the space $W^{-1/p,p}(\partial\Omega)$ and its dual space $W^{1/p,p'}(\partial\Omega)$. Given a Banach space B with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X, *i.e.*,

$$B' \perp X = \{f \in B',\ \forall v \in X, \langle f, v \rangle = 0\} = (B/X)'.$$

Finally, as usual, $C > 0$ denotes a generic constant the value of which may change from line to line and even at the same line.

2 A class of weighted Sobolev spaces and preliminaries

We introduce the weight function

$$\rho(\boldsymbol{x}) = \left(1 + r^2\right)^{1/2}.$$

For $\alpha \in \mathbb{R}$, we define

$$W^{0,p}_\alpha(\Omega) = \{u \in \mathscr{D}'(\Omega), \quad \rho^\alpha u \in L^p(\Omega)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{W^{0,p}_\alpha(\Omega)} = \|\rho^\alpha u\|_{L^p(\Omega)}.$$

For any $m \in \mathbb{N} \setminus \{0\}$ and $\alpha \in \mathbb{R}$, we set

$$k = k(m, p, \alpha) = \begin{cases} -1, & \text{if} \quad 3/p + \alpha \notin \{1, \ldots, m\}, \\ m - 3/p - \alpha, & \text{if} \quad 3/p + \alpha \in \{1, \ldots, m\}. \end{cases}$$

We define the weighted Sobolev space:

$$\begin{aligned} W^{m,p}_\alpha(\Omega) = \{&u \in \mathscr{D}'(\Omega); \\ &\forall\, \boldsymbol{\lambda} \in \mathbb{N}^3 : 0 \leq |\boldsymbol{\lambda}| \leq k,\ \rho^{\alpha - m + |\boldsymbol{\lambda}|} (\ln(2 + r^2))^{-1} \partial^{\boldsymbol{\lambda}} u \in L^p(\Omega); \\ &\forall\, \boldsymbol{\lambda} \in \mathbb{N}^3 :\ k + 1 \leq |\boldsymbol{\lambda}| \leq m, \rho^{\alpha - m + |\boldsymbol{\lambda}|} \partial^{\boldsymbol{\lambda}} u \in L^p(\Omega)\}, \end{aligned}$$

which is a Banach space when endowed with its natural norm:

$$\|u\|_{W^{m,p}_\alpha(\Omega)} = \left(\sum_{0\le|\boldsymbol{\lambda}|\le k} ||\rho^{\alpha-m+|\boldsymbol{\lambda}|}(\ln(2+r^2))^{-1}\partial^{\boldsymbol{\lambda}} u||^p_{L^p(\Omega)} + \sum_{k+1\le|\boldsymbol{\lambda}|\le m} ||\rho^{\alpha-m+|\boldsymbol{\lambda}|}\partial^{\boldsymbol{\lambda}} u||^p_{L^p(\Omega)}\right)^{1/p}.$$

We define the semi-norm

$$|u|_{W^{m,p}_\alpha(\Omega)} = \left(\sum_{|\boldsymbol{\lambda}|=m} \|\rho^\alpha \partial^{\boldsymbol{\lambda}} u\|_{L^p(\Omega)}\right)^{1/p}.$$

Let us give an example of such space that will be often used in the remaining of the paper. For $m = 1$, we have

$$W^{1,p}_\alpha(\Omega) = \left\{u \in \mathscr{D}'(\Omega);\ \rho^{\alpha-1}\, u \in L^p(\Omega),\ \ \rho^\alpha\, \nabla\, u \in L^p(\Omega)\right\} \quad \text{if} \quad 3/p+\alpha \neq 1$$

and

$$W^{1,p}_\alpha(\Omega) = \left\{u \in \mathscr{D}'(\Omega);\ \rho^{\alpha-1}(\ln(2+r^2))^{-1}\, u \in L^p(\Omega),\ \ \rho^\alpha\, \nabla\, u \in L^p(\Omega)\right\} \quad \text{if} \quad 3/p+\alpha = 1.$$

Observe that the logarithmic weight only appears for the critical case $3/p+\alpha \in \{1,\dots,m\}$.

We shall now give some basic properties of those spaces. For more details, the reader can refer to [5, 6, 18]. The space $\mathscr{D}(\overline{\Omega})$ is dense in $W^{m,p}_\alpha(\Omega)$. For any $\boldsymbol{\lambda} \in \mathbb{N}^3$, the mapping

$$u \in W^{m,p}_\alpha(\Omega) \to \partial^{\boldsymbol{\lambda}} u \in W^{m-|\boldsymbol{\lambda}|,p}_\alpha(\Omega) \tag{2.2}$$

is continuous.

If $m \in \mathbb{N}\setminus\{0\}$ and $3/p+\alpha \neq 1$, we have the following continuous embedding:

$$W^{m,p}_\alpha(\Omega) \hookrightarrow W^{m-1,p}_{\alpha-1}(\Omega). \tag{2.3}$$

Let j be defined as follow:

$$j = \begin{cases} [m-3/p-\alpha], & \text{if } 3/p+\alpha \notin \mathbb{Z}^-, \\ m-3/p-\alpha-1, & \text{otherwise.} \end{cases} \tag{2.4}$$

Then $\mathscr{P}_j$ is the space of all polynomials included in $W^{m,p}_\alpha(\Omega)$. Moreover, the following Poincaré-type inequality holds (see [6]):

$$\forall\, u \in W^{m,p}_\alpha(\Omega), \quad \inf_{\mu\in\mathscr{P}_{j'}} \|u+\mu\|_{W^{m,p}_\alpha(\Omega)} \le C|u|_{W^{m,p}_\alpha(\Omega)}, \tag{2.5}$$

where $j' = \min(j, m-1)$. Inequality (2.5) is the reason of choosing the weight functions in the definition of $W^{m,p}_\alpha(\Omega)$. All the local properties of the space $W^{m,p}_\alpha(\Omega)$ coincide with those of the standard Sobolev space $W^{m,p}(\Omega)$. Hence, it also satisfies the usual trace theorems on the boundary $\partial\Omega$. Therefore, we can define the space

$$\mathring{W}_{\alpha}^{m,p}(\Omega) = \left\{u \in W_{\alpha}^{m,p}(\Omega),\ \gamma_0 u = 0, \gamma_1 u = 0, ..., \gamma_{m-1} u = 0\right\}.$$

If Ω is the whole space $\mathbb{R}^3$, then we have $\mathring{W}_{\alpha}^{m,p}(\Omega) = W_{\alpha}^{m,p}(\mathbb{R}^3)$. The space $\mathscr{D}(\Omega)$ is dense in $\mathring{W}_{\alpha}^{m,p}(\Omega)$. Therefore the dual space of $\mathring{W}_{\alpha}^{m,p}(\Omega)$, denoted by $W_{-\alpha}^{-m,p'}(\Omega)$ is a space of distributions. Moreover, we have the following Poincaré-type inequality:

$$\forall u \in \mathring{W}_{\alpha}^{m,p}(\Omega), \quad \|u\|_{W_{\alpha}^{m,p}(\Omega)} \leq C|u|_{W_{\alpha}^{m,p}(\Omega)}. \tag{2.6}$$

We recall some weighted asymptotic properties proved in [1]. To that end, let us introduce first the following norm:

$$\|u(R,.)\|_{L^p(\partial B_1)} := R^{-2/p}\left(\int_{\partial B_R} |u(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p} = \left(\int_0^{2\pi}\int_0^{\pi} |u(R,\theta,\varphi)|^p \sin\theta d\theta d\varphi\right)^{1/p} \tag{2.7}$$

where (R,θ,φ) are the spherical coordinates of $\boldsymbol{x}$. Now let $\alpha \in \mathbb{Z}$ and $R > 1$ be a real number such that $\overline{\Omega'} \subset B_R$. Then there exists $C > 0$ such that for any $u \in W_{\alpha}^{1,p}(\Omega)$, we have

$$\begin{aligned} &\|u(R,.)\|_{L^p(\partial B_1)} \leq CR^{1-3/p-\alpha}\|u\|_{W_{\alpha}^{1,p}(\Omega)} \quad \text{if} \quad 3/p+\alpha \neq 1, \\ &\|u(R,.)\|_{L^p(\partial B_1)} \leq C\ln\left(2+R^2\right)\|u\|_{W_{\alpha}^{1,p}(\Omega)} \quad \text{if} \quad 3/p+\alpha = 1. \end{aligned} \tag{2.8}$$

The above asymptotic properties yield the following result that will be often used in the sequel.

Proposition 2.1. *Let α, β be real numbers. Let λ be a polynomial that belongs to $W_{\alpha}^{1,p}(\Omega) + W_{\beta}^{1,q}(\Omega)$. Then λ belongs to $\mathscr{P}_{[\gamma]}$ where $\gamma = \max\left(1-3/p-\alpha, 1-3/q-\beta\right)$.*

This proposition is proved in Appendix A.

The main tools of this paper are the following isomorphism results on the Laplace operator which summarize Theorems 6.6, 9.5, 9.9 of [5].

Theorem 2.2. *Let α and p satisfy*

$$\alpha \in \mathbb{Z}, \quad 3/p+\alpha \notin \mathbb{Z}^- \quad \textit{and} \quad 3/p'-\alpha \notin \mathbb{Z}^-. \quad (\boldsymbol{H})$$

Then the Laplace operator defined by

$$\Delta : W_{\alpha}^{1,p}(\mathbb{R}^3)/\mathscr{P}_{[1-3/p-\alpha]}^{\Delta} \to W_{\alpha}^{-1,p}(\mathbb{R}^3)\perp\mathscr{P}_{[1-3/p'+\alpha]}^{\Delta} \tag{2.9}$$

is an isomorphism.

Assume moreover that $3/p+\alpha \neq 1$ and $3/p'-\alpha \neq 1$, then for any interger $m \geq 1$, the Laplace operators defined by

$$\Delta : W_{\alpha+m}^{1+m,p}(\mathbb{R}^3)/\mathscr{P}_{[1-3/p-\alpha]}^{\Delta} \to W_{\alpha+m}^{-1+m,p}(\mathbb{R}^3)\perp\mathscr{P}_{[1-3/p'+\alpha]}^{\Delta} \tag{2.10}$$

and

$$\Delta : W^{1-m,p'}_{-\alpha-m}(\mathbb{R}^3)/\mathscr{P}^{\Delta}_{[1-3/p'+\alpha]} \to W^{-1-m,p'}_{-\alpha-m}(\mathbb{R}^3)\perp\mathscr{P}^{\Delta}_{[1-3/p-\alpha]} \tag{2.11}$$

are isomorphisms.

Remark 1.

1. The above isomorphism results are also valid for real values of α satisfying at least ($\boldsymbol{H}$). But for the sake of simplicity, we restrict ourselves to $\alpha \in \mathbb{Z}$.

2. Observe that if $\alpha < -1+3/p'$, then $\mathscr{P}^{\Delta}_{[1-3/p'+\alpha]} = \{0\}$ and there are no orthogonality conditions in (2.9) and (2.10). In the sequel, due to these orthogonality conditions, apart from assumption ($\boldsymbol{H}$), additional assumptions on α and p will be required for the vector potential results in weighted Sobolev spaces.

3. We recall that if $u \in \mathscr{S}'(\mathbb{R}^3)$ satisfies $\Delta u = 0$ in $\mathbb{R}^3$, then u is a polynomial.

We end this section by introducing the spaces that will be used to study the vector potential problem. We first recall that for any vector field $\boldsymbol{v} = (v_1, v_2, v_3)$, the curl of $\boldsymbol{v}$ is defined by

$$\mathbf{curl}\,\boldsymbol{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right).$$

Next note that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\,\boldsymbol{v} = \mathbf{grad}\,(\mathrm{div}\,\boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}.$$

This leads to the following definitions. For $\alpha \in \mathbb{Z}$, we define

$$H^p_\alpha(\mathbf{curl},\Omega) = \left\{\boldsymbol{v} \in W^{0,p}_\alpha(\Omega);\, \mathbf{curl}\,\boldsymbol{v} \in W^{0,p}_{\alpha+1}(\Omega)\right\},$$

$$H^p_\alpha(\mathrm{div},\Omega) = \left\{\boldsymbol{v} \in W^{0,p}_\alpha(\Omega);\, \mathrm{div}\,\boldsymbol{v} \in W^{0,p}_{\alpha+1}(\Omega)\right\}$$

and we set

$$X^p_\alpha(\Omega) = H^p_\alpha(\mathbf{curl},\Omega) \cap H^p_\alpha(\mathrm{div},\Omega).$$

These spaces are respectively endowed with the norms

$$\|\boldsymbol{v}\|_{H^p_\alpha(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^p_{W^{0,p}_\alpha(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^p_{W^{0,p}_{\alpha+1}(\Omega)}\right)^{1/p},$$

$$\|\boldsymbol{v}\|_{H^p_\alpha(\mathrm{div},\Omega)} = \left(\|\boldsymbol{v}\|^p_{W^{0,p}_\alpha(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^p_{W^{0,p}_{\alpha+1}(\Omega)}\right)^{1/p}$$

and

$$X^p_\alpha(\Omega) = \left(\|\boldsymbol{v}\|^p_{W^{0,p}_\alpha(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^p_{W^{0,p}_{\alpha+1}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^p_{W^{0,p}_{\alpha+1}(\Omega)}\right)^{1/p}.$$

These definitions will also be used when the exterior domain Ω is replaced by the whole space $\mathbb{R}^3$.
Observe that $\mathscr{D}(\overline{\Omega})$ is dense in $H^p_\alpha(\mathrm{div},\Omega)$ and in $H^p_\alpha(\mathbf{curl},\Omega)$. For the proof, one can use the same arguments as for the proof of the density of $\mathscr{D}(\overline{\Omega})$ in $W^{m,p}_\alpha(\Omega)$(see [18]). Therefore, denoting by $\boldsymbol{n}$ the unit normal vector to the boundary $\partial\Omega$ pointing outside Ω, if $\boldsymbol{v}$ belongs to $H^p_\alpha(\mathrm{div},\Omega)$, then $\boldsymbol{v}$ has normal trace $\boldsymbol{v}\cdot\boldsymbol{n}$ in $W^{-1/p,p}(\partial\Omega)$, where $W^{-1/p,p}(\partial\Omega)$ denotes the dual space of $W^{1-1/p',p'}(\partial\Omega)$. By the same way, if $\boldsymbol{v}$ belongs to $H^p_\alpha(\mathbf{curl},\Omega)$, then $\boldsymbol{v}$ has a tangential trace $\boldsymbol{v}\times\boldsymbol{n}$ that belongs to $W^{-1/p,p}(\partial\Omega)$. Similarly as in bounded domain, we have the trace theorems: for each $\alpha\in\mathbb{Z}$, there exists $C>0$ such that

$$\forall \boldsymbol{v}\in H^p_\alpha(\mathrm{div},\Omega),\ ||\boldsymbol{v}\cdot\boldsymbol{n}||_{W^{-1/p,p}(\partial\Omega)}\leq C||\boldsymbol{v}||_{H^p_\alpha(\mathrm{div},\Omega)},$$

$$\forall \boldsymbol{v}\in H^p_\alpha(\mathbf{curl},\Omega),\ ||\boldsymbol{v}\times\boldsymbol{n}||_{W^{-1/p,p}(\partial\Omega)}\leq C||\boldsymbol{v}||_{H^p_\alpha(\mathbf{curl},\Omega)}.$$

Moreover the following Green's formulas hold. If $3/p'-\alpha\neq 1$, then for any $\boldsymbol{v}\in H^p_\alpha(\mathrm{div},\Omega)$ and $\varphi\in W^{1,p'}_{-\alpha}(\Omega)$, we have

$$\langle \boldsymbol{v}\cdot\boldsymbol{n},\varphi\rangle_{\partial\Omega}=\int_\Omega \boldsymbol{v}\cdot\nabla\varphi\,d\boldsymbol{x}+\int_\Omega\varphi\,\mathrm{div}\,\boldsymbol{v}\,d\boldsymbol{x} \tag{2.12}$$

and for any $\boldsymbol{v}\in H^p_\alpha(\mathbf{curl},\Omega)$ and $\varphi\in W^{1,p'}_{-\alpha}(\Omega)$, we have

$$\langle \boldsymbol{v}\times\boldsymbol{n},\boldsymbol{\varphi}\rangle_{\partial\Omega}=\int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,d\boldsymbol{x}-\int_\Omega\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}. \tag{2.13}$$

We finally introduce two subspaces of $X^p_\alpha(\Omega)$:

$$X^p_{\alpha,N}(\Omega)=\left\{\boldsymbol{v}\in X^p_\alpha(\Omega),\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}\ on\ \partial\Omega\right\},$$

and

$$X^p_{\alpha,T}(\Omega)=\left\{\boldsymbol{v}\in X^p_\alpha(\Omega);\boldsymbol{v}\cdot\boldsymbol{n}=0\ on\ \partial\Omega\right\}.$$

3 Auxiliary problems

In this section, we solve in weighted Sobolev spaces two problems that we need in order to solve the vector potential problem and to establish weighted Sobolev inequalities: the divergence and the Laplace problem.

3.1 The divergence operator

We start by recalling a lifting boundary result (see [4, Lemma 3.3] or [12]).

Lemma 3.1. *Let $\mathscr{O}\subset\mathbb{R}^3$ be a bounded domain with boundary $\partial\mathscr{O}$ Lipschitz-continuous. Let $\boldsymbol{g}\in W^{1/p',p}(\partial\mathscr{O})$ be a given function such that*

$$\int_{\partial\mathscr{O}}\boldsymbol{g}\cdot\boldsymbol{n}\,ds=0.$$

Then there exists $\boldsymbol{u} \in W^{1,p}(\mathscr{O})$ *such that*

$$\operatorname{div} \boldsymbol{u} = 0 \quad \textit{in} \quad \mathscr{O} \quad \textit{and} \quad \boldsymbol{u} = \boldsymbol{g} \quad \textit{on} \quad \partial\mathscr{O}. \tag{3.14}$$

Moreover we have

$$\inf_{\boldsymbol{v} \in V^{1,p}(\mathscr{O})} \|\boldsymbol{u} + \boldsymbol{v}\|_{W^{1,p}(\mathscr{O})} \leq C \|\boldsymbol{g}\|_{W^{1/p',p}(\partial\mathscr{O})},$$

where $C > 0$ *is independent of* $\boldsymbol{u}$ *and* $\boldsymbol{g}$ *and* $V^{1,p}(\mathscr{O}) = \{\boldsymbol{v} \in \mathring{W}^{1,p}(\mathscr{O}), \operatorname{div} \boldsymbol{v} = 0\}$.

We introduce the space

$$V^{m,p}_{\alpha}(\Omega) = \left\{\boldsymbol{v} \in \mathring{W}^{m,p}_{\alpha}(\Omega), \quad \operatorname{div} \boldsymbol{v} = 0\right\}. \tag{3.15}$$

Proposition 3.2. *Let* Ω *be an exterior domain of* $\mathbb{R}^3$ *with boundary* $\partial\Omega$ *of class* $\mathscr{C}^{1,1}$ *if* $p \neq 2$ *and Lipschitz-continuous if* $p = 2$*. Let* α *and* p *satisfy (**H**). Then the following divergence operators are isomorphisms:*

$$\operatorname{div} \quad : \mathring{W}^{1,p}_{\alpha}(\Omega)/V^{1,p}_{\alpha}(\Omega) \to W^{0,p}_{\alpha}(\Omega) \quad \textit{if} \quad \alpha < 3/p',$$

$$\operatorname{div} \quad : \mathring{W}^{1,p}_{\alpha}(\Omega)/V^{1,p}_{\alpha}(\Omega) \to W^{0,p}_{\alpha}(\Omega) \perp \mathbb{R} \quad \textit{if} \quad \alpha > 3/p'.$$

Proof - First observe that if $\boldsymbol{u} \in \mathring{W}^{1,p}_{\alpha}(\Omega)$, then $\boldsymbol{u} \in H^{p}_{\alpha-1}(\Omega, \operatorname{div})$ and since $3/p' \neq \alpha$ the Green formula (2.12) holds. Thus owing to the boundary condition of $\boldsymbol{u}$, we have

$$\forall \varphi \in W^{1,p'}_{1-\alpha}(\Omega), \quad \int_{\Omega} \varphi \operatorname{div} \boldsymbol{u} \, d\boldsymbol{x} = -\int_{\Omega} \boldsymbol{u} \cdot \nabla\varphi \, d\boldsymbol{x}.$$

Furthermore, if $\alpha > 3/p'$, then the constants belong to $W^{1,p'}_{1-\alpha}(\Omega)$ which implies that we can take $\varphi = 1$ and this yields

$$\int_{\Omega} \operatorname{div} \boldsymbol{u} \, d\boldsymbol{x} = 0.$$

Hence, if $\alpha > 3/p'$, then $\operatorname{div} \boldsymbol{u}$ must be orthogonal to constants.

Next the divergence operator is clearly linear and continuous. It is also injective by construction. It remains to prove that the operator is onto. Let z be in $W^{0,p}_{\alpha}(\Omega)$ if $\alpha < 3/p'$ or $W^{0,p}_{\alpha}(\Omega) \perp \mathbb{R}$ if $\alpha > 3/p'$. Extending z by zero in Ω', then the extended function $\tilde{z}$ belongs to $W^{0,p}_{\alpha}(\mathbb{R}^3)$ if $\alpha < 3/p'$ or $W^{0,p}_{\alpha}(\mathbb{R}^3) \perp \mathbb{R}$ if $\alpha > 3/p'$. Since α and p satisfy (**H**), then there exists $\tilde{\boldsymbol{\varphi}} \in W^{1,p}_{\alpha}(\mathbb{R}^3)$ such that

$$\operatorname{div} \tilde{\boldsymbol{\varphi}} = \tilde{z} \quad \text{in} \quad \mathbb{R}^3,$$

(see [1, Proposition 2.1]). Now let $R > 0$ be a real number large enough such that $\overline{\Omega'} \subset B_R$. Then the restriction of $\tilde{\boldsymbol{\varphi}}$ to $\partial\Omega_R$ belongs to $W^{1/p',p}(\partial\Omega_R)$. Set now

$$\boldsymbol{g} = \begin{cases} \tilde{\boldsymbol{\varphi}} & \text{on} \quad \partial\Omega \\ \boldsymbol{0} & \text{on} \quad \partial B_R. \end{cases}$$

Then $\boldsymbol{g}$ belongs to $W^{1/p',p}(\partial\Omega_R)$ and

$$\int_{\partial\Omega_R} \boldsymbol{g} \cdot \boldsymbol{n}\, d\boldsymbol{s} = \int_{\partial\Omega} \tilde{\boldsymbol{\varphi}} \cdot \boldsymbol{n}\, d\boldsymbol{x} = \int_{\Omega'} \operatorname{div} \tilde{\boldsymbol{\varphi}}\, d\boldsymbol{x} = 0.$$

It follows from Lemma 3.1 that there exists $\boldsymbol{\psi} \in W^{1,p}(\Omega_R)$ such that

$$\operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega_R \quad \text{and} \quad \boldsymbol{\psi} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega_R.$$

Extending $\boldsymbol{\psi}$ by zero outside B_R, then the extended vector field still denoted $\boldsymbol{\psi}$ belongs to $W^{1,p}_{\alpha}(\Omega)$. Let $\boldsymbol{u} = \tilde{\boldsymbol{\varphi}} - \boldsymbol{\psi} \in \mathring{W}^{1,p}_{\alpha}(\Omega)$, then we have $\operatorname{div} \boldsymbol{u} = z$ in Ω. $\square$

As already mentioned in the proof, this result is an extension to exterior domains of a result proved in the whole space $\mathbb{R}^3$ (see [1]).

3.2 The Laplace's equation

3.2.1 The Dirichlet problem for the Laplace operator

In this section, we propose to solve the Laplace equation with Dirichlet boundary condition: given f in $W^{-1,p}_{\alpha}(\Omega)$ and g in $W^{1/p',p}(\partial\Omega)$, find u in $W^{1,p}_{\alpha}(\Omega)$ solution of:

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = g \quad \text{on} \quad \partial\Omega. \tag{3.16}$$

The case $\alpha = 0$, $1 < p < \infty$ has been studied in [6] and the case $\alpha \in \mathbb{Z}$, $p = 2$ has been studied in [16].
We start by giving the definition of the kernel of the Laplace operator for any integer $\alpha \in \mathbb{Z}$:

$$\mathscr{A}^{\Delta}_{\alpha,p}(\Omega) = \Big\{ \chi \in W^{1,p}_{\alpha}(\Omega);\ \Delta\chi = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \chi = 0 \quad \text{on} \quad \partial\Omega \Big\}.$$

In contrast with a bounded domain, the Dirichlet problem for the Laplace operator with zero data can have nontrivial solutions in an exterior domain; it depends upon the exponent of the weight. We recall the characterizations proved in [16] and [6] respectively.

Proposition 3.3. *(Giroire [16]). Let Ω be an exterior domain of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Assume $\alpha \in \mathbb{Z}$. Then*

$$\mathscr{A}^{\Delta}_{\alpha,2}(\Omega) = \left\{ v(q) - q, \quad q \in \mathscr{P}^{\Delta}_{-\alpha-1} \right\},$$

where $v(q) \in W_0^{1,2}(\Omega)$ *is the unique solution of the Dirichlet problem*

$$\Delta v(q) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad v(q) = q \quad \text{on} \quad \partial\Omega. \tag{3.17}$$

In particular, $\mathscr{A}^{\Delta}_{\alpha,2}(\Omega) = \{0\}$ *if* $\alpha \geq 0$.

Proposition 3.4. *(Amrouche et al. [6]). Let* Ω *be an exterior domain of* $\mathbb{R}^3$ *with boundary* $\partial\Omega$ *of class* $\mathscr{C}^{1,1}$ *if* $p \neq 2$ *and Lipschitz-continuous if* $p = 2$. *Then*

$$\mathscr{A}^{\Delta}_{0,p}(\Omega) = \left\{ v(q) - q, \quad q \in \mathscr{P}^{\Delta}_{[1-3/p]} \right\},$$

where $v(q) \in W_0^{1,2}(\Omega) \cap W_0^{1,p}(\Omega)$ *is the unique solution of* (3.17). *In particular* $\mathscr{A}^{\Delta}_{0,p}(\Omega) = \{0\}$ *if* $1 < p < 3$.

We generalize the two previous results by characterizing the kernel $\mathscr{A}^{\Delta}_{\alpha,p}(\Omega)$ for $\alpha \in \mathbb{Z}$ and $1 < p < \infty$. The proof is given in Appendix B.

Proposition 3.5. *Let* Ω *be an exterior domain of* $\mathbb{R}^3$ *with boundary* $\partial\Omega$ *of class* $\mathscr{C}^{1,1}$ *if* $p \neq 2$ *and Lipschitz-continuous if* $p = 2$. *Assume* $\alpha \in \mathbb{Z}$. *Then*

$$\mathscr{A}^{\Delta}_{\alpha,p}(\Omega) = \left\{ v(q) - q, \quad q \in \mathscr{P}^{\Delta}_{[1-3/p-\alpha]} \right\}.$$

where $v(q) \in W_0^{1,2}(\Omega) \cap W_\alpha^{1,p}(\Omega)$ *is the unique solution of* (3.17). *In particular,* $\mathscr{A}^{\Delta}_{\alpha,p}(\Omega) = \{0\}$ *if* $\alpha > 1 - 3/p$.

Our second step is to solve the following harmonic Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial\Omega. \tag{3.18}$$

Theorem 3.6. *Let* Ω *be an exterior domain of* $\mathbb{R}^3$ *with boundary* $\partial\Omega$ *of class* $\mathscr{C}^{1,1}$ *if* $p \neq 2$ *and Lipschitz-continuous if* $p = 2$. *Let* α *and* p *satisfy (**H**). Assume moreover that* $\alpha < 0$. *Then for any* g *in* $W^{1/p',p}(\partial\Omega)$, *Problem* (3.18) *has a solution* u *in* $W_\alpha^{1,p}(\Omega)$.

The proof of this theorem is given in Appendix C. As a consequence, we can prove the following theorem for any $\alpha \in \mathbb{Z}$.

Theorem 3.7. *Let* Ω *be an exterior domain of* $\mathbb{R}^3$ *with boundary* $\partial\Omega$ *of class* $\mathscr{C}^{1,1}$ *if* $p \neq 2$ *and Lipschitz-continuous if* $p = 2$. *Let* α *and* p *satisfy (**H**). Then for any* f *in* $W_\alpha^{-1,p}(\Omega)$ *and* g *in* $W^{1/p',p}(\partial\Omega)$ *such that*

$$\forall \lambda \in \mathscr{A}^{\Delta}_{-\alpha,p'}(\Omega), \quad \langle f, \lambda \rangle_{W_\alpha^{-1,p}(\Omega) \times \mathring{W}_{-\alpha}^{1,p'}(\Omega)} = \left\langle g, \frac{\partial \lambda}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p',p}(\partial\Omega) \times W^{-1/p',p'}(\partial\Omega)}, \tag{3.19}$$

problem (3.16) *has a unique solution* u *in* $W_\alpha^{1,p}(\Omega)/\mathscr{A}^{\Delta}_{\alpha,p}(\Omega)$. *In addition, there exists a constant* C, *independent of* u, f *and* g, *such that*

$$\|u\|_{W_\alpha^{1,p}(\Omega)/\mathscr{A}^{\Delta}_{\alpha,p}(\Omega)} \leq C \left(\|f\|_{W_\alpha^{-1,p}(\Omega)} + \|g\|_{W^{1/p',p}(\partial\Omega)} \right).$$

Proof - The case $\alpha = 0$ is proved in [6] so we focus on the case $\alpha \neq 0$.

1. We first prove that (3.19) is a necessary condition. Let λ be in $\mathscr{A}^{\Delta}_{-\alpha,p'}(\Omega)$. Note that for any open bounded domain $\mathscr{O} \subset \Omega$, λ belongs to $W^{1,p'}(\mathscr{O})$ and $\Delta\lambda$ to $L^{p'}(\mathscr{O})$. Then for any $\psi \in \mathscr{D}(\overline{\Omega})$, the following Green formula holds:

$$\int_{\Omega} \psi \Delta\lambda \, d\boldsymbol{x} - \int_{\Omega} \Delta\psi\lambda \, d\boldsymbol{x} = \left\langle \psi, \frac{\partial\lambda}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p',p}(\partial\Omega)\times W^{-1/p',p'}(\partial\Omega)} - \left\langle \frac{\partial\psi}{\partial \boldsymbol{n}}, \lambda \right\rangle_{\partial\Omega}.$$

Since $\lambda \in \mathscr{A}^{\Delta}_{-\alpha,p'}(\Omega)$, this is reduced to

$$-\int_{\Omega} \Delta\psi\lambda \, d\boldsymbol{x} = \left\langle \psi, \frac{\partial\lambda}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p',p}(\partial\Omega)\times W^{-1/p',p'}(\partial\Omega)}.$$

Let now $u \in W^{1,p}_{\alpha}(\Omega)$ and $g \in W^{1/p',p}(\partial\Omega)$ satisfy (3.16). Then, due to the density of $\mathscr{D}(\overline{\Omega})$ in $W^{1,p}_{\alpha}(\Omega)$, we deduce (3.19).

2. For the resolution of (3.16), the proof is made of three steps.

   - Step 1: the case $\alpha < 0$ and $g = 0$.
   The first point is to extend f by zero in Ω'. Since f belongs to $W^{-1,p}_{\alpha}(\Omega)$, then there exists a function $\boldsymbol{F}$ in $W^{0,p}_{\alpha}(\Omega)$ such that $f = \operatorname{div} \boldsymbol{F}$ in Ω (see [2, Proposition 1.3]). Let $\widetilde{\boldsymbol{F}}$ denote the extension by zero of $\boldsymbol{F}$ in Ω' and set $\widetilde{f} = \operatorname{div} \widetilde{\boldsymbol{F}}$. Then $\widetilde{f} \in W^{-1,p}_{\alpha}(\mathbb{R}^3)$ is an extension of f. Next since $\alpha < 0$, the polynomials space $\mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}$ is reduced to $\{0\}$. Using the fact that α and p satisfy ($\boldsymbol{H}$), we deduce from Theorem 2.2 that there exists $\widetilde{w}$ in $W^{1,p}_{\alpha}(\mathbb{R}^3)$ such that

   $$-\Delta\, \widetilde{w} = \widetilde{f} \quad \text{in} \quad \mathbb{R}^3.$$

   We denote by w the restriction of $\widetilde{w}$ to Ω. Thanks to Theorem 3.6, there exists $\xi \in W^{1,p}_{\alpha}(\Omega)$ satisfying

   $$\Delta\xi = 0 \quad \text{in} \quad \Omega, \quad \xi = -w \quad \text{on} \quad \partial\Omega.$$

   Setting $u = w + \xi$, then $u \in W^{1,p}_{\alpha}(\Omega)$ and satisfies problem (3.16) with $g = 0$.

   - Step 2: the case $\alpha > 0$ and $g = 0$.
   It follows from the previous case that the Laplace operator defined by

   $$\Delta : \mathring{W}^{1,p'}_{-\alpha}(\Omega)/\mathscr{A}^{\Delta}_{-\alpha,p'}(\Omega) \longmapsto W^{-1,p'}_{-\alpha}(\Omega)$$

   is an isomorphism. Therefore, by duality, the Laplace operator

   $$\Delta : \mathring{W}^{1,p}_{\alpha}(\Omega) \longmapsto W^{-1,p}_{\alpha}(\Omega) \perp \mathscr{A}^{\Delta}_{-\alpha,p'}(\Omega)$$

   is also an isomorphism

- Step 3: the general case.

  Let R be chosen so that $\overline{\Omega'}$ is contained in B_R. Let $v \in W^{1,p}(\Omega_R)$ be the lifting function of g satisfying:

$$\Delta\, v = 0 \quad \text{in} \quad \Omega_R, \quad v = g \quad \text{on} \quad \partial\Omega, \quad v = 0 \quad \text{on} \quad \partial B_R. \tag{3.20}$$

  Extending v by zero outside B_R and still denoted by v the extended function, problem (3.16) is equivalent to

$$-\Delta\, z = f + \Delta\, v \quad \text{in} \quad \Omega, \quad z = 0 \quad \text{on} \quad \partial\Omega. \tag{3.21}$$

  If $\alpha < -1+3/p'$, then thanks to Proposition 3.5, $\mathscr{A}^{\Delta}_{-\alpha,p'}(\Omega) = \{0\}$ and as $f+\Delta\, v$ belongs to $W^{-1,p}_{\alpha}(\Omega)$, it follows from steps 1 and 2 that problem (3.21) has a solution $z \in W^{1,p}_{\alpha}(\Omega)$. Hence $u = v + z$ belongs to $W^{1,p}_{\alpha}(\Omega)$ and satisfies problem (3.16). Uniqueness follows from the definition of the kernel $\mathscr{A}^{\Delta}_{\alpha,p}(\Omega)$. If $\alpha \geq -1+3/p'$, then, using (3.19) and the properties of v, one can easily prove that $f+\Delta v$ is orthogonal to $\mathscr{A}^{\Delta}_{-\alpha,p'}(\Omega)$ and we can again use the previous steps to end the proof.
  □

Remark 2. Assume Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$, assume that f is in $W^{0,p}_{\alpha}(\Omega)$ and g in $W^{1+1/p',p}(\partial\Omega)$. If α and p satisfy (***H***) and the additional assumptions $3/p+\alpha \neq 1$ and $3/p+\alpha \neq 2$, then one can prove that the solution u discussed in Theorem 3.7 (where α is replaced by $\alpha-1$) belongs to $W^{2,p}_{\alpha}(\Omega)/\mathscr{A}^{\Delta}_{\alpha-1,p}(\Omega)$. In addition, there exists a constant C, independent of u, f and g, such that

$$||u||_{W^{2,p}_{\alpha}(\Omega)/\mathscr{A}^{\Delta}_{\alpha-1,p}(\Omega)} \leq C\Big(||f||_{W^{0,p}_{\alpha}(\Omega)} + ||g||_{W^{1+1/p',p}(\partial\Omega)}\Big).$$

This regularity result is obtained via an adequate partition of unity that allows to split u into the solution of a Laplace problem in $\mathbb{R}^3$, solved by Theorem 2.2 (isomorphism (2.10) with $m=1$ and α replaced by $\alpha-1$) and the solution of a Dirichlet problem for the Laplace operator in a bounded domain.

3.2.2 The harmonic Neumann problem

In this part we consider the following Laplace equation: For g given in $W^{-1/p,p}(\partial\Omega)$, find u solution of:

$$\Delta\, u = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \partial\Omega. \tag{3.22}$$

We define the kernel of the Laplace operator with Neumann boundary condition:

$$\mathscr{N}^{\Delta}_{\alpha,p}(\Omega) = \Big\{\chi \in W^{1,p}_{\alpha}(\Omega), \quad \Delta\chi = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial\chi}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \partial\Omega\Big\}.$$

The characterizations below are proved in [16] and in [6] respectively.

Proposition 3.8. *(Giroire [16]). Let Ω be an exterior domain of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Assume $\alpha \in \mathbb{Z}$. We have*

$$\mathcal{N}^{\Delta}_{\alpha,2}(\Omega) = \left\{ w(q) - q, \quad q \in \mathcal{P}^{\Delta}_{-1-\alpha} \right\},$$

where $w(q) \in W^{1,2}_0(\Omega) \cap W^{1,2}_\alpha(\Omega)$ is the unique solution of the Neumann problem

$$\Delta\, w(q) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial w(q)}{\partial \boldsymbol{n}} = \frac{\partial q}{\partial \boldsymbol{n}} \quad \text{on} \quad \partial\Omega. \tag{3.23}$$

In particular $\mathcal{N}^{\Delta}_{-1,2}(\Omega) = \mathbb{R}$ and $\mathcal{N}^{\Delta}_{\alpha,2}(\Omega) = \{0\}$, If $\alpha \geq 0$.

Proposition 3.9. *(Amrouche et al. [6]). Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathcal{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. We have*

$$\mathcal{N}^{\Delta}_{0,p}(\Omega) = \mathcal{P}_{[1-3/p]}.$$

The next proposition characterizes the kernel $\mathcal{N}^{\Delta}_{\alpha,p}(\Omega)$ for $\alpha \in \mathbb{Z}$ and the proof is given in Appendix D.

Proposition 3.10. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathcal{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Assume $\alpha \in \mathbb{Z}$. We have*

$$\mathcal{N}^{\Delta}_{\alpha,p}(\Omega) = \left\{ w(q) - q, \quad q \in \mathcal{P}^{\Delta}_{[1-3/p-\alpha]} \right\},$$

where $w(q) \in W^{1,2}_0(\Omega) \cap W^{1,p}_\alpha(\Omega)$ is the unique solution of (3.23). In particular, $\mathcal{N}^{\Delta}_{\alpha,p}(\Omega) = \{0\}$ if $\alpha > 1 - 3/p$ and $\mathcal{N}^{\Delta}_{\alpha,p}(\Omega) = \mathbb{R}$ if $-3/p < \alpha \leq 1 - 3/p$.

The two theorems below solve the harmonic Neumann problem in weighted spaces.

Theorem 3.11. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathcal{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Let α and p satisfy (**H**). Assume moreover that $\alpha < -1 + 3/p'$. Then for any g in $W^{-1/p,p}(\partial\Omega)$, problem (3.22) has a solution u in $W^{1,p}_\alpha(\Omega)$ unique up to an element of $\mathcal{N}^{\Delta}_{\alpha,p}(\Omega)$. Moreover we have the estimate*

$$\|u\|_{W^{1,p}_\alpha(\Omega)/\mathcal{N}^{\Delta}_{\alpha,p}(\Omega)} \leq C \|g\|_{W^{-1/p,p}(\partial\Omega)}.$$

Theorem 3.12. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathcal{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Let α and p satisfy (**H**). Assume moreover that $\alpha \geq -1 + 3/p'$. Then for any g in $W^{-1/p,p}(\partial\Omega)$, problem (3.22) has a solution u in $W^{1,p}_\alpha(\Omega)$ if and only if*

$$\forall\, v \in \mathcal{N}^{\Delta}_{-\alpha,p'}(\Omega), \quad \langle g, v\rangle_{\partial\Omega} = 0. \tag{3.24}$$

The solution u is unique up to an element of $\mathcal{N}^{\Delta}_{\alpha,p}(\Omega)$. Moreover we have the estimate

$$\|u\|_{W^{1,p}_\alpha(\Omega)/\mathcal{N}^{\Delta}_{\alpha,p}(\Omega)} \leq C \|g\|_{W^{-1/p,p}(\partial\Omega)}.$$

Remark 3.

1. Note that these results have been proved by Specovius-Neugebauer [22] for exterior domains with boundaries of class at least $\mathscr{C}^2$. The solution of the harmonic Neumann problem is represented as a single layer potential and weighted estimates on the solution are established. We propose proofs with a different approach making use of known results established in [6], [16] and Theorem 2.2. The proofs can be seen in Appendices E and F.

2. Assume Ω be an exterior domain with boundary of class $\mathscr{C}^{1,1}$ and assume that the datum g belongs to $W^{1/p',p}(\partial\Omega)$. If α and p satisfy ($\boldsymbol{H}$) and the additional assumptions $3/p+\alpha \neq 1$ and $3/p+\alpha \neq 2$, then as in Remark 2, the solution u of Problem (3.22) discussed in Theorem 3.11 and Theorem 3.12 belongs to $W^{2,p}_{\alpha}(\Omega)$.

4 The Vector potential operator

In this section we study the vector potential problem (1.1). We begin with the whole space $\mathbb{R}^3$ and then extend the results to exterior domains.

4.1 The vector potential in $\mathbb{R}^3$

We start by introducing the following spaces:

$$\begin{aligned} \mathscr{G}_k &= \left\{\nabla q,\ q \in \mathscr{P}^{\Delta}_{k+1}\right\} \\ G^{m,p}_{\alpha} &= \left\{\nabla q,\ q \in W^{m+1,p}_{\alpha}(\mathbb{R}^3)/\mathscr{P}_{j'}\right\} \end{aligned}$$

where $j' = \min(0, j)$ and where j is defined in (2.4). For $m \in \mathbb{Z}$, we recall the space

$$V^{m,p}_{\alpha}(\mathbb{R}^3) = \left\{\boldsymbol{v} \in W^{m,p}_{\alpha}(\mathbb{R}^3), \quad \operatorname{div} \boldsymbol{v} = 0\right\}.$$

We finally recall a useful lemma whose proof can be found in [13].

Lemma 4.1. *(Girault [13]). Let $k \geq 1$ be an integer, $\boldsymbol{q}$ be a polynomial in $\mathscr{P}_{k-1}$ such that* $\operatorname{div} \boldsymbol{q} = 0$ *and r be a polynomial in $\mathscr{P}_{k-1}$. Then there exists a unique polynomial $\boldsymbol{\lambda}$ in $\mathscr{P}_k/\mathscr{G}_k$ such that*

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\lambda} = \boldsymbol{q} \ \ \textit{and} \ \ \operatorname{div} \boldsymbol{\lambda} = r.$$

The mapping $(\boldsymbol{q}, r) \to \boldsymbol{\lambda}$ is linear.

Theorem 4.2. *Let α and p satisfy ($\boldsymbol{H}$). Assume, moreover that $\alpha < 1 + 3/p'$. Let $\boldsymbol{u}$ be in $V_{\alpha}^{0,p}(\mathbb{R}^3)$. Then there exists a unique vector potential $\boldsymbol{\psi} \in W_{\alpha}^{1,p}(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}$ such that*

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \textit{and} \quad \operatorname{div}\boldsymbol{\psi} = 0.$$

Moreover, we have

$$\|\boldsymbol{\psi}\|_{W_{\alpha}^{1,p}(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}} \leq C\|\boldsymbol{u}\|_{W_{\alpha}^{0,p}(\mathbb{R}^3)}.$$

Proof - We shall prove that the **curl** operator is an isomorphism from $V_{\alpha}^{1,p}(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}$ onto $V_{\alpha}^{0,p}(\mathbb{R}^3)$. The **curl** operator is clearly linear and continuous. Let us prove that it is injective. Let $\boldsymbol{u}$ be in $V_{\alpha}^{0,p}(\mathbb{R}^3)$. Assume that there exist two vector potentials $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ that belong to $V_{\alpha}^{1,p}(\mathbb{R}^3)$ satisfying $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}_1 = \mathbf{curl}\,\boldsymbol{\psi}_2$. Set $\boldsymbol{\varphi} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2$, then

$$\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0} \quad \text{and} \quad \operatorname{div}\boldsymbol{\varphi} = \mathbf{0}. \tag{4.25}$$

This implies that $\Delta\boldsymbol{\varphi} = \mathbf{0}$ and therefore, $\boldsymbol{\varphi}$ belongs to $\mathscr{P}_{[1-3/p-\alpha]}$. From (4.25) and Lemma 4.1, $\boldsymbol{\varphi}$ belongs to $\mathscr{G}_{[1-3/p-\alpha]}$. We shall now prove that the **curl** operator is onto. Let $\boldsymbol{u} \in V_{\alpha}^{0,p}(\mathbb{R}^3)$, then it follows that $\mathbf{curl}\,\boldsymbol{u}$ belongs to $W_{\alpha}^{-1,p}(\mathbb{R}^3)$.

- Assume first that $\alpha < -1 + 3/p'$, then thanks to Theorem 2.2, there exists $\boldsymbol{\psi} \in W_{\alpha}^{1,p}(\mathbb{R}^3)$ such that

$$-\Delta\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{u} \quad \text{in} \quad \mathbb{R}^3. \tag{4.26}$$

  Furthermore, since $\operatorname{div}\boldsymbol{u} = 0$, then, on the one hand, we have $\mathbf{curl}\,\boldsymbol{\psi} - \boldsymbol{u} \in W_{\alpha}^{0,p}(\mathbb{R}^3)$ satisfies $\Delta(\mathbf{curl}\,\boldsymbol{\psi} - \boldsymbol{u}) = \mathbf{0}$ which shows that $\mathbf{curl}\,\boldsymbol{\psi} - \boldsymbol{u}$ is a polynomial of $\mathscr{P}_{[-3/p-\alpha]}$ and, on the other hand, we also have $\operatorname{div}(\mathbf{curl}\,\boldsymbol{\psi} - \boldsymbol{u}) = \mathbf{0}$. Besides, Equality (4.26) implies that $\Delta(\operatorname{div}\boldsymbol{\psi}) = 0$ which yields that $\operatorname{div}\boldsymbol{\psi}$ is a polynomial of $\mathscr{P}_{[-3/p-\alpha]}$. Therefore, according to Lemma 4.1, there exists a polynomial $\boldsymbol{\lambda}$ in $\mathscr{P}_{[1-3/p-\alpha]}$ such that

$$\mathbf{curl}\,\boldsymbol{\lambda} = \mathbf{curl}\,\boldsymbol{\psi} - \boldsymbol{u} \quad \text{and} \quad \operatorname{div}\boldsymbol{\lambda} = \operatorname{div}\boldsymbol{\psi}.$$

  Hence a vector potential candidate is $\boldsymbol{\psi} - \boldsymbol{\lambda}$.

- Assume next that $-1 + 3/p' \leq \alpha < 3/p'$. Then $\mathbf{curl}\,\boldsymbol{u}$ is obviously orthogonal to $\mathscr{P}_0$ and we can again solve (4.26).

- Assume finally $3/p' < \alpha < 1 + 3/p'$, then $\mathbf{curl}\,\boldsymbol{u}$ is orthogonal to $\mathscr{P}_1$. Indeed, we have

$$\forall \boldsymbol{q} \in \mathscr{P}_1, \quad \langle \mathbf{curl}\,\boldsymbol{u}, \boldsymbol{q} \rangle_{W_{\alpha}^{-1,p}(\mathbb{R}^3) \times W_{-\alpha}^{1,p'}(\mathbb{R}^3)} = -\langle \boldsymbol{u}, \mathbf{curl}\,\boldsymbol{q} \rangle_{W_{\alpha}^{0,p}(\mathbb{R}^3) \times W_{-\alpha}^{0,p'}(\mathbb{R}^3)}.$$

  Besides, for any $\boldsymbol{q} \in \mathscr{P}_1$, there exists $p \in \mathscr{P}_1$ such that $\mathbf{curl}\,\boldsymbol{q} = \nabla p$. It follows that

$$\langle \boldsymbol{u}, \mathbf{curl}\,\boldsymbol{q} \rangle_{W_{\alpha}^{0,p}(\mathbb{R}^3) \times W_{-\alpha}^{0,p'}(\mathbb{R}^3)} = \langle \boldsymbol{u}, \nabla p \rangle_{W_{\alpha}^{0,p}(\mathbb{R}^3) \times W_{-\alpha}^{0,p'}(\mathbb{R}^3)} = \langle \operatorname{div}\boldsymbol{u}, p \rangle_{W_{\alpha}^{-1,p}(\mathbb{R}^3) \times W_{-\alpha}^{1,p'}(\mathbb{R}^3)} = 0$$

since $u \in V_\alpha^{1,p}(\mathbb{R}^3)$. Then, again we can solve (4.26). $\square$

Theorem 4.3. *Let α and p satisfy (**H**). Assume moreover that $\alpha \geq 1 + 3/p'$. Let $\boldsymbol{u}$ be in $V_\alpha^{0,p}(\mathbb{R}^3)$ satisfy*

$$\langle \mathbf{curl}\, \boldsymbol{u}, q \rangle_{W_\alpha^{-1,p}(\mathbb{R}^3) \times W_{-\alpha}^{1,p'}(\mathbb{R}^3)} = 0, \quad \forall q \in \mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}. \tag{4.27}$$

Then there exists a unique $\boldsymbol{\psi} \in W_\alpha^{1,p}(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}$ such that

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \textit{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \quad \textit{in} \quad \mathbb{R}^3.$$

Conversely, any function $\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi}$ with $\boldsymbol{\psi}$ in $V_\alpha^{1,p}(\mathbb{R}^3)$ satisfies (4.27).

Proof - Since $\mathbf{curl}\, \boldsymbol{u}$ satisfies the compatibility condition (4.27), then, for the proof of the existence of the unique vector potential $\boldsymbol{\psi}$ in $V_\alpha^{1,p}(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}$, one can apply Theorem 2.2 and therefore one can proceed as in the proof of Theorem 4.2. Let us now prove the converse. If $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}$ with $\boldsymbol{\psi} \in V_\alpha^{1,p}(\mathbb{R}^3)$, then we have,

$$\mathbf{curl}\, \boldsymbol{u} = \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{\psi} = -\Delta \boldsymbol{\psi} + \nabla(\operatorname{div} \boldsymbol{\psi}).$$

It follows that $\boldsymbol{\psi}$ is solution of the Laplace equation

$$\Delta \boldsymbol{\psi} = -\mathbf{curl}\, \boldsymbol{u} \ \text{ in } \mathbb{R}^3.$$

Hence, in view of Theorem 2.2, $\mathbf{curl}\, \boldsymbol{u}$ belongs to $W_\alpha^{-1,p}(\mathbb{R}^3) \perp \mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}$. In other words, $\mathbf{curl}\, \boldsymbol{u}$ satisfies (4.27). $\square$

Remark 4. The main drawback of Theorem 4.3 is that the necessary compatibility condition (4.27) is difficult to satisfy in most applications. The last theorem of the subsection states that the existence and the uniqueness of the vector potential in the case $\alpha \geq 1 + 3/p'$ can still be valid once condition (4.27) is removed. In return, this unique vector potential is not necessarily of divergence free. We first need a lemma on characterizations of duals of some weighted spaces.

Lemma 4.4. *Let α and p satisfy (**H**). Assume moreover that $\alpha \geq 1 + 3/p'$. Then we have*

$$\left(V_{-\alpha}^{-1,p'}(\mathbb{R}^3) \right)' = W_\alpha^{1,p}(\mathbb{R}^3)/G_\alpha^{1,p}$$

and

$$\left(V_{-\alpha}^{0,p'}(\mathbb{R}^3)/\mathscr{G}_{[-3/p'+\alpha]} \right)' = V_\alpha^{0,p}(\mathbb{R}^3).$$

This lemma is proved in Appendix G. As a consequence, we can state the following.

Theorem 4.5. *Let α and p satisfy (**H**). Assume moreover that $\alpha \geq 1 + 3/p'$. Let $\boldsymbol{u}$ be in $V_\alpha^{0,p}(\mathbb{R}^3)$. Then there exists a unique $\boldsymbol{\psi} \in W_\alpha^{1,p}(\mathbb{R}^3)/G_\alpha^{1,p}$ such that*

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \textit{in} \quad \mathbb{R}^3.$$

Proof - Proceeding as in the proof of Theorem 4.2 with the use of the isomorphism of the Laplace operator (G.55), we can easily prove that the **curl** operator defined by

$$\mathbf{curl} : V_{-\alpha}^{0,p'}(\mathbb{R}^3)/\mathscr{G}_{[-3/p'+\alpha]} \mapsto V_{-\alpha}^{-1,p'}(\mathbb{R}^3)$$

is an isomorphism. By duality, the **curl** operator defined by

$$\mathbf{curl} : \left(V_{-\alpha}^{-1,p'}(\mathbb{R}^3)\right)' \mapsto \left(V_{-\alpha}^{0,p'}(\mathbb{R}^3)/\mathscr{G}_{[-3/p'+\alpha]}\right)'$$

is an isomorphism. Then Lemma 4.4 ends the proof. □

4.2 The vector potential in an exterior domain

For α in $\mathbb{Z}$, we define the spaces

$$Y_{\alpha,N}^{p}(\Omega) = \left\{\boldsymbol{w} \in X_{\alpha,N}^{p}(\Omega);\ \operatorname{div} \boldsymbol{w} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{w} = \mathbf{0} \quad \text{in } \Omega\right\}. \tag{4.28}$$

and

$$Y_{\alpha,T}^{p}(\Omega) = \left\{\boldsymbol{w} \in X_{\alpha,T}^{p}(\Omega);\ \operatorname{div} \boldsymbol{w} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{w} = \mathbf{0} \quad \text{in } \Omega\right\}. \tag{4.29}$$

As a consequence of Section 3 and the vector potential problem in the whole space, we have the following characterizations of these spaces.

Proposition 4.6. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Let α and p satisfy (**H**). Then*

$$Y_{\alpha,N}^{p}(\Omega) = \left\{\nabla\big(w(q) - q\big), \quad q \in \mathscr{P}_{[1-3/p-\alpha]}^{\Delta}\right\},$$

where $w(q)$ is the unique solution in $W_0^{1,2}(\Omega) \cap W_{\alpha}^{1,p}(\Omega)$ of problem (3.17). In particular, $Y_{\alpha,N}^{p}(\Omega) = \{\mathbf{0}\}$ if $\alpha > 1 - 3/p$.

Proof - The proof follows the ideas of Girault-Giroire-Sequeira in [15]. Let $\boldsymbol{w}$ be in $Y_{\alpha,N}^{p}(\Omega)$. Then since Ω' is simply-connected and under the assumptions on α and p, thanks to Theorem 4.2 and Theorem 4.5, there exists a $\chi \in W_{\alpha}^{1,p}(\Omega)$, unique up to an additive constant, such that $\boldsymbol{w} = \nabla\chi$. But $\boldsymbol{w} \times \boldsymbol{n} = \mathbf{0}$, hence, χ is constant on $\partial\Omega$ (we recall that $\partial\Omega$ is a connected boundary) and we choose the additive constant in χ so that $\chi = 0$ on $\partial\Omega$. Thus $\chi \in W_{\alpha}^{1,p}(\Omega)$ satisfies the following problem:

$$\Delta\chi = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \chi = 0 \quad \text{on} \quad \partial\Omega.$$

Thus the proposition follows from the characterization of the kernel $\mathscr{A}_{\alpha,p}^{\Delta}(\Omega)$. Now, to end the proof we shall prove that $\nabla(w(q) - q)$ belongs to $Y_{\alpha,N}^{p}(\Omega)$. But this is a simple consequence of the definition of q and $w(q)$. □

Proposition 4.7. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Let α and p satisfy (**H**). Then*

$$Y^p_{\alpha,T}(\Omega) = \left\{\nabla\big(w(q) - q\big), \quad q \in \mathscr{P}^{\Delta}_{[1-3/p-\alpha]}\right\},$$

where $w(q)$ is the unique solution in $W^{1,2}_0(\Omega) \cap W^{1,p}_\alpha(\Omega)$ of problem (3.23). In particular, $Y^p_{\alpha,T}(\Omega) = \{\mathbf{0}\}$ if $\alpha > -3/p$.

Proof - We proceed as in the proof of Proposition 4.6. If $\boldsymbol{w}$ belongs to $\in Y^p_{\alpha,T}(\Omega)$, then the fact that Ω' is simply-connected and under the assumptions on α and p, there exists a $\chi \in W^{1,p}_\alpha(\Omega)$, unique up to an additive constant, such that $\boldsymbol{w} = \nabla\chi$. But $\boldsymbol{w}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ thus $\chi \in W^{1,p}_\alpha(\Omega)$ satisfies the following problem:

$$\Delta\chi = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial\chi}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \partial\Omega.$$

Thus the proposition follows from the characterization of the kernel $\mathscr{N}^{\Delta}_{\alpha,p}(\Omega)$. Finally the fact that $\nabla(w(q) - q)$ belongs to $Y^p_{\alpha,T}(\Omega)$ is straightforward. $\square$

Our first theorem does not require the uniqueness of the vector potential and holds for any exponent of the weight satisfying (**H**).

Theorem 4.8. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Let α and p satisfy (**H**). Then for any function $\boldsymbol{v} \in H^p_\alpha(\mathrm{div},\Omega)$ that satisfies*

$$\mathrm{div}\,\boldsymbol{v} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \langle\boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\partial\Omega} = 0,$$

there exists $\boldsymbol{\psi}$ in $W^{1,p}_\alpha(\Omega)$ such that

$$\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega.$$

Proof - Let us solve the following Neumann problem in the bounded domain Ω':

$$\Delta\theta = 0 \quad \text{in} \quad \Omega' \quad \text{and} \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = -\boldsymbol{v}\cdot\boldsymbol{n} \quad \text{on} \quad \partial\Omega.$$

Since $\langle\boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\partial\Omega} = 0$, this problem has a solution θ in $W^{1,p}(\Omega')$ and there exists a constant C independent of $\boldsymbol{v}$ such that

$$||\theta||_{W^{1,p}(\Omega')} \leq C||\boldsymbol{v}\cdot\boldsymbol{n}||_{W^{-1/p,p}(\partial\Omega)}.$$

Let us take $\boldsymbol{w} = \nabla\theta$ in Ω' and $\boldsymbol{w} = \boldsymbol{v}$ in Ω. Then $\boldsymbol{w}$ belongs to $H^p_\alpha(\mathrm{div},\mathbb{R}^3)$ and $\mathrm{div}\,\boldsymbol{w} = 0$ in $\mathbb{R}^3$. Indeed let

$\varphi \in \mathscr{D}(\mathbb{R}^3)$, then we have

$$\begin{aligned}\langle \operatorname{div} \boldsymbol{w}, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^3)\times\mathscr{D}(\mathbb{R}^3)} &= -\int_{\mathbb{R}^3} \boldsymbol{w}\cdot\nabla\varphi\, d\boldsymbol{x} \\ &= -\int_{\Omega} \boldsymbol{v}\cdot\nabla\varphi\, d\boldsymbol{x} - \int_{\Omega'} \nabla\theta\cdot\nabla\varphi\, d\boldsymbol{x} \\ &= -\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi\rangle_{\partial\Omega} - \left\langle \frac{\partial\theta}{\partial\boldsymbol{n}}, \varphi\right\rangle_{\partial\Omega} = 0.\end{aligned}$$

Moreover we have

$$\|\boldsymbol{w}\|_{H^p_\alpha(\operatorname{div},\mathbb{R}^3)} \le \|\boldsymbol{v}\|_{H^p_\alpha(\operatorname{div},\Omega)}.$$

The proof is now splitted into two cases.

- The case $\alpha < 1 + 3/p'$. Thanks to Theorem 4.2, we deduce that $\boldsymbol{w}$ has a unique vector potential $\boldsymbol{\psi}$ in $W^{1,p}_\alpha(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}$ that satisfies

$$\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

  Then the restriction of $\boldsymbol{\psi}$ to Ω is the required vector potential.

- The case $\alpha \ge 1 + 3/p'$. In order to apply Theorem 4.3, $\mathbf{curl}\,\boldsymbol{w}$ needs to satisfy the compatibility condition (4.27). The goal is then to modify the extension $\boldsymbol{w}$ so that it satisfies (4.27). Observe first that due to Theorem 4.5, there exists a vector potential $\boldsymbol{\psi} \in W^{1,p}_\alpha(\mathbb{R}^3)$ such that $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{\psi}$ in $\mathbb{R}^3$. But of course, this vector potential is not necessarily divergence free. Let now D_1 and D_2 be two balls such that $D_1 \subset D_2 \subset \Omega'$ and let $\mu \in \mathscr{D}(\mathbb{R}^3)$ be a cut-off function such that

$$0 \le \mu \le 1, \quad \mu = 1 \quad \text{in} \quad D_1, \quad \operatorname{supp}\mu \subset D_2.$$

  We introduce the mapping

$$q \mapsto \left(\int_{\Omega'} \mu(\boldsymbol{x})\, q(\boldsymbol{x})^2\, d\boldsymbol{x}\right)^{1/2} \tag{4.30}$$

  which is a norm on $\mathscr{P}_k$ for any $k \in \mathbb{N}$. Consider next the following problem: find $\boldsymbol{\lambda} \in \mathscr{P}_{[-1-3/p'+\alpha]}$ solution of

$$\forall \boldsymbol{q} \in \mathscr{P}_{[-1-3/p'+\alpha]}, \quad \int_{\Omega'} \mu(\boldsymbol{x})\,\boldsymbol{\lambda}(\boldsymbol{x})\cdot\boldsymbol{q}(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\mathbb{R}^3} \boldsymbol{\psi}(\boldsymbol{x})\cdot\boldsymbol{q}(\boldsymbol{x})\,d\boldsymbol{x}. \tag{4.31}$$

  The bilinear form $(\boldsymbol{q}_1, \boldsymbol{q}_2) \mapsto \int_{\Omega'} \mu(\boldsymbol{x})\,\boldsymbol{q}_1(\boldsymbol{x})\cdot\boldsymbol{q}_2(\boldsymbol{x})\,d\boldsymbol{x}$ is continuous and coercive with respect to the norm defined in (4.30). Therefore, thanks to Lax-Milgram lemma, problem (4.31) has a unique solution $\boldsymbol{\lambda} \in \mathscr{P}_{[-1-3/p'+\alpha]}$. Hence for any $\boldsymbol{q} \in \mathscr{P}_{[1-3/p'+\alpha]}$, we have $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{q} \in \mathscr{P}_{[-1-3/p'+\alpha]}$, so we can write

$$\int_{\Omega'} \mu(\boldsymbol{x})\,\boldsymbol{\lambda}(\boldsymbol{x})\cdot\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{q}(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\mathbb{R}^3} \boldsymbol{\psi}(\boldsymbol{x})\cdot\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{q}(\boldsymbol{x})\,d\boldsymbol{x}$$

which implies

$$\int_{\Omega'} \textbf{curl curl } \big(\mu(\boldsymbol{x})\boldsymbol{\lambda}(\boldsymbol{x})\big)\cdot \boldsymbol{q}(\boldsymbol{x})\, d\boldsymbol{x} = \langle \textbf{curl } \boldsymbol{w}, \boldsymbol{q}\rangle_{W^{-1,p}_{\alpha}(\mathbb{R}^3)\times W^{1,p'}_{-\alpha}(\mathbb{R}^3)}.$$

Setting now $\widetilde{\boldsymbol{w}} = \boldsymbol{w} - \textbf{curl}\,(\mu\boldsymbol{\lambda})$, then $\widetilde{\boldsymbol{w}} \in H^p_\alpha(\mathrm{div},\mathbb{R}^3)$ is an extension of $\boldsymbol{v}$ that satisfies div $\widetilde{\boldsymbol{w}} = 0$ and condition (4.27). Then Theorem 4.3 ends the proof. □

In order to ensure the uniqueness of the vector potential, we shall impose either the tangential component of the trace or the normal component of the trace to vanish. This will lead to additional constraints on the exponent of the weight function. We start with a result where the vanishing of the tangential component of the trace is imposed.

Theorem 4.9. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Let α and p satisfy ($\boldsymbol{H}$). Assume moreover that $\alpha < 3/p'$, $3/p+\alpha \neq 1$ and $3/p+\alpha \neq 2$. Then each function $\boldsymbol{v}$ in $H^p_\alpha(\mathrm{div},\Omega)$ that satisfies*

$$\mathrm{div}\,\boldsymbol{v} = 0 \quad in \quad \Omega \quad and \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad on \quad \partial\Omega$$

has a unique vector potential $\boldsymbol{\psi}$ in $W^{1,p}_\alpha(\Omega)/Y^p_{\alpha-1,N}(\Omega)$ such that

$$\boldsymbol{v} = \textbf{curl}\,\boldsymbol{\psi}, \quad \mathrm{div}\,\boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{\psi}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega. \tag{4.32}$$

Moreover it depends continuously on $\boldsymbol{v}$:

$$||\boldsymbol{\psi}||_{W^{1,p}_\alpha(\Omega)/Y^p_{\alpha-1,N}(\Omega)} \leqslant C||\boldsymbol{v}||_{W^{0,p}_\alpha(\Omega)}. \tag{4.33}$$

Proof - Let us prove the existence. Since $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, we can extend $\boldsymbol{v}$ by zero in Ω' and the extended function denoted by $\widetilde{\boldsymbol{v}}$ belongs to $H^p_\alpha(\mathrm{div},\mathbb{R}^3)$ and satisfies $\mathrm{div}\,\widetilde{\boldsymbol{v}} = 0$ in $\mathbb{R}^3$. Under the assumptions ($\boldsymbol{H}$) and $\alpha < 3/p'$, we can apply Theorem 4.2 which implies that $\widetilde{\boldsymbol{v}}$ has a vector potential $\boldsymbol{z} \in W^{1,p}_\alpha(\mathbb{R}^3)$. In addition, we have

$$||\boldsymbol{z}||_{W^{1,p}_\alpha(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}} \leqslant C||\widetilde{\boldsymbol{v}}||_{W^{0,p}_\alpha(\mathbb{R}^3)}. \tag{4.34}$$

Now, the fact that $\textbf{curl}\,\boldsymbol{z} = \boldsymbol{0}$ in Ω' implies that, $\boldsymbol{z} = \nabla\overline{\theta}$ in Ω' for some function $\overline{\theta} \in W^{2,p}(\Omega')/\mathbb{R}$. Let θ be a representative of $\overline{\theta}$. Then (4.34) yields

$$||\theta||_{W^{2,p}(\Omega')} \leqslant C||\boldsymbol{z}||_{W^{1,p}(\Omega')/\mathscr{G}_{[1-3/p-\alpha]}} \leqslant C||\boldsymbol{v}||_{W^{0,p}_\alpha(\Omega)}. \tag{4.35}$$

As a consequence, the trace of θ on $\partial\Omega$ satisfies

$$||\theta_{|\partial\Omega}||_{W^{2-1/p,p}(\partial\Omega)} \leqslant C||\theta||_{W^{2,p}(\Omega')} \leqslant C||\boldsymbol{v}||_{W^{0,p}_\alpha(\Omega)}. \tag{4.36}$$

In order to satisfy the boundary condition in (4.32), we solve the following Dirichlet problem for Laplace:

$$\Delta\chi = 0 \quad \text{in} \quad \Omega, \quad \chi = \theta \quad \text{on} \quad \partial\Omega. \tag{4.37}$$

Since $\theta|_{\partial\Omega}$ belongs to $W^{2-1/p,p}(\partial\Omega)$, under the assumptions on α and p , it follows from Theorem 3.7 and Remark 2 that this problem has a solution χ in $W^{2,p}_{\alpha}(\Omega)$ unique up to an element of $\mathscr{A}^{\Delta}_{\alpha-1,p}(\Omega)$ such that

$$||\chi||_{W^{2,p}_{\alpha}(\Omega)/\mathscr{A}^{\Delta}_{\alpha-1,p}(\Omega)} \leq C||\theta|_{\partial\Omega}||_{W^{2-1/p,p}(\partial\Omega)}. \tag{4.38}$$

Now, every solution χ of problem (4.37) satisfies $\nabla\chi\times\boldsymbol{n} = \boldsymbol{z}\times\boldsymbol{n}$ on $\partial\Omega$. Indeed, for each tangent vector $\boldsymbol{\tau}$ along $\partial\Omega$ we have $\nabla\chi\cdot\boldsymbol{\tau} = \nabla\theta\cdot\boldsymbol{\tau} = \boldsymbol{z}\cdot\boldsymbol{\tau}$. It follows that $\nabla\chi-(\nabla\chi\cdot\boldsymbol{n})\cdot\boldsymbol{n} = z-(z\cdot\boldsymbol{n})\cdot\boldsymbol{n}$ which implies $\nabla\chi\times\boldsymbol{n} = \boldsymbol{z}\times\boldsymbol{n}$ on $\partial\Omega$.

Therefore it remains to set $\boldsymbol{\psi} = \boldsymbol{z}-\nabla\chi$ is the required vector potential of $\boldsymbol{v}$. Equality (4.33) follows from (4.34), (4.36) and (4.38). Finally, the uniqueness follows immediately from the definition of the space $Y^{p}_{\alpha-1,N}(\Omega)$. □

We end this section with a result where we impose the normal component of the trace of the vector potential to vanish.

Theorem 4.10. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$ if $p\neq 2$ and Lipschitz-continuous if $p=2$. Let α and p satisfy (**H**). Assume moreover that $\alpha<1+3/p'$, $3/p+\alpha\neq 1$ and $3/p+\alpha\neq 2$. Then each function $\boldsymbol{v}\in H^{p}_{\alpha}(\mathrm{div},\Omega)$ that satisfies*

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\partial\Omega} = 0,$$

has a unique vector $\boldsymbol{\psi}$ in $W^{1,p}_{\alpha}(\Omega)/Y^{p}_{\alpha-1,T}(\Omega)$ such that

$$\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\psi}, \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in}\ \Omega \quad \text{and} \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega. \tag{4.39}$$

Moreover we have

$$\|\boldsymbol{\psi}\|_{W^{1,p}_{\alpha}(\Omega)/Y^{p}_{\alpha-1,T}(\Omega)} \leqslant C\|\boldsymbol{v}\|_{W^{0,p}_{\alpha}(\Omega)}. \tag{4.40}$$

Proof - Let $\boldsymbol{\varphi}$ be the vector potential of $\boldsymbol{v}$ constructed in Theorem 4.8 and let us solve the following problem

$$\Delta z = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial z}{\partial\boldsymbol{n}} = \boldsymbol{\varphi}\cdot\boldsymbol{n} \quad \text{on} \quad \partial\Omega. \tag{4.41}$$

Since $\boldsymbol{\varphi}\in W^{1,p}_{\alpha}(\Omega)$ and $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω we deduce that $\boldsymbol{\varphi}\cdot\boldsymbol{n}$ belongs to $W^{1/p',p}(\partial\Omega)$. The proof now is splitted into two cases.

(i) The case $\alpha<3/p'$. With this assumption, $\mathscr{N}^{\Delta}_{1-\alpha,p'}(\Omega)$ is reduced to $\{0\}$ which implies that $\boldsymbol{\varphi}\cdot\boldsymbol{n}$ has no compatibility condition to satisfy. Therefore applying Theorem 3.11 problem (4.41) has a solution z

in $W^{1,p}_{\alpha-1}(\Omega)$.

(ii) The case $3/p' \leq \alpha < 1+3/p'$. It follows from Proposition 3.10 that $\mathcal{N}^{\Delta}_{1-\alpha,p'}(\Omega) = \mathbb{R}$. Thus problem (4.41) does not have a solution in $W^{1,p}_{\alpha-1}(\Omega)$ unless $\boldsymbol{\varphi}\cdot\boldsymbol{n}$ satisfies the necessary condition $\langle\boldsymbol{\varphi}\cdot\boldsymbol{n},1\rangle_{\partial\Omega} = 0$. But, for any $w \in W^{1,p'}_{1-\alpha}(\Omega)$ the following Green's formula holds:

$$\langle\boldsymbol{\varphi}\cdot\boldsymbol{n}, w\rangle_{\partial\Omega} = \int_{\Omega} \varphi\cdot\nabla w \, d\,x. \tag{4.42}$$

Since $3/p' \leq \alpha < 1+3/p'$, the constants belong to $W^{1,p'}_{1-\alpha}(\Omega)$ then (4.42) is still valid for $w = 1$ and thus we have

$$\langle\boldsymbol{\varphi}\cdot\boldsymbol{n},1\rangle_{\partial\Omega} = 0.$$

It follows from Theorem 3.12 that there exists z in $W^{1,p}_{\alpha-1}(\Omega)$ solution of problem (4.41).

In both cases, thanks to regularity results (see Remark 3) we deduce that z belongs to $W^{2,p}_{\alpha}(\Omega)$. Therefore it remains to set $\boldsymbol{\psi} = \boldsymbol{\varphi} - \nabla z$ is the required vector potential of $\boldsymbol{v}$ and it satisfies (4.40).
The proof of uniqueness follows the same lines as in the proof of Theorem 4.9. □

5 Applications of the vector potential results

We state here some important results that are obtained as consequences of the vector potential results.

5.1 Weighted Sobolev's inequalities

Theorem 5.1. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Let α and p satisfy (**H**). Assume moreover that $\alpha < 3/p'$, $3/p+\alpha \neq 1$ and $3/p+\alpha \neq 2$. Then the space $X^p_{\alpha-1,N}(\Omega)$ is continuously imbedded in $W^{1,p}_{\alpha}(\Omega)$. In addition for any $\boldsymbol{\varphi} \in X^p_{\alpha-1,N}(\Omega)$, we have*

$$\|\boldsymbol{\varphi}\|_{W^{1,p}_{\alpha}(\Omega)} \leq C\left\{||\boldsymbol{\varphi}||^p_{W^{0,p}_{\alpha-1}(\Omega)} + ||\mathrm{div}\,\boldsymbol{\varphi}||^p_{W^{0,p}_{\alpha}(\Omega)} + ||\mathbf{curl}\,\boldsymbol{\varphi}||^p_{W^{0,p}_{\alpha}(\Omega)}\right\}^{1/p}. \tag{5.43}$$

Proof - We shall use a partition of unity. Let $R > 0$ be a real number large enough so that $\overline{\Omega'} \subset B_R$. Let λ and μ be two scalar, nonnegative functions in $C^{\infty}(\mathbb{R}^3)$ that satisfy

$$\forall\ x \in B_R,\quad \lambda(x) = 1,\quad \mathrm{supp}\,\lambda \subset B_{R+1},\quad \forall\ x \in \mathbb{R}^3,\quad \lambda(x)+\mu(x) = 1.$$

Then for any $\boldsymbol{\varphi} \in X^p_{\alpha-1,N}(\Omega)$ we can write $\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi} + \mu\boldsymbol{\varphi}$, where $\lambda\boldsymbol{\varphi}$ has its support in Ω_{R+1} and $\mu\boldsymbol{\varphi}$ in C_R. Consider first $\mu\boldsymbol{\varphi}$. Its domain C_R is simply-connected and has a very smooth and connected boundary ∂C_R. Next, $\mathrm{div}(\mu\boldsymbol{\varphi}) = \mu\,\mathrm{div}\,\boldsymbol{\varphi} + \boldsymbol{\varphi}\cdot\nabla\mu$ belongs to $W^{0,p}_{\alpha}(C_R)$. Therefore, applying Proposition 3.2, there exists $\boldsymbol{w}$ in

$W_\alpha^{1,p}(C_R)$, that vanishes on ∂C_R, such that

$$\operatorname{div} \boldsymbol{w} = \operatorname{div}(\mu \boldsymbol{\varphi}) \quad \text{in} \quad \Omega$$

and

$$\begin{aligned} \|\boldsymbol{w}\|_{W_\alpha^{1,p}(C_R)} &\leqslant C\|\operatorname{div}(\mu \boldsymbol{\varphi})\|_{W_\alpha^{0,p}(C_R)} \\ &\leqslant C\left(\|\operatorname{div} \boldsymbol{\varphi}\|_{W_\alpha^{0,p}(C_R)} + \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(B_{R+1}\setminus B_R)}\right). \end{aligned}$$

Let $\boldsymbol{\psi} = \mu\boldsymbol{\varphi} - \boldsymbol{w} - \nabla z(\lambda)$, where $z(\lambda) \in \mathscr{A}^{\Delta}_{\alpha-1,p}(C_R)$. Then

$$\operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in} \quad C_R, \quad \boldsymbol{\psi} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \partial C_R.$$

Set $\boldsymbol{v} = \operatorname{\mathbf{curl}} \boldsymbol{\psi}$, then $\boldsymbol{v} = \mu \operatorname{\mathbf{curl}} \boldsymbol{\varphi} + \nabla \mu \times \boldsymbol{\varphi} - \operatorname{\mathbf{curl}} \boldsymbol{w}$ belongs to $W_\alpha^{0,p}(C_R)$, and

$$\|\boldsymbol{v}\|_{W_\alpha^{0,p}(C_R)} \leqslant C\left(\|\boldsymbol{w}\|_{W_\alpha^{1,p}(C_R)} + \|\operatorname{\mathbf{curl}} \boldsymbol{\varphi}\|_{W_\alpha^{0,p}(C_R)} + \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(B_{R+1}\setminus B_R)}\right).$$

In addition, the function $\boldsymbol{v}$ satisfies

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad C_R \quad \text{and} \quad \boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \partial C_R.$$

Then the uniqueness of the vector potential defined by Theorem 4.9 implies that $\boldsymbol{\psi}$ belongs to $W_\alpha^{1,p}(C_R)/Y^p_{\alpha-1,N}(C_R)$ and

$$\|\boldsymbol{\psi}\|_{W_\alpha^{1,p}(C_R)/Y^p_{\alpha-1,N}(C_R)} \leqslant C\|\boldsymbol{v}\|_{W_\alpha^{1,p}(C_R)}.$$

In turn, this implies that $\mu\boldsymbol{\varphi}$ belongs to $W_\alpha^{1,p}(C_R)$ and

$$\|\mu\boldsymbol{\varphi}\|_{W_\alpha^{1,p}(C_R)} \leqslant C\left(\|\operatorname{div} \boldsymbol{\varphi}\|_{W_\alpha^{0,p}(C_R)} + \|\operatorname{\mathbf{curl}} \boldsymbol{\varphi}\|_{W_\alpha^{0,p}(C_R)} + \|\boldsymbol{\varphi}\|_{L^p(B_{R+1}\setminus B_R)}\right). \tag{5.44}$$

Finally, consider $\lambda\boldsymbol{\varphi}$. Its domain Ω_{R+1} is of class $C^{1,1}$. Next, $\lambda\boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0}$ on $\partial\Omega_{R+1}$, because $\lambda\boldsymbol{\varphi}$ vanishes on ∂B_{R+1} and $\boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0}$ on $\partial\Omega$. Applying [8] (Theorem 3.2), implies that $\lambda\boldsymbol{\varphi}$ belongs to $\boldsymbol{W}^{1,p}(\Omega_{R+1})$ and

$$\|\lambda\boldsymbol{\varphi}\|_{W^{1,p}(\Omega_{R+1})} \leqslant C(\|\operatorname{div}(\lambda\boldsymbol{\varphi})\|_{L^p(\Omega_{R+1})} + \|\operatorname{\mathbf{curl}}(\lambda\boldsymbol{\varphi})\|_{L^p(\Omega_{R+1})} + \|\lambda\boldsymbol{\varphi}\|_{L^p(\Omega_{R+1})}). \tag{5.45}$$

Combining these two results, we derive that $\boldsymbol{\varphi}$ belongs to $W_\alpha^{1,p}(\Omega)$ and (5.43) follows from (5.44) and (5.45). □

Our second imbedding result is stated in the following theorem:

Theorem 5.2. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Assume moreover that $\alpha < 1 + 3/p'$, $3/p + \alpha \neq 1$ and $3/p + \alpha \neq 2$. Then the space $X^p_{\alpha-1,T}(\Omega)$*

is continuously imbedded in $W^{1,p}_\alpha(\Omega)$. *In addition there exists a constant C such that for any* $\boldsymbol{\varphi} \in X^p_{\alpha-1,T}(\Omega)$,

$$\|\boldsymbol{\varphi}\|_{W^{1,p}_\alpha(\Omega)} \leq C\left\{||\boldsymbol{\varphi}||^p_{W^{0,p}_{\alpha-1}(\Omega)} + ||\operatorname{div}\boldsymbol{\varphi}||^p_{W^{0,p}_\alpha(\Omega)} + ||\mathbf{curl}\,\boldsymbol{\varphi}||^p_{W^{0,p}_\alpha(\Omega)}\right\}^{1/p}. \tag{5.46}$$

We skip the proof as it is the same as the one of Theorem 5.1. The only diffrence is that here we need the second type of vector potential proved in Theorem 4.10.

5.2 The Helmholtz decomposition

Another application of the vector potential problem is the Helmholtz decomposition. We start with a result in the whole space $\mathbb{R}^3$ and we state an extention to exterior domains.

Theorem 5.3. *Let* α *and* p *satisfy (**H**). Assume moreover that* $\alpha < 3/p'$ *and let* $\boldsymbol{g} \in W^{0,p}_\alpha(\mathbb{R}^3)$. *Then* $\boldsymbol{g}$ *has the unique decomposition*

$$\boldsymbol{g} = \nabla\varphi + \mathbf{curl}\,\boldsymbol{\psi},$$

where $\varphi \in W^{1,p}_\alpha(\mathbb{R}^3)/\mathscr{P}^\Delta_{[1-3/p-\alpha]}$ *and* $\boldsymbol{\psi} \in V^{1,p}_\alpha(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}$. *Moreover, we have the estimate*

$$\|\boldsymbol{\varphi}\|_{W^{1,p}_\alpha(\mathbb{R}^3)/\mathscr{P}^\Delta_{[1-3/p-\alpha]}} + \|\boldsymbol{\psi}\|_{W^{1,p}_\alpha(\mathbb{R}^3)/\mathscr{G}_{[1-3/p-\alpha]}} \leq C\|\boldsymbol{g}\|_{W^{0,p}_\alpha(\mathbb{R}^3)}. \tag{5.47}$$

Proof - Since $\boldsymbol{g} \in W^{0,p}_\alpha(\mathbb{R}^3)$, then $\operatorname{div}\boldsymbol{g} \in W^{-1,p}_\alpha(\mathbb{R}^3)$. Next since $\alpha < 3/p'$, the polynomial space $\mathscr{P}_{[1-3/p'+\alpha]}$ contains at most constants. Therefore $\operatorname{div}\boldsymbol{g}$ is orthogonal to $\mathscr{P}_{[1-3/p'+\alpha]}$. Thanks to Theorem 2.2, there exists a unique $\varphi \in W^{1,p}_\alpha(\mathbb{R}^3)/\mathscr{P}^\Delta_{[1-3/p-\alpha]}$ such that

$$\Delta\varphi = \operatorname{div}\boldsymbol{g} \quad \text{in} \quad \mathbb{R}^3.$$

Then by construction $\boldsymbol{g} - \nabla\varphi \in W^{0,p}_\alpha(\mathbb{R}^3)$ and $\operatorname{div}(\boldsymbol{g} - \nabla\varphi) = 0$ in $\mathbb{R}^3$. Therefore from Theorem 4.2, there exists a unique vector potential $\boldsymbol{\psi} \in V^{1,p}_\alpha(\Omega)/\mathscr{G}_{[1-3/p-\alpha]}$ such that $\boldsymbol{g} - \nabla\varphi = \mathbf{curl}\,\boldsymbol{\psi}$. □

Theorem 5.4. *Let* Ω *be an exterior domain of* $\mathbb{R}^3$ *with boundary* $\partial\Omega$ *of class* $\mathscr{C}^{1,1}$ *if* $p \neq 2$ *and Lipschitz-continuous if* $p = 2$. *Let* α *and* p *satisfy (**H**). Assume moreover that* $\alpha < 3/p'$, $3/p + \alpha \neq 1$, $3/p + \alpha \neq 2$ *and let* $\boldsymbol{g} \in W^{0,p}_\alpha(\Omega)$. *Then* $\boldsymbol{g}$ *has the decomposition*

$$\boldsymbol{g} = \nabla\varphi + \mathbf{curl}\,\boldsymbol{\psi},$$

where $\varphi \in W^{1,p}_\alpha(\Omega)$ *and* $\boldsymbol{\psi} \in V^{1,p}_\alpha(\Omega)$, *such that* $\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0}$ *on* $\partial\Omega$.
In addition, there exists $\lambda \in \mathscr{N}^\Delta_{\alpha,p}(\Omega)$ *and* $q \in \mathscr{P}^\Delta_{[1-3/p-\alpha]}$ *such that*

$$\|\varphi + \lambda + q\|_{W^{1,p}_\alpha(\Omega)} + \|\boldsymbol{\psi}\|_{W^{1,p}_\alpha(\Omega)/Y^p_{\alpha-1,N}(\Omega)} \leq C||\boldsymbol{g}||_{W^{0,p}_\alpha(\Omega)}.$$

Proof - Let us extend $\boldsymbol{g}$ by zero in Ω' and let $\widetilde{\boldsymbol{g}}$ denote the extended function. Then $\widetilde{\boldsymbol{g}} \in W^{0,p}_\alpha(\mathbb{R}^3)$ and

$\operatorname{div}\widetilde{\boldsymbol{g}} \in W^{-1,p}_{\alpha}(\mathbb{R}^3)$. Since $\alpha < 3/p'$, then Theorem 2.2 implies that there exists a unique function $\widetilde{v} \in W^{1,p}_{\alpha}(\mathbb{R}^3)/\mathscr{P}^{\Delta}_{[1-3/p-\alpha]}$ such that

$$\Delta\widetilde{v} = \operatorname{div}\widetilde{\boldsymbol{g}} \quad \text{in} \quad \mathbb{R}^3,$$

$$\|\widetilde{v}\|_{W^{1,p}_{\alpha}(\mathbb{R}^3)/\mathscr{P}^{\Delta}_{[1-3/p-\alpha]}} \leq C\|\widetilde{\boldsymbol{g}}\|_{W^{0,p}_{\alpha}(\mathbb{R}^3)} = C\|\boldsymbol{g}\|_{W^{0,p}_{\alpha}(\Omega)}.$$

Then $\widetilde{\boldsymbol{g}} - \nabla\widetilde{v} \in W^{0,p}_{\alpha}(\mathbb{R}^3)$ and $\operatorname{div}\big(\widetilde{\boldsymbol{g}} - \nabla\widetilde{v}\big) = 0$ in $\mathbb{R}^3$. Therefore $\big(\widetilde{\boldsymbol{g}} - \nabla\widetilde{v}\big)\cdot\boldsymbol{n}$ belongs to $W^{-1/p,p}(\partial\Omega)$. Now let us solve the following problem

$$\Delta w = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial w}{\partial \boldsymbol{n}} = \big(\widetilde{\boldsymbol{g}} - \nabla\widetilde{v}\big)\cdot\boldsymbol{n} \quad \text{on} \quad \partial\Omega. \tag{5.48}$$

The proof now is splitted into two cases.

(i) The case $\alpha < -1 + 3/p'$. With this assumption, $\big(\widetilde{\boldsymbol{g}} - \nabla\widetilde{v}\big)\cdot\boldsymbol{n}$ has no compatibility condition to satisfy since $\mathcal{N}^{\Delta}_{-\alpha,p'}(\Omega)$ is reduced to $\{0\}$. Therefore applying Theorem 3.11 problem (5.48) has a solution w in $W^{1,p}_{\alpha}(\Omega)$.

(ii) The case $-1 + 3/p' \leq \alpha < 3/p'$. It follows from Proposition 3.10 that $\mathcal{N}^{\Delta}_{-\alpha,p'}(\Omega) = \mathbb{R}$ and proceeding as in the proof of Theorem 4.10, we can prove that

$$\langle\big(\widetilde{\boldsymbol{g}} - \nabla\widetilde{v}\big)\cdot\boldsymbol{n}, 1\rangle_{\partial\Omega} = 0.$$

Therefore applying Theorem 3.12 there exists $w \in W^{1,p}_{\alpha}(\Omega)$ solution of problem (5.48).

In both cases we have

$$\|w\|_{W^{1,p}_{\alpha}(\Omega)/\mathcal{N}^{\Delta}_{\alpha,p}(\Omega)} \leq C\|\boldsymbol{g}\|_{W^{0,p}_{\alpha}(\Omega)}.$$

Let v be the restriction of $\widetilde{v}$ to Ω, then setting $\varphi = v + w$ and $\boldsymbol{z} = \boldsymbol{g} - \nabla\varphi$ then $\boldsymbol{z}$ belongs to $W^{0,p}_{\alpha}(\Omega)$, $\operatorname{div}\boldsymbol{z} = 0$ in Ω and $\boldsymbol{z}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. It follows from Theorem 4.9 that there exists a unique vector potential $\boldsymbol{\psi}$ in $W^{1,p}_{\alpha}(\Omega)/Y^{p}_{\alpha-1,N}(\Omega)$ such that

$$\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\psi}, \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad \boldsymbol{\psi}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega. \quad \square$$

Theorem 5.5. *Let Ω be an exterior domain of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^{1,1}$ if $p \neq 2$ and Lipschitz-continuous if $p = 2$. Let α and p satisfy (**H**). Assume moreover that $\alpha < 3/p'$, $3/p + \alpha \neq 1$, $3/p + \alpha \neq 2$ and let $\boldsymbol{g} \in W^{0,p}_{\alpha}(\Omega)$. Then $\boldsymbol{g}$ has the decomposition*

$$\boldsymbol{g} = \nabla\varphi + \mathbf{curl}\,\boldsymbol{\psi},$$

where $\varphi \in W^{1,p}_{\alpha}(\Omega)$ and $\boldsymbol{\psi} \in V^{1,p}_{\alpha}(\Omega)$, such that $\boldsymbol{\psi}\cdot\boldsymbol{n} = \boldsymbol{0}$ on $\partial\Omega$.

In addition, there exists $\lambda \in \mathcal{N}^{\Delta}_{\alpha,p}(\Omega)$ *and* $q \in \mathscr{P}^{\Delta}_{[1-3/p-\alpha]}$ *such that*

$$\|\varphi + \lambda + q\|_{W^{1,p}_{\alpha}(\Omega)} + \|\boldsymbol{\psi}\|_{W^{1,p}_{\alpha}(\Omega)/Y^{p}_{\alpha-1,T}(\Omega)} \leq C\|\boldsymbol{g}\|_{W^{0,p}_{\alpha}(\Omega)}.$$

Proof - For the proof, one can use the same arguments as for the proof of Theorem 5.4. Note that since the vector field $\boldsymbol{z}$, constructed in the previous proof, belongs to $W^{0,p}_{\alpha}(\Omega)$, satisfies $\operatorname{div}\boldsymbol{z} = 0$ in Ω and $\boldsymbol{z}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, then it also satisfies $\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\partial\Omega} = 0$. From here, Theorem 4.10 yields the existence of a unique vector potential $\boldsymbol{\psi}$ in $W^{1,p}_{\alpha}(\Omega)/Y^{p}_{\alpha-1,T}(\Omega)$ such that

$$\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\psi}, \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega. \quad \square$$

6 Conclusions

This paper solves the div-curl problem in exterior domains. In order to describe the asymptotic behaviour at infinity of the solutions, we set the problem in weighted L^p spaces. We proved the existence of vector potentials that may grow or decay at infinity. To insure the uniqueness of the vector potential, we impose either the tangential component of the trace or the normal component of the trace to vanish on the boundary of the domain. Moreover we also impose a sufficient growth at infinity of the vector potential.

In this paper, we have only considered an exterior domain Ω that is the complement of a simply-connected bounded obstacle $\overline{\Omega'}$. The results stated here can be extended and adapted to Ω' that is not necessarily connected, but has a finite number of connected components where each of them has a connected boundary.

Another possible extension of this paper is the extension to exterior domains where the first two Betti numbers do not vanish. The study has been done for bounded domains in [8, 19]. As far as we know, one of the first study in exterior domains has been done in [21] where the vector potential is assumed to be regular enough (at least of class $\mathscr{C}^1$).

Appendices

A Proof of Proposition 2.1

Let us assume $3/p + \alpha \neq 1$ and $3/q + \beta \neq 1$. The proofs of the other cases are similar. Next without loss of generality we may assume that $q > p$. Now let λ be a polynomial in $W^{1,p}_{\alpha}(\Omega) + W^{1,q}_{\beta}(\Omega)$. Then there exist $u \in W^{1,p}_{\alpha}(\Omega)$ and $v \in W^{1,q}_{\beta}(\Omega)$ such that $\lambda = u + v$. Let $R > 1$ be a real number such that $\overline{\Omega'} \subset B_R$. Then we can write

$$\begin{aligned}\|\lambda(R,.)\|_{L^p(\partial B_1)} &\leq \|u(R,.)\|_{L^p(\partial B_1)} + \|v(R,.)\|_{L^p(\partial B_1)}\\ &\leq \|u(R,.)\|_{L^p(\partial B_1)} + C\|v(R,.)\|_{L^q(\partial B_1)}.\end{aligned}$$

The second inequality holds due to the fact that $q > p$. Now, thanks to (2.8), we obtain

$$\|\lambda(R,.)\|_{L^p(\partial B_1)} \leq C\left(R^{1-3/p-\alpha} + R^{1-3/q-\beta}\right) \leq CR^{\gamma},$$

where $\gamma = \max\left(1-3/p-\alpha, 1-3/q-\beta\right)$. Assume now that $\lambda(R,.) = R^j$ for an integer j. Then we have

$$\|\lambda(R,.)\|_{L^p(\partial B_1)} = (4\pi)^{1/p} R^j.$$

Hence, this shows that if λ is a polynomial, its degree can not exceed γ.

B Proof of Proposition 3.5

Let z be in $\mathscr{A}^{\Delta}_{\alpha,p}(\Omega)$. Then for any bounded domain $\mathscr{O}$ such that $\overline{\mathscr{O}} \subset \Omega$, z belongs to $W^{1,p}(\mathscr{O})$ and Δz belongs to $L^p(\mathscr{O})$. Moreover, due to inner regularity results z also belongs to $W^{1,2}(\mathscr{O})$ and Δz also belongs to $L^2(\mathscr{O})$. Thus $\dfrac{\partial z}{\partial \boldsymbol{n}}$ belongs to $W^{-1/p,p}(\partial\Omega) \cap W^{-1/2,2}(\partial\Omega)$. Let us extend z by zero in Ω'. The extended function, denoted by $\tilde{z}$ belongs to $W^{1,p}_{\alpha}(\mathbb{R}^3)$ and for any $\varphi \in \mathscr{D}(\mathbb{R}^3)$, we have

$$\langle \Delta\tilde{z}, \varphi\rangle = -\left\langle \frac{\partial z}{\partial \boldsymbol{n}}, \varphi\right\rangle_{\partial\Omega},$$

where $\langle .,.\rangle_{\partial\Omega}$ not only denotes the duality pairing between $W^{-1/p,p}(\partial\Omega)$ and $W^{1/p,p'}(\partial\Omega)$ but also the duality pairing between $W^{-1/2,2}(\partial\Omega)$ and $W^{1/2,2}(\partial\Omega)$. This shows that $\Delta\tilde{z}$ belongs to $W^{-1,p}_{\alpha}(\mathbb{R}^3) \cap W^{-1,2}_{0}(\mathbb{R}^3)$ and has a compact support. Then, thanks to Theorem 2.2, there exists a unique $\tilde{v} \in W^{1,2}_{0}(\mathbb{R}^3)$ satisfying

$$\Delta\tilde{v} = \Delta\tilde{z} \quad \text{in} \quad \mathbb{R}^3.$$

Hence $\tilde{v} - \tilde{z}$ is a harmonic tempered distribution and we deduce that $\tilde{v} = \tilde{z} + q$ where q is a polynomial of $W^{1,2}_{0}(\mathbb{R}^3) + W^{1,p}_{\alpha}(\mathbb{R}^3)$. But since there are no polynomials in the space $W^{1,2}_{0}(\mathbb{R}^3)$, thanks to Proposition 2.1, q belongs to $\mathscr{P}_{[1-3/p-\alpha]} \subset W^{1,p}_{\alpha}(\mathbb{R}^3)$. Thus, $\tilde{v}$ belongs to $W^{1,2}_{0}(\mathbb{R}^3) \cap W^{1,p}_{\alpha}(\mathbb{R}^3)$, its restriction to Ω belongs to $W^{1,2}_{0}(\Omega) \cap W^{1,p}_{\alpha}(\Omega)$ and satisfies (3.17). The fact that $\mathscr{A}^{\Delta}_{\alpha,p}(\Omega) = \{0\}$ if $\alpha > 1-3/p$ follows from the fact that in this case, $\mathscr{P}^{\Delta}_{[1-3/p-\alpha]} = \{0\}$ and from the uniqueness of v.

C Proof of Theorem 3.6

Let R be chosen so that $\overline{\Omega'}$ is contained in B_R. Let v be the lifting function of g satisfying:

$$\Delta\, v = 0 \quad \text{in} \quad \Omega_R, \quad v = g \quad \text{on} \quad \partial\Omega, \quad v = 0 \quad \text{on} \quad \partial B_R.$$

This set of equations defines a unique function v in $W^{1,p}(\Omega_R)$. Extending v by zero outside B_R, the extended function, still denoted by v, belongs to $W^{1,p}_{\beta}(\Omega)$ for any $\beta \in \mathbb{R}$, so in particular, v belong to $W^{1,p}_{\alpha}(\Omega)$. Then problem (3.18) is equivalent to

$$-\Delta z = \Delta v \quad \text{in} \quad \Omega, \quad z = 0 \quad \text{on} \quad \partial\Omega, \tag{C.49}$$

where, using the same arguments as in the proof of Proposition 3.5, Δv belongs to $W_\alpha^{-1,p}(\Omega)$ and has a bounded support. The proof now is splitted into two cases.

(i) The case $p > 3/2$. Owing to the support of Δv, Δv is also in $W_0^{-1,p}(\Omega)$. Applying Theorem 2.10 of [6], problem (C.49) has a solution z in $W_0^{1,p}(\Omega)$. Note that in the case $p > 3/2$, we have $\mathscr{A}^{\Delta}_{0,p'}(\Omega) = \{0\}$ which implies that Δz has no compatibility condition to satisfy. We thus deduce that $u = z + v \in W_0^{1,p}(\Omega)$ is a solution of problem (3.18). Finally since $\alpha < 0$, then u also belongs to $W_\alpha^{1,p}(\Omega)$.

(ii) The case $1 < p \le 3/2$. Due to inner regularity results, for any bounded domain $\mathscr{O}$ such that $\overline{\mathscr{O}} \subset \Omega_R \cup \partial B_R$, $v \in W^{1,2}(\mathscr{O})$ and $\Delta v \in L^2(\mathscr{O})$ and consequently $\dfrac{\partial v}{\partial \boldsymbol{n}}$ belongs to $W^{-1/2,2}(\partial\Omega)$. Extending v by zero outside B_R and still denoting by v the extended function, then Δv belongs to $W_0^{-1,2}(\Omega)$ and thus there exists w in $W_0^{1,2}(\Omega)$ such that

$$-\Delta w = \Delta v \quad \text{in} \quad \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega.$$

Let us extend w by zero in Ω' and let $\widetilde{w}$ denote the extended function. Then $\Delta \widetilde{w}$ belongs to $W_0^{-1,2}(\mathbb{R}^3)$ and has a bounded support and since $1 < p \le 3/2$, this implies that $\Delta \widetilde{w}$ also belongs to $W_\alpha^{-1,p}(\mathbb{R}^3)$. Next since $\alpha < 0$, then thanks to Theorem 2.2, there exists $\widetilde{z}$ in $W_\alpha^{1,p}(\mathbb{R}^3)$ such that

$$\Delta \widetilde{z} = \Delta \widetilde{w} \quad \text{in} \quad \mathbb{R}^3.$$

Hence $\widetilde{z} - \widetilde{w}$ is a harmonic tempered distribution and therefore a polynomial of $\mathscr{P}^{\Delta}_{[1-\alpha-3/p]} \subset W_\alpha^{1,p}(\mathbb{R}^3)$. Thus w belongs to $W_\alpha^{1,p}(\Omega)$ and $u = w + v \in W_\alpha^{1,p}(\Omega)$ is the required solution to problem (3.18).

D Proof of Proposition 3.10

The proof uses similar ideas than the proof of the characterization of the kernel $\mathscr{A}^{\Delta}_{\alpha,p}(\Omega)$. Let z be in $\mathscr{N}^{\Delta}_{\alpha,p}(\Omega)$ and let $z' \in W^{1,p}(\Omega')$ be the unique solution of

$$\Delta z' = 0 \quad \text{in} \quad \Omega', \quad z' = z \quad \text{on} \quad \partial\Omega.$$

Note that $\dfrac{\partial z'}{\partial \boldsymbol{n}}$ belongs to $W^{-1/p,p}(\partial\Omega)$. Next due to inner regularity results, for any open set $\mathscr{O}$ such that $\overline{\mathscr{O}} \subset \Omega'$, we have in particular $z' \in W^{1,2}(\mathscr{O})$ and $\Delta z' \in L^2(\mathscr{O})$. This shows that $\dfrac{\partial z'}{\partial \boldsymbol{n}}$ also belongs to $W^{-1/2,2}(\partial\Omega)$.

Set now

$$\widetilde{z} = \begin{cases} z & \text{in} \quad \Omega \\ z' & \text{in} \quad \Omega' \end{cases} .$$

Then $\widetilde{z}$ belongs to $W^{1,p}_{\alpha}(\mathbb{R}^3)$ and due to the previous arguments, $\Delta\widetilde{z}$ belongs to $W^{-1,2}_{0}(\mathbb{R}^3) \cap W^{-1,p}_{\alpha}(\mathbb{R}^3)$ and has a compact support. Therefore there exists a unique $\widetilde{\omega} \in W^{1,2}_{0}(\mathbb{R}^3)$ satisfying

$$\Delta\widetilde{\omega} = \Delta\widetilde{z} \quad \text{in} \quad \mathbb{R}^3.$$

Since there are no polynomials in $W^{1,2}_{0}(\mathbb{R}^3)$, thanks to Proposition 2.1, $\widetilde{w} = \widetilde{z} + q$ where $q \in \mathscr{P}^{\Delta}_{[1-3/p-\alpha]}$. This shows that $\widetilde{\omega}$ also belongs to $W^{1,p}_{\alpha}(\mathbb{R}^3)$ and its restriction to Ω denoted by ω belongs to $W^{1,p}_{\alpha}(\Omega)$ and satisfies (3.23).

Now if $\alpha > 1 - 3/p$, then $\mathscr{P}_{[1-3/p-\alpha]} = \{0\}$ and as a result of the uniqueness of ω, $\mathscr{N}^{\Delta}_{\alpha,p}(\Omega) = \{0\}$. By the same way, if $-3/p < \alpha \leq 1 - 3/p$, then $\mathscr{P}_{[1-3/p-\alpha]} = \mathbb{R}$ and as a result $\mathscr{N}^{\Delta}_{\alpha,p}(\Omega) = \mathbb{R}$.

E Proof of Theorem 3.11

The proof is splitted into two cases.

(i) The case $p > 2$. Applying Proposition 3.5 of [6], Problem (3.22) has a solution u in $W^{1,2}_{0}(\Omega) \cap W^{1,p}_{0}(\Omega)$. It remains now to prove that u belongs to $W^{1,p}_{\alpha}(\Omega)$. Let $u' \in W^{1,2}(\Omega') \cap W^{1,p}(\Omega')$ be the unique solution of

$$\Delta u' = 0 \quad \text{in} \quad \Omega', \quad u' = u \quad \text{on} \quad \partial\Omega. \tag{E.50}$$

Set now

$$\widetilde{u} = \begin{cases} u & \text{in} \quad \Omega, \\ u' & \text{in} \quad \Omega'. \end{cases} \tag{E.51}$$

Then $\widetilde{u}$ belongs to $W^{1,2}_{0}(\mathbb{R}^3) \cap W^{1,p}_{0}(\mathbb{R}^3)$, $\Delta\widetilde{u}$ belongs to $W^{-1,2}_{0}(\mathbb{R}^3) \cap W^{-1,p}_{0}(\mathbb{R}^3)$ and has a compact support. Therefore, $\Delta\widetilde{u}$ also belongs to $W^{-1,p}_{\alpha}(\mathbb{R}^3)$. Since $\alpha < -1 + 3/p'$, then thanks to Theorem 2.2, there exists $\widetilde{z} \in W^{1,p}_{\alpha}(\mathbb{R}^3)$ such that

$$\Delta\widetilde{z} = \Delta\widetilde{u} \quad \text{in} \quad \mathbb{R}^3.$$

This shows that $\widetilde{z} - \widetilde{u}$ is a harmonic tempered distribution and therefore it is a polynomial that belongs in particular to $W^{1,2}_{0}(\mathbb{R}^3) + W^{1,p}_{\alpha}(\mathbb{R}^3)$. Thanks to Proposition 2.1, $\widetilde{z} - \widetilde{u}$ belongs to $W^{1,p}_{\alpha}(\mathbb{R}^3)$ which implies that u belongs $W^{1,p}_{\alpha}(\Omega)$.

(ii) The case case $1 < p \leq 2$. Let R be chosen so that $\overline{\Omega'}$ is contained in B_R. Let v be the lifting function of g satisfying:

$$\Delta v = 0 \quad \text{in} \quad \Omega_R, \quad \frac{\partial v}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \partial\Omega, \quad v = 0 \quad \text{on} \quad \partial B_R.$$

This set of equations defines a unique function v in $W^{1,p}(\Omega_R)$ (see for example [17]). Extending v by zero outside B_R, the extended function, still denoted by v, belongs to $W_\alpha^{1,p}(\Omega)$. Then problem (3.22) is equivalent to

$$-\Delta w = \Delta v \quad \text{in} \quad \Omega, \quad \frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \partial\Omega, \tag{E.52}$$

where Δv belongs to $W_\alpha^{-1,p}(\Omega)$ and has a bounded support. Using inner regularity results, for any bounded domain $\mathscr{O}$ such that $\overline{\mathscr{O}} \subset \Omega_R \cup \partial B_R$, $v \in W^{1,2}(\mathscr{O})$ and $\Delta v \in L^2(\mathscr{O})$. Then using an adequate partition of unity of Ω, one can show that Δv belongs to $L^2(\Omega)$ and since the support of Δv is bounded, Δv also belongs to $W_1^{0,2}(\Omega)$. Therefore, there exists w in $W_0^{1,2}(\Omega)$ satisfying (E.52) (see [16] or [14]). Let $\widetilde{w} \in W_0^{1,2}(\mathbb{R}^3)$ denote the extended function constructed in (E.51), where $\omega' \in W^{1,2}(\Omega')$ is the unique solution of (E.50). Then $\Delta \widetilde{w}$ belongs to $W_0^{-1,2}(\mathbb{R}^3)$, has a bounded support and since $1 < p \leq 2$, it follows that $\Delta \widetilde{w}$ also belongs to $W_\alpha^{-1,p}(\mathbb{R}^3)$. Since $\alpha < -1 + 3/p'$, then thanks to Theorem 2.2, there exists $\widetilde{z}$ in $W_\alpha^{1,p}(\mathbb{R}^3)$ such that

$$\Delta \widetilde{z} = \Delta \widetilde{w} \quad \text{in} \quad \mathbb{R}^3.$$

Hence $\widetilde{z} - \widetilde{w}$ is a harmonic tempered distribution and therefore a polynomial of $\mathscr{P}^{\Delta}_{[1-3/p-\alpha]} \subset W_\alpha^{1,p}(\mathbb{R}^3)$. Thus w belongs to $W_\alpha^{1,p}(\Omega)$ and $u = w + v \in W_\alpha^{1,p}(\Omega)$ is the required solution of problem (3.18).

F Proof of Theorem 3.12

To prove that condition (3.24) is necessary, one can proceed as in the proof of Theorem 3.7. Let us prove that condition (3.24) is sufficient. The proof follows the same line as in Theorem 3.11. The only difference here is that in order to use Theorem 2.2, the distributions $\Delta\widetilde{u}$ and $\Delta\widetilde{\omega}$ defined in Theorem 3.11 must be orthogonal to the polynomials of $\mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}$. We shall prove this for $\Delta\widetilde{u}$. So let us prove that for any $q \in \mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}$, we have

$$\langle \Delta\widetilde{u}, q \rangle_{W_\alpha^{-1,p}(\mathbb{R}^3) \times W_{-\alpha}^{1,p'}(\mathbb{R}^3)} = 0.$$

Using Green formula, this amounts to prove that for any $q \in \mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}$

$$\left\langle \frac{\partial u'}{\partial \boldsymbol{n}} - g, q \right\rangle_{\partial\Omega} = 0.$$

- On the one hand, in Ω', for any $q \in \mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}$, we can write

$$\int_{\Omega'} \Delta u' \, q \, d\boldsymbol{x} = 0 = \int_{\Omega'} u' \, \Delta q \, d\boldsymbol{x} + \left\langle \frac{\partial u'}{\partial \boldsymbol{n}}, q \right\rangle_{\partial\Omega} - \left\langle \widetilde{u}, \frac{\partial q}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p,p}(\partial\Omega) \times W^{-1/p,p'}(\partial\Omega)}.$$

  Since $\Delta q = 0$, we get

$$\left\langle \frac{\partial u'}{\partial \boldsymbol{n}}, q \right\rangle_{\partial\Omega} = \left\langle \widetilde{u}, \frac{\partial q}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p,p}(\partial\Omega) \times W^{-1/p,p'}(\partial\Omega)}. \tag{F.53}$$

- On the other hand, in Ω, for any $v \in W^{1,p'}_{-\alpha}(\Omega)$ such that $\Delta v = 0$ in Ω, we can write

$$\int_\Omega \Delta u\, w\, d\boldsymbol{x} = 0 = \left\langle \frac{\partial u}{\partial \boldsymbol{n}}, v \right\rangle_{\partial\Omega} - \left\langle \widetilde{u}, \frac{\partial v}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p,p}(\partial\Omega)\times W^{-1/p,p'}(\partial\Omega)},$$

  which implies

$$\langle g, v\rangle_{\partial\Omega} = \left\langle \widetilde{u}, \frac{\partial v}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p,p}(\partial\Omega)\times W^{-1/p,p'}(\partial\Omega)}. \tag{F.54}$$

Now, for any $q \in \mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}$, we can write

$$\begin{aligned}\left\langle \frac{\partial u'}{\partial \boldsymbol{n}} - g, q \right\rangle_{\partial\Omega} &= \left\langle \frac{\partial u'}{\partial \boldsymbol{n}}, q \right\rangle_{\partial\Omega} - \langle g, q\rangle_{\partial\Omega} \\ &= \left\langle \frac{\partial u'}{\partial \boldsymbol{n}}, q \right\rangle_{\partial\Omega} + \langle g, v(q) - q\rangle_{\partial\Omega} - \langle g, v(q)\rangle_{\partial\Omega},\end{aligned}$$

where $v(q) \in W^{1,2}_0(\Omega) \cap W^{1,p'}_{-\alpha}(\Omega)$ is the unique solution of (3.23). Since by definition $v(q) - q$ belongs to $\mathscr{N}^{\Delta}_{-\alpha,p'}(\Omega)$, then thanks to condition (3.24), we have $\langle g, v(q) - q\rangle_{\partial\Omega} = 0$. Therefore, using (F.53), (F.54) and the fact that $\dfrac{\partial v(q)}{\partial \boldsymbol{n}} = \dfrac{\partial q}{\partial \boldsymbol{n}}$, we get

$$\begin{aligned}\left\langle \frac{\partial u'}{\partial \boldsymbol{n}} - g, q \right\rangle_{\partial\Omega} &= \left\langle \frac{\partial u'}{\partial \boldsymbol{n}}, q \right\rangle_{\partial\Omega} - \langle g, v(q)\rangle_{\partial\Omega}, \\ &= \left\langle \widetilde{u}, \frac{\partial q}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p,p}(\partial\Omega)\times W^{-1/p,p'}(\partial\Omega)} - \left\langle \widetilde{u}, \frac{\partial v(q)}{\partial \boldsymbol{n}} \right\rangle_{W^{1/p,p}(\partial\Omega)\times W^{-1/p,p'}(\partial\Omega)} \\ &= 0.\end{aligned}$$

Similar arguments hold to prove that $\Delta\widetilde{w}$ is orthogonal to $\mathscr{P}^{\Delta}_{[1-3/p'+\alpha]}$.

G Proof of Lemma 4.4

(i) Let us prove that

$$\left(V^{-1,p'}_{-\alpha}(\mathbb{R}^3)\right)' = W^{1,p}_{\alpha}(\mathbb{R}^3)/G^{1,p}_{\alpha}.$$

The first point is to prove that $G^{1,p}_{\alpha}$ is a closed subspace of $W^{1,p}_{\alpha}(\mathbb{R}^3)$. To that end, let us show that the divergence operator defined by

$$\mathrm{div} : W^{-1,p'}_{-\alpha}(\mathbb{R}^3)/V^{-1,p'}_{-\alpha}(\mathbb{R}^3) \longmapsto W^{-2,p'}_{-\alpha}(\mathbb{R}^3)$$

is an isomorphism. The above div operator is clearly linear and injective. To prove that it is onto take z in $W^{-2,p'}_{-\alpha}(\mathbb{R}^3)$. Observe next that under the assumptions on α and p, we have $3/p + \alpha \neq 2$ and

$3/p' - \alpha \neq 0$. Then thanks to Theorem 2.2 (isomorphism (2.11)), the Laplace operator defined by

$$\Delta : W^{0,p'}_{-\alpha}(\mathbb{R}^3)/\mathscr{P}^{\Delta}_{[-3/p'+\alpha]} \mapsto W^{-2,p'}_{-\alpha}(\mathbb{R}^3) \perp \mathscr{P}^{\Delta}_{[2-3/p-\alpha]} \tag{G.55}$$

is an isomorphism. The fact that $\alpha \geq 1 + 3/p'$ implies that $2 - 3/p - \alpha \leq -2 < 0$. Hence $\mathscr{P}^{\Delta}_{[2-3/p-\alpha]} = \{0\}$ and there exists $v \in W^{0,p'}_{-\alpha}(\mathbb{R}^3)$ such that

$$\Delta v = z \quad \text{in} \quad \mathbb{R}^3.$$

Then $\boldsymbol{u} = \nabla v \in W^{-1,p'}_{-\alpha}(\mathbb{R}^3)$ satisfies

$$\operatorname{div} \boldsymbol{u} = z \quad \text{in} \quad \mathbb{R}^3.$$

Thus the div operator defined above is indeed an isomorphism. Now by duality, the gradient operator defined by

$$\nabla : W^{2,p}_{\alpha}(\mathbb{R}^3) \mapsto W^{1,p}_{\alpha}(\mathbb{R}^3) \perp V^{-1,p'}_{-\alpha}(\mathbb{R}^3)$$

is an isomorphism. This shows that $G^{1,p}_{\alpha}$ is a closed subspace of $W^{1,p}_{\alpha}(\mathbb{R}^3)$. It follows that we can write

$$V^{-1,p'}_{-\alpha}(\mathbb{R}^3) = W^{-1,p'}_{-\alpha}(\mathbb{R}^3) \perp G^{1,p}_{\alpha}$$

which implies that

$$\left(V^{-1,p'}_{-\alpha}(\mathbb{R}^3)\right)' = \left(W^{-1,p'}_{-\alpha}(\mathbb{R}^3) \perp G^{1,p}_{\alpha}\right)' = W^{1,p}_{\alpha}(\mathbb{R}^3)/G^{1,p}_{\alpha}.$$

(ii) Let us prove that

$$\left(V^{0,p'}_{-\alpha}(\mathbb{R}^3)/\mathscr{G}_{[-3/p'+\alpha]}\right)' = V^{0,p}_{\alpha}(\mathbb{R}^3).$$

Thanks to Theorem 2.2, the Laplace operator defined by

$$\Delta : W^{1,p'}_{-\alpha}(\mathbb{R}^3)/\mathscr{P}^{\Delta}_{[1-3/p'+\alpha]} \mapsto W^{-1,p'}_{-\alpha}(\mathbb{R}^3) \perp \mathscr{P}^{\Delta}_{[1-3/p-\alpha]}$$

is an isomorphism. Therefore, under the assumption on α and p and proceeding as in (i), we can prove that $G^{0,p}_{\alpha}$ is a closed subspace of $W^{0,p}_{\alpha}(\mathbb{R}^3)$. Then we can write

$$W^{0,p'}_{-\alpha}(\mathbb{R}^3) \perp G^{0,p}_{\alpha} = V^{0,p'}_{-\alpha}(\mathbb{R}^3)$$

which implies that

$$\left(V^{0,p'}_{-\alpha}(\mathbb{R}^3)\right)' = \left(W^{0,p'}_{-\alpha}(\mathbb{R}^3) \perp G^{0,p}_{\alpha}\right)' = W^{0,p}_{\alpha}(\mathbb{R}^3)/G^{0,p}_{\alpha}.$$

Now for any $v \in W^{0,p}_{\alpha}(\mathbb{R}^3) \perp \mathscr{G}_{[-3/p'+\alpha]}$, $\operatorname{div} v \in W^{-1,p}_{\alpha}(\mathbb{R}^3) \perp \mathscr{P}_{[1-3/p'+\alpha]}$, then according to Theorem 2.2, there exists $w \in W^{1,p}_{\alpha}(\mathbb{R}^3)$ such that

$$\Delta w = \operatorname{div} v \quad \text{in} \quad \mathbb{R}^3.$$

This shows that one can identify the space $(W^{0,p}_\alpha(\mathbb{R}^3)\perp\mathscr{G}_{[-3/p'+\alpha]})/G^{0,p}_\alpha$ with $V^{0,p}_\alpha(\mathbb{R}^3)$ on the space $V^{0,p'}_{-\alpha}(\mathbb{R}^3)$. Moreover thanks again to Theorem 2.2, the space $G^{0,p}_\alpha$ is orthogonal to $\mathscr{G}_{[-3/p'+\alpha]}$ and by definition the space $V^{0,p}_\alpha(\mathbb{R}^3)$ is also orthogonal to $\mathscr{G}_{[-3/p'+\alpha]}$. Therefore, summarizing, we can write

$$\begin{aligned}\left(V^{0,p'}_{-\alpha}(\mathbb{R}^3)/\mathscr{G}_{[-3/p'+\alpha]}\right)' &= \left(V^{0,p'}_{-\alpha}(\mathbb{R}^3)\right)'\perp\mathscr{G}_{[-3/p'+\alpha]}\\ &= \left(W^{0,p}_\alpha(\mathbb{R}^3)/G^{0,p}_\alpha\right)\perp\mathscr{G}_{[-3/p'+\alpha]}\\ &= \left(W^{0,p}_\alpha(\mathbb{R}^3)\perp\mathscr{G}_{[-3/p'+\alpha]}\right)/G^{0,p}_\alpha\\ &= V^{0,p}_\alpha(\mathbb{R}^3).\end{aligned}$$

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