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## **$C^k$ SURFACE APPROXIMATION FROM SURFACE PATCHES**

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**Abstract.** *We study the problem of constructing a smooth approximant from a finite set of patches given on a surface defined by an equation  $x_3 = f(x_1, x_2)$ . As an approximant of  $f$ , a discrete smoothing spline belonging to a suitable piecewise polynomial space is proposed. Error results and numerical results are given.*

**Keywords :** Spline approximation, numerical integration, convergence.

**AMS Subject Classification :** 65D07, 41A15, 65D17, 41A63, 65D30, 65N30.

**Running title :** Surface approximation from surface patches

## **0 Introduction**

The problem of constructing a surface from given patches on this surface appears, for instance, in geophysics or geology processes like migration of time-maps or depth-maps as shown in [4]. Classical algorithms used to solve this class of problems usually select points on the patches to define a Lagrange dataset, and subsequently make use of classical spline functions (e.g. de Boor [7], Laurent [22], Schumaker [26]), bivariate spline (Lai and Schumaker [20], [21], Schumaker and von Golitschek[15]), or spline functions in Hilbert spaces (see for instance Duchon [13], Arcangéli [5]). In the available literature, to our knowledge there are classical methods that explicitly take into account the continuous aspect of the data (which consist of surface patches). Meanwhile, an alternative approach has been developed by Le Méhauté [23] using surface's extensions: A. Le Méhauté proposes a rational scheme interpolating a Taylor field of order 1 (or a function and its first partial derivatives) given on the edge of a triangle  $T$  with acute angles. This approach is very interesting but difficult to use on real data sets, and it does not perform well to get a  $C^1$  (or higher) approximant. That is why we propose to use a penalized least square approach, as done for

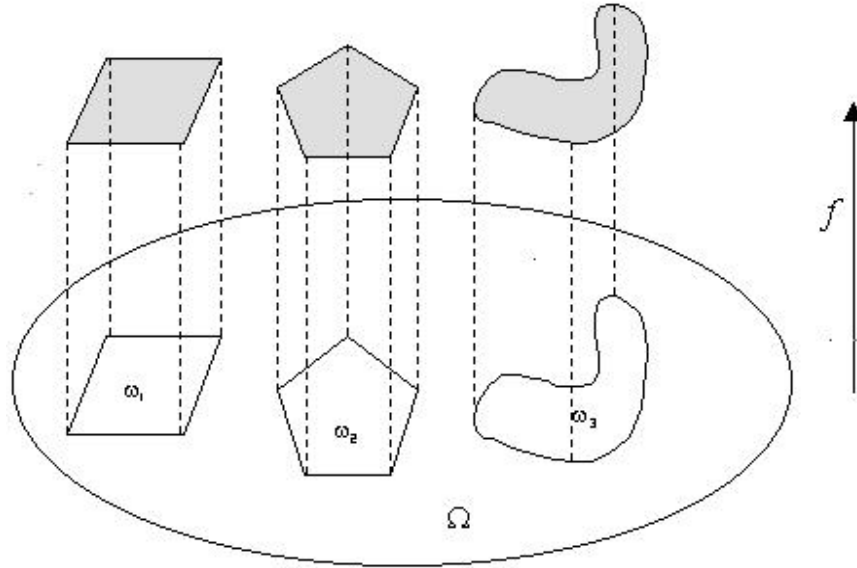


Figure 1: Example of an open set  $\Omega$  with three subsets  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ . From these three surface patches, the goal is to propose a method to reconstruct the surface on  $\Omega$ .

instance in Apprato and Arcangéli [1], in von Golitschek and Schumaker [15], or in Manzanilla [24].

In this paper, our purpose is to devise an approximation method which takes into account the original aspect of the data. To do that, we use a fidelity criterion to the data of integral type related to the  $L^2$ -norm of these data. It leads to the following abstract formulation: from a finite set of open subsets  $\omega_j, j = 1, \dots, N$  in the closure of a bounded non empty open set  $\Omega \subset \mathbb{R}^n$  (see Fig. 1) and from a function  $f$  defined on  $\omega = \bigcup_{j=1}^N \omega_j$ , construct a regular function  $\Phi$  on  $\Omega$ , approximating  $f$  on  $\omega$ .

In practice,  $n = 2$  and  $\omega$  is the finite union of pairwise disjoint open subsets  $\omega_j, j = 1, \dots, N$  of  $\Omega$ , which are called patches (see again Fig. 1).

**Remark 1:** As a fit criterion, we have chosen the  $L^2$ -norms (which define a distance). Of course, it would have been possible to use the  $L^1$ -norms (the fidelity criterion to the data being a minimization of the volume located between the patches and the approximant), but, in order to obtain a functional with "good properties" we have chosen a  $L^2$ -norm approach. This choice permits to study the problem in Hilbert spaces ( $L^1$ -norm only defines reflexive Banach

spaces).

The paper is organized as it follows: we first introduce some notations and state an existence-uniqueness result. In Section 1, we focus on the quadrature formula used to approximate the fidelity criterion to the data. Section 2 is devoted to the proof of the convergence theorem which is the main result of the paper. We give numerical results in Section 3.

We assume that:

- $\Omega$  is a non-empty, connected, bounded open set in  $\mathbb{R}^n$ , with Lipschitz-continuous boundary (in the sense of Nečas [25]);
- for an integer  $j, j = 1, \dots, N$ , let  $\omega_j$  be a non-empty connected subset in  $\mathbb{R}^n$ ;
- $k \in \mathbb{N}$  and  $m > \frac{n}{2}$ ;
- and for simplicity,  $f$  is the restriction on  $\omega$  of a function  $f$  defined on  $\Omega$ , given on  $\omega \left( = \bigcup_{j=1}^N \omega_j \right)$  and unknown on  $\Omega \setminus \omega$ , that belongs to the Sobolev space  $H^m(\Omega)$ . Finally, we impose the condition that the approximant  $\Phi$  belongs to  $H^m(\Omega) \cap C^k(\overline{\Omega})$ , where  $\overline{\Omega}$  is the closure of  $\Omega$ . The main interest of such a regularity for  $\Phi$  is that it allows one to obtain a final surface that can later be used directly as an input model in a different application, such as ray tracing, image synthesis or numerical simulation (e.g., Komatitsch and Tromp [19]).

The problem of the approximation of  $f$  on  $\omega$  is a fitting problem on each surface patches:  $\{(x, f(x)) : x \in \omega\}$ .

When  $m > k + \frac{n}{2}$ , the corresponding interpolation problem  $\Phi|_{\omega} = f|_{\omega}$  has an infinity of solutions because, in this case,  $H^m(\Omega) \hookrightarrow C^k(\overline{\Omega})$ . We can obtain a solution using Duchon's theory (see [13]). Unfortunately, Duchon's theory leads to linear systems whose order increases rapidly with the number of data points, which makes the method inefficient in the case of large datasets. We can obtain another solution using the following (interpolation) method: we introduce the convex set

$$K = \{v \in H^m(\Omega), v|_{\omega} = f|_{\omega}\}.$$

Then we consider the following interpolation problem on the data patches: find  $\sigma \in K$  such that

$$\forall v \in K, |\sigma|_{m,\Omega} \leq |v|_{m,\Omega}, \quad (1)$$

where

$$|v|_{m,\Omega} = \left( \sum_{|\alpha|=m} \int_{\Omega} (\partial^{\alpha} v(x))^2 dx \right)^{1/2},$$

with  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \sum_{i=1}^n \alpha_i, x = (x_i)_{i=1}^n$  and  $\partial^{\alpha} v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$ .

Let  $L^2(\omega)$  be equipped with the usual norm

$$\|v\|_{0,\omega} = \left( \sum_{j=1}^N \int_{\omega_j} v^2(x) dx \right)^{1/2}.$$

Then, Gout [16] has established the existence and uniqueness of the problem of equation (1) using compactness argument (see Necas [25]) to establish the equivalence of the usual norm of  $H^m(\Omega)$  denoted by  $\|u\|_{m,\Omega} = \left( \sum_{|\alpha| \leq m} \int_{\Omega} (\partial^\alpha v)^2 dx \right)^{1/2}$  and the norm

$$\| \|u\| = \left( \|u\|_{0,\omega}^2 + |u|_{m,\Omega}^2 \right)^{1/2}. \quad (2)$$

Then, as  $K$  is a convex, closed and non-empty set in  $H^m(\Omega)$ , the solution  $\sigma$  of (1) is nothing but the unique element of minimal norm  $\| \cdot \|$ .

Hence we could take the solution  $\Phi = \sigma$  when  $m > k + \frac{n}{2}$ . Unfortunately, it is often impossible to compute  $\sigma$  using a discretization of the problem (1), because in a finite dimensional space, it is generally not possible to satisfy an infinity of interpolation conditions. Therefore, to take into account the continuous aspect of the data  $(f|_{\omega})$ , we instead choose to define the approximant  $\Phi$  as a fitting surface.

We propose to construct a "smoothing  $D^m - Spline$ ", as defined by Arcangeli [5], that will be discretized in a suitable piecewise polynomial space. In order to do that, we introduce the functional  $J_\varepsilon$  defined on  $H^m(\Omega)$  by

$$J_\varepsilon(v) = \|v - f\|_{0,\omega}^2 + \varepsilon |v|_{m,\Omega}^2, \quad (3)$$

where  $\varepsilon > 0$  is a classical smoothing parameter. The key idea here is that the fidelity criterion to the data  $\|v - f\|_{0,\omega}^2$  honors their continuous aspect.

We now need to numerically estimate this term, which is done using a quadrature formula. In this regard, the approach is quite different from more classical techniques that usually simply make use of a large number of data points on  $\omega$  in order to solve the fitting problem.

In this paper,  $C$  denotes a generic positive constant and may take different values at different occurrences.

## 1 Approximation of $\| \cdot \|_{0,\omega}$ .

In this section, we propose a quadrature formula to approximate  $\|\cdot\|_{0,\omega}$  with certain order of approximation.

We introduce a bounded subset  $E$  in  $\mathbb{R}_+^*$  for which 0 is an accumulation point, and, for any  $\eta \in E$  and for an integer  $j, j = 1, \dots, N$ , a set  $\{\zeta_i\}_{1 \leq i \leq L}$  of  $L = L(\eta, j)$  distinct points  $\zeta_i = \zeta_i(\eta, j)$  of  $\overline{\omega}_j$  such that

$$\left( \max_{i,k=1,\dots,L} \delta(\zeta_i, \zeta_k) \right) \text{ depends on } \eta,$$

where  $\delta$  is the Euclidean distance in  $\mathbb{R}^n$  and the  $\zeta_i$  are the nodes of a numerical integration formulae. We also introduce a set  $\{\lambda_i\}_{1 \leq i \leq L}$  of real numbers  $\lambda_i = \lambda_i(\eta, j) > 0$ .

Then we define, for any  $\eta \in E$  and any  $v \in C^0(\overline{\omega}_j)$ ,

$$\ell_j^\eta(v) = \sum_{i=1}^L \lambda_i v(\zeta_i), \quad j = 1, \dots, N,$$

and, for any  $v \in C^0(\overline{\omega})$ ,

$$\ell^\eta(v) = \sum_{j=1}^N \ell_j^\eta(v), \quad j = 1, \dots, N.$$

In all that follows, we assume that there exists  $C, t > 0$ , such that, for any  $\eta \in E$ , and for any  $v \in H^m(\Omega)$ ,

$$\left| \ell_j^\eta(v^2) - \int_{\omega_j} v^2 dx \right| \leq C \eta^t \|v\|_{m,\Omega}^2, \quad j = 1, \dots, N. \quad (4)$$

For simplicity we shall write  $v$  instead of  $v|_{\omega_j}$  or  $v|_{\omega}$  and we shall consider  $\ell^\eta$  as a linear continuous form defined in  $C^0(\overline{\Omega})$ , or in  $C^0(\overline{\Omega}')$  for all open sets  $\Omega'$  such that  $\Omega \subset \Omega'$ .

**Remark 2:** When the hypothesis (4) is satisfied, the relation  $\|v\|_{0,\omega}^2 \sim \ell^\eta(v^2)$  gives us an abstract numerical integration formula for  $\|\cdot\|_{0,\omega}$ . In order to obtain the convergence of this integration formula when  $\eta$  tends to 0, the nodes  $\zeta_i$  must satisfy

$$\max_{1 \leq i \leq L} \min_{1 \leq k \leq L, k \neq i} \delta(\zeta_i, \zeta_k) \leq \eta,$$

where  $\delta(\cdot, \cdot)$  represents the Euclidean distance in  $\mathbb{R}^n$ .

Let us assume, for example, that  $\omega$  is a polyhedron. One can introduce a suitable triangulation  $\mathcal{T}_\eta$  on  $\overline{\omega}$  by means of  $n$ -simplexes  $T$  of diameter  $\leq \eta$ , and a  $P_{m-1}$  integration formula on each  $T$

$$\int_T v(x) dx \simeq \text{meas}(T) \sum_{i=1}^N \gamma_i v(\zeta_{iT}),$$

where the  $(\lambda_i)_{i=1,\dots,N}$  and the  $(\gamma_i)_{i=1,\dots,N}$  are classically the weights and the nodes of the integration formula. In that context, Arcangéli and al. [6] showed that when  $m \geq n$ , the relation (4) is satisfied with  $t = m$  and

$$\ell^n(v) = \sum_{T \in \mathcal{T}_n} (\text{meas } T) \sum_{i=1}^N \gamma_i v(\zeta_{iT}).$$

In practice, when  $n = 2$ , one can use the  $P_3$ - exact formula on a triangle  $T$  with vertices  $(a_i)_{i=1,2,3}$ , with mid-points of the sides  $(b_i)_{i=1,2,3}$ , and with barycenter  $c$  :

$$\int_T v \, dx \simeq \text{meas}(T) \left[ \frac{1}{20} \sum_{i=1}^3 v(a_i) + \frac{2}{15} \sum_{i=1}^3 v(b_i) + \frac{9}{20} v(c) \right].$$

Thus, for any  $\eta \in E$  and for any  $v \in C^0(\bar{\omega})$ , we obtain

$$\ell^n(v) = \sum_{T \in \mathcal{T}_n} \text{meas}(T) \left[ \frac{1}{20} \sum_{i=1}^3 v(a_i) + \frac{2}{15} \sum_{i=1}^3 v(b_i) + \frac{9}{20} v(c) \right].$$

When  $\omega$  is not a polyhedron, one can use quadrature formulas adapted to the geometry of  $\omega$  (e.g. Gout and Guessab [17], [18]). ■

## 2 Discrete Smoothing $D^m$ -Spline.

In order to compute a discrete approximant  $\Phi$ , we could use Bézier-polynomials space or any other finite dimensional space. We select a finite element representation of  $\Phi$ , which allows us to obtain a very small sparse linear system (see Ciarlet [8] and [9] for more details) and an easier study of the convergence of the approximation.

Let  $H$  be a bounded subset in  $\mathbb{R}_+^*$  for which 0 is an accumulation point, let  $\tilde{\Omega}$  be a bounded polygonal open set in  $\mathbb{R}^n$  such that  $\Omega \subset \tilde{\Omega}$ , and, for any  $h \in H$ , let  $\tilde{\mathcal{T}}_h$  be a triangulation on  $\tilde{\Omega}$  by means of elements  $K$  whose diameter  $h_K$  are  $\leq h$  and let  $\tilde{V}_h$  be a finite element space constructed on  $\tilde{\mathcal{T}}_h$  such that

$$\tilde{V}_h \text{ is a finite- dimensional subspace of } H^m(\tilde{\Omega}) \cap C^k(\tilde{\Omega}). \quad (5)$$

Furthermore, to study the convergence of the approximation, we assume that there exists a family of operators  $(\tilde{\Pi}_h)_{h \in H} \subset \mathcal{L}(H^m(\tilde{\Omega}), \tilde{V}_h)$  satisfying:

$$\left\{ \begin{array}{l} \text{i) } \exists C > 0 \text{ such that for any } h \in H, \text{ for any } l = 0, \dots, m-1, \\ \text{and for any } v \in H^m(\tilde{\Omega}), \quad |v - \tilde{\Pi}_h v|_{l, \tilde{\Omega}} \leq Ch^{m-l} |v|_{m, \tilde{\Omega}}, \\ \text{ii) } \text{For any } v \in H^m(\tilde{\Omega}), \quad \lim_{h \rightarrow 0} |v - \tilde{\Pi}_h v|_{m, \tilde{\Omega}} = 0. \end{array} \right. \quad (6)$$

The condition (6) does not need the classical hypothesis of regularity of the finite element method :  $H^m(\tilde{\Omega}) \hookrightarrow C^s(\tilde{\Omega})$  where  $s$  is the maximal order of the derivatives appearing in the definition of the degrees of freedom of the generic finite element of  $(\tilde{V}_h)_{h \in H}$ , but it is assumed that

$$\text{the family } (\tilde{T}_h)_{h \in H} \text{ is regular (see Ciarlet- Raviart [10]).} \quad (7)$$

Moreover, the condition (6) needs the following hypothesis: the generic finite element  $(K, P_K, \Sigma_K)$  of the family  $(\tilde{V}_h)_{h \in H}$  satisfies the inclusion  $P_K \supset P_m(K)$ .

In fact, **i**) only uses the inclusion  $P_{m-1}(K) \subset P_K$ , and a property of uniformity of the generic finite element of  $(\tilde{V}_h)_{h \in H}$ , which is satisfied in the usual cases (see Clément [11] and Strang [27]).

**Remark 3:** In most problems one would want to solve in practice, the value of  $m$  would be either 2 or 3, allowing one to get either a  $C^1$  or a  $C^2$  approximant. When  $m = 2$ , the finite elements used to solve the problem could typically be classical elements of class  $C^1$  or  $C^2$ , such as the Argyris or the Bell triangle (see Zenisek [29], Manzanilla [24]), or the Bogner-Fox-Schmit quadrangle (e.g., Ciarlet [8], Apprato and al. [2]). When  $m = 3$ , one could use the same finite element of class  $C^2$  as for  $m = 2$ . When  $m > 3$ , one could generalize the Bogner-Fox-Schmit quadrangle into a finite element of class  $C^{m-1}$ . Other elements, such as isoparametric finite elements (Ciarlet [8]) or rational finite elements (Wachspress [28]) could also be used. Isoparametric finite elements are useful to impose boundary conditions, but this is not usually a critical problem in the context of surface approximation. On the other hand, the use of rational finite elements is expensive, therefore we choose to use the Bogner-Fox-Schmit quadrangle of class  $C^1$ , which allows us to obtain a  $C^1$ -approximant. Note that in certain classes of interpolation problems, each data point must also be a node of the finite element grid, in which case the use of triangles, as opposed to quadrangles, greatly facilitates the creation of a suitable finite element mesh to numerically solve the problem. This is not the case in a surface fitting problem (data points do not need to be linked with the geometry of the finite element grid), and as rectangles are less expensive than triangles, we will use the Bogner-Fox-Schmit rectangles. In all cases, we could verify that the conditions (5) and (6) are satisfied under (7). ■

Now, for any  $h \in H$ , we consider the subset  $\Omega_h$  (see Fig. 2) defined by:

$$\left\{ \begin{array}{l} \Omega_h \text{ is the inside of the union of the rectangles } K \text{ of } \mathcal{T}_h \\ \text{such that } K \cap \Omega \neq \emptyset. \end{array} \right. \quad (8)$$



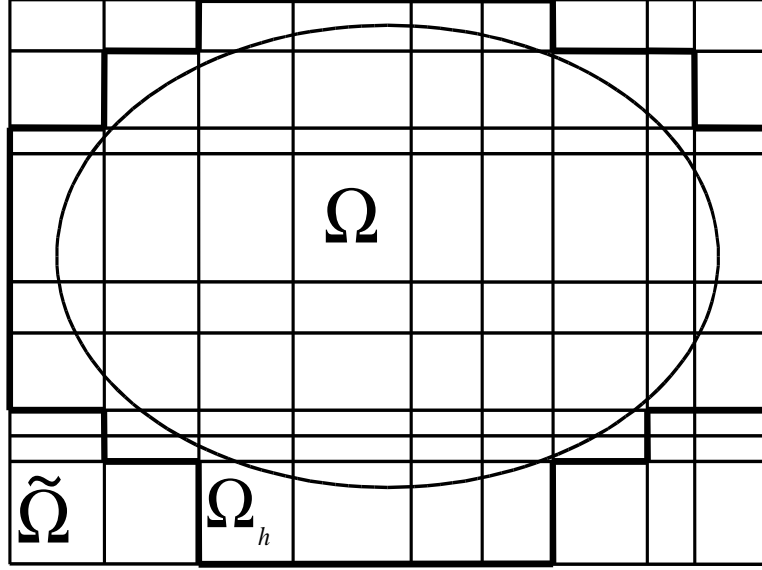


Figure 2: Definition of the sets  $\Omega$ ,  $\Omega_h$  and  $\tilde{\Omega}$ .

It is clear that the family  $(\Omega_h)_{h \in H}$  satisfies the following relations

$$\forall h \in H, \Omega \subset \Omega_h \subset \tilde{\Omega}, \quad (9)$$

$$\lim_{h \rightarrow 0} \text{meas}(\Omega_h \setminus \tilde{\Omega}) = 0. \quad (10)$$

For any  $h \in H$ , we define  $V_h$  as it follows:

$$V_h \text{ is the vector space of the restrictions to } \Omega_h \text{ of the functions of } \tilde{V}_h. \quad (11)$$

For any  $\varepsilon > 0$ , any  $h \in H$  and any  $\eta \in E$ , we consider the minimization problem of finding  $\sigma_{\varepsilon, h}^\eta \in V_h$  satisfying, for any  $v_h \in V_h$ ,

$$J_{\varepsilon, h}^\eta(\sigma_{\varepsilon, h}^\eta) \leq J_{\varepsilon, h}^\eta(v_h), \quad (12)$$

where  $J_{\varepsilon, h}^\eta$  is the functional defined by

$$J_{\varepsilon, h}^\eta(v_h) = \ell^\eta \left[ (v_h - f)^2 \right] + \varepsilon |v_h|_{m, \Omega_h}^2.$$

Then, we consider the variational problem of finding  $\sigma_{\varepsilon, h}^\eta \in V_h$  satisfying,

$$\forall v_h \in V_h, \ell^\eta(\sigma_{\varepsilon, h}^\eta v_h) + \varepsilon (\sigma_{\varepsilon, h}^\eta, v_h)_{m, \Omega_h} = \ell^\eta(f v_h), \quad (13)$$

where  $(u, v)_{m, \Omega_h} = \sum_{|\alpha|=m} \int_{\Omega_h} \partial^\alpha u(x) \partial^\alpha v(x) dx$ . Then, we have the following theorem.

**Theorem 2.1.** *We assume that  $\Omega$ ,  $\omega$ ,  $m$  and  $f$  are defined as in section 0 and that hypotheses (4), (5), (8), and (11) are satisfied, then, for any  $\varepsilon > 0$ , any  $h \in H$ , there exists  $\eta_0 > 0$  such that for any  $\eta \in E$ ,  $\eta \leq \eta_0$ , the problems (12) and (13) have the same unique solution  $\sigma_{\varepsilon, h}^\eta$ .*

*Proof.* This proof is divided into two parts.

1) Using compactness arguments (see Nečas [25]), we show, under the relation:

$$\text{for any } p \in P_{m-1}(\overline{\Omega}_h), p|_\omega = 0 \implies p \equiv 0,$$

that the function  $[\![\cdot]\!]_h$  defined on  $H^m(\Omega_h)$  by

$$[\![v]\!]_h = \left( \|v\|_{0, \omega}^2 + |v|_{m, \Omega_h}^2 \right)^{1/2}$$

is a norm on  $H^m(\Omega_h)$  which is equivalent to the usual norm

$$\|v\|_{m, \Omega_h} = \left( \sum_{|\alpha| \leq m} \int_{\Omega_h} (\partial^\alpha v)^2 dx \right)^{1/2}.$$

2) As a consequence of the definition of  $\ell^\eta$ , the symmetric bilinear form

$$(u_h, v_h) \longmapsto \ell^\eta(u_h v_h) + \varepsilon (u_h, v_h)_{m, \Omega_h}$$

is continuous on  $V_h \times V_h$ . Likewise, this form is  $V_h$ -elliptic for  $\eta$  small enough because using (4) we have

$$\begin{aligned} \ell^\eta(v_h^2) + \varepsilon |v_h|_{m, \Omega_h}^2 &\geq \|v_h\|_{0, \omega}^2 - C\eta^t \|v_h\|_{m, \Omega}^2 + \varepsilon |v_h|_{m, \Omega_h}^2, \\ &\geq \min(1, \varepsilon) [\![v_h]\!]_h^2 - C\eta^t \|v_h\|_{m, \Omega}^2, \\ &\geq (C' \min(1, \varepsilon) - C\eta^t) \|v_h\|_{m, \Omega_h}^2, \end{aligned} \quad (14)$$

where  $C'$  is a constant related to the equivalence of norms ( $[\![\cdot]\!]_h$  and  $\|\cdot\|_{m, \Omega_h}$ ).

Let us assume (see also [6] for more details) that there exists  $\beta > 0$ , such that

$\forall \varepsilon > 0, \forall \eta \in E, \frac{\eta^t}{\min(1, \varepsilon)} < \beta$ . Taking  $\beta = \frac{C'}{C}$ , there exists  $C'' > 0$  such that

$\ell^\eta(v_h^2) + \varepsilon |v_h|_{m, \Omega_h}^2 \geq C'' \|v_h\|_{m, \Omega_h}^2$ . Then, the Lax-Milgram lemma gives the result. ■

The function  $\sigma_{\varepsilon, h}^\eta$  is called the  $V_h$ -discrete smoothing  $D^m$ -spline of  $f$  relative to  $\omega$ ,  $\eta$  and  $\varepsilon$ .

*Remark 4.* Denoting by  $M = M(h)$  the dimension of  $V_h$  and by  $(\varphi_j)_{1 \leq j \leq M}$  a basis of  $V_h$ , we set

$$\sigma_{\varepsilon, h}^\eta = \sum_{j=1}^M \alpha_j \varphi_j,$$

with  $\alpha_j \in \mathbb{R}$ ,  $1 \leq j \leq M$ . Introducing the matrices  $\mathcal{A} = (\ell^\eta(\varphi_i \varphi_j))_{1 \leq i, j \leq M}$ ,  $\mathcal{R} = ((\varphi_i, \varphi_j)_{m, \Omega_h})_{1 \leq i, j \leq M}$  and  $\mathcal{F} = (\ell^\eta(f \varphi_i))_{1 \leq i \leq M}$ , we see that (13) is equivalent to the problem:

$$\text{find } \alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_M) \in \mathbb{R}^M \text{ solution of } (\mathcal{A} + \varepsilon \mathcal{R}) \alpha = \mathcal{F}.$$

Then, we can take as an approximation of  $f$  the function  $\Phi = \sigma_{\varepsilon, h}^\eta|_\Omega$  which, using the hypotheses (5), (8) and (11), belongs to  $H^m(\Omega) \cap C^k(\overline{\Omega})$ . We have to know in which sense  $\Phi$  is an approximation of  $f$ . The next result proves the convergence of  $\sigma_{\varepsilon, h}^\eta$  to the solution  $\sigma$  of (1) and gives an error result between  $\sigma_{\varepsilon, h}^\eta$  and  $f$  on  $\omega$ . Other convergence results are given in Apprato and Gout [3] and in Gout [16] when the number of data patches increases to infinity.

**Theorem 2.2.** *Under the hypotheses of Theorem 2.1, if moreover we assume that (6) is satisfied, then the solution  $\sigma_{\varepsilon, h}^\eta$  of (12) and (13) satisfies:*

$$(i) \quad \lim_{\varepsilon \rightarrow 0, h^{2m}/\varepsilon \rightarrow 0, \eta^t/\varepsilon < \beta} \left\| \sigma_{\varepsilon, h}^\eta - \sigma \right\|_{m, \Omega} = 0, \text{ where } \sigma \text{ is the solution of (1)}$$

and where  $\beta$  is introduced in Theorem 2.1.

(ii) *There exists a positive constant  $C$  such that*

$$\left\| \sigma_{\varepsilon, h}^\eta - f \right\|_{0, \omega}^2 \leq C (h^{2m} + \eta^t o(1) + \varepsilon), \text{ when } \varepsilon \rightarrow 0, h^{2m}/\varepsilon \rightarrow 0, \eta^t/\varepsilon < \beta.$$

*Proof:*

The proof of point (i) will be split into four steps.

• Let  $\sigma$  be the unique solution of (1). We have  $\sigma|_\omega = f|_\omega$  and

$$\left\| \sigma_{\varepsilon, h}^\eta - f \right\|_{0, \omega}^2 = \left\| \sigma_{\varepsilon, h}^\eta - \sigma \right\|_{0, \omega}^2.$$

We obtain using (4)

$$\left\| \sigma_{\varepsilon, h}^\eta - f \right\|_{0, \omega}^2 \leq \ell^\eta \left( \left( \sigma_{\varepsilon, h}^\eta - \sigma \right)^2 \right) + C \eta^t \left\| \sigma_{\varepsilon, h}^\eta - \sigma \right\|_{m, \Omega}^2. \quad (15)$$

Then, from (12) we have

$$\forall v_h \in V_h, \ell^\eta \left( \left( \sigma_{\varepsilon, h}^\eta - \sigma \right)^2 \right) + \varepsilon \left| \sigma_{\varepsilon, h}^\eta \right|_{m, \Omega_h}^2 \leq \ell^\eta \left( (v_h - \sigma)^2 \right) + \varepsilon |v_h|_{m, \Omega_h}^2. \quad (16)$$

Let  $\tilde{\sigma}$  be a  $m$ -extension of  $\sigma$  on  $\tilde{\Omega}$ , taking (with (6))  $v_h = \tilde{\Pi}_h \tilde{\sigma} \in H^m(\tilde{\Omega})$ , we obtain

$$\ell^\eta \left( \left( \sigma_{\varepsilon,h}^\eta - \sigma \right)^2 \right) \leq \ell^\eta \left( \left( \tilde{\Pi}_h \tilde{\sigma} - \sigma \right)^2 \right) + \varepsilon \left| \tilde{\Pi}_h \tilde{\sigma} \right|_{m,\Omega_h}^2, \quad (17)$$

but from (6) **ii**), there exists  $h_0 \in H$  such that

$$\forall h \leq h_0, \quad \left| \tilde{\Pi}_h \tilde{\sigma} \right|_{m,\Omega_h}^2 \leq \left| \tilde{\Pi}_h \tilde{\sigma} \right|_{m,\tilde{\Omega}}^2 \leq C |\tilde{\sigma}|_{m,\tilde{\Omega}}^2 \leq C \|\sigma\|_{m,\Omega}^2. \quad (18)$$

Moreover, using (4) we have

$$\ell^\eta \left( \left( \tilde{\Pi}_h \tilde{\sigma} - \sigma \right)^2 \right) \leq \left\| \tilde{\Pi}_h \tilde{\sigma} - \sigma \right\|_{0,\omega}^2 + C\eta^t \left\| \tilde{\Pi}_h \tilde{\sigma} - \sigma \right\|_{m,\Omega}^2,$$

and finally, with (6) **i**)

$$\left\| \tilde{\Pi}_h \tilde{\sigma} - \sigma \right\|_{0,\omega} \leq \left\| \tilde{\Pi}_h \tilde{\sigma} - \tilde{\sigma} \right\|_{0,\tilde{\Omega}} \leq Ch^m |\tilde{\sigma}|_{m,\tilde{\Omega}}.$$

Thus, using (6), we get

$$\ell^\eta \left( \left( \tilde{\Pi}_h \tilde{\sigma} - \sigma \right)^2 \right) \leq Ch^{2m} |\tilde{\sigma}|_{m,\tilde{\Omega}}^2 + C\eta^t o(1), \quad h \rightarrow 0. \quad (19)$$

From (17) and (19), we deduce that

$$\ell^\eta \left( \left( \sigma_{\varepsilon,h}^\eta - \sigma \right)^2 \right) \leq Ch^{2m} |\tilde{\sigma}|_{m,\tilde{\Omega}}^2 + C\eta^t o(1) + \varepsilon \left| \tilde{\Pi}_h \tilde{\sigma} \right|_{m,\Omega_h}^2, \quad h \rightarrow 0, \quad (20)$$

and therefore, using (15), we have

$$\begin{aligned} \left\| \sigma_{\varepsilon,h}^\eta - \sigma \right\|_{0,\omega}^2 &\leq Ch^{2m} |\tilde{\sigma}|_{m,\tilde{\Omega}}^2 + C\eta^t o(1) + \varepsilon \left| \tilde{\Pi}_h \tilde{\sigma} \right|_{m,\Omega_h}^2 + C\eta^t \left\| \sigma_{\varepsilon,h}^\eta - \sigma \right\|_{m,\Omega}^2 \\ &\leq C \left( h^{2m} + \eta^t o(1) \right) + \varepsilon \left| \tilde{\Pi}_h \tilde{\sigma} \right|_{m,\Omega_h}^2 + C\eta^t \left\| \sigma_{\varepsilon,h}^\eta - \sigma \right\|_{m,\Omega}^2, \quad h \rightarrow 0. \end{aligned} \quad (21)$$

Likewise, the relations (16) with  $v_h = \tilde{\Pi}_h \tilde{\sigma}$ , (18) and (19) involve that

$$\left| \sigma_{\varepsilon,h}^\eta - \sigma \right|_{m,\Omega}^2 \leq C \left( \frac{h^{2m}}{\varepsilon} + \frac{\eta^t}{\varepsilon} o(1) + \|\sigma\|_{m,\Omega}^2 \right), \quad h \rightarrow 0. \quad (22)$$

We finally obtain, using (21) and (22):

$$\left\| \left\| \sigma_{\varepsilon,h}^\eta - \sigma \right\| \right\|^2 \leq C(\varepsilon + 1) \left( \frac{h^{2m}}{\varepsilon} + \frac{\eta^t}{\varepsilon} o(1) + \|\sigma\|_{m,\Omega}^2 \right) + C\eta^t \left\| \sigma_{\varepsilon,h}^\eta - \sigma \right\|_{m,\Omega}^2, \quad h \rightarrow 0 \quad (23)$$

where  $|||\cdot|||$  is defined in (2). Because the norm  $|||\cdot|||$  is equivalent to the usual norm  $\|\cdot\|_{m,\Omega}$  in  $H^m(\Omega)$ , we deduce from (23) that

$$(1 - C\eta^t) \left\| \sigma_{\varepsilon,h}^\eta - \sigma \right\|_{m,\Omega}^2 \leq C(\varepsilon + 1) \left( \frac{h^{2m}}{\varepsilon} + \frac{\eta^t}{\varepsilon} o(1) + \|\sigma\|_{m,\Omega}^2 \right), \quad h \rightarrow 0.$$

Let  $\eta'$  be a positive constant less than  $\left(\frac{1}{C}\right)^{\frac{1}{t}}$ . Therefore, the family  $(\sigma_{\varepsilon,h}^\eta)_{\eta,\varepsilon,h}$  is bounded in  $H^m(\Omega)$  if

$$\varepsilon \rightarrow 0, \quad \frac{h^{2m}}{\varepsilon} \rightarrow 0, \quad \eta \leq \eta', \quad \frac{\eta^t}{\varepsilon} \leq \beta. \quad (24)$$

The previous result involves that  $H^m(\Omega)$  contains a sequence  $(\sigma_{\varepsilon_n, h_n}^{\eta_n})_{n \in \mathbb{N}^*}$  extracted from the family  $(\sigma_{\varepsilon,h}^\eta)$  and a function  $\sigma^* \in H^m(\Omega)$  such that:

$$\begin{cases} \sigma^* = \text{weak} \lim_{n \rightarrow +\infty} \sigma_{\varepsilon_n, h_n}^{\eta_n} \text{ in } H^m(\Omega), \\ \text{with } \lim_{n \rightarrow +\infty} \varepsilon_n = \lim_{n \rightarrow +\infty} \frac{h_n^{2m}}{\varepsilon_n} = 0, (\eta_n) \subset (0, \eta') \text{ and } \left(\frac{\eta_n^t}{\varepsilon_n}\right) \subset (0, \beta). \end{cases} \quad (25)$$

• Now, we have to show that  $\sigma^* = \sigma$ . Let us consider the relations (20) and (22) applied to the sequence  $(\sigma_{\varepsilon_n, h_n}^{\eta_n})_n$ :

$$\begin{cases} \exists C > 0, \\ \left| \sigma_{\varepsilon_n, h_n}^{\eta_n} \right|_{m,\Omega}^2 \leq C \left( \frac{h_n^{2m}}{\varepsilon_n} + \frac{\eta_n^t}{\varepsilon_n} o(1) \right) + \left| \tilde{\Pi}_{h_n} \tilde{\sigma} \right|_{m,\Omega_{h_n}}^2, \quad n \rightarrow +\infty \\ \ell^{\eta_n} \left( \left( \sigma_{\varepsilon_n, h_n}^{\eta_n} - \sigma \right)^2 \right) \leq C h_n^{2m} |\tilde{\sigma}|_{m,\tilde{\Omega}} + C \eta_n^t o(1) + \varepsilon_n \left| \tilde{\Pi}_{h_n} \tilde{\sigma} \right|_{m,\Omega_{h_n}}^2, \quad n \rightarrow +\infty. \end{cases} \quad (26)$$

But, for  $n$  large enough, we have, knowing that  $\lim_{n \rightarrow +\infty} \left| \tilde{\Pi}_{h_n} \tilde{\sigma} \right|_{m,\Omega_{h_n}} = |\sigma|_{m,\Omega}$ ,

$$\left| \tilde{\Pi}_{h_n} \tilde{\sigma} \right|_{m,\Omega_{h_n}}^2 \leq |\sigma|_{m,\Omega}^2 + \mu^2, \quad \mu > 0 \text{ arbitrarily small,}$$

and then it comes that

$$\left| \sigma_{\varepsilon_n, h_n}^{\eta_n} \right|_{m,\Omega}^2 \leq C \left( \frac{h_n^{2m}}{\varepsilon_n} + \frac{\eta_n^t}{\varepsilon_n} o(1) \right) + |\sigma|_{m,\Omega}^2 + \mu^2, \quad n \rightarrow +\infty. \quad (27)$$

From (4), we have

$$\left\| \sigma_{\varepsilon_n, h_n}^{\eta_n} - \tilde{\sigma} \right\|_{0,\omega}^2 \leq \ell^{\eta_n} \left( \left( \sigma_{\varepsilon_n, h_n}^{\eta_n} - \tilde{\sigma} \right)^2 \right) + C \eta_n^t \left\| \sigma_{\varepsilon_n, h_n}^{\eta_n} - \tilde{\sigma} \right\|_{m,\Omega}^2,$$

thus, using (26), and because the sequence  $(\sigma_{\varepsilon_n, h_n}^{\eta_n})$  is bounded in  $H^m(\Omega)$ , we obtain

$$\left\| \sigma_{\varepsilon_n, h_n}^{\eta_n} - \tilde{\sigma} \right\|_{0, \omega}^2 \leq C \varepsilon_n \left( \frac{h_n^{2m}}{\varepsilon_n} + \frac{\eta_n^t}{\varepsilon_n} o(1) + |\sigma|_{m, \Omega}^2 + \mu^2 + \frac{\eta_n^t}{\varepsilon_n} \right), \quad n \rightarrow +\infty. \quad (28)$$

Therefore, from (27) and (28), we finally have:

$$\begin{cases} |\sigma^*|_{m, \Omega}^2 \leq \liminf_{n \rightarrow +\infty} \left| \sigma_{\varepsilon_n, h_n}^{\eta_n} \right|_{m, \Omega}^2 \leq |\sigma|_{m, \Omega}^2 + \mu^2, \\ \|\sigma - \sigma^*\|_{0, \omega}^2 \leq \liminf_{n \rightarrow +\infty} \left\| \sigma_{\varepsilon_n, h_n}^{\eta_n} - \tilde{\sigma} \right\|_{0, \omega}^2 \leq 0. \end{cases} \quad (29)$$

It comes from (29), as  $\mu$  is arbitrarily small

$$\begin{cases} \sigma^* \in H^m(\Omega), \\ |\sigma^*|_{m, \Omega} \leq |\sigma|_{m, \Omega}, \\ \sigma = \sigma^* \text{ on } \omega. \end{cases}$$

And thus  $\sigma = \sigma^*$  in  $H^m(\Omega)$  by Theorem 0.1.

- We now have to show that  $\lim_{n \rightarrow +\infty} \sigma_{\varepsilon_n, h_n}^{\eta_n} = \sigma$  in  $H^m(\Omega)$ . Using the compact embedding of  $H^m(\Omega)$  in  $H^{m-1}(\Omega)$  and knowing that  $\sigma = \sigma^*$  in  $H^m(\Omega)$ , we have  $\sigma_{\varepsilon_n, h_n}^{\eta_n} \rightharpoonup \sigma$  in  $H^{m-1}(\Omega)$ . Furthermore, for any  $n \in \mathbb{N}$ ,

$$\left| \sigma_{\varepsilon_n, h_n}^{\eta_n} - \sigma \right|_{m, \Omega}^2 \leq \left| \sigma_{\varepsilon_n, h_n}^{\eta_n} \right|_{m, \Omega}^2 + |\sigma|_{m, \Omega}^2 - 2 \left( \sigma_{\varepsilon_n, h_n}^{\eta_n}, \sigma \right)_{m, \Omega}.$$

Therefore, using (27), we obtain

$$\left| \sigma_{\varepsilon_n, h_n}^{\eta_n} - \sigma \right|_{m, \Omega}^2 \leq C \left( \frac{h_n^{2m}}{\varepsilon_n} + \frac{\eta_n^t}{\varepsilon_n} o(1) \right) + 2 |\sigma|_{m, \Omega}^2 + \mu^2 - 2 \left( \sigma_{\varepsilon_n, h_n}^{\eta_n}, \sigma \right)_{m, \Omega}, \quad n \rightarrow +\infty$$

and when  $n \rightarrow +\infty$ , it comes because  $\sigma_{\varepsilon_n, h_n}^{\eta_n} \rightharpoonup \sigma$  in  $H^m(\Omega)$  that,

$$\lim_{n \rightarrow +\infty} \left| \sigma_{\varepsilon_n, h_n}^{\eta_n} - \sigma \right|_{m, \Omega}^2 \leq 2 |\sigma|_{m, \Omega}^2 + \mu^2 - 2 (\sigma, \sigma)_{m, \Omega} = \mu^2,$$

and the result follows because  $\mu$  is arbitrarily small.

- To achieve this proof, we now assume that

$$\lim \left\| \sigma_{\varepsilon, h}^{\eta} - \sigma \right\|_{m, \Omega}^2 \neq 0$$

with  $\varepsilon \rightarrow 0$ ,  $\frac{h^{2m}}{\varepsilon} \rightarrow 0$ ,  $\frac{\eta^t}{\varepsilon} \leq \beta$ . It means there exists a sequence  $(d'_n, \eta'_n, \varepsilon'_n, h'_n)_{n \in \mathbb{N}}$

such that  $\lim_{n \rightarrow +\infty} \varepsilon'_n = \lim_{n \rightarrow +\infty} \frac{h_n'^{2m}}{\varepsilon'_n} = 0$ ,  $\frac{\eta_n'^t}{\varepsilon'_n} \leq \beta$  satisfying for any  $n \in \mathbb{N}^*$ ,

$$\left\| \sigma_{\varepsilon'_n, h'_n}^{d'_n, \eta'_n} - f \right\|_{m, \Omega} > \alpha.$$

But such a sequence is bounded in  $H^m(\Omega)$  and using the previous argument, we reach a contradiction.

Point (ii) is an immediate consequence of (21) and point (i), taking into account that  $\sigma_{i\omega} = f_{i\omega}$ . ■

### 3 Numerical results

The CPU time for each of the following examples was less than 10 seconds (on a PC Intel Pentium III 500 Mhz, 128 Megabytes).

We have chosen two functions  $f$  and  $g$  defined by:

$$f(x, y) = e^{-(3x-1)^2 - (3y-1)^2},$$

and

$$g(x, y) = \frac{3}{4}e^{(-\frac{1}{4}(9x-2)^2 - \frac{1}{4}(9y-2)^2)} + \frac{3}{4}e^{(-\frac{1}{49}(9x+1)^2 - \frac{1}{16}(9y+1)^2)} \\ + \frac{1}{2}e^{(-\frac{1}{4}(9x-7)^2 - \frac{1}{4}(9y-3)^2)} - \frac{1}{5}e^{(-(9x-4)^2 - (9y-7)^2)}.$$

The open  $\Omega$  is  $]0, 1[ \times ]0, 1[$ . We use a finite element grid which divides  $\Omega$  into four rectangles. The chosen generic finite element is the Bogner-Fox-Schmit rectangle of class  $C^1$ . Therefore the associated finite element space  $V_h$  as a dimension of  $V_h = 36$ .

For the numerical integration on the given subsets, we use the  $P_3$  formula on triangles:

$$\int_T u dx \simeq meas(T) \left( \frac{1}{20} \sum_{i=1}^3 u(a_i) + \frac{2}{15} \sum_{i=1}^3 u(b_i) + \frac{9}{20} u(c) \right),$$

as introduced in the remark 2.

We take  $\varepsilon = 10^{-6}$ . To our knowledge, there is no mathematical method to optimize this choice. In (Deshpande and Girard [12], Girard [14]), cross validation methods are studied to solve a similar kind of problem.

For the function  $f$  (see Fig. 3), the data corresponding to the first example are reproduced in Figure 4. Only one surface patch  $\omega_1$  is used and the triangulation is made using 32 triangles (total of 32 distinct vertices for  $\omega$ , see Fig. 5). For the second example, data are reproduced in Figure 7. In this case,  $\omega$  has three connected components  $(\omega_1, \omega_2, \omega_3)$  and the triangulation is made using 2

triangles with a common side per  $\omega_{i:i=1,2,3}$  (total of 12 distinct vertices for  $\omega$ , see Fig. 8). We show the three surface patches. The corresponding approximant surface are given, for the first example, in Figure 6, and for the second one in Figure 9.

For the function  $g$  (see Fig. 10), we have used the same kind of data as for the function  $f$ : one surface patch (see Fig. 11) or three surface patches (see Fig. 13). The corresponding approximating surfaces are displayed in Figs. 12 and 14, respectively.

In all cases, we have evaluated the approximant on a regular  $40 \times 40$  grid of points.

To estimate the accuracy of the method, we evaluate the quadratic error on  $\Omega$  (for the approximant and a usual  $D^m$ -spline (e.g. [5])) based upon the classical formula:

$$Q_{error} \left( \bigcup_i (x_i, y_i, z_i) \right) = \left( \sum_{i=1}^{1600} (\tilde{z}_i - z_i)^2 / \sum_{i=1}^{1600} z_i^2 \right)$$

where  $z_i$  represents the  $z$ -data value, and where  $\tilde{z}_i$  is the  $z$ -approximant for the same  $(x_i, y_i)$ .

For the usual  $D^m$ -spline, the Lagrange data are the nodes of the triangulation, and of course, we have chosen exactly the same finite element grid (and the Bogner-Fox-Schmit finite element of class  $C^1$ ).

Function $f$	$Q_{error} \left( \bigcup_i (x_i, y_i, z_i) \right)$	Data
<i>approximant</i>	$1.84 \cdot 10^{-3}$	$\omega_1$ : 128 data points
<i><math>D^m</math> Spline</i>	$5.17 \cdot 10^{-3}$	$\omega_1$ : 128 data points
<i>approximant</i>	$8.91 \cdot 10^{-3}$	$\omega_{i:i=1,2,3}$ : 33 data points
<i><math>D^m</math> Spline</i>	$1.07 \cdot 10^{-2}$	$\omega_{i:i=1,2,3}$ : 33 data points

Table 1.

Function $g$	$Q_{error} \left( \bigcup_i (x_i, y_i, z_i) \right)$	Data
<i>approximant</i>	$1.21 \cdot 10^{-2}$	$\omega_1$ : 128 data points
<i><math>D^m</math> Spline</i>	$1.13 \cdot 10^{-2}$	$\omega_1$ : 128 data points
<i>approximant</i>	0.18	$\omega_{i:i=1,2,3}$ : 33 data points
<i><math>D^m</math> Spline</i>	0.24	$\omega_{i:i=1,2,3}$ : 33 data points

Table 2



Considering the dataset corresponding to the first triangulation (32 triangles), such values are considered very good in the context of surface approximation. Let us note that we obtain the same error order if we use more data points and a larger finite element space : for instance, using 1152 data points in  $\omega_1$ , and using 81 ( $= 9 \times 9$ ) quadrangular BFS finite elements, we obtain a quadratic error of  $1.4 \cdot 10^{-3}$  on  $\Omega$  for the approximant of the function  $f$ .

The second triangulation (6 triangles) illustrates the method in the case of fewer data points.

For the reader interested in learning more about possible applications of this technique to realistic cases, numerical examples to real geophysical data (large datasets up to several hundreds of thousands data points coming from an old glacial valley located in the Vallée d'Ossau, Pyrénées Mountains, France) are given in [4]. The regularity obtained, which can be  $C^0$ ,  $C^1$ , or higher, allows to describe the topography of real geophysical surfaces accurately. Future work will focus on investigating automatic methods to choose the finite element grid and, when necessary, the local refinement, as well as on the choice of quadrature formula with better order of approximation.

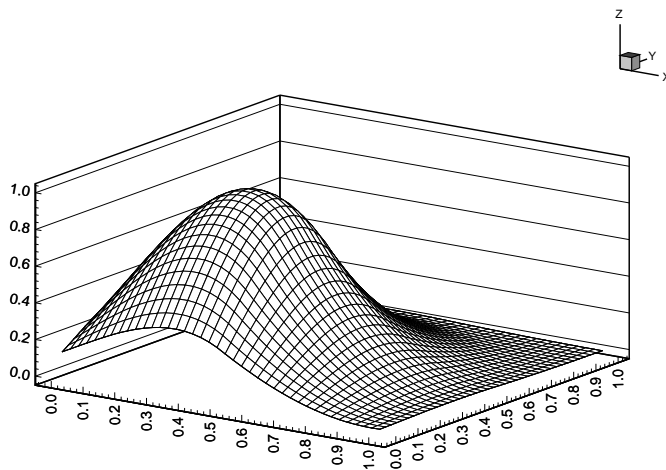


Figure 3: Function  $f$ .

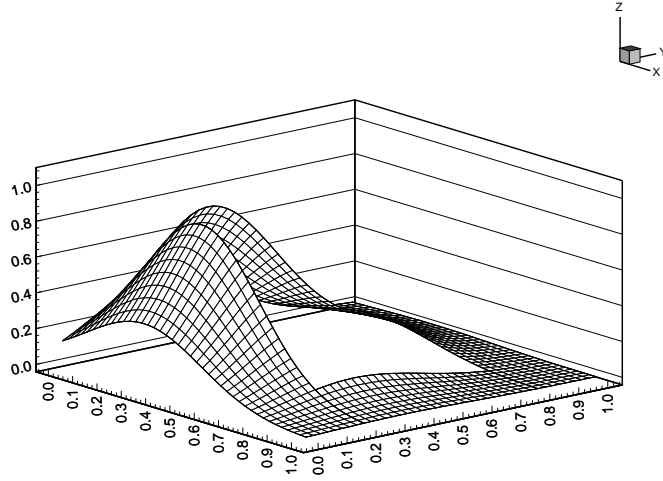


Figure 4: Data corresponding to  $f_{\omega_1}$ .

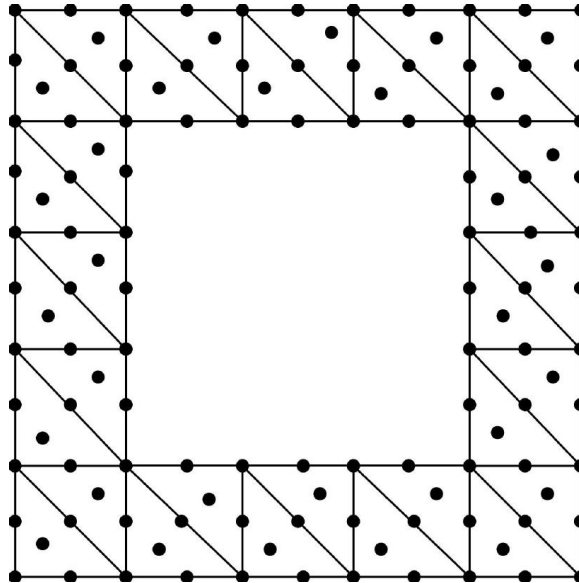


Figure 5: Triangulation of  $\omega_1$ .

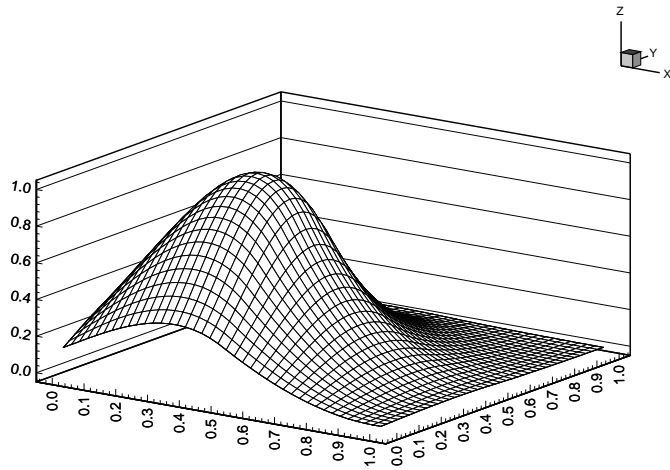


Figure 6:  $C^1$  approximant.

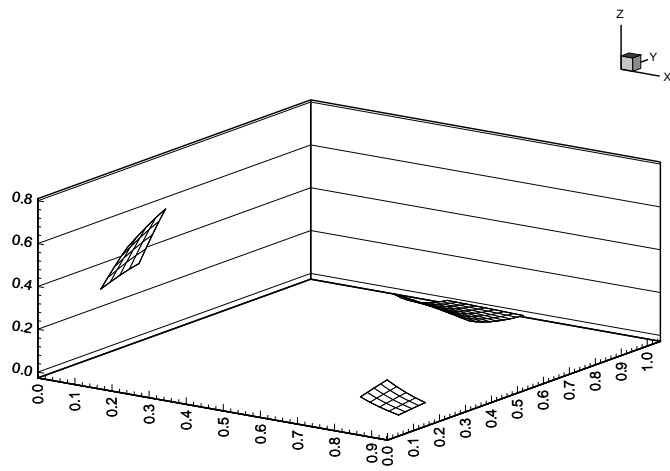


Figure 7: Data corresponding to  $f_{|\omega_{1,2,3}}$ .

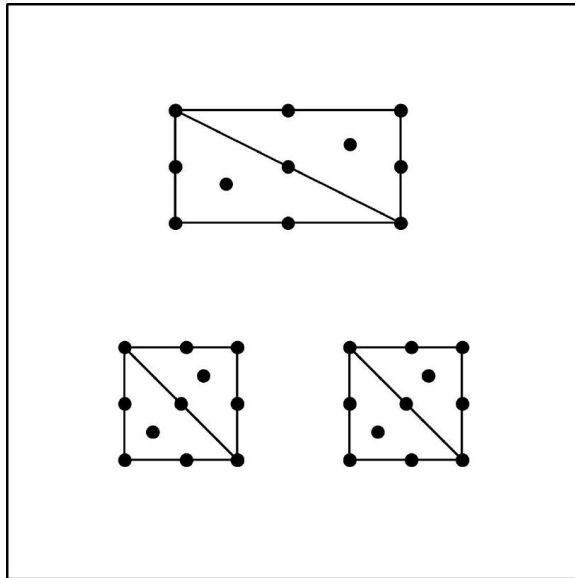


Figure 8: Triangulation of  $\omega_{1,2,3}$ .

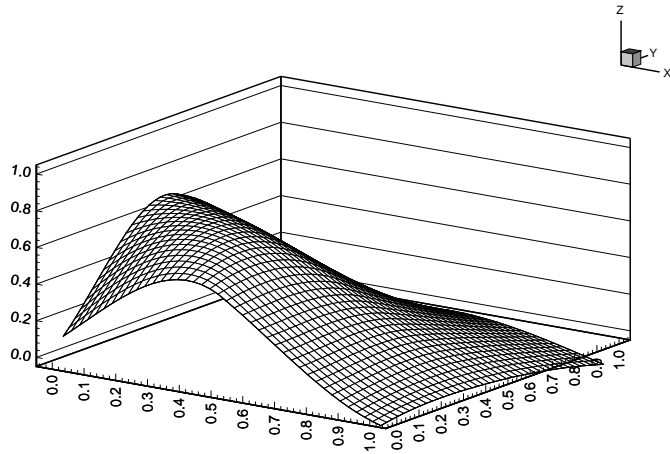


Figure 9:  $C^1$  approximant.

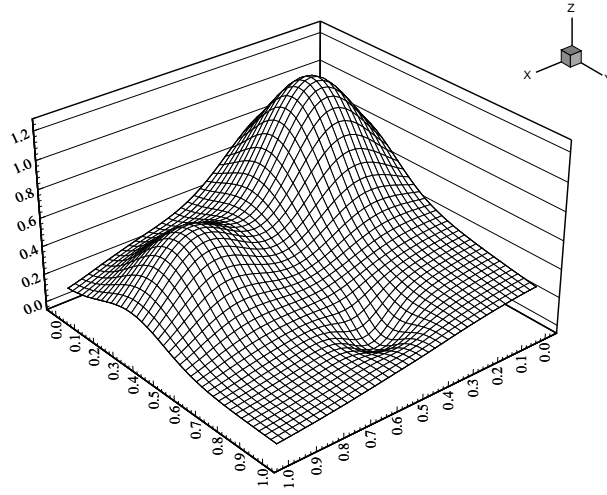


Figure 10: Function  $g$ .

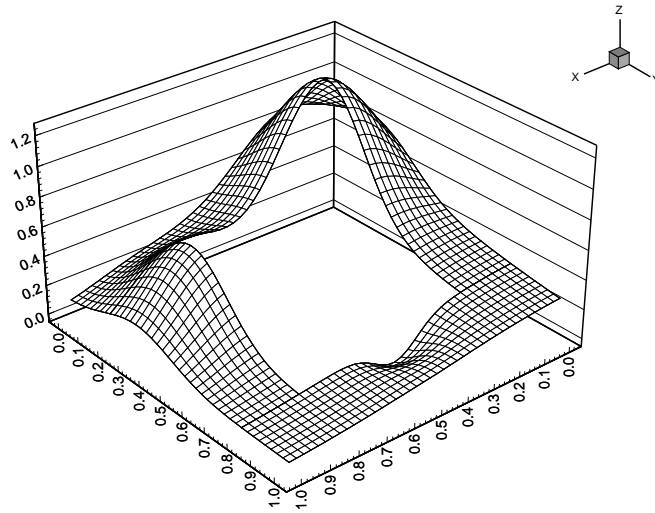


Figure 11: Data corresponding to  $g_{\omega_1}$ .

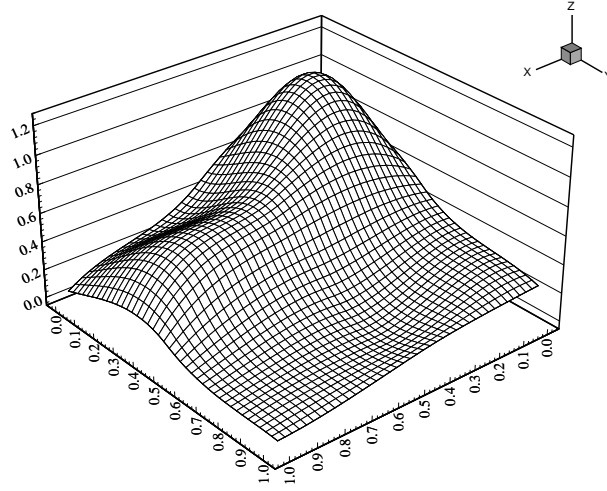


Figure 12:  $C^1$  approximant.

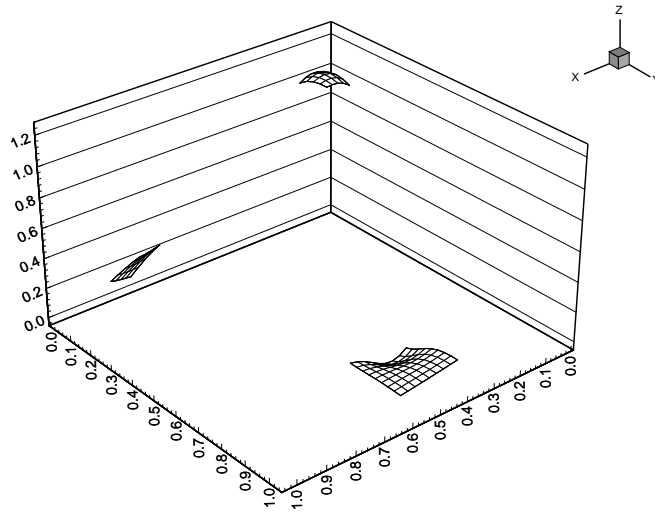


Figure 13: Data corresponding to  $g_{l\omega_{1,2,3}}$ .

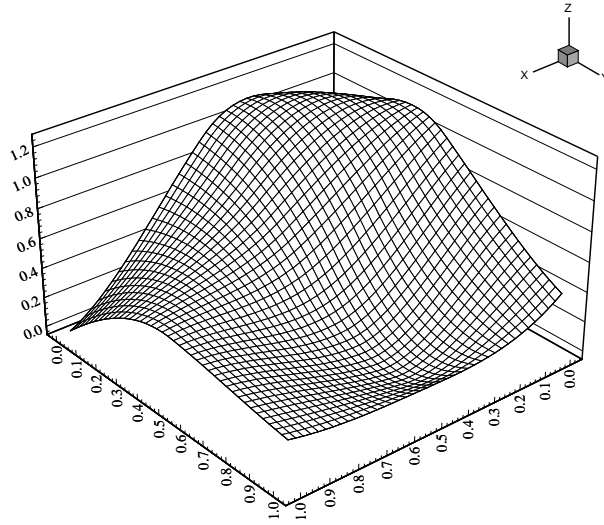


Figure 14:  $C^1$  approximant.

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