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# BLOWUP FOR THE B- FAMILY EQUATION

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ABSTRACT. In this paper we consider the b-Family Equations on the  $\mathbb{T}$   
 $u_t + u_{txx} + (b + 1)uu_x = bu_xu_{xx} + uu_{xxx} = 0$ , that for appropriate values of  $b$   
 includes well known models, such as Camassa - Holm equation or the Degasperis -  
 Procesi equation. We establish a local-in-space blowup criterion

## 1. BLOWUP FOR THE NON PERIODIC B- FAMILY EQUATION

Many literature is write about non periodic and periodic Camassa - Holm (C-H) and Degasperis- Procesi equations. We can see that the Camassa - Holm equation is a bi Hamiltonian model for waves in shallow water while the Degasperis - Procesi equation was discovered in research of an integrable equation similar that (C-H). Both equation played a role in the study about water waves. The basic question for this type the Cauchy problem are : " Local Well-posed" and if possible take the time interval in an arbitrary manner When the answer is negative then one expects give a estimate about time of lifespan of the solution. In this sense is our contribution the study of the conditions enough for that the time of lifespan is finite. Let us focus in the next periodic Cauchy problem

$$(1.1) \quad \begin{cases} u_t + u_{txx} + (b + 1)uu_x = bu_xu_{xx} + uu_{xxx}, & x \in \mathbb{T}, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{T}. \end{cases}$$

, In the study of the EDPs that describe the motion of the water waves, (1.1) is know as periodic B-Family equation, where  $b$  is a real parameter and the fluid velocity  $u(x, t)$  is defined on the torus  $\mathbb{T}$ . If we denote  $y = u - u_{xx}$  (momentum density). we can rewrite (1.1) as

$$\begin{cases} y_t + uy_x + byu_x = 0, & x \in \mathbb{T}, \quad t > 0, \\ y_0(x) = u_0 - u_{0,xx}(x), & x \in \mathbb{T}. \end{cases}$$

The B- family equation can be derived as the family of asymptotically equivalent shallow water equation. When  $b = 2$  and  $b = 3$  (1.1) recover (C-H) and (D-P) periodic reciprocally. The advantage of these values of "b" is the fact that are only values which (1.1) is completely integrable. Thanks to this, there is a considerable amount about the blow up criteria and global existence criteria of (1.1). With respect blow up criteria is usually non local with relation of the variable  $x$ . The reader unfamiliar with the subtleties of research in waves in shallow water, may not understand what it means "non local with relation of the  $v$  variable spatial". In ([4],[3]) the condition for which have the scenario blow up is by perturbation the initial data  $u_0$  in some region in  $\mathbb{T}$  or  $\mathbb{R}$ . As in our last article [1], we would like use similar tools but in B family equation then the aim is only make a perturbation the initial data in one point the next way: if there exist  $x_0$  such that

$u_{0_x}(x_0) \leq -\beta_b |u_0(x_0)|$  or  $u_{0_x}(x_0) \geq \beta_b |u_0(x_0)|$ , where  $\beta_b$  is positive constant depending of  $b$  then we are in scenario blow up for (1.1).

The interesting think about the criterion "local blow up" is that recover many non local blow up criterion . Then this short article could be seen as "How feasible is to use the tools developed in [1], for the  $B$  periodic family equation?".

Let starts to give the framework ( local Well-posed, global existence, nonlocal criterion blow up). After we show the main theorem. Finally anything comments (non periodic b equation, open question). It is convenient to rewrite the periodic Cauchy problem (1.1) in the following weak form

$$(1.2) \quad \begin{cases} u_t + uu_x + \partial_x p * \left[ \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right] = 0, & x \in \mathbb{T}, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{T} \\ u(t, x) = u(t, x + 1) \quad t \geq 0, \quad x \in \mathbb{T} \end{cases}$$

were

$$(1.3) \quad p(x) = \frac{\cosh(x - [x] - \frac{1}{2})}{2 \sinh(\frac{1}{2})},$$

is the fundamental solution of the operator  $1 - \partial_x^2$  and  $[\cdot]$  stands for the integer part of  $x \in \mathbb{R}$ . The next theorem we help us.

**Definition 1.1.** If  $u \in C([0, T], H^s(\mathbb{T})) \cap C^1((0, T^*), H^{s-1}(\mathbb{T}))$ , with  $s > \frac{3}{2}$  satisfies (1.2) then  $u$  is called a strong solution to (1.2). If  $u$  is a strong solution on  $[0, T)$  for every  $T > 0$ , then is called global strong solution of (1.2).

If  $u_0 \in H^s(\mathbb{T})$ ,  $s > \frac{3}{2}$ , we can applying the theorem of Kato[5] and we have the Following Local Well-posed result of (1.2).

**Theorem 1.2** (See [3]). *For any constant  $b$ , given  $u_0 \in H^s(\mathbb{T})$ ,  $s > \frac{3}{2}$ , then there exist a maximal  $T^* = T^*(\|u_0\|_{H^s}) > 0$  and a unique strong solution  $u$  to (1.2), such that*

$$(1.4) \quad u = u(\cdot, u_0) \in C([0, T^*), H^s(\mathbb{T})) \cap C^1((0, T^*), H^{s-1}(\mathbb{T})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping  $u_0 \mapsto u(\cdot, u_0) : H^s(T) \rightarrow C([0, T^*]; H^s(T)) \cap C^1([0, T^*]; H^{s-1}(T))$  is continuous.

More precisely, the maximal  $T$  in Theorem may be chosen independent of  $s$  in the following sense:

**Theorem 1.3** (See [3]). *If  $u = u(\cdot, u_0) \in C([0, T^*), H^s(\mathbb{T})) \cap C^1((0, T^*), H^{s-1}(\mathbb{T}))$  to (1.2) and  $u_0 \in H^{s'}(\mathbb{T})$  for some  $s' \neq s$ ,  $s' > \frac{3}{2}$ , then  $u = u(\cdot, u_0) \in C([0, T^*), H^{s'}(\mathbb{T})) \cap C^1((0, T^*), H^{s'-1}(\mathbb{T}))$  and with same  $T^*$ . In particular, if  $u_0 \in \cap_{s \geq 0} H^s$ , then  $u \in C([0, T^*), H^\infty(\mathbb{T}))$ .*

Also, we specify the blow up scenario for  $B$ - family equation.

**Theorem 1.4** (See [3]). *Assume  $b \in \mathbb{R}$  and  $u_0 \in H^s(\mathbb{T})$ ,  $s > \frac{3}{2}$ . Then blow up of the strong solution  $u = u(\cdot, u_0)$  in finite time occurs if only if*

$$(1.5) \quad \lim_{t \rightarrow T^*} \inf \{ (2b - 1) \inf_{x \in \mathbb{R}} [u_x(t, x)] \} = -\infty$$

Before presenting our contribution, we will present the known blow up theorems with respect to (1.2)

**Theorem 1.5** (See [3]). *Let  $\frac{5}{3} < b \leq 3$  and  $\int_{\mathbb{T}} (u'_0)^3(x) dx < 0$ . Assume that  $u_0 \in H^s(\mathbb{T}), s > \frac{3}{2}, u_0 \not\equiv 0$ , and the corresponding solution  $u(t)$  (1.2) has a zero for any time  $t \geq 0$ . Then, the solution  $u(t)$  of the (1.2) blow up finite time*

Next blow up theorem used the fact that if  $u(x, t)$  is a solution to (1.2) with initial datum  $u_0$ , then  $-u(t, -x)$  is also a solution to (1.2) with initial datum  $-u_0(-x)$ . Hence due to the uniqueness of the solutions, the solution to (1.2) is odd as long as the initial datum  $u_0(x)$  is odd. See [3]

**Theorem 1.6** (See [3]). *Let  $1 < b \leq 3$  and  $u_0 \in H^s(\mathbb{T})$   $s > \frac{3}{2}$  be odd and nonzero. If  $u'_0(0) \leq 0$ , then the corresponding solution of (1.2) blow up in finite time.*

Here we starting the goal of this short article. Let us introduce the next definitions.

**Definition 1.7.** For any real  $\beta$ , let us consider the 1-periodic function

$$(1.6) \quad w(x) = p(x) + \beta \partial_x p(x)$$

where  $p$  is the kernel introduced in (1.2) and  $\partial_x p$  denotes the distributional derivative on  $\mathbb{R}$ , that agrees in this case with the classical a.e pointwise derivative on  $\mathbb{R} \setminus \mathbb{Z}$ . The non-negativity condition  $w \geq 0$  is equivalent to the inequality  $\cosh(1/2) \geq \pm \beta \sinh(1/2)$ , i.e., to the condition

$$(1.7) \quad -\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}.$$

Throughout this section, we will work under the above condition on  $\beta$ .

**Definition 1.8.**

$$(1.8) \quad E_\beta = \{u \in L^1_{loc}(0, 1) : \|u\|_{E_\beta}^2 = \int_0^1 w(x)(u^2 + u_x^2)(x) dx < \infty\},$$

where the derivative is understood in the distributional sense. Notice that  $E_\beta$  agrees with the classical Sobolev space  $H^1(0, 1)$  when  $|\beta| < \frac{e+1}{e-1}$ , as in this case  $w$  is bounded and bounded away from 0, and the two norms  $\|\cdot\|_{E_\beta}$  and  $\|\beta\|_{H^1}$  are equivalent. The situation is different for  $\beta = \pm \frac{e+1}{e-1}$  as  $E_\beta$  is strictly larger than  $H^1(0, 1)$  in this case. Indeed, we have

$$(1.9) \quad w(x) = \frac{2e}{(e-1)^2} \sinh(x), \quad x \in (0, 1), \quad \left( \text{if } \beta = \frac{e+1}{e-1} \right).$$

The elements of  $E_{(e+1)/(e-1)}$ , after modification on a set of measure zero, are thus continuous on  $(0, 1]$ , but may be unbounded for  $x \rightarrow 0^+$  (for instance  $|\log(x/2)|^{1/3} \in E_{(e+1)/(e-1)}$ ). In the same way,

$$(1.10) \quad w(x) = \frac{2e}{(e-1)^2} \sinh(1-x), \quad x \in (0, 1), \quad \left( \text{if } \beta = -\frac{e+1}{e-1} \right).$$

After modification on a set of measure zero, the elements of  $E_{-(e+1)/(e-1)}$  are continuous on  $[0, 1)$ , but may be unbounded for  $x \rightarrow 1^-$ .

**Definition 1.9.** Let the closed subspace  $E_{\beta,0}$  of  $E_\beta$  defined (with slightly abusive notation) as follows:

$$\begin{aligned} E_{\beta,0} &= H_0^1(0,1) = \{u \in H^1(0,1) : u(0) = u(1) = 0\}, \quad \text{if } |\beta| < \frac{e+1}{e-1}, \\ E_{\beta,0} &= \{u \in E_\beta : u(1) = 0\}, \quad \text{if } \beta = \frac{e+1}{e-1} \\ E_{\beta,0} &= \{u \in E_\beta : u(0) = 0\}, \quad \text{if } \beta = -\frac{e+1}{e-1}. \end{aligned}$$

Equivalently,  $E_{\beta,0}$  could be defined as the closure of  $C_c^\infty(0,1)$  in  $E_\beta$ . This is of course for  $|\beta| < \frac{e+1}{e-1}$ . For  $\beta = \pm \frac{e+1}{e-1}$  our claim follows from the next lemma.

**Lemma 1.10** (See [2]). *Let  $\beta = \pm \frac{e+1}{e-1}$  and  $u \in E_{\beta,0}$ .*

*Then there is a sequence  $(u_n) \subset C_c^\infty(0,1)$  such that  $\|u_n - u\|_{E_\beta} \rightarrow 0$ .*

*Proof.* This demonstration is found in [2]. □

Thus the elements of  $E_{\beta,0}$  satisfy to the weighted Poincaré inequality below:

**Lemma 1.11.** *For all  $|\beta| \leq \frac{e+1}{e-1}$ , there exists a constant  $C > 0$  such that*

$$(1.11) \quad \forall v \in E_{\beta,0} \int_0^1 w(x) v^2(x) dx \leq C \int_0^1 w(x) v_x^2(x) dx.$$

*Proof.* This demonstration is found in [2]. □

Now, we start preparing some notations.

**Definition 1.12.** For any real constant  $b \neq 1$  and  $\beta$ , let  $J(b, \beta) \geq -\infty$ , defined by

$$(1.12) \quad J(b, \beta) = \inf \left\{ \int_0^1 (p + \beta \partial_x p) \left( \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) dx; u \in H^1(0,1), u(0) = u(1) = 1 \right\}$$

and

$$(1.13) \quad \beta_b = \inf \left\{ \beta > 0 : \beta^2 + \frac{2}{|b-1|} \left( J(b, \beta) - \frac{b}{2} \right) \geq 0 \right\}.$$

Notice that a priori  $0 \leq \beta_b \leq +\infty$ , as the set on the right-hand side could be empty. Then, with this definitions we can give the main results in this paper.

**Theorem 1.13.** *Let  $b \in ]1, 3]$  be such that  $\beta_b$  is finite. Let  $u_0 \in H^s(\mathbb{T})$  be with  $s > \frac{3}{2}$  and assume that there exist  $x_0 \in \mathbb{T}$ , such that*

$$(1.14) \quad u_0'(x_0) < -\beta_b |u_0(x_0)|$$

*then the corresponding solution  $u$  of (1.2) in  $C([0, T^*), H^s(\mathbb{T})) \cap C^1([0, T^*), H^{s-1}(\mathbb{T})$  arising from  $u_0$  blows up in finite time. Moreover, the maximal time  $T^*$  is estimated by*

$$(1.15) \quad T^* \leq \left( \frac{2}{b-1} \right) \frac{1}{\sqrt{(u_0'(x_0))^2 - \beta_b^2 u_0^2(x_0)}}$$

For the proof of the theorem 1.13, we need the following propositions

**Proposition 1.1.** *Let  $b \leq 3$  then we have*

$$(1.16) \quad J(b, \beta) > -\infty \Leftrightarrow \begin{cases} |\beta| \leq \frac{e+1}{e-1} \\ \frac{b}{3-b} > -\frac{1}{C_\beta}, \end{cases}$$

where  $C_\beta > 0$  is the best Poincaré constant in inequality (1.11).

*Proof.* Putting  $u = v + 1$  and observing that  $\int_0^1 w(x) dx = 1$ , we see that

$$(1.17) \quad J(b, \beta) = \frac{b}{2} + \inf\{T(v) : v \in H_0^1(0, 1)\},$$

where

$$(1.18) \quad T(v) = \int_0^1 w(x) \left( \frac{b}{2}(v^2 + 2v) + \left( \frac{3-b}{2} \right) v_x^2 \right) (x) dx$$

Assume that  $J(b, \beta) > -\infty$ . Then to show  $|\beta| \leq \frac{e+1}{e-1}$ , we refer to the proof of proposition 3.3. in [2]. To prove the second inequality, we only have to treat the case  $b < 0$ . Applying the inequality

$$(1.19) \quad \int_0^1 w(x) \left( \frac{b}{2}(n^2 v^2 + 2nv) + \left( \frac{3-b}{2} \right) n^2 v_x^2 \right) (x) dx \geq J(b, \beta) - \frac{b}{2},$$

valid for all  $v \in H_0^1(0, 1)$  and all  $n \in \mathbb{N}$  and letting  $n \rightarrow \infty$ , we get

$$\int_0^1 w(x) \left( \frac{b}{2} v^2 + \left( \frac{3-b}{2} \right) v_x^2 \right) (x) dx \geq 0.$$

We deduce:

$$\int_0^1 w(x) v^2(x) dx \leq -\frac{3-b}{b} \int_0^1 w(x) v_x^2(x) dx$$

Then we get  $\frac{b}{3-b} \geq -\frac{1}{C_\beta}$ . By a similar argument used in the proposition 3.3. in [2] we can said that the inequality is strict.

Conversely, assume that  $|\beta| \leq \frac{e+1}{e-1}$ . By the weighted Poincaré inequality (1.11), we can consider an equivalent norm in  $E_{\beta,0}$ , as

$$(1.20) \quad \|v\|_{E_{\beta,0}} = \int_0^1 w(x) v_x(x) dx.$$

As  $\frac{b}{3-b} \geq -\frac{1}{C_\beta}$ , the symmetric bilinear form

$$(1.21) \quad B(u, v) = \int_0^1 w(x) \left( \frac{b}{2} uv + \left( \frac{3-b}{2} \right) u_x v_x \right) (x) dx$$

is coercive on the Hilbert space  $E_{\beta,0}$ . Applying the Lax-Milgram theorem yields the existence and uniqueness of a minimizer  $\hat{v} \in E_{\beta,0}$  for the functional  $T$ .

But  $H_1^0(0, 1) \subset E_{\beta,0}$ , so in particular, we get  $J(b, \beta) > -\infty$ . Moreover, if  $|\beta| < \frac{e+1}{e-1}$ , then recalling  $E_{\beta,0} = H_0^1(0, 1)$  we see that  $J(b, \beta)$  is in fact a minimum, achieved at  $\hat{u} = 1 + \hat{v} \in H^1(0, 1)$ .  $\square$

Looking at the last proposition, one might ask: what is the reason for the restriction  $b \leq 3$ . The answer is given by the following lemma.

**Lemma 1.14.** *Let  $b > 3$ , then  $J(b, \beta) = -\infty$ , for all  $\beta \in \mathbb{R}$ .*

*Proof.* As a necessary condition for  $J(b, \beta) > -\infty$  is that  $|\beta| \leq \frac{e+1}{e-1}$ , then we will take  $\beta$  of this way. Let

$$(1.22) \quad u_n(x) = 1 + \frac{1}{2} \sin(n2\pi x) \Rightarrow u_n'(x) = n\pi \cos(n2\pi x),$$

In fact, for each  $n \in \mathbb{N}$   $u_n \in H^1(0, 1)$ ,  $u_n(1) = u_n(0) = 1$ . Thus there is a constant  $c_1 > 0$  independent of  $n$ , such that

$$\forall n \in \mathbb{N} \quad 0 \leq \frac{b}{2} \int_0^1 w(x) u_n^2(x) dx \leq c_1,$$

and

$$\frac{3-b}{2} \int_0^1 w(x) (u_n')^2(x) dx \rightarrow -\infty,$$

because  $b > 3$  and then  $J(b, \beta) = -\infty$ . □

The next lemma provides some useful information on  $J(b, \beta)$ .

**Lemma 1.15.** *The function  $(b, \beta) \mapsto J(b, \beta) \in \mathbb{R} \cup \{-\infty\}$  defined for all  $(b, \beta) \in \mathbb{R}^2$  is concave with respect to each one of its variables and is even with respect to the variable  $\beta$ . Also  $\forall (b, \beta) \in \mathbb{R}^2$ ,  $-\infty \leq J(b, \frac{e+1}{e-1}) \leq J(b, \beta) \leq J(b, 0) \leq \frac{b}{2}$ .*

*Proof.* The proof is similar that the proposition 3.4. in [2] □

Next lemma motivates the introduction of quantity  $J(b, \beta)$  in relation with B-equation.

**Proposition 1.2.** *Let  $(\alpha, \beta) \in \mathbb{R}^2$  and  $u \in H^1(\mathbb{T})$ , we get*

$$\forall x \in \mathbb{T}, \quad (p + \beta \partial_x p) * \left( \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) (x) \geq J(b, \beta) u^2(x)$$

*Proof.* Let  $\alpha = \alpha(b, \beta)$  be some constant. Because of the invariance under translation, we get that

$$(1.23) \quad (p + \beta \partial_x p) * \left( \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) (x) \geq \alpha u^2(x)$$

holds true for all  $u \in H^1(\mathbb{T})$  and all  $x \in \mathbb{T}$  if and only if

$$(1.24) \quad (p + \beta \partial_x p) * \left( \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) (1) \geq \alpha u^2(1)$$

holds true for all  $u \in H^1(\mathbb{T})$ . But on the interval  $]0, 1[$ ,  $(p + \beta \partial_x p)(1-x) = (p - \beta \partial_x p)(x)$ . Then we get

$$(1.25) \quad (p + \beta \partial_x p) * \left( \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) (1) = \int_0^1 (p - \beta \partial_x p) \left( \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) (x) dx.$$

Normalizing to obtain  $u(1) = 1$ , we get that the best constant  $\alpha$  in inequality (1.23) satisfies  $\alpha = J(b, -\beta) = J(b, \beta)$ . □

Next proposition provides an a priori estimate of  $J(b, \beta)$ , when  $b \in [-1, 3]$ .

**Proposition 1.3.** *Let  $-1 \leq b \leq 3$ ,  $|\beta| \leq \frac{e+1}{e-1}$ . Then*

$$(p \pm \beta \partial_x p) * \left( \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) \geq \begin{cases} \delta_b u^2, & \text{if } |\beta| \leq 1 \\ \frac{\delta_b}{2} [(e+1) - |\beta|(e-1)], & \text{if } 1 \leq |\beta| \leq \frac{e+1}{e-1} \end{cases}$$

where

$$(1.26) \quad \delta_b = \frac{\sqrt{3-b}}{4} \left( \sqrt{3(1+b)} - \sqrt{3-b} \right).$$

*Remark 1.16.* Notice that for  $0 \leq b \leq 3$ , the constant  $\delta_b \geq 0$ .

*Proof.* It is sufficient to consider the case  $0 \leq \beta \leq \frac{e+1}{e-1}$ . we make the convolution estimates for  $(p + \beta \partial_x p)$ , the convolution estimates for  $(p - \beta \partial_x p)$  being similar. First observe that:

$$(1.27) \quad \forall x \in \mathbb{R} \quad p(x) = \frac{e^{x-\frac{1}{2}-[x]}}{4 \sinh \frac{1}{2}} + \frac{e^{-x+\frac{1}{2}-[x]}}{4 \sinh \frac{1}{2}} =: p_1(x) + p_2(x).$$

We start with the estimate of  $p_1 * (a^2 u^2 + u_x^2)(1)$ , with  $a \in \mathbb{R}$  to be determined later. We get

$$\begin{aligned} p_1 * (a^2 u^2 + u_x^2)(1) &= \frac{1}{4 \sinh(\frac{1}{2})} \int_0^1 e^{\frac{1}{2}-\xi} (a^2 u^2 + u_x^2)(\xi) dx \\ &\geq \frac{-a}{4 \sinh(\frac{1}{2})} \int_0^1 e^{\frac{1}{2}-\xi} (2u u_x)(\xi) dx \\ &= \frac{-a}{4 \sinh(\frac{1}{2})} (e^{\frac{-1}{2}} - e^{\frac{1}{2}}) u^2(1) - \frac{1}{4 \sinh(\frac{1}{2})} \int_0^1 e^{\frac{1}{2}-\xi} a u^2 dx \\ &= \frac{a}{2} u^2(1) - p_1 * (a u^2)(1). \end{aligned}$$

Hence

$$p_1 * ((a^2 + a) u^2 + u_x^2)(1) \geq \frac{a}{2} u^2(1),$$

and because of the invariance under translations, we get

$$(1.28) \quad p_1 * ((a^2 + a) u^2 + u_x^2) \geq \frac{a}{2} u^2.$$

Similarly:

$$\begin{aligned} p_2 * (a^2 u^2 + u_x^2)(1) &= \frac{1}{4 \sinh(\frac{1}{2})} \int_0^1 e^{\xi-\frac{1}{2}} (a^2 u^2 + u_x^2)(\xi) dx \\ &\geq \frac{a}{4 \sinh(\frac{1}{2})} \int_0^1 e^{\xi-\frac{1}{2}} (2u u_x)(\xi) dx \\ &= \frac{a}{4 \sinh(\frac{1}{2})} (e^{\frac{1}{2}} - e^{\frac{-1}{2}}) u^2(1) - \frac{1}{4 \sinh(\frac{1}{2})} \int_0^1 e^{\xi-\frac{1}{2}} a u^2 dx \\ &= \frac{a}{2} u^2(1) - p_2 * (a u^2)(1). \end{aligned}$$

Hence, again using the invariance under translations, we get

$$(1.29) \quad p_2 * ((a^2 + a) u^2 + u_x^2) \geq \frac{a}{2} u^2.$$



Choose  $a$  such that  $a^2 + a = \frac{b}{3-b}$ , if  $-1 \leq b < 3$  (if  $b = 3$ , the proposition is trivial and there is nothing to prove). We get:

$$(1.30) \quad p_1 * \left( \frac{b}{2}u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) \geq \frac{\delta_b}{2}u^2,$$

$$(1.31) \quad p_2 * \left( \frac{b}{2}u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) \geq \frac{\delta_b}{2}u^2.$$

(1.32)

Now, from the identity  $p = p_1 + p_2$  and  $\partial_x p = p_1 - p_2$ , that holds both in the distributional and in the point wise sense in  $\mathbb{R} \setminus \mathbb{Z}$ , we get

$$(1.33) \quad p + \beta \partial_x = (1 + \beta)p_1 + (1 - \beta)p_2.$$

If  $0 \leq \beta \leq 1$ , from (1.30) and (1.33), we deduce

$$(1.34) \quad (p + \beta \partial_x) * \left( \frac{b}{2}u^2 + \left( \frac{3-b}{2}u_x \right) \right) \geq [(1 + \beta) + (1 - \beta)] \frac{\delta_b}{2}u^2 = \delta_b u^2.$$

Notice that (1.34) holds for  $-1 \leq b \leq 3$ , as for this range the equation  $a^2 + a = \frac{b}{3-b}$  can be solved. If  $1 \leq \beta \leq \frac{e+1}{e-1}$ , we observe that we have the point wise estimate:

$$(1.35) \quad p(x) \leq e p_1(x), \forall x \in (0,1).$$

Hence,

$$(1.36) \quad p + \beta \partial_x = (1 + \beta)p_1 - (\beta - 1)p_2$$

$$(1.37) \quad \geq [(e + 1) - \beta(e - 1)]p_1$$

We deduce, using (1.29):

$$(1.38) \quad \forall 1 \leq \beta \leq \frac{e+1}{e-1} \quad (p + \beta \partial_x) * \left( \frac{b}{2}u^2 + \left( \frac{3-b}{2}u_x \right) \right) \geq [(e + 1) - \beta(e - 1)] \frac{\delta_b}{2}u^2$$

□

*Remark 1.17.* If  $-1 \leq b \leq 3$ ,  $|\beta| \leq \frac{e+1}{e-1}$ , then  $J(b, \beta) \geq \delta_b$ .

*Proof. Theorem 1.13* Applying a simple density argument, we only need to show that the above theorem with some  $s \geq 3$ . Here without loss generality we can suppose that  $u_0 \in H^3(\mathbb{T})$ . We thus obtain a unique solution of (1.2), defined in some nontrivial interval  $[0, T[$ , and such that  $u \in C([0, T[, H^3(\mathbb{T})) \cap C^1([0, T[, H^2(\mathbb{T}))$ . The starting point is the analysis of the flow map  $q(t, x)$  of (1.2)

$$(1.39) \quad \begin{cases} q_t(t, x) = u(t, q(t, x)) & x \in \mathbb{T}, \quad t \in [0, T^*), \\ q(0, x) = x, & x \in \mathbb{T}. \end{cases}$$

As  $u \in C^1([0, T[, H^2(\mathbb{T}))$ , we can see that  $u$  and  $u_x$  are continuous on  $[0, T[ \times \mathbb{T}$  and  $x \mapsto u(t, x)$  is Lipschitz, uniformly with respect to  $t$  in any compact time interval in  $[0, T[$ . Then the flow map  $q(t, x)$  is well defined by (1.39) in the time interval  $[0, T[$  and

$q \in C^1([0, T[ \times \mathbb{R}, \mathbb{R})$ . Differentiating (1.2) with respect to the  $x$  variable and applying the identity  $\partial_x^2 p * f = p * f - f$

$$u_{tx} + uu_{xx} = \frac{b}{2}u^2 - \left(\frac{b-1}{2}\right)u_x^2 - p * \left[\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right].$$

Let us introduce the two  $C^1$  functions of the time variable depending on  $\beta$ . The constant  $\beta$ , will be chosen later on

$$f(t) = (-u_x + \beta u)(t, q(t, x_0)) \quad \text{and} \quad g(t) = -(u_x + \beta u)(t, q(t, x_0)).$$

Using 1.39 and differentiating with respect  $t$ , we get

$$\begin{aligned} \frac{df}{dt}(t) &= [(-u_{tx} - uu_{xx}) + \beta(u_t + uu_x)](t, q(t, x_0)) \\ &= -\frac{b}{2}u^2 + \left(\frac{b-1}{2}\right)u_x^2 + (p - \beta \partial_x p) * \left[\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right](t, q(t, x_0)), \end{aligned}$$

and

$$\begin{aligned} \frac{dg}{dt}(t) &= [(-u_{tx} - uu_{xx}) - \beta(u_t + uu_x)](t, q(t, x_0)) \\ &= -\frac{b}{2}u^2 + \left(\frac{b-1}{2}\right)u_x^2 + (p + \beta \partial_x p) * \left[\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right](t, q(t, x_0)). \end{aligned}$$

Let us first consider  $b \in ]1, 3]$ . From the de definition of  $\beta_b$  (1.13) and the condition  $\beta_b < \infty$ , we deduce that there exist  $\beta \geq 0$  such that

$$(1.40) \quad \beta^2 \geq \frac{2}{b-1} \left( \frac{b}{2} - J(b, \beta) \right).$$

Applying the convolution estimate of (1.2) and the fact that  $J(b, \beta) = J(b, -\beta)$ .

$$\begin{aligned} \frac{df}{dt}(t) &\geq \left(\frac{b-1}{2}\right)u_x^2 + \left(J(b, -\beta) - \frac{b}{2}\right)u^2(t, q(t, x_0)) \\ &\geq \frac{b-1}{2}(u_x^2 - \beta^2 u^2)(t, q(t, x_0)) \\ &= \frac{b-1}{2}[f(t)g(t)] \end{aligned}$$

In the same way,

$$\begin{aligned} \frac{dg}{dt}(t) &\geq \left(\frac{b-1}{2}\right)u_x^2 + \left(J(b, \beta) - \frac{b}{2}\right)u^2(t, q(t, x_0)) \\ &\geq \frac{b-1}{2}(u_x^2 - \beta^2 u^2)(t, q(t, x_0)) \\ &= \frac{b-1}{2}[f(t)g(t)]. \end{aligned}$$

The assumption  $u'_0(x_0) < -\beta_b |u_0(x_0)|$  guarantees that we may choose  $\beta$  satisfying (1.40) with  $\beta - \beta_b > 0$  is small enough in a way that

$$u'_0(x_0) < -\beta |u_0(x_0)|$$

For such a choice  $\beta$  we have  $f(0) > 0$  and  $g(0) > 0$ . The Blow up of  $u$  will rely on the following basic property.

**Lemma 1.18.** *Let  $0 < T^* \leq \infty$  and  $f, g \in C^1([0, T^*[, \mathbb{R})$  be such that, for some constant  $c > 0$  and all  $t \in [0, T^*[$ ,*

$$\begin{aligned}\frac{df}{dt}(t) &\geq cf(t)g(t) \\ \frac{dg}{dt}(t) &\geq cf(t)g(t).\end{aligned}$$

*If  $f(0) > 0$  and  $g(0) > 0$ , then*

$$T^* \leq \frac{1}{c\sqrt{f(0)g(0)}}.$$

□

## 2. ESTIMATES OF $\beta_b$

Theorem 1.13 is meaningful only if  $b$  is such that  $\beta_b < \infty$ . Then we propose three estimates which allowed know for that  $b \in ]1, 3]$ ,  $\beta_b$  is finite. We start with the a priori estimates and after we used the properties of  $J(b, \beta)$  for found the optimal result.

**2.1. estimates 1.** Let  $0 \leq \beta \leq \frac{e+1}{e-1}$  and  $1 < b \leq 3$ , then we have the elementary estimate

$$(p \pm \beta \partial_x p) * \left( \frac{b}{2} u^2 + \left( \frac{3-b}{2} \right) u_x^2 \right) \geq 0.$$

Here we return to the definition of  $\beta_b$

$$\beta_b = \inf \left\{ \beta > 0 : \beta^2 + \frac{2}{|b-1|} \left( J(b, \beta) - \frac{b}{2} \right) \geq 0 \right\},$$

then a sufficient condition that allowed that  $\beta < \infty$ , is :

$$(2.1) \quad |\beta| \geq \sqrt{\frac{b}{b-1}}.$$

Thus, if we consider the function  $b \mapsto \sqrt{\frac{b}{b-1}}$ , we deduce that when  $0 \leq \beta \leq 1$ , (2.1) don't have solution. While if  $1 < \beta \leq \frac{e+1}{e-1}$ , we have that if  $3 \geq b \geq \frac{(\frac{e+1}{e-1})^2}{(\frac{e+1}{e-1})^2 - 1} \approx 1.271$ , then  $\beta_b < +\infty$ .

**2.2. estimates 2.** Here we have used the proposition 1.26 and by the definition of  $\beta_b$ , we deduce that a sufficient condition for that  $\beta_b < +\infty$  is:

$$(2.2) \quad \beta^2 + \frac{2}{b-1} \left( \delta_b - \frac{b}{2} \right) \geq 0,$$

if  $0 \leq \beta \leq 1$ , while if  $1 \leq \beta \leq \frac{e+1}{e-1}$ , a sufficient condition for that  $\beta_b < +\infty$ .

$$(2.3) \quad \beta^2 + \frac{2}{b-1} \left( [(e+1) + \beta(e-1)] \frac{\delta_b}{2} - \frac{b}{2} \right) \geq 0,$$

where

$$\delta_b = \frac{\sqrt{3-b}}{4} \left( \sqrt{3(1+b)} - \sqrt{3-b} \right).$$

Thus if we studied the function  $b \mapsto \sqrt{\frac{2}{b-1} \left( \frac{b}{2} - \delta_b \right)}$ , we deduce that when  $0 \leq \beta \leq 1$ , (2.2) don't have solution for  $b \neq 2$  and for  $b = 2$ , we found  $\beta_2 = 1$ , we found the known result

for Camassa Holm equation (see [2]).

While if  $1 < \beta \leq \frac{e+1}{e-1}$ , we get

$$(2.4) \quad P(\beta) = \beta^2 + \beta \delta_b \left( \frac{e+1}{b-1} \right) + \left( \delta_b \left( \frac{e+1}{b-1} \right) - \frac{b}{b-1} \right) \geq 0$$

**2.3. estimates 3.** In this part we used the in this part, we use the properties of  $J(b, \beta)$  which is described in the lemma 1.15. Let  $b \in [1, 3]$ , we used the calculus made in [2], with the next relation : If  $|\beta| \leq \frac{e+1}{e-1}$ .

$$J(b, \beta) = \begin{cases} \frac{3-b}{2} I \left( \frac{b}{3-b}, \beta \right), & \text{if } b \neq 3 \\ \frac{3}{2} \inf \left\{ \int_0^1 w(x) u^2 dx; u \in H^1(0, 1), u(0) = u(1) = 1 \right\}, & \text{if } b = 3. \end{cases}$$

where  $I(\alpha, \beta)$  is as in [2]. If  $b \neq 3$ , by the estimates in [2], is easy arrive to

$$w(x) = p(x) + \frac{e+1}{e-1} \quad \text{and} \quad \partial_x p(x) = \frac{2e}{(e-1)^2} \sinh x, \quad x \in (0, 1)$$

and

$$\begin{aligned} J \left( b, \frac{e+1}{e-1} \right) &= \frac{3-b}{2} I \left( \frac{b}{3-b}, \frac{e+1}{e-1} \right) \\ &= \frac{3-b}{4e} (e+1)^2 \frac{P'_{v(b)}}{P_{v(b)}}(\cosh 1) \end{aligned}$$

where

$$v(b) = -\frac{1}{2} + \frac{1}{2} \cdot \sqrt{1 + 4 \cdot \left( \frac{b}{3-b} \right)} \in \{z \in \mathbb{C} : \Im(z) \geq 0\}.$$

and  $P_{v(b)}$  is Legendre function of the first kind, of the degree  $v(b)$ , defined in [2]. Thus by lemma 1.15, we have that  $J(b, \beta) > J \left( b, \frac{e+1}{e-1} \right)$ . Then a sufficient condition for that  $\beta_b < +\infty$ .

$$\begin{aligned} \beta^2 + \frac{2}{b-1} \left( J \left( b, \frac{e+1}{e-1} \right) - \frac{b}{2} \right) &\geq 0, \\ \beta^2 + \frac{3-b}{b-1} \left( I \left( b, \frac{e+1}{e-1} \right) - \frac{b}{3-b} \right) &\geq 0, \end{aligned}$$

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