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Abstract

We extend the Hybrid High-Order method introduced by the authors for the Poisson problem to problems with heterogeneous/anisotropic diffusion. The cornerstone is a local discrete gradient reconstruction from element- and face-based polynomial degrees of freedom. Optimal error estimates are proved.

Résumé

Méthodes hybrides d'ordre élevé pour des problèmes à diffusion variable sur des maillages généraux. Nous étendons la méthode hybride d'ordre élevé conçue par les auteurs pour le problème de Poisson à des problèmes de diffusion hétérogène/anisotrope. La pierre angulaire est une reconstruction locale du gradient discret à partir des degrés de liberté polynomiaux sur les éléments et les faces. On établit des estimations d'erreur optimales.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, denote an open, bounded, polytopic domain. Let $f \in L^2(\Omega)$ and, for a subset $X \subset \overline{\Omega}$, denote by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ the inner product and norm in $L^2(X)$, respectively. We focus on the following variable-diffusion problem: Find $u \in U_0 := H^1_0(\Omega)$ such that

$$(\boldsymbol{\kappa}\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_\Omega = (f, v)_\Omega \qquad \forall v \in U_0, \tag{1}$$

where $\boldsymbol{\kappa}$ is a bounded, tensor-valued function in Ω, taking symmetric values with lowest eigenvalue uniformly bounded from below away from zero. Owing to the Lax–Milgram Lemma, problem (1) is well-posed.

The approximation of diffusive problems on general polytopic meshes has received an increasing attention lately. Several low-order methods have been developed; see, e.g., [1, 2] and references therein. Recently, high-order methods have also become available; we mention the high-order Mimetic Finite Difference (MFD) schemes [3, 4], the Virtual Element Method [5], the Mixed High-Order method [6], and the Hybrid High-Order (HHO) methods [7, 8]. For the latter, the degrees of freedom (DOFs) are scalar-valued polynomials at mesh elements and faces up to some degree $k \geq 0$ (as for the MFD schemes in [4]), and the construction hinges on (i) a local discrete gradient reconstruction of order k and (ii) a least-squares local penalty that weakly enforces the matching between element- and face-based DOFs while preserving the order of the gradient reconstruction. This design leads to optimal energy- and L^2-norm error estimates; cf. [7] for the Poisson problem ($\boldsymbol{\kappa}$ being the identity tensor in (1)) and [8] for (quasi-incompressible) linear elasticity.

The purpose of the present work is to extend the HHO method of [7] to the variable-diffusion problem (1). The key idea is to modify the gradient reconstruction so as to account for the diffusion tensor $\boldsymbol{\kappa}$. Then, adapting the ideas of [7], we prove stability of the discrete problem and derive optimal error estimates. We make the reasonable assumption that there is a partition P_Ω of Ω so that $\boldsymbol{\kappa}$ is piecewise Lipschitz. For simplicity of exposition, we also assume that $\boldsymbol{\kappa}$ is a piecewise polynomial; otherwise, an additional quadrature error has to be accounted for. In applications from the geosciences, $\boldsymbol{\kappa}$ can often be taken piecewise constant.

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2. Discrete setting and local gradient reconstruction

We consider admissible mesh sequences in the sense of [9, Sect. 1.4]. Each mesh $\mathcal{T}_h$ in the sequence is a finite collection $\{T\}$ of nonempty, disjoint, open, polytopic elements such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ and $h = \max_{T\in\mathcal{T}_h} h_T$ (with h_T the diameter of T), and there is a matching simplicial submesh of $\mathcal{T}_h$ with locally equivalent mesh size and which is shape-regular in the usual sense. For all $T \in \mathcal{T}_h$, the faces of T are collected in the set $\mathcal{F}_T$. In an admissible mesh sequence, $\mathrm{card}(\mathcal{F}_T)$ is uniformly bounded, the usual discrete and multiplicative trace inequalities hold on element faces, and the L^2-orthogonal projector onto polynomial spaces enjoys optimal approximation properties on each mesh element. Let a polynomial degree $k \geq 0$ be fixed. For all $T \in \mathcal{T}_h$, we define the local space of DOFs as $\mathsf{U}_T^k := \mathbb{P}_d^k(T) \times \left\{\times_{F\in\mathcal{F}_T}\mathbb{P}_{d-1}^k(F)\right\}$, where $\mathbb{P}_d^k(T)$ (resp., $\mathbb{P}_{d-1}^k(F)$) is spanned by the restrictions to T (resp., F) of d-variate (resp., $(d{-}1)$-variate) polynomials of total degree $\leq k$. In what follows, $A \lesssim B$ denotes the inequality $A \leq CB$ with positive constant C independent of the polynomial degree k, the meshsize h, and the diffusion tensor $\boldsymbol{\kappa}$. We assume that each mesh $\mathcal{T}_h$ in the sequence is compatible with the partition P_Ω associated with the diffusion tensor. We denote by $\kappa_T^\flat$ and $\kappa_T^\sharp$ the lowest and largest eigenvalue of $\boldsymbol{\kappa}$ in T, respectively, and we introduce the local heterogeneity/anisotropy ratio $\rho_T := \kappa_T^\sharp/\kappa_T^\flat \geq 1$. In what follows, we explicitly track the dependency of the bounds on the ratio ρ_T. To avoid the profileration of symbols, we assume that for all $T \in \mathcal{T}_h$, the Lipschitz constant of $\boldsymbol{\kappa}$ in T, say L_T^κ, satisfies $L_T^\kappa \lesssim \kappa_T^\sharp$.

For all $T \in \mathcal{T}_h$, we define the local gradient reconstruction operator $\boldsymbol{G}_T^k : \mathsf{U}_T^k \to \boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T)$ such that, for all $\mathsf{v} := \left(\mathsf{v}_T, (\mathsf{v}_F)_{F\in\mathcal{F}_T}\right) \in \mathsf{U}_T^k$ and all $w \in \mathbb{P}_d^{k+1}(T)$,

$$(\boldsymbol{\kappa}\boldsymbol{G}_T^k\mathsf{v}, \boldsymbol{\nabla}w)_T = (\boldsymbol{\kappa}\boldsymbol{\nabla}\mathsf{v}_T, \boldsymbol{\nabla}w)_T + \sum_{F\in\mathcal{F}_T}(\mathsf{v}_F - \mathsf{v}_T, \boldsymbol{\nabla}w\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF})_F, \tag{2}$$

which can be computed by solving a local (well-posed) Neumann problem in $\mathbb{P}_d^{k+1}(T)$. We next introduce the potential reconstruction operator $p_T^k : \mathsf{U}_T^k \to \mathbb{P}_d^{k+1}(T)$ such that, for all $\mathsf{v} \in \mathsf{U}_T^k$, $\boldsymbol{\nabla}p_T^k\mathsf{v} := \boldsymbol{G}_T^k\mathsf{v}$ and $\int_T p_T^k\mathsf{v} := \int_T \mathsf{v}_T$ ($p_T^k\mathsf{v}$ is well-defined since $\boldsymbol{G}_T^k\mathsf{v} \in \boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T)$). Finally, we define the local interpolation operator $\mathsf{I}_T^k : H^1(T) \to \mathsf{U}_T^k$ such that, for all $v \in H^1(T)$, $\mathsf{I}_T^k v := \left(\pi_T^k v, (\pi_F^k v)_{F\in\mathcal{F}_T}\right)$, where π_T^k and π_F^k are the L^2-orthogonal projectors onto $\mathbb{P}_d^k(T)$ and $\mathbb{P}_{d-1}^k(F)$, respectively.

Lemma 2.1 (Approximation properties for $p_T^k\mathsf{I}_T^k$)**.** *The following holds for all $v \in H^{k+2}(T)$ with $\alpha = 1/2$ if $\boldsymbol{\kappa}$ is piecewise constant and $\alpha = 1$ in the general case:*

$$\|v - p_T^k\mathsf{I}_T^k v\|_T + h_T^{1/2}\|v - p_T^k\mathsf{I}_T^k v\|_{\partial T} + h_T\|\boldsymbol{\nabla}(v - p_T^k\mathsf{I}_T^k v)\|_T + h_T^{3/2}\|\boldsymbol{\nabla}(v - p_T^k\mathsf{I}_T^k v)\|_{\partial T} \lesssim \rho_T^\alpha h_T^{k+2}\|v\|_{H^{k+2}(T)}. \tag{3}$$

Proof. Let $v \in H^{k+2}(T)$. A direct calculation using (2), the definitions of p_T^k and I_T^k, and integration by parts shows that, for all $w \in \mathbb{P}_d^{k+1}(T)$,

$$(\boldsymbol{\kappa}\boldsymbol{\nabla}(v - p_T^k\mathsf{I}_T^k v), \boldsymbol{\nabla}w)_T = ((\boldsymbol{\kappa} - \bar{\boldsymbol{\kappa}}_T)\boldsymbol{\nabla}(v - \pi_T^k v), \boldsymbol{\nabla}w)_T - \sum_{F\in\mathcal{F}_T}(\pi_F^k v - \pi_T^k v, \boldsymbol{\nabla}w\cdot(\boldsymbol{\kappa} - \bar{\boldsymbol{\kappa}}_T)\cdot\boldsymbol{n}_{TF})_F,$$

where $\bar{\boldsymbol{\kappa}}_T$ denotes the mean-value of $\boldsymbol{\kappa}$ in T. Note that the right-hand side vanishes if $\boldsymbol{\kappa}$ is piecewise constant. In the general case, owing to the assumptions on $\boldsymbol{\kappa}$ and using the approximation properties of the L^2-orthogonal projectors along with a discrete trace inequality for $\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}w\|_F$, we infer that

$$|(\boldsymbol{\kappa}\boldsymbol{\nabla}(v - p_T^k\mathsf{I}_T^k v), \boldsymbol{\nabla}w)_T| \lesssim L_T^\kappa h_T h_T^k\|v\|_{H^{k+1}(T)}\|\boldsymbol{\nabla}w\|_T \lesssim \kappa_T^\sharp h_T^{k+1}\|v\|_{H^{k+1}(T)}\|\boldsymbol{\nabla}w\|_T. \tag{4}$$

We now observe that

$$\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}(v - p_T^k\mathsf{I}_T^k v)\|_T^2 = (\boldsymbol{\kappa}\boldsymbol{\nabla}(v - p_T^k\mathsf{I}_T^k v), \boldsymbol{\nabla}(v - \pi_T^{k+1}v))_T + (\boldsymbol{\kappa}\boldsymbol{\nabla}(v - p_T^k\mathsf{I}_T^k v), \boldsymbol{\nabla}(\pi_T^{k+1}v - p_T^k\mathsf{I}_T^k v))_T. \tag{5}$$

Denote by $\mathfrak{T}_1$ and $\mathfrak{T}_2$ the addends on the right-hand side of (5). Using the Cauchy–Schwarz inequality and the approximation properties of π_T^{k+1}, we obtain $|\mathfrak{T}_1| \lesssim \|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}(v - p_T^k\mathsf{I}_T^k v)\|_T(\kappa_T^\sharp)^{1/2}h_T^{k+1}\|v\|_{H^{k+2}(T)}$.

When $\boldsymbol{\kappa}$ is piecewise constant, $\mathfrak{T}_2$ vanishes, so that using Young's inequality yields $\|\boldsymbol{\nabla}(v - p_T^k \mathsf{I}_T^k v)\|_T \le (\kappa_T^\flat)^{-1/2}\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}(v - p_T^k \mathsf{I}_T^k v)\|_T \lesssim \rho_T^{1/2} h_T^{k+1}\|v\|_{H^{k+2}(T)}$. In the general case, using (4) with $w = (\pi_T^{k+1}v - p_T^k \mathsf{I}_T^k v)$ and since $\|\boldsymbol{\nabla}(\pi_T^{k+1}v - p_T^k \mathsf{I}_T^k v)\|_T = \|\boldsymbol{\nabla}\pi_T^{k+1}(v - p_T^k \mathsf{I}_T^k v)\|_T \lesssim \|\boldsymbol{\nabla}(v - p_T^k \mathsf{I}_T^k v)\|_T$ owing to the H^1-stability of the projector π_T^{k+1}, we infer that $|\mathfrak{T}_2| \lesssim \rho_T^{1/2}(\kappa_T^\sharp)^{1/2} h_T^{k+1}\|v\|_{H^{k+1}(T)}\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}(v - p_T^k \mathsf{I}_T^k v)\|_T$, which leads to the estimate on $\|\boldsymbol{\nabla}(v - p_T^k \mathsf{I}_T^k v)\|_T$ in (3). The other terms in (3) are then bounded as in [7, Lemma 3]. $\square$

Remark 1 ($\alpha = 0$). *It is also possible to take $\alpha = 0$ whenever, for all $T \in \mathcal{T}_h$, the eigenvectors of $\boldsymbol{\kappa}_{|T}$ are constant and its eigenvalues satisfy, with obvious notation, $|\lambda(x) - \bar{\lambda}_T| \lesssim h_T \lambda(x)$ for all $x \in T$.*

3. Discrete problem and stability

For all $T \in \mathcal{T}_h$, we introduce the local bilinear forms a_T and s_T on $\mathsf{U}_T^k \times \mathsf{U}_T^k$ such that

$$a_T(\mathsf{u},\mathsf{v}) := (\boldsymbol{\kappa}\boldsymbol{G}_T^k\mathsf{u}, \boldsymbol{G}_T^k\mathsf{v})_T + s_T(\mathsf{u},\mathsf{v}), \quad s_T(\mathsf{u},\mathsf{v}) := \sum_{F\in\mathcal{F}_T} \frac{\kappa_F}{h_F}(\pi_F^k(\mathsf{u}_F - P_T^k\mathsf{u}), \pi_F^k(\mathsf{v}_F - P_T^k\mathsf{v}))_F, \tag{6}$$

with $\kappa_F := \|\boldsymbol{n}_{TF}\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF}\|_{L^\infty(F)}$ and the local potential reconstruction $P_T^k : \mathsf{U}_T^k \to \mathbb{P}_d^{k+1}(T)$ such that $P_T^k\mathsf{v} := \mathsf{v}_T + (p_T^k\mathsf{v} - \pi_T^k p_T^k\mathsf{v})$. We define the global space of DOFs by patching interface values, so that $\mathsf{U}_h^k := \left\{\times_{T\in\mathcal{T}_h}\mathbb{P}_d^k(T)\right\} \times \left\{\times_{F\in\mathcal{F}_h}\mathbb{P}_{d-1}^k(F)\right\}$, and, for all $T \in \mathcal{T}_h$, we denote by $\mathsf{L}_T : \mathsf{U}_h^k \to \mathsf{U}_T^k$ the restriction operator that maps the global DOFs in U_h^k to the corresponding local DOFs in U_T^k. The discrete problem consists in seeking $\mathsf{u}_h \in \mathsf{U}_{h,0}^k := \left\{\mathsf{v}_h = \left((\mathsf{v}_T)_{T\in\mathcal{T}_h}, (\mathsf{v}_F)_{F\in\mathcal{F}_h}\right) \in \mathsf{U}_h^k \mid \mathsf{v}_F \equiv 0\ \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\}$ such that

$$a_h(\mathsf{u}_h,\mathsf{v}_h) := \sum_{T\in\mathcal{T}_h} a_T(\mathsf{L}_T\mathsf{u}_h, \mathsf{L}_T\mathsf{v}_h) = \sum_{T\in\mathcal{T}_h}(f,\mathsf{v}_T)_T =: l_h(\mathsf{v}_h) \qquad \forall \mathsf{v}_h \in \mathsf{U}_{h,0}^k. \tag{7}$$

To analyze the stability of the discrete problem, we introduce the following seminorm on U_T^k:

$$\|\mathsf{v}\|_{\boldsymbol{\kappa},T}^2 := \|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}\mathsf{v}_T\|_T^2 + \sum_{F\in\mathcal{F}_T}\frac{\kappa_F}{h_F}\|\mathsf{v}_F - \mathsf{v}_T\|_F^2, \tag{8}$$

and we set $\|\mathsf{v}_h\|_{\boldsymbol{\kappa},h}^2 := \sum_{T\in\mathcal{T}_h}\rho_T^{-1}\|\mathsf{L}_T\mathsf{v}_h\|_{\boldsymbol{\kappa},T}^2$ for all $\mathsf{v}_h \in \mathsf{U}_h^k$. Observe that $\|\cdot\|_{\boldsymbol{\kappa},h}$ is a norm on $\mathsf{U}_{h,0}^k$.

Lemma 3.1 (Stability)**.** *The following inequalities hold for all $\mathsf{v} \in \mathsf{U}_T^k$:*

$$\rho_T^{-1}\|\mathsf{v}\|_{\boldsymbol{\kappa},T}^2 \lesssim a_T(\mathsf{v},\mathsf{v}) \lesssim \rho_T\|\mathsf{v}\|_{\boldsymbol{\kappa},T}^2. \tag{9}$$

Consequently, $\|\mathsf{v}_h\|_{\boldsymbol{\kappa},h}^2 \lesssim a_h(\mathsf{v}_h,\mathsf{v}_h)$ for all $\mathsf{v}_h \in \mathsf{U}_h^k$ and problem (7) is well-posed.

Proof. We adapt the proof of [7, Lemma 4]. Concerning the face terms, we obtain

$$\sum_{F\in\mathcal{F}_T}\frac{\kappa_F}{h_F}\|\mathsf{v}_F - \mathsf{v}_T\|_F^2 \le s_T(\mathsf{v},\mathsf{v}) + \rho_T\|\boldsymbol{\kappa}^{1/2}\boldsymbol{G}_T^k\mathsf{v}\|_T^2, \quad s_T(\mathsf{v},\mathsf{v}) \lesssim \sum_{F\in\mathcal{F}_T}\frac{\kappa_F}{h_F}\|\mathsf{v}_F - \mathsf{v}_T\|_F^2 + \rho_T\|\boldsymbol{\kappa}^{1/2}\boldsymbol{G}_T^k\mathsf{v}\|_T^2. \tag{10}$$

To compare $\|\boldsymbol{\kappa}^{1/2}\boldsymbol{G}_T^k\mathsf{v}\|_T$ and $\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}\mathsf{v}_T\|_T$, we observe that, for all $w \in \mathbb{P}_d^{k+1}(T)$ and all $F \in \mathcal{F}_T$,

$$\|\boldsymbol{\nabla}w\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF}\|_F^2 \le (|\boldsymbol{n}_{TF}\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF}|, |\boldsymbol{\nabla}w\cdot\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}w|)_F \lesssim \frac{\kappa_F}{h_F}\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}w\|_T^2, \tag{11}$$

where we have used the Cauchy–Schwarz inequality for $\boldsymbol{\kappa}$, the definition of κ_F, and a discrete trace inequality. Taking $w = \mathsf{v}_T$ in the definition (2) of $\boldsymbol{G}_T^k\mathsf{v}$ yields $\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}\mathsf{v}_T\|_T^2 = (\boldsymbol{\kappa}\boldsymbol{G}_T^k\mathsf{v}, \boldsymbol{\nabla}\mathsf{v}_T)_T - \sum_{F\in\mathcal{F}_T}(\mathsf{v}_F - \mathsf{v}_T, \boldsymbol{\nabla}\mathsf{v}_T\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF})_F$. Hence, using (11), a discrete trace inequality for $\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}\mathsf{v}_T\|_F$, the first bound in (10), $\rho_T \ge 1$, and Young's inequality yields

$$\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}\mathsf{v}_T\|_T^2 \lesssim \|\boldsymbol{\kappa}^{1/2}\boldsymbol{G}_T^k\mathsf{v}\|_T^2 + \sum_{F\in\mathcal{F}_T}\frac{\kappa_F}{h_F}\|\mathsf{v}_F - \mathsf{v}_T\|_F^2 \lesssim \rho_T\|\boldsymbol{\kappa}^{1/2}\boldsymbol{G}_T^k\mathsf{v}\|_T^2 + s_T(\mathsf{v},\mathsf{v}).$$

Moreover, since $\|\boldsymbol{\kappa}^{1/2}\boldsymbol{G}_T^k\mathsf{v}\|_T = \sup_{w\in\mathbb{P}_d^{k+1}(T)}\frac{(\boldsymbol{\kappa}\boldsymbol{G}_T^k\mathsf{v}, \boldsymbol{\nabla}w)_T}{\|\boldsymbol{\kappa}^{1/2}\boldsymbol{\nabla}w\|_T}$ and proceeding similarly leads to $\|\boldsymbol{\kappa}^{1/2}\boldsymbol{G}_T^k\mathsf{v}\|_T \lesssim \|\mathsf{v}\|_{\boldsymbol{\kappa},T}$. Combining the above bounds yields (9), and the rest of the proof is straightforward. $\square$

4. Error analysis

Theorem 4.1 (Energy-error estimate)**.** *Let $u \in U_0$ solve (1) and let $\mathsf{u}_h \in \mathsf{U}_{h,0}^k$ solve (7). Assume that $u_{|T} \in H^{k+2}(T)$ for all $T \in \mathcal{T}_h$. Then, letting $\widehat{\mathsf{u}}_h := \left((\pi_T^k u)_{T\in\mathcal{T}_h}, (\pi_F^k u)_{F\in\mathcal{F}_h}\right) \in \mathsf{U}_{h,0}^k$ and, recalling the definition of α from Lemma 2.1, the following holds with consistency error $\mathcal{E}_h(\mathsf{v}_h) := a_h(\widehat{\mathsf{u}}_h, \mathsf{v}_h) - l_h(\mathsf{v}_h)$:*

$$\|\widehat{\mathsf{u}}_h - \mathsf{u}_h\|_{\boldsymbol{\kappa},h} \lesssim \sup_{\mathsf{v}_h\in\mathsf{U}_{h,0}^k,\,\|\mathsf{v}_h\|_{\boldsymbol{\kappa},h}=1} \mathcal{E}_h(\mathsf{v}_h) \lesssim \left\{ \sum_{T\in\mathcal{T}_h} \kappa_T^\sharp \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}. \tag{12}$$

Proof. We adapt the proof of [7, Theorem 8]. The first inequality in (12) is an immediate consequence of Lemma 3.1. Proceeding as in [7] with $\check{u}_T := p_T^k \mathsf{L}_T \widehat{\mathsf{u}}_h = p_T^k \mathsf{I}_T^k(u_{|T})$ and $\mathsf{v}_h \in \mathsf{U}_{h,0}^k$ with $\|\mathsf{v}_h\|_{\boldsymbol{\kappa},h} = 1$ leads to

$$\mathcal{E}_h(\mathsf{v}_h) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{\kappa}\boldsymbol{\nabla}(\check{u}_T - u), \boldsymbol{\nabla}\mathsf{v}_T)_T + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} (\mathsf{v}_F - \mathsf{v}_T, (\boldsymbol{\nabla}\check{u}_T - \boldsymbol{\nabla}u)\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF})_F + \sum_{T\in\mathcal{T}_h} s_T(\mathsf{L}_T\widehat{\mathsf{u}}_h, \mathsf{L}_T\mathsf{v}_h).$$

Denote by $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ the three terms on the right-hand side. Combining the results of Lemmas 2.1 and 3.1, we infer that $|\mathfrak{T}_1 + \mathfrak{T}_2|^2 \lesssim \sum_{T\in\mathcal{T}_h} \kappa_T^\sharp \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2$. Moreover, since $s_T(\mathsf{L}_T\widehat{\mathsf{u}}_h, \mathsf{L}_T\mathsf{v}_h) \le s_T(\mathsf{L}_T\widehat{\mathsf{u}}_h, \mathsf{L}_T\widehat{\mathsf{u}}_h)^{1/2} s_T(\mathsf{L}_T\mathsf{v}_h, \mathsf{L}_T\mathsf{v}_h)^{1/2}$, proceeding as in [7] for the first factor, and using the second bound in (10) for the second factor yields $|\mathfrak{T}_3|^2 \lesssim \sum_{T\in\mathcal{T}_h} \kappa_T^\sharp \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2$. □

Finally, adapting the proof of [7, Theorem 10] leads to the following L^2-norm error estimate.

Theorem 4.2 (L^2-error estimate)**.** *Assume elliptic regularity for problem (1) in the form $\|z\|_{H^2(\Omega)} \lesssim \|g\|_\Omega$ for all $g \in L^2(\Omega)$ and $z \in U_0$ solving (1) with data g. Assume $f \in H^{k+\delta}(\Omega)$ with $\delta = 0$ for $k \ge 1$ and $\delta = 1$ for $k = 0$. Then, using the same notation as in Theorem 4.1, and defining the piecewise polynomial functions $\widehat{u}_h$ and u_h such that $\widehat{u}_{h|T} = \pi_T^k u$ and $u_{h|T} = \mathsf{u}_T$ for all $T \in \mathcal{T}_h$, the following holds:*

$$\|\widehat{u}_h - u_h\|_\Omega \lesssim |(\kappa^\sharp)^{1/2}\rho^{1/2+\alpha}h|_{\ell^\infty} \left\{ \sum_{T\in\mathcal{T}_h} \kappa_T^\sharp \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2} + h^{k+2}\|f\|_{H^{k+\delta}(\Omega)},$$

where $|(\kappa^\sharp)^{1/2}\rho^{1/2+\alpha}h|_{\ell^\infty} := \max_{T\in\mathcal{T}_h} (\kappa_T^\sharp)^{1/2} \rho_T^{1/2+\alpha} h_T$.

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