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Olivier Ley, Vinh Duc Nguyen

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GRADIENT BOUNDS FOR NONLINEAR DEGENERATE PARABOLIC EQUATIONS AND APPLICATION TO LARGE TIME BEHAVIOR OF SYSTEMS

OLIVIER LEY AND VINH DUC NGUYEN

ABSTRACT. We obtain new oscillation and gradient bounds for the viscosity solutions of fully nonlinear degenerate elliptic equations where the Hamiltonian is a sum of a sublinear and a superlinear part in the sense of Barles and Souganidis (2001). We use these bounds to study the asymptotic behavior of weakly coupled systems of fully nonlinear parabolic equations. Our results apply to some “asymmetric systems” where some equations contain a sublinear Hamiltonian whereas the others contain a superlinear one. Moreover, we can deal with some particular case of degenerate systems using a generalization of the strong maximum principle.

1. INTRODUCTION

One of the main result of this work is to obtain new results about the large time behavior of the solution $u(x, t) = (u_1(x, t), \dots, u_m(x, t))$ of the fully nonlinear parabolic system

$$(1.1) \quad \begin{cases} \frac{\partial u_i}{\partial t} + \sup_{\theta \in \Theta} \{-\text{trace}(A_{\theta_i}(x)D^2u_i) + H_{\theta_i}(x, Du_i)\} + \sum_{j=1}^m d_{ij}u_j = 0, \\ (x, t) \in \mathbb{T}^N \times (0, +\infty), \quad 1 \leq i \leq m, \\ u_i(x, 0) = u_{0i}(x), \quad x \in \mathbb{T}^N, \end{cases}$$

in the periodic setting (\mathbb{T}^N denotes the flat torus $\mathbb{R}^N/\mathbb{Z}^N$), where the equations are linearly coupled through a matrix $D = (d_{ij})_{ij}$ which is always assumed to be monotone and irreducible, see Section 3 for the definitions. The set Θ is a compact metric space, the diffusion matrices $A_{\theta_i}(x) = \sigma_{\theta_i}\sigma_{\theta_i}(x)^T$ with $\sigma_{\theta_i} \in W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_N)$ and $u_{0i} \in C(\mathbb{T}^N)$. The precise assumptions on the Hamiltonians $H_{\theta_i} \in C(\mathbb{T}^N \times \mathbb{R}^N)$, which is the point in this work will be explained below. We prove the following asymptotic behavior of the solution,

$$(1.2) \quad u_i(x, t) + c_i t \rightarrow v_i(x) \quad \text{uniformly in } \mathbb{T}^N \text{ as } t \rightarrow \infty, \quad 1 \leq i \leq m,$$

where $c = (c_1, \dots, c_m) \in \mathbb{R}^m$ and $v = (v_1, \dots, v_m) \in W^{1,\infty}(\mathbb{T}^N)$ are solutions of the so-called ergodic problem

$$(1.3) \quad \sup_{\theta \in \Theta} \{-\text{trace}(A_{\theta_i}(x)D^2v_i) + H_{\theta_i}(x, Dv_i)\} + \sum_{j=1}^m d_{ij}v_j = c_i, \quad x \in \mathbb{T}^N, 1 \leq i \leq m.$$

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Let us give immediately one of the most striking application of our result. We are able to prove the convergence of *asymmetric systems* like

$$(1.4) \quad \begin{cases} \frac{\partial u_1}{\partial t} - \text{trace}(A_1(x)D^2u_1) + \langle b_1(x), Du_1 \rangle + \ell_1(x) + u_1 - u_2 = 0, \\ \frac{\partial u_2}{\partial t} - \text{trace}(A_2(x)D^2u_2) + |Du_2|^2 + \ell_2(x) - u_1 + u_2 = 0. \end{cases}$$

where A_1 is uniformly elliptic and A_2 may be degenerate. The name asymmetric means that the different equations can have different nature as above: the first one contains a sublinear Hamiltonian and is uniformly elliptic whereas the second one contains a superlinear one and may be degenerate. The general framework will be explained more precisely below.

In order to study the large time behavior of systems, we start by proving new estimates for the viscosity solutions of a large class of fully nonlinear degenerate elliptic (scalar) equations

$$(1.5) \quad \epsilon \phi^\epsilon + \sup_{\theta \in \Theta} \{-\text{trace}(A_\theta(x)D^2\phi^\epsilon) + H_\theta(x, D\phi^\epsilon)\} = 0, \quad x \in \mathbb{T}^N, \quad \epsilon > 0.$$

We are interested in oscillation bounds

$$(1.6) \quad \text{osc}(\phi^\epsilon) := \sup_{\mathbb{T}^N} \phi^\epsilon - \inf_{\mathbb{T}^N} \phi^\epsilon \leq K$$

and gradient bounds

$$(1.7) \quad |D\phi^\epsilon|_\infty \leq K$$

which are *independent of ϵ* . This latter condition is crucial to be able to send ϵ to 0 in (1.5) in order to solve the ergodic problem (1.3), which is a first step when trying to prove (1.2).

Barles and Souganidis [7] obtained the first results concerning both estimates like (1.7) and the asymptotic behavior (1.2) when $m = 1$ (scalar equations). When there is no control θ in the equation and the diffusion matrix $A(x) = I$ (actually, some extensions to $A(x, p)$ are mentioned in the paper), (1.7) was proved in [7] in two contexts. The first one is for *sublinear Hamiltonians*, i.e., for Hamiltonians with a sublinear growth with respect to the gradient. A typical example is

$$H(x, p) = \langle b(x), p \rangle + \ell(x), \quad b \in C(\mathbb{T}^N; \mathbb{R}^N), \ell \in C(\mathbb{T}^N).$$

The second context is for *superlinear Hamiltonians*. The precise assumption is more involved (see (2.41)) and designed to allow the use of weak Bernstein-type arguments ([2]). The most important example is Hamiltonians with a superlinear growth with respect to the gradient

$$(1.8) \quad H(x, p) = a(x)|p|^{1+\alpha} + \ell(x), \quad \alpha > 0, a, \ell \in C(\mathbb{T}^N) \text{ and } a > 0.$$

As a consequence of these bounds together with the strong maximum principle, they were able to obtain the convergence (1.2) for the solution of

$$\frac{\partial u}{\partial t} - \Delta u + H(x, Du) = 0, \quad (x, t) \in \mathbb{T}^N \times (0, +\infty)$$

both in the sublinear and superlinear case.

Using the extension of viscosity solutions to monotone systems of parabolic equations by Ishii and Koike [20], it is not difficult to adapt the results of [7] to the case of *sublinear* systems (1.1) (i.e., systems for which all the Hamiltonians are sublinear in the sense of [7]) in the one hand, and to the case of *superlinear* systems. A more striking issue is the case of what we call *asymmetric systems* which are systems containing both sublinear and superlinear Hamiltonians. The main difficulty is that the proofs of (1.7) in [7] are completely different in

the two contexts. In the case of sublinear Hamiltonians, the uniform ellipticity of the equation allows to control the sublinear terms following some ideas of Ishii and Lions [21]. Whereas the superlinear Hamiltonian can control every term in the other case; in particular the equation may be degenerate. When dealing with asymmetric systems, a unified proof is needed.

Our first goal is to obtain the gradient bound (1.7) for (1.5) when

$$H_\theta = \underline{H}_\theta + \overline{H}_\theta,$$

where H_θ is a sublinear Hamiltonian, i.e, having a sublinear growth,

$$|\overline{H}_\theta(x, p)| \leq C(1 + |p|),$$

and \overline{H}_θ is a superlinear one. An important feature, which will be crucial when dealing with asymmetric systems, is that we allow \underline{H}_θ or/and \overline{H}_θ to be zero which is natural for \underline{H}_θ but strange for \overline{H}_θ . We propose two definitions for superlinear Hamiltonians, one when (1.5) is uniformly elliptic, see (2.9), which generalizes slightly the one of [7] and a stronger one, see (2.37), which allows to deal with degenerate equations (1.5). Both of them include of course typical superlinear Hamiltonians like (1.8) and \overline{H}_θ may be zero in some cases. We refer the reader to Section 2.4 for comments and examples.

To prove (1.7) in the case of superlinear Hamiltonians, we need to use weak Bernstein-type arguments as in [7]. They rely on an exponential change of function $e^{w^\epsilon} = \phi^\epsilon - \min_{\mathbb{T}^N} \phi^\epsilon + 1$ which allows to take advantage of the superlinear property of the Hamiltonian. So, to make a proof in the case of a sum of a sub- and superlinear Hamiltonian, such a change of function should be performed. But this makes some quadratic terms $|\sigma_\theta(x) Dw^\epsilon|^2$ appear which are hard to control using the ellipticity when the superlinear Hamiltonian vanishes. To overcome this difficulty, we first prove the oscillation bound (1.6). It helps to localize the proof and conclude. It is worth mentioning that our proof of the oscillation bound works in very general settings. For instance, it is sufficient that $\frac{H_\theta(x, p)}{|p|} \rightarrow +\infty$ as $|p| \rightarrow +\infty$ uniformly with respect to x, θ , which is a new result interesting by itself.

The next step is to extend the gradient bounds (1.7) to approximate asymmetric systems

$$\epsilon \phi_i^\epsilon + \sup_{\theta \in \Theta} \left\{ -\text{trace}(A_{\theta i}(x) D^2 \phi_i^\epsilon) + H_{\theta i}(x, D \phi_i^\epsilon) \right\} + \sum_{j=1}^m d_{ij} \phi_j^\epsilon = 0, \quad x \in \mathbb{T}^N, 1 \leq i \leq m.$$

An immediate consequence is that we can solve the ergodic problem (1.3) extending the classical by now proofs of [24, 1] to the case of our systems. We then prove the convergence (1.2). As in [7], the proof of the convergence is based on the strong maximum principle but let us mention that we establish a new version of the strong maximum principle for irreducible systems which allows some degeneracy of the equations, see Theorem 3.5 for details. In particular, the result holds for the degenerate system (1.4); See Section 3.6 for more examples and discussions.

Let us turn to an overview of related results in the literature. The ideas of the proof of gradient bounds in viscosity theory using the uniform ellipticity of the equation are due to Ishii and Lions [21], see also [13, 3] and the references therein. Gradient bounds for superlinear-type Hamiltonians can be found in Lions [23] and [2], see also Lions and Souganidis [25]. These ideas were used in Barles and Souganidis [7] as explained above. Our approach is mainly based on this latter work. For superlinear Hamiltonians satisfying $H(x, p) \geq a(|p|^m - 1)$, $a, m > 1$, some Hölder or gradient estimates were obtained in Capuzzo Dolcetta et al. [10], Barles [4], Cardaliaguet [11]. Recently, oscillations and Hölder bounds for nonlinear degenerate parabolic

equations were proved in Cardaliaguet and Sylvestre [12] but the bounds depends on the L^∞ norm of the solution.

The large time behavior of such kind of nonlinear equations or systems in the periodic setting were extensively studied. For Hamilton-Jacobi equations (the totally degenerate case when $A_\theta \equiv 0$), we refer the reader to [28, 17, 6, 15, 5] and the references therein. In this framework, the gradient bounds are not a difficult step but the proof of the convergence is more delicate since one does not have any strong maximum principle. Such kind of results were extended to systems of Hamilton-Jacobi equations in [9, 27, 29, 26]. For second order nonlinear equations, the asymptotics results of [7] were recently generalized in [22] to some superlinear degenerate equations which are totally degenerate on some subset Σ of \mathbb{T}^N and uniformly parabolic outside Σ using the gradient bound (Theorem 2.6) and some strong maximum principle type ideas. Similar results for uniformly convex degenerate equations were established in Cagnetti et al. [8] using a different approach based on a nonlinear adjoint method [16]. Their results also apply to systems with uniformly convex quadratic Hamiltonians with quite general degeneracy assumptions since the proof is not based on strong maximum principle-type arguments. However, such a method does not seem to be applicable for fully nonlinear equations and the system in [8] is not asymmetric.

The paper is organized as follows. The bounds (1.6)-(1.7) are established in Section 2 for the scalar equation (1.5) with $H_\theta = \underline{H}_\theta + \overline{H}_\theta$, firstly when the equation is uniformly elliptic and secondly in some degenerate cases. Some examples of applications are collected in Section 2.4. Section 3 is devoted to systems. The assumptions on the coupling are recalled in Section 3.1. The gradient bounds for asymmetric systems are obtained in Section 3.2 and a new strong maximum principle is obtained in Section 3.3. Then the ergodic problem is solved and the main application of large time behavior of asymmetric systems is investigated. The section ends with some examples of applications. Several results are collected in the appendix. In particular, since the equations under consideration do not satisfy the classical assumptions in viscosity solutions (due to the possibly superlinear growth of the Hamiltonian for instance), we recall several versions of the comparison principle which apply in our case. Finally a control theoretical interpretation is given.

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2. GRADIENT BOUNDS FOR NONLINEAR PARABOLIC EQUATIONS

For $\epsilon > 0$, we consider the approximated equation

$$(2.1) \quad \epsilon \phi^\epsilon + \sup_{\theta \in \Theta} \left\{ -\text{trace}(A_\theta(x) D^2 \phi^\epsilon) + H_\theta(x, D\phi^\epsilon) \right\} = 0, \quad x \in \mathbb{T}^N,$$

where Θ is a compact metric space. We introduce the following steady assumptions

$$(2.2) \quad \begin{cases} A_\theta = \sigma_\theta \sigma_\theta^T, \quad \sigma_\theta \in C(\mathbb{T}^N; \mathcal{M}_N), \\ |\sigma_\theta(x)| \leq C \text{ and } |\sigma_\theta(x) - \sigma_\theta(y)| \leq C|x - y|, \quad x, y \in \mathbb{T}^N, \theta \in \Theta. \end{cases}$$

$$(2.3) \quad \begin{cases} H_\theta \in C(\mathbb{T}^N \times \mathbb{R}^N), \quad \sup_{x \in \mathbb{T}^N, \theta \in \Theta} |H_\theta(x, 0)| \leq C, \\ \text{for all } R > 0, \text{ there exists a modulus of continuity } m_R \text{ such that} \\ |H_\theta(x, p) - H_\theta(y, p)| \leq m_R(|x - y|), \quad x, y \in \mathbb{T}^N, |p| \leq R, \theta \in \Theta. \end{cases}$$

Assumption (2.2) is very classical and (2.3) is natural in control theory and is automatically satisfied, e.g., when the Hamiltonians are continuous with respect to (x, p, θ) since Θ is compact.

2.1. A general result for the oscillation. We first show that the oscillation of the solution of (2.1) is uniformly bounded under a very general hypothesis. This result is interesting by itself.

$$(2.4) \quad \left\{ \begin{array}{l} H_\theta \text{ satisfies (2.3) and there exists } L > 1 \text{ such that for all } x, y \in \mathbb{T}^N, \\ \text{if } |p|=L, \text{ then } H_\theta(x, p) \geq |p| \left[H_\theta(y, \frac{p}{|p|}) + |H_\theta(\cdot, 0)|_\infty + N^{3/2} |(\sigma_\theta)_x|_\infty^2 \right] \text{ for } \theta \in \Theta. \end{array} \right.$$

Notice that (2.4) is satisfied when

$$\limsup_{|p| \rightarrow +\infty} \frac{H_\theta(x, p)}{|p|} = +\infty \text{ uniformly with respect to } x, \theta,$$

see Section 2.4.

Lemma 2.1. *Assume (2.2) and (2.4). Let ϕ^ϵ be a continuous solution of (2.1) and let $\phi^\epsilon(x_\epsilon) = \min \phi^\epsilon$. Then*

$$\phi^\epsilon(x) - \phi^\epsilon(x_\epsilon) \leq L|x - x_\epsilon| \quad \text{for all } x \in \mathbb{T}^N,$$

where L is the constant (independent of ϵ) which appears in (2.4).

An immediate consequence is

$$\text{osc}(\phi^\epsilon) := \max \phi^\epsilon - \min \phi^\epsilon \leq \sqrt{NL}.$$

Proof of Lemma 2.1. For simplicity, we skip the ϵ superscript for ϕ^ϵ writing ϕ instead. The constant L which appears below is the one of (2.4). Consider

$$M = \max_{x, y \in \mathbb{T}^N} \{ \phi(x) - L\phi(y) + (L-1) \min \phi - L|x - y| \}.$$

We are done if $M \leq 0$. Otherwise, the above positive maximum is achieved at (\bar{x}, \bar{y}) with $\bar{x} \neq \bar{y}$. Notice that the continuity of ϕ is crucial at this step. The theory of second order viscosity solutions (see [13] and Lemma 2.4) yields, for every $\varrho > 0$, the existence of $(p, X) \in \bar{J}^{2,+} \phi(\bar{x})$ and $(p/L, Y/L) \in \bar{J}^{2,-} \phi(\bar{y})$, $p = L \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$, such that

$$-\text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) \geq -LN^{3/2}|(\sigma_\theta)_x|_\infty^2 + O(\varrho)$$

and

$$\left\{ \begin{array}{l} \epsilon \phi(\bar{x}) + \sup_{\theta \in \Theta} \{ -\text{trace}(A_\theta(\bar{x})X) + H_\theta(\bar{x}, p) \} \leq 0, \\ \epsilon \phi(\bar{y}) + \sup_{\theta \in \Theta} \{ -\text{trace}(A_\theta(\bar{y}) \frac{Y}{L}) + H_\theta(\bar{y}, \frac{p}{L}) \} \geq 0. \end{array} \right.$$

It follows

$$\epsilon(\phi(\bar{x}) - L\phi(\bar{y})) + \inf_{\theta \in \Theta} \{ -\text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) + H_\theta(\bar{x}, p) - LH_\theta(\bar{y}, \frac{p}{L}) \} \leq 0.$$

We have

$$\epsilon(\phi(\bar{x}) - L\phi(\bar{y})) > -(L-1)\epsilon \min \phi \geq \sup_{\theta \in \Theta} \{ -L|H_\theta(\cdot, 0)|_\infty \}$$

since, for all θ , $\epsilon \min \phi \leq |H_\theta(\cdot, 0)|_\infty$. Combining all the above information, we get

$$\inf_{\theta \in \Theta} \left\{ H_\theta(\bar{x}, p) - L \left[H_\theta(\bar{y}, \frac{p}{L}) + |H_\theta(\cdot, 0)|_\infty + N^{3/2} |(\sigma_\theta)_x|_\infty^2 \right] \right\} < 0.$$

Applying (2.4) yields a contradiction. \square

2.2. Gradient bounds for uniformly elliptic equations. In this section, we suppose that (2.1) is uniformly elliptic

$$(2.5) \quad \text{there exists } \nu > 0 \text{ such that } A_\theta(x) \geq \nu I, \quad x \in \mathbb{T}^N, \theta \in \Theta.$$

In this case, we consider Hamiltonians under the special form $H_\theta = \underline{H}_\theta + \overline{H}_\theta$ where \underline{H}_θ is a sublinear Hamiltonian and \overline{H}_θ is of superlinear type in some sense defined below. This form will be useful later to deal with asymmetric systems. We rewrite (2.1) as

$$(2.6) \quad \epsilon \phi^\epsilon + \sup_{\theta \in \Theta} \left\{ -\text{trace}(A_\theta(x) D^2 \phi^\epsilon) + \underline{H}_\theta(x, D\phi^\epsilon) + \overline{H}_\theta(x, D\phi^\epsilon) \right\} = 0, \quad x \in \mathbb{T}^N.$$

We say that the \underline{H}_θ 's are *sublinear Hamiltonian* (uniformly with respect with θ) if

$$(2.7) \quad \underline{H}_\theta \text{ satisfies (2.3) and } |\underline{H}_\theta(x, p)| \leq C(1 + |p|), \quad (x, p) \in \mathbb{T}^N \times \mathbb{R}^N, \theta \in \Theta.$$

We assume the following *superlinear* type assumptions for \overline{H}_θ . The first one is needed to obtain an oscillation for the solution and the second one is a stronger assumption to get the gradient bound.

$$(2.8) \quad \begin{cases} \overline{H}_\theta \text{ satisfies (2.3) and there exists } C \geq 0, L, \mu > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, |p| \geq L \text{ and } \theta \in \Theta, \overline{H}_\theta(x, p) - \mu \overline{H}_\theta(y, \frac{p}{\mu}) \geq -C|p|. \end{cases}$$

$$(2.9) \quad \begin{cases} \overline{H}_\theta \text{ satisfies (2.3) and there exists } C \geq 0, L > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, |p| \geq L, \theta \in \Theta, \text{ and } \mu \geq 1 + L|x - y|, \\ \overline{H}_\theta(x, p) - \mu \overline{H}_\theta(y, \frac{p}{\mu}) \geq -C|p|. \end{cases}$$

From the inequality in (2.7), we see that \underline{H}_θ has a sublinear growth in the classical sense. But, let us point out that the *superlinear* \overline{H}_θ may be zero in (2.8) and (2.9). This fact will allow to treat some cases of asymmetric systems later. We chose to keep the terminology *superlinear* since (2.9) is a consequence of the superlinear assumption

$$(2.10) \quad \begin{cases} \overline{H}_\theta \in W_{\text{loc}}^{1, \infty}(\mathbb{T}^N \times \mathbb{R}^N) \text{ satisfies (2.3) and there exists } L > 1 \text{ such that} \\ \text{for a.e. } x \in \mathbb{T}^N, |p| \geq L, L [(\overline{H}_\theta)_{pp} - \overline{H}_\theta] - |(\overline{H}_\theta)_x| \geq 0 \end{cases}$$

introduced in [7]. Moreover, (2.9) is fulfilled for the typical superlinear Hamiltonian $\overline{H}_\theta(x, p) = a(x)|p|^{1+\alpha} + \ell(x)$, $\alpha > 0$, $a > 0$ we have in mind. We refer the reader to Section 2.4 for more discussions and examples showing that our assumptions are quite general.

We state the main result of this section.

Theorem 2.2. *Assume (2.2), (2.5), (2.7) and (2.9). For all $\epsilon > 0$, there exists a unique continuous viscosity solution $\phi^\epsilon \in C(\mathbb{T}^N)$ of (2.6) and a constant $K > 0$ independent of ϵ such that*

$$|D\phi^\epsilon|_\infty \leq K.$$

The proof relies on two important lemmas. The first one establishes a uniform bound for the oscillation and the second one improves this bound into a gradient bound. We first state and prove the lemmas and then give the proof of the theorem.

Lemma 2.3. *Under the hypotheses of Theorem 2.2, where (2.9) could be replaced by the weaker condition (2.8), there exists a constant $K > 0$ (independent of ϵ) such that, if ϕ^ϵ is a continuous solution of (2.6), then*

$$\text{osc}(\phi^\epsilon) := \max \phi^\epsilon - \min \phi^\epsilon \leq K.$$

Proof of Lemma 2.3. For simplicity, we skip the ϵ superscript for ϕ^ϵ writing ϕ instead.

1. *Construction of a concave function.* Consider the function $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$(2.11) \quad \Psi(s) = \frac{A_1}{A_2}(1 - e^{-A_2 s})$$

where $A_1, A_2 > 0$ will be chosen later. It is straightforward to see that Ψ is a C^∞ concave increasing function satisfying $\Psi(0) = 0$ and, for all $r > 0$ and $s \in [0, r]$,

$$(2.12) \quad \Psi'' + A_2 \Psi' = 0, \quad A_1 e^{-A_2 r} = \Psi'(r) \leq \Psi'(s) \leq \Psi'(0) = A_1.$$

2. *Viscosity inequalities.* We fix $r = \text{diameter}(\mathbb{T}^N) = \sqrt{N}$ and consider

$$M_\mu = \max_{x, y \in \mathbb{T}^N} \{\phi(x) + (\mu - 1) \min \phi - \mu \phi(y) - \Psi(|x - y|)\}.$$

If $M_\mu \leq 0$ then the lemma holds with $K = A_1/A_2$. From now on, we argue by contradiction assuming that the maximum is positive and achieved at (\bar{x}, \bar{y}) with $\bar{x} \neq \bar{y}$.

The theory of second order viscosity solutions yields, for every $\varrho > 0$, the existence of $(p, X) \in \bar{J}^{2,+} \phi(\bar{x}), (p/\mu, Y/\mu) \in \bar{J}^{2,-} \phi(\bar{y})$ such that

$$(2.13) \quad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varrho A^2,$$

with

$$(2.14) \quad q = \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}, \quad p = \Psi'(|\bar{x} - \bar{y}|)q, \quad B = \frac{1}{|\bar{x} - \bar{y}|}(I - q \otimes q),$$

$$(2.15) \quad A = \Psi'(|\bar{x} - \bar{y}|) \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \Psi''(|\bar{x} - \bar{y}|) \begin{pmatrix} q \otimes q & -q \otimes q \\ -q \otimes q & q \otimes q \end{pmatrix}$$

and the following viscosity inequalities hold for the solution ϕ of (2.6),

$$\begin{cases} \epsilon \phi(\bar{x}) + \sup_{\theta \in \Theta} \{-\text{trace}(A_\theta(\bar{x})X) + \underline{H}_\theta(\bar{x}, p) + \overline{H}_\theta(\bar{x}, p)\} \leq 0, \\ \epsilon \phi(\bar{y}) + \sup_{\theta \in \Theta} \{-\text{trace}(A_\theta(\bar{y})\frac{Y}{\mu}) + \underline{H}_\theta(\bar{y}, \frac{p}{\mu}) + \overline{H}_\theta(\bar{y}, \frac{p}{\mu})\} \geq 0. \end{cases}$$

It follows

$$(2.16) \quad \epsilon \phi(\bar{x}) - \epsilon \mu \phi(\bar{y}) + \inf_{\theta \in \Theta} \{-\text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) \\ + \underline{H}_\theta(\bar{x}, p) - \mu \underline{H}_\theta(\bar{y}, \frac{p}{\mu}) + \overline{H}_\theta(\bar{x}, p) - \mu \overline{H}_\theta(\bar{y}, \frac{p}{\mu})\} \leq 0.$$

3. *Trace estimates.* We have the following estimates which will be useful in the sequel, see for instance [21, 7, 3]. A proof is given in the Appendix.

Lemma 2.4. *Under assumption (2.2),*

$$-\text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) \geq -N|(\sigma_\theta)_x|_\infty^2 |\bar{x} - \bar{y}| \Psi'(|\bar{x} - \bar{y}|) + O(\varrho).$$

If, in addition, (2.5) holds, then

$$(2.17) \quad -\text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) \geq -4\nu\Psi''(|\bar{x} - \bar{y}|) - \tilde{C}\Psi'(|\bar{x} - \bar{y}|) + O(\varrho),$$

where $\tilde{C} = \tilde{C}(N, \nu, |\sigma_\theta|_\infty, |(\sigma_\theta)_x|_\infty)$ is given by (4.3).

4. *End of the proof.* Let $\theta \in \Theta$. At \hat{x} such that $\phi(\hat{x}) = \max \phi$, we get

$$\epsilon \min \phi + \underline{H}_\theta(\hat{x}, 0) + \overline{H}_\theta(\hat{x}, 0) \leq \epsilon \max \phi + \sup_\theta \{\underline{H}_\theta(\hat{x}, 0) + \overline{H}_\theta(\hat{x}, 0)\} \leq 0.$$

It follows

$$(2.18) \quad \epsilon \min \phi \leq \sup_{\theta \in \Theta} |(\underline{H}_\theta + \overline{H}_\theta)(\cdot, 0)|_\infty =: \mathcal{H}_0.$$

So we obtain

$$\epsilon(\phi(\bar{x}) - \mu\phi(\bar{y})) > -(\mu - 1)\epsilon \min \phi \geq -(\mu - 1)\mathcal{H}_0.$$

We have

$$|\underline{H}_\theta(\bar{x}, p) - \mu \underline{H}_\theta(\bar{y}, \frac{p}{\mu})| \leq 2C(1 + |p|) = 2C(1 + \Psi'(|\bar{x} - \bar{y}|)),$$

from (2.7) and

$$-\text{tr}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) \geq -4\nu\Psi''(|\bar{x} - \bar{y}|) - \tilde{C}\Psi'(|\bar{x} - \bar{y}|) + O(\varrho)$$

from Lemma 2.4 (2.17). Choosing

$$A_1 = Le^{A_2\sqrt{N}},$$

we obtain $|p| = \Psi'(|\bar{x} - \bar{y}|) \geq L$ from (2.12), (2.14) and

$$\overline{H}_\theta(\bar{x}, p) - \mu \overline{H}_\theta(\bar{y}, \frac{p}{\mu}) \geq -C\Psi'(|\bar{x} - \bar{y}|)$$

from (2.8). Using these estimates in (2.16) noticing that everything is uniform with respect to $\theta \in \Theta$ and sending ϱ to 0, we have

$$(2.19) \quad -4\nu\Psi'' - (\tilde{C} + 3C)\Psi' - (\mu - 1)\mathcal{H}_0 - 2C \leq 0.$$

Recalling that $\Psi'' + A_2\Psi' = 0$ by (2.12), we obtain a contradiction with (2.19) with the choice

$$A_2 = \frac{1}{4\nu} \left(\tilde{C} + 3C + \frac{(\mu - 1)\mathcal{H}_0 + 2C + 1}{L} \right).$$

This ends the proof. □

Lemma 2.5. *Under the hypotheses of Theorem 2.2. Let ϕ^ϵ be a continuous viscosity solution of (2.6) and define*

$$\exp(w^\epsilon) = \phi^\epsilon - \min_{\mathbb{T}^N} \phi^\epsilon + 1.$$

Then, there exists a constant K (independent of ϵ) such that $|Dw^\epsilon|_\infty \leq K$.

Proof of Lemma 2.5. For simplicity, we skip the ϵ superscript.

1. *New equation for w .* The function w solves

$$(2.20) \quad \epsilon e^{-w(x)}(\min \phi - 1) + \epsilon + \sup_{\theta \in \Theta} \left\{ -\text{tr}(A_\theta D^2 w) + \underline{G}_\theta(x, w, Dw) + \overline{G}_\theta(x, w, Dw) \right\} = 0,$$

where

$$\underline{G}_\theta(x, w, p) = e^{-w} \underline{H}_\theta(x, e^w p) - |\sigma_\theta(x)^T p|^2, \quad \overline{G}_\theta(x, w, p) = e^{-w} \overline{H}_\theta(x, e^w p).$$

2. *Definition of the test function.* We define Ψ as in (2.11) and we show later in Step 5 that we can choose r, A_1, A_2 such that

$$(2.21) \quad \Psi(r) = \frac{A_1}{A_2}(1 - e^{-A_2 r}) = \text{osc}(\phi).$$

Notice that, from Lemma 2.3, $\text{osc}(\phi)$ is bounded independently of ϵ . Consider

$$(2.22) \quad \max_{x, y \in \mathbb{T}^N} \{w(x) - w(y) - \Psi(|x - y|)\}.$$

If this maximum is nonpositive, for all $x, y \in \mathbb{T}^N$, we have

$$w(x) - w(y) \leq \Psi(|x - y|) \leq A_1|x - y|,$$

where the latter inequality follows from the concavity of Ψ . This yields the desired result.

From now on, we argue by contradiction, assuming that the maximum in (2.22) is positive and achieved at $(\bar{x}, \bar{y}) \in \mathbb{T}^N \times \mathbb{T}^N$. Notice that $\bar{x} \neq \bar{y}$ because the maximum is positive and

$$|\bar{x} - \bar{y}| < r$$

thanks to the choice of r in (2.21). The continuity of w is crucial in this step.

3. *Viscosity inequalities for (2.20).* Writing the viscosity inequalities as in Step 2 of the proof of Lemma 2.3, we obtain

$$\begin{aligned} \epsilon e^{-w(\bar{x})}(\min \phi - 1) + \epsilon + \sup_{\theta \in \Theta} \left\{ -\text{trace}(A_\theta(\bar{x})X) + \underline{G}_\theta(\bar{x}, w(\bar{x}), p) + \overline{G}_\theta(\bar{x}, w(\bar{x}), p) \right\} &\leq 0, \\ \epsilon e^{-w(\bar{y})}(\min \phi - 1) + \epsilon + \sup_{\theta \in \Theta} \left\{ -\text{trace}(A_\theta(\bar{y})Y) + \underline{G}_\theta(\bar{y}, w(\bar{y}), p) + \overline{G}_\theta(\bar{y}, w(\bar{y}), p) \right\} &\geq 0. \end{aligned}$$

Therefore,

$$(2.23) \quad \begin{aligned} &\epsilon(e^{-w(\bar{x})} - e^{-w(\bar{y})})(\min \phi - 1) \\ &+ \inf_{\theta \in \Theta} \left\{ -\text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) + \underline{G}_\theta(\bar{x}, w(\bar{x}), p) - \underline{G}_\theta(\bar{y}, w(\bar{y}), p) \right. \\ &\quad \left. + \overline{G}_\theta(\bar{x}, w(\bar{x}), p) - \overline{G}_\theta(\bar{y}, w(\bar{y}), p) \right\} \leq 0. \end{aligned}$$

The end of the proof consists in reaching a contradiction in the above inequality.

4. *Estimates of the terms in (2.23).* From (2.18) and the fact that $w(\bar{x}) > w(\bar{y}) \geq 0$, we have

$$(2.24) \quad \epsilon(e^{-w(\bar{x})} - e^{-w(\bar{y})})(\min \phi - 1) > -\mathcal{H}_0.$$

From now on, we fix $\theta \in \Theta$.

From Lemma 2.4 (2.17), we have

$$(2.25) \quad -\text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) \geq -4\nu\Psi''(|\bar{x} - \bar{y}|) - \tilde{C}\Psi'(|\bar{x} - \bar{y}|) + O(\rho).$$

Using (2.2), (2.7) and recalling that $|p| = \Psi'(|\bar{x} - \bar{y}|)$, we get

$$(2.26) \quad \begin{aligned} & |\underline{G}_\theta(\bar{x}, w(\bar{x}), p) - \underline{G}_\theta(\bar{y}, w(\bar{y}), p)| \\ & \leq |e^{-w(\bar{x})} \underline{H}_\theta(\bar{x}, e^{w(\bar{x})} p)| + |e^{-w(\bar{y})} \underline{H}_\theta(\bar{y}, e^{w(\bar{y})} p)| + \|\sigma_\theta(\bar{x})^T p\|^2 - \|\sigma_\theta(\bar{y})^T p\|^2 \\ & \leq 2C(1 + \Psi'(|\bar{x} - \bar{y}|)) + 2C^2|\bar{x} - \bar{y}|\Psi'^2(|\bar{x} - \bar{y}|). \end{aligned}$$

We now estimate $\overline{G}_\theta(\bar{x}, w(\bar{x}), p) - \overline{G}_\theta(\bar{y}, w(\bar{y}), p)$ using (2.9). Setting $P := e^{w(\bar{x})} p$ and $\mu := e^{w(\bar{x}) - w(\bar{y})}$, we have

$$(2.27) \quad \overline{G}_\theta(\bar{x}, w(\bar{x}), p) - \overline{G}_\theta(\bar{y}, w(\bar{y}), p) = e^{-w(\bar{x})} \left(\overline{H}_\theta(\bar{x}, P) - \mu \overline{H}_\theta(\bar{y}, \frac{P}{\mu}) \right).$$

We will prove in Step 5 that it is possible to choose A_1, A_2, r in (2.21) such that

$$(2.28) \quad |P| = |e^{w(\bar{x})} p| \geq |p| = \Psi'(|\bar{x} - \bar{y}|) \geq \Psi'(r) = A_1 e^{-A_2 r} \geq L,$$

where L is the constant which appears in (2.9). Since the maximum in (2.22) is positive and Ψ is concave, we get

$$(2.29) \quad \mu \geq 1 + w(\bar{x}) - w(\bar{y}) > 1 + \Psi(|\bar{x} - \bar{y}|) \geq 1 + \Psi'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}| \geq 1 + L|\bar{x} - \bar{y}|.$$

Notice that

$$e^{w(\bar{x})} \leq \text{osc}(\phi) + 1 \quad \text{and} \quad |p| \leq \Psi'(0) = A_1.$$

Therefore

$$(2.30) \quad L \leq |p| \leq A_1 = C(\nu, L) \quad \text{and} \quad 1 + L|\bar{x} - \bar{y}| \leq \mu \leq \text{osc}(\phi) + 1,$$

see Step 5. It follows that we can apply (2.9) to (2.27) to get

$$(2.31) \quad \overline{G}_\theta(\bar{x}, w(\bar{x}), p) - \overline{G}_\theta(\bar{y}, w(\bar{y}), p) \geq -C e^{-w(\bar{x})} |P| = -C \Psi'(|\bar{x} - \bar{y}|).$$

Plugging (2.24), (2.25), (2.26) and (2.31) in (2.23) noticing that the estimates are uniform with respect to $\theta \in \Theta$ and letting $\varrho \rightarrow 0$, we obtain

$$-4\nu\Psi'' - (\tilde{C} + 3C)\Psi' - 2C^2 r \Psi'^2 - \mathcal{H}_0 - 2C < 0.$$

Recall that \mathcal{H}_0 defined by (2.18) is bounded by some constant C thanks to (2.3). Since for all $s \in [0, r]$, $\Psi'' + A_2\Psi' = 0$ and $s\Psi'(s) \leq \Psi(s) \leq \Psi(r)$, we can rewrite the above estimate as

$$(2.32) \quad \left(4\nu A_2 - (\tilde{C} + 3C + 2C^2\Psi(r)) \right) \Psi' - 3C < 0.$$

5. *The choice of suitable constants.* The last step is to show that it is possible to set the constants r, A_1, A_2 in order that (2.21), (2.28) hold and to obtain a contradiction in (2.32). Of course, up to increase L at the beginning, we may assume that

$$(2.33) \quad L > \text{osc}(\phi).$$

We set

$$A_2 = \frac{1}{4\nu}(\tilde{C} + 6C + 2C^2L) \quad \text{and} \quad A_1 = L \exp(A_2\sqrt{N}).$$

From the choice of A_1 , we have $\Psi'(r) = A_1 e^{-A_2 r} \geq L e^{A_2(\sqrt{N} - r)} \geq L$ and (2.28) holds. On the one hand,

$$\Psi(\sqrt{N}) = \frac{L}{A_2}(e^{A_2\sqrt{N}} - 1) \geq L\sqrt{N} > \text{osc}(\phi)$$

by (2.33). On the other hand, $\Psi(r) \rightarrow 0$ as $r \rightarrow 0$. It follows that there exists $r = r(A_1, A_2) \in (0, \sqrt{N})$ such that (2.21) holds. It is then straightforward to check that we obtain a contradiction in (2.32).

Finally, we obtain the result with $K = A_1$. \square

We turn to the proof of Theorem 2.2. To use the previous lemmas, we need first to build a *continuous* viscosity solution of (2.6). It is straightforward to build a discontinuous viscosity solution for (2.6) using Perron's method for viscosity solutions of second order equations, see [19, 13]. The continuity of this solution usually follows from a strong comparison principle for (2.6), i.e., a comparison principle between USC viscosity subsolutions and LSC viscosity supersolutions. But, classical structure assumptions on the first order nonlinearity $\underline{H}_\theta + \overline{H}_\theta$ like Lipschitz continuity with respect to the x -variable, do not hold here. It follows that a strong comparison result may not hold. We are only able to compare a discontinuous sub- or supersolution with a Lipschitz continuous solution, see Theorem 4.1 in the Appendix. That is why, we need another approach inspired from [7] to build a continuous solution of (2.6). It is based on a truncation of the nonlinearity. Another natural approach would be to use the classical regularity theory for uniformly elliptic equations but it would not apply for degenerate equations we will consider in the next section.

Proof of Theorem 2.2. We fix $\epsilon > 0$ and skip the ϵ -dependence in the proof for simplicity.

1. *Truncated Hamiltonian.* For all $n > 0$, we define the continuous Hamiltonian

$$(2.34) \quad H_{\theta n} := \underline{H}_{\theta n} + \overline{H}_{\theta n}$$

with

$$\underline{H}_{\theta n}(x, p) \text{ (resp. } \overline{H}_{\theta n}(x, p)) = \begin{cases} \underline{H}_\theta(x, p) \text{ (resp. } \overline{H}_\theta(x, p)) & \text{if } |p| \leq n, \\ \underline{H}_\theta(x, n \frac{p}{|p|}) \text{ (resp. } \overline{H}_\theta(x, n \frac{p}{|p|})) & \text{if } |p| \geq n. \end{cases}$$

2. *Construction of a continuous viscosity solution for the truncated equation.* We have a strong comparison principle between discontinuous solutions for (2.6) where $\underline{H}_\theta, \overline{H}_\theta$ are replaced with $\underline{H}_{\theta n}, \overline{H}_{\theta n}$ respectively, see Theorem 4.2. By Perron's method, see [13, 19], there exists a continuous viscosity solution ϕ_n .

3. *Uniform Lipschitz bound for solutions of the truncated equation.* We first notice that $\underline{H}_{\theta n}$ satisfies (2.7) with the same constant C as \underline{H}_θ . Moreover, if we choose n bigger than L which appears in (2.8), then, by Lemma 2.3, we obtain a bound for the oscillation of ϕ_n which is independent of n, ϵ . Moreover, if n is chosen bigger than the right hand side of (2.30), then (2.9) hold for $\overline{H}_{\theta n}$ with the same constants as for \overline{H}_θ for all p satisfying (2.30). As noticed in the proof of Lemma 2.5, it is enough to obtain a gradient bound K for w_n defined by

$$\exp(w_n) = \phi_n - \min_{\mathbb{T}^N} \phi_n + 1.$$

The crucial point is that K does not depend on n since the constants in (2.7), (2.8), (2.9) are the same for all $\underline{H}_{\theta n}, \overline{H}_{\theta n}$. The uniform bound for the oscillation of ϕ_n yields a L^∞ bound for w_n which is independent of n, ϵ . It follows

$$(2.35) \quad |D\phi_n|_\infty \leq K \exp(\sqrt{N}K).$$

4. *Convergence of ϕ_n .* In addition to (2.35), from (4.5), we have a L^∞ bound for ϕ_n which is independent of n . Then, by Ascoli-Arzelà's theorem, we obtain that, up to a subsequence, $\phi_n \rightarrow \phi$ in $C(\mathbb{T}^N)$ and $\phi = \phi^\epsilon$ is Lipschitz continuous with $|D\phi|_\infty \leq Ke^{\sqrt{N}K}$, which is

independent of ϵ . Noticing that $\underline{H}_{\theta_n}, \overline{H}_{\theta_n} \rightarrow \underline{H}_\theta, \overline{H}_\theta$ in $C(\mathbb{T}^N \times \mathbb{R}^N)$, by the stability result for viscosity solution, we conclude that ϕ is a Lipschitz continuous viscosity solution of (2.6).

5. *Uniqueness.* The uniqueness of ϕ in the class of continuous viscosity solutions relies on a comparison principle between the Lipschitz solution ϕ with any continuous solution $\tilde{\phi}$ of (2.6), see Theorem 4.1 in the Appendix. \square

2.3. Gradient bounds for degenerate elliptic equations. We now consider degenerate elliptic equations, i.e., (2.5) does not necessarily hold. We give two results. The first one applies when the equations may be degenerate everywhere. In this case, we suppose that the sublinear Hamiltonian $\underline{H}_\theta \equiv 0$ in (2.6) and we reinforce the assumptions (2.8)-(2.9) in order that the superlinear Hamiltonian \overline{H}_θ controls all the terms in the equation. In the second result, we deal with a mixed case where the equation is uniformly elliptic for some controls θ and degenerate for the others.

The assumptions are

$$(2.36) \quad \left\{ \begin{array}{l} \overline{H}_\theta \text{ satisfies (2.3) and there exists } L, \mu > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, |p| \geq L \text{ and } \theta \in \Theta, \\ \overline{H}_\theta(x, p) - \mu \overline{H}_\theta(y, \frac{p}{\mu}) \geq (\mu - 1) (|\overline{H}_\theta(\cdot, 0)|_\infty + N|(\sigma_\theta)_x|_\infty^2 |p|), \end{array} \right.$$

$$(2.37) \quad \left\{ \begin{array}{l} \overline{H}_\theta \text{ satisfies (2.3) and there exists } L > 1 \text{ such that} \\ \text{for all } x, y \in \mathbb{T}^N, |p| \geq L \text{ and } \theta \in \Theta \text{ and } \mu \geq 1 + L|x - y|, \\ \overline{H}_\theta(x, p) - \mu \overline{H}_\theta(y, \frac{p}{\mu}) \geq (\mu - 1) (|\overline{H}_\theta(\cdot, 0)|_\infty + N|(\sigma_\theta)_x|_\infty^2 |p| + 2|(\sigma_\theta)_x|_\infty |\sigma_\theta|_\infty |p|). \end{array} \right.$$

As for the uniformly elliptic case, the first assumption is needed to get the oscillation bound and the stronger one to obtain the gradient bound. Some discussion about these assumptions and examples are given in Section 2.4.

Theorem 2.6. *Assume (2.2), (2.37) and suppose that $\underline{H}_\theta \equiv 0$ for all θ . For all $\epsilon > 0$, there exists a unique continuous viscosity solution $\phi^\epsilon \in C(\mathbb{T}^N)$ of (2.6) and a constant $K > 0$ independent of ϵ such that*

$$|D\phi^\epsilon|_\infty \leq K.$$

For the second result, we consider the case where the control space Θ is split into two smaller control spaces Θ_1, Θ_2 such that the hypotheses of Theorem 2.2 hold in Θ_1 and the hypotheses of Theorem 2.6 hold in Θ_2 . We introduce

$$(2.38) \quad (2.5)-(2.7)-(2.9) \text{ hold for } \theta \in \Theta_1.$$

$$(2.39) \quad \underline{H}_\theta \equiv 0 \text{ for } \theta \in \Theta_2 \text{ and (2.37) holds for } \theta \in \Theta_2.$$

Theorem 2.7. *Assume (2.2) and suppose that $\Theta = \Theta_1 \cup \Theta_2$ and (2.38), (2.39) hold. For all $\epsilon > 0$, there exists a unique continuous viscosity solution $\phi^\epsilon \in C(\mathbb{T}^N)$ of (2.6) and a constant $K > 0$ independent of ϵ such that*

$$|D\phi^\epsilon|_\infty \leq K.$$

When $\Theta_1 = \mathbb{T}^N \rightarrow \Theta_2 = \emptyset$ (respectively $\Theta_2 = \mathbb{T}^N \rightarrow \Theta_1 = \emptyset$) then Theorem 2.7 reduces to Theorem 2.2 (respectively Theorem 2.6).

The proof of the above theorems are similar to the one of Theorem 2.2, so we skip it. It relies on the auxiliary Lemmas 2.3 and 2.5 where (2.8)-(2.9) are replaced by (2.36)-(2.37) for Theorem 2.6 and by (2.38)-(2.39) for Theorem 2.7. The proofs of the auxiliary lemmas follow the same lines, the changes occur in the estimates of the terms in (2.16) and (2.23), so we only rewrite Step 4 of the proof of Lemma 2.5 where the main changes occur.

Proof of Lemma 2.5 for Theorem 2.6 under assumptions (2.37). We estimate the different terms appearing in (2.23). We set $P := e^{w(\bar{x})}p$ and $\mu := e^{w(\bar{x})-w(\bar{y})}$ and we recall that, if the maximum is positive in (2.22) and with a suitable choice of A_1, A_2, r in (2.11), then

$$|P| = |e^{w(\bar{x})}p| \geq L > 1 \text{ and } \mu > 1 + |p||\bar{x} - \bar{y}| \geq 1 + L|\bar{x} - \bar{y}|$$

(see (2.28) and (2.29)). From (2.18), we get

$$(2.40) \quad \begin{aligned} \epsilon(e^{-w(\bar{x})} - e^{-w(\bar{y})})(\min \phi - 1) &> e^{-w(\bar{x})}(1 - \mu)\epsilon(\min \phi - 1) \\ &\geq -e^{-w(\bar{x})}(\mu - 1)|\overline{H}_\theta(\cdot, 0)|_\infty. \end{aligned}$$

From Lemma 2.4 (2.17), we have

$$-\text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) \geq -e^{-w(\bar{x})}(\mu - 1)N|(\sigma_\theta)_x|_\infty^2|P| + O(\varrho).$$

Since $\underline{H}_\theta \equiv 0$, we have

$$\begin{aligned} \underline{G}_\theta(\bar{x}, w(\bar{x}), p) - \underline{G}_\theta(\bar{y}, w(\bar{y}), p) &= -|\sigma_\theta(\bar{x})^T p|^2 + |\sigma_\theta(\bar{y})^T p|^2 \\ &\geq -2|\sigma_\theta|_\infty|(\sigma_\theta)_x|_\infty|\bar{x} - \bar{y}||p|^2 \\ &\geq -2e^{-w(\bar{x})}(\mu - 1)|\sigma_\theta|_\infty|(\sigma_\theta)_x|_\infty|P|. \end{aligned}$$

As far as the superlinear Hamiltonians are concerned, we have

$$\overline{G}_\theta(\bar{x}, w(\bar{x}), p) - \overline{G}_\theta(\bar{y}, w(\bar{y}), p) = e^{-w(\bar{x})} \left(\overline{H}_\theta(\bar{x}, P) - \mu \overline{H}_\theta(\bar{y}, \frac{P}{\mu}) \right).$$

Plugging the previous estimates in (2.23) and applying (2.37), we reach a contradiction with the strict inequality in (2.40) which is uniform with respect to $\theta \in \Theta$ and $\varrho \rightarrow 0$. This ends the proof. \square

Proof of Lemma 2.5 for Theorem 2.7 under assumptions (2.39). To estimate the different terms appearing in (2.23), it is enough to reproduce Step 4 of the proof of Lemma 2.5 for $\theta \in \Theta_1$ and the proof above for $\theta \in \Theta_2$. Since all the constants which appear are uniform with respect to θ , we reach a contradiction for any $\theta \in \Theta = \Theta_1 \cup \Theta_2$. \square

2.4. Comments on the assumptions and examples.

Example 2.8. (Hamiltonian satisfying (2.4)) If $\limsup_{|p| \rightarrow +\infty} \frac{H_\theta(x, p)}{|p|} = +\infty$ uniformly with respect to x, θ , then (2.4) holds. For instance, if $H_\theta(x, p) = a_\theta(x)\sin(h(|p|))|p|^{1+\alpha} + \ell_\theta(x)$, where $\alpha > 0$, a_θ, ℓ_θ, h are continuous for all θ , $\bar{a} \geq a_\theta \geq \underline{a} > 0$, $\bar{\ell} \geq \ell_\theta$ and $\limsup_{r \rightarrow +\infty} \sin(h(r)) > 0$, then (2.4) holds.

Lemma 2.9.

- (i) A sublinear Hamiltonian \underline{H}_θ satisfying (2.7) satisfies (2.8).

- (ii) If H_θ is continuous and there exists $\alpha \geq 0$, $A \geq a > 0$, $B, L \geq 0$ such that, for all $x \in \mathbb{T}^N$, $|p| \geq L$, $\theta \in \Theta$,

$$A|p|^\alpha + B|p| \geq H_\theta(x, p) \geq a|p|^\alpha - B|p|,$$

then (2.8) holds

Proof of Lemma 2.9. (i) Using (2.7), we have, for $\mu = 2$ and $|p| > 1$,

$$\underline{H}_\theta(x, p) - \mu \underline{H}_\theta(y, \frac{p}{\mu}) \geq -C(1 + |p|) - C\mu(1 + |\frac{p}{\mu}|) \geq -3C(1 + |p|) \geq -6C|p|.$$

- (ii) When $0 \leq \alpha \leq 1$, then (2.7) holds. For $\alpha > 1$,

$$H_\theta(x, p) - \mu H_\theta(y, \frac{p}{\mu}) \geq (a - \mu^{1-\alpha}A)|p|^\alpha - 2B|p| \geq -2B|p|$$

when $\mu \geq (A/a)^{1/(\alpha-1)}$. □

Notice that no regularity assumption (except continuity) is needed to obtain the oscillation bound. To obtain a gradient bound, we need to reinforce (2.8)-(2.36) into (2.9)-(2.37). These new assumptions contain a kind of regularity assumption with respect to (x, p) . Actually, (2.9) is very close to (2.10) which is introduced in [7] and (2.37) is very close to

$$(2.41) \left\{ \begin{array}{l} \overline{H}_\theta \in W_{\text{loc}}^{1,\infty}(\mathbb{T}^N \times \mathbb{R}^N) \text{ satisfies (2.3) and there exists } L \text{ such that} \\ \text{if } |p| \geq L, \text{ then, for a.e. } (x, p) \in \mathbb{T}^N \times \mathbb{R}^N, \theta \in \Theta, \\ L [(\overline{H}_\theta)_{pp} - \overline{H}_\theta - |\overline{H}_\theta(\cdot, 0)|_\infty - N|(\sigma_\theta)_x|_\infty^2|p| - 2|\sigma_\theta|_\infty|(\sigma_\theta)_x|_\infty|p|] - |(\overline{H}_\theta)_x| \geq 0, \end{array} \right.$$

which is an extension of (2.10) for (x, θ) -dependent degenerate diffusion matrices. In (2.10)-(2.41), the Hamiltonian is supposed to be locally Lipschitz with respect to (x, p) . When the Hamiltonians are locally Lipschitz, we can prove that (2.9)-(2.37) and (2.10)-(2.41) are equivalent but our assumptions allow to deal with some non Lipschitz continuous Hamiltonians as shown in the following example.

Example 2.10. (A non Lipschitz continuous Hamiltonian satisfying (2.9)) Let $H(x, p) = |p|^2 + h(x, p)$ with h continuous bounded. For all $x, y \in \mathbb{T}^N$, $p \in \mathbb{R}^N$ and $\mu > 1$, we have

$$H(x, p) - \mu H(y, \frac{p}{\mu}) = (1 - \frac{1}{\mu})|p|^2 + h(x, p) - \mu h(y, \frac{p}{\mu}) \geq -(1 + \mu)|h|_\infty.$$

From (2.30), we see that (2.9) has to hold only for bounded $\mu \geq 1 + L|x - y|$ and $|p| \geq L > 1$. So (2.9) holds.

We turn to some examples of sub- and superlinear Hamiltonians.

Example 2.11. (Typical sublinear Hamiltonians from optimal control problem)

$$\underline{H}_\theta(x, p) = \langle b_\theta(x), p \rangle + \ell_\theta(x),$$

where $b_\theta, \ell_\theta \in C(\mathbb{T}^N; \mathbb{R})$ and $|b_\theta(x)|, |\ell_\theta(x)| \leq C$. Such a \underline{H}_θ satisfies (2.7). Notice that the Lipschitz continuity is not needed in the proof of the gradient bound.

Example 2.12. (Typical superlinear Hamiltonian)

$$\overline{H}_\theta(x, p) = a_\theta(x)|p|^{1+\alpha_\theta} + b_\theta(x)|p| + c_\theta(x),$$

with $\overline{\alpha} \geq \alpha_\theta \geq \underline{\alpha} > 1$, $a_\theta, b_\theta, c_\theta \in C(\mathbb{T}^N)$ such that there exist $C, \underline{a} > 0$ such that

$$|a_\theta|, |b_\theta|, |c_\theta| \leq C \text{ and } a_\theta \geq \underline{a}, \quad \theta \in \Theta.$$

Then \overline{H}_θ satisfies (2.4) as soon as σ_θ satisfies (2.2). If moreover

$$|(a_\theta)_x|_\infty, |(b_\theta)_x|_\infty, |(c_\theta)_x|_\infty \leq C, \quad \theta \in \Theta,$$

then \overline{H}_θ satisfies (2.9) and (2.37) as soon as (2.2) holds. It follows that the gradient bound of Theorem 2.2 holds for (2.6) even if σ_θ is degenerate (i.e., (2.5) does not hold).

Example 2.13. $H(x, p) = |B(x)p|^k + \langle b(x), p \rangle + \ell(x)$ with $B \in C(\mathbb{T}^N; \mathcal{M}_N)$, $b \in C(\mathbb{T}^N; \mathbb{R}^N)$, $\ell \in C(\mathbb{T}^N)$ satisfies (2.9) if

$$|b_x|_\infty, |B_x|_\infty \leq C \quad \text{and} \quad B(x)B(x)^T > 0.$$

Example 2.14. Let define $\hat{H}(p) = \hat{H}(|p|)$ radial by $\hat{H}(0) = 0$ and, for all $t \in [n, n+1]$, $\hat{H}(t) = (n+1)t - n(n+1)/2$. We notice that $\hat{H} \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)$ and $\hat{H}_p(p)p - \hat{H} = 0$ a.e. It follows that $H(x, p) = \hat{H}(p) + \ell(x)$ satisfies (2.9) if ℓ is continuous but does not satisfy (2.37) (even if ℓ is Lipschitz continuous). The Hamiltonian $H(x, p) = \underline{H}(x, p) + \hat{H}(p) + \ell(x)$ where \underline{H} satisfies (2.7) fulfills the assumptions of Theorem 2.2 if the diffusion matrix satisfies (2.2)-(2.5) but such a case cannot be handled with the results of [7].

Example 2.15. (Example including controls and mixing degenerate and nondegenerate A_θ) Set $\Theta = \{0, 1\}$ and

$$\begin{aligned} A_0 \text{ satisfies (2.2), } \quad \underline{H}_0 &\equiv 0, \quad \overline{H}_0(x, p) = a(x)|p|^2 + \ell_0(x), \\ A_1(x) &\equiv I, \quad \underline{H}_1(x, p) = |p|F(x, \frac{p}{|p|}) + \ell_1(x), \quad \overline{H}_1 \equiv 0, \end{aligned}$$

where a, F, ℓ_0, ℓ_1 are continuous and $a(x) > 0$. Then (2.38)-(2.39) hold for $\Theta_1 = \{1\}$ and $\Theta_2 = \{0\}$.

3. ASYMMETRIC SYSTEMS

3.1. Preliminaries on systems and coupling matrices. We consider the weakly coupled system

$$(3.1) \quad \epsilon \phi_i^\epsilon + \sup_{\theta \in \Theta} \{-\text{trace}(A_{\theta i}(x)D^2\phi^\epsilon) + H_{\theta i}(x, D\phi^\epsilon)\} + \sum_{j=1}^m d_{ij}\phi_j^\epsilon = 0, \quad x \in \mathbb{T}^N, \quad 1 \leq i \leq m,$$

where, for all $1 \leq i \leq m$, $A_{\theta i}$ and $H_{\theta i}$ satisfy the Assumptions (2.2)-(2.3). We consider monotone fully coupled systems, i.e., the following assumptions hold.

$$(3.2) \quad D = (d_{ij})_{ij} \text{ is monotone: } d_{ii} \geq 0, \quad d_{ij} \leq 0 \text{ for } i \neq j \quad \text{and} \quad \sum_{j=1}^m d_{ij} = 0 \text{ for all } i.$$

$$(3.3) \quad D \text{ is irreducible:}$$

for all subset $\mathcal{I} \subsetneq \{1, \dots, m\}$, there exists $i \in \mathcal{I}$ and $j \notin \mathcal{I}$ such that $d_{ij} \neq 0$.

Roughly speaking, when the coupling matrix is irreducible, it means that the system cannot be split into smaller systems.

Lemma 3.1. ([9]) *Suppose that $m \geq 2$ and $D = (d_{ij})_{1 \leq i, j \leq m}$ satisfies (3.2) and (3.3). Then D is degenerate of rank $m - 1$, $\ker(D) = \text{span}\{(1, \dots, 1)\}$ and the real part of each nonzero complex eigenvalue of D is positive. Moreover there exists $\Lambda = (\Lambda_1, \dots, \Lambda_m) \in \mathbb{R}^m$ with positive components $\Lambda_i > 0$ such that $D^T \Lambda = 0$.*

From now on, we will assume that all the systems are coupled with a matrix $D = (d_{ij})_{ij}$ satisfying (3.2) and (3.3). In particular, thanks to Lemma 3.1, for $m \geq 2$, we can define

$$\lambda_D := \max_{1 \leq i \leq m} \frac{1}{\Lambda_i} \sum_{j=1}^m \Lambda_j.$$

If $m = 1$ (scalar equations), then $D = 0$ and we set $\lambda_D := 0$.

3.2. Gradient bounds for systems. The aim is to obtain uniform gradient bounds (i.e., independent of ϵ) for (3.1) when $H_{\theta_i} = \underline{H}_{\theta_i} + \overline{H}_{\theta_i}$, $1 \leq i \leq m$, satisfy (2.38)-(2.39) extended to the case of systems. We define

$$\mathcal{H} := \sup_{x \in \mathbb{T}^N, \theta \in \Theta, 1 \leq j \leq m} |H_{\theta_j}(x, 0)|.$$

and introduce, for all $1 \leq i \leq m$,

$$(3.4) \quad \begin{cases} \Theta = \Theta_{1i} \cup \Theta_{2i}, \\ \text{for } \theta \in \Theta_{1i}, (2.5)-(2.7)-(2.9) \text{ hold for } A_{\theta_i}, \underline{H}_{\theta_i}, \overline{H}_{\theta_i} \\ \text{for } \theta \in \Theta_{2i}, \underline{H}_{\theta_i} \equiv 0 \text{ and (2.37) holds for } \overline{H}_{\theta_i} \text{ with } |\overline{H}_{\theta_i}(\cdot, 0)| \text{ replaced by } (2\lambda_D + 1)\mathcal{H}. \end{cases}$$

This assumption is technical but includes asymmetric systems for which some equations appear with sublinear Hamiltonians while the others appear with superlinear Hamiltonians. To deal with such ‘‘mixed’’ systems is one the main goal of this work. Notice also that (3.4) reduces to (2.38)-(2.39) when $m = 1$ (scalar equations). Some examples are given in Section 3.6.

Theorem 3.2. *Assume (2.2) and (3.4). There exists a unique continuous viscosity solution $\phi^\epsilon = (\phi_1^\epsilon, \dots, \phi_m^\epsilon) \in C(\mathbb{T}^N)^m$ of (3.1) and there exists $K > 0$ depending only on the H_{θ_i} 's and D such that*

$$|D\phi_i^\epsilon|_\infty \leq K \quad \text{for all } \epsilon > 0, 1 \leq i \leq m.$$

Similarly to the case of scalar equations, the result consists of two main steps: first, we prove an uniform bound for the oscillation and we then improve it to a uniform gradient bound. The key lemmas are

Lemma 3.3. *Under the hypotheses of Theorem 3.2, and let $\phi^\epsilon = (\phi_1^\epsilon, \dots, \phi_m^\epsilon) \in C(\mathbb{T}^N)^m$ be a solution of (3.1). There exists $K > 0$ depending only on the H_{θ_i} 's and D such that*

$$\text{osc}(\phi_i^\epsilon) \leq K \quad \text{for all } \epsilon > 0, 1 \leq i \leq m.$$

We skip the proof of Lemma 3.3 since it is similar to the case of scalar equations.

Lemma 3.4. *Under the hypotheses of Theorem 3.2, let ϕ^ϵ be a continuous viscosity solution of (3.1) and define w^ϵ by*

$$\exp(w_i^\epsilon) = \phi_i^\epsilon - \min_{\mathbb{T}^N} \phi_i^\epsilon + 1 \quad \text{for all } \epsilon, i.$$

Then, there exists a constant K independent of ϵ such that $|Dw_i^\epsilon|_\infty \leq K$ for all ϵ, i .

Proof of Theorem 3.2. It is sufficient to see that the proof of Theorem 2.2 can be extended to systems like (3.1). For $1 \leq i \leq m$, we define $H_{\theta_{in}}$ like in (2.34). By Perron's method for systems, see [20] and Theorem 4.2, there exists a continuous viscosity solution ϕ_n of (3.1) where the H_{θ_i} are replaced with the $H_{\theta_{in}}$'s. It is now possible to apply Lemma 3.4 to ϕ_n

to obtain a gradient bound which is independent of ϵ, n . We conclude as in the proof of Theorem 2.2. \square

Proof of Lemma 3.4. The proof is almost the same as the one for scalar equations. The main change is the estimate (3.7). So we only write few important steps.

1. Set $m_i = \min_{\mathbb{T}^N} \phi_i$, it is clear that (w_1, \dots, w_m) solves

$$\sup_{\theta \in \Theta} \left\{ -\text{trace}(A_{\theta_i}(x)D^2w_i) + \underline{G}_{\theta_i}(x, w_i, Dw_i) + \overline{G}_{\theta_i}(x, w_i, Dw_i) \right\} + b_i(w_i) + e^{-w_i} \sum_{j=1}^m d_{ij}e^{w_j} = 0,$$

where

$$\begin{aligned} \underline{G}_{\theta_i}(x, w, p) &= e^{-w} \underline{H}_{\theta_i}(x, e^w p) - |\sigma_{\theta_i}(x)^T p|^2, \quad \overline{G}_{\theta_i}(x, w, p) = e^{-w} \overline{H}_{\theta_i}(x, e^w p), \\ b_i(w) &= e^{-w} \left[\sum_{j=1}^m d_{ij} m_j + \epsilon(m_i - 1) \right] + \epsilon. \end{aligned}$$

Consider

$$(3.5) \quad \max_{\mathbb{T}^N \times \mathbb{T}^N, 1 \leq i \leq m} \{w_i(x) - w_i(y) - \Psi(|x - y|)\},$$

where Ψ is of the form in (2.11) with A_1, A_2 to be chosen later. We are done if the maximum is nonpositive.

2. Otherwise, the maximum is positive and hence is achieved at $\bar{x} \neq \bar{y}$ for some $1 \leq i \leq m$. So we can write the viscosity inequalities for the i th equation at this point. For every $\varrho > 0$, there exist $(p, X) \in \overline{J}^{2,+} w_i(\bar{x})$, and $(p, Y) \in \overline{J}^{2,-} w_i(\bar{y})$ such that (2.13)-(2.14)-(2.15) hold and (2.16) is replaced by

$$(3.6) \quad \inf_{\theta \in \Theta} \{ \underline{G}_{\theta_i} + \overline{G}_{\theta_i} \} + \mathcal{B} + \mathcal{C} \leq 0,$$

where

$$\begin{aligned} \underline{G}_{\theta_i} &= -\text{tr}(A_{\theta_i}(\bar{x})X - A_{\theta_i}(\bar{y})Y) + \underline{G}_{\theta_i}(\bar{x}, w_i(\bar{x}), p) - \underline{G}_{\theta_i}(\bar{y}, w_i(\bar{y}), p), \\ \overline{G}_{\theta_i} &= \overline{G}_{\theta_i}(\bar{x}, w_i(\bar{x}), p) - \overline{G}_{\theta_i}(\bar{y}, w_i(\bar{y}), p), \\ \mathcal{B} &= b_i(w_i(\bar{x})) - b_i(w_i(\bar{y})), \\ \mathcal{C} &= \sum_{j=1}^m d_{ij} e^{w_j(\bar{x}) - w_i(\bar{x})} - \sum_{j=1}^m d_{ij} e^{w_j(\bar{y}) - w_i(\bar{y})}. \end{aligned}$$

Since the maximum in (3.5) is positive, we have $\mathcal{C} \geq 0$. Next, we claim that

$$(3.7) \quad \sum_{j=1}^m d_{ij} m_j + \epsilon m_i \leq (2\lambda_D + 1)\mathcal{H},$$

see Step 6 for a proof. It follows easily that

$$\mathcal{B} \geq -(2\lambda_D + 1)\mathcal{H}.$$

To get a contradiction in (3.6), we distinguish two cases: "uniformly elliptic" (Step 3) and "degenerate elliptic" (step 4) depending on $\theta \in \Theta = \Theta_{1i} \cup \Theta_{2i}$.

3. If $\theta \in \Theta_{1i}$, then we use the arguments of Step 4 in the proof of Lemma 2.5. We obtain

$$\begin{aligned} & \underline{\mathcal{G}}_{\theta i} + \overline{\mathcal{G}}_{\theta i} + \mathcal{B} + \mathcal{C} \\ & \geq -4\nu\Psi'' - (\tilde{C} + 2C + \frac{C \operatorname{osc}(\phi_i)}{L})\Psi' - 2C - (2\lambda_D + 1)\mathcal{H}, \end{aligned}$$

where L is the constant appearing in (2.9) and $\operatorname{osc}(\phi_i)$ is bounded by Lemma 3.3. With a suitable choice of A_1, A_2 in the definition of Ψ , the right-hand side is positive (uniformly with respect to $\theta \in \Theta_{1i}$).

4. If $\theta \in \Theta_{2i}$, we shall control the 0th order terms using the superlinear Hamiltonian. That is the reason why we need to reinforce slightly (2.37) in (3.4). We repeat readily the arguments of the proof of Lemma 2.5 in Section 2.3 (case of degenerate elliptic equations) estimating $\overline{\mathcal{G}}_{\theta i}$ using now (3.4). We obtain that $\underline{\mathcal{G}}_{\theta i} + \overline{\mathcal{G}}_{\theta i} + \mathcal{B} + \mathcal{C} > 0$ uniformly with respect to $\theta \in \Theta_{2i}$.

5. Steps 3 and 4 lead to a contradiction in (3.6). So, the maximum (3.5) cannot be positive.

6. It remains to prove (3.7). Let x_k such that $\phi_k(x_k) = \min_{\mathbb{T}^N} \phi_k$, we get from (3.1) that

$$\begin{aligned} & - \sum_{j=1}^m d_{kj} \phi_j(x_k) = \epsilon \phi_k(x_k) + \sup_{\theta \in \Theta} \{ -\operatorname{trace}(A_{\theta i}(x_k) D^2 \phi^\epsilon) + H_{\theta i}(x_k, 0) \} \\ & \leq \epsilon \phi_k(x_k) + \sup_{\theta \in \Theta} H_{\theta i}(x_k, 0) \leq 2\mathcal{H}. \end{aligned}$$

Then

$$- \sum_{k \neq i} \Lambda_k \sum_{j=1}^m d_{kj} m_j \leq - \sum_{k \neq i} \Lambda_k \sum_{j=1}^m d_{kj} \phi_j(x_k) \leq 2\mathcal{H} \sum_{k=1}^m \Lambda_k,$$

where the Λ_k 's are given by Lemma 3.1. Using this lemma 3.1 again, we have

$$\sum_{k=1}^m \Lambda_k \sum_{j=1}^m d_{kj} m_j = 0.$$

It follows

$$\Lambda_i \sum_{j=1}^m d_{ij} m_j \leq 2\mathcal{H} \sum_{k=1}^m \Lambda_k.$$

This implies easily (3.7). □

3.3. Strong maximum principle for systems. The following extension of the strong maximum principle to parabolic systems is a crucial ingredient in the proof of the large time behavior.

Theorem 3.5. *Assume that, for all $1 \leq i \leq m$, $\theta \in \Theta$, $A_{\theta i} \geq \nu_i(x)I$ with $\nu_i \in C(\mathbb{T}^N, \mathbb{R})$, $\nu_i \geq 0$ and*

$$(3.8) \quad \bigcup_{1 \leq i \leq m} \{x \in \mathbb{T}^N : \nu_i(x) > 0\} = \mathbb{T}^N.$$

If u is a continuous viscosity subsolution of

$$(3.9) \quad \begin{cases} \frac{\partial u_i}{\partial t} + \inf_{\theta \in \Theta} \{ -\operatorname{trace}(A_{\theta i}(x) D^2 u_i) \} - C|Du_i| + \sum_{j=1}^m d_{ij} u_j = 0, & x \in \mathbb{T}^N \times (0, +\infty), \\ u_i(x, 0) = u_{0i}(x), & x \in \mathbb{T}^N, \end{cases}$$

which attains a maximum at $(\bar{x}, \bar{t}) \in \mathbb{T}^N \times (0, +\infty)$, then u is constant.

Remark 3.6.

- (i) A new feature of the above result is that one only need a partial nondegeneracy condition for the diffusion matrices A_i in the following sense. At each point of \mathbb{T}^N , there exists at least one equation such that $A_i(x)$ is nondegenerate. Due to the irreducibility of the coupling, this is enough to conclude that all the components u_i are constant everywhere in \mathbb{T}^N . It can be interpreted using optimal control as follows. When considering the stochastic control problem associated with the equation in (3.9), it means that the controlled process visits any open set of \mathbb{T}^N almost surely for any open time interval (see Section 4.4 for the control interpretation).
- (ii) This result contains, as a particular case, stationary systems.

Proof of Theorem 3.5.

1. Suppose that

$$(3.10) \quad M := \sup_{(x,t) \in \mathbb{T}^N \times [0, +\infty), 1 \leq j \leq m} u_j(x, t) = u_i(\bar{x}, \bar{t}), \quad \bar{t} > 0.$$

We do a formal calculation. A rigorous one can be made using classical viscosity techniques. At the point (\bar{x}, \bar{t}) , we have $\frac{\partial u_i}{\partial t} = 0$, $Du_i = 0$ and $-D^2u_i \geq 0$. From the i th equation, we obtain

$$\sum_{j=1}^m d_{ij} u_j(\bar{x}, \bar{t}) \leq 0.$$

From (3.10) and (3.2), the opposite inequality is also true. Therefore, for all $1 \leq j \leq m$ such that $d_{ij} \neq 0$, we get $u_j(\bar{x}, \bar{t}) = u_i(\bar{x}, \bar{t})$. By irreducibility (3.3), there exists at least such a j . It follows that (\bar{x}, \bar{t}) is a maximum point of u_j in $\mathbb{T}^N \times [0, +\infty)$. We can repeat the same argument for the j th equation and describe the whole set of index to obtain that (\bar{x}, \bar{t}) is a maximum point of all u_j , $1 \leq j \leq m$, in $\mathbb{T}^N \times [0, +\infty)$.

2. We set $\Omega_j = \{x \in \mathbb{T}^N : \nu_j(x) > 0\}$. Notice that Ω_j is an open subset of \mathbb{T}^N . By (3.8) and the open character of Ω_j , there exists a set of balls $\{B_p\}_{p \in \mathbb{N}}$ such that $\mathbb{T}^N = \bigcup_{p \in \mathbb{N}} B_p$ and for all $p \in \mathbb{N}$, there exists j such that $\bar{B}_p \subset \Omega_j$. Since \mathbb{T}^N is compact, we can extract a finite covering of the form

$$(3.11) \quad \mathbb{T}^N = \bigcup_{p=1}^n B_p.$$

It follows that there exist p, k such that $\bar{x} \in \bar{B}_p \subset \Omega_k$. By continuity of ν_k and compactness, we have

$$\inf_{y \in \bar{B}_p, |\xi|=1, \theta \in \Theta} \langle A_{\theta k}(y)\xi, \xi \rangle \geq \inf_{y \in \bar{B}_p} \nu_k(y) =: \nu > 0.$$

It follows that the k th equation is uniformly parabolic in B_p . Moreover, the maximum in (3.10) is also a maximum on $B_p \times [0, +\infty)$. We set $u_j^M = u_j - M$ for all j . Then $u_j^M \leq 0$. Using (3.2),

we have

$$\begin{aligned}
& \frac{\partial u_k^M}{\partial t} + \inf_{\theta \in \Theta} \{-\text{trace}(A_{\theta k} D^2 u_k^M)\} - C|Du_k^M| + d_{kk}u_k^M \\
& \leq \frac{\partial u_k^M}{\partial t} + \inf_{\theta \in \Theta} \{-\text{trace}(A_{\theta k} D^2 u_k^M)\} - C|Du_k^M| + \sum_{j=1}^m d_{kj}u_j^M \\
& = \frac{\partial u_k}{\partial t} + \inf_{\theta \in \Theta} \{-\text{trace}(A_{\theta k} D^2 u_k^M)\} - C|Du_k| + \sum_{j=1}^m d_{kj}u_j \leq 0 \quad \text{in } B_p.
\end{aligned}$$

By the strong maximum principle for viscosity solutions of single parabolic equations (see, e.g., [14]), we obtain $u_k^M \equiv 0$ in $B_p \times [0, +\infty)$. Therefore, the maximum in (3.10) is achieved for $i = k$ at every $(y, t) \in B_p \times [0, +\infty)$. Coming back to Step 1, we obtain $u_j(y, t) \equiv M$ for all j and $(y, t) \in B_p \times [0, +\infty)$.

3. From the previous steps, we obtain that $u_j \equiv M$ for all j and $(x, t) \in B_p \times [0, +\infty)$. Now, by (3.11), there exist p' and x' such that $x' \in B_p \cap B_{p'}$. It follows that the u_j 's achieve their maximum over $B_{p'} \times [0, +\infty)$ at (x', \bar{t}) . Repeating Step 2, we conclude that the u_j 's are constant in $B_{p'} \times [0, +\infty)$. Since $\cup_{1 \leq p \leq n} B_p$ is an open finite covering of \mathbb{T}^N , we conclude that $u_j \equiv M$ for all j and $(x, t) \in \mathbb{T}^N \times [0, +\infty)$. \square

3.4. Ergodic problem. The uniform gradient bound established for (3.1) allows us to solve the ergodic problem.

The following assumption is used to “linearize” the system in order to apply the strong maximum principle Theorem 3.5.

$$(3.12) \quad H_{\theta i}(x, \cdot) \in W_{\text{loc}}^{1, \infty}(\mathbb{R}^N) \text{ uniformly for } x \in \mathbb{T}^N \text{ and } \theta \in \Theta, 1 \leq i \leq m.$$

Theorem 3.7 (Ergodic problem). *Assume (2.2) and (3.4). Then, there exists a solution $(c, v) \in \mathbb{R}^m \times W^{1, \infty}(\mathbb{T}^N)^m$ of (1.3). The ergodic constant c is unique and $c = (c_1, \dots, c_1)$. If, in addition, the $H_{\theta i}$'s satisfy (3.12), then v is unique up to an additive constant vector.*

For $m = 1$, we find the classical results for scalar equations (see [24, 7]).

Proof of Theorem 3.7. Let ϕ^ϵ be the continuous solution of (3.1) given by Theorem 3.2. We first claim that there exists a constant C' independent of ϵ such that

$$(3.13) \quad \left| \sum_{j=1}^m d_{ij} \phi_j^\epsilon(x) \right| \leq C', \quad x \in \mathbb{T}^N, 1 \leq i \leq m.$$

Indeed, introducing \bar{x}_i as the maximum point of ϕ_i , thanks to (2.3), (4.5) and Theorem 3.2, we have

$$\sum_{j=1}^m d_{ij} \phi_j^\epsilon(x) = \sum_{j=1}^m d_{ij} \phi_j^\epsilon(\bar{x}_i) + \sum_{j=1}^m d_{ij} (\phi_j^\epsilon(x) - \phi_j^\epsilon(\bar{x}_i)) \leq C + K \sum_{j=1}^m |d_{ij}| |x - \bar{x}_i| \leq C'.$$

The lower bound is established in the same way by introducing \underline{x}_i as the minimum point of ϕ_i .

Fix $x^* \in \mathbb{T}^N$ and set $\rho_i^\epsilon := \sum_{j=1}^m d_{ij} \phi_j^\epsilon(x^*)$ and $v_i^\epsilon(x) := \phi_i^\epsilon(x) - \phi_i^\epsilon(x^*)$ for all $1 \leq i \leq m$. From Theorem 3.2, (4.5) and (3.13), using Ascoli-Arzelà's theorem, there exists a sequence $\epsilon_k \rightarrow 0$ so that

$$v_i^{\epsilon_k} \rightarrow v_i, \quad \epsilon_k \phi_i^{\epsilon_k} \rightarrow -c_i, \quad \rho_i^{\epsilon_k} \rightarrow \rho_i,$$

uniformly on \mathbb{T}^N as $k \rightarrow \infty$ for some $v_i \in W^{1,\infty}(\mathbb{T}^N)$, $c_i, \rho_i \in \mathbb{R}$. Note that c_i is a constant independent of x^* while v_i, ρ_i depend on x^* .

Now, we rewrite (3.1) as

$$(3.14) \quad \epsilon \phi_i^\epsilon + \sup_{\theta \in \Theta} \{-\text{trace}(A_{\theta_i}(x)D^2v_i^\epsilon) + H_{\theta_i}(x, Dv_i^\epsilon)\} + \sum_{j=1}^m d_{ij}v_j^\epsilon + \rho_i^\epsilon = 0, \quad 1 \leq i \leq m.$$

Passing to the limit along the subsequence ϵ_k , by stability, we obtain that $v = (v_1, \dots, v_m) \in W^{1,\infty}(\mathbb{T}^N)^m$ is solution of

$$(3.15) \quad \sup_{\theta \in \Theta} \{-\text{trace}(A_{\theta_i}(x)D^2v_i) + H_{\theta_i}(x, Dv_i)\} + \sum_{j=1}^m d_{ij}v_j + \rho_i = c_i, \quad 1 \leq i \leq m.$$

Multiplying (3.14) by ϵ and passing to the limit along the subsequence ϵ_k , we get

$$-\sum_{j=1}^m d_{ij}c_j = 0, \quad 1 \leq i \leq m.$$

Therefore $c = (c_1, \dots, c_m) \in \ker(D)$. From Lemma 3.1, $c = (c_1, \dots, c_1)$ for some $c_1 \in \mathbb{R}$. Set $\rho^\epsilon = (\rho_1^\epsilon, \dots, \rho_m^\epsilon)$ and with the vector $\Lambda \in \mathbb{R}^m$ given in Lemma 3.1, we have

$$\langle \Lambda, \rho^\epsilon \rangle = \sum_{i=1}^m \Lambda_i \rho_i^\epsilon = \sum_{j=1}^m \phi_j^\epsilon(x^*) \sum_{i=1}^m d_{ij} \Lambda_i = 0$$

since $D^T \Lambda = 0$. Passing to the limit $\epsilon_k \rightarrow 0$, we get $\langle \Lambda, \rho \rangle = 0$ where $\rho = (\rho_1, \dots, \rho_m)$. From Lemma 3.1, it follows that the image $\text{im}(D)$ of D is Λ^\perp . Therefore, there exists $\tilde{\rho} \in \mathbb{R}^m$ such that $D\tilde{\rho} = \rho$. Then, the system (3.15) can be rewritten as

$$\sup_{\theta \in \Theta} \{-\text{trace}(A_{\theta_i}(x)D^2v_i) + H_{\theta_i}(x, Dv_i)\} + \sum_{j=1}^m d_{ij}(v_j(x) + \tilde{\rho}_i) = c_1, \quad 1 \leq i \leq m.$$

We conclude by setting $\tilde{v}_i(x) = v_i(x) + \tilde{\rho}_i$ for $1 \leq i \leq m$.

We finally mention that we can easily see the uniqueness of the ergodic constant by the comparison principle for (1.1) (Theorem 4.3). We claim that the uniqueness up to constant of solutions comes from the strong maximum principle (Theorem 3.5). Indeed, let v, \tilde{v} be Lipschitz continuous solutions of (1.3) (the constant c is the same by the above). Since $|Dv|_\infty, |D\tilde{v}|_\infty \leq K$ for some K and $H_{\theta_i} \in W^{1,\infty}(\overline{B}(0, K))$ (uniformly with respect to θ, i), classical arguments in viscosity solutions imply that $v - \tilde{v}$ is a viscosity subsolution of the stationary version of (3.9) in \mathbb{T}^N . Therefore $v = \tilde{v} + C$ where $C \in \mathbb{R}^m$ is a constant. \square

3.5. Large time behavior result. We first give a general result and then apply it to asymmetric systems.

Theorem 3.8. *(Large time behavior) Suppose that (3.8), (3.12) hold. Suppose there exists a viscosity solution $u \in C(\mathbb{T}^N \times [0, +\infty))^m$ of (1.1), a solution $((c_1, \dots, c_1), v) \in \mathbb{R}^m \times W^{1,\infty}(\mathbb{T}^N)^m$ of the ergodic problem (1.3) and $K > 0$ such that*

$$|Du_i(\cdot, t)|_\infty \leq K, \quad t \geq 0, \quad 1 \leq i \leq m.$$

Then, there exists a constant ℓ such that

$$u(x, t) + (c_1, \dots, c_1)t + \ell \rightarrow v(x) \quad \text{uniformly as } t \rightarrow +\infty.$$

Proof of Theorem 3.8. Several parts of this proof are inspired by [7].

1. Since $|Du_i(\cdot, t)|_\infty \leq K$, we deduce from Theorem 4.3 that u is the unique continuous viscosity solution of (1.1).

From (3.2), setting $c = (c_1, \dots, c_1)$, we see that $v^\pm(x, t) = v(x) - ct \pm (C, \dots, C)$ is a super and subsolution, respectively, of (1.1) for C large enough. Thanks to the comparison principle,

$$v_i(x) - C \leq u_i(x, t) + ct \leq v_i(x) + C, \quad x \in \mathbb{T}^N, \quad 1 \leq i \leq m.$$

It follows that $u_i(x, t) + ct$ is bounded; by the change of function $u \rightarrow u + ct$, we may assume without loss of generality that u is bounded and the ergodic constant c is 0.

Set $m(t) = \max_{x \in \mathbb{T}^N, 1 \leq i \leq m} (u_i(x, t) - v_i(x))$. The comparison principle claims that m is non-increasing and hence, $m(t) \rightarrow \ell$ as $t \rightarrow \infty$.

2. A by-product of Step 1 is that $\{u(\cdot, t), t > 0\}$ is relatively compact in $W^{1,\infty}(\mathbb{T}^N)^m$. So we can extract a sequence, $t_j \rightarrow +\infty$ such that $u(\cdot, t_j) \rightarrow \bar{u} \in W^{1,\infty}(\mathbb{T}^N)^m$. Applying Theorem 4.3 for (1.1), we obtain

$$|u_i(x, t + t_j) - u_i(x, t + t_p)| \leq \max_{y \in \mathbb{T}^N, 1 \leq j \leq m} |u_j(y, t_j) - u_j(y, t_p)|, \quad x \in \mathbb{T}^N, \quad t \geq 0, \quad j, p \in \mathbb{N},$$

which proves that $(u(\cdot, \cdot + t_j))_j$ is a Cauchy sequence in $C(\mathbb{T}^N \times [0, +\infty))^m$. We call u_∞ its limit. Notice, on the one hand, that $|Du_{\infty i}(\cdot, t)|_\infty \leq K$ for all i, t and, on the other hand, by stability, u_∞ is solution of (1.1) with initial data \bar{u} .

3. Using the uniform convergence of $u_i(\cdot, t + t_j)$, we pass to the limit with respect to j in

$$m(t + t_j) = \max_{i,x} (u_i(x, t + t_j) - v_i(x))$$

to obtain $\ell = \max_{i,x} (u_{\infty i}(x, t) - v_i(x))$ for any $t > 0$.

4. Since u_∞ is solution of (1.1) and v is solution of (1.3) and, up to increase K , both are K -Lipschitz continuous in x , thanks to the Lipschitz continuity of H_{θ_i} with respect to p , see (3.12), we obtain that $u_\infty - v$ is subsolution of (3.9). The strong maximum principle (Theorem 3.5), then implies

$$\ell = u_{\infty i}(x, t) - v_i(x) \quad (x, t) \in \mathbb{T}^N \times [0, +\infty), \quad 1 \leq i \leq m.$$

Noticing that $\ell + v(x)$ does not depend on the choice of subsequences, we obtain $u_i(x, t) - \ell - v_i(x) \rightarrow 0$ uniformly in x as $t \rightarrow \infty$, for all $i \in \{1, \dots, m\}$. \square

We apply the previous result for the particular systems we studied in Section 3.

Corollary 3.9. *(Large time behavior) Assume (3.8), (3.12) and suppose that the assumptions of Theorem 3.2 are in force. Then, for any initial condition $u_0 \in W^{1,\infty}(\mathbb{T}^N)^m$,*

- (i) *The system (1.1) has a unique viscosity solution $u \in C(\mathbb{T}^N \times [0, +\infty))^m$ and there exists $K > 0$ such that $|Du_i(\cdot, t)|_\infty \leq K$ for all $t \geq 0, 1 \leq i \leq m$.*
- (ii) *There exist a unique ergodic constant $c = (c_1, \dots, c_1) \in \mathbb{R}$ and $v \in W^{1,\infty}(\mathbb{T}^N)^m$ solution of (1.3) such that $u(x, t) + ct \rightarrow v(x)$ uniformly as $t \rightarrow +\infty$.*

Remark 3.10. Remember that the classical assumptions on the Hamiltonians in (1.1) do not hold in order to have comparison and uniqueness of viscosity solutions. It is why, we prove first the existence of a (Lipschitz) continuous in space viscosity solution of (1.1). To this purpose, we need the initial condition to be Lipschitz continuous though this regularity assumption is not necessary to obtain the long time behavior.

Proof of Corollary 3.9.

1. We introduce the truncated evolutive system (1.1) in $\mathbb{T}^N \times [0, +\infty)$ with the $H_{\theta_{in}}$ defined by (4.7). From Theorem 4.3 and Perron's method, there exists a unique continuous viscosity solution u_n in $\mathbb{T}^N \times [0, +\infty)$.

In the following, M is a constant which may vary line to line but is independent of n .

2. Applying Theorem 3.7 with the $H_{\theta_{in}}$'s, we obtain the existence of $(c_n, v_n) \in \mathbb{R}^m \times W^{1,\infty}(\mathbb{T}^N)^m$. Since the $H_{\theta_{in}}$'s satisfy (3.4) with constants independent of n for large n , and $c_n \rightarrow c$ when $n \rightarrow +\infty$, we obtain from Theorem 3.2 that $|v_n|_\infty, |Dv_n|_\infty \leq M$.

3. Noticing that $v_n(x) - c_n t$ satisfies (1.1) with the $H_{\theta_{in}}$'s, by Theorem 4.3, we obtain

$$v_{ni}(x) - M \leq u_{ni}(x, t) + c_{ni}t \leq v_{ni}(x) + M, \quad (x, t) \in \mathbb{T}^N \times [0, +\infty), \quad 1 \leq i \leq m.$$

Therefore $|u_n(\cdot, t) + c_n t|_\infty, \text{osc}(u_n(\cdot, t)) \leq M$.

4. We claim that $|Du_n(\cdot, t)|_\infty \leq M$. To prove this fact, we repeat the proof of Lemma 3.4 for the evolutive system (1.1) with the $H_{\theta_{in}}$'s. Let $T > 0$ and consider

$$\max_{(x,y) \in (\mathbb{T}^N)^2, t \in [0, T], 1 \leq i \leq m} \{w_{ni}(x, t) - w_{ni}(y, t) - \psi(|x - y|)\},$$

where $e^{w_{ni}(x, t)} = u_{ni}(x, t) - \min_{\mathbb{T}^N} u_{ni}(\cdot, t) + 1$, and ψ is given by (2.11). If the maximum is nonpositive we are done. Otherwise it is positive and achieved at some $(\bar{x}, \bar{y}, \bar{t}, i)$ with $\bar{x} \neq \bar{y}$ and we can write the viscosity inequalities for the i th equation, see [13, Theorem 8.3]: for every $\varrho > 0$, there exist $(a, p, X) \in \bar{P}^{2,+} w_i(\bar{x}, \bar{t})$ and $(b, p, Y) \in \bar{P}^{2,-} w_i(\bar{y}, \bar{t})$ such that (2.16) holds since $a - b = 0$. We achieve a contradiction as in the proof of Lemma 3.4.

5. From Ascoli-Arzelà's theorem and the stability result for systems, by letting $n \rightarrow +\infty$, we obtain a continuous viscosity solution u of (1.1) in $\mathbb{T}^N \times [0, +\infty)$, with $|Du(\cdot, t)|_\infty \leq M$. This solution is unique thanks to Theorem 4.3. This ends the proof of (i). The proof of (ii) is an immediate consequence of Theorems 3.7 and 3.8. \square

3.6. Examples. We give some examples satisfying the hypotheses of our convergence result, Corollary 3.9. For systems with superlinear Hamiltonians only we do not need any diffusion matrices to be uniformly elliptic everywhere.

Example 3.11. (Degenerate systems with superlinear Hamiltonians) Consider

$$\begin{cases} \frac{\partial u_1}{\partial t} + \sup_{\theta \in \Theta} \{-d_{\theta_1}^2(x) \Delta u_1\} + a_1(x) |Du_1|^{1+\alpha_1} + u_1 - u_2 = f_1(x), & x \in \mathbb{T}^N \times (0, +\infty), \\ \frac{\partial u_2}{\partial t} + \sup_{\theta \in \Theta} \{-d_{\theta_2}^2(x) \Delta u_2\} + a_2(x) |Du_2|^{1+\alpha_2} + u_2 - u_1 = f_2(x), & x \in \mathbb{T}^N \times (0, +\infty), \\ u_1(x, 0) = u_{01}(x), \quad u_2(x, 0) = u_{02}(x), & x \in \mathbb{T}^N, \end{cases}$$

where $d_{\theta_i}, a_i, f_i, u_{0i}$ are Lipschitz continuous, $\alpha_i > 0$ and $a_i > 0$. Suppose moreover that for any $x \in \mathbb{T}^N$, we have either $\inf_{\theta \in \Theta} d_{\theta_1}(x) > 0$ or $\inf_{\theta \in \Theta} d_{\theta_2}(x) > 0$ (this implies (3.8)). So, the hypotheses of Corollary 3.9 hold.

When there is at least one sublinear Hamiltonian, we need that the diffusion matrix of the same equation is uniformly elliptic to be able to control this Hamiltonian. But we permit that other diffusion matrices to be degenerate everywhere.

Example 3.12. (Degenerate asymmetric systems) In the following system, the first equation is of sublinear type while the second is of superlinear type.

$$\begin{cases} \frac{\partial u_1}{\partial t} - \text{trace}(A_1 D^2 u_1) + a_1(x)|Du_1| + u_1 - u_2 = f_1(x), & x \in \mathbb{T}^N \times (0, +\infty), \\ \frac{\partial u_2}{\partial t} - \text{trace}(A_2 D^2 u_2) + a_2(x)|Du_2|^{1+\alpha} + u_2 - u_1 = f_2(x), & x \in \mathbb{T}^N \times (0, +\infty), \\ u_1(x, 0) = u_{01}(x), \quad u_2(x, 0) = u_{02}(x), & x \in \mathbb{T}^N, \end{cases}$$

where $a_1, f_1 \in C(\mathbb{T}^N)$, a_2, f_2, u_{01}, u_{02} are Lipschitz continuous, $\alpha > 0$ and $a_2 > 0$. If we suppose moreover that $A_1 > 0$ in \mathbb{T}^N (notice that we only assume $A_2 \geq 0$), then the hypotheses of Corollary 3.9 hold.

4. APPENDIX

4.1. Proof of Lemma 2.4. From (2.13), for every $\zeta, \xi \in \mathbb{R}^N$, we have

$$\langle X\zeta, \zeta \rangle - \langle Y\xi, \xi \rangle \leq \Psi' \langle \zeta - \xi, B(\zeta - \xi) \rangle + \Psi'' \langle \zeta - \xi, (q \otimes q)(\zeta - \xi) \rangle + O(\varrho).$$

We estimate $\text{trace}(A_\theta(\bar{x})X)$ and $\text{trace}(A_\theta(\bar{y})Y)$ using two orthonormal bases (e_1, \dots, e_N) and $(\tilde{e}_1, \dots, \tilde{e}_N)$ in the following way:

$$\begin{aligned} T := \text{trace}(A_\theta(\bar{x})X - A_\theta(\bar{y})Y) &= \sum_{i=1}^N \langle X\sigma_\theta(\bar{x})e_i, \sigma_\theta(\bar{x})e_i \rangle - \langle Y\sigma_\theta(\bar{y})\tilde{e}_i, \sigma_\theta(\bar{y})\tilde{e}_i \rangle \\ &\leq \sum_{i=1}^N \Psi' \langle \zeta_i, B\zeta_i \rangle + \Psi'' \langle \zeta_i, (q \otimes q)\zeta_i \rangle + O(\varrho) \\ (4.1) \qquad \qquad \qquad &\leq \Psi'' \langle \zeta_1, (q \otimes q)\zeta_1 \rangle + \sum_{i=1}^N \Psi' \langle \zeta_i, B\zeta_i \rangle + O(\varrho), \end{aligned}$$

where we set $\zeta_i = \sigma_\theta(\bar{x})e_i - \sigma_\theta(\bar{y})\tilde{e}_i$ and noticing that $\Psi'' \langle \zeta_i, (q \otimes q)\zeta_i \rangle = \Psi'' \langle \zeta_i, q \rangle^2 \leq 0$ since Ψ is concave.

We now build suitable bases in two following cases. In the case where σ_θ could be degenerate, we choose any orthonormal basis such that $e_i = \tilde{e}_i$. It follows

$$\begin{aligned} T &\leq \sum_{i=1}^N \Psi' \langle (\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{y}))e_i, B(\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{y}))e_i \rangle + O(\varrho) \\ &\leq \Psi' N |\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{y})|^2 |B| + O(\varrho) \\ &\leq \Psi' N |(\sigma_\theta)_x|_\infty^2 |\bar{x} - \bar{y}| + O(\varrho) \end{aligned}$$

from (2.2) and since $|B| \leq 1/|\bar{x} - \bar{y}|$.

When (2.5) holds, i.e., $A_\theta(x) \geq \nu I$ for every x , we can set

$$e_1 = \frac{\sigma_\theta(\bar{x})^{-1}q}{|\sigma_\theta(\bar{x})^{-1}q|}, \quad \tilde{e}_1 = -\frac{\sigma_\theta(\bar{y})^{-1}q}{|\sigma_\theta(\bar{y})^{-1}q|}, \quad \text{where } q \text{ is given by (2.14).}$$

If e_1 and \tilde{e}_1 are collinear, then we complete the basis with orthogonal unit vectors $e_i = \tilde{e}_i \in e_1^\perp$, $2 \leq i \leq N$. Otherwise, in the plane $\text{span}\{e_1, \tilde{e}_1\}$, we consider a rotation \mathcal{R} of angle $\frac{\pi}{2}$ and define

$$e_2 = \mathcal{R}e_1, \quad \tilde{e}_2 = -\mathcal{R}\tilde{e}_1.$$

Finally, noticing that $\text{span}\{e_1, e_2\}^\perp = \text{span}\{\tilde{e}_1, \tilde{e}_2\}^\perp$, we can complete the orthonormal basis with unit vectors $e_i = \tilde{e}_i \in \text{span}\{e_1, e_2\}^\perp$, $3 \leq i \leq N$.

From (2.5), we have

$$(4.2) \quad \nu \leq \frac{1}{|\sigma_\theta(x)^{-1}q|^2} \leq |\sigma_\theta|_\infty^2 \leq C^2.$$

It follows

$$\langle \zeta_1, (q \otimes q)\zeta_1 \rangle = \left(\frac{1}{|\sigma_\theta(\bar{x})^{-1}q|} + \frac{1}{|\sigma_\theta(\bar{y})^{-1}q|} \right)^2 \geq 4\nu.$$

From (2.14), we deduce $Bq = 0$. Therefore

$$\langle \zeta_1, B\zeta_1 \rangle = 0.$$

For $3 \leq i \leq N$, using (2.2),

$$\langle \zeta_i, B\zeta_i \rangle = \langle (\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{y}))e_i, B(\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{y}))e_i \rangle \leq |(\sigma_\theta)_x|_\infty^2 |\bar{x} - \bar{y}| \leq \sqrt{N}C^2.$$

since $|B| \leq 1/|\bar{x} - \bar{y}|$ and $|\bar{x} - \bar{y}| \leq \sqrt{N}$. We have

$$|\zeta_2| = |(\sigma_\theta(\bar{x}) - \sigma_\theta(\bar{y}))\mathcal{R}e_1 + \sigma_\theta(\bar{y})\mathcal{R}(e_1 + \tilde{e}_1)| \leq C|\bar{x} - \bar{y}| + C|e_1 + \tilde{e}_1|.$$

It remains to estimate

$$\begin{aligned} |e_1 + \tilde{e}_1| &\leq \frac{1}{|\sigma_\theta(\bar{x})^{-1}q|} |\sigma_\theta(\bar{x})^{-1}q - \sigma_\theta(\bar{y})^{-1}q| + |\sigma_\theta(\bar{y})^{-1}q| \left| \frac{1}{|\sigma_\theta(\bar{x})^{-1}q|} - \frac{1}{|\sigma_\theta(\bar{y})^{-1}q|} \right| \\ &\leq \frac{2|\sigma_\theta|_\infty |(\sigma_\theta)_x|_\infty}{\nu} |\bar{x} - \bar{y}| = \frac{2C^2}{\nu} |\bar{x} - \bar{y}|, \end{aligned}$$

from (4.2) and $|(\sigma_\theta^{-1})_x|_\infty \leq |(\sigma_\theta)_x|_\infty/\nu$.

From (4.1), we finally obtain the conclusion $T \leq 4\nu\Psi'' + \tilde{C}\Psi' + O(\varrho)$ where

$$(4.3) \quad \tilde{C} = \tilde{C}(N, \nu, |\sigma_\theta|_\infty, |(\sigma_\theta)_x|_\infty) := C^2\sqrt{N}(N - 2 + (1 + \frac{2C^2}{\nu})^2) \text{ with } C \text{ given by (2.2).}$$

4.2. Comparison principles for stationary systems. The following results are stated in [7] in the case of a scalar equation without control. We extend them to systems by considering for any $\epsilon > 0$, the approximated system

$$(4.4) \quad \epsilon\phi_i^\epsilon + \sup_{\theta \in \Theta} \{-\text{trace}(A_{\theta i}(x)D^2\phi_i^\epsilon) + H_{\theta i}(x, D\phi_i^\epsilon)\} + \sum_{j=1}^m d_{ij}\phi_j^\epsilon = 0, \quad 1 \leq i \leq m.$$

Theorem 4.1. *Suppose that $\sigma_{\theta i}$ satisfies (2.2) and $H_{\theta i}$ satisfies (2.3). Let $\psi^\epsilon \in USC(\mathbb{T}^N)^m$ and $\phi^\epsilon \in LSC(\mathbb{T}^N)^m$ be respectively a viscosity subsolution and supersolution of (4.4). Assume that either ψ^ϵ or $\phi^\epsilon \in W^{1,\infty}(\mathbb{T}^N)^m$. Then $\psi_i \leq \phi_i$ in \mathbb{T}^N , for all i .*

A useful consequence of Theorem 4.1 is that, for any viscosity solution ϕ^ϵ of (4.4),

$$(4.5) \quad |\epsilon\phi_i^\epsilon|_\infty \leq \sup_{\theta \in \Theta, 1 \leq i \leq m} |H_{\theta i}(\cdot, 0)|_\infty =: \mathcal{H}, \quad 1 \leq i \leq m,$$

which is finite by (2.3). Indeed, it is sufficient to notice that, from (3.2), $\phi^\pm = \pm\mathcal{H}/\epsilon$ are respectively Lipschitz continuous sub- and supersolution of (4.4).

Proof of Theorem 4.1. We only give a sketch of proof since it is classical. We suppose without loss of generality that $\sup_i |D\psi_i|_\infty =: L < \infty$. Set

$$M_0 = \max_{x \in \mathbb{T}^N, 1 \leq j \leq m} \{\psi_j(x) - \phi_j(x)\},$$

$$M_\alpha = \max_{x, y \in \mathbb{T}^N, 1 \leq j \leq m} \{\psi_j(x) - \phi_j(y) - \alpha^2 |x - y|^2\} = \psi_i(\bar{x}) - \phi_j(\bar{y}) - \alpha^2 |\bar{x} - \bar{y}|^2.$$

We have some classical estimates

$$\alpha^2 |\bar{x} - \bar{y}|^2 \xrightarrow{\alpha \rightarrow \infty} 0, \quad |p| \leq L \text{ where } p = 2\alpha^2(\bar{x} - \bar{y}), \quad \limsup_{\alpha \rightarrow \infty} M_\alpha = M_0.$$

After subtracting write the viscosity inequalities at (\bar{x}, \bar{y}) of (4.4), we get

$$\begin{aligned} \epsilon(\psi_i(\bar{x}) - \phi_i(\bar{y})) + \inf_{\theta} \{-\text{trace}(A_{\theta i}(\bar{x})X - A_{\theta i}(\bar{y})Y) + H_{\theta i}(\bar{x}, p) - H_{\theta i}(\bar{y}, p)\} \\ + \sum_{j=1}^m d_{ij}(\psi_j(\bar{x}) - \phi_j(\bar{y})) \leq 0. \end{aligned}$$

By (3.2) and $\psi_j(\bar{x}) - \phi_j(\bar{y}) \leq \psi_i(\bar{x}) - \phi_i(\bar{y})$, we get $\sum_{j=1}^m d_{ij}(\psi_j(\bar{x}) - \phi_j(\bar{y})) \geq 0$.

Moreover $-\text{trace}(A_{\theta i}(\bar{x})X - A_{\theta i}(\bar{y})Y) \geq o_\alpha(1)$, which is independent of θ . And (2.3) gives

$$(4.6) \quad \inf_{\theta} \{H_{\theta i}(\bar{x}, p) - H_{\theta i}(\bar{y}, p)\} = o_\alpha(1).$$

Therefore $\epsilon(\psi_i(\bar{x}) - \phi_i(\bar{y})) \leq o_\alpha(1)$, this yields $M_0 \leq 0$ as desired. \square

In the proofs of Theorems 2.2 and 3.2, we need a discontinuous comparison result for truncated system when $H_{\theta i}$ in (4.4) is replaced by

$$(4.7) \quad H_{\theta in} := \begin{cases} H_{\theta i}(x, p) & \text{if } |p| \leq n, \\ H_{\theta i}(x, n \frac{p}{|p|}) & \text{if } |p| \geq n. \end{cases}$$

Due to the truncation in the gradient variable, we do not need to assume that the sub- or supersolution is Lipschitz continuous. The proof follows the same lines as above up to replace (4.6) with

$$|\inf_{\theta} \{H_{\theta in}(\bar{x}, p) - H_{\theta in}(\bar{y}, p)\}| \leq \sup_{\xi \in \bar{B}(0, n)} \sup_{\theta} \{|H_{\theta in}(\bar{x}, \xi) - H_{\theta in}(\bar{y}, \xi)|\} \leq o_\alpha(1).$$

We obtain

Theorem 4.2. *Assume that $\sigma_{\theta i}$ satisfies (2.2) and $H_{\theta i}$ satisfies (2.3). Let $\psi \in USC(\mathbb{T}^N)^m$ and $\phi \in LSC(\mathbb{T}^N)^m$ be respectively a viscosity subsolution and supersolution of (4.4) with $H_{\theta in}$ defined by (4.7). Then $\psi_i \leq \phi_i$ in \mathbb{T}^N for all i .*

4.3. Comparison for the evolutive problem.

Theorem 4.3. *Assume that $\sigma_{\theta i}$ satisfies (2.2) and $H_{\theta i}$ satisfies (2.3). Let $u \in USC(\mathbb{T}^N \times [0, +\infty))^m$ and $v \in LSC(\mathbb{T}^N \times [0, +\infty))^m$ be respectively a viscosity subsolution and supersolution of (1.1). Assume that either $u(\cdot, t)$ or $v(\cdot, t) \in W^{1, \infty}(\mathbb{T}^N)^m$ uniformly in $t \in [0, +\infty)$. Then*

$$(4.8) \quad u_i(x, t) - v_i(x, t) \leq \sup_{y \in \mathbb{T}^N, 1 \leq j \leq m} (u_j(y, 0) - v_j(y, 0)), \quad (x, t) \in \mathbb{T}^N \times [0, +\infty), \quad 1 \leq i \leq m.$$

Theorem 4.4. *Assume that $\sigma_{\theta i}$ satisfies (2.2) and $H_{\theta i}$ satisfies (2.3). Let $u \in USC(\mathbb{T}^N \times [0, +\infty))^m$ and $v \in LSC(\mathbb{T}^N \times [0, +\infty))^m$ be respectively a viscosity subsolution and supersolution of (1.1) with $H_{\theta in}$ defined by (4.7). Then (4.8) holds.*

The proofs are easy adaptations of the proofs of Theorems 4.1 and 4.2 in the case of degenerate parabolic systems.

4.4. Control-theoretic interpretation. We can give an interpretation of weakly coupled systems as dynamic programming equations of hybrid systems with pathwise stochastic trajectories with random switching, see [18].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space, W_t be a \mathcal{F}_t -adapted standard N -Brownian motion such that $W_0 = 0$ a.s. Consider the controlled random evolution process (X_t, ν_t) with dynamics

$$\begin{cases} dX_t = b_{\theta_t \nu_t}(X_t)dt + \sigma_{\theta_t \nu_t}(X_t)dW_t, & t > 0, \\ (X_0, \nu_0) = (x, i) \in \mathbb{T}^N \times \{1, \dots, m\}, \end{cases}$$

where the control law $\theta_t : [0, \infty) \rightarrow \Theta$ is a measurable function (Θ is a compact metric space), $(x, \theta) \mapsto b_{\theta i}(x) \in \mathbb{R}^N$ and $(x, \theta) \mapsto \sigma_{\theta i}(x) \in \mathcal{M}_N$ are measurable and, for all θ, i , the maps $b_{\theta i}, \sigma_{\theta i} = f$ satisfy

$$(4.9) \quad |f(x)| \leq C, \quad |f(x) - f(y)| \leq C|x - y|, \quad x, y \in \mathbb{T}^N.$$

For every θ_t and matrix $G = (\gamma_{ij})_{i,j}$ satisfying $\sum_{j \neq i} \gamma_{ij} = 1$ for $i \neq j$ and $\gamma_{ii} = -1$, there exists a \mathcal{F}_t -adapted solution (X_t, ν_t) , where $X_t : [0, \infty) \rightarrow \mathbb{T}^N$ is piecewise C^1 and ν_t is a continuous-time Markov chain with state space $\{1, \dots, m\}$ and probability transitions given by

$$\mathbb{P}\{\nu_{t+\Delta t} = j \mid \nu_t = i\} = \gamma_{ij}\Delta t + o(\Delta t)$$

for $j \neq i$.

We introduce the value functions of the optimal control problems

$$u_i(x, t) = \inf_{\theta_t \in L^\infty([0, t], \Theta)} \mathbb{E}_{x, i} \left\{ \int_0^t \ell_{\theta_s \nu_s}(X_s) ds + u_{0, \nu_t}(X_t) \right\}, \quad i = 1, \dots, m,$$

where $\mathbb{E}_{x, i}$ denote the expectation of a trajectory starting at x in the mode i , $\ell_{\theta i}, u_{0, i} : \mathbb{T}^N \rightarrow \mathbb{R}$ are continuous and $(x, \theta) \mapsto \ell_{\theta i}(x) \in \mathbb{R}$ is measurable and satisfies (4.9).

At least formally, the functions u_i 's satisfy the system

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sup_{\theta \in \Theta} \left\{ -\frac{1}{2} \text{trace}(\sigma_{\theta i}(x) \sigma_{\theta i}(x)^T D^2 u_i) - \langle b_{\theta i}(x), D u_i \rangle - \ell_{\theta i}(x) \right\} + \sum_{j \neq i} \gamma_{ij} (u_i - u_j) = 0 \\ u_i(x, 0) = u_{0, i}(x) \end{cases}$$

$(x, t) \in \mathbb{T}^N \times (0, +\infty)$, $1 \leq i \leq m$. This system has the form (1.1) by setting $A_{\theta i} = \frac{1}{2} \sigma_{\theta i} \sigma_{\theta i}^T$, $H_{\theta i}(x, p) = -\langle b_{\theta i}(x), p \rangle - \ell_{\theta i}(x)$, $d_{ii} = \sum_{j \neq i} \gamma_{ij} = 1$ and $d_{ij} = -\gamma_{ij}$ for $j \neq i$.

The assumptions (2.2), (2.3) and (3.2) are clearly satisfied. In this case, $H_{\theta i}$ is sublinear and satisfies (2.7).

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CORRESPONDING AUTHOR, IRMAR, INSA DE RENNES, 35708 RENNES, FRANCE
E-mail address: olivier.ley@insa-rennes.fr

IRMAR, INSA DE RENNES, 35708 RENNES, FRANCE
E-mail address: vinh.nguyen@insa-rennes.fr