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MEAN CURVATURE BOUNDS AND EIGENVALUES OF ROBIN LAPLACIANS

KONSTANTIN PANKRASHKIN AND NICOLAS POPOFF

Abstract. We consider the Laplacian with attractive Robin boundary conditions,

$$Q_\alpha^\Omega u = -\Delta u, \quad \frac{\partial u}{\partial n} = \alpha u \text{ on } \partial\Omega,$$

in a class of bounded smooth domains $\Omega \in \mathbb{R}^\nu$; here n is the outward unit normal and $\alpha > 0$ is a constant. We show that for each $j \in \mathbb{N}$ and $\alpha \to +\infty$, the jth eigenvalue $E_j(Q_\alpha^\Omega)$ has the asymptotics

$$E_j(Q_\alpha^\Omega) = -\alpha^2 - (\nu-1)H_{\max}(\Omega)\,\alpha + \mathcal{O}(\alpha^{2/3}),$$

where $H_{\max}(\Omega)$ is the maximum mean curvature at $\partial\Omega$. The discussion of the reverse Faber-Krahn inequality gives rise to a new geometric problem concerning the minimization of $H_{\max}$. In particular, we show that the ball is the strict minimizer of $H_{\max}$ among the smooth star-shaped domains of a given volume, which leads to the following result: if B is a ball and Ω is any other star-shaped smooth domain of the same volume, then for any fixed $j \in \mathbb{N}$ we have $E_j(Q_\alpha^B) > E_j(Q_\alpha^\Omega)$ for large α. An open question concerning a larger class of domains is formulated.

1. Introduction

Let $\Omega \subset \mathbb{R}^\nu$, $\nu \geq 2$, be a bounded, connected, C^3 smooth domain; such domains will be called *admissible* through the text. For $\alpha \in \mathbb{R}$, denote by Q_α^Ω the self-adjoint operator in $L^2(\Omega)$ acting as $Q_\alpha^\Omega u = -\Delta u$ on the functions $u \in H^2(\Omega)$ satisfying the Robin boundary conditions

$$\frac{\partial u}{\partial n} = \alpha u \text{ on } S := \partial\Omega,$$

where n is the outward pointing unit normal vector at S.

More precisely, Q_α^Ω is the self-adjoint operator in $L^2(\Omega)$ associated with the quadratic form q_α^Ω defined on the domain $\mathcal{D}(q_\alpha^\Omega) = H^1(\Omega)$ by

$$q_\alpha^\Omega(u,u) = \int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_S |u|^2 \mathrm{d}S,$$

where $\mathrm{d}S$ stands for the $(\nu-1)$-dimensional Hausdorff measure on S. We remark that the boundary S is not necessarily connected.

Throughout the paper, for a lower semibounded self-adjoint operator Q with a compact resolvent we denote by $E_j(Q)$ its jth eigenvalue when numbered in non-decreasing order and counted according to the multiplicities. The aim of the paper is to discuss some relations between the geometry of Ω and the eigenvalues $E_j(Q_\alpha^\Omega)$ in the asymptotic regime $\alpha \to +\infty$.

The first order asymptotics of the low lying eigenvalues has been investigated in the previous papers [LOS98, LZ04, LP08, DK10] and it was shown

that for any fixed $j \in \mathbb{N}$ one has

$$E_j(Q_\alpha^\Omega) = -\alpha^2 + o(\alpha^2), \quad \alpha \to +\infty.$$

Some results concerning the convergence of higher eigenvalues were obtained in [CCH13]. We mention also several works dealing with non-smooth boundaries [HePan14, LP08, McC11]. Only few works are dedicated to the study of subsequent terms in the asymptotic expansion. In the two-dimensional case ($\nu = 2$) it was shown [EMP14, Pan13] that for any fixed $j \in \mathbb{N}$ one has

$$E_j(Q_\alpha^\Omega) = -\alpha^2 - \gamma_{\max}\alpha + \mathcal{O}(\alpha^{2/3}), \quad \alpha \to +\infty, \tag{1}$$

where $\gamma_{\max}$ is the maximum curvature at the boundary. Some results for the principal eigenvalue of balls and spherical shells in arbitrary dimension were obtained in [FK14]. Our first aim is to find an analog of (1) for general domains in higher dimensions, in particular we wonder what is the quantity which plays the role of $\gamma_{\max}$ in dimension $\nu \geq 3$.

Let us introduce the necessary notation. As mentioned above, $s \mapsto n(s)$ is the Gauss map on S, i.e. $n(s)$ is the outward pointing unit normal vector at $s \in S$. Denote

$$L_s := \mathrm{d}n(s) : T_sS \to T_sS$$

the shape operator at $s \in S$. Its eigenvalues $\kappa_1, \dots, \kappa_{\nu-1}$ are the so-called principal curvatures at s, and the mean curvature $H(s)$ at s is defined by

$$H(s) = \frac{\kappa_1(s) + \dots + \kappa_{\nu-1}}{\nu - 1} \equiv \frac{1}{\nu - 1} \operatorname{tr} L_s.$$

Due to the assumption $S \in C^3$ we have at least $H \in C^1(S)$, and we will denote

$$H_{\max} \equiv H_{\max}(\Omega) := \max_{s \in S} H(s).$$

Theorem 1. *Let $\nu \geq 2$ and $\Omega \subset \mathbb{R}^\nu$ be an admissible domain, then for any fixed $j \in \mathbb{N}$ one has the asymptotics*

$$E_j(Q_\alpha^\Omega) = -\alpha^2 - (\nu - 1)H_{\max}(\Omega)\alpha + \mathcal{O}(\alpha^{2/3}) \text{ as } \alpha \to +\infty. \tag{2}$$

If, additionally, the boundary is C^4, then the remainder estimate $\mathcal{O}(\alpha^{2/3})$ can be replaced by $\mathcal{O}(\alpha^{1/2})$

The proof is given in Section 2.

The result of Theorem 1 can be used to discuss some questions related to the so-called reverse Faber-Krahn inequality. It was conjectured in [Bar77] that for any $\alpha > 0$ among the domains Ω of the same volume it is the ball which maximizes the first eigenvalue of Q_α^Ω. In such a setting, the conjecture appears to be false as the very recent counterexample of [FK14] shows: if B is a ball of radius $r > 0$ and Ω is a spherical shell of the same volume, then for large α one has $E_1(Q_\alpha^B) < E_1(Q_\alpha^\Omega)$. This result is easily visible with the help of our result: one has $H_{\max}(B) = 1/r$ and $H_{\max}(\Omega) = 1/R$ with R being the outer radius of Ω. Therefore, $H_{\max}(B) < H_{\max}(\Omega)$, which gives the sought inequality. However, some positive results are available for restricted classes of domains and special ranges of α. In particular, the conjecture is true for domains which are in a sense close to a ball [Bar77, FNT13] as well as for small α and two-dimensional domains [FK14]. (We remark that the

situation for negative α is completely different: the ball minimizes the first eigenvalue for any $\alpha < 0$, see [Bos86, Dan06, BG14]).

With the help of Theorem 1 one can easily see that the study of Faber-Krahn-optimal domains for large α leads to the purely geometric question: to minimize $H_{\max}(\Omega)$ in a given class of domains. It seems that such a problem setting is new and may have its own interest. Some aspects of this problem will be discussed in Section 3. In particular, we show the following lower bound, see Subsection 3.3:

Theorem 2. *Let $\Omega \subset \mathbb{R}^\nu$ be a bounded, star-shaped, C^2 smooth domain, then*

$$H_{\max}(\Omega) \geq \left(\frac{\operatorname{Vol} B_\nu}{\operatorname{Vol}\Omega}\right)^{1/\nu} \tag{3}$$

where B_ν is the unit ball in $\mathbb{R}^\nu$. Moreover, the above lower bound is an equality if and only if Ω is a ball.

By combining the two theorems we arrive at the following spectral result, which can viewed as an asymptotic version of the reverse Faber-Krahn inequality for star-shaped domains:

Corollary 3. *Let $\Omega \subset \mathbb{R}^\nu$ be admissible, star-shaped, different from a ball. Let B be a ball in $\mathbb{R}^\nu$ with $\operatorname{Vol} B = \operatorname{Vol}\Omega$. For any $j \in \mathbb{N}$ there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$ there holds the strict inequality*

$$E_j(Q_\alpha^B) > E_j(Q_\alpha^\Omega).$$

It would be interesting to understand whether Theorem 2 and hence Corollary 3 can be extended to a larger class of domains:

Open question. Let $\Omega \in \mathbb{R}^\nu$ be a bounded, C^2 smooth domain with a *connected boundary* and let B be a ball of the same volume. Do we have $H_{\max}(\Omega) \geq H_{\max}(B)$?

At last we remark that one of our motivations for the present work came from various analogies between the Robin problem and the Neumann magnetic laplacian in two dimensions, in particular, a certain analogue of (1) holds for the magnetic case [BS98, HeMo01]. However, a magnetic field has a non-trivial interaction with the boundary in higher dimension [HeMo04], and the comparison seems to stop here as our result shows.

2. Asymptotics of the eigenvalues

This section is dedicated to the proof of Theorem 1. The scheme of the proof is close to that of [EMP14, Pan13] and is based on the Dirichlet-Neumann bracketing in suitably chosen tubular neighborhoods of S and on an explicit spectral analysis of some one-dimensional operators. We recall the max-min principle for the eigenvalues of self-adjoint operators, see e.g. [Sch12, Sec. 12.1]: if Q is a lower semibounded self-adjoint operator with a compact resolvent in a Hilbert space $\mathcal{H}$ and q is its quadratic form, then for any $j \in \mathbb{N}$ we have

$$E_j(Q) = \max_{\psi_1,\dots,\psi_{j-1}\in\mathcal{H}} \min_{\substack{u\in\mathcal{D}(q),\, u\neq 0\\ u\perp\psi_1,\dots,\psi_{j-1}}} \frac{q(u,u)}{\langle u,u\rangle}.$$

2.1. **Dirichlet-Neumann bracketing.** For $\delta > 0$ denote

$$\Omega_\delta := \big\{x \in \Omega : \inf_{s\in S} |x-s| < \delta\big\}, \quad \Theta_\delta := \Omega \setminus \overline{\Omega_\delta}.$$

Denote by $q_\alpha^{\Omega,N,\delta}$ and $q_\alpha^{\Omega,D,\delta}$ the quadratic forms defined by the same expression as q_α^Ω but acting respectively on the domains

$$\mathcal{D}(q_\alpha^{\Omega,N,\delta}) = H^1(\Omega_\delta) \cup H^1(\Theta_\delta), \quad \mathcal{D}(q_\alpha^{\Omega,D,\delta}) = \widetilde{H}^1_0(\Omega_\delta) \cup H^1_0(\Theta_\delta),$$
$$\widetilde{H}^1_0(\Omega_\delta) := \{f \in H^1(\Omega_\delta) : \, f = 0 \text{ at } \partial\Omega_\delta \setminus S\}$$

and denote by $Q_\alpha^{\Omega,N,\delta}$ and $Q_\alpha^{\Omega,D,\delta}$ the associated self-adjoint operators in $L^2(\Omega)$. Due to the inclusions

$$\mathcal{D}(q_\alpha^{\Omega,D,\delta}) \subset \mathcal{D}(q_\alpha^\Omega) \subset \mathcal{D}(q_\alpha^{\Omega,N,\delta})$$

and according to the max-min principle, for each $j \in \mathbb{N}$ we have the inequalities

$$E_j(Q_\alpha^{\Omega,N,\delta}) \le E_j(Q_\alpha^\Omega) \le E_j(Q_\alpha^{\Omega,D,\delta}).$$

Furthermore, we have the representations

$$Q_\alpha^{\Omega,N,\delta} = B_\alpha^{\Omega,N,\delta} \oplus (-\Delta)^N_{\Theta_\delta}, \quad Q_\alpha^{\Omega,D,\delta} = B_\alpha^{\Omega,D,\delta} \oplus (-\Delta)^D_{\Theta_\delta},$$

where $B_\alpha^{\Omega,N,\delta}$ and $B_\alpha^{\Omega,D,\delta}$ are the self-adjoint operators in $L^2(\Omega_\delta)$ associated with the respective quadratic forms

$$b_\alpha^{\Omega,\star,\delta}(u,u) = \int_{\Omega_\delta} |\nabla u|^2 dx - \alpha \int_S |u|^2 dS, \quad \star \in \{N, D\},$$
$$\mathcal{D}(b_\alpha^{\Omega,N,\delta}) = H^1(\Omega_\delta), \quad \mathcal{D}(b_\alpha^{\Omega,D,\delta}) = \widetilde{H}^1_0(\Omega_\delta),$$

and $(-\Delta)^N_{\Theta_\delta}$ and $(-\Delta)^D_{\Theta_\delta}$ denote respectively the Neumann and the Dirichlet Laplacian in Θ_δ. As both Neumann and Dirichlet Laplacians are non-negative, we have the inequalities

$$E_j(B_\alpha^{\Omega,N,\delta}) \le E_j(Q_\alpha^\Omega) \le E_j(B_\alpha^{\Omega,D,\delta}) \text{ for all } j \text{ with } E_j(B_\alpha^{\Omega,D,\delta}) < 0. \tag{4}$$

We remark that these inequalities are valid for any value of δ.

2.2. **Change of variables.** In order to study the eigenvalues of the operators $B_\alpha^{\Omega,N,\delta}$ and $B_\alpha^{\Omega,D,\delta}$ we proceed first with a change variables in Ω_δ with small δ. The computations below are very similar to those performed in [CEK04] for a different problem.

It is a well-known result of the differential geometry that for δ sufficiently small the map Φ defined by

$$\Sigma := S \times (0,\delta) \ni (s,t) \mapsto \Phi(s,t) = s - tn(s) \in \Omega_\delta$$

is a diffeomorphism. The metric G on Σ induced by this embedding is

$$G = g \circ (I_s - tL_s)^2 + \mathrm{d}t^2, \tag{5}$$

where $I_s : T_sS \to T_sS$ is the identity map, and g is the metric on S induced by the embedding in $\mathbb{R}^\nu$. The associated volume form $\mathrm{d}\Sigma$ on Σ is

$$\mathrm{d}\Sigma = |\det G|^{1/2} \mathrm{d}s\, \mathrm{d}t = \varphi(s,t) |\det g|^{1/2} \mathrm{d}s\, \mathrm{d}t = \varphi\, \mathrm{d}S\, \mathrm{d}t,$$

where

$$\mathrm{d}S = |\det g|^{1/2} \mathrm{d}s$$

is the induced $(\nu-1)$-dimensional volume form on S, and the weight φ is given by

(6)
$$\varphi(s,t) := \big|\det(I_s - tL_s)\big| = 1 - t\,\mathrm{tr}\,L_s + p(s,t)t^2 \equiv 1-(\nu-1)H(s)t + p(s,t)t^2,$$

with p being a polynomial in t with C^1 coefficients depending on s.

Consider the unitary map

$$U : L^2(\Omega_\delta) \to L^2(\Sigma, \mathrm{d}\Sigma), \quad Uf = f \circ \Phi,$$

and the quadratic forms

$$c_\alpha^\star(f,f) = b_\alpha^{\Omega,\star,\delta}(U^{-1}f, U^{-1}f) \quad \mathcal{D}(c_\alpha^\star) = U\mathcal{D}(b_\alpha^{\Omega,\star,\delta}), \quad \star \in \{N, D\}.$$

We have then

$$c_\alpha^N(u,u) = \int_\Sigma G^{jk}\overline{\partial_j u}\partial_k u\,\mathrm{d}\Sigma_\alpha - \alpha\int_S |u(s,0)|^2\mathrm{d}S, \quad \mathcal{D}(c_\alpha^N) = H^1(\Sigma),$$
$$c_\alpha^D(u,u) = \text{the restriction of } c_\alpha^N \text{ to } \mathcal{D}(c_\alpha^D) = \widetilde{H}_0^1(\Sigma),$$

with

$$\widetilde{H}_0^1(\Sigma) := \big\{f \in H^1(\Sigma) : f(\cdot,\delta) = 0\big\}, \quad (G^{jk}) := G^{-1}.$$

2.3. **Estimates of the metrics.** We remark that due to (5) we can estimate, with some $0 < C_- < C_+$,

$$C_- g^{-1} + \mathrm{d}t^2 \le G^{-1} \le C_+ g^{-1} + \mathrm{d}t^2, \quad (g^{\rho\mu}) := g^{-1},$$

which shows that we have the form inequalities

$$c_\alpha^- \le c_\alpha^N \quad \text{and} \quad c_\alpha^D \le c_\alpha^+ \tag{7}$$

with

$$c_\alpha^-(u,u) := C_-\int_\Sigma g^{\rho\mu}\,\overline{\partial_\rho u}\,\partial_\mu u\,\mathrm{d}\Sigma + \int_\Sigma |\partial_t u|^2\mathrm{d}\Sigma - \alpha\int_S |u(s,0)|^2\mathrm{d}S,$$
$$\mathcal{D}(c_\alpha^-) = \mathcal{D}(c_\alpha^N) = H^1(\Sigma),$$
$$c_\alpha^+(u,u) := C_+\int_\Sigma g^{\rho\mu}\,\overline{\partial_\rho u}\,\partial_\mu u\,\mathrm{d}\Sigma + \int_\Sigma |\partial_t u|^2\mathrm{d}\Sigma - \alpha\int_S |u(s,0)|^2\mathrm{d}S,$$
$$\mathcal{D}(c_\alpha^+) = \mathcal{D}(c_\alpha^D) = \widetilde{H}_0^1(\Sigma).$$

In particular, if C_α^- and C_α^+ are the self-adjoint operators acting in $L^2(\Sigma, \mathrm{d}\Sigma)$ and associated with the forms c_α^- and c_α^+ respectively, then it follows from (4) and (7) that

$$E_j(C_\alpha^-) \le E_j(Q_\alpha^\Omega) \le E_j(C_\alpha^+) \text{ for all } j \text{ with } E_j(C_\alpha^+) < 0. \tag{8}$$

In order to remove the weight φ in $\mathrm{d}\Sigma$ we introduce another unitary transform

$$V : L^2\big(S \times (0,\delta), \mathrm{d}S\,\mathrm{d}t\big) \equiv L^2(\Sigma, \mathrm{d}S\,\mathrm{d}t) \to L^2(\Sigma, \mathrm{d}\Sigma),$$
$$(Vf)(s,t) = \varphi(s,t)^{-1/2} f(s,t)$$

and the quadratic forms

$$d_\alpha^\pm(f,f) = c_\alpha^\pm(Vf, Vf), \quad \mathcal{D}(d_\alpha^\pm) = V^{-1}\mathcal{D}(c_\alpha^\pm).$$

We remark that due to (6) we can choose δ sufficiently small to have the representation

$$\varphi(s,t)^{-1/2} = 1 + m(s)\,t + t^2 P(s,t), \quad P \in C^1(\overline{\Sigma}), \quad \varphi > 0 \text{ in } \overline{\Sigma}, \tag{9}$$

where we denote for the sake of brevity

$$m(s) := \frac{\nu - 1}{2} H(s).$$

We can write

$$\partial_\mu (Vu) = \varphi^{-1/2} \partial_\mu u + u\, \partial_\mu \varphi^{-1/2},$$

and

$$\begin{aligned} \int_\Sigma g^{\rho\mu} \overline{\partial_\rho (Vu)} \partial_\mu (Vu)\, \mathrm{d}\Sigma &= \int_\Sigma g^{\rho\mu} \overline{\partial_\rho (Vu)} \partial_\mu (Vu)\, \varphi\, \mathrm{d}S\mathrm{d}t \\ &= \int_\Sigma \Big[\varphi^{-1} g^{\rho\mu} \overline{\partial_\rho u} \partial_\mu u + g^{\rho\mu} \big(\overline{u} \partial_\rho \varphi^{-1/2} \big) \varphi^{-1/2} \partial_\mu u \\ &\quad + g^{\rho\mu} \varphi^{-1/2} \overline{\partial_\rho u} \big(u \partial_\mu \varphi^{-1/2} \big) + |u|^2 g^{\rho\mu} \overline{\partial_\rho \varphi^{-1/2}} \partial_\mu \varphi^{-1/2} \Big] \varphi\, \mathrm{d}S\mathrm{d}t. \end{aligned} \tag{10}$$

As $(g^{\rho\mu})$ is positive definite, using the Cauchy-Schwarz inequality we estimate

$$\begin{aligned} &\left| \int_\Sigma g^{\rho\mu} \left(\overline{u} \partial_\rho \varphi^{-1/2} \right) \left(\varphi^{-1/2} \partial_\mu u \right) \varphi\, \mathrm{d}S\mathrm{d}t \right|^2 \\ &\le \int_\Sigma \left(g^{\rho\mu}\, \overline{\varphi^{-1/2} \partial_\rho u}\, \varphi^{-1/2} \partial_\mu u \right) \varphi\, \mathrm{d}S\, \mathrm{d}t \cdot \int_\Sigma |u|^2 \Big(g^{\rho\mu}\, \overline{\partial_\rho \varphi^{-1/2}}\, \partial_\mu \varphi^{-1/2} \Big) \varphi\, \mathrm{d}S\, \mathrm{d}t \\ &= \int_\Sigma g^{\rho\mu}\, \overline{\partial_\rho u}\, \partial_\mu u\, \mathrm{d}S\, \mathrm{d}t \cdot \int_\Sigma W |u|^2\, \mathrm{d}S\, \mathrm{d}t, \quad W := \varphi\, g^{\rho\mu}\, \overline{\partial_\rho \varphi^{-1/2}}\, \partial_\mu \varphi^{-1/2}. \end{aligned}$$

Substituting this inequality into (10) we obtain

$$\begin{aligned} \int_\Sigma g^{\rho\mu}\, \overline{\partial_\rho u}\, \partial_\mu u\, \mathrm{d}S\, \mathrm{d}t - \int_\Sigma W |u|^2 \mathrm{d}S\, \mathrm{d}t &\le \int_\Sigma g^{\rho\mu}\, \overline{\partial_\rho (Vu)}\, \partial_\mu (Vu)\, \mathrm{d}\Sigma \\ &\le 2 \int_\Sigma g^{\rho\mu}\, \overline{\partial_\rho u}\, \partial_\mu u\, \mathrm{d}S\, \mathrm{d}t + 2 \int_\Sigma W |u|^2 \mathrm{d}S\, \mathrm{d}t. \end{aligned}$$

Denoting

$$C_0 := \|W\|_{L^\infty(\Sigma)}$$

we arrive finally at

$$\begin{aligned} \int_\Sigma g^{\rho\mu}\, \overline{\partial_\rho u}\, \partial_\mu u\, \mathrm{d}S\, \mathrm{d}t - C_0 \int_\Sigma u|^2 \mathrm{d}S\mathrm{d}t &\le \int_\Sigma g^{\rho\mu} \overline{\partial_\rho (Vu)} \partial_\mu (Vu)\, \mathrm{d}\Sigma \\ &\le 2 \int_\Sigma g^{\rho\mu}\, \overline{\partial_\rho u}\, \partial_\mu u\, \mathrm{d}S\, \mathrm{d}t + 2C_0 \int_\Sigma |u|^2 \mathrm{d}S\, \mathrm{d}t. \end{aligned}$$

On the other hand, as follows from (9),

$$\partial_t \varphi^{-1/2} = m + tR \text{ with } R \in C^1(\overline{\Sigma}),$$

and we have

$$\int_\Sigma \big|\partial_t(Vu)\big|^2 \mathrm{d}\Sigma = \int_\Sigma \varphi\Big|\varphi^{-1/2}\partial_t u + (m+tR)u\Big|^2 \mathrm{d}S\mathrm{d}t$$
$$= \int_\Sigma |\partial_t u|^2 \mathrm{d}S\mathrm{d}t + \int_\Sigma (m+tR)\varphi^{1/2}\partial_t|u|^2\mathrm{d}S\mathrm{d}t + \int_\Sigma (m+tR)^2\varphi|u|^2\mathrm{d}S\mathrm{d}t.$$

By setting

$$B_1 := \big\|(m+tR)^2\varphi\big\|_{L^\infty(\Sigma)}$$

we have

$$-B_1\int_\Sigma |u|^2\mathrm{d}S\,\mathrm{d}t \le \int_\Sigma (m+tR)^2\varphi|u|^2\mathrm{d}S\,\mathrm{d}t \le B_1\int_\Sigma |u|^2\mathrm{d}S\,\mathrm{d}t.$$

Furthermore, using the integration by parts we arrive at

$$\int_\Sigma (m+tR)\varphi^{1/2}\partial_t|u|^2\mathrm{d}S\,\mathrm{d}t = \int_S\Big(\int_0^\delta (m+tR)\varphi^{1/2}\partial_t|u|^2\mathrm{d}t\Big)\mathrm{d}S$$
$$= \int_S \varphi(s,\delta)^{1/2}\big(m(s)+\delta R(s,\delta)\big)\big|u(s,\delta)\big|^2\mathrm{d}S - \int_S m(s)\big|u(s,0)\big|^2\mathrm{d}S$$
$$- \int_\Sigma \partial_t\big((m+tR)\varphi^{1/2}\big)|u|^2\mathrm{d}S\,\mathrm{d}t.$$

Set

$$\beta := \big\|\varphi(\cdot,\delta)^{1/2}\big(m(\cdot)+\delta R(\cdot,\delta)\big)\big\|_{L^\infty(S)}, \quad B_2 := \Big\|\partial_t\big((m+tR)\varphi^{1/2}\big)\Big\|_{L^\infty(\Sigma)}.$$

By summing up all the terms we see that for any $u \in H^1\big(S\times(0,\delta)\big)$ we have the minoration

$$d_\alpha^-(u,u) \ge A_-\int_{S\times(0,\delta)} g^{\rho\mu}\,\overline{\partial_\rho u}\,\partial_\mu u\,\mathrm{d}S\,\mathrm{d}t + \int_{S\times(0,\delta)}|\partial_t u|^2\mathrm{d}S\,\mathrm{d}t$$
$$+ B_-\int_{S\times(0,\delta)}|u|^2\mathrm{d}S\,\mathrm{d}t - \int_S\big(\alpha+m(s)\big)|u(s,0)|^2\mathrm{d}S - \beta\int_S|u(s,\delta)|^2\mathrm{d}S$$

with

$$A_- := C_-, \quad B_- := -C_-C_0 - B_1 - B_2.$$

Using the quantity

$$m_{\max} := \max_{s\in S} m(s) \equiv \frac{\nu-1}{2}H_{\max},$$

and the inequality

$$-\big(\alpha+m(s)\big) \ge -(\alpha+m_{\max}), \quad s\in S,$$

we estimate finally, in the sense of forms,

$$k_\alpha^- \le d_\alpha^-$$

with

$$\begin{aligned} k_\alpha^-(u,u) :=& A_- \int_{S\times(0,\delta)} g^{\rho\mu}\overline{\partial_\rho u}\partial_\mu u \,\mathrm{d}S\,\mathrm{d}t + \int_{S\times(0,\delta)} |\partial_t u|^2 \mathrm{d}S\,\mathrm{d}t \\ &+ B_- \int_{S\times(0,\delta)} |u|^2 \mathrm{d}S\,\mathrm{d}t - (\alpha + m_{\max}) \int_S |u(s,0)|^2 \mathrm{d}S \\ &- \beta \int_S |u(s,\delta)|^2 \mathrm{d}S, \\ \mathcal{D}(k_\alpha^-) =& H^1\big(S\times(0,\delta)\big). \end{aligned}$$

We also have the majoration

$$d_\alpha^+ \le k_\alpha^+$$

with

$$\begin{aligned} k_\alpha^+(u,u) =& A_+ \int_{S\times(0,\delta)} g^{\rho\mu}\,\overline{\partial_\rho u}\,\partial_\mu u \,\mathrm{d}S\,\mathrm{d}t + \int_{S\times(0,\delta)} |\partial_t u|^2 \mathrm{d}S\,\mathrm{d}t \\ &+ B_+ \int_{S\times(0,\delta)} |u|^2 \mathrm{d}S\,\mathrm{d}t - \int_S \big(\alpha + m(s)\big)|u(s,0)|^2 \mathrm{d}S, \\ \mathcal{D}(k_\alpha^+) =& \widetilde{H}_0^1\big(S\times(0,\delta)\big) := \big\{ f \in H^1\big(S\times(0,\delta)\big) : f(\cdot,\delta) = 0 \big\}, \end{aligned}$$

where

$$A_+ := 2C_+ > 0, \quad B_+ := 2C_+B_0 + B_1 + B_2.$$

Denote by $K_\alpha^\pm$ the self-adjoint operators in $L^2\big(S\times(0,\delta), \mathrm{d}S\mathrm{d}t\big)$ associated with the respective forms $k_\alpha^\pm$ then, due to the preceding constructions, for any $j \in \mathbb{N}$ we have the inequalities

$$E_j(K_\alpha^-) \le E_j(C_\alpha^-), \quad E_j(C_\alpha^+) \le E_j(K_\alpha^+),$$

which implies, due to (8),

$$E_j(K_\alpha^-) \le E_j(Q_\alpha^\Omega) \le E_j(K_\alpha^+) \text{ for all } j \text{ with } E_j(K_\alpha^+) < 0. \tag{11}$$

We remark that in what precedes the parameter δ was chosen sufficiently small but fixed.

2.4. **Analysis of the reduced operators and lower bound.** Let us analyze the eigenvalues of K_α^-. We remark that it can be represented as

$$K_\alpha^- = (A_- L + B_-) \otimes 1 + 1 \otimes T_\alpha^-,$$

where L is the (positive) Laplace-Beltrami operator on S and acting in $L^2(S, \mathrm{d}S)$ and T_α^- is the self-adjoint operator in $L^2(0,\delta)$ associated with the quadratic form

$$\tau_\alpha^-(f,f) = \int_0^\delta \big|f'(t)\big|^2 \mathrm{d}t - (\alpha + m_{\max})\big|f(0)\big|^2 - \beta\big|f(\delta)\big|^2, \quad \mathcal{D}(\tau_\alpha^-) = H^1(0,\delta).$$

Clearly, the eigenvalues of $A_-L + B_-$ do not depend on α. On the other hand, the eigenvalues of T_α^- in the limit $\alpha \to +\infty$ were already analyzed in the previous works, and one has the following result, see e.g. [Pan13, Lemma 3] or [EMP14, Lemma 2.5]:

$$E_1(T_\alpha^-) = -(\alpha + m_{\max})^2 + o(1) \text{ and } E_2(T_\alpha^-) \ge 0 \text{ as } \alpha \to +\infty.$$

It follows that for any fixed $j \in \mathbb{N}$ and $\alpha \to +\infty$ we have

$$\begin{aligned} E_j(K_\alpha^-) &= E_1(T_\alpha^-) + A_- E_j(L) + B_- = -(\alpha + m_{\max})^2 + \mathcal{O}(1) \\ &= -\alpha^2 - 2m_{\max}\alpha + \mathcal{O}(1) = -\alpha^2 - (\nu - 1)H_{\max}\,\alpha + \mathcal{O}(1). \end{aligned} \tag{12}$$

Combining this with (11) provides the lower bound of (2).

2.5. **Construction of quasi-modes and upper bound.** It remains to analyze the eigenvalues of K_α^+. Let us show first an additional auxiliary estimate.

Lemma 4. *Let T_α^+ denote the self-adjoint operator in $L^2(0,\delta)$ associated with the quadratic form*

$$\begin{aligned} \tau_\alpha^+(f,f) &:= \int_0^\delta \big|f'(t)\big|^2 \mathrm{d}t - (\alpha + m_{\max})\big|f(0)\big|^2, \\ \mathcal{D}(\tau_\alpha^+) &= \big\{f \in H^1(0,\delta) : \ f(\delta) = 0\big\}. \end{aligned}$$

Then

$$E_1(T_\alpha^+) = -(\alpha + m_{\max})^2 + o(1) \ \text{ and } E_2(T_\alpha^+) \geq 0 \ \text{ as } \alpha \to +\infty. \tag{13}$$

Furthermore, if ψ_α is a normalized eigenfunction for the eigenvalue $E_1(T_\alpha^+)$, then

$$\big|\psi_\alpha(0)\big|^2 = \mathcal{O}(\alpha) \ \text{ as } \alpha \to +\infty. \tag{14}$$

Proof. The asymptotics (13) were already obtained in the previous works, see e.g. [Pan13, Lemma 4] or [EMP14, Lemma 2.4]. Furthermore, as the operator in question is just the Laplacian with suitable boundary conditions, the lowest eigenfunction can be represented as

$$\psi_\alpha(t) = C_1(\alpha)e^{kt} + C_2(\alpha)e^{-kt}, \quad k := \sqrt{-E_1(T_\alpha^+)}.$$

This function must satisfy the Dirichlet boundary condition at $t = \delta$, which gives

$$C_1(\alpha) = -e^{-2k\delta}C_2(\alpha) = \mathcal{O}(e^{-2\delta\alpha})C_2(\alpha).$$

Therefore, for large α we have

$$\begin{aligned} 1 = \|\psi_\alpha\|^2 &= \int_0^\delta \big|C_1(\alpha)e^{kt} + C_2(\alpha)e^{-kt}\big|^2 \mathrm{d}t \\ &= \big|C_1(\alpha)\big|^2 \frac{e^{2k\delta} - 1}{2k} + \big(\overline{C_1(\alpha)}C_2(\alpha) + C_1(\alpha)\overline{C_2(\alpha)}\big)\delta + \big|C_2(\alpha)\big|^2 \frac{1 - e^{-2k\delta}}{2k} \\ &= \frac{1}{2k}\big|C_2(\alpha)\big|^2\big(1 + o(1)\big), \end{aligned}$$

from which we infer

$$\big|C_2(\alpha)\big|^2 = 2k + o(k) = \mathcal{O}(\alpha), \quad \big|C_1(\alpha)\big|^2 = \mathcal{O}(ke^{-4\delta\alpha}) = \mathcal{O}(\alpha e^{-4\delta\alpha}),$$

and, finally,

$$\big|\psi_\alpha(0)\big|^2 = \big|C_1(\alpha)\big|^2 + \big(\overline{C_1(\alpha)}C_2(\alpha) + C_1(\alpha)\overline{C_2(\alpha)}\,\big) + \big|C_2(\alpha)\big|^2 = \mathcal{O}(\alpha). \ \square$$

Now let us pick a point $s_0 \in S$ with $m(s_0) = m_{\max}$. Take an arbitrary function $v \in C^\infty\big([0,+\infty)\big)$ which is not identically zero and such that $v'(0) = 0$ and that $v(x) = 0$ for $x \geq 1$. Consider the functions

$$u_\alpha(s,t) = v\big(\varepsilon^{-1} d(s,s_0)\big)\psi_\alpha(t),$$

where d means the geodesic distance on S and $\varepsilon = \varepsilon(\alpha) = o(1)$ as $\alpha \to +\infty$ will be determined later. One easily checks that for $\alpha \to +\infty$ one has $u_\alpha \in \widetilde{H}^1_0\big(S \times (0,\delta)\big)$ and that

$$\theta(\alpha) := \int_S \Big|v\big(\varepsilon^{-1} d(s,s_0)\big)\Big|^2 \mathrm{d}S \equiv \|u_\alpha\|^2 = c_0 \varepsilon^{\nu-1} + o(\varepsilon^{\nu-1}), \quad c_0 > 0, \tag{15}$$

$$\int_{S\times(0,\delta)} g^{\rho\mu} \overline{\partial_\rho u_\alpha} \partial_\mu u_\alpha \mathrm{d}S\mathrm{d}t = \mathcal{O}(\varepsilon^{\nu-3}).$$

Furthermore,

$$\begin{aligned}
&\int_{S\times(0,\delta)} |\partial_t u_\alpha|^2 \mathrm{d}S\mathrm{d}t - \int_S \big(\alpha + m(s)\big)|u_\alpha(s,0)|^2 \mathrm{d}S \\
&\quad = \int_{S\times(0,\delta)} |\partial_t u_\alpha|^2 \mathrm{d}S\mathrm{d}t \\
&\qquad - (\alpha + m_{\max}) \int_S |u_\alpha(s,0)|^2 \mathrm{d}S + \int_S \big(m_{\max} - m(s)\big)|u_\alpha(s,0)|^2 \mathrm{d}S \\
&\quad = \int_S \Big|v\big(\varepsilon^{-1} d(s,s_0)\big)\Big|^2 \Big(\int_0^\delta \big|\psi'_\alpha(t)\big|^2 \mathrm{d}t - (\alpha + m_{\max})\big|\psi_\alpha(0)\big|^2 \Big) \mathrm{d}S \\
&\qquad + \big|\psi_\alpha(0)\big|^2 \int_S \big(m_{\max} - m(s)\big)\Big|v\big(\varepsilon^{-1} d(s,s_0)\big)\Big|^2 \mathrm{d}S \\
&\quad = E_1(T^+_\alpha) \int_S \Big|v\big(\varepsilon^{-1} d(s,s_0)\big)\Big|^2 \int_0^\delta \big|\psi_\alpha(t)\big|^2 \mathrm{d}t\mathrm{d}S \\
&\qquad + \big|\psi_\alpha(0)\big|^2 \int_S \big(m_{\max} - m(s)\big)\Big|v\big(\varepsilon^{-1} d(s,s_0)\big)\Big|^2 \mathrm{d}S \\
&\quad = -(\alpha + m_{\max})^2 \theta(\alpha) + o\big(\theta(\alpha)\big) \\
&\qquad + \mathcal{O}(\alpha) \int_S \big(m_{\max} - m(s)\big)\Big|v\big(\varepsilon^{-1} d(s,s_0)\big)\Big|^2 \mathrm{d}S;
\end{aligned}$$

on the last step we used (13) and (14). As $m \in C^1$, one has, for $d(s,s_0)$ sufficiently small,

$$m_{\max} - m(s) \leq c_1 d(s,s_0), \quad c_1 > 0. \tag{16}$$

It follows that

$$\int_S \big(m_{\max} - m(s)\big)\Big|v\big(\varepsilon^{-1} d(s,s_0)\big)\Big|^2 \mathrm{d}S \leq c_1 \varepsilon \int_S \Big|v\big(\varepsilon^{-1} d(s,s_0)\big)\Big|^2 \mathrm{d}S = \mathcal{O}\big(\varepsilon\theta(\alpha)\big).$$

Finally, using (15),

$$\begin{aligned}
\frac{k^+_\alpha(u_\alpha,u_\alpha)}{\|u_\alpha\|^2} &= \frac{-(\alpha + m_{\max})^2\theta(\alpha) + \mathcal{O}\big(\theta(\alpha)\big) + \mathcal{O}(\varepsilon^{\nu-3}) + \mathcal{O}\big(\alpha\varepsilon\theta(\alpha)\big)}{\theta(\alpha)} \\
&= -(\alpha + m_{\max})^2 + \mathcal{O}(1) + \mathcal{O}(\varepsilon^{-2}) + \mathcal{O}(\alpha\varepsilon) \\
&= -\alpha^2 - 2m_{\max}\alpha + \mathcal{O}(\varepsilon^{-2} + \alpha\varepsilon).
\end{aligned}$$

In order to optimize we take $\varepsilon = \alpha^{-1/3}$, which gives

$$\frac{k_\alpha^+(u_\alpha, u_\alpha)}{\|u_\alpha\|^2} \leq -\alpha^2 - 2m_{\max}\alpha + \mathcal{O}\big(\alpha^{2/3}\big). \tag{17}$$

Now let us choose j functions $v_1, \dots, v_j \in C^\infty\big([0, +\infty)\big)$ having disjoint supports, non identically zero, and such that $v_i'(0) = 0$ and that $v_i(x) = 0$ for $x \geq 1$ and any $i = 1, \dots, j$. Set

$$u_{i,\alpha}(s,t) = v_i\big(\varepsilon^{-1} d(s, s_0)\big)\psi_\alpha(t).$$

One easily checks that

$$k_\alpha^+(u_{i,\alpha}, u_{l,\alpha}) = 0 \text{ for } i \neq l,$$

and the preceding computations give the estimate

$$\frac{k_\alpha^+(u_{i,\alpha}, u_{i,\alpha})}{\|u_{i,\alpha}\|^2} \leq -\alpha^2 - 2m_{\max}\alpha + \mathcal{O}\big(\alpha^{2/3}\big).$$

Therefore, by the max-min principle we have

$$E_j(K_\alpha^+) \leq -\alpha^2 - 2m_{\max}\alpha + \mathcal{O}\big(\alpha^{2/3}\big) = -\alpha^2 - (\nu - 1)H_{\max}\,\alpha + \mathcal{O}\big(\alpha^{2/3}\big).$$

By inserting this estimate into (11) we get the upper bound of (2).

If $S \in C^4$, then the remainder estimate can be slightly improved. Namely, in this case we have $m \in C^2$. As s_0 is a local extremum, instead of (16) one can use

$$m_{\max} - m(s) \leq c_1 d(s, s_0)^2, \quad c_1 > 0, \tag{18}$$

and the choice $\varepsilon := \alpha^{-1/4}$ transforms (17) into

$$\frac{k_\alpha^+(u_\alpha, u_\alpha)}{\|u_\alpha\|^2} \leq -\alpha^2 - 2m_{\max}\alpha + \mathcal{O}\big(\alpha^{1/2}\big).$$

and then leads to

$$E_j(K_\alpha^+) \leq -\alpha^2 - 2m_{\max}\alpha + \mathcal{O}\big(\alpha^{1/2}\big) = -\alpha^2 - (\nu - 1)H_{\max}\,\alpha + \mathcal{O}\big(\alpha^{1/2}\big).$$

This finishes the proof of Theorem 1.

3. Estimates for the mean curvature

3.1. **Mean curvature and comparison of eigenvalues.** The result of Theorem 1 provides, at least for large α, a geometric point of view at various inequalities involving the Robin eigenvalues.

For example, as shown in [GS07], for all $\alpha > 0$ one has $E_1(Q_\alpha^\Omega) < -\alpha^2$. For large α this has a simple interpretation in view of Theorem 1: one always has $H_{\max}(\Omega) > 0$.

Furthermore, in view of Theorem 1 one has a simple condition allowing one to compare the Robin eigenvalues of two different domains for large α:

Corollary 5. *Let Ω_1 and Ω_2 be two admissible domains in $\mathbb{R}^\nu$. Assume that*

$$H_{\max}(\Omega_1) < H_{\max}(\Omega_2),$$

then for any fixed $j \in \mathbb{N}$ there exists $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$ there holds

$$E(Q_\alpha^{\Omega_2}) < E(Q_\alpha^{\Omega_1}).$$

For example, another result obtained in [GS07, Th. 2.4] is as follows: if Ω satisfies a uniform sphere condition of radius r and B is a ball of radius $r/2$, then $E(Q^\Omega_\alpha) \geq E(Q^B_\alpha)$ for all $\alpha > 0$. For large α this can be improved as follows:

Proposition 6. *Let $\Omega \subset \mathbb{R}^\nu$ be an admissible domain satisfying the uniform sphere condition of radius $R > 0$ and let B be a ball of radius $r < R$. Then for any $j \in \mathbb{N}$ one can find $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$ there holds $E_j(Q^\Omega_\alpha) > E_j(Q^B_\alpha)$.*

Proof. The uniform sphere condition of radius R at s implies the inequality $\max_j \kappa_j(s) \leq 1/R$, and due to $H(s) \leq \max_j \kappa_j(s)$ we have $H(s) \leq 1/R$ and $H_{\max}(\Omega) \leq 1/R < 1/r = H_{\max}(B)$, and the assertion follows from Corollary 5. □

In the following subsections we are going to discuss the problem of minimizing $H_{\max}$ among domains of a fixed volume. Corollary 5 shows that this geometric problem is closely related to the reverse Faber-Krahn inequality as discussed in the introduction. We remark that for a given value of $\operatorname{Vol}\Omega$ the value $H_{\max}(\Omega)$ can be made arbitrarily small in the class of admissible domains, as the example of thin spherical shells shows. We are going to show that a positive lower bound still exists in a smaller class in which the ball has the extremal property.

3.2. **Local minimizers.** We would like to show first that the balls are the only possible local minimizers of the problem. Our argument is based on the so-called Alexandrov theorem [Ale62]: Let $\Omega \subset \mathbb{R}^\nu$ be a bounded, C^2 smooth, connected domain such that the mean curvature at the boundary is constant, then Ω is a ball.

Proposition 7. *Let $\Omega \subset \mathbb{R}^\nu$ be a C^k smooth ($k \geq 2$), compact, connected domain, different from a ball. Then for any $\varepsilon > 0$ there exists another C^k smooth, compact, connected domain $\Omega_\varepsilon \subset \mathbb{R}^\nu$ with the following properties:*

- $\operatorname{Vol}\Omega_\varepsilon = \operatorname{Vol}\Omega$,
- *the boundary of Ω_ε is diffeomorphic to that of Ω,*
- *the Hausdorff distance between Ω_ε and Ω is smaller than ε,*
- $H_{\max}(\Omega_\varepsilon) < H_{\max}(\Omega)$.

Proof. By Alexandrov's theorem, the mean curvature H on $\partial\Omega$ is non-constant, so we can find a point $s_* \in \partial\Omega$ with $H(s_*) < H_{\max}(\Omega)$. By applying a suitable rotation, without loss of generality we may assume that for $x = (x', x_\nu)$, $x' = (x_1, \ldots, x_{\nu-1})$, close to s_* the condition $x \in \Omega$ is equivalent to $x_\nu < \varphi(x')$, where φ is a C^k function defined in a ball $B \subset \mathbb{R}^{\nu-1}$, and hence the boundary $\partial\Omega$ near s_* coincides with the graph of φ. By the choice of s_* and using the continuity of H we may additionally assume that B is chosen in such a way there exists $\delta > 0$ such that

$$H(s) \leq H_{\max}(\Omega) - \delta \text{ for any point } s \in S_B := \big\{\big(x', \varphi(x')\big) : x' \in B\big\}. \tag{19}$$

Choose any non-negative $\psi \in C^\infty_c(B)$ which is not identically zero. If $\varepsilon_0 > 0$ is sufficiently small, then for all $\varepsilon \in [0, \varepsilon_0)$, one has the inclusion

$$\Theta_\varepsilon := \big\{(x', x_\nu) : x' \in B,\, \varphi(x') - \varepsilon\psi(x') < x_\nu < \varphi(x')\big\} \subset \Omega,$$

and we set $\widetilde{\Omega}_\varepsilon := \Omega \setminus \overline{\Theta_\varepsilon}$. By construction we have, for $\varepsilon \in (0, \varepsilon_0)$,

$$\operatorname{Vol} \widetilde{\Omega}_\varepsilon = \operatorname{Vol} \Omega - \varepsilon \int_B \psi(x') dx' < \operatorname{Vol} \Omega, \tag{20}$$

and the Hausdorff distance between $\widetilde{\Omega}_\varepsilon$ and Ω is at most $\|\psi\|_{L^\infty(B)} \cdot \varepsilon$.

Furthermore,

$$\partial\widetilde{\Omega}_\varepsilon := (\partial\Omega \setminus S_B) \cup S_B^\varepsilon \text{ with } S_B^\varepsilon := \big\{ \big(x', \varphi(x') - \varepsilon\psi(x')\big) : x' \in B \big\},$$

and S_B^ε is diffeomorph to S_B (as the both are graphs of smooth functions). Furthermore, as ψ has a compact support, S_B^ε coincides with S_B near the boundary, and, finally, $\partial\widetilde{\Omega}_\varepsilon$ is diffeomorph to $\partial\Omega$ for any $\varepsilon \in (0, \varepsilon_0)$.

For $x' \in B$, we denote by $H(x', \varepsilon)$ the mean curvature of $\partial\widetilde{\Omega}_\varepsilon$ at the point $\big(x', \varphi(x') - \varepsilon\psi(x')\big)$, then $\big\|H(\cdot, \varepsilon) - H(\cdot, 0)\big\|_{L^\infty(B)} = \mathcal{O}(\varepsilon)$, and in virtue of (19) we can find $\varepsilon_1 \in (0, \varepsilon_0)$ such that for any $\varepsilon \in (0, \varepsilon_1)$ we have $H(x', \varepsilon) < H_{\max}(\Omega)$ for all $x' \in B$, which gives $H_{\max}(\widetilde{\Omega}_\varepsilon) = H_{\max}(\Omega)$.

Now set

$$\Omega_\varepsilon := \Big(\frac{\operatorname{Vol} \Omega}{\operatorname{Vol} \widetilde{\Omega}_\varepsilon}\Big)^{1/\nu} \widetilde{\Omega}_\varepsilon,$$

then $\operatorname{Vol} \Omega_\varepsilon = \operatorname{Vol} \Omega$, and

$$H_{\max}(\Omega_\varepsilon) = \Big(\frac{\operatorname{Vol} \widetilde{\Omega}_\varepsilon}{\operatorname{Vol} \Omega}\Big)^{1/\nu} H_{\max}(\widetilde{\Omega}_\varepsilon) \equiv \Big(\frac{\operatorname{Vol} \widetilde{\Omega}_\varepsilon}{\operatorname{Vol} \Omega}\Big)^{1/\nu} H_{\max}(\Omega) < H_{\max}(\Omega).$$

The boundary of Ω_ε is clearly diffeomorph to that of $\widetilde{\Omega}_\varepsilon$ and, consequently, to that of Ω, and using (20) one easily checks that that the Hausdorff distance between Ω_ε and Ω is $\mathcal{O}(\varepsilon)$. □

3.3. **Proof of Theorem 2.** We now prove Theorem 2. Without loss of generality we assume that Ω is star-shaped with respect to the origin. Let us introduce a function p on S by $p(s) := s \cdot n(s)$, where $\cdot$ stands for the scalar product in $\mathbb{R}^\nu$. The function p is called sometimes the support function. We will use two integral identities related to Ω. First, due to the divergence theorem we have

$$\operatorname{Vol} \Omega = \frac{1}{\nu} \int_S p \, \mathrm{d}S.$$

Furthermore, we have the so-called Minkowski formula

$$\operatorname{Area} S = \int_S p \, H \, \mathrm{d}S,$$

see [Hsi54, Theorem 1]; remark that in [Hsi54], the mean curvature M_1 corresponds to $(-H)$ with our notation.

As Ω is star-shaped, we have $p \geq 0$, and we deduce

$$\operatorname{Area} S \leq H_{\max}(\Omega) \int_S p \, \mathrm{d}S = \nu H_{\max}(\Omega) \operatorname{Vol} \Omega,$$

which gives

$$H_{\max}(\Omega) \geq \frac{1}{\nu} \frac{\operatorname{Area} S}{\operatorname{Vol} \Omega}. \tag{21}$$

Due to the classical isoperimetric inequality we have

$$\operatorname{Area} S \geq \nu \Big(\frac{\operatorname{Vol} B_\nu}{\operatorname{Vol} \Omega}\Big)^{1/\nu} \operatorname{Vol} \Omega, \tag{22}$$

and the equality holds only if Ω is a ball [BZ88, §2.10]. Substituting (22) into (21) we obtain

$$H_{\max}(\Omega) \geq \left(\frac{\operatorname{Vol} B_\nu}{\operatorname{Vol}\Omega}\right)^{1/\nu}. \tag{23}$$

If one has the equality in (23), then automatically one has the equality in (22), which gives in turn that Ω is a ball. On the other hand, is Ω is a ball of radius $r > 0$, then $H_{\max}(\Omega) = 1/r$ and $\operatorname{Vol}\Omega = r^\nu \operatorname{Vol} B_\nu$, and one has the equality in (23).

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