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# A Wronskian Approach to the Real $\tau$ -Conjecture

Pascal Koiran<sup>a</sup>, Natacha Portier<sup>a</sup>, Sébastien Tavenas<sup>a</sup>

<sup>a</sup>*LIP - UMR 5668 ENS Lyon - CNRS - UCBL - INRIA, Université de Lyon, École Normale Supérieure de Lyon*

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## Abstract

According to the real  $\tau$ -conjecture, the number of real roots of a sum of products of sparse univariate polynomials should be polynomially bounded in the size of such an expression. It is known that this conjecture implies a superpolynomial lower bound on the arithmetic circuit complexity of the permanent.

In this paper, we use the Wronskian determinant to give an upper bound on the number of real roots of sums of products of sparse polynomials of a special form. We focus on the case where the number of distinct sparse polynomials is small, but each polynomial may be repeated several times. We also give a deterministic polynomial identity testing algorithm for the same class of polynomials.

Our proof techniques are quite versatile; they can in particular be applied to some sparse geometric problems that do not originate from arithmetic circuit complexity. The paper should therefore be of interest to researchers from these two communities (complexity theory and sparse polynomial systems).

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## 1. Introduction

The complexity of the permanent polynomial

$$\text{per}(x_{11}, \dots, x_{nn}) = \sum_{\sigma \in S_n} \prod_{i=1}^n x_{i\sigma(i)}$$

is one of the central open problems in complexity theory. It is widely believed that the permanent is not computable by arithmetic circuits of size polynomial in  $n$ . This problem can be viewed as an algebraic version of the P versus NP problem [27, 7].

It is known that this much coveted lower bound for the permanent would follow from a so-called *real  $\tau$ -conjecture* for sums of products of sparse

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*Email addresses:* `pascal.koiran@ens-lyon.fr` (Pascal Koiran),  
`natacha.portier@ens-lyon.fr` (Natacha Portier), `sebastien.tavenas@ens-lyon.fr`  
(Sébastien Tavenas)

univariate polynomials [18]. Those are polynomials in  $\mathbb{R}[x]$  of the form  $\sum_{i=1}^k \prod_{j=1}^m f_{ij}(x)$ , where the sparse polynomials  $f_{ij}$  have at most  $t$  monomials. According to the real  $\tau$ -conjecture, the number of real roots of such an expression should be polynomially bounded in  $k$ ,  $m$  and  $t$ . The original  $\tau$ -conjecture by Shub and Smale [26] deals with integer roots of arbitrary (constant-free) straight-line programs.

As a first step toward the real  $\tau$ -conjecture, Grenet, Koiran, Portier and Strozecki [10] considered the family of sums of products of powers of sparse polynomials. Those polynomials are of the form

$$\sum_{i=1}^k \prod_{j=1}^m f_j^{\alpha_{i,j}}. \quad (1)$$

They are best viewed as sums of products of sparse polynomials where the total number  $m$  of distinct sparse polynomials is “small”, but each polynomial may be repeated several times. In particular, if one can find a  $(kt)^{O(1)}2^{O(m)}$  upper bound on the number of real roots, then it will imply the real- $\tau$  conjecture in the case where the number of distinct sparse polynomials is logarithmically bounded. The upper bound on the number of real roots obtained in [10] is polynomial in  $t$ , but exponential in  $m$  and doubly exponential in  $k$ .

Bounds on the number of real zeros for systems of sparse polynomials were extensively studied by Khovanskiĭ [17] in his “fewnomial theory”. His results imply an upper bound exponential in  $k$ ,  $m$  and  $t$ . In this article, we will give a bound of order  $t^{O(k^2m)}$ , thereby removing the double exponential from [10] while staying polynomial in  $t$ . Moreover, our results extend well to some other families of functions. In particular, they extend a result from Avendaño [4] on the intersection of a sparse plane curve and a line. He gave a linear bound on the number of roots for polynomials of the form  $\sum_{i=1}^k x^{\alpha_i}(ax+b)^{\beta_i}$  where  $\alpha_i$  and  $\beta_i$  are integers and gave an example proving that his linear bound does not apply for non-integer powers. Our result gives a polynomial upper bound for the wider family (1) where the polynomials  $f_j$  are of bounded degrees and the  $\alpha_{i,j}$  are real exponents.

In addition to bounds on the number of real roots, we also give a deterministic identity testing algorithm for polynomials of the form (1). The running time of our algorithm is polynomial in  $t$ , in the bit size of coefficients and of the powers  $(\alpha_{i,j})$  and exponential in  $k$  and  $m$ . Polynomial Identity Testing (PIT) is a very well-studied problem. The Schwartz-Zippel lemma yields a randomized algorithm for PIT, but the existence of an efficient deterministic algorithm is an outstanding open problem. Connections between circuit lower bounds and deterministic PIT algorithms were discovered in 1980 by Heintz and Schnorr [14], then more recently by Kabanets and Impagliazzo [16], by Aaronson and van Melkebeek [1] and by Agrawal [2]. Recently, many deterministic PIT algorithms have been found for several restricted models (see e.g. the two surveys [3, 25]). In particular, a deter-

ministic PIT algorithm for functions of the form of (1) has already been given in [10]. Their algorithm is polynomial in  $t$ , exponential in  $m$  but doubly exponential in  $k$  whereas we give a new algorithm which is only exponential in  $k$ .

We now present our main technical tools. Finding the roots of a product of polynomials is easy: it is the union of the roots of the corresponding polynomials. But finding the roots of a sum is difficult: for example how to bound the number of real roots of  $fg + 1$  where  $f$  and  $g$  are  $t$ -sparse? It is an open question to decide if this bound is linear in  $t$ . Our main tool in this paper to tackle the sum is the Wronskian. We recall that the Wronskian of a family of functions  $f_1, \dots, f_k$  is the determinant of the matrix of their derivatives of order 0 up to  $k - 1$ . More formally,

$$W(f_1, \dots, f_k) = \det \left( \left( f_j^{(i-1)} \right)_{1 \leq i, j \leq k} \right).$$

The Wronskian is useful especially for its connection to linear independence (more on this in the next section). Another classical and very useful tool is Descartes' rule of signs:

**Lemma 1** (Strong rule of signs). *Let  $f = \sum_{i=1}^t a_i x^{\alpha_i}$  be a polynomial such that  $\alpha_1 < \alpha_2 < \dots < \alpha_t$  and  $a_i$  are nonzero real numbers. Let  $N$  be the number of sign changes in the sequence  $(a_1, \dots, a_t)$ . Then the number of positive real roots of  $f$  is bounded by  $N$ .*

In this article, we will use a weak form of this lemma.

**Lemma 2** (Weak rule of signs). *Let  $f = \sum_{i=1}^t a_i x^{\alpha_i}$  be a polynomial such that  $\alpha_1 < \alpha_2 < \dots < \alpha_t$  and  $a_i$  are nonzero real numbers. Then the number of positive real roots of  $f$  is bounded by  $t - 1$ . Moreover, the result is also true in the case where the exponents  $\alpha_i$  are real.*

In their book [23], Pólya and Szegő gave a generalization of the strong rule of sign using the Wronskian. Some relations were already known between Wronskians and the sparse polynomials (c.f. for example [12], [11] and [13]). We show in Theorem 7 that bounding the number of roots of the Wronskian yields a bound on the number of roots of the corresponding sum. In general, the Wronskian may seem more complicated than the sum of the functions, but for the families studied in this paper it can be factorized more easily (Theorems 12 and 13).

The paper is organized as follows. The main results of Section 2 are Theorems 7, 8 and 9, which bound the number of roots of sums as a function of the number of roots of the Wronskian. Then, in Section 3, we apply these results to particular families of polynomials. The main applications that we have in mind are to polynomials of the form (1), and to the polynomials studied by Avendaño. We give in Section 4 some PIT algorithms for polynomials of the form (1). The proof of Theorem 9 will be given in Section 5.

And finally, we show in Section 6 that our method is optimal in a precise sense. Some of the proofs are postponed to the Appendix.

## 2. Zeros of the Wronskian as an upper bound

Let us recall that for a finite family of real functions  $f_1, \dots, f_k$  sufficiently differentiable, the Wronskian is defined by

$$W(f_1, \dots, f_k) = \det \left( \left( f_j^{(i-1)} \right)_{1 \leq i, j \leq k} \right).$$

We will use the following properties of the Wronskian.

**Lemma 3.** *Let  $f_1, \dots, f_k$  and  $g$  be  $k - 1$  times differentiable real functions. Then,  $W(gf_1, \dots, gf_k) = g^k W(f_1, \dots, f_k)$ .*

As a corollary:

**Lemma 4.** *Let  $f_1, \dots, f_k$  be  $k - 1$  times differentiable real functions and let  $I$  be an interval where they do not vanish.*

*Then, over  $I$ , we have  $W(f_1, \dots, f_k) = (f_1)^k W \left( \left( \frac{f_2}{f_1} \right)', \dots, \left( \frac{f_k}{f_1} \right)' \right)$ .*

These results can be found in [23] (ex. 57, 58 in Part 7). Notice that the Wronskian of a linearly dependent family of functions is identically zero (if a family is dependent then the family of the derivatives is also dependent with the same coefficients). But the converse is not necessarily true. Peano, then Bôcher, found counterexamples [21, 22, 5] (see [8] for a history of these results). However, Bôcher [6] proved that this converse becomes true if the functions are analytic [15].

**Lemma 5.** *If  $f_1, \dots, f_k$  are analytic functions, then  $(f_i)$  is linearly dependent if and only if  $W(f_1, \dots, f_k) = 0$ .*

**Definition 6.** *For every function  $g$  and interval  $I$ , we will denote  $Z_I(g)$  the number of distinct real roots of  $g$  over  $I$ . We just write  $Z(g)$  when the interval is clear from the context.*

Throughout the paper we consider only intervals that are not reduced to a single point (and we allow unbounded intervals). The next theorem is in fact implied, in the analytic case, by Voorhoeve and Van Der Poorten's result [28] (see below, Theorem 21 and the following paragraph for more precisions). In Theorem 7 we only assume that the  $f$  are sufficiently differentiable.

**Theorem 7.** *Let  $k$  be a non zero integer. Let  $f_i$  be  $k$  functions  $k - 1$  times differentiable in an interval  $I$  such that for all  $i \leq k$ , the Wronskian  $W(f_1, \dots, f_i)$  does not have any zero over  $I$ .*

*If the real constants  $a_1, \dots, a_k$  are not all equal to 0,  $a_1 f_1 + a_2 f_2 + \dots + a_k f_k$  has at most  $k - 1$  real zeros over  $I$  counted with multiplicity.*

*Proof.* We show this result by induction on  $k$ . If  $k = 1$ , then,  $f_1 = W(f_1)$  does not have any zero. Moreover,  $a_1$  is not zero. So,  $a_1 f_1$  has no zeros.

For some  $k \geq 2$ , let us suppose that the property is true for all linear combinations of size  $k - 1$ . Denote  $z$  the number of zeros of  $a_1 f_1 + \dots + a_k f_k$ . If  $a_2 = a_3 = \dots = a_k = 0$ , then  $a_1 \neq 0$  and  $a_1 f_1 + a_2 f_2 + \dots + a_k f_k = a_1 f_1$  has no zero, and the conclusion of the theorem holds true. Otherwise,  $a_1 + \frac{a_2 f_2}{f_1} + \dots + \frac{a_k f_k}{f_1}$  has  $z$  zeros (since  $f_1 = W(f_1)$  does not have any zero by hypothesis). By application of Rolle's Theorem,  $a_2 \left(\frac{f_2}{f_1}\right)' + \dots + a_k \left(\frac{f_k}{f_1}\right)'$  has at least  $z - 1$  zeros over  $I$ .

Function  $f_1$  does not have any root in  $I$ , so the functions  $\left(\frac{f_2}{f_1}\right)', \dots, \left(\frac{f_k}{f_1}\right)'$  are  $k - 2$  times differentiable. Moreover, for all  $2 \leq i \leq k$ ,  $W\left(\left(\frac{f_2}{f_1}\right)', \dots, \left(\frac{f_i}{f_1}\right)'\right) = W(f_1, \dots, f_i)/f_1^i$  does not have any roots over  $I$ . Since the coefficients  $a_2, \dots, a_k$  are not all zero, by induction hypothesis  $a_2 \left(\frac{f_2}{f_1}\right)' + \dots + a_k \left(\frac{f_k}{f_1}\right)'$  has at most  $k - 2$  zeros. Hence  $a_1 f_1 + a_2 f_2 + \dots + a_k f_k$  has at most  $k - 1$  zeros by Rolle's theorem.  $\square$

The following theorem gives us a method to find upper bounds on the number of roots. We will show in Section 6 that it is sometimes tight.

**Theorem 8.** *Let  $f_1, \dots, f_k$  be analytic functions on an interval  $I$ . If the real constants  $a_1, \dots, a_k$  are not all equal to 0,*

$$Z(a_1 f_1 + \dots + a_k f_k) \leq \left(1 + \sum_{i=1}^k Z(W(f_1, \dots, f_i))\right) k - 1. \quad (2)$$

*More precisely, if  $\Upsilon = \{x \in I \mid \exists i \leq k, W(f_1, \dots, f_i)(x) = 0\}$  is finite, then*

$$Z(a_1 f_1 + \dots + a_k f_k) \leq (1 + |\Upsilon|) k - 1. \quad (3)$$

*Moreover, the inequalities still hold if on the left side, zeros which are not zero of one of the Wronskians  $W(f_1, \dots, f_i)$  are counted with multiplicity.*

*Proof.* We will directly prove the more precise version (3). If  $a_1 f_1 + \dots + a_k f_k$  is the zero polynomial, then the family is linearly dependent and so the Wronskian  $W(f_1, \dots, f_k)$  is also the zero polynomial. This means that  $\Upsilon = I$  is infinite and the inequality is verified.

Otherwise,  $a_1 f_1 + \dots + a_k f_k$  has a finite number of zeros. We have  $\Upsilon = \bigcup_{i=1}^k Z(W(f_1, \dots, f_i))$ . So,  $|\Upsilon| \leq \sum_{i=1}^k |Z(W(f_1, \dots, f_i))|$  and we will prove (3). The set  $I \setminus \Upsilon$  is an union of  $|\Upsilon| + 1$  intervals. Let  $J$  be one of these intervals. With Theorem 7, we get  $Z_J(a_1 f_1 + \dots + a_k f_k) \leq k - 1$ . So  $a_1 f_1 + \dots + a_k f_k$  has at most  $(1 + |\Upsilon|)(k - 1)$  zeros over  $I \setminus \Upsilon$  and at most  $(1 + |\Upsilon|)(k - 1) + |\Upsilon|$  zeros over  $I$ .  $\square$

In Section 5, we will prove the following variation on Theorem 8:

**Theorem 9.** *Let  $f_1, \dots, f_k$  be analytic linearly independent functions on an interval  $I$ . Then,*

$$Z(f_1 + \dots + f_k) \leq k - 1 + Z(W_k) + Z(W_{k-1}) + 2 \sum_{j=1}^{k-2} Z(W_j).$$

In most applications, this result yields a better bound than Theorems 7 and 8.

### 3. Applications

In this section, we prove Theorem 12 which bounds the number of zeros of the polynomials of the form (1). The given bound improves both Grenet, Koiran, Portier and Strozecki's result [10] and the bound implied by Khovanskii's fewnomial theory [17]. At the end of this section, we also extend Avendaño's result to real exponents. We saw before (in Section 2) that the number of zeros of a linear combination of real functions can be bounded as a function of the number of zeros of their Wronskians. As a result, it remains to bound the number of zeros of Wronskians of polynomials of the form  $\prod_{j=1}^m f_j^{\alpha_{i,j}}$ . Such a Wronskian has few zeros thanks to a nice factorization property: after factoring out some high powers, we are left with a determinant whose entries are low-degree polynomials (or sparse polynomials, depending on the model under consideration). It is then straightforward to bound the number of real roots of this determinant.

#### 3.1. Derivative of a power

We use ultimately vanishing sequences of integer numbers, i.e., infinite sequences of integers which have only finitely many nonzero elements. We denote this set  $\mathbb{N}^{(\mathbb{N})}$ . For any positive integer  $p$ , let  $\mathcal{S}_p = \{(s_1, s_2, \dots) \in \mathbb{N}^{(\mathbb{N})} \mid \sum_{i=1}^{\infty} i s_i = p\}$  (so for each  $p$ , this set is finite). Then if  $s$  is in  $\mathcal{S}_p$ , we observe that for all  $i \geq p + 1$ , we have  $s_i = 0$ . Moreover for any  $p$  and any  $s = (s_1, s_2, \dots) \in \mathbb{N}^{(\mathbb{N})}$ , we will denote  $|s| = \sum_{i=1}^{\infty} s_i$  (the sum makes sense because it is finite).

**Lemma 10.** *Let  $p$  be a positive integer and  $\alpha \geq p$  be a real number. Then*

$$(f^\alpha)^{(p)} = \sum_{s \in \mathcal{S}_p} \left[ \beta_{\alpha,s} f^{\alpha-|s|} \prod_{k=1}^p (f^{(k)})^{s_k} \right]$$

where  $(\beta_{\alpha,s})$  are some constants.

The order of differentiation of a monomial  $\prod_{k=1}^p (f^{(k)})^{s_k}$  is  $\sum_{k=1}^p k s_k$ . The order of differentiation of a differential polynomial is the maximal order of its monomials. For example: if  $f$  is a function, the total order of differentiation of  $f^3 (f')^2 (f^{(4)})^3 + 3ff'$  is  $\max(3 * 0 + 2 * 1 + 3 * 4, 0 * 1 + 1 * 1) = 14$ .

Lemma 10 just means that the  $p$ -th derivative of a power  $\alpha$  of a function  $f$  is a linear combination of terms such that each term is a product of derivatives of  $f$  of total degree  $\alpha$  and of total order of differentiation  $p$ .

*Proof.* In the following,  $e_i$  is the sequence  $(0, 0, \dots, 0, 1, 0, 0, \dots)$  where the 1 appears at the  $i^{\text{th}}$  coordinate. We show this lemma by induction over  $p$ . If  $p = 1$ , then  $(f^\alpha)' = \alpha f' f^{\alpha-1}$ . That is the basis case since  $\mathcal{S}_1 = \{(1, 0, 0, \dots)\}$ . We notice that  $\beta_{\alpha, (1, 0, \dots)} = \alpha$ .

Let us suppose that the lemma is true for a fixed  $p$ . By induction hypothesis, we have

$$\begin{aligned} (f^\alpha)^{(p+1)} &= \left( \sum_{s \in \mathcal{S}_p} \beta_{\alpha, s} f^{\alpha-|s|} \prod_{k=1}^p (f^{(k)})^{s_k} \right)' \\ &= g_1 + g_2 \end{aligned}$$

where

$$\begin{aligned} g_1 &= \sum_{s \in \mathcal{S}_p} \beta_{\alpha, s} (f^{\alpha-|s|})' \left( \prod_{k=1}^p (f^{(k)})^{s_k} \right) \\ g_2 &= \sum_{s \in \mathcal{S}_p} \beta_{\alpha, s} f^{\alpha-|s|} \left( \prod_{k=1}^p (f^{(k)})^{s_k} \right)'. \end{aligned}$$

By rewriting each term, we get

$$\begin{aligned} g_1 &= \sum_{s \in \mathcal{S}_p} \beta_{\alpha, s} (\alpha - |s|) f' f^{\alpha-|s|-1} \prod_{k=1}^p (f^{(k)})^{s_k} \\ &= \sum_{\substack{s \in \mathcal{S}_p \\ s' = s + e_1}} \beta_{\alpha, s} (\alpha - |s'| + 1) f^{\alpha-|s'|} \prod_{k=1}^p (f^{(k)})^{s'_k} \\ g_2 &= \sum_{s \in \mathcal{S}_p} \beta_{\alpha, s} f^{\alpha-|s|} \sum_{j=1}^p s_j f^{(j+1)} (f^{(j)})^{s_j-1} \prod_{k \neq j} (f^{(k)})^{s_k} \\ &= \sum_{j=1}^p \left[ \sum_{\substack{s \in \mathcal{S}_p \\ s_j \neq 0 \\ s' = s - e_j + e_{j+1}}} \beta_{\alpha, s' + e_j - e_{j+1}} f^{\alpha-|s'|} (s'_j + 1) \prod_{k=1}^{p+1} (f^{(k)})^{s'_k} \right]. \end{aligned}$$



If  $s$  is in  $\mathcal{S}_p$ , then  $s + e_1 \in \mathcal{S}_{p+1}$  and if moreover  $s_j \neq 0$  then  $s - e_j + e_{j+1} \in \mathcal{S}_{p+1}$ . So the result is proved and the constants  $\beta$  are defined by:  $\beta_{\alpha,(1,0,0,\dots)} = \alpha$ ; and if  $s \in \mathcal{S}_p$  with  $p > 1$ , then

$$\begin{aligned} \beta_{\alpha,s} &= \mathbb{1}_{s_1 \neq 0}(\alpha - |s| + 1)\beta_{\alpha,(s_1-1,s_2,s_3,\dots)} \\ &\quad + \sum_{\substack{2 \leq j \leq p \\ s_j \neq 0}} (s_{j-1} + 1)\beta_{\alpha,(s_1,\dots,s_{j-1},s_{j-1}+1,s_{j-1},s_{j+1},\dots)}. \end{aligned}$$

□

### 3.2. Several models

In [10], the authors gave an  $t^{O(m2^k)}$  bound on the number of distinct real roots of polynomials of the form  $f = \sum_{i=1}^k a_i \prod_{j=1}^m f_j^{\alpha_{i,j}}$ , where the  $f_j$  are polynomials with at most  $t$  monomials. We improve their result in Theorem 12 using our results on the Wronskian from Section 2.

**Lemma 11.** *Let  $M$  be a set of  $T$  monomials and  $f_1, \dots, f_s$  be polynomials whose monomials are in  $M$ . For every formal monomial  $P$  in the  $s^2$  variables  $f_1, f_1', \dots, f_1^{s-1}, f_2, f_2', \dots, f_s^{s-1}$  of degree  $d$  and of order of differentiation  $e$ , the number of monomials in  $x$  of  $P(f_1, f_1', \dots, f_s^{s-1})(x)$  is bounded by  $\binom{d+T-1}{T-1}$ . More precisely, the set of these monomials is included in a set  $E_{d,e}$  of size at most  $\binom{d+T-1}{T-1}$  which does not depend on  $P$ .*

*Proof.* Let  $M^d$  be the set of monomials which are the product of  $d$  not necessarily distinct monomials of  $M$ . The cardinal of this set is bounded by the cardinal of the set of multisets of size  $d$  of elements in  $M$ , that is  $\binom{T+d-1}{T-1}$ . It is easy to see that we can take the set  $E_{d,e}$  defined as the set of monomials of  $x^{-e}M^d$ . Its cardinal is bounded by the cardinality of  $M^d$ . □

**Theorem 12.** *Let  $f = \sum_{i=1}^k a_i \prod_{j=1}^m f_j^{\alpha_{i,j}}$  be a non identically zero function such that each  $f_j$  is a polynomial with at most  $t$  monomials and such that  $a_i \in \mathbb{R}$  and  $\alpha_{i,j} \in \mathbb{N}$ . Then,  $Z_{\mathbb{R}}(f) \leq 4ktm + 4(e(1+t))^{\frac{mk^2}{2}} = O(t^{\frac{mk^2}{2}})$ .*

*Moreover, if  $I$  is a real interval such that for all  $j$ ,  $f_j(I) \subseteq ]0, +\infty[$  (which ensures  $f$  is defined on  $I$ ), then the result is still true for real (possibly negative) powers  $\alpha_{i,j}$ , i.e.,  $Z_I(f) \leq 4ktm + 4(e(1+t))^{\frac{mk^2}{2}}$ .*

*Proof.* Let  $N$  an integer such that for all  $i$  and  $j$ , we have  $\alpha_{i,j} + N > 0$ . Let us consider  $\tilde{f} = \sum_{i=1}^k a_i g_i$  where  $g_i = \prod_{j=1}^m f_j^{\alpha_{i,j} + N + k}$ . Note that  $\tilde{f} = f \cdot \prod_{j=1}^m f_j^{N+k}$ . We are going to bound the number of zeros of  $\tilde{f}$ . In both cases (whether  $\alpha_{i,j}$  are integer or real numbers), the functions  $g_i$  are analytic in  $I$ . Furthermore, we can assume without loss of generality that the family

$(g_i)$  is linearly independent. Indeed, if it is not the case, we can consider a basis of the family  $(g_i)$  and write  $\tilde{f}$  in this basis. Then we can suppose that all  $a_i$  are non-zero, otherwise, we remove these terms from the sum. We want to bound the number of zeros of  $W(g_1, \dots, g_s)$  for all  $s \leq k$  to conclude with Theorem 9. We know that for  $1 \leq u, v \leq s$

$$g_u^{(v-1)} = \sum_{\substack{r_1, r_2, \dots, r_m \\ r_1 + \dots + r_m = v-1}} \prod_{j=1}^m \left( f_j^{\alpha_{u,j} + N + k} \right)^{(r_j)}. \quad (4)$$

We use now Lemma 10 and we simplify the notation by writing  $\beta_{u,j,s}$  instead of  $\beta_{\alpha_{u,j} + N + k, s}$ .

$$\begin{aligned} g_u^{(v-1)} &= \sum_{\substack{r_1, r_2, \dots, r_m \\ r_1 + \dots + r_m = v-1}} \prod_{j=1}^m \left[ \sum_{s \in \mathcal{S}_{r_j}} \beta_{u,j,s} f_j^{\alpha_{u,j} + N + k - |s|} \prod_{k=1}^{r_j} \left( f_j^{(k)} \right)^{s_k} \right] \\ &= \left( \prod_{j=1}^m f_j^{\alpha_{u,j} + N} \right) \left( \prod_{j=1}^m f_j^{k-v+1} \right) T_{u,v} \left( \left( f_p^{(q-1)} \right)_{1 \leq p, q \leq s} \right). \end{aligned} \quad (5)$$

with :

$$T_{u,v} \left( \left( f_p^{(q-1)} \right)_{1 \leq p, q \leq s} \right) = \sum_{\substack{r_1, r_2, \dots, r_m \\ r_1 + \dots + r_m = v-1}} \prod_{j=1}^m \left[ \sum_{s \in \mathcal{S}_{r_j}} \beta_{u,j,s} f_j^{v-1-|s|} \prod_{k=1}^{r_j} \left( f_j^{(k)} \right)^{s_k} \right].$$

The polynomial  $T_{u,v}$  is homogeneous of total degree  $(v-1)m$  with respect to the  $s^2$  variables  $\left( f_p^{(q-1)} \right)_{1 \leq p, q \leq s}$  and each of its terms is of differentiation order  $v-1$ .

Then, we notice that, in (5), the first parenthesis does not depend on  $v$  and the second one on  $u$ . We get

$$W(g_1, \dots, g_s) = \left( \prod_{i=1}^s \prod_{j=1}^m f_j^{\alpha_{i,j} + N + k - i + 1} \right) \det \left( \left( T_{u,v} \left( \left( f_p^{(q-1)} \right)_{1 \leq p, q \leq s} \right) \right)_{u, v \leq s} \right).$$

Hence,

$$Z(W(g_1, \dots, g_s)) \leq \left( \sum_{j=1}^m Z(f_j) \right) + Z \left( \det \left( T_{u,v} \left( \left( f_p^{(q-1)} \right)_{1 \leq p, q \leq s} \right) \right) \right). \quad (6)$$

We are now going to bound the number of monomials in  $x$  of  $\det(T_{u,v})$ . We saw that  $T_{u,v}$  is a homogeneous polynomial of degree  $(v-1)m$  with respect to the  $s^2$  variables  $\left( f_p^{(q-1)} \right)_{1 \leq p, q \leq s}$  and of order of differentiation  $v-1$ .

Moreover, as the family  $(g_i)$  is linearly independent and as these functions are analytic, the Wronskian is not identically zero (Lemma 5). So  $\det(T_{u,v})$  is a linear combination, with respect to the variables  $\left(f_p^{(q-1)}\right)_{1 \leq p, q \leq s}$ , of monomials of degree exactly  $\sum_{v=1}^s (v-1)m = m \binom{s}{2}$  and of order of differentiation  $\binom{s}{2}$ . By Lemma 11, the monomials in  $x$  of each term of  $\det(T_{u,v})$  are in the set  $E_{\binom{s}{2}m, \binom{s}{2}}$ . Consequently, the number of monomials in  $x$  of  $\det(T_{u,v})$  is bounded by the cardinal of  $E_{\binom{s}{2}m, \binom{s}{2}}$ , i.e. by  $\binom{m \binom{s}{2} + mt - 1}{mt - 1}$ . Descartes' rule of signs (Lemma 2) gives

$$Z \left( \det_{u,v \leq s} (T_{u,v}) \right) \leq 2 \binom{m \binom{s}{2} + mt - 1}{mt - 1} - 1. \quad (7)$$

We have now all the tools to prove the theorem. We have:

$$Z(f) \leq Z \left( \sum_{i=1}^k a_i g_i \right).$$

By Theorem 9:

$$Z(f) \leq k - 1 + 2 \sum_{s=1}^k Z(W(g_1, \dots, g_s)).$$

Using formula (6):

$$Z(f) \leq k - 1 + 2k \left( \sum_{j=1}^m Z(f_j) \right) + 2 \sum_{s=1}^k Z \left( \det_{u,v \leq s} \left( T_{u,v} \left( (f_p^{(q-1)})_{p,q \leq s} \right) \right) \right).$$

By Descartes' rule,  $\sum_{j=1}^m Z(f_j) \leq (2t-1)m$ . We can then apply (7) to obtain the inequality

$$Z(f) \leq k - 1 + 2k(2t-1)m + 2 \sum_{s=1}^k \left( 2 \binom{m \binom{s}{2} + mt - 1}{mt - 1} - 1 \right).$$

Finally, we use the well known bound:  $\binom{n}{k} \leq (en/k)^k$

$$\begin{aligned} Z(f) &\leq k - 1 + 4ktm - 2km - 2k + 4 + 4 \sum_{s=2}^k \left( e \left( 1 + \frac{mt-1}{m \binom{s}{2}} \right) \right)^{m \binom{s}{2}} \\ &\leq 4ktm + 4(e(1+t))^{\frac{mk^2}{2}}. \end{aligned}$$

□

Using polynomials of small degrees instead of sparse polynomials, the same argument gives a polynomial bound.

**Theorem 13.** *Let  $f = \sum_{i=1}^k a_i \prod_{j=1}^m f_j^{\alpha_{i,j}}$  where  $f$  is not null, the  $f_j$  are of degrees bounded by  $d$  and such that the  $a_i$  are reals and the  $\alpha_{i,j}$  are integers. Then,  $Z_{\mathbb{R}}(f) \leq \frac{1}{3}k^3md + 2kmd + k \sim \frac{k^3md}{3}$*

*Moreover, if  $I$  is a real interval such that for all  $j$ ,  $f_j(I) \subseteq \mathbb{R}^{+**}$  (which ensures  $f$  is defined on  $I$ ), then the result is always true for real powers  $\alpha_{i,j}$ , i.e.  $Z_I(f) \leq \frac{1}{3}k^3md + 2kmd + k \sim \frac{k^3md}{3}$ .*

*Proof.* In the proof of Theorem 12, we saw that  $\det(T_{u,v})$  is a homogeneous polynomial of degree  $m\binom{s}{2}$  in the  $s^2$  variables  $f_1, f_1', \dots, f_s^{(s-1)}$ . So, it is of degree  $md\binom{s}{2}$  in the variable  $x$ . Moreover, as the family  $(g_i)$  is linearly independent and as these functions are analytic, the Wronskian is not identically zero (Lemma 5). In Equation (6), the first term is bounded by  $md$  and the second one by  $md\binom{s}{2}$ . By Theorem 9<sup>1</sup>, the number of zeros of  $\sum_{i=1}^k a_i g_i$  is bounded by  $(\frac{1}{3}k^3md + 2kmd + k)$ .  $\square$

Avendaño studied the case  $f = \sum_{i=1}^k x^{\alpha_i}(ax + b)^{\beta_i}$  where  $\alpha_i$  and  $\beta_i$  are integers [4]. He found an upper bound linear in  $k$  for the number of roots. But he showed also that his bound is false in the case of real powers. We find here a polynomial bound which works also for real powers.

**Corollary 14.** *Let  $f = \sum_{i=1}^k c_i x^{\alpha_i}(ax + b)^{\beta_i}$ . Let  $I$  be the interval  $\{x \in \mathbb{R} | x > 0 \wedge ax + b > 0\}$ . Then  $Z_I(f) \leq \frac{2}{3}k^3 + 5k$ .*

Li, Rojas and Wang [19](Lemma 2) showed that polynomials  $f = \sum_{i=1}^k a_i \prod_{j=1}^m (c_j X + d_j)^{\alpha_{i,j}}$  where coefficients  $a_i, b_j, c_j$  and exponents  $\alpha_{i,j}$  are real, have at more  $m + m^2 + \dots + m^{k-1}$  zeros. Our result improves this bound:

**Corollary 15.** *Let  $f = \sum_{i=1}^k a_i \prod_{j=1}^m (c_j x + d_j)^{\alpha_{i,j}}$  where the coefficients  $a_i, c_j, d_j$  and the exponents  $\alpha_{i,j}$  are real numbers. On the interval  $I = \{x \in \mathbb{R} | \forall j, c_j x + d_j > 0\}$  we have  $Z_I(f) = O(mk^3)$ .*

Another corollary was suggested to us by Maurice Rojas. In [19], Li, Rojas and Wang bound, when it is finite, the number of intersection between a trinomial curve and a  $t$ -sparse curve by  $2^t - 2$ . We improve here their result. The main idea is to make a change of variables and reduce to the case where  $f$  is affine. This may introduce rational exponents, even if the original system has integer coefficients only.

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<sup>1</sup>Using Theorem 8 instead of Theorem 9 would multiply our upper bound by a  $O(k)$  factor.

**Corollary 16.** *Let  $f$  be a non-zero bivariate trinomial and  $g$  be a bivariate  $t$ -sparse polynomial. Then the number of positive intersections between these two curves is infinite or bounded by  $\frac{2}{3}t^3 + 5t$ .*

*Furthermore, the result still holds if the coefficients are real.*

*Proof.* Let  $f(X, Y) = c_1 X^{\gamma_1} Y^{\delta_1} + c_2 X^{\gamma_2} Y^{\delta_2} + c_3 X^{\gamma_3} Y^{\delta_3}$  (with  $c_3 \neq 0$ ) and  $g(X, Y) = \sum_{i=1}^t a_i X^{\alpha_i} Y^{\beta_i}$ . On  $(\mathbb{R}^{+\star})^2$ , the zeros of  $f$  are the same than the zeros of  $c_1 + c_2 X^{\gamma_2 - \gamma_1} Y^{\delta_2 - \delta_1} + c_3 X^{\gamma_3 - \gamma_1} Y^{\delta_3 - \delta_1}$ . Then we can and will assume that  $\gamma_1 = \delta_1 = 0$ .

- First case: there exists  $r \in \mathbb{R}^*$  such that  $(\gamma_3, \delta_3) = r \cdot (\gamma_2, \delta_2)$ . In this case, we can put  $A = X^{\gamma_2} Y^{\delta_2}$ , then on  $\mathbb{R}^{+\star}$ :

$$f = 0 \Leftrightarrow c_1 + c_2 A + c_3 A^r = 0.$$

By Descartes' rule of signs (Lemma 2): the last equation has at most two real positive solutions which will be denoted  $s_1$  and  $s_2$ . So the system is equivalent to the following:

$$\begin{cases} Y = \left(\frac{s_1}{X^{\gamma_2}}\right)^{\frac{1}{\delta_2}} \text{ or } Y = \left(\frac{s_2}{X^{\gamma_2}}\right)^{\frac{1}{\delta_2}} \\ g(X, Y) = 0. \end{cases}$$

We obtain

$$g(X, Y) = \sum_{i=1}^t a_i s_1^{\frac{\beta_i}{\delta_2}} X^{\alpha_i - \frac{\beta_i \gamma_2}{\delta_2}} \text{ or } \sum_{i=1}^t a_i s_2^{\frac{\beta_i}{\delta_2}} X^{\alpha_i - \frac{\beta_i \gamma_2}{\delta_2}}.$$

Again by Lemma 2, the number of positive solutions is infinite or bounded by  $2t - 2 \leq \frac{2}{3}t^3 + 5t$ .

- Second case: the family  $((\gamma_2, \delta_2), (\gamma_3, \delta_3))$  is linearly independent. We can define  $A = X^{\gamma_2} Y^{\delta_2}$  and  $B = X^{\gamma_3} Y^{\delta_3}$ , then on  $\mathbb{R}^{+\star}$ :

$$\begin{aligned} f = 0 &\Leftrightarrow c_1 + c_2 A + c_3 B = 0 \\ &\Leftrightarrow B = -\frac{c_1}{c_3} - \frac{c_2}{c_3} A. \end{aligned}$$

Let us define  $\Delta = \det \begin{vmatrix} \gamma_2 & \gamma_3 \\ \delta_2 & \delta_3 \end{vmatrix} \neq 0$ . Then  $X = A^{\frac{\delta_3}{\Delta}} B^{-\frac{\delta_2}{\Delta}}$  and  $Y = A^{-\frac{\gamma_3}{\Delta}} B^{\frac{\gamma_2}{\Delta}}$ . Hence,

$$g(X, Y) = 0 \Leftrightarrow \sum_{i=1}^t a_i A^{\frac{\alpha_i \delta_3 - \beta_i \gamma_3}{\Delta}} B^{\frac{-\alpha_i \delta_2 + \beta_i \gamma_2}{\Delta}} = 0.$$

The number of solutions corresponds to the number of roots of the polynomial:

$$\sum_{i=1}^t a_i A^{\frac{\alpha_i \delta_3 - \beta_i \gamma_3}{\Delta}} \left(-\frac{c_1}{c_3} - \frac{c_2}{c_3} A\right)^{\frac{-\alpha_i \delta_2 + \beta_i \gamma_2}{\Delta}} = 0.$$

By Corollary 14, the number of positive roots is infinite (if the polynomial is identically zero) or bounded by  $\frac{2}{3}t^3 + 5t$ .

□

#### 4. Some Algorithms for Polynomial Identity Testing

A PIT algorithm takes a polynomial as input, and decides whether the polynomial is identically equal to zero. There are two classical forms for these algorithms: blackbox and whitebox. For the first one, the input is given by a blackbox. And in the second case, the input is given by a circuit. These two types of algorithms are not comparable in our case since, if we have a circuit, we cannot always evaluate it efficiently on an input because the circuit may be of high degree.

##### 4.1. Blackbox PIT algorithms

The bounds on real roots of Theorem 12 immediately give a blackbox PIT algorithm for some families of polynomials.

**Corollary 17.** *Let  $f = \sum_{i=1}^k a_i \prod_{j=1}^m f_j^{\alpha_{i,j}}$  be a function such that  $f_j$  is a polynomial with at most  $t$  monomials and such that  $a_i \in \mathbb{R}$  and  $\alpha_{i,j} \in \mathbb{N}$ . Then, there is a blackbox PIT algorithm which makes only  $1 + 4ktm + 4(e(1+t))^{\frac{mk^2}{2}}$  queries.*

*Proof.* We consider the algorithm which tests if the polynomial outputs zero on the  $1 + 4ktm + 4(e(1+t))^{\frac{mk^2}{2}}$  first integers. By Theorem 12, this set is a hitting set, that is, if the polynomial is not zero, then at least one of these integers will not be a root of the polynomial. □

**Corollary 18.** *Let  $f = \sum_{i=1}^k a_i \prod_{j=1}^m f_j^{\alpha_{i,j}}$  where each  $f_j$  is of degree bounded by  $d$  and such that the  $a_i$  are reals and the  $\alpha_{i,j}$  are integers. Then, there is a blackbox PIT algorithm which makes only  $1 + \frac{1}{3}k^3md + 2kmd + k$  queries.*

*Proof.* We apply Theorem 13. □

##### 4.2. A Whitebox PIT algorithm

Results for whiteboxes are more complicated. They rely on the link between Wronskians and linear independence. In this section, we prove the following proposition:

**Proposition 19.** *Let  $f = \sum_{i=1}^k a_i \prod_{j=1}^m f_j^{\alpha_{i,j}}$  where  $f_j$  is a polynomial with at most  $t$  monomials, the  $a_i$  are integers and the  $\alpha_{i,j}$  are non-negative integers. Let  $C$  be an upper bound on the degrees of the  $f_j$ , on the bit size of their*

coefficients as well as on the bit size of the coefficients  $a_i$  and of the exponents  $\alpha_{i,j}$ . Then, there is a whitebox PIT algorithm which decides if  $f$  is zero in time  $\tilde{O}\left(C2^{4mk^2 \log t}\right)$ .

First, we will need an algorithm for testing if some Wronskians are identically zero or not. We now describe an algorithm which takes as inputs functions  $h_1 = \prod_{j=1}^m (f_j)^{\alpha_{1,j}}, \dots, h_l = \prod_{j=1}^m (f_j)^{\alpha_{l,j}}$  (given by sequences  $(f_j)_{1 \leq j \leq m}$  and  $(\alpha_{i,j})_{1 \leq i \leq l, 1 \leq j \leq m}$ ) and which outputs the leading coefficient of the Wronskian  $W(h_1, \dots, h_l)$  if this determinant is not identically zero, and outputs zero otherwise. We will use the notation  $f(n) = \tilde{O}(g(n))$ . It is a shorthand for  $f(n) = O(g(n) \log^k g(n))$  for some constant  $k$ .

**Proposition 20.** *There is an algorithm which on the input  $(f_j)_{j \leq m}$ ,  $(\alpha_{i,j})_{i \leq l, j \leq m}$  outputs the leading coefficient of the Wronskian of*

$$\left( \prod_{j=1}^m (f_j)^{\alpha_{1,j}}, \dots, \prod_{j=1}^m (f_j)^{\alpha_{l,j}} \right)$$

*if the Wronskian is not identically zero and which outputs zero otherwise. This algorithm runs in time  $\tilde{O}\left(C2^{4ml^2 \log t}\right)$ .*

*Proof.* As in the proof of Theorem 12, we in fact compute the Wronskian of  $(g_1, \dots, g_l)$  where  $g_i = \prod_{j=1}^m (f_j)^{\alpha_{i,j} + l}$ . To get the correct leading coefficient, we can just notice that:

$$W(g_1, \dots, g_l) = W(h_1, \dots, h_l) \prod_{j=1}^m (f_j)^{l^2}.$$

Hence, we want to compute the Wronskian of  $(g_1, \dots, g_l)$ . Again as in the proof of Theorem 12, we factorize each column  $u$  by  $\prod_{j=1}^m (f_j)^{\alpha_{u,j}}$  and each row  $v$  by  $\prod_{j=1}^m (f_j)^{l-v+1}$ . We will denote the resulting matrix by  $M$ . The entries of this matrix are polynomials.

According to Lemma 29 in Appendix Appendix A, we can compute the expanded polynomial in one cell  $(v, u)$  of  $M$  in time  $\tilde{O}(2^{vm} t^{mv} v^m C \log l)$ .

Then, computing all entries of the matrix which is of size  $(l \times l)$  needs  $\tilde{O}(2^{lm} t^{ml} l^m C)$  operations. Next, we have to compute the determinant of this matrix. We are going to compute this determinant directly by expanding it as a sum of  $l!$  products. This computation takes time  $\tilde{O}\left(C2^{4ml^2 \log t}\right)$  by Lemma 30 in Appendix Appendix A.

If the determinant is zero, it means that the Wronskian is zero, then the algorithm outputs zero. Otherwise, for computing the leading coefficient, we have to multiply the coefficient we got by the leading coefficient of  $\frac{(\prod_{u=1}^l \prod_{j=1}^m (f_j)^{\alpha_{u,j}}) (\prod_{v=1}^l \prod_{j=1}^m (f_j)^{l-v+1})}{\prod_{j=1}^m (f_j)^{l^2}}$ . This operation takes  $\tilde{O}(Cml(C+l))$

operations since we can compute the product of  $n$  integers of size  $s$  in time  $\tilde{O}(ns)$ .  $\square$

Second, we will also need the following algorithm: if  $W(h_1, \dots, h_l) \neq 0$  and  $W(h_1, \dots, h_{l+1}) = 0$  then find  $a_1, \dots, a_{l+1}$  such that  $a_1 h_1 + \dots + a_l h_l = h_{l+1}$  (these constants exist according to Lemma 5). So for each  $i \in \mathbb{N}$ ,  $a_1 h_1^{(i)} + \dots + a_l h_l^{(i)} = h_{l+1}^{(i)}$ . We can compute the  $a_j$  using Cramer's formula. As a result, for each  $1 \leq j \leq l$  we have:

$$a_j = \frac{\begin{vmatrix} h_1 & \cdots & h_{j-1} & h_{l+1} & h_{j+1} & \cdots & h_l \\ h'_1 & \cdots & h'_{j-1} & h'_{l+1} & h'_{j+1} & \cdots & h'_l \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1^{(l-1)} & \cdots & h_{j-1}^{(l-1)} & h_{l+1}^{(l-1)} & h_{j+1}^{(l-1)} & \cdots & h_l^{(l-1)} \end{vmatrix}}{W(h_1, \dots, h_l)} \\ = \frac{lc(W(h_1, \dots, h_{j-1}, h_{l+1}, h_{j+1}, \dots, h_l))}{lc(W(h_1, \dots, h_l))}$$

where  $lc(W(h_1, \dots, h_l))$  is the leading coefficient of the Wronskian for the family  $(h_1, \dots, h_l)$ . The previous algorithm (Proposition 20) computes these coefficients, so we can compute the  $(a_j)$  in time  $\tilde{O}\left(C2^{4ml^2 \log t}\right)$ .

Finally, with such algorithms, we just need to go from  $a_1 h_1$  to  $a_1 h_1 + \dots + a_k h_k$ . Each time we add a  $h_i$ , either it is linearly independent and we add it to the current basis or it is dependent, and we write it in the current basis. At the end,  $a_1 h_1 + \dots + a_k h_k$  is expressed as a linear combination of basis functions. We just have to check if all coefficients are zero to conclude if this function is identically zero or not. This completes the proof of Proposition 19.

## 5. An improved upper bound

In this section, we give a proof of Theorem 9. First, we point out that Voorhoeve and van der Poorten's paper [28] contains a result similar to Theorem 8, except that all zeros are counted with multiplicity.

**Theorem 21.** [Voorhoeve and van der Poorten, 1975] *Let  $f_1, \dots, f_k$  be real analytic functions over an interval  $I$ . Then,*

$$N(f_1 + \dots + f_k) \leq k - 1 + \sum_{j=1}^{k-2} N(W_j) + \sum_{j=1}^k N(W_j)$$

where  $N(f)$  is the number of roots of  $f$  on  $I$  counted with multiplicities.

This result immediately implies Theorem 7 in the analytic case. However in our applications we will have to not consider the multiplicity of roots.



Indeed, the bounds in Theorem 12 and 13 do not depend on the exponents  $\alpha_{i,j}$ . If we counted multiplicities, the resulting bound would depend on the  $\alpha_{i,j}$ . Using some ideas of the proof of this theorem, we can nevertheless improve equation (2).

We will denote  $W_i = W(f_1, f_2, \dots, f_i)$  for  $i \geq 1$  when the family  $(f_1, \dots, f_i)$  is clear from the context. Finally, we define  $W_0 = 1$ .

In addition to Lemma 5, several connections between the Wronskian and the linear combination of the functions are known. We will use a result of Frobenius [9, 24, 20]:

**Lemma 22.** *Let  $f_i$  a family of analytic functions. Let  $R_i$  be the family of functions defined by:*

$$R_0 = f_1 + \dots + f_k$$

$$R_{i+1} = \frac{W_{i+1}^2}{W_i} \left( \frac{R_i}{W_{i+1}} \right)'.$$

*Then the functions  $R_i$  are analytic and  $R_{k-1} = W_k$ .*

*Proof of Theorem 9.* Let  $R_i$  be the family of analytic functions defined by:

$$R_0 = f_1 + \dots + f_k$$

$$R_{i+1} = \frac{W_{i+1}^2}{W_i} \left( \frac{R_i}{W_{i+1}} \right)'.$$

We will prove by induction that for all  $1 \leq i \leq k-1$ , the analytic function  $R_i$  has at least  $Z(f_1 + \dots + f_k) - i - Z(W_i) - 2 \sum_{j=1}^{i-1} Z(W_j)$  roots on  $I$ . That yields the theorem with  $i = k-1$  and Lemma 22.

If  $i = 0$ , then  $R_0 = f_1 + \dots + f_k$  and  $R_0$  has exactly (and so at least)  $Z(f_1 + \dots + f_k)$  zeros.

Suppose that the property is verified for a particular  $i$  such that  $i \leq k-2$ . We will denote  $m_x(F)$  the multiplicity of the root  $x$  in  $F$  for all  $x \in \mathbb{R}$  and analytic function  $F$ . We define four values:

- $Z_i^+$  is the number of  $x \in \mathbb{R}$  such that  $m_x(R_i) > m_x(W_{i+1}) > 0$ .
- $Z_i^-$  is the number of  $x \in \mathbb{R}$  such that  $m_x(R_i) = m_x(W_{i+1}) > 0$ .
- $Z_i^{->}$  is the number of  $x \in \mathbb{R}$  such that  $m_x(W_{i+1}) > m_x(R_i) > 0$ .
- $Z_i^{-0}$  is the number of  $x \in \mathbb{R}$  such that  $m_x(W_{i+1}) > 0 = m_x(R_i)$ .

We have:  $Z(W_{i+1}) = Z_i^+ + Z_i^- + Z_i^{-0} + Z_i^{->}$ .

We know by Lemma 22 that  $R_i$  is indeed analytic. Then by induction hypothesis, the fraction  $\frac{R_i}{W_{i+1}}$  has at least

$$Z(f_1 + \dots + f_k) - i - Z(W_i) - 2 \left( \sum_{j=1}^{i-1} Z(W_j) \right) - Z_i^- - Z_i^{->}$$

roots and at most  $Z_i^{->} + Z_i^{-0}$  poles. By Rolle's Theorem, the number of zeros of  $\left(\frac{R_i}{W_{i+1}}\right)'$  is at least

$$\left[ Z(f_1 + \dots + f_k) - i - Z(W_i) - 2 \left( \sum_{j=1}^{i-1} Z(W_j) \right) - Z_i^- - Z_i^{->} \right] - [Z_i^{->} + Z_i^{-0} + 1].$$

So, the number of zeros of  $R_{i+1} = \frac{W_{i+1}^2}{W_i} \left(\frac{R_i}{W_{i+1}}\right)'$  is at least

$$\left[ Z(f_1 + \dots + f_k) - i - Z(W_i) - 2 \left( \sum_{j=1}^{i+1} Z(W_j) \right) - Z_i^- - Z_i^{->} \right] - [Z_i^{->} + Z_i^{-0} + 1] + Z_i^{->} - Z(W_i).$$

We used here that if  $x$  is such that  $0 < m_x(R_i) < m_x(W_{i+1})$  then  $-m_x(W_{i+1}) < m_x(R_i) - m_x(W_{i+1}) < 0$  and so  $m_x\left(W_{i+1}^2 \left(\frac{R_i}{W_{i+1}}\right)'\right) \geq m_x(W_{i+1}) - 1 > 0$ . Hence

$$Z(R_{i+1}) \geq Z(f_1 + \dots + f_k) - (i+1) - Z(W_{i+1}) - 2 \left( \sum_{j=1}^i Z(W_j) \right).$$

□

## 6. Optimality of Theorem 8

Recall that in Theorem 8, it was proved that

$$Z(a_1 f_1 + \dots + a_k f_k) \leq (1 + |\Upsilon|)k$$

with  $\Upsilon = \bigcup_{1 \leq i \leq k} Z(W(f_1, \dots, f_i))$ . It will be shown in Theorem 26 that this theorem is quite optimal in the sense that for arbitrarily large values of  $|\Upsilon|$  and  $k$ , we can find functions  $f_1, \dots, f_k$  and coefficients  $a_1, \dots, a_k$  such that

$$Z(a_1 f_1 + \dots + a_k f_k) \geq (1 + |\Upsilon|)(k - 1).$$

We begin by proving a technical lemma.

**Lemma 23.** *Let  $f$  be a non-constant polynomial. There exists, for  $0 \leq i \leq q$ , rational functions  $F_{i,q}$  such that if we define the function  $h_{p,i} = \frac{p!}{(p-i)!} (f')^i f^{p-i}$ , then we have:*

1. for all  $q \geq 0$ , we have  $F_{q,q} = 1$

2. for all  $0 \leq i \leq q$ , the rational function  $[(f')^q F_{i,q}]$  is a polynomial
3. For all  $q \geq 0$ , for all  $p \geq 1$  we have:  $(f^p)^{(q)} = \sum_{i=0}^q h_{p,i} F_{i,q}$ .

The main point is that  $F_{i,q}$  does not depend on  $p$ .

*Proof.* We define  $F_{i,q}$  by induction on  $q$ . If  $q = 0$ , let us define  $F_{0,0} = 1$ . Then, we have  $f^p = h_{p,0}$  and  $[(f')^0 F_{0,0}] = 1$ .

We suppose now that  $F_{i,q'}$  are defined for all  $i, q'$  such that  $0 \leq i \leq q' \leq q$ . Let us define  $F_{i,q+1}$ . We have:

$$\begin{aligned} (h_{p,i})' &= \left[ \frac{p!}{(p-i)!} (f')^i f^{p-i} \right]' \\ &= \frac{p!}{(p-i)!} [(p-i)(f')^{i+1} f^{p-i-1} + i f'' (f')^{i-1} f^{p-i}] \\ &= h_{p,i+1} + \left( i \frac{f''}{f'} \right) h_{p,i}. \end{aligned}$$

So,

$$\begin{aligned} (f^p)^{(q+1)} &= \left( \sum_{i=0}^q h_{p,i} F_{i,q} \right)' \\ &= \sum_{i=0}^q \left[ h_{p,i} (F_{i,q})' + \left( h_{p,i+1} + \left( i \frac{f''}{f'} \right) h_{p,i} \right) F_{i,q} \right] \\ &= h_{p,0} (F_{0,q})' + \left[ \sum_{i=1}^q h_{p,i} \left( (F_{i,q})' + F_{i,q} \left( i \frac{f''}{f'} \right) + F_{i-1,q} \right) \right] \\ &\quad + h_{p,q+1} F_{q,q}. \end{aligned}$$

We can then define

$$\begin{aligned} F_{0,q+1} &= (F_{0,q})' \\ \text{for } 1 \leq i \leq q, \quad F_{i,q+1} &= (F_{i,q})' + F_{i,q} \left( i \frac{f''}{f'} \right) + F_{i-1,q} \\ \text{and } F_{q+1,q+1} &= F_{q,q} = 1. \end{aligned}$$

Then, we have (1) and (3) by construction. Finally, (2) is verified, since by induction hypothesis:

$$(f')^{q+1} F'_{i,q} = [f' ((f')^q F_{i,q})]' - (q+1) f' ((f')^q F_{i,q}) \text{ is a polynomial.}$$

□

We are going to show that the zeros of  $W(f^{\alpha_1+k}, \dots, f^{\alpha_s+k}(x))$  are either zeros of  $f$  or zeros of  $f'$ .

**Lemma 24.** *Let  $f$  be an analytic non-constant function in an interval  $I$  and  $\alpha_0, \dots, \alpha_k$  be  $k$  pairwise distinct integers (with  $k \geq 1$ ). Then*

$$\left\{ x \in I \mid \exists s \leq k, W \left( f^{\alpha_1+k}, \dots, f^{\alpha_s+k} \right) (x) = 0 \right\} \subseteq \{ x \in I \mid (ff')(x) = 0 \}.$$

*Proof.* Let us consider  $f^{\alpha_1+k}, \dots, f^{\alpha_k+k}$ . First suppose that this family is linearly dependent. This means that there exist some constants  $(a_1, \dots, a_k) \in \mathbb{R}^k \setminus \{(0, 0, \dots, 0)\}$  such that

$$\sum_{i=1}^k a_i f^{\alpha_i+k} = 0 \text{ on } I. \quad (8)$$

But the integers  $(\alpha_i + k)$  are all distinct so the polynomial  $P(Y) = \sum_{i=1}^k a_i Y^{\alpha_i+k}$  is not zero. Hence  $P(Y)$  has a finite number of roots. By (8),  $\text{Im}(f)$  is included in the (finite) set of roots of  $P$ . Nevertheless, as  $f$  is continuous, by the intermediate value theorem  $\text{Im}(f)$  is a real interval. So  $\text{Im}(f)$  is a singleton. This contradicts the hypothesis that  $f$  is not constant. Therefore, this family is linearly independent.

Let  $\Delta$  be the matrix defined by  $\Delta_{i,j} = (f^{\alpha_i+k})^{(j-1)}$ . By Lemma 23, we get  $\Delta_{i,j} = \sum_{l=0}^{j-1} h_{\alpha_i+k,l} F_{l,j-1}$ , i.e. in terms of matrix product:

$$\Delta = [h_{\alpha_i+k,l-1}]_{1 \leq i,l \leq s} [F_{l-1,j-1} \mathbb{1}_{l \leq j}]_{1 \leq l,j \leq s}.$$

The second matrix of the product is an upper triangular matrix whose entries on the main diagonal are 1 and so its determinant is 1. Then,

$$\det \left( (\Delta_{i,j})_{1 \leq i,j \leq s} \right) = \det \left( (h_{\alpha_i+k,j-1})_{1 \leq i,j \leq s} \right).$$

Finally,  $h_{\alpha_i+k,j-1} = \frac{(\alpha_i+k)!}{(\alpha_i+k-j+1)!} [f^{\alpha_i}] \left[ (f')^{j-1} f^{k-j+1} \right]$ . The first bracket does not depend on  $j$  and the second one on  $i$ . Consequently,

$$\det (h_{\alpha_i+k,j-1}) = \left[ f^{\sum_{l=1}^s \alpha_l} \right] \left[ (f')^{\binom{s}{2}} f^{(k-s+1)s + \binom{s}{2}} \right] \det \left( \frac{(\alpha_i+k)!}{(\alpha_i+k-j+1)!} \right).$$

Then, for all  $x$  in  $I$ :

$$\begin{aligned} W \left( f^{\alpha_1+k}, \dots, f^{\alpha_s+k} \right) (x) = 0 &\Leftrightarrow \det (h_{\alpha_i+k,j-1}) (x) = 0 \quad (9) \\ \Rightarrow \left\{ f(x) = 0 \text{ or } f'(x) = 0 \text{ or } \det \left( \frac{(\alpha_i+k)!}{(\alpha_i+k-j+1)!} \right) = 0 \right\}. \end{aligned}$$

If  $\det \left( \frac{(\alpha_i+k)!}{(\alpha_i+k-j+1)!} \right) = 0$ , as it does not depend on  $x$ , the function  $\det (h_{\alpha_i+k,j-1})$  vanishes for all  $x$  and so the Wronskian is zero over  $I$ . But

as the functions  $f^{\alpha_i+k}$  are analytic, they would be linearly dependent by Lemma 5. That contradicts the hypothesis. Consequently,

$$\left\{x \in I \mid \exists s \leq k, W\left(f^{\alpha_1+k}, \dots, f^{\alpha_s+k}\right) = 0\right\} \subseteq \left\{x \in I \mid (ff')(x) = 0\right\}.$$

□

As a byproduct, we give another proof of the weak version of Descartes' rule of signs (Lemma 2). Let  $g = \sum_{i=1}^k a_i x_i^{\alpha_i}$ . We need to show that the number of distinct real roots is bounded by  $2k - 1$ . We can use the result of Lemma 24 with  $f(x) = x$ . In this case  $g = \sum_{i=1}^k a_i x^{\alpha_i}$ . We get

$$\Upsilon = \left\{x \in I \mid \exists s \leq k, W\left(f^{\alpha_1+k}, \dots, f^{\alpha_s+k}\right) = \{0\}\right\} \subseteq \left\{x \in I \mid (ff')(x) = 0\right\} \\ \subseteq \{0\}$$

So, Theorem 8 gives:  $Z(f) \leq 2k - 1$ . A similar proof of Lemma 1 appears in [23] (Part V, exercise 90).

In Lemma 24, it can be seen that the converse of the implication (9) is true as soon as  $f'$  really appears as a factor of  $\det\left((h_{\alpha_i+k, j-1})_{1 \leq i, j \leq s}\right)$ . It is the case when  $\binom{s}{2}$  is different from zero, that is to say when  $s \geq 2$ . This implies the following result.

**Lemma 25.** *Let  $f$  be an analytic non-constant function in an interval  $I$  and  $\alpha_0, \dots, \alpha_k$  be  $k$  pairwise distinct integers with the condition  $k \geq 2$ . Then*

$$\left\{x \in I \mid \exists s \leq k, W\left(f^{\alpha_1+k}, \dots, f^{\alpha_s+k}\right)(x) = 0\right\} = \left\{x \in I \mid (ff')(x) = 0\right\}.$$

We have now all the tools to prove the main result of the section: the optimality of Theorem 8.

**Theorem 26.** *Let  $\Upsilon = \{x \in I \mid \exists i \leq k, W(f_1, \dots, f_i)(x) = 0\}$  as in Theorem 8. For every  $k$  and  $p$ , there exists a function  $g = \sum_{i=1}^k a_i f^{\alpha_i}$  such that  $\alpha_i$  are positive integers,  $f$  is a polynomial such that  $|\Upsilon| \geq p$  and such that  $g$  has at least  $(1 + |\Upsilon|)(k - 1) + Z(f)$  zeros.*

*Proof.* Let  $h = x \prod_{i=1}^{k-1} (x^2 - i^2)$ . This polynomial is  $k$ -sparse and has  $2k - 1$  distinct real roots:  $-k + 1 < \dots < -1 < 0 < 1 < k - 1$ .

$$\text{Let } f = k \prod_{i=1}^{1 + \left\lceil \frac{p+1}{2} \right\rceil} (x - 2i).$$

Then, we just have to verify that  $g = h \circ f$  has the required properties.

We have  $g(x) = 0$  if and only if  $f(x) \in [-k + 1, k - 1] \cap \mathbb{Z}$ . But for  $y$  an odd integer, we have  $|f(y)| > k - 1$  and for  $y$  an even integer between 2 and  $2 + 2 \left\lceil \frac{p+1}{2} \right\rceil$ , we have  $f(y) = 0$ . By the Intermediate Value Theorem,  $g$  has at least  $k - 1$  zeros over each interval  $(n, n + 1)$  with  $1 \leq n \leq 2 + 2 \left\lceil \frac{p+1}{2} \right\rceil$ . So

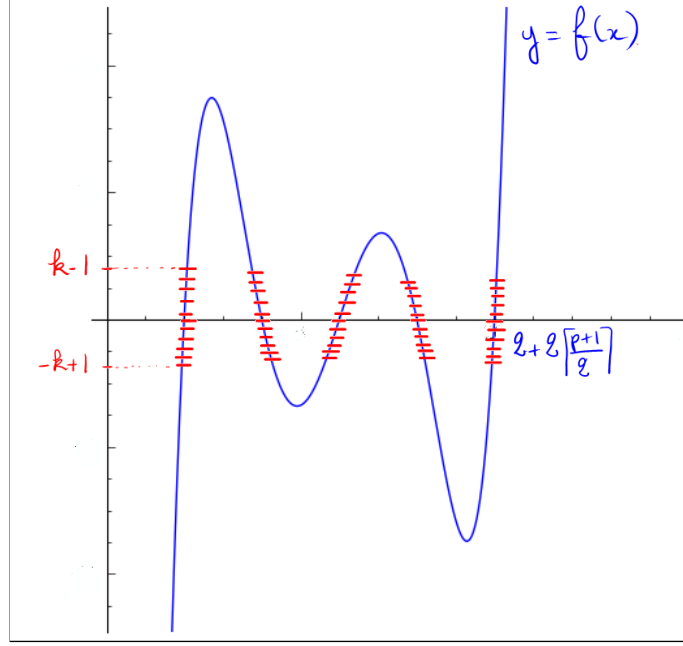


Figure 1: roots of  $g = h \circ f$  in the proof of Theorem 26

$$\begin{aligned}
 Z(g) &= 2 \left( 1 + \left\lceil \frac{p+1}{2} \right\rceil \right) (k-1) + \left( 1 + \left\lceil \frac{p+1}{2} \right\rceil \right) \\
 &= (2k-1) \left( 1 + \left\lceil \frac{p+1}{2} \right\rceil \right).
 \end{aligned} \tag{10}$$

Rolle's Theorem ensures that for two roots of  $f$ , there exists a root of  $f'$  which is strictly between both roots of  $f$ . Hence,  $Z(ff') \geq 2Z(f) - 1 = 1 + 2 \left\lceil \frac{p+1}{2} \right\rceil$ . Considering the degree of  $ff'$ , we find  $Z(ff') = 1 + 2 \left\lceil \frac{p+1}{2} \right\rceil$ .

Besides,  $f$  is not constant so by Lemma 25,  $|\Upsilon| = Z(ff')$ . Hence,

$$|\Upsilon| = Z(ff') = 1 + 2 \left\lceil \frac{p+1}{2} \right\rceil. \tag{11}$$

We can verify that the hypothesis  $|\Upsilon| \geq p$  is true. Finally, equations (10) and (11) show that  $Z(g) \geq (|\Upsilon| + 1)(k-1) + Z(f)$ .  $\square$

In the proof of Theorem 26, the roots of all  $W(f_1, \dots, f_i)$  are included in the zeros of  $W(f_1, \dots, f_k)$ . So, it could be possible to improve both Theorem 8 and Theorem 9 by proving the following proposition.

**Open question.** *Let  $f_1, \dots, f_k$  be analytic functions on an infinite interval*

$I$  and  $a_1, \dots, a_k$  be non-zero real constants. Is the inequality

$$Z(a_1 f_1 + \dots + a_k f_k) \leq k - 1 + \sum_{i=1}^k Z(W(f_1, \dots, f_i))$$

always true?

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Saugata Basu pointed out to one of the authors (P.K.) that Pólya and Szegő's book could be relevant and Maurice Rojas pointed out Voorhoeve's papers and the fact that Theorem 13 implies Corollary 16. In the introduction, we mentioned the problem of bounding the number of real solutions to an equation of the form  $fg + 1 = 0$ . This question was raised by Arkadev Chattopadhyay.

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## Appendix A. Proof of lemmas for Section 4

In this section we prove two lemmas (29 and 30) needed for the proof of Proposition 20. These proofs are elementary but somewhat technical.

In the following, we will compute additions of  $n$  integers of size  $s$  in time  $O(n(s + \log n))$  and products of  $n$  integers of size  $s$  in time  $\tilde{O}(ns)$ .

We begin by bounding the complexity of expanding a product of sparse polynomials.

**Lemma 27.** *We consider a product of  $\mu$   $\tau$ -sparse polynomials  $P_1, \dots, P_\mu$  of degrees bounded by  $\gamma$  with integer coefficients of size bounded by  $\gamma$ . This product can be expanded in time  $\tilde{O}(\tau^\mu \gamma)$ . Moreover, the size of new coefficients is bounded by  $\mu\gamma + \mu \log \tau$ .*

*Proof.* For expanding such a product, we compute one by one each monomial of the sum and we store the coefficients of these  $\tau^\mu$  new monomials. For computing one coefficient, we have three things to do. We have to compute its degree (sum of  $\mu$  integers of size  $\gamma$ ) in time  $O(\mu(\gamma + \log \mu))$ , its coefficient (product of  $\mu$  integers of size  $\gamma$ ) in time  $\tilde{O}(\mu\gamma)$  and we add together the monomials with the same exponent. At the end, at most  $\tau^\mu$  coefficients will be added together to form a given monomial, so the size of the coefficient is bounded by  $\mu\gamma + \mu \log \tau$ . Hence, as we add coefficients one by one, at each step we have to add an integer of size  $\mu\gamma$  by one of size at most  $\mu\gamma + \mu \log \tau$ . Each term of the sum takes time

$$\begin{aligned} & \tilde{O}(\mu(\gamma + \log \mu) + \mu\gamma + (\mu\gamma + \mu \log \tau)) \\ &= \tilde{O}(\mu\gamma + \mu \log(\mu\tau)). \end{aligned}$$

Therefore, computing all coefficients takes time

$$\begin{aligned} & \tilde{O}(\tau^\mu(\mu\gamma + \mu \log(\mu\tau))) \\ &= \tilde{O}(\tau^\mu \gamma). \end{aligned}$$

□

Theorem 12 uses some constants  $\beta_{\alpha,s}$  which have been defined in Lemma 10. We will need to compute them.

**Lemma 28.** *For every  $p$  in  $\mathbb{N}$ , we have  $|\mathcal{S}_p| \leq 2^{p-1}$ . For every  $\alpha, p$  in  $\mathbb{N}$  and for every  $l$  in  $\mathcal{S}_p$ ,  $0 \leq \beta_{\alpha,s} \leq (p^2 + \alpha)^p$ .*

*Furthermore, for every  $\alpha, p$  in  $\mathbb{N}$  we can compute all  $\beta_{\alpha,s}$  with  $s \in \mathcal{S}_q$  and  $q \leq p$  in time  $\tilde{O}(2^p \log \alpha)$ .*

*Proof.* We showed in the proof of Lemma 10 that  $\beta_{\alpha,(1,0,0,\dots)} = \alpha$  and if  $s \in \mathcal{S}_p$  with  $p \neq 1$ , then

$$\begin{aligned} \beta_{\alpha,s} &= \mathbb{1}_{s_1 \neq 0}(\alpha - |s| + 1)\beta_{\alpha,(s_1-1,s_2,s_3,\dots)} \\ &\quad + \sum_{\substack{j: 2 \leq j \leq p \\ s_j \neq 0}} (s_{j-1} + 1)\beta_{\alpha,(s_1,\dots,s_{j-1},s_{j-1}+1,s_{j-1},s_{j+1},\dots)}. \end{aligned} \quad (\text{A.1})$$

However, in the formula above, the sequences  $(s_1 - 1, s_2, s_3, \dots)$  and  $(s_1, \dots, s_{j-1}, s_{j-1} + 1, s_j - 1, s_{j+1}, \dots)$  fall in  $\mathcal{S}_{p-1}$ . Let us denote  $M_{\alpha,p} = \max_{s \in \mathcal{S}_p} |\beta_{\alpha,s}|$ .

Hence, if  $p \neq 1$ , (A.1) implies,

$$M_{\alpha,p} \leq (p^2 + \alpha)M_{\alpha,p-1}.$$

Since,  $M_{\alpha,1} = \alpha$ , we get by induction  $\beta_{\alpha,s} \leq (p^2 + \alpha)^{p-1}\alpha$ .

For computing these constants, we notice that:

$$\begin{aligned} |\mathcal{S}_{p+1}| &= |\{s \in \mathcal{S}_{p+1} | s_1 \neq 0\}| + |\{s \in \mathcal{S}_{p+1} | s_1 = 0\}| \\ &\leq 2|\mathcal{S}_p|. \end{aligned}$$

The inequality comes from the two surjective functions:

$$\begin{aligned} \mathcal{S}_p &\rightarrow \{s \in \mathcal{S}_{p+1} | s_1 \neq 0\} & \text{and} & & \mathcal{S}_p &\rightarrow \{s \in \mathcal{S}_{p+1} | s_1 = 0\} \\ s &\mapsto (s_1 + 1, s_2, \dots) & & & s &\mapsto (0, s_2, \dots, s_{s_1}, s_{1+s_1} + 1, s_{2+s_1}, \dots). \end{aligned}$$

Hence by induction,  $|\mathcal{S}_p| \leq 2^{p-1}$  and  $|\bigcup_p \mathcal{S}_p| = O(2^p)$ . Then, if  $s \in \mathcal{S}_p$  is fixed, for computing  $\beta_{\alpha,s}$  with (A.1), we need to compute  $p$  products of an integer of size  $p \log(p^2 + \alpha)$  by an integer of size  $\log p$  or  $\log \alpha$  (in time  $\tilde{O}(p \log(p^2 + \alpha))$ ) and a sum of all these products in time  $\tilde{O}(p^2 \log(p^2 + \alpha))$ . Finally computing all constants  $\beta_{\alpha,s}$  with  $s \in \mathcal{S}_q$  and  $q \leq p$  needs time  $\tilde{O}(2^p p^2 \log(p^2 + \alpha))$ . That proves the lemma.  $\square$

We can now prove the two intermediate lemmas of Proposition 20. For the following, we keep the notations of Proposition 20.

**Lemma 29.** *Computing the expanded polynomial of a cell  $(v, u)$  in the matrix  $M$  takes time  $\tilde{O}(2^{vm} t^{mv} v^m C \log l)$ . Coefficients are of size bounded by  $\tilde{O}(mv C \log tl)$  and degrees of size bounded by  $Cmv$ .*

*Proof.* We also keep the notations of Theorem 12. Each cell  $(v, u)$  corresponds to the polynomial

$$\begin{aligned} T_{u,v} \left( (f_p^{(q-1)})_{1 \leq p, q \leq l} \right) &= \sum_{\substack{r_1, r_2, \dots, r_m \\ r_1 + \dots + r_m = v-1}} \prod_{j=1}^m \left[ \sum_{s \in \mathcal{S}_{r_j}} \beta_{u,j,s} f_j^{v-1-|s|} \prod_{k=1}^{r_j} (f_j^{(k)})^{s_k} \right] \\ &= \sum_{\substack{r_1, r_2, \dots, r_m \\ r_1 + \dots + r_m = v-1}} \sum_{\substack{s^1, s^2, \dots, s^m \\ s^i \in \mathcal{S}_{r_i}}} \left[ \left( \prod_{j=1}^m \beta_{u,j,s^j} \right) \left( \prod_{j=1}^m f_j^{v-1-|s^j|} \prod_{k=1}^{r_j} (f_j^{(k)})^{(s^j)_k} \right) \right]. \end{aligned}$$

The first sum is a size at most  $v^m$  and the second one of size bounded by  $2^{vm}$  (Lemma 28). For computing one term, first, we need to compute  $\prod_{j=1}^m \beta_{u,j,s^j} = \prod_{j=1}^m \beta_{\alpha_{u,j}+l,s^j}$  which is a product of  $m$  integers, each one of size  $v \log(v^2 + 2^C + l)$  (since  $\alpha \leq 2^C$ ). It is done in time  $\tilde{O}(mvC \log l)$ .

Now, we want to develop the formula with respect to  $x$ . We saw in the proof of Theorem 12 that each monomial with respect to the  $l^2$  variables  $\left(f_p^{(q-1)}\right)_{1 \leq p, q \leq l}$  is of total degree  $m(v-1)$ . We consider one monomial with respect to  $\left(f_p^{(q-1)}\right)_{1 \leq p, q \leq l}$ . It is a product of  $m(v-1)$   $t$ -sparse polynomials with respect to the variable  $x$ . By Lemma 27, this product (the second parenthesis in the formula) can be expanded in time  $\tilde{O}(t^{m(v-1)}C)$  and coefficients are of size  $Cm(v-1) + m(v-1) \log t$ . Then each coefficient is first, multiplied by the corresponding coefficient  $\prod_{j=1}^m \beta_{u,j,s^j}$  in time

$$\begin{aligned} & \tilde{O}(\max\{Cm(v-1) + m(v-1) \log t, mv \log(v^2 + 2^C + l)\}) \\ & = \tilde{O}(mvC \log tl), \end{aligned}$$

that gives an integer of size at most  $\tilde{O}(mvC \log tl)$ . Second, it is added to the stored coefficient corresponding to the same monomial in time

$$\begin{aligned} & \tilde{O}(mvC \log tl + \log(v^m 2^{vm})) \\ & = \tilde{O}(mvC \log tl). \end{aligned}$$

To conclude, computing the cell takes time

$$\begin{aligned} & \tilde{O}\left(v^m 2^{vm} \left(mvC \log l + \left(t^{m(v-1)}C + t^{m(v-1)}(mvC \log tl + mvC \log tl)\right)\right)\right) \\ & = \tilde{O}\left(2^{vm} v^m t^{m(v-1)} C \log l\right). \end{aligned}$$

That completes the proof of the lemma.  $\square$

**Lemma 30.** *Assume that the entries of  $M$  are given in expanded form (i.e., as sums of monomials). Computing the determinant of  $M$  takes time  $\tilde{O}\left(C 2^{4ml^2} \log t\right)$ .*

*Proof.* For each one of the  $(l!)$  permutations, we expand the corresponding polynomial. Each cell has at most  $2^{ml} l^m t^{ml}$  monomials. Hence, each permutation corresponds to a product of size  $l$  of  $(2^{ml} l^m t^{ml})$ -sparse polynomials. Powers are bounded by  $Cml$  and coefficient sizes are bounded by  $\tilde{O}(mlC \log t)$ . By Lemma 27, each permutation can be computed in time

$$\begin{aligned} & \tilde{O}\left(2^{ml^2} l^{ml} t^{ml^2} mlC \log t\right) \\ & = \tilde{O}\left(2^{3ml^2} \log t C\right) \end{aligned}$$

and size of coefficients is bounded by

$$\begin{aligned} & \tilde{O}(ml^2 C \log t + l \log(2^{ml} l^m t^{ml})) \\ & = \tilde{O}(ml^2 C \log t). \end{aligned}$$

For computing the whole determinant, we compute permutations one-by-one, adding each time new coefficients to the one computed before. This is done in time

$$\begin{aligned} & \tilde{O}\left(l^l \left(2^{3ml^2 \log t} C + 2^{ml^2} l^{ml} t^{ml^2} ml^2 C \log t\right)\right) \\ &= \tilde{O}\left(2^{4ml^2 \log t} C\right). \end{aligned}$$

□