# On Alexander-Conway polynomials of two-bridge links

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**Abstract**

We consider Conway polynomials of two-bridge links as Euler continuant polynomials. As a consequence, we obtain new and elementary proofs of classical Murasugi's 1958 alternating theorem and Hartley's 1979 trapezoidal theorem. We give a modulo 2 congruence for links, which implies the classical Murasugi's 1971 congruence for knots. We also give sharp bounds for the coefficients of Euler continuants and deduce bounds for the Alexander polynomials of two-bridge links. These bounds improve and generalize those of Nakanishi-Suketa'96. We easily obtain some bounds for the roots of the Alexander polynomials of two-bridge links. This is a partial answer to Hoste's conjecture on the roots of Alexander polynomials of alternating knots.

*Keywords:* Euler continuant polynomial, two-bridge link, Conway polynomial, Alexander polynomial
*2010 MSC:* 57M25, 11C08

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## 1. Introduction

In this paper, we consider the Conway polynomial of a two-bridge link as an Euler continuant polynomial. We study the problem of determining whether a given polynomial is the Conway polynomial of a two-bridge link (or knot), or equivalently, if it is an Euler continuant polynomial. For small degrees, this problem can be solved by an exhaustive search of possible two-bridge links. Here, we give necessary conditions on the coefficients of the polynomial, which can be tested for high degree polynomials.

In section 2 we present Euler continuant polynomials and give some properties of their coefficients. We show their relations with the Fibonacci polynomials $f_k$ defined

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by:

$$f_0 = 0, f_1 = 1, f_{n+2}(z) = zf_{n+1}(z) + f_n(z).$$

In section 3, we recall the definitions of two-bridge links and we present the description of the Conway polynomial of a two-bridge link as an extended Euler continuant polynomial. We obtain a characterization of modulo 2 two-bridged Conway polynomials.

**Theorem 3.5.** *Let $\nabla(z) \in \mathbf{Z}[z]$ be the Conway polynomial of a two-bridge link (or knot). There exists a Fibonacci polynomial $f_D(z)$ such that $\nabla(z) \equiv f_D(z) \pmod 2$.*

We give a simple method (Algorithm 3.6) that determines the integer $D$ such that $\nabla(z) \equiv f_D(z) \pmod 2$. This is used to test when $\nabla(z) \equiv 1 \pmod 2$, which is a necessary condition to be a two-bridge Lissajous knot.

In section 4, we find inequalities for the coefficients of the Conway polynomials of two-bridge links denoted by

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k} z^{m-2k}.$$

**Theorem 4.1.** *For $k \geq 0$,*

$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m| \,.$$

*If equality holds for some integer $0 < k < \lfloor \frac{m}{2} \rfloor$, then it holds for all integers $0 \leq k \leq \lfloor \frac{m}{2} \rfloor$. In this case, the link is isotopic to the link $C(2, 2, \ldots, 2)$, or to the torus link $\mathrm{T}(2, m) = C(2, -2, \ldots, (-1)^{m-1}2)$, up to mirror symmetry.*

When $|c_m| \neq 1$, we have the following sharper bounds:

**Theorem 4.4.** *Let $g \geq 1$ be the greatest prime divisor of $c_m$, and let $k \neq 0$. Then*

$$|c_{m-2k}| \leq \left( \binom{m-k-1}{k} + \tfrac{1}{g} \left( \binom{m-k-1}{k-1} - 1 \right) \right) |c_m| + 1.$$

*Equality holds for $C(2g, 2, \ldots, 2)$ and $C(2g, -2, 2, \ldots, (-1)^{m-1}2)$.*

In section 5, we apply our results to the Alexander polynomials. Our modulo 2 congruence of Theorem 3.5 provides a simple proof of a congruence of Murasugi [22] for periodic knots (two-bridge knots have period two). Moreover, we deduce a congruence for the Hosokawa polynomials of two-bridge links (Corollary 5.5). Then, we obtain a simple proof of both the Murasugi alternating theorem [23, 21], and the Hartley trapezoidal theorem [7] (see also [9]) using the trapezoidal property:

**Theorem 4.6.** *Let $K$ be a two-bridge link (or knot). Let*

$$\nabla_K = c_m \Big( \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m-2i+1} \Big), \quad \alpha_0 = 1$$

*be its Conway polynomial written in the Fibonacci basis. Then we have*

1. $\alpha_j \geq 0,\ j = 0, \ldots, \lfloor \frac{m}{2} \rfloor$.
2. *If* $\alpha_i = 0$ *for some* $i > 0$ *then* $\alpha_j = 0$ *for* $j \geq i$.

We conclude this section with bounds for the coefficients of the Alexander poynomial. These bounds improve those of Nakanishi and Suketa for the Alexander polynomials of two-bridge knots (see [24, theorems 2 and 3]). Moreover, they are sharp and hold for any $k$.

We prove that the conditions on Conway coefficients are sharper than the conditions on the Alexander coefficients deduced from them.

In section 6, we conclude our paper with the following convexity conjecture:

**Conjecture 6.2**. *Let* $P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ *be the Alexander polynomial of a two-bridge knot. Then there exists an integer* $k \leq n$ *such that* $(a_0, \ldots, a_k)$ *is convex and* $(a_k, \ldots, a_n)$ *is concave.*

We have verified this conjecture for all two-bridge knots with 20 crossings or fewer.

We also deduce some bounds for the roots of Alexander polynomials of two-bridge links (or knots) from the properties of Euler continuant polynomials. This gives a partial answer to the Hoste conjecture 6.3.

## 2. Extended Euler continuant polynomial

We define the extended Euler continuant polynomial $D_m(b_1, \ldots, b_m)(z)$ for $m \geq 1$ as the determinant of the tridiagonal matrix

$$\begin{pmatrix} b_1 z & -1 & 0 & \ldots & 0 \\ 1 & b_2 z & -1 & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & & -1 \\ 0 & \ldots & 0 & 1 & b_m z \end{pmatrix}. \tag{1}$$

If $D_{-1} = 0$ and $D_0 = 1$, then the polynomials $D_i$ satisfy the recurrence relation

$$D_k = b_k z D_{k-1} + D_{k-2}. \tag{2}$$

When $z = 1$, this is the classical Euler continuant polynomial (see [14]).

When all the $b_i$ are equal to 1, we obtain the Fibonacci polynomials defined by

$$f_0 = 0, f_1 = 1, f_{n+2}(z) = z f_{n+1}(z) + f_n(z),\ n \in \mathbf{Z}. \tag{3}$$

Let us recall some basic facts about Fibonacci polynomials.

**Lemma 2.1.** *For $m \geq 0$:*

$$f_{m+1}(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} z^{m-2k}.$$

Proof. By induction on $m$. The result is clear for $m = 1$ and for $m = 2$. Let us suppose the result true for $m - 1$ and $m$. By induction, the coefficient of $z^{m-2k}$ is $\binom{m-1-k}{k}$ in $zf_m(z)$, and $\binom{m-1-k}{k-1}$ in $f_{m-1}(z)$. Consequently, the coefficient of $z^{m-2k}$ in $f_{m+1}(z)$ is

$$\binom{m-1-k}{k} + \binom{m-1-k}{k-1} = \binom{m-k}{k}.$$

**Remark 2.2.** *This means that the Fibonacci polynomials can be read on the diagonals of Pascal's triangle. When $z = 1$, we recover the classical Lucas identity*

$$F_m = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k},$$

*where $F_m$ are the Fibonacci numbers ($F_0 = 0,\ F_1 = 1, \ldots, F_{n+1} = F_n + F_{n-1}$).*

We shall need the following explicit notation for Euler continuant polynomials:

$$D_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k}(b_1, \ldots, b_m) z^{m-2k}. \tag{4}$$

We obtain some properties of $c_{m-2k}(b_1, \ldots, b_m)$, considered as a polynomial in the $m$ variables $b_1, \ldots, b_m$.

**Proposition 2.3.** *Let $\mathcal{M}$ be the set of all monomials $\dfrac{b_1 \cdots b_m}{\prod_{h=1}^{k} b_{i_h} b_{i_h+1}}$, where $k \neq 0$ and $i_h + 1 < i_{h+1}$. Let $\mathcal{M}_j$ be the subset of all monomials of $\mathcal{M}$ that are relatively prime to $b_j$. Then we have*

1. *The polynomial $c_{m-2k}(b_1, \ldots, b_m)$ is the sum of all monomials of $\mathcal{M}$.*
2. *The set $\mathcal{M}$ has $\binom{m-k}{k}$ elements.*
3. *The monomials of $\mathcal{M}$ do not have a common divisor except 1.*
4. *The number of elements of $\mathcal{M}_j$ is at least $\binom{m-1-k}{k-1}$.*
5. *If $m \geq 4$, then the monomials of $\mathcal{M}_j$ do not have a common divisor except 1.*

Proof. 1. This is a classical property of the Euler continuant (see [14]).

2. This number is $c_{m-2k}(1,1,\ldots,1)$, which is a coefficient of the Fibonacci polynomial

$$f_{m+1}(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k}(1,1,\ldots,1) z^{m-2k} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} z^{m-2k}.$$

3. For every integer $i \le m$, there is an element of $\mathcal{M}$ which is not divisible by $b_i$. Hence the GCD of the elements of $\mathcal{M}$ is 1.
4. Let $1 \le j \le m$ and $\mathbf{b} = (1,\ldots,1,0,1,\ldots,1)$ where $b_j = 0$, and $b_k = 1$ for $k \ne j$. Let us define the polynomials $g_n$, for $n \le m$ by $g_n(z) = D_n(\mathbf{b})(z)$. The number of elements of $\mathcal{M}_j$ is the coefficient $c_{m-2k}(\mathbf{b})$ of $g_m(z)$.

   If $j = 1$, then we have $g_1 = 0$, $g_2 = 1$. Then, an easy induction shows that $g_n = zg_{n-1} + g_{n-2}$ is the Fibonacci polynomial $g_n = f_{n-1}$.

   If $j > 1$, then we have

   $$g_1 = f_2, \ \ldots, \ g_{j-1} = f_j, \ g_j = f_{j-1}, \quad \text{and} \quad g_{n+1} = zg_n + g_{n-1} \quad \text{if} \quad n \ge j.$$

   Let us write $p(z) \succeq q(z)$ when each coefficient of $p$ is greater than or equal to the corresponding coefficient of $q$. We have $f_{k+2} \succeq f_k$, and therefore $g_{j+1} = zf_{j-1} + f_j \succeq zf_{j-1} + f_{j-2} = f_j$. Then a simple induction shows that $g_m \succeq f_{m-1}$, and consequently $c_{m-2k}(\mathbf{b}) \ge \binom{m-1-k}{k-1}$.
5. Since $m \ge 4$, for every $i \ne j$, there is a monomial which is not divisible by $b_i$. Consequently, the GCD of the elements of $\mathcal{M}_j$ is 1.

## 3. Conway polynomials of two-bridge links

A two-bridge knot (or link) admits a diagram in Conway's normal form. This form, denoted by $C(a_1, a_2, \ldots, a_n)$ where $a_i$ are integers, is explained by Figure 1 (see [4], [18, pp. 407-408] or [23, p. 187]).

The number of twists is denoted by the integer $|a_i|$, and the sign of $a_i$ is defined as follows: if $i$ is odd, then the right twist is positive, if $i$ is even, then the right twist is negative. In Figure 1 the $a_i$ are positive (the $a_1$ first twists are right twists).

The two-bridge links are classified by their Schubert fractions (see [25])

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}} = [a_1, \ldots, a_n], \quad \alpha > 0.$$

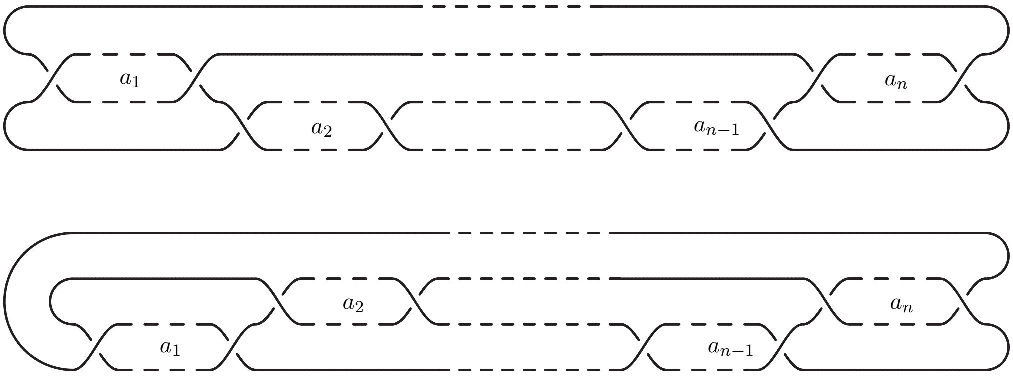


Figure 1: Conway's normal forms ($n$ odd or even)

We shall denote by $S(\dfrac{\alpha}{\beta})$ a two-bridge link with Schubert fraction $\dfrac{\alpha}{\beta}$. The two-bridge links $S(\dfrac{\alpha}{\beta})$ and $S(\dfrac{\alpha'}{\beta'})$ are equivalent if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} (\text{mod } \alpha)$.

The two-bridge link $S(\dfrac{\alpha}{-\beta})$ is the mirror image of the two-bridge link $S(\dfrac{\alpha}{\beta})$. The integer $\alpha$ is odd for a knot, and even for a two-component link.

**Example 3.1.** *The torus link* $\mathrm{T}(2, m) = S(m)$ *is a knot iff* $m$ *is odd. We have the continued fraction expansion of length* $m-1$*:* $\dfrac{m}{m-1} = [2, -2, \ldots, (-1)^{m-2}2]$. *Consequently* $C(2, -2, \ldots, (-1)^{m-2}2)$ *is a diagram of the mirror-image of* $\mathrm{T}(2, m)$

**Remark 3.2.** *When* $\alpha\beta$ *is even, one shows (see [13, p. 26], [15, 11]) that there is a unique continued fraction expansion* $\dfrac{\alpha}{\beta} = [2b_1, 2b_2, \ldots, 2b_m]$, $b_i \in \mathbf{Z} - \{0\}$. *It means that any oriented two-bridge link can be put in the form shown in Figure 2. It will be denoted by* $C(2b_1, 2b_2, \ldots, 2\,b_m)$*, including the indicated orientation. This is a two-component link if and only if* $m$ *is odd.*

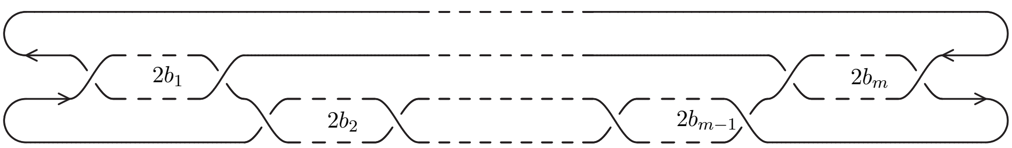


Figure 2: Oriented two-bridge links ($m$ odd)

The Conway polynomial $\nabla_K(z) \in \mathbf{Z}[z]$ is a polynomial invariant of the oriented link $K$ (see [5]). When $K$ is a two-bridge link its Conway polynomial $\nabla_m$ is given by the following method (see [26] and [5, Th. 8.7.4]):

**Theorem 3.3 ([26, 5]).** *Let us consider the oriented two-bridge link*

$$C(2b_1, -2b_2, \ldots, (-1)^{m-1}2b_m).$$

*Its Conway polynomial $\nabla_m(z)$ is the Euler continuant polynomial $D_m(b_1, \ldots, b_m)(z)$.*

**Example 3.4 (The torus links).** *The Conway polynomial of the torus link* $\mathrm{T}(2, m)$ *is the Fibonacci polynomial $f_m(z)$ (see [12, 17]).*

Consequently, the following result gives in fact a characterization of modulo 2 Conway polynomials of two-bridge links.

**Theorem 3.5.** *Let $\nabla(z) \in \mathbf{Z}[z]$ be the Conway polynomial of a two-bridge link (or knot). There exists a Fibonacci polynomial $f_d(z)$ such that $\nabla(z) \equiv f_d(z) \pmod 2$.*

Proof. Let us write $(a, b) \equiv (c, d) \pmod 2$ when $a \equiv c \pmod 2$ and $b \equiv d \pmod 2$. We will show by induction on $m$ that there exist integers $d$ and $e = \pm 1$ such that $(\nabla_{m-1}, \nabla_m) \equiv (f_{d-e}, f_d) \pmod 2$.

The result is true for $m = 0$ as $(\nabla_{-1}, \nabla_0) = (0, 1) = (f_0, f_1)$, that is $d = e = 1$.

Suppose that $(\nabla_{m-1}, \nabla_m) \equiv (f_{d-e}, f_d) \pmod 2$, with $e = \pm 1$ for some $m \geq 0$. Then we have $\nabla_{m+1} = b_{m+1} z \nabla_m + \nabla_{m-1}$.

If $b_{m+1} \equiv 0 \pmod 2$ then $\nabla_{m+1} \equiv \nabla_{m-1} \equiv \nabla_{d-e} \pmod 2$ and $(\nabla_m, \nabla_{m+1}) \equiv (f_d, f_{d-e})$.

If $b_{m+1} \equiv 1 \pmod 2$ then $\nabla_{m+1} \equiv z f_d + f_{d-e} \equiv f_{d+e} \pmod 2$. Consequently $(\nabla_m, \nabla_{m+1}) \equiv (f_d, f_{d+e})$.

We thus deduce a fast algorithm for the determination of the integer $d$ such that $\nabla_m \equiv f_d \pmod 2$, see also [3].

**Algorithm 3.6.** *Let us define the sequences of integers $e_i$ and $d_i$, $i = 0, \ldots, m$, by*

$$e_0 = 1, \; d_0 = 1, \; e_{i+1} = -(-1)^{b_{i+1}} e_i, \; d_{i+1} = d_i + e_{i+1}.$$

*Then we have $\nabla_m(z) \equiv f_d(z) \pmod 2$ where $d = |d_m|$.*

**Remark 3.7.** *Let us consider the two-bridge link* $K = C(2b_1, -2b_2, \ldots, (-1)^{m-1}2b_m)$. *From [28], the crossing number* $N$ *of* $K$ *is* $2\sum_{i=1}^{m}|b_i| - \#\{i, b_ib_{i+1} < 0\} \geq m+1$. *We deduce that one computes* $d$ *such that* $\nabla_K \equiv f_d \pmod 2$ *in* $\mathcal{O}(N)$ *steps. On the other hand, the equality* $m = N - 1$ *is obtained when we consider the torus link* $\mathrm{T}(2, m+1) = S(m+1)$.

Jones, Przytycki, and Lamm proved that the Conway polynomial of a two-bridge Lissajous knot satisfies the congruence $\nabla(z) \equiv 1 \pmod 2$, that is $d = 1$ (see [8, 19]). Using Algorithm 3.6 we deduce the number of two-bridge knots with a Conway polynomial congruent to 1 modulo 2 (see Table 1 and compare [2]).

| Crossing Number | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| Two-bridge | 1 | 1 | 2 | 3 | 7 | 12 | 24 | 45 | 91 | 176 |
| $\nabla(z) \equiv 1$ | 0 | 0 | 1 | 1 | 2 | 4 | 8 | 13 | 26 | 51 |

| Crossing Number | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|
| Two-bridge | 352 | 693 | 1387 | 2752 | 5504 | 10965 | 21931 | 43776 | 87552 | 174933 |
| $\nabla(z) \equiv 1$ | 97 | 185 | 365 | 705 | 1369 | 2675 | 5233 | 10211 | 20011 | 39221 |

Table 1: The number of two-bridge knots, and two-bridge knots with Conway polynomial congruent to 1 modulo 2.

## 4. Inequalities for Conway Polynomials

We shall write the Conway polynomial of a two-bridge link

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k} z^{m-2k}.$$

**Theorem 4.1.** *For* $k \geq 0$,

$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m| .$$

*If equality holds for some integer* $0 < k < \lfloor \frac{m}{2} \rfloor$, *then it holds for all integers* $0 \leq k \leq \lfloor \frac{m}{2} \rfloor$. *In this case, the link is isotopic to the link* $C(2, 2, \ldots, 2)$, *or to the torus link* $\mathrm{T}(2, m) = C(2, -2, \ldots, (-1)^{m-1}2)$, *up to mirror symmetry.*

Proof. Let $K$ be the two-bridge link $C(2b_1, -2b_2, \ldots, (-1)^{m-1}2b_m)$. By Theorem 3.3, $\nabla_K = D_m(b_1, \ldots, b_m)$. By Proposition 2.3, the number of monomials of $c_{m-2k}(b_1, \ldots, b_m)$ is $\binom{m-k}{k}$. The result follows since no monomial is greater than $|c_m| = |b_1 \cdots b_m|$.

If the equality holds for some positive integer $k < \lfloor \frac{m}{2} \rfloor$, we deduce that each monomial $\prod_{h=1}^{k} b_{i_h} b_{i_h+1}$ is equal to the same value $\varepsilon = \pm 1$ and then $|b_i| = 1$, $i = 1, \ldots, m$.

Let $i$ be an integer $1 \le i \le m-2$. Let us consider the $\lfloor \frac{m}{2} \rfloor$ disjoint sets $\{2j-1, 2j\}$, $j = 1, \ldots, \lfloor \frac{m}{2} \rfloor$. There are $\lfloor \frac{m}{2} \rfloor - 2$ member of this family that do not intersect $\{i, i+1, i+2\}$. Let us consider $k-1$ of these and the corresponding product $p = \prod_{h=1}^{k-1} b_{2i_h-1} b_{2i_h}$. From $p \cdot b_i b_{i+1} = p \cdot b_{i+1} b_{i+2} = \varepsilon$, we deduce that $b_i = b_{i+2}$ which concludes the proof.

**Example 4.2.** *The knot* $10_{145}$ *has Conway polynomial* $\nabla = 1 + 5z^2 + z^4$*. We have* $\nabla \equiv f_5 \pmod 2$*, but* $\nabla$ *does not satisfy the condition* $|c_2| \le 3$*, and then* $10_{145}$ *is not a two-bridge knot.*

*The knot* $11n109$ *has Conway polynomial* $1 + 6z^2 + z^4 - z^6$*. It satisfies the bounds of Theorem 4.1:* $|c_2| \le 6$, $|c_4| \le 5$*, but not the equality condition:* $c_2 = 6$ *whereas* $c_4 \ne 5$*. Consequently,* $11n109$ *is not a two-bridge knot.*

We shall use the following lemma, which generalizes the inequality $a + b \le ab + 1$, valid for positive integers (see also [24]).

**Lemma 4.3.** *Let* $p_i$, $i \in \mathcal{S}$ *be relatively prime divisors of* $p = x_1 x_2 \cdots x_m$ *in* $\mathbf{Q}[x_1, \ldots, x_m]$*. Let* $\mathbf{b} = (b_1, \ldots, b_m)$ *be a m-tuple of positive integers. Then*

$$\sum_{i \in \mathcal{S}} p_i(\mathbf{b}) \le \Big(\mathrm{card}(\mathcal{S}) - 1\Big) p(\mathbf{b}) + 1. \tag{5}$$

Proof. We do not suppose the $p_i$ distinct. Let us prove the result by induction on $k = \mathrm{card}(\mathcal{S})$. The result is clear if $k = 1$, we have $p_1 = \pm 1$, and the inequality is $\pm 1 \le 1$.

If all the $p_i = 1$, the result is clear. Otherwise, let $x_h$ be a divisor of some $p_i$.

Let $\mathcal{S}_1 = \{i \in \mathcal{S} : \ x_h | p_i\}$, and $\mathcal{S}_2 = \mathcal{S} - \mathcal{S}_1$. We have $k = k_1 + k_2$, where $k_j = \mathrm{card}(\mathcal{S}_j)$. Let $q_j = \mathrm{GCD}\{p_i, i \in \mathcal{S}_j\}$, then $q_1$ and $q_2$ are coprime, and $q_1 q_2$ is a divisor of $p$.

By induction we obtain for $j = 1, 2$:

$$\sum_{i \in \mathcal{S}_j} p_i(\mathbf{b}) \le q_j(\mathbf{b}) \Big( (k_j - 1) \frac{p(\mathbf{b})}{q_j(\mathbf{b})} + 1 \Big) = (k_j - 1) p(\mathbf{b}) + q_j(\mathbf{b}).$$

Adding these two inequalities we get

$$\begin{aligned} \sum_{i \in \mathcal{S}} p_i(\mathbf{b}) &\le (k_1 + k_2 - 1) p(\mathbf{b}) + q_1(\mathbf{b}) + q_2(\mathbf{b}) - p(\mathbf{b}) \\ &\le (k_1 + k_2 - 1) p(\mathbf{b}) + q_1(\mathbf{b}) q_2(\mathbf{b}) - p(\mathbf{b}) + 1, \end{aligned}$$

which proves the result, since $q_1(\mathbf{b}) q_2(\mathbf{b}) \le p(\mathbf{b})$.

With this lemma we can prove:

**Theorem 4.4.** *Let $\nabla_m(z)=\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor} c_{m-2k}z^{m-2k}$ be the Conway polynomial of a two-bridge link. Let $g\geq 1$ be the greatest prime divisor of $c_m$, and let $k\neq 0$. Then*

$$|c_{m-2k}|\leq \Big(\binom{m-k-1}{k}+\tfrac{1}{g}(\binom{m-k-1}{k-1}-1)\Big)\,|c_m|+1.$$

*Equality holds for the links $C(2g,2,\dots,2)$ and $C(2g,-2,2,\dots,(-1)^{m-1}2)$.*

Proof. Let $K$ be the two-bridge link $C(2b_1,-2b_2,\dots,(-1)^{m-1}2b_m)$. Then $\nabla_K(z)=D_m(\mathbf{b})=\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor} c_{m-2k}z^{m-2k}$ where $\mathbf{b}=(b_1,\dots,b_m)$. If $k=1$, there are $m-1$ monomials in the polynomial $c_{m-2}(\mathbf{b})$, by Proposition 2.3. Then, using Lemma 4.3 and the notation $|\mathbf{b}|=(|b_1|,\dots,|b_m|)$, we get

$$|c_{m-2}|=|c_{m-2}(\mathbf{b})|\leq c_{m-2}(|\mathbf{b}|)\leq(m-2)c_m(|\mathbf{b}|)+1=(m-2)\,|c_m|+1.$$

Now, suppose $k\geq 2$. Let $g$ be the greatest prime divisor of the integer $c_m=b_1\cdots b_m$, and suppose that $g\mid b_j$. Let $N$ be the number of monomials of $c_{m-2k}(b_1,\dots,b_m)$ that are prime to the monomial $b_j$. By Proposition 2.3, these monomials are relatively prime, and $N\geq\binom{m-1-k}{k-1}$.

Using Lemma 4.3 we obtain: $\sum_{p_i\in\mathcal{M}_j}p_i(\mathbf{b})\leq(N-1)\dfrac{|c_m|}{|b_j|}+1$ and then

$$\begin{aligned}
|c_{m-2k}|=\Big|\textstyle\sum_{p_i\in\mathcal{M}}p_i(\mathbf{b})\Big| &\leq \Big(\frac{N-1}{g}+(\binom{m-k}{k}-N)\Big)\,|c_m|+1\\
&= \Big(\binom{m-k}{k}-N(1-\tfrac{1}{g})-\tfrac{1}{g}\Big)\,|c_m|+1\\
&\leq \Big(\binom{m-k}{k}-\binom{m-1-k}{k-1}(1-\tfrac{1}{g})-\tfrac{1}{g}\Big)\,|c_m|+1\\
&= \Big(\binom{m-1-k}{k}+\tfrac{1}{g}(\binom{m-1-k}{k-1}-1)\Big)\,|c_m|+1.
\end{aligned}$$

For $\mathbf{b}=(g,1,\dots,1)$ we obtain $N=\binom{m-1-k}{k-1}$, $c_m=g$, and $c_{m-2k}=g\binom{m-1-k}{k}+\binom{m-1-k}{k-1}$, and equality holds throughout.

For $\mathbf{b}=(g,-1,1,\dots,(-1)^{m-1})$ we get $c_{m-2k}=(-1)^{\lfloor\frac{m}{2}\rfloor+k}\Big(g\binom{m-1-k}{k}+\binom{m-1-k}{k-1}\Big)$.

**Example 4.5.** *The knot $13n3010$ has Conway polynomial $\nabla=1+10\,z^2+4\,z^4-2\,z^6$. It satisfies all conditions of Theorems 4.1 and 3.5 but not those of Theorem 4.4.*

Now, we will express the Conway polynomials in terms of Fibonacci polynomials, and show that their coefficients are alternating.

**Theorem 4.6.** *Let $K$ be a two-bridge link (or knot). Let*

$$\nabla_K = c_m\Big(\sum_{i=0}^{\lfloor \frac{m}{2}\rfloor}(-1)^i\alpha_i f_{m-2i+1}\Big),\quad \alpha_0=1$$

*be its Conway polynomial expressed in the Fibonacci basis. Then we have*

1. $\alpha_j \geq 0,\ j=0,\dots,\lfloor \frac{m}{2}\rfloor$.
2. *If $\alpha_i=0$ for some $i>0$ then $\alpha_j=0$ for $j\geq i$.*

Proof. Let $K$ be the two-bridge link $C(2b_1,-2b_2,\dots,(-1)^{m-1}2b_m)$, then $\nabla_K(z) = D_m(b_1,\dots,b_m)=\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor} c_{m-2k}z^{m-2k}$.

We have $\nabla_0=f_1$, $\nabla_1=b_1f_2$, $\nabla_2=b_1b_2\Big(f_3-(1-\frac{1}{b_1b_2})f_1\Big)$.

Let us show by induction that if

$$\nabla_m = b_1\cdots b_m\Big(\sum_{i=0}^{\lfloor \frac{m}{2}\rfloor}(-1)^i\alpha_i f_{m+1-2i}\Big),\ \nabla_{m-1}=b_1\cdots b_{m-1}\Big(\sum_{i=0}^{\lfloor \frac{m-1}{2}\rfloor}(-1)^i\beta_i f_{m-2i}\Big)$$

then $\alpha_j\geq\beta_j\geq 0$, and if $\alpha_i=0$ for some $i$, then $\alpha_j=0$ for $j\geq i$.

The result is true for $m=2$ from the expressions of $\nabla_1$ and $\nabla_2$. Using $zf_{m+1-2i} = f_{m+2-2i}-f_{m-2i}$ and $\nabla_{m+1}=b_{m+1}z\nabla_m+\nabla_{m-1}$, we deduce that

$$\nabla_{m+1} = b_1\cdots b_{m+1}\Big(\sum_{i=0}^{\lfloor \frac{m+1}{2}\rfloor}(-1)^i\gamma_i f_{m+2-2i}\Big),$$

where $\gamma_0=1$ and

$$\gamma_i \ = \alpha_i+(\alpha_{i-1}-\beta_{i-1})+(1-\tfrac{1}{b_mb_{m+1}})\beta_{i-1},\ i=1,\dots,\lfloor \tfrac{m+1}{2}\rfloor. \tag{6}$$

As $|b_mb_{m+1}|\geq 1$, we deduce by induction that $\gamma_i\geq\alpha_i\geq 0$.

Furthermore, if $\gamma_i=0$, then by Formula (6) $\alpha_i=0$, and then, by induction, $\alpha_j=\beta_j=0$ for $j\geq i$. Finally, by Formula (6), we get $\gamma_j=0$ for $j\geq i$.

## 5. Applications to the Alexander polynomial

In this section, we will see that our necessary conditions on the Euler continuant polynomials imply analogous necessary conditions on both Conway coefficients and Alexander coefficients of two-bridge knots and links. These conditions are improvements of the classical results.

The Conway and the Alexander polynomials of a knot $K$ will be denoted by

$$\nabla_K(z) = 1 + \tilde{c}_1 z^2 + \cdots + \tilde{c}_n z^{2n}$$

and

$$\Delta_K(t) = a_0 - a_1(t + t^{-1}) + \cdots + (-1)^n a_n (t^n + t^{-n}).$$

The Alexander polynomial $\Delta_K(t)$ is deduced from the Conway polynomial:

$$\Delta_K(t) = \nabla_K\Big(t^{1/2} - t^{-1/2}\Big).$$

It is often normalized so that $a_n$ is positive. Thanks to this formula, it is not difficult to deduce the Alexander polynomial from the Conway polynomial. If we use the Fibonacci basis, it is even easier to deduce the Conway polynomial of a knot from its Alexander polynomial.

**Lemma 5.1.** *If* $z = t^{1/2} - t^{-1/2}$, *and* $n \in \mathbf{Z}$ *is an integer, then we have the identity*

$$f_{n+1}(z) + f_{n-1}(z) = (t^{1/2})^n + (-t^{-1/2})^n,$$

*where* $f_k(z)$ *are the Fibonacci polynomials.*

Proof. Let $A = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$ be the (polynomial) Fibonacci matrix. If $z = t^{1/2} - t^{-1/2}$, the eigenvalues of $A$ are $t^{1/2}$ and $-t^{-1/2}$, and consequently $\operatorname{tr} A^n = (t^{1/2})^n + (-t^{-1/2})^n$. On the other hand, we have $A^n = \begin{bmatrix} f_{n+1}(z) & f_n(z) \\ f_n(z) & f_{n-1}(z) \end{bmatrix}$, and then $\operatorname{tr} A^n = f_{n+1}(z) + f_{n-1}(z)$.

From Lemma 5.1, we immediately deduce:

**Proposition 5.2.** *Let the Laurent polynomial* $P(t)$ *be defined by*

$$P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n (t^n + t^{-n}).$$

*We have*

$$P(t) = \sum_{k=0}^{n} (-1)^k (a_k - a_{k+1}) f_{2k+1}(z),$$

*where* $z = t^{1/2} - t^{-1/2}$, *and* $a_{n+1} = 0$.

We deduce a useful formula (by substituting $a_0 = \ldots = a_n = 1$).

$$f_{2n+1}(t^{1/2} - t^{-1/2}) = (t^n + t^{-n}) - (t^{n-1} + t^{1-n}) + \cdots + (-1)^n. \tag{7}$$

Then, we deduce a simple proof of an elegant criterion due to Murasugi ([22, 3])

**Corollary 5.3 (Murasugi (1971)).** *Let* $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ *be the Alexander polynomial of a two-bridge knot. There exists an integer* $k \le n$ *such that* $a_0, a_1, \ldots, a_k$ *are odd, and* $a_{k+1}, \ldots, a_n$ *are even.*

Proof. If $K$ is a two-bridge knot, its Conway polynomial is a modulo 2 Fibonacci polynomial $f_{2k+1}$ by Theorem 3.5. By Proposition 5.2 we have $f_{2k+1}(t^{1/2} - t^{-1/2}) = (t^k + t^{-k}) - (t^{k-1} + t^{1-k}) + \cdots + (-1)^k$, and the result follows.

**Remark 5.4.** *This congruence may be used as a simple criterion to prove that some knots cannot be two-bridge knots. There is a more efficient criterion by Kanenobu [10, 27] using the Jones and Q polynomials.*

We also deduce an analogous result for two-component links (see also [3, p. 186])

**Corollary 5.5 (Modulo 2 Hosokawa polynomials of two-bridge links).** *Let* $\Delta(t) = (t^{1/2} - t^{-1/2})\Big(a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})\Big)$ *be the Alexander polynomial of a two-component two-bridge link. Then all the coefficients* $a_i$ *are even or there exists an integer* $k \le n$ *such that* $a_k, a_{k-2}, a_{k-4}, \ldots$ *are odd, and the other coefficients are even.*

Proof. If $K$ is a two-component two-bridge link, its Conway polynomial is an odd Fibonacci polynomial modulo 2, that is of the form $f_{2h}(z)$. An easy induction shows that

$$f_{4k}(t^{1/2} - t^{-1/2}) = (t^{1/2} - t^{-1/2})(u_1 + u_3 + \cdots + u_{2k-1})$$

and

$$f_{4k+2}(t^{1/2} - t^{-1/2}) = (t^{1/2} - t^{-1/2})(1 + u_2 + \cdots + u_{2k}),$$

where $u_j = t^j + t^{-j}$, and the result follows.

Theorem 4.6 implies both Murasugi and Hartley theorems for two-bridge knots.

**Theorem 5.6 (Murasugi (1958), Hartley (1979)).** *Let*

$$\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),\ a_n > 0$$

*be the Alexander polynomial of a two-bridge knot. There exists an integer* $k \le n$ *such that* $a_0 = a_1 = \ldots = a_k > a_{k+1} > \ldots > a_n$.

PROOF. Let $K$ be a two-bridge knot and $\nabla(z) = \alpha_0 f_1 - \alpha_1 f_3 + \cdots + (-1)^n \alpha_n f_{2n+1}$ be its Conway polynomial written in the Fibonacci basis. By Theorem 4.6, $\alpha_n \alpha_k \geq 0$ for all $k$, and if $\alpha_i = 0$ for some $i$ then $\alpha_j = 0$ for $j \leq i$.

Let $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),\ a_n > 0$ be the Alexander polynomial of $K$. We have $\Delta(t) = \varepsilon \nabla(t^{1/2} - t^{-1/2})$, where $\varepsilon = \pm 1$, and then, by Proposition 5.2, $\varepsilon \alpha_k = a_k - a_{k+1}$. We deduce that $\varepsilon \alpha_n = a_n > 0$, and then $a_k - a_{k+1} = \varepsilon \alpha_k \geq 0$ for all $k$. Consequently we obtain $a_0 \geq a_1 \geq \ldots \geq a_n > 0$.

Furthermore, if $a_k = a_{k-1}$ for some $k$, then $\alpha_{k-1} = 0$, and consequently $\alpha_{j-1} = 0$ for all $j \leq k$. This implies that for all $j \leq k$, $a_j = a_{j-1}$, which concludes the proof.

Now, we shall give explicit formulas for Alexander coefficients in terms of Conway coefficients.

**Proposition 5.7.** *Let $Q(z) = \tilde{c}_0 + \tilde{c}_1 z^2 + \cdots + \tilde{c}_n z^{2n}$ be a polynomial. We have*

$$Q(t^{1/2} - t^{-1/2}) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),$$

*where*

$$a_{n-j} = \sum_{k=0}^{j} (-1)^{n-k} \tilde{c}_{n-k} \binom{2n-2k}{j-k}. \tag{8}$$

PROOF. It is sufficient to prove Formula (8) for the monomials $Q(z) = z^{2m}$. Let us consider $u_i = t^i + t^{-i}$. By the binomial formula we have

$$\left(t^{1/2} - t^{-1/2}\right)^{2m} = \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} u_{m-k} + (-1)^m \binom{2m}{m}.$$

and then $a_{n-j} = (-1)^m \binom{2m}{h}$ where $m - h = n - j$. On the other hand, the proposed formula asserts

$$a_{n-j} = \sum_{k=0}^{j} (-1)^{n-k} \tilde{c}_{n-k} \binom{2n-2k}{j-k} = (-1)^m \binom{2m}{h} \quad \text{where} \quad h = m + j - n,$$

which is the same result.

**Remark 5.8.** *Considering the Fibonacci polynomials $f_{2n+1} = \sum_{k=0}^{n} \binom{2n-k}{k} z^{2n-2k}$, Formulas (7) and (8) give the identity*

$$1 = \sum_{k=0}^{j} (-1)^k \binom{2n-k}{k} \binom{2n-2k}{j-k},\ n, j \geq 0.$$

**Remark 5.9.** *Fukuhara [6] gives a converse formula for the $\tilde{c}_k$ in terms of the $a_k$,*

$$\tilde{c}_{n-j} = \sum_{k=0}^{j} (-1)^{n-k} a_{n-k} \tfrac{2n-2k}{2n-j-k} \tbinom{2n-j-k}{2n-2j}.$$

From the bounds we obtained for Conway coefficients we can deduce a simple proof of the Nakanishi–Suketa bounds ([24, Th. 1, 2]) for the Alexander coefficients.

**Theorem 5.10 (Nakanishi–Suketa (1993)).** *Let*

$$\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),\ a_n > 0$$

*be the Alexander polynomial of a two-bridge knot. We have the following sharp inequalities:*

1. $0 < a_{n-j} \le a_n \Big( \sum_{k=0}^{j} \binom{2n-2k}{j-k} \binom{2n-k}{k} \Big)$.
2. $2a_n - 1 \le a_{n-1} \le (4n-2)a_n + 1$.

Proof. We first have $a_i > 0$ from Theorem 5.6.

1. Using Formula (8) and Theorem 4.1, we obtain

$$a_{n-j} \le \sum_{k=0}^{j} |\tilde{c}_{n-k}| \tbinom{2n-2k}{j-k} \le a_n \sum_{k=0}^{j} \tbinom{2n-k}{k} \tbinom{2n-2k}{j-k}. \tag{9}$$

2. We have $|\tilde{c}_{n-1}| \le \binom{2n-2}{1} |\tilde{c}_n| + 1$ by Theorem 4.4, and $a_{n-1} = \tilde{c}_{n-1} - \binom{2n}{1}\tilde{c}_n$ by Proposition 5.7. We thus deduce

$$a_{n-1} \le \tbinom{2n}{1} |\tilde{c}_n| + \tbinom{2n-2}{1} |\tilde{c}_n| + 1 = (4n-2)a_n + 1. \tag{10}$$

   We also have

$$a_{n-1} \ge \tbinom{2n}{1} |\tilde{c}_n| - |\tilde{c}_{n-1}| \ge \tbinom{2n}{1} |\tilde{c}_n| - \tbinom{2n-2}{1} |\tilde{c}_n| - 1 = 2a_n - 1.$$

The upper bounds (9) and (10) are attained by the knots $C(2, 2, \ldots, 2)$.

We also have the following sharp bound, which improves the Nakanishi–Suketa third bound ([24, Th. 3])

**Theorem 5.11.** *Let*

$$\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),\ a_n > 0$$

*be the Alexander polynomial of a two-bridge knot. If $a_n \ne 1$, then $a_{n-2} \le (8n^2 - 15n + 8)a_n + 2n - 1$. This bound is sharp.*

Proof. From Proposition 5.7 and Theorem 4.4, we get

$$\begin{array}{rcl} |a_{n-2}| & \leq & \binom{2n}{2}|\tilde{c}_n| + \binom{2n-2}{1}|\tilde{c}_{n-1}| + \binom{2n-4}{0}|\tilde{c}_{n-2}| \\ & \leq & \binom{2n}{2}|\tilde{c}_n| + \binom{2n-2}{1}(\binom{2n-2}{1}|\tilde{c}_n| + 1) + \left(\binom{2n-3}{2} + \frac{1}{g}(\binom{2n-3}{1} - 1)\right)|\tilde{c}_n| + 1 \\ & = & (8n^2 - 16n + 10 + \frac{2(n-2)}{g})\,|a_n| + 2n - 1. \end{array}$$

If $a_n \neq 1$ then $g \geq 2$, and we obtain

$$|a_{n-2}| \leq |a_n|\,(8n^2 - 15n + 8) + 2n - 1. \tag{11}$$

This bound is attained for the knot $C(4, 2, 2, 2, \ldots, 2)$.

**Example 5.12.** *Let us consider the Conway polynomial* $\nabla_K(z) = 1 + 8z^2 + 3z^4 - z^6$ *of the knot* $K = 13n1862$ *(see [1]). It does not verify the bounds of Theorem 4.1, and thus it is not a two-bridge knot. Nevertheless, its Alexander polynomial* $\Delta_K(t) = 23 - 19(t + 1/t) + 9(t^2 + 1/t^2) - (t^3 + 1/t^3)$ *satisfies the bounds of Nakanishi and Suketa, and also the conditions of Murasugi and Hartley. This example shows that the conditions on the Conway coefficients are stronger than the conditions on the Alexander coefficient deduced from them.*

**Remarks 5.13.** *The inequality in the statement of Theorem 5.11 can be improved:*

1. *If* $g \geq 3$ *then*

$$a_{n-2} \leq (8n^2 - 16n + 10 + \frac{2(n-2)}{g})a_n + 2n - 1.$$

2. *For* $j = 3$ *we obtain*

$$\begin{array}{rcl} a_{n-3} & \leq & 2/3\,(2\,n-3)\left(8\,n^2 - 24\,n + 25\right)a_n + \frac{(3\,n-5)(2\,n-5)}{g}a_n + n\,(2\,n-3) \\ & \leq & 1/6\left(64\,n^3 - 270\,n^2 + 413\,n - 225\right)a_n + n\,(2\,n-3)\,. \end{array}$$

3. *Since the inequalities on Conway coefficients are simpler and stronger, we shall not give the inequalities on Alexander coefficients for* $j \geq 4$. *Furthermore, if we want to apply our bounds to the Alexander polynomials, we first compute*

$$\tilde{c}_{n-j} = \sum_{k=0}^{j}(-1)^{n-k}a_{n-k}\frac{2n-2k}{2n-j-k}\binom{2n-j-k}{2n-2j},$$

*using Remark 5.9 and test if* $|\tilde{c}_{n-j}| \leq \binom{2n-j}{j}|\tilde{c}_n|$, *which is stronger than the inequality (9), or if* $|\tilde{c}_{n-j}| \leq \left(\binom{2n-j-1}{j} + \frac{1}{g}(\binom{2n-j-1}{j-1} - 1)\right)|c_n| + 1$. *The cost of these evaluations is less than the cost of the evaluations of the inequalities of Theorem 5.10. They are also sharper.*

The following example shows an infinite family of polynomials satisfying all the necessary conditions except the equality case of Theorem 4.1.

**Example 5.14.** *Consider the polynomial* $P(z) = f_{m+1}(z) - 2dz^2,\ m = 4n \geq 4,\ d \neq 0$. *All its coefficients, except one, satisfy* $c_{m-2k} = \binom{m-k}{k}$. *By Theorem 4.1, it is not the Conway polynomial of a two-bridge knot. Hence, the corresponding Alexander polynomial*

$$\Delta(t) = 4d + 1 - (2d+1)u_1 + u_2 - u_3 + \cdots + u_{2n},$$

*where* $u_i = t^i + t^{-i}$, *is not the Alexander polynomial of a two-bridge knot. Nevertheless, it satisfies all the necessary conditions of Hartley and Murasugi. If* $0 < d < \frac{1}{2}n(n+1)$, *it also satisfies the bounds of Theorems 4.1 and 4.4, and then the Nakanishi–Suketa bounds.*

## 6. Conjectures

We observed a trapezoidal property for the Conway polynomials of two-bridged links with 20 or fewer crossings (their number is 131 839).

**Conjecture 6.1.** *Let* $\nabla_m = c_m\Big(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor}(-1)^i \alpha_i f_{m+1-2i}\Big),\ \alpha_0 = 1$, *be the Conway polynomial of a two-bridge link (or knot) written in the Fibonacci basis. Then there exists* $n \leq \lfloor \frac{m}{2} \rfloor$ *such that*

$$0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n, \quad \alpha_n \geq \alpha_{n+1} \geq \cdots \geq \alpha_{\lfloor \frac{m}{2} \rfloor} \geq 0.$$

If this conjecture was true, it would imply the following property of Alexander polynomials:

**Conjecture 6.2.** *Let* $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ *be the Alexander polynomial of a two-bridge knot. Then there exists an integer* $k \leq n$ *such that* $(a_0, \ldots, a_k)$ *is convex and* $(a_k, \ldots, a_n)$ *is concave.*

This last result is a consequence of Conjecture 6.1 and Proposition 5.2. As usual we say that the sequence $(a_i)$ is convex (resp. concave) if $(a_i + a_{i+2} - 2a_{i+1})$ is nonnegative (resp. nonpositive). It is shown in [24] that the sequence $(a_j)$ is not necessarily convex.

The following conjecture is attributed to Hoste:

**Conjecture 6.3 (Hoste).** *If* $z \in \mathbf{C}$ *is a root of the Alexander polynomial of an alternating knot, then* $\operatorname{Re} z > -1$.

This conjecture is shown to be true in some peculiar cases (see [20, 29]). As a direct consequence of the definition of Euler continuant polynomials, we show that:

**Theorem 6.4.** *Let $K$ be a two-bridge link (or knot). Let $\alpha$ be a root of the Alexander polynomial $\Delta_K$, then $-\dfrac{3}{2} < \operatorname{Re}\alpha < 3+2\sqrt{2}$. If $\alpha$ is real then $3-2\sqrt{2} < \alpha < 3+2\sqrt{2}$.*

Proof. Let $K$ be a two-bridge link. $\nabla_K$ is an Euler continuant polynomial $D_m(b_1, \ldots, b_m)$. If $z$ is a root of $\nabla_K$, then the determinant in Formula (1) is equal to 0.

By the classical theorems of Gershgörin, there exists $i$ such that $|b_i z| < 2$. We thus deduce that $|z| < 2$.

Let $\alpha$ be a root of $\Delta_K$. Then $z = \alpha^{1/2} - \alpha^{-1/2}$ is a root of $\nabla_K$ and we have the relation $P(\alpha, z) = \alpha^2 - (z^2+2)\alpha + 1 = 0$. Eliminating $z$ between $P$ and $|z| < 2$, we obtain that $\alpha = x + iy$ satisfies $R(x, y) < 0$ where

$$R = x^4 + 2\,x^2y^2 + y^4 - 4\,x^3 - 4\,xy^2 - 10\,x^2 - 14\,y^2 - 4\,x + 1.$$

An easy computation shows that the curve $R = 0$ has vertical tangents at the four

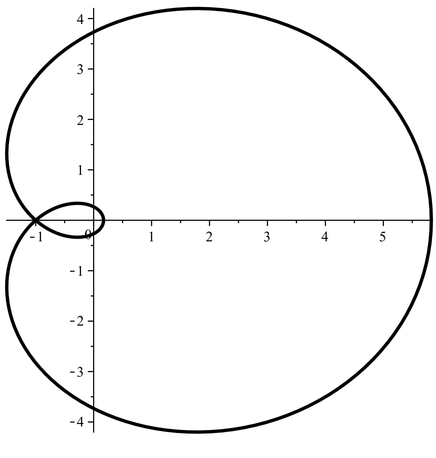


Figure 3: Region ($R < 0$) containing the roots of Alexander polynomials of two-bridge links.

points:

$$(-\frac{3}{2}, \pm\frac{\sqrt{7}}{2}),\ (3 \pm 2\sqrt{2}, 0).$$

Suppose that $\alpha$ is real. Then $z^2 = \alpha + 1/\alpha - 2$ is real and $\operatorname{Discr}(P) = z^2(z^2+4) \geq 0$. We thus deduce that $z$ is real and belongs to $(-2, 2)$. We thus have $\alpha \in (3-2\sqrt{2}, 3+2\sqrt{2})$.

Theorem 6.4 is an improvement of a Theorem of Lyubich and Murasugi [20]. We subsequently found that it was independently obtained by Stoimenow, but later [30]. It should be improved by a careful study of the tridiagonal determinant $D_m$ in Formula (1).


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