Multiple Bernoulli series and volumes of moduli spaces of flat bundles over surfaces

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Abstract

Using Szenes formula for multiple Bernoulli series, we explain how to compute Witten series associated to classical Lie algebras. Particular instances of these series compute volumes of moduli spaces of flat bundles over surfaces, and also certain multiple zeta values.

Keywords: multiple Bernoulli series, moduli spaces of flat connections, multiple zeta values

Introduction

Let V be a finite dimensional real vector space, and Λ a lattice in V. We denote the dual of Λ by Γ.

We consider a finite sequence of vectors Φ lying in Λ, and let $\Gamma_{\mathrm{reg}}(\Phi) = \{\gamma \in \Gamma | \ \langle \phi, \gamma \rangle \neq 0, \text{ for all } \phi \in \Phi\}$ be the set of regular elements in Γ relative to Φ.

In this paper we compute

$$\mathcal{B}(\Phi, \Lambda)(v) = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(\Phi)} \frac{e^{\langle 2i\pi v, \gamma \rangle}}{\prod_{\phi \in \Phi} \langle 2i\pi\phi, \gamma \rangle}, \tag{0.0.1}$$

a function on the torus V/Λ. This sum, if not absolutely convergent, has a meaning as a generalized function. If Φ generates V, then $\mathcal{B}(\Phi, \Lambda)$ is *piecewise polynomial* (see [12]).

For example, for $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ with standard lattice $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, if we choose $\Phi = [e_1, e_1, e_2, e_1 + e_2, e_1 - e_2]$, then

$$\mathcal{B}(\Phi, \Lambda)(v_1 e_1 + v_2 e_2) = \sum_{n_1, n_2}{}' \frac{e^{2i\pi(v_1 n_1 + v_2 n_2)}}{(2i\pi n_1)^2 (2i\pi n_2)(2i\pi(n_1 + n_2))(2i\pi(n_1 - n_2))},$$

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where the summation $\sum'$ means that we sum only over the integers n_1 and n_2 such that $n_1 n_2(n_1+n_2)(n_1-n_2) \neq 0$. The expression for $\mathcal{B}(\Phi,\Lambda)(v_1e_1+v_2e_2)$ as a piecewise polynomial function of v_1 and v_2 (of degree 5) is given in Section 2, Equation (2.6.2).

We call $\mathcal{B}(\Phi,\Lambda)$ the *multiple Bernoulli series* associated to Φ and Λ. Multiple Bernoulli series have been extensively studied by A. Szenes ([12],[13]). They are natural generalizations of Bernoulli series: for $V=\mathbb{R}\omega$, $\Lambda=\mathbb{Z}\omega$ and $\Phi_k = [\omega,\omega,\ldots,\omega]$, where ω is repeated k times with $k>0$, the function

$$\mathcal{B}(\Phi_k,\Lambda)(t\omega) = \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^k}$$

is equal to $-\frac{1}{k!}B(k,\{t\})$ where $B(k,t)$ denotes the k^{th} Bernoulli polynomial in variable t, and $\{t\}=t-[t]$ is the fractional part of t. If $k=2g$ and $t=0$, due to the symmetry $n\to -n$,

$$\mathcal{B}(\Phi_{2g},\Lambda)(0) = 2\frac{1}{(2i\pi)^{2g}}\zeta(2g).$$

From the residue theorem in one variable, for $k>0$,

$$\sum_{n\in\mathbb{Z},\ n\neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^k} = \mathrm{Res}_{z=0}\Big(\frac{1}{z^k}e^{zt}\frac{1}{1-e^z}\Big).$$

Szenes multidimensional residue formula (cf. Theorem 1.27) is the generalization of this formula to higher dimensions, and it is the tool that we use for computing $\mathcal{B}(\Phi,\Lambda)(v)$ as a piecewise polynomial function.

A particular but crucial instance of multiple Bernoulli series is when Λ is the coroot lattice of a compact connected simple Lie group G, and Φ is comprised of positive coroots of G. The series $\mathcal{B}(\Phi_{2g-2},\Lambda)$, where the argument Φ_{2g-2} refers to taking elements of Φ with multiplicity $2g-2$, appeared in the work of E. Witten ([16], §3), where Witten shows that its value at $v=0$ (up to a scalar depending on G and g) is the symplectic volume of the moduli space of flat G-connections on a Riemann surface of genus g. Similarly, for a collection of regular elements $\mathbf{v}=\{v_1,\ldots,v_s\}$ of the Cartan Lie algebra of G, certain linear combinations of $\mathcal{B}(\Phi_{2g-2+s},\Lambda)$ at some particular values (depending on $\mathbf{v}$) is the symplectic volume of the moduli space of flat G-connections on a Riemann surface of genus g with s boundary components, around which the holonomy is determined by $\mathbf{v}$. Then, its dependence on $\mathbf{v}$ is piecewise polynomial.

Multiple Bernoulli series have also been studied by P.E. Gunnells and R. Sczech ([5]) in view of applications to zeta functions of real number fields. Explicit computations of volumes of moduli spaces of flat bundles on Riemann surfaces are also obtained in [5]. Yet, the techniques they use is a generalization of the continued fraction algorithm and it is different from ours.

Y. Komori, K. Matsumoto and H. Tsumura ([6],[7],[8],[9]) studied the restriction of the series (0.0.1), by summing it over the cone of dominant regular weights of a semi-simple Lie group G, and defined a function $\zeta(\mathbf{s}, v, G)$ (cf. Section 5.2). They also obtained relations between these functions over $\mathbb{Q}$. When Λ is the coroot lattice of a compact connected simple Lie group G and the sequence Φ is the set of its positive coroots with equal even multiplicity for long roots and (possibly different) equal even multiplicity for short roots, due to the Weyl group symmetry, the summation $\mathcal{B}(\Phi, \Lambda)(0)$ over the full (regular) weight lattice is just (up to multiplication by an appropriate power of (2π)) Komori-Matsumoto-Tsumura zeta function $\zeta(\mathbf{s}, 0, G)$. Thus, the value of $\zeta(\mathbf{s}, 0, G)$ (up to a certain power of (2π)) is a rational number which can be computed explicitly, we give examples of such computations. As it is observed in [6], some instances of $\zeta(\mathbf{s}, v, G)$ also compute certain multiple zeta values. In the last part of the article we give various such computations of multiple zeta values using $\mathcal{B}(\Phi, \Lambda)$.

Here is the outline of individual sections.

In Section 1, we recall a formula due to A. Szenes, which allows an efficient computation of $\mathcal{B}(\Phi, \Lambda)$.

In Section 2, we give an outline of an algorithm that efficiently computes the needed ingredients of this formula for classical root systems. We also give several simple examples.

In Section 3, we show how this applies to the symplectic volume of the moduli space of flat G-connections on a Riemann surface of genus g with s boundary components. We obtain an expression for the symplectic volume by taking the limit of the Verlinde formula. We then show that our formula thus obtained coincides with that of Witten (including the constants) given in terms of the Riemannian volumes of G and T. We also give examples of these functions.

In Sections 4 and 5, we give several examples and tables of Witten volumes, which include some examples from [6], [7], [8] and [9]. We give an idea of computational limitation of our algorithm (written as a simple Maple program) in terms of the rank of the group G and the number of elements in Φ. Following Komori-Matsumoto-Tsumura, we also give some examples of rational multiple zeta values. To compute more examples, our Maple program is available on the webpage of the last author.

Finally, in the appendix, for completeness, we include a slightly modified proof of Szenes formula.

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1. Szenes formula for multiple Bernoulli series

1.1. Functions on the complement of hyperplanes

In this subsection U is an r-dimensional complex vector space. We denote its dual by V. If $\phi \in V$, we denote by $H_\phi = \{u \in U; \langle \phi, u\rangle = 0\}$ the hyperplane in U determined by ϕ.

We recall briefly some structure theorems for the ring of rational functions that are regular on the complement of a union of hyperplanes [4].

Let $\mathcal{H} = \{H_1, \ldots, H_N\}$ be a set of hyperplanes in U. Then, we may choose $\phi_k \in V$ such that $H_k = H_{\phi_k}$; the element ϕ_k is called an equation of H_k. Clearly, an equation ϕ_k is not unique, it is determined up to a non-zero scalar multiple. Consider $U_\mathcal{H} := \{u \in U; \langle \phi_k, u\rangle \neq 0 \text{ for all } k\}$, an open subset of U; an element of $U_\mathcal{H}$ will be called *regular.*

We denote by $S(V)$ the symmetric algebra of V and identify it with the ring of polynomial functions on U. Let $\mathcal{R}_\mathcal{H}$ denote the ring of rational functions on U that are regular on $U_\mathcal{H}$, that is, the ring generated by $S(V)$ together with inverses of the linear forms ϕ_k defining $\mathcal{H}$.

The ring $\mathcal{D}(U)$ of differential operators on U with polynomial coefficients acts on $\mathcal{R}_\mathcal{H}$. In particular, U operates on $\mathcal{R}_\mathcal{H}$ by differentiation. We denote by $\partial(U)\mathcal{R}_\mathcal{H}$ the subspace of $\mathcal{R}_\mathcal{H}$ obtained by differentiation.

In the particular case that V is one dimensional and $\mathcal{H} = \{0\}$, the ring $\mathcal{R}_\mathcal{H}$ is the ring of Laurent polynomials $\mathbb{C}[z, z^{-1}]$, and the function z^i, for $i \neq -1$, is obtained as a derivative $\frac{d}{dz}\frac{1}{i+1}z^{i+1}$. Thus $\mathcal{R}_\mathcal{H} = \frac{d}{dz}\mathcal{R}_\mathcal{H} \oplus \mathbb{C}z^{-1}$.

If $f = \sum_n a_n z^n$ is an element of $\mathbb{C}[z, z^{-1}]$, we denote by $\mathrm{Res}_{z=0}\, f$ the coefficient a_{-1} of z^{-1} in the expression of f. The linear form $\mathrm{Res}_{z=0}$ is characterized by the fact that it vanishes on $\frac{d}{dz}\mathcal{R}_\mathcal{H}$ and takes the value a_{-1} on $\mathbb{C}z^{-1}$.

By analogy to the one dimensional case demonstrated above, a linear functional on $\mathcal{R}_\mathcal{H}$ vanishing on $\partial(U)\mathcal{R}_\mathcal{H}$ will be called a 'residue'.

Let us thus analyze the space $\mathcal{R}_\mathcal{H}$ modulo $\partial(U)\mathcal{R}_\mathcal{H}$.

Consider a set $\Phi^{eq} := \{\phi_1, \phi_2, \ldots, \phi_N\}$ of equations for $\mathcal{H}$. A subset σ of Φ^{eq} will be called a *basis* if the elements ϕ_k in σ form a basis of V. We denote by $\mathfrak{B}(\Phi^{eq})$ the set of such subsets σ. A subset ν of Φ^{eq} will be called *generating* if the elements ϕ_k in ν generate the vector space V.

Definition 1.1. *Let $\sigma := \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \in \mathfrak{B}(\Phi^{eq})$. The element*

$$f_\sigma(z) := \frac{1}{\prod_{k=1}^r \alpha_k(z)}$$

of $\mathcal{R}_\mathcal{H}$ is called a 'simple fraction'. We denote by $\mathcal{S}_\mathcal{H}$ the subspace of $\mathcal{R}_\mathcal{H}$ generated by the elements f_σ, $\sigma \in \mathfrak{B}(\Phi^{eq})$.

Let $\nu = [\alpha_1, \ldots, \alpha_k]$ be a sequence of k elements of Φ^{eq} and $\mathbf{n} = [n_1, n_2, \ldots, n_k]$ be a sequence of positive integers. We define

$$\theta(\nu, \mathbf{n}) = \frac{1}{\alpha_1^{n_1} \cdots \alpha_k^{n_k}}.$$

We denote by $\mathcal{G}_{\mathcal{H}}$ the subspace of $\mathcal{R}_{\mathcal{H}}$ generated by the elements $\theta(\nu, \mathbf{n})$ where ν is generating.

As the notation suggests, the spaces $\mathcal{R}_{\mathcal{H}}$, $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{G}_{\mathcal{H}}$ depend only on $\mathcal{H}$. The term simple fraction comes from the fact that if $\sigma = \{\phi_1, \phi_2, \ldots, \phi_r\}$ is a basis, then we can choose coordinates z_i on U so that $\phi_i(z) = z_i$, so that for this system of coordinates $f_\sigma(z) = \frac{1}{\prod_{i=1}^r z_i}$.

We recall the following 'partial fraction' decomposition result from [4].

Lemma 1.2. *Let ν be a subset of Φ^{eq} generating a t dimensional subspace of V. Then $\theta(\nu, \mathbf{n})$ may be written as a linear combination of elements $\theta(\sigma, \mathbf{m}) = \frac{1}{\alpha_{i_1}^{m_1} \cdots \alpha_{i_t}^{m_t}}$ where $\sigma := \{\alpha_{i_1}, \ldots, \alpha_{i_t}\}$ is a subset of ν consisting of t independent elements and $\mathbf{m} = \{m_1, \ldots, m_t\}$ a sequence of positive integers.*

Example 1.3. $\dfrac{1}{z_1 z_2 (z_1 + z_2)} = \dfrac{1}{z_1 (z_1 + z_2)^2} + \dfrac{1}{z_2 (z_1 + z_2)^2}.$

Theorem 1.4. *(Brion-Vergne [4])*

$$\mathcal{R}_{\mathcal{H}} = \partial(U) \mathcal{R}_{\mathcal{H}} \oplus \mathcal{S}_{\mathcal{H}}.$$

The projector $\mathbf{R} : \mathcal{R}_{\mathcal{H}} \to \mathcal{S}_{\mathcal{H}}$ will be called the *total residue.* In view of this theorem, a residue is just a linear form on $\mathcal{S}_{\mathcal{H}}$.

In the case that $\mathcal{H}$ is the set of hyperplanes with equations the positive coroots of a simple compact Lie group G, the dimension of $\mathcal{S}_{\mathcal{H}}$ is given by the product of exponents of G [11]. In Section 2, we will give an explicit basis for $\mathcal{S}_{\mathcal{H}}$ for simple Lie algebras of type A, B and C (which defines the same set of hyperplanes as B) with dual basis consisting of iterated residues.

1.2. Szenes polynomial

In this section and for the rest of the article, V will denote a **real** vector space of dimension r.

Let U be the dual vector space of V. Let Λ be a lattice in V with dual lattice Γ in U.

Let $\mathcal{H} := \{H_1, H_2, \ldots, H_N\}$ be a real arrangement of hyperplanes in U. We say that Λ and $\mathcal{H}$ are *compatible* if the hyperplanes in $\mathcal{H}$ are rational with respect to Λ, that is, they can be defined by equations $\phi_k \in \Lambda$. If Λ' is another lattice commensurable with Λ, then Λ' and $\mathcal{H}$ are also compatible.

Thus we now consider a lattice Λ and a real arrangement of hyperplanes $\mathcal{H} = \{H_1, H_2, \ldots, H_N\}$ in U rational with respect to Λ.

We choose $\Phi^{eq} = \{\phi_1, \phi_2, \ldots, \phi_N\}$, a set of defining equations for $\mathcal{H}$, with each ϕ_i in Λ. We sometimes refer to $\mathcal{H}$ only via its set of equations Φ^{eq} and write $\mathcal{H} = \cup\{\phi_k = 0\}$.

We denote the complex arrangement defined by $\cup\{\phi_k = 0\}$ in $U_{\mathbb{C}}$ with the same letter $\mathcal{H}$, and denote by $U_{\mathcal{H}} = \{\prod_k \phi_k \neq 0\}$ the corresponding open subset of $U_{\mathbb{C}}$.

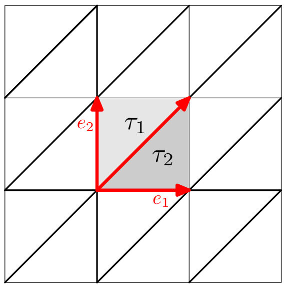

Figure 1: Topes for Example 1.5

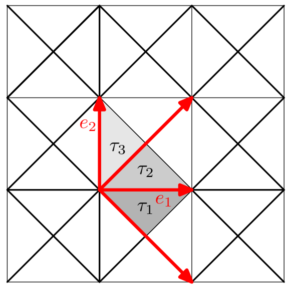

Figure 2: Topes for Example 1.6

An *admissible hyperplane* W in V (for the system $\mathcal{H}$) is an hyperplane generated by $(r-1)$ linearly independent elements ϕ_k of Φ^{eq}. Such an hyperplane will also be called an (admissible) *wall.* An admissible *affine* wall is a translate of a wall by an element of Λ.

An element $v \in V$ is called *regular* for $(\mathcal{H}, \Lambda)$ if v is not on any affine wall (we will simply say that v is regular). The meaning of the word regular is thus different for elements $v \in V$ (v is not on any affine wall) and $u \in U_{\mathbb{C}}$ (u is such that $\prod_k \langle \phi_k, u \rangle \neq 0$). However, it will be clear what regular means in the context.

A *tope* τ is a connected component of the complement of all affine hyperplanes. Thus a tope τ is a connected open subset of V consisting of regular elements. We denote the set of topes by $\mathcal{T}(\mathcal{H}, \Lambda)$. As the notation indicates, $\mathcal{T}(\mathcal{H}, \Lambda)$ does not depend on the choice of equations for $\mathcal{H}$.

Example 1.5. Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Let U be its dual with basis $\{e^1, e^2\}$. We express $z \in U_{\mathbb{C}}$ as $z = z_1 e^1 + z_2 e^2$, and consider the set of hyperplanes $\mathcal{H} = \{\{z_1 = 0\}, \{z_2 = 0\}, \{z_1 + z_2 = 0\}\}$ with the set of equations $\Phi^{eq} = \{e_1, e_2, e_1 + e_2\}$. Figure 1 depicts topes associated to this pair.

Example 1.6. With V and Λ as above, we now consider the set of hyperplanes $\mathcal{H} = \{\{z_1 = 0\}, \{z_2 = 0\}, \{z_1 + z_2 = 0\}, \{z_1 - z_2 = 0\}\}$ with the set of equations $\Phi^{eq} = \{e_1, e_2, e_1 + e_2, e_1 - e_2\}$. Figure 2 depicts topes associated to this pair.

We denote by $V_{reg}(\mathcal{H}, \Lambda)$ (or simply V_{reg}) the set of $(\mathcal{H}, \Lambda)$ regular elements of V. It is an open subset of V which is the disjoint union of all topes.

A locally constant function on V_{reg} is a function on V_{reg} which is constant on each tope. A piecewise polynomial function on V_{reg} is a function on V_{reg} which is given by a polynomial expression on each tope.

If $t \in \mathbb{R}$, we denote by $[t]$ the integral part of t, and by $\{t\} = t - [t]$ the fractional part of t. If $\gamma \in \Gamma$ vanishes on an admissible hyperplane W, and c is a constant, then the function $v \to \{\langle \gamma, v \rangle + c\}$ is piecewise polynomial (piecewise linear) and is periodic with respect to Λ. Szenes residue formula provides an algorithm to describe Bernoulli series in terms of these basic functions.

Definition 1.7. *Let $\mathcal{M}_{\mathcal{H}}$ be the space of functions h/Q where Q is a product of linear forms belonging to Φ^{eq}, and h a holomorphic function defined in a neighborhood of 0 in $U_{\mathbb{C}}$.*

We define the space $\hat{\mathcal{R}}_{\mathcal{H}}$ as the space of functions $\hat{h}/Q$ where $\hat{h} = \sum_{k=0}^{\infty} P_k$ is a formal power series and Q is a product of linear forms belonging to Φ^{eq}.

Taking the Taylor series $\hat{h}$ of h at 0 defines an injective map from $\mathcal{M}_{\mathcal{H}}$ to $\hat{\mathcal{R}}_{\mathcal{H}}$. The projector $\mathbf{R}$ from $\mathcal{R}_{\mathcal{H}}$ to $\mathcal{S}_{\mathcal{H}}$ extends to $\hat{\mathcal{R}}_{\mathcal{H}}$. Indeed $\mathbf{R}$ vanishes outside the homogeneous components of degree $-r$ of the graded space $\mathcal{R}_{\mathcal{H}}$. Thus if h/Q is an element in $\mathcal{M}_{\mathcal{H}}$, with Q a product of N elements of Φ^{eq}, we take the Taylor series $[h]_{N-r}$ of h up to order $N-r$, and define $\mathbf{R}(\frac{h}{Q}) = \mathbf{R}(\frac{[h]_{N-r}}{Q})$. For example, the equality

$$\frac{e^{zt}}{e^z - 1} = \frac{1}{z}(z\frac{e^{zt}}{e^z - 1})$$

identifies the function $\frac{e^{zt}}{e^z-1}$ to an element of $\mathcal{M}_{\mathcal{H}}$ with $\mathcal{H} = \{0\}$. Note that each homogeneous term of the Taylor series expansion

$$z\frac{e^{zt}}{e^z - 1} = \sum_{k=0}^{\infty} B(k,t)\frac{z^k}{k!},$$

where $B(k,t)$ is the k^{th} Bernoulli polynomial in t, is thus a polynomial in t.

Let $f \in \mathcal{R}_{\mathcal{H}}$, $z \in U_{\mathcal{H}}$ and $\gamma \in \Gamma$. Then if z is small, $2i\pi\gamma - z$ is still a regular element of $U_{\mathbb{C}}$. Now consider the series

$$S(f,z,v) = \sum_{\gamma\in\Gamma} f(2i\pi\gamma - z)e^{\langle v, 2i\pi\gamma\rangle}.$$

In the case that f above decreases sufficiently quickly at infinity, the series $S(f,z,v)$ is absolutely convergent and defines a continuous function of v. In general, as $f \in \mathcal{R}_{\mathcal{H}}$ is of at most polynomial growth, $\sum_{\gamma\in\Gamma} f(2i\pi\gamma - z)e^{\langle v, 2i\pi\gamma\rangle}$ is the Fourier series of a generalized function on V/Λ.

Multiplying $S(f,z,v)$ by the exponential $e^{-\langle z,v\rangle}$ we introduce the following generalized function.

Definition 1.8. *Let $f \in \mathcal{R}_{\mathcal{H}}$, $z \in U_{\mathcal{H}}$ and small. We define the generalized function $A^{\Lambda}(f)(z,v)$ of v by*

$$A^{\Lambda}(f)(z,v) = \sum_{\gamma\in\Gamma} f(2i\pi\gamma - z)e^{\langle v, 2i\pi\gamma - z\rangle}.$$

The meaning of $A^{\Lambda}(f)$ is clear : average the function $z \mapsto f(-z)e^{-\langle v,z\rangle}$ over $2i\pi\Gamma$ in order to obtain a function on the complex torus $U_{\mathbb{C}}/2i\pi\Gamma$. We consider $A^{\Lambda}(f)(z,v)$ as a generalized function of $v \in V$ with coefficients meromorphic functions of z on $U_{\mathbb{C}}/2i\pi\Gamma$. In fact, as we shall see in Proposition 1.11, when f is in $\mathcal{S}_{\mathcal{H}}$, the convergence of the series $\sum_{\gamma\in\Gamma} f(2i\pi\gamma - z)e^{\langle v, 2i\pi\gamma\rangle}$ holds in the sense of the Fourier series of an L^2- periodic function of $v \in V/\Lambda$, and

$$v \mapsto e^{-\langle v,z\rangle}\left(\sum_{\gamma\in\Gamma} f(2i\pi\gamma - z)e^{\langle v,2i\pi\gamma\rangle}\right) = A^\Lambda(f)(z,v)$$

is a locally constant function of $v \in V_{reg}$ with values in $\mathcal{M}_{\mathcal{H}}$.

Note the covariance relation. For $\lambda \in \Lambda$,

$$A^\Lambda(f)(z, v+\lambda) = e^{-\langle\lambda,z\rangle} A^\Lambda(f)(z,v). \tag{1.8.1}$$

It is easy to compare $A^\Lambda(f)(z,v)$ when we change the lattice Λ.

Lemma 1.9. *Let $f \in \mathcal{R}_{\mathcal{H}}$. If $\Lambda^1 \subset \Lambda^2$, then*

$$A^{\Lambda^2}(f)(z,v) = |\Lambda^2/\Lambda^1|^{-1} \sum_{\lambda\in\Lambda^2/\Lambda^1} A^{\Lambda^1}(f)(z, v+\lambda). \tag{1.9.1}$$

Proof. Denote the dual of Λ^i by Γ^i. Then,

$$A^{\Lambda^1}(f)(z, v+\lambda) = \sum_{\gamma\in\Gamma^1} f(2i\pi\gamma - z)e^{\langle v+\lambda, 2i\pi\gamma - z\rangle},$$

and the sum over $\lambda \in \Lambda^2/\Lambda^1$ of $e^{\langle\lambda,2i\pi\gamma\rangle}$ is zero except when $\gamma \in \Gamma^2$. $\square$

Example 1.10. Let $V = \mathbb{R}$, $\Lambda = \mathbb{Z}$, $z \in \mathbb{C} \setminus \{0\}$ and small, and $f(z) = \frac{1}{z}$. Then,

$$A^\Lambda(f)(z,v) = \sum_{n\in\mathbb{Z}} \frac{e^{v(2i\pi n - z)}}{(2i\pi n - z)} = e^{-zv}\sum_{n\in\mathbb{Z}} \frac{e^{2i\pi n v}}{(2i\pi n - z)}.$$

This series is not absolutely convergent, but the oscillatory factor $e^{2i\pi nv}$ insures the convergence in the distributional sense as a function of v. The L^2-expansion of the periodic function $v \mapsto \frac{e^{(v-[v])z}}{1-e^z}$ is

$$\sum_{n\in\mathbb{Z}}\left(\int_0^1 \frac{e^{(v-[v])z}}{1-e^z} e^{-2i\pi nv} dv\right) e^{2i\pi nv} = \sum_{n\in\mathbb{Z}}\left(\int_0^1 \frac{e^{v(z-2i\pi n)}}{1-e^z} dv\right) e^{2i\pi nv}$$

$$= \sum_{n\in\mathbb{Z}} \frac{e^{(z-2i\pi n)} - 1}{(1-e^z)(z - 2i\pi n)} e^{2i\pi nv} = \sum_{n\in\mathbb{Z}} \frac{1}{(2i\pi n - z)} e^{2i\pi nv}.$$

Thus

$$A^\Lambda(f)(z,v) = \frac{e^{-[v]z}}{1-e^z}. \tag{1.10.1}$$

We see in this one dimensional example that $A^\Lambda(f)(z,v)$ is a locally constant function of v. In general, we have the following proposition.

Proposition 1.11. *If $f \in \mathcal{S}_{\mathcal{H}}$, the function $v \to A^\Lambda(f)(z,v)$ is a locally constant function on V_{reg}, with values in $\mathcal{M}_{\mathcal{H}}$.*

We prove the above proposition by computing $A^\Lambda(f)(z,v)$ explicitly for a simple fraction $f = f_\sigma$. Then the result will follow for any $f \in \mathcal{S}_\mathcal{H}$.

Recall that the set of equations Φ^{eq} is a subset of Λ. Let $\sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be an element of $\mathfrak{B}(\Phi^{eq})$, with α_k belonging to Λ. Let $Q_\sigma := \oplus_{k=1}^r [0,1)\alpha_k$ be the semi-open parallelepiped spanned by σ. For a regular element v in V, we define $T(v,\sigma)$ to be the set of elements $\lambda \in \Lambda$ such that $v + \lambda \in Q_\sigma$. This set depends only on the tope τ where v belongs, hence we denote it by $T(\tau,\sigma)$.

Let Λ_σ be the sublattice of Λ generated by the elements in the basis σ. Then the set $T(\tau,\sigma)$ contains exactly Λ/Λ_σ elements.

Proposition 1.12. *If $v \in \tau$ and $\sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \in \mathfrak{B}(\Phi^{eq})$,*

$$A^\Lambda(f_\sigma)(z,v) = \frac{1}{|\Lambda/\Lambda_\sigma|} \sum_{\lambda \in T(\tau,\sigma)} \frac{e^{\langle \lambda, z\rangle}}{\prod_{i=1}^r (1 - e^{\langle \alpha_i, z\rangle})}.$$

Proof. If $\Lambda = \Lambda_\sigma$, the formula reduces to the one dimensional case. Otherwise, we use Lemma 1.9 and the covariance relation (1.8.1). $\square$

Thus the dependence of $A^\Lambda(f_\sigma)(z,v)$ on v is only via the tope τ where it belongs. Therefore, for any $f \in \mathcal{S}_\mathcal{H}$, the function $A^\Lambda(f)(z,v)$ is locally constant on V_{reg}. Hence the claim in Proposition 1.11.

Example 1.13. We return to Example 1.5. To describe $A^\Lambda(f)(z,v)$ on V_{reg}, it suffices to give its expression on topes τ_1 and τ_2 (see Figure 1). This is true since any element of V_{reg} can be translated to τ_1 or τ_2 by an element of Λ, and then one uses the covariance relation (1.8.1).

We choose $\sigma = \{e_1, e_1+e_2\}$ as a basis of Φ^{eq}, and write $z = z_1e^1 + z_2e^2$. Then $f_\sigma(z) = \frac{1}{z_1(z_1+z_2)}$ is in $S_\mathcal{H}$. If $v \in \tau_1$, we have $A^\Lambda(f_\sigma)(z,v) = \frac{e^{z_1}}{(1-e^{z_1})(1-e^{z_1+z_2})}$, while if $v \in \tau_2$, then $A^\Lambda(f_\sigma)(z,v) = \frac{1}{(1-e^{z_1})(1-e^{z_1+z_2})}$.

For $v \in V_{reg}$, denote by $Z^\Lambda(v) : \mathcal{S}_\mathcal{H} \to \mathcal{M}_\mathcal{H}$ the map

$$(Z^\Lambda(v)f)(z) = A^\Lambda(f)(z,v).$$

This operator is locally constant. We denote its value on τ by $Z^\Lambda(\tau)$. Hence $(Z^\Lambda(\tau)f)(z) = A^\Lambda(f)(z,v)$ for any choice of $v \in \tau$.

We now define a piecewise polynomial function of v associated to a function $g(z)$ in $\mathcal{R}_\mathcal{H}$. If $v \in V$ and $g \in \mathcal{R}_\mathcal{H}$, then $g_v(z) := g(z)e^{\langle z,v\rangle}$ is a function in $\mathcal{M}_\mathcal{H}$ depending on v. Now suppose $v \in V_{reg}$, and consider the map $\mathcal{S}_\mathcal{H} \to \mathcal{M}_\mathcal{H}$ which associates to $f \in \mathcal{S}_\mathcal{H}$ the function $g(z)e^{\langle z,v\rangle}(Z^\Lambda(v)f)(z)$. We project back this function on $\mathcal{S}_\mathcal{H}$ using the projector $\mathbf{R}$. Thus the map

$$f(z) \mapsto \mathbf{R}\left(g(z)e^{\langle z,v\rangle}(Z^\Lambda(v)f)(z)\right) \tag{1.13.1}$$

is a map from $\mathcal{S}_\mathcal{H}$ to $\mathcal{S}_\mathcal{H}$ depending on v. As $\mathcal{S}_\mathcal{H}$ is finite dimensional, we can take the trace of this operator, and thus obtain a function of $v \in V_{reg}$. Let us record this definition.

Definition 1.14. *Let $g \in \mathcal{R}_{\mathcal{H}}$. Define the function $P(\mathcal{H}, \Lambda, g)$ on $V_{reg}(\mathcal{H}, \Lambda)$ by*

$$P(\mathcal{H}, \Lambda, g)(v) := Tr_{\mathcal{S}_{\mathcal{H}}} \left(\mathbf{R}\, g_v\, Z^{\Lambda}(v) \right).$$

Let us see that $P(\mathcal{H}, \Lambda, g)(v)$ is a polynomial function of v on each tope τ. Indeed, to compute $P(\mathcal{H}, \Lambda, g)(v)$ using (1.13.1), we have to compute the total residue of functions $g(z)e^{\langle z,v\rangle} A^{\Lambda}(f_i)(z, v)$ with f_i varying over a basis of $\mathcal{S}_{\mathcal{H}}$. If $v \in \tau$, then $A^{\Lambda}(f_i)(z, v) = Z^{\Lambda}(\tau) f_i(z)$ is constant in v. So when v stays in a tope τ, the dependence of $g(z)e^{\langle z,v\rangle} A^{\Lambda}(f_i)(z, v) = g(z)e^{\langle z,v\rangle}(Z^{\Lambda}(\tau) f_i)(z)$ on v is via $e^{\langle z,v\rangle}$, and the map $\mathbf{R}$ involves only the Taylor series of this function up to some order. Thus we have associated to $g \in \mathcal{R}_{\mathcal{H}}$ (and Λ) a piecewise polynomial function $P(\mathcal{H}, \Lambda, g)$ on V_{reg}.

It is easy to compare piecewise polynomial functions $P(\mathcal{H}, \Lambda, g)$ associated to different lattices: Observe that if $\Lambda^1 \subset \Lambda^2$, then $V_{reg}(\mathcal{H}, \Lambda^2) \subset V_{reg}(\mathcal{H}, \Lambda^1)$. Then, using Lemma 1.9, we have

$$P(\mathcal{H}, \Lambda^2, g)(v) = |\Lambda^2/\Lambda^1|^{-1} \sum_{\lambda \in \Lambda^2/\Lambda^1} P(\mathcal{H}, \Lambda^1, g)(v + \lambda). \tag{1.14.1}$$

Our next aim is to compute the piecewise polynomial function $P(\mathcal{H}, \Lambda, g)$ using residues. We need more definitions.

An ordered basis of Φ^{eq} is a sequence $[\alpha_1, \alpha_2, \ldots, \alpha_r]$ of elements of Φ^{eq} such that the underlying set is in $\mathfrak{B}(\Phi^{eq})$. We denote the set of ordered bases of Φ^{eq} by $\overrightarrow{\mathfrak{B}}(\Phi^{eq})$.

Let $\overrightarrow{\sigma} = [\alpha_1, \alpha_2, \ldots, \alpha_r] \in \overrightarrow{\mathfrak{B}}(\Phi^{eq})$. Then, to this data, one associates an iterated residue functional $\mathrm{Res}^{\overrightarrow{\sigma}}$ on $\mathcal{R}_{\mathcal{H}}$ as follows. For $z \in U_{\mathbb{C}}$, let $z_j = \langle z, \alpha_j \rangle$. Then a function f in $\mathcal{R}_{\mathcal{H}}$ can be expressed as a function $f(z_1, z_2, \ldots, z_r)$. We define

$$\mathrm{Res}^{\overrightarrow{\sigma}}(f) := \mathrm{Res}_{z_1=0}(\mathrm{Res}_{z_2=0} \cdots (\mathrm{Res}_{z_r=0}\, f(z_1, z_2, \ldots, z_r)) \cdots).$$

Clearly $\mathrm{Res}^{\overrightarrow{\sigma}}(f_\sigma) = 1$. Moreover, the functional $\mathrm{Res}^{\overrightarrow{\sigma}}$ factors through the canonical projection $\mathbf{R} : \mathcal{R}_{\mathcal{H}} \to \mathcal{S}_{\mathcal{H}}$, that is, $\mathrm{Res}^{\overrightarrow{\sigma}} = \mathrm{Res}^{\overrightarrow{\sigma}}\,\mathbf{R}$.

Definition 1.15. *A diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq})$ is a subset $\overrightarrow{\mathcal{D}}$ of $\overrightarrow{\mathfrak{B}}(\Phi^{eq})$ such that the set of simple fractions f_σ, $\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}$, forms a basis of $\mathcal{S}_{\mathcal{H}}$:*

$$\mathcal{S}_{\mathcal{H}} = \oplus_{\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}} \mathbb{C} f_\sigma$$

and the dual basis to the basis $\{f_\sigma, \overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}\}$ of $\mathcal{S}_{\mathcal{H}}$ is the set of linear forms $\mathrm{Res}^{\overrightarrow{\sigma}}$, that is, $\mathrm{Res}^{\overrightarrow{\tau}}(f_\sigma) = \delta_\sigma^\tau$, for $\overrightarrow{\sigma}, \overrightarrow{\tau} \in \overrightarrow{\mathcal{D}}$.

A total order on Φ^{eq} allows us to construct the set of Orlik-Solomon bases (see [4]), which provides diagonal basis of $\mathcal{S}_{\mathcal{H}}$. However we will also use some other diagonal subsets.

If $B : \mathcal{S}_{\mathcal{H}} \to \mathcal{M}_{\mathcal{H}}$ is an operator, the trace of the operator $A := \mathbf{R}B$ is thus

$$Tr(A) := \sum_{\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}} \mathrm{Res}^{\overrightarrow{\sigma}}\, B f_\sigma.$$

Definition 1.16. *Let $g \in \mathcal{R}_{\mathcal{H}}$, τ a connected component of V_{reg}, and $v \in \tau$. We denote by $P(\mathcal{H}, \Lambda, g, \tau)(v)$ the polynomial function on V such that*

$$P(\mathcal{H}, \Lambda, g)(v) = P(\mathcal{H}, \Lambda, g, \tau)(v).$$

We may give a more explicit formula for the polynomial $P(\mathcal{H}, \Lambda, g, \tau)(v)$ using a diagonal subset $\overrightarrow{\mathcal{D}}$.

Proposition 1.17. *Let $g \in \mathcal{M}_{\mathcal{H}}$ and $\tau \in \mathcal{T}(\mathcal{H}, \Lambda)$ be a tope. Let $\overrightarrow{\mathcal{D}}$ be a diagonal subset of $\overrightarrow{\mathcal{B}}(\Phi^{eq})$. Then*

$$P(\mathcal{H}, \Lambda, g, \tau)(v) = \sum_{\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}} \operatorname{Res}^{\overrightarrow{\sigma}} \left(e^{\langle z, v \rangle} g(z) Z^{\Lambda}(\tau)(f_\sigma)(z) \right).$$

Furthermore $Z^{\Lambda}(\tau)(f_\sigma)(z)$ is given explicitly by Proposition 1.12. Thus, in principle, the above formula allows us to compute $P(\mathcal{H}, \Lambda, g)$.

It is important to remark that the determination of a diagonal subset $\overrightarrow{\mathcal{D}}$ depends essentially only on the system of hyperplanes $\mathcal{H}$ and not on the choice of Φ^{eq}. The difficulties in writing an algorithm for $P(\mathcal{H}, \Lambda, g)$ lies in the description of a diagonal subset $\overrightarrow{\mathcal{D}}$, and also for each $\sigma \in \overrightarrow{\mathcal{D}}$, in the computation of $A^{\Lambda}(f_\sigma)$. The difficulty of this last computation depends on the lattice Λ.

Definition 1.18. *Let $\Phi^{eq} \subset \Lambda$. A basis $\sigma \in \mathfrak{B}(\Phi^{eq})$ is called unimodular with respect to Λ if $\Lambda_\sigma = \Lambda$. A set Φ^{eq} is called unimodular, if any basis $\sigma \in \mathfrak{B}(\Phi^{eq})$ is unimodular.*

Let $\sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a basis. For $1 \leq i \leq r$, the linear form $v \to c_i^\sigma(v)$ is the coefficient of v with respect to α_i. Now, consider the function $\{t\} = t - [t]$; on each open interval $\tau =]n, n+1[$ it coincides with the linear function $t \mapsto (t-n)$. We express $v = \sum_{i=1}^r c_i^\sigma(v)\alpha_i$, then $v - \sum_{i=1}^r [c_i^\sigma(v)]\alpha_i = \sum_{i=1}^r \{c_i^\sigma(v)\}\alpha_i$ is in Q_σ. Thus, if σ is a unimodular basis, the set $T(\tau, \sigma)$ contains exactly the element $\lambda = -\sum_{i=1}^r [c_i^\sigma(v)]\alpha_i$ (which depends only on the tope τ where v lies).

Corollary 1.19. *Let σ be a unimodular basis in $\mathfrak{B}(\Phi^{eq})$. Let $v \in \tau$, and $\lambda = -\sum_{i=1}^r [c_i^\sigma(v)]\alpha_i$. Then*

$$A^{\Lambda}(f_\sigma)(z, v) = \frac{e^{\langle \lambda, z \rangle}}{\prod_{i=1}^r (1 - e^{\langle \alpha_i, z \rangle})}.$$

It may happen that even when the system Φ^{eq} is not unimodular for Λ, we can choose $\overrightarrow{\mathcal{D}}$ to consist of unimodular bases. In particular, using Proposition 1.17, we can give an explicit algorithm for computing the piecewise polynomial function $P(\mathcal{H}, \Lambda, g)$ for classical root systems in the form of a 'step polynomial'. Let us define what this means.

Definition 1.20. *Let $\overrightarrow{\mathcal{D}}$ be a subset of $\overrightarrow{\mathcal{B}}(\Phi^{eq})$. We denote by Step($\overrightarrow{\mathcal{D}}$) the algebra of functions on V generated by the piecewise linear functions $v \to \{c_i^\sigma(v)\}$ with σ running over $\overrightarrow{\mathcal{D}}$ and $1 \leq i \leq r$. An element of the algebra Step($\overrightarrow{\mathcal{D}}$) is called a step polynomial (associated to $\overrightarrow{\mathcal{D}}$).*

It is clear that a step polynomial is a periodic function on V, which is expressed by a polynomial formula on each tope.

Proposition 1.21. *Suppose $\overrightarrow{\mathcal{D}}$ is a diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq})$ consisting of unimodular basis (with respect to Λ). Then, for $g \in \mathcal{G}_{\mathcal{H}}$, the piecewise polynomial function $P(\mathcal{H}, \Lambda, g)$ belongs to the algebra $Step(\overrightarrow{\mathcal{D}})$.*

Proof. This is clear, as we have the formula

$$P(\mathcal{H}, \Lambda, g)(v) = \sum_{\overrightarrow{\sigma} \in \overrightarrow{\mathcal{D}}} \operatorname{Res}^{\overrightarrow{\sigma}} g(z) e^{\sum_{i=1}^{r} \{c_i^{\sigma}(v)\} \langle \alpha_i, z \rangle} \frac{1}{\prod_{i=1}^{r} (1 - e^{\langle \alpha_i, z \rangle})}, \tag{1.21.1}$$

and the dependence on v is through the Taylor series expansion (in z) of $e^{\sum_{i=1}^{r} \{c_i^{\sigma}(v)\} \langle \alpha_i, z \rangle}$ up to some order. □

1.3. Multiple Bernoulli series

We return to our main object of study: the multiple Bernoulli series.

Let V, Λ and $\mathcal{H}$ be as before. We denote by $\Gamma \subset U$ the dual lattice to Λ, and by $\Gamma_{reg}(\mathcal{H})$ the set $\Gamma \cap U_{\mathcal{H}}$. If $\gamma \in \Gamma_{reg}(\mathcal{H})$, a function g in $\mathcal{R}_{\mathcal{H}}$ is defined on $2i\pi\gamma$.

Definition 1.22. *If $g \in \mathcal{R}_{\mathcal{H}}$, the generalized function $\mathcal{B}(\mathcal{H}, \Lambda, g)(v)$ on V is defined by*

$$\mathcal{B}(\mathcal{H}, \Lambda, g)(v) = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(\mathcal{H})} g(2i\pi\gamma) e^{2i\pi \langle v, \gamma \rangle}.$$

The above series converges in the space of generalized functions on V.

We state some obvious properties of $\mathcal{B}(\mathcal{H}, \Lambda, g)$, with which we can compare it over commensurable lattices.

Lemma 1.23. *If $\Lambda^1 \subset \Lambda^2$, then*

$$\mathcal{B}(\mathcal{H}, \Lambda^2, g)(v) = |\Lambda^2/\Lambda^1|^{-1} \sum_{\lambda \in \Lambda^2/\Lambda^1} \mathcal{B}(\mathcal{H}, \Lambda^1, g)(v + \lambda). \tag{1.23.1}$$

If we dilate a lattice Λ by ℓ, and if g is homogeneous of degree h, we clearly have

$$\mathcal{B}(\mathcal{H}, \ell\Lambda, g)(v) = \ell^{-h} \mathcal{B}(\mathcal{H}, \Lambda, g)(\frac{v}{\ell}). \tag{1.23.2}$$

Definition 1.24. *A generalized function b on V is called piecewise polynomial relative to $\mathcal{H}$ and Λ if it is locally L^1, and if, for any τ in $\mathcal{T}(\mathcal{H}, \Lambda)$, there exists a polynomial function b^{τ} on V such that the restriction of b to τ coincides with the restriction of the polynomial b^{τ} to τ.*

As an L^1-function is entirely determined by its restriction to V_{reg}, we will not distinguish between piecewise polynomial generalized functions on V and piecewise polynomial functions on V_{reg} as defined in the preceding section. One should be careful that, the restriction of a piecewise polynomial generalized

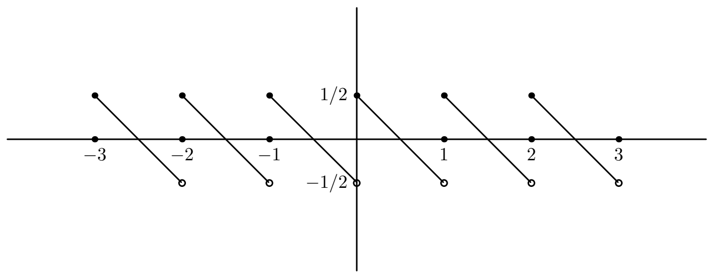

Figure 3: Graph of $\mathcal{B}(\{0\}, \mathbb{Z}, 1/z)(v) = \frac{1}{2} - \{v\}$

function to any tope is polynomial, however, the converse is not true. For example, the δ function of the lattice Λ restricts to 0 on any tope, but is not a piecewise polynomial generalized function, as it is not locally L^1.

Any function f in $\mathcal{R}_{\mathcal{H}}$ is of the form pg, with $g \in \mathcal{G}_{\mathcal{H}}$ and p a polynomial. Moreover, the function $v \mapsto \mathcal{B}(\mathcal{H}, \Lambda, pg)(v)$ is obtained from $\mathcal{B}(\mathcal{H}, \Lambda, g)(v)$ by differentiation (in the distribution sense). Thus, the computation of $\mathcal{B}(\mathcal{H}, \Lambda, f)$ for f in $\mathcal{R}_{\mathcal{H}}$ can be reduced to that of $\mathcal{B}(\mathcal{H}, \Lambda, g)$ for g in $\mathcal{G}_{\mathcal{H}}$. The following proposition follows from calculations in dimension one, Lemma 1.2 and formulae of Lemma 1.23.

Proposition 1.25. *If $f \in \mathcal{R}_{\mathcal{H}}$, the restriction of $\mathcal{B}(\mathcal{H}, \Lambda, f)$ to any tope τ is given by a polynomial function.*

Furthermore, if $f \in \mathcal{G}_{\mathcal{H}}$, the generalized function $\mathcal{B}(\mathcal{H}, \Lambda, f)$ is a piecewise polynomial generalized function.

Let us emphasize on the subtle difference between the conditions $f \in \mathcal{R}_{\mathcal{H}}$ and $f \in \mathcal{G}_{\mathcal{H}}$.

Consider $f = 1$ in the one dimensional space $U_{\mathbb{C}}$ and $\mathcal{H} = \{0\}$. The function f is not in $\mathcal{G}_{\mathcal{H}}$. For $v \in V$, $\mathcal{B}(\mathcal{H}, \Lambda, f)(v) = \sum_{n \neq 0} e^{2i\pi nv} = -1 + \sum_{n \in \mathbb{Z}} e^{2i\pi nv}$, which, on any tope, is the constant function equal to -1. However, it has some singular part $\delta_{\mathbb{Z}}$, and it is not locally L^1. In contrast, if $f = \frac{1}{z}$, the generalized function $\mathcal{B}(\mathcal{H}, \Lambda, f)(v) = \sum_{n \neq 0} \frac{e^{2i\pi nv}}{2i\pi n}$ is locally L^1 and equal to the piecewise polynomial function $-B(1, \{v\}) = 1/2 - \{v\}$ (see Figure 3).

Definition 1.26. *Let $f \in \mathcal{R}_{\mathcal{H}}$. Given a tope τ in $\mathcal{T}(\mathcal{H}, \Lambda)$, we denote by $\mathcal{B}(\mathcal{H}, \Lambda, f, \tau)$ the polynomial function on V which coincides with $\mathcal{B}(\mathcal{H}, \Lambda, f)$ on the tope τ.*

Recall the piecewise polynomial function $P(\mathcal{H}, \Lambda, f)$ on $V_{reg}(\mathcal{H}, \Lambda)$ as given in Definition 1.14.

Theorem 1.27. *(Szenes) Let $f \in \mathcal{R}_{\mathcal{H}}$. On $V_{\mathrm{reg}}(\mathcal{H}, \Lambda)$, we have*

$$\mathcal{B}(\mathcal{H}, \Lambda, f) = P(\mathcal{H}, \Lambda, f).$$

For completeness, we give a proof of this theorem in the Appendix.

Our Maple program computes, given data $\mathcal{H}$, Λ, f, where $\mathcal{H}$ is the hyperplane arrangement associated to a classical root system, a piecewise polynomial function on V in terms of step polynomials. Naturally, we can also evaluate this function at any point $v \in V_{reg}$.

We return to the definition of multiple Bernoulli series as given in the introduction: V is a vector space with a lattice Λ, and Φ a **list** of elements in Λ. Associated to this data, we defined,

$$\mathcal{B}(\Phi, \Lambda)(v) = \sum_{\gamma \in \Gamma_{\mathrm{reg}}(\Phi)} \frac{e^{\langle 2i\pi v, \gamma \rangle}}{\prod_{\phi \in \Phi} \langle 2i\pi \phi, \gamma \rangle},$$

where Γ is the dual of Λ, and $\Gamma_{\mathrm{reg}}(\Phi) = \{\gamma \in \Gamma; \langle \phi, \gamma \rangle \neq 0 \text{ for all } \phi \in \Phi\}$ are its regular elements.

Consider $\mathcal{H} = \cup_{\phi \in \Phi}\{\phi = 0\}$ (some elements of the list Φ might define the same hyperplane) and $g(z) = 1/\prod_{\phi \in \Phi} \langle \phi, z \rangle$, then

$$\mathcal{B}(\Phi, \Lambda)(v) = \mathcal{B}(\mathcal{H}, \Lambda, g)(v).$$

We will also call the functions $\mathcal{B}(\mathcal{H}, \Lambda, g)(v)$ multiple Bernoulli series.

2. Classical root systems

Let G be a simple, connected, simply connected, compact Lie group of rank r with maximal torus T. We denote the Lie algebra of T and G by $\mathfrak{t}$ and $\mathfrak{g}$ respectively. Then the complexification $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}) \subset \mathfrak{h}^*$ be the set of roots; we denote the root lattice by Q and its dual, the coweight lattice, by $\check{P}$. For $\alpha \in R$, H_α denotes the associated *coroot*; the coroot lattice is denoted by $\check{Q}$. The weight lattice is $P = \{\lambda \in \mathfrak{h}^*; \lambda(H_\alpha) \in \mathbb{Z}, \forall \alpha \in R\}$; a *regular* weight $\lambda \in P^{reg}$ is such that $\lambda(H_\alpha) \neq 0$ for all H_α. We denote by $\mathfrak{h}_{\mathbb{R}} := \sum_\alpha \mathbb{R} H_\alpha$ the real span of coroots.

In this section we have $V = \mathfrak{h}_{\mathbb{R}}$, and its dual $\mathfrak{h}_{\mathbb{R}}^*$ is denoted by U as before. We follow the notation of Bourbaki for root data.

2.1. Diagonal subsets

To compute multiple Bernoulli series associated to classical root systems we need to construct explicit diagonal bases for the corresponding $\mathcal{S}_{\mathcal{H}}$. Such bases can be constructed by an algorithmic procedure, based on Orlik-Solomon construction [10]. However in some cases one can describe a diagonal subset $\overrightarrow{\mathcal{D}}$ of $\overrightarrow{\mathcal{B}}(\Phi^{eq})$ whose associated simple fractions form a basis for $\mathcal{S}_{\mathcal{H}}$ in a direct way. We now demonstrate this.

2.1.1. The system of type A_r

Let $n = r + 1$. We consider $\mathbb{R}^n$ with standard basis $\{e_i\}$; denote the dual basis by $\{e^i\}$. Let $A_r := [(e_i - e_j);\ 1 \le i < j \le n]$ be the root system of type A and rank r, and $V = \{v = \sum_{i=1}^n v_i e^i;\ \sum_{i=1}^n v_i = 0\}$.

Let $z = \sum_{i=1}^n z^i e_i$ be in $U_{\mathbb{C}}$ (hence $\sum_{i=1}^n z^i = 0$) and let $\mathcal{H}_r^A$ be the system of hyperplanes in $U_{\mathbb{C}}$ given by $\mathcal{H}_r^A = \cup_{1\le i<j\le n}\{z^i - z^j = 0\}$. We take the set $\Phi^{eq}(A_r) = \{e^i - e^j; 1 \le i < j \le n\}$ of positive coroots as equations of $\mathcal{H}_r^A$.

One way to find a diagonal basis of $\mathcal{S}_{\mathcal{H}_r^A}$ is as follows.

Let $\Sigma = [e^1 - e^2, e^2 - e^3, \ldots, e^r - e^{r+1}]$ be the set of simple coroots. For a permutation w, we denote by $\overrightarrow{\sigma}_w = [e^{w(i)} - e^{w(i+1)},\ i = 1, \ldots, r]$. Then $\overrightarrow{\sigma}_w$ is an ordered basis associated to w, and the corresponding simple fraction is $f_w(z) := \dfrac{1}{\prod_{i=1}^r (z^{w(i)} - z^{w(i+1)})}$.

Let W_r be the subgroup of the Weyl group Σ_{r+1} of permutations of $\{e^1, e^2, \ldots, e^{r+1}\}$ leaving the last element $e^{r+1} = e^n$ fixed. Recall the following result (see for example Baldoni-Vergne [2] for a proof).

Proposition 2.1. *The set $\overrightarrow{\mathcal{D}}_W$ consisting of ordered bases $\overrightarrow{\sigma}_w$ for $w \in W_r$ is a diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq}(A_r))$.*

We use the above basis in our Maple program. We now give another interesting diagonal subset.

Consider a sequence $\overrightarrow{\sigma} = [\alpha_2, \alpha_3, \ldots, \alpha_n]$ where $\alpha_i = e^i - e^j$ with $j < i$. That is, $\alpha_2 = e^2 - e^1$, $\alpha_3 = e^3 - e^2$ or $e^3 - e^1$, $\alpha_4 = e^4 - e^3$, or $e^4 - e^2$, or $e^4 - e^1$, etc. Clearly, $\overrightarrow{\sigma}$ is in $\overrightarrow{\mathfrak{B}}(\Phi^{eq}(A_r))$. We call such $\overrightarrow{\sigma}$ a $f$lag basis; there are $r!$ such sequences $\overrightarrow{\sigma}$.

Lemma 2.2. *The set $\overrightarrow{\mathcal{D}}(A_r)$ consisting of flag bases is a diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq}(A_r))$.*

We only need to prove that if $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$ are two flag bases, then $\mathrm{Res}^{\overrightarrow{\sigma}} f_\tau = 0$ unless $\overrightarrow{\sigma} = \overrightarrow{\tau}$. But this is evident.

2.1.2. Systems of type B_r or C_r

We consider $V = \mathbb{R}^r$ with standard basis $\{e^i\}$. Let $B_r = [\pm e_i, \pm(e_i \pm e_j),\ 1 \le i \le r,\ 1 \le i < j \le r]$ be the root system of type B and rank r.

Let $C_r = [\pm 2e_i, \pm(e_i \pm e_j),\ 1 \le i \le r,\ 1 \le i < j \le r]$ be the root system of type C and rank r.

As roots of the systems of type B and C are proportional, the system of hyperplanes in $U = \mathfrak{h}_{\mathbb{R}}^*$ defined by coroots of B and C are the same, and we denote that system (and the corresponding complex arrangement) by $\mathcal{H}_r^{BC}$. More precisely, let $z = \sum_{i=1}^r z^i e_i$ in $U_{\mathbb{C}}$, then the system of hyperplanes $\mathcal{H}_r^{BC}$ in $U_{\mathbb{C}}$ is given by $\mathcal{H}_r^{BC} = \cup_{1\le i<j\le r}\{z^i \pm z^j = 0\} \cup \cup_{1\le i\le r}\{z^i = 0\}$.

We take the set $\Phi^{eq}(BC_r) = \cup_{1\le i<j\le r}\{e^i \pm e^j = 0\} \cup \cup_{1\le i\le r}\{e^i = 0\}$ as equations of $\mathcal{H}_r^{BC}$.

We now define a flag basis $\overrightarrow{\sigma}$ of $\Phi^{eq}(BC_r)$. This is a basis of the form $\overrightarrow{\sigma} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$ of r elements of $\Phi^{eq}(BC_r)$ so that $\alpha_i = e^i$, or $e^i - e^j$ or

$e^i + e^j$ with $j < i$. That is, $\alpha_1 = e^1$, $\alpha_2 = e^2$ or $e^2 - e^1$, or $e^2 + e^1$, $\alpha_3 = e^3$ or $e^3 - e^2$, or $e^3 + e^2$, or $e^3 - e^1$, or $e^3 + e^1$, etc. Clearly, there are $1 \cdot 3 \cdot 5 \cdots (2r-1)$ such sequences $\overrightarrow{\sigma}$.

Lemma 2.3. *The set $\overrightarrow{\mathcal{D}}(BC_r)$ consisting of flag bases is a diagonal subset of $\overrightarrow{\mathfrak{B}}(\Phi^{eq}(BC_r))$.*

Proof. We first prove, by induction on r, that simple fractions f_b associated to a flag basis b generate $\mathcal{S}_{\mathcal{H}_r^{BC}}$. We use the identities

$$\frac{1}{(x_r - x_i)} \frac{1}{(x_r + x_i)} = \frac{1}{(x_r + x_i)} \frac{1}{2x_r} + \frac{1}{(x_r - x_i)} \frac{1}{2x_r},$$

$$\frac{1}{x_r} \frac{1}{(x_r + x_i)} = -\frac{1}{(x_r + x_i)} \frac{1}{x_i} + \frac{1}{x_r} \frac{1}{x_i}, \quad \frac{1}{x_r} \frac{1}{(x_r - x_i)} = \frac{1}{(x_r - x_i)} \frac{1}{x_i} - \frac{1}{x_r} \frac{1}{x_i},$$

to reduce to the case where a simple fraction f_b contains a linear form of type e^r, or $e^r + e^i$ or $e^r - e^i$ in the denominator, but not any two at the same time. Then, by induction on r, we see that the simple fractions f_b associated to flag basis b generates the space $\mathcal{S}_{\mathcal{H}_r^{BC}}$. The dual property on the elements of $\overrightarrow{\mathcal{D}}(BC_r)$ is evident. ☐

Remark 2.4. *Although the system $\Phi^{eq}(BC_r)$ is not unimodular for the lattice $\Lambda = \oplus \mathbb{Z} e^i$, we see that any $\overrightarrow{\sigma}$ in the set $\overrightarrow{\mathcal{D}}(BC_r)$ above is unimodular, so that the computation of $Z^\Lambda(\tau)(f_\sigma)$ is easy.*

2.1.3. The system of type D_r

We consider $V = \mathbb{R}^r$ with standard basis $\{e^i\}$. Let $D_r = [\pm(e_i \pm e_j); 1 \leq i < j \leq r]$ be the root system of type D and rank r. Let $z = \sum_{i=1}^r z^i e_i$ in $U_{\mathbb{C}}$. We consider the system of hyperplanes in $U_{\mathbb{C}}$

$$\mathcal{H}_r^D = \cup_{1 \leq i < j \leq r} \{z^i \pm z^j = 0\}.$$

The dimension of $\mathcal{S}_{\mathcal{H}_r^D}$ is known to be $1 \cdot 3 \cdot 5 \cdots (2r-3)(r-1)$. However, we did not find a nice diagonal basis for $\mathcal{S}_{\mathcal{H}_r^D}$. We proceed as follows: The set $U_{\mathcal{H}_r^D}$ of regular elements for $\mathcal{H}_r^D$ contains $U_{\mathcal{H}_r^{BC}}$. Indeed, for any z in $U_{\mathcal{H}_r^D}$, we have $z^i \pm z^j \neq 0$, but z^i may equal zero. We define the set

$$U_{k,r} := \{z^k = 0,\ z^i \pm z^j \neq 0 \text{ for } 1 \leq i < j \leq r \text{ and } z^i \neq 0 \text{ for } 1 \leq i \leq r,\ i \neq k\}.$$

Then, the following disjoint decomposition $U_{\mathcal{H}_r^D} = U_{\mathcal{H}_r^{BC}} \bigcup \cup_{k=1}^r U_{k,r}$ holds. The set $U_{k,r}$ is clearly isomorphic to the set $U_{\mathcal{H}_{r-1}^{BC}}$ in rank $r-1$ via the map i_k which inserts a zero coordinate in position k, and hence,

$$U_{\mathcal{H}_r^D} = U_{\mathcal{H}_r^{BC}} \bigcup \cup_{k=1}^r i_k(U_{\mathcal{H}_{r-1}^{BC}}). \tag{2.4.1}$$

The above decomposition allows us to reduce calculations in systems of type D to those of systems of type B or C.

2.2. Calculations of multiple Bernoulli series for type A

We use the same notation as in Section 2.1.1.

Let $Q_A \subset U$ be the root lattice generated by A_r, and $P_A \subset U$ be the weight lattice. Then P_A is generated by Q_A and $e_1 - \frac{1}{r+1}(e_1 + e_2 + \cdots + e_{r+1})$. The cardinality of P_A/Q_A is $r+1$.

Let Γ be a lattice such that $Q_A \subset \Gamma \subset P_A$. We denote by $\Gamma_{reg} = \Gamma \cap U_{\mathcal{H}_r^A}$ the set of regular elements in Γ. Let $\Lambda \subset V$ be the dual lattice to Γ, and let $\mathbf{s} = [s_\alpha]$ be a list of exponents. Define

$$g_{\mathbf{s}}^A(z) = \frac{1}{\prod_{\alpha>0} \langle H_\alpha, z\rangle^{s_\alpha}},$$

where the set $\{H_\alpha, \alpha > 0\}$ is the set of positive coroots $\Phi^{eq}(A_r)$. If $v \in V$,

$$\mathcal{B}(\mathcal{H}_r^A, \Lambda, g_{\mathbf{s}}^A)(v) = \sum_{\gamma \in \Gamma_{reg}} \frac{e^{2i\pi\langle v,\gamma\rangle}}{\prod_{\alpha>0}(2i\pi\langle H_\alpha, \gamma\rangle)^{s_\alpha}}.$$

If we use the diagonal basis $\overrightarrow{\mathcal{D}}_W$ for $\Phi^{eq}(A_r)$ as defined in Proposition 2.1, then it consists of elements $[e^{w(1)} - e^{w(2)}, \ldots, e^{w(r)} - e^{r+1}]$ where w is an element of W_r. Thus if we express $v = \sum_{i=1}^r v_i(e^i - e^{r+1})$, the algebra $Step(\overrightarrow{\mathcal{D}}_W)$ consists of functions $\{\sum_I v_i\}$ where I runs over subsets of $\{1, 2, \ldots, r\}$.

We now discuss two simple cases, where Γ is either the weight lattice P_A, or the root lattice Q_A.

2.2.1. Bernoulli series for the weight lattice

The dual of the weight lattice P_A is the coroot lattice $\check{Q}_A$ generated by simple coroots H_α, and the system $\Phi^{eq}(A_r)$ of equations (the positive coroots) is unimodular with respect to $\check{Q}_A$. Thus $\mathcal{B}(\mathcal{H}_r^A, \check{Q}_A, g_{\mathbf{s}}^A)(v)$ is a piecewise polynomial function of degree $\sum_\alpha s_\alpha$ and lies in the algebra $Step(\overrightarrow{\mathcal{D}}_W)$.

Our program then gives $\mathcal{B}(\mathcal{H}_r^A, \check{Q}_A, g_{\mathbf{s}}^A)(v)$ as a polynomial expression in $\{\sum_I v_i\}$. It also computes numerically the value of this function at any point v.

Example 2.5. Consider the root system of type A_2. For $\mathbf{s} = [1, 1, 1]$,

$$\mathcal{B}(\mathcal{H}_2^A, \check{Q}_A, g_{\mathbf{s}}^A)(v) = \sum_{\substack{m,n\in\mathbb{Z} \\ mn(m+n)\neq 0}} \frac{e^{2i\pi m v_1 - 2i\pi n v_3}}{(2i\pi m)(2i\pi n)(2i\pi(m+n))}. \tag{2.5.1}$$

where $v = v_1e^1 + v_2e^2 + v_3e^3$ with $v_1 + v_2 + v_3 = 0$. Denoting the fractional part of t with $\{t\} \in [0, 1[$, we obtain that $P(v_1, v_2, v_3) := \mathcal{B}(\mathcal{H}_2^A, \check{Q}_A, g_{\mathbf{s}}^A)(v)$ is equal to

$$\frac{1}{6}(\{v_2\}-\{v_1\})(\{v_1\}^2-3\{v_1+v_2\}\{v_1\}+\{v_2\}\{v_1\}+3\{v_1+v_2\}-1-3\{v_1+v_2\}\{v_2\}+\{v_2\}^2). \tag{2.5.2}$$

We remark that the series (2.5.1) is not absolutely convergent, but the sum has a meaning and is a piecewise polynomial function.

Let us give some numerical examples.

Consider A_2 with $\mathbf{s} = [10, 10, 10]$, $v_1 = v_2 = 0$. Then, $\mathcal{B}(\mathcal{H}_2^A, \check{Q}_A, g_{\mathbf{s}}^A)(0)$ is

$$\sum_{m,n}{}' \frac{1}{(2i\pi m)^{10}(2i\pi n)^{10}(2i\pi(m+n))^{10}} = -\frac{27739097}{417467193212109927669143961600000}.$$

Consider A_4. We list the exponents with respect to the following order on the roots $[e_1-e_2, e_1-e_3, e_1-e_4, e_1-e_5, e_2-e_3, e_2-e_4, e_2-e_5, e_3-e_4, e_3-e_5, e_4-e_5]$. For $\mathbf{s} = [6, 6, 6, 6, 4, 2, 2, 2, 2, 2]$ and $v = [0, 0, 0, 0, 0]$,

$$\mathcal{B}(\mathcal{H}_4^A, \check{Q}_A, g_{\mathbf{s}}^A)(0) = \frac{1}{(2i\pi)^{38}} \times$$

$$\sum{}' \frac{1}{m_1^6 m_2^4 m_3^2 m_4^2 (m_1+m_2)^6 (m_1+m_2+m_3)^6 (m_1+m_2+m_3+m_4)^6 (m_2+m_3)^2 (m_2+m_3+m_4)^2 (m_3+m_4)^2}$$

$$= \frac{66581757}{2081416538897698301902069565296214016000000000},$$

while for $\mathbf{s} = [4, 4, 4, 4, 4, 4, 4, 4, 4, 4]$ and $v = [0, 0, 0, 0, 0]$, we obtain

$$\mathcal{B}(\mathcal{H}_4^A, \check{Q}_A, g_{\mathbf{s}}^A)(0) = \frac{3998447009863}{193188341191020986049682108358620340866252800000000000}.$$

2.2.2. Bernoulli series for the root lattice

Let $\xi = \sum_{j=1}^{r}(e^j - e^{r+1})$. Then a system of representatives for $\check{P}_A/\check{Q}_A$ consists of the elements $\{\lambda_j = \frac{j}{r+1}\xi \; 0 \leq j \leq r\}$. Using Formula (1.23.1),

$$\mathcal{B}(\mathcal{H}_r^A, \check{P}_A, g_{\mathbf{s}}^A)(v) = \frac{1}{r+1}\sum_{j=0}^{r} \mathcal{B}(\mathcal{H}_r^A, \check{Q}_A, g_{\mathbf{s}}^A)(v + \lambda_j).$$

Hence, we obtain an expression for $\mathcal{B}(\mathcal{H}_r^A, \check{P}_A, g_{\mathbf{s}}^A)$ in terms of the functions $\{(\sum_I v_i) + c/(r+1)\}$ where c are integers between 0 and r.

Example 2.6. With the notation of Example 2.5, we now compute $\mathcal{B}(\mathcal{H}_2^A, \check{P}_A, g_{\mathbf{s}}^A)(v)$ for $v = v_1 e^1 + v_2 e^2 + v_3 e^3$ with $v_1 + v_2 + v_3 = 0$. We express $v = v_1(e^1 - e^2) - v_3(e^2 - e^3)$. For exponents $\mathbf{s} = [1, 1, 1]$,

$$\mathcal{B}(\mathcal{H}_2^A, \check{P}_A, g_{\mathbf{s}}^A)(v) = \sum_{\substack{m,n\in\mathbb{Z} \\ (2m-n)(2n-m)(m+n)\neq 0}} \frac{e^{2i\pi m(v_1-v_2)+2i\pi n(v_2-v_3)}}{(2i\pi(2m-n))(2i\pi(2n-m))(2i\pi(m+n))}$$

$$= \frac{1}{3}\left(P(v_1, v_2, v_3) + P(v_1 + \frac{1}{3}, v_2 + \frac{1}{3}, v_3 - \frac{2}{3}) + P(v_1 + \frac{2}{3}, v_2 + \frac{2}{3}, v_3 - \frac{4}{3})\right),$$

where P is the piecewise polynomial function given in Equation (2.5.2).

2.3. Calculation of multiple Bernoulli series for types B and C

We use the same notation as in Section 2.1.2.

We now consider $\mathcal{H}_r^{BC} = \cup_{1\leq i\leq r}\{z^i = 0\} \cup \cup_{1\leq i<j\leq r}\{z^i \pm z^j = 0\}$, a system of hyperplanes in $U_{\mathbb{C}}$.

Let Λ be a lattice commensurable with $\oplus \mathbb{Z}e^i$, with dual lattice Γ. Denote simply by $\Gamma_{reg} = \Gamma_{\mathrm{reg}}(\mathcal{H}_r^{BC})$. If $g \in \mathcal{R}_{\mathcal{H}_r^{BC}}$,

$$\mathcal{B}(\mathcal{H}_r^{BC}, \Lambda, g)(v) = \sum_{\gamma\in\Gamma_{\mathrm{reg}}} g(2i\pi\gamma)e^{2i\pi\langle v,\gamma\rangle}. \tag{2.6.1}$$

2.3.1. Root system C_r

Let P_C be the weight lattice of the root system C_r. We thus have the coroot lattice $\check{Q}_C = \oplus_{i=1}^r \mathbb{Z}e^i$. Let $\mathbf{s} = [s_\alpha]$ be a list of exponents and let

$$g_{\mathbf{s}}^C(z) = \frac{1}{\prod_{\alpha>0}\langle H_\alpha, z\rangle^{s_\alpha}}.$$

Here $\{H_\alpha,\ \alpha > 0\}$ are positive coroots of the system C_r, which are explicitly $\{e^i, 1 \leq i \leq r, (e^i \pm e^j), 1 \leq i < j \leq r\}$. Clearly, the function $g_{\mathbf{s}}^C$ belongs to $\mathcal{R}_{\mathcal{H}_r^{BC}}$. If $v \in V$,

$$\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v) = \sum_{\gamma\in(P_C)_{reg}} \frac{e^{2i\pi\langle v,\gamma\rangle}}{\prod_{\alpha>0}(2i\pi\langle H_\alpha,\gamma\rangle)^{s_\alpha}}.$$

The function $\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)$ is a piecewise polynomial function on V of degree $\sum_\alpha s_\alpha$. We use the diagonal basis constructed in Section 2.1.2 to compute it.

Let us now compute the example given in the introduction, which corresponds to C_2 with exponents $\mathbf{s} = [2, 1, 1, 1]$ ordered in accordance with the order $[2e_1, 2e_2, e_1 + e_2, e_1 - e_2]$ of positive roots (so that $[e^1, e^2, e^1 + e^2, e^1 - e^2]$ is the corresponding order in positive coroots). Express $v = v_1e^1 + v_2e^2$, then

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v) = {\sum_{m,n}}' \frac{e^{2i\pi m v_1 + 2i\pi n v_2}}{(2i\pi m)^2(2i\pi n)(2i\pi(m+n))(2i\pi(m-n))}.$$

This piecewise polynomial function $Q(v_1, v_2) := \mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v)$ is given by

$$\begin{aligned}
Q(v_1,v_2) = &-\frac{1}{160}\{-v_2+v_1\}^5 - \frac{1}{48}\{v_1\}^2 + \frac{1}{24}\{v_2+v_1\}^3\{v_2\} - \frac{1}{48}\{v_2+v_1\}^4\{v_2\} \\
&+ \frac{1}{24}\{v_1\}^3 - \frac{1}{48}\{v_2+v_1\}^2\{v_2\} - \frac{1}{960}\{v_2+v_1\} + \frac{1}{96}\{v_2+v_1\}^2 - \frac{1}{96}\{v_2+v_1\}^3 - \frac{1}{48}\{v_1\}^4 \\
&- \frac{1}{192}\{v_2+v_1\}^4 + \frac{1}{960}\{-v_2+v_1\} + \frac{1}{96}\{-v_2+v_1\}^2 - \frac{1}{32}\{-v_2+v_1\}^3 + \frac{5}{192}\{-v_2+v_1\}^4 \\
&+ \frac{1}{24}\{v_1\}^2\{v_2\} - \frac{1}{12}\{v_1\}^3\{v_2\} + \frac{1}{24}\{v_1\}^4\{v_2\} + \frac{1}{160}\{v_2+v_1\}^5 + \frac{1}{24}\{-v_2+v_1\}^3\{v_2\} \\
&- \frac{1}{48}\{-v_2+v_1\}^4\{v_2\} - \frac{1}{48}\{-v_2+v_1\}^2\{v_2\}.
\end{aligned} \tag{2.6.2}$$

Let us demonstrate what happens on a tope. Figure 2 depicts topes associated to the pair $\Phi^{eq}(BC_2) = \{e^1, e^2, e^1 + e^2, e^1 - e^2\}$ and $\Lambda = \mathbb{Z}e^1 \oplus \mathbb{Z}e^2$. For example, on $\tau_2 = \{v_1 > 0,\, v_2 > 0,\, v_1 > v_2,\, v_1 + v_2 < 1\}$, the piecewise polynomial function $\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v)$ coincides with the polynomial

$$\begin{aligned} Q_{\tau_2}(v_1, v_2) = & \frac{1}{8}(-\frac{1}{60}v_2 + \frac{1}{2}{v_1}^2 v_2 - {v_1}^3 v_2 + \frac{1}{6}{v_2}^2 - v_1 {v_2}^2 + v_1 {v_2}^3 + {v_1}^2 {v_2}^2 \\ & - \frac{1}{6}{v_2}^3 + \frac{1}{6}{v_2}^4 + \frac{1}{2}{v_1}^4 v_2 - {v_1}^2 {v_2}^3 - \frac{7}{30}{v_2}^5) \end{aligned} \tag{2.6.3}$$

We give some more numerical examples with different exponents. For example, we may compute with exponents $\mathbf{s} = [s_1, s_2, s_3, s_4]$ corresponding to the order $[2e_1, 2e_2, e_1 + e_2, e_1 - e_2]$ of roots, and $v = [v_1, v_2]$. Then,

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(v_1 e^1 + v_2 e^2) = \sum_{m,n}{}' \frac{e^{2i\pi m v_1 + 2i\pi n v_2}}{(2i\pi m)^{s_1}(2i\pi n)^{s_2}(2i\pi(m+n))^{s_3}(2i\pi(m-n))^{s_4}}$$

$$= \begin{cases} \frac{810650239}{132316540312500} & if \quad \mathbf{s}\text{=[2,2,1,1]} \quad and \quad \text{v=[1/5, 1/19]} \\ \frac{470361104388547613016364599941}{15291744295791972509433253459771267822387 20} & if \quad \mathbf{s}\text{=[2,3,4,5]} \quad and \quad \text{v=[1/7, 1/17]} \end{cases}$$

2.3.2. Root system B_r

Let $\Gamma = P_B$ be the weight lattice of the root system B_r, with its dual the coroot lattice $\check{Q}_B$. Let $\mathbf{s} = [s_\alpha]$ be a list of exponents. We define,

$$g_{\mathbf{s}}^B(z) = \frac{1}{\prod_{\alpha>0} \langle H_\alpha, z\rangle^{s_\alpha}},$$

where $\{H_\alpha,\ \alpha > 0\}$ are positive coroots of the system B_r. If $v \in V$,

$$\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) = \sum_{\gamma \in (P_B)_{reg}} \frac{e^{2i\pi\langle v, \gamma\rangle}}{\prod_{\alpha>0}(2i\pi\langle H_\alpha, \gamma\rangle)^{s_\alpha}}.$$

Clearly, as long coroots of B are twice the short coroots of C, and short coroots of B are long coroots of C, we have $g_{\mathbf{s}}^B = c_1 g_{\mathbf{s}}^C$ where $c_1 = \frac{1}{2^{s_{2e_1} + \cdots + s_{2e_r}}}$. Then $2\check{Q}_C$ is a sublattice of $\check{Q}_B$ of index 2^{r-1}, and a set of representatives is given, for example, by

$$F := \{0, e^{i_1} + e^{i_2} + \cdots + e^{i_k}, 1 \le i_1 < i_2 < \cdots < i_k,\ k = 2j,\ 1 \le j \le [r/2]\}.$$

We then use comparison Formulae (1.23.1) and (1.23.2). Since $g_{\mathbf{s}}^C$ is homogeneous of degree $-\sum_\alpha s_\alpha$, we obtain

$$\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) = \frac{1}{2^{r-1}} c_2 \left(\sum_{\lambda \in F} \mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(\frac{v+\lambda}{2}) \right)$$

where $c_2 = 2^{\sum_{1 \le i < j \le r} s_{e_i - e_j} + s_{e_i + e_j}}$. In particular, if $\mathbf{s} = [m, \ldots, m]$, then $c_2 = 2^{r(r-1)m}$.

For example, for B_2, we compute for $v = v_1 e^1 + v_2 e^2$ and exponents $\mathbf{s} = [2,1,1,1]$ ordered with respect to the order $[e_1 - e_2, e_2, e_1 + e_2, e_1]$ of roots,

$$(2i\pi)^5 \mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) = \sum_{m_1,m_2}{}' \frac{e^{2i\pi((m_1+1/2\,m_2)v_1+1/2\,m_2v_2)}}{m_1^2 m_2\,(2m_1+\,m_2)\,(m_2+m_1)}$$

where the symbol $\sum'$ means we sum over the integers m_1, m_2 with $(2m_1+\,m_2)\,(m_2+m_1)\,m_2m_1 \neq 0$. We obtain

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) = 2\left(Q(\frac{v_1}{2}, \frac{v_2}{2}) + Q(\frac{v_1+1}{2}, \frac{v_2+1}{2})\right)$$

where $Q = \mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)$ is given in Equation (2.6.2).

In particular, for $u = [1/15, 1/30]$ we obtain

$$\mathcal{B}(\mathcal{H}_2^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(u) = \frac{-276037}{5832000000}.$$

For B_3, we compute for $v = v_1e^1 + v_2e^2 + v_3e^3$ and $\mathbf{s} = [1,1,1,1,1,1,1,1,1]$:

$$(2i\pi)^9 \mathcal{B}(\mathcal{H}_3^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) =$$

$$\sum_{m_1,m_2,m_3}{}' \frac{e^{2i\pi((m_1+m_2+1/2\,m_3)v_1+(m_2+1/2\,m_3)v_2+1/2\,m_3v_3}}{(2m_1+2m_2+m_3)\,(2m_2+\,m_3)\,m_3m_1\,(m_1+2m_2+m_3)\,(m_1+m_2)\,(m_1+m_2+m_3)\,m_2\,(m_2+m_3)}.$$

We obtain

$$\mathcal{B}(\mathcal{H}_3^{BC}, \check{Q}_B, g_{\mathbf{s}}^B)(v) =$$

$$2^4\left(S(\frac{v_1}{2}, \frac{v_2}{2}, \frac{v_3}{2}) + S(\frac{v_1+1}{2}, \frac{v_2+1}{2}, \frac{v_3}{2}) + S(\frac{v_1+1}{2}, \frac{v_2}{2}, \frac{v_3+1}{2}) + S(\frac{v_1}{2}, \frac{v_2+1}{2}, \frac{v_3+1}{2})\right)$$

where $S = \mathcal{B}(\mathcal{H}_3^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)$ is a piecewise polynomial that is too long to be included here.

2.4. Calculation of multiple Bernoulli series for type D

We follow the same notation as in Section 2.1.3.

Recall the system of hyperplanes $\mathcal{H}_r^D = \cup_{i,j}\{z^i \pm z^j = 0, 1 \le i < j \le r\}$. Let $\mathbf{s} = [s_\alpha]$ be a list of exponents; the ordering of elements in the list $\mathbf{s}$ is taken to match the following ordering $[e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_r, e_2 - e_3, \ldots, e_1 + e_2, \ldots, e_{r-1} + e_r]$ of positive roots of the system D_r. We define

$$g_{\mathbf{s}}^D(z) = \frac{1}{\prod_{\alpha>0} \langle H_\alpha, z\rangle^{s_\alpha}},$$

where $\{H_\alpha,\ \alpha > 0\}$ are positive coroots of D_r.

We embed the list of roots of D_r in to the list of roots of B_r by writing the short roots e_i of B_r at the very end of the list. We denote by $\mathbf{S} = [s_\alpha, 0, \ldots, 0]$ the list obtained from $\mathbf{s}$ by adjoining r zeros to its end. Then, by construction, we have $g_{\mathbf{s}}^D(z) = g_{\mathbf{S}}^B(z)$.

We now associate to the list $\mathbf{s}$ a list of exponents $\mathbf{s}_k$ for the system B_{r-1}. In $\mathbf{s}$ we eliminate the position corresponding to the roots $e_i \pm e_k$ for $i < k$, and

$e_k \pm e_i$ for $i > k$. Then we assign the value $s_{e_i+e_k} + s_{e_i-e_k}$ to the exponent corresponding to root e_i of B_{r-1} for $i < k$, similarly we assign the value $s_{e_k+e_i} + s_{e_k-e_i}$ to the exponent corresponding to the root e_i of B_{r-1} for $i > k$.

We also let $i_k(v)$ to be the vector with $r-1$ coordinates obtained from $v = \sum_{i=1}^r v_i e^i$ by putting $v_k = 0$.

Let $\Gamma = P_D$ be the weight lattice of D and $\check{Q}_D$ the dual lattice generated by the coroots. Since P_D is the weight lattice of the simply connected group $\mathrm{Spin}(2r)$, $\gamma = \sum_{i=1}^r \gamma^i e_i$ is in P_D if $\gamma^i \pm \gamma^j \in \mathbb{Z}$ and $P_D = P_B$. Consider the intersection of P_D with the hyperplane $z^k = 0$. Then, we see that this intersection is isomorphic to the weight lattice of a system $C_{r-1,k}$ of type C, rank $r-1$, embedded in C of rank r with simple roots $\{e_1-e_2, e_2-e_3, \dots, e_{k-1}-e_{k+1}, e_{k+1}-e_{k+2}, \dots, e_r\}$.

Using the decomposition (2.4.1), we decompose the set of regular elements of the lattice P_D as a disjoint union of the set of regular elements of the lattice P_B and the set of regular elements of the lattice P_C of rank $r-1$. In particular, if $\gamma \in (P_D)_{reg}$ and $\gamma_k = 0$, then

$$g_{\mathbf{s}}^D(\gamma) = \frac{1}{\prod\limits_{\gamma_i \pm \gamma_j \neq 0,\, \gamma_k = 0} (\gamma_i - \gamma_j)^{s_{e_i-e_j}} (\gamma_i + \gamma_j)^{s_{e_i+e_j}}} = c_k g_{\mathbf{s}_k}^C(i_k(\gamma))$$

with $c_k = (-1)^{\sum_{j=k+1}^r s_{e_k-e_j}}$. In particular, if $\mathbf{s} = [m, \dots, m]$, then $c_k = (-1)^{(r-k)m}$. Thus, we can compute multiple Bernoulli series for the system of type D_r by using computations for types B_r and C_{r-1} with appropriate exponents. More explicitly,

$$\begin{aligned}\mathcal{B}(\mathcal{H}_r^D, \check{Q}_D, g_{\mathbf{s}}^D)(v) &= \sum_{\gamma \in (P_D)_{reg}} \frac{e^{2i\pi\langle v,\gamma\rangle}}{\prod_{\alpha>0}(2i\pi\langle H_\alpha, \gamma\rangle)^{s_\alpha}} \\ &= \mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_B, g_{\mathbf{S}}^B)(v) + \sum_{k=1}^r c_k \mathcal{B}(\mathcal{H}_{r-1}^{BC}, \check{Q}_{C_{r-1,k}}, g_{\mathbf{s}_k}^C)(i_k(v)).\end{aligned}$$

3. Witten formula for volumes of moduli spaces of flat connections on surfaces

As in Section 2, let G be a simple, connected, simply connected, compact Lie group of rank r with maximal torus T. For $g_1, g_2 \in G$, we denote by $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ the commutator of g_1 and g_2. Let Σ be a compact connected oriented surface of genus g and let $p := \cup_j\{p_j\}$ be a set of s points on Σ. Let $\mathcal{C} := (\mathcal{C}_j)$ be a set of s conjugacy classes in G. We consider the representation variety

$$\mathcal{M}(G, g, s, \mathcal{C}) := \{(a, c) \in G^{2g} \times \mathcal{C};\ \prod_{i=1}^g [a_{2i-1}, a_{2i}] = \prod_{j=1}^s c_j\}/G.$$

If the adjoint orbits $\mathcal{C}_j$ are generic, $\mathcal{M}(G, g, s, \mathcal{C})$ is an orbifold of real dimension $(2g-2)\dim G + s \dim G/T$. It parameterizes the set of flat G-valued connections on $\Sigma - p$, with holonomy around p_j belonging to the conjugacy class $\mathcal{C}_j$ modulo gauge equivalence. As shown by Atiyah-Bott [1], once a G-invariant inner product on $\mathfrak{g}$ is chosen, the manifold $\mathcal{M}(G, g, s, \mathcal{C})$ carries a natural symplectic form, and thus a natural volume form. E. Witten gave a formula for the volume of $\mathcal{M}(G, g, s, \mathcal{C})$ that we recall.

Let R^+ be a choice of positive roots; denote the highest root of R by θ. Let $\mathfrak{h}_+ := \{h \in \mathfrak{h}_{\mathbb{R}};\ \alpha(h) \geq 0 \text{ for all } \alpha \in R^+\}$ be the positive chamber (closed) in $\mathfrak{h}_{\mathbb{R}}$, and $\mathfrak{A} := \{h \in \mathfrak{h}_+;\ \theta(h) \leq 1\}$ be the fundamental alcove. An element of $\mathfrak{A}$ is said to be *regular* if it lies strictly inside the alcove. Let $W = W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ be the Weyl group (identified with $N_G(T)/T$).

We now give the Witten formula.

Let $\mathbf{a} = \{a_1, a_2, \ldots, a_s\}$ be a set of regular elements in $\mathfrak{A} \subset \mathfrak{h}_+$. Let $\mathcal{C}_j$ be the adjoint orbit of $\exp(a_j)$; we denote the collection of orbits $\mathcal{C}_j$ by $\mathcal{C}$. Consider the function on $\mathfrak{h}^*$ given by

$$N_{\mathbf{a}}(\lambda) = \prod_{j=1}^{s} \sum_{w \in W} \varepsilon(w) e^{\langle w a_j, \lambda \rangle}.$$

Let $\Phi = \Phi(G)$ be the list of positive coroots H_α. Define

$$W(\Phi(G), P, g, s)(\mathbf{a}) := \sum_{\gamma \in P^{reg}} \frac{N_{\mathbf{a}}(2i\pi\gamma)}{\prod_{H_\alpha \in \Phi} (2i\pi \langle H_\alpha, \gamma \rangle)^{2g-2+s}}.$$

The above expression is always meaningful as a generalized function of the parameters a_j. If $s = 0$, this formula has to be understood as

$$W(\Phi(G), P, g) = \sum_{\gamma \in P^{reg}} \frac{1}{\prod_{H_\alpha \in \Phi} (2i\pi \langle H_\alpha, \gamma \rangle)^{2g-2}}$$

which is meaningful if $g \geq 2$.

Interchanging the sum and the product, $N_{\mathbf{a}}(\lambda)$ may be expressed as

$$N_{\mathbf{a}}(\lambda) = \sum_{(w_1, w_2, \ldots, w_s) \in W^s} \prod_{j=1}^{s} \varepsilon(w_j) e^{\sum_{j=1}^{s} \langle w_j a_j, \lambda \rangle}.$$

Hence the function $W(\Phi(G), P, g, s)(\mathbf{a})$ can be expressed as a sum over W^s with signs of Bernoulli series $\mathcal{B}(\Phi_{2g-2+s}, \check{Q})(\sum_j w_j a_j)$. Here, as before, Φ_{2g-2+s} means that each coroot in Φ is taken with multiplicity $2g - 2 + s$. As is well known, the series $W(\Phi(G), P, g, s)(\mathbf{a})$ computes the symplectic volume of $\mathcal{M}(G, g, s, \mathcal{C})$ up to a scalar factor, which we will give in the next section.

Let us now recall the normalization of the volume as the limit of the Verlinde formula. To demonstrate this we need some more notation.

Let $(\ |\ )$ denote the G-invariant symmetric form on $\mathfrak{g}_{\mathbb{C}}$ normalized such that $(H_\theta | H_\theta) = 2$. We will use the same notation for the restricted form on $\mathfrak{h}$, and the induced form on $\mathfrak{h}^*$. We call $(\ |\ )$ the *basic invariant form.* It is positive definite on $\mathfrak{h}_{\mathbb{R}}$, and negative definite on $\mathfrak{t}$.

Let $\check{h} := \rho(H_\theta) + 1$ be the dual Coxeter number, where ρ is the half sum of positive roots. Let $Q_{long} \subset Q$ be the lattice spanned by long roots. The basic invariant form identifies $\mathfrak{h}_\mathbb{R}$ and ${\mathfrak{h}_\mathbb{R}}^*$; under this isomorphism the coroot lattice $\check{Q}$ is identified to Q_{long}. Let q be the index of Q_{long} in Q, and let f be the index of Q in P. The center of G is denoted by $Z = Z(G)$.

For a positive integer ℓ, define the set

$$P_\ell := \{\mu \in P \cap \mathfrak{h}_+^*; \mu(H_\theta) \leq \ell\}.$$

An element of P_ℓ is said to be a weight of *level* ℓ. We denote by P'_ℓ the subset of P_ℓ consisting of elements μ satisfying $\mu(H_\theta) < \ell$ and $\mu(H_\alpha) > 0$ for any simple root α. By definition of $\check{h}$, there is a bijection between sets P_ℓ and $P'_{\ell+\check{h}}$ via $\mu \mapsto \mu + \rho$.

Consider the maximal torus T of G with Lie algebra $\mathfrak{t}$. If $t = \exp X \in T$, with $X \in \mathfrak{t}$, and α is a root (which takes imaginary values on $\mathfrak{t}$), we denote by $e^\alpha(t) = e^{\langle \alpha, X\rangle}$. Let $\Delta(t) = \prod_{\alpha \in R}(e^\alpha(t) - 1)$. An element t of T is said to be regular if $\Delta(t) \neq 0$. Denote by T_ℓ the subgroup of elements t of T such that $e^\alpha(t)$ is $\ell + \check{h}$ root of unity for each long root α. We denote the set of regular elements in T_ℓ by T_ℓ^{reg}.

We now give the Verlinde formula.

Consider a set $\underline{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ with $\lambda_i \in P_\ell$. Then to this collection $\underline{\lambda}$, the group G, and a nonnegative integer g, is associated a vector space $\mathcal{V}(G, g, s, \underline{\lambda}, \ell)$ (see [15]), called the space of conformal blocks, whose dimension is given by the Verlinde formula $V(G, \underline{\lambda}, g, \ell)$:

$$V(G, \underline{\lambda}, g, \ell) = (fq)^{g-1}(\ell + \check{h})^{r(g-1)} \sum_{t \in T_\ell^{\mathrm{reg}}/W} \frac{\chi_{V(\underline{\lambda})}(t)}{\Delta(t)^{g-1}}.$$

Above r is the rank of G, $V(\underline{\lambda}) = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_s}$ where V_{λ_i} denotes the simple $\mathfrak{g}$ module with highest weight λ_i, and χ_{V_λ} denotes the character of V_λ. By Weyl character formula $\chi_{V_\lambda} = J(e^{\lambda+\rho})/J(e^\rho)$ where $J(e^\nu) = \displaystyle\sum_{w \in W} \varepsilon(w) e^{w\nu}$.

We remark that if $\sum_{i=1}^s \lambda_i$ is not in the root lattice, then $V(G, \underline{\lambda}, g, \ell)$ is zero.

Under the isomorphism given by the basic invariant form, an element a lying in $\mathfrak{A} \subset \mathfrak{h}_+$ defines an element $\hat{a}$ of $\mathfrak{h}_+^*$. We now consider a collection $\{a_1, a_2, \ldots, a_s\}$ of rational elements in $\mathfrak{A}$, that is, each a_j lies in the dense subset $\mathfrak{A} \cap (\check{Q} \otimes \mathbb{Q}) \subset \mathfrak{A}$. We may choose ℓ large enough so that each $\lambda_j := \ell \hat{a}_j$ is a weight; which then lies in P_ℓ. We furthermore choose ℓ so that $\sum_{j=1}^s \lambda_j$ is in the root lattice and consider the space of conformal blocks $\mathcal{V}(G, g, s, \underline{\lambda}, \ell)$ associated to this collection $\underline{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$. We can dilate simultaneously the weights λ_j and the level ℓ by a factor k. Then, the function

$$k \mapsto \dim(\mathcal{V}(G, g, s, [k\lambda_1, k\lambda_2, \cdots, k\lambda_s], k\ell))$$

is a quasi-polynomial in k of degree $m = \dim(G)(g-1) + s|R^+|$, the complex dimension of the moduli space $\mathcal{M}(G, g, s, \mathcal{C})$. The volume computes the highest term of this quasi-polynomial. More precisely,

$$\mathrm{vol}(\mathcal{M}(G,g,s,\mathcal{C}))=\lim_{k\to\infty}(\ell k)^{-m}dim(\mathcal{V}(G,g,s,[k\ell\hat{a}_1,\dots,k\ell\hat{a}_s],k\ell).$$

Proposition 3.1. *Let* $\mathbf{a}=\{a_1,\cdots,a_s\}$ *be a collection of regular rational elements in* $\mathfrak{A}$*. Let* $\mathrm{vol}(G,g)(\mathbf{a})$ *denote the symplectic volume of the moduli space* $\mathcal{M}(G,g,s,\mathcal{C})$*. Then,*

$$\mathrm{vol}(G,g)(\mathbf{a})=(fq)^{g-1}\frac{|Z|}{|W|}\epsilon_G^{p(2g-2+s)}(-1)^{(g-1)|\Phi(G)|}W(\Phi(G),P,g,s)(\mathbf{a}),$$

where p *is the number of short positive roots of* G*, and* $\epsilon_G=2$ *for any simple Lie group except for* G_2*, where* $\epsilon_G=3$*.*

We recall that for simply laced groups $p=0$ since all roots are considered as long.

Proof. Choose ℓ so that each $\lambda_j:=\ell\hat{a}_j$ lies in P_ℓ and $\sum_{j=1}^s\lambda_j$ is a root. Then,

$$\mathrm{vol}(G,g)(\mathbf{a})=\lim_{k\to\infty}\frac{1}{(k\ell)^m}V(G,k\underline{\lambda},g,k\ell),$$

where $m=\dim(G)(g-1)+s|R^+|$ is the dimension of $\mathcal{M}(G,g,s,\mathcal{C})$ and

$$V(G,k\underline{\lambda},g,k\ell)=(fq)^{g-1}(k\ell+\check{h})^{r(g-1)}\sum_{t\in T^{\mathrm{reg}}_{k\ell}/W}\frac{\chi_{V(k\underline{\lambda})}(t)}{\Delta(t)^{g-1}}.$$

An element $\mu\in P_\ell$ determines a unique regular element $h_\mu\in\mathfrak{A}$, the image of $\frac{\mu+\rho}{\ell+\check{h}}$ under the identification given by the basic invariant form. Denote the image of h_μ under the exponential map by $t_\mu\in T_{\mathbb{C}}$. The set $\{t_\mu:\mu\in P_\ell\}$ form a set of representatives for T^{reg}_ℓ/W. Using also the bijection between sets P_ℓ and $P'_{\ell+\check{h}}$ given by $\mu\mapsto\mu+\rho$,

$$\begin{aligned}V(G,k\underline{\lambda},g,k\ell)&=(fq)^{g-1}(k\ell+\check{h})^{r(g-1)}\sum_{\mu+\rho\in P'_{k\ell+\check{h}}}\frac{\chi_{V(k\underline{\lambda})}(t_\mu)}{\Delta(t_\mu)^{g-1}}\\&=(fq)^{g-1}(k\ell+\check{h})^{r(g-1)}|W|^{-1}\sum_{\mu+\rho\in W\cdot P'_{k\ell+\check{h}}}\frac{\prod_{j=1}^sJ(e^{k\lambda_j+\rho})(t_\mu)}{(J(e^\rho)(t_\mu))^s\Delta(t_\mu)^{g-1}}\\&=\frac{(fq)^{g-1}}{|W|}(k\ell+\check{h})^{r(g-1)}(-1)^{|\Delta^+|(g-1)}\sum_{\mu+\rho\in W\cdot P'_{k\ell+\check{h}}}\frac{\prod_{j=1}^sJ(e^{k\lambda_j+\rho})(t_\mu)}{(J(e^\rho)(t_\mu))^{2g-2+s}}.\end{aligned}$$

The second line above follows from the fact that both $\chi_{V(\lambda)}(t)$ and $\Delta(t)$ are W-invariant. The third line follows from the second by the identity

$$\Delta(t)=J(e^\rho)(t)\overline{J(e^\rho)(t)}=(-1)^{|R^+|}(J(e^\rho)(t))^2. \tag{3.1.1}$$

We now analyze $V(G,k\underline{\lambda},g,k\ell)$ as k gets large.

The expression $\prod_{j=1}^{s} J(e^{k\lambda_j+\rho})(t_\mu)$ in the summand is equal to

$$\prod_{j=1}^{s}\sum_{w\in W}\varepsilon(w)e^{w(k\lambda_j+\rho)}(t_\mu)=\prod_{j=1}^{s}\sum_{w\in W}\varepsilon(w)\exp\left(2i\pi\left(\mu+\rho|\frac{w(k\ell\hat{a}_j+\rho)}{k\ell+\check{h}}\right)\right).$$

Now as k gets large, the expression $\exp\left(2i\pi\left(\mu+\rho|\frac{w(k\ell\hat{a}_j+\rho)}{k\ell+\check{h}}\right)\right)$ approaches to $\exp(2i\pi(\mu+\rho|w\hat{a}_j))$. Observe also that the set $W\cdot P'_{k\ell+\check{h}}$ approaches P^{reg}. Denote an element $\mu+\rho$ of this limiting set by γ. Hence, $\prod_{j=1}^{s} J(e^{k\lambda_j+\rho})(t_\mu)$ approaches $\prod_{j=1}^{s}\sum\limits_{w\in W}\varepsilon(w)e^{\langle 2i\pi\gamma,wa_j\rangle}=N_{\mathbf{a}}(2i\pi\gamma)$.

Now we analyze the denominator of the summand,

$$\frac{1}{J(e^{\rho})(t_\mu)}=\frac{1}{\prod_{\alpha>0}(e^{\alpha/2}(t_\mu)-e^{-\alpha/2}(t_\mu))}=\frac{1}{\prod_{\alpha>0}2i\sin(\pi\frac{(\alpha|\mu+\rho)}{k\ell+\check{h}})}.$$

This expression explodes at each central vertex and the contribution from each, as k gets large, is $\frac{(k\ell+\check{h})^{|R^+|}}{\prod_{\alpha>0}2i\pi(\alpha|\mu+\rho)}$. Also observe that, for $z\in Z(G)$ both T_ℓ^{reg} and $\Delta(t)$ is invariant under $t\mapsto tz$. Moreover, since $\sum_{i=1}^{s}\lambda_i$ is in the root lattice by construction, $\chi_{V(k\underline{\lambda})}(t)$ is also invariant. Therefore, we may add all these equal contributions from central vertices. (see also Remark 5.8. [14]). Hence, we get that the expression

$$\sum_{\mu+\rho\in W\cdot P'_{k\ell+\check{h}}}\frac{\prod_{j=1}^{s}J(e^{k\lambda_j+\rho})(t_\mu)}{(J(e^{\rho})(t_\mu))^{2g-2+s}}$$

approaches

$$|Z(G)|(k\ell+\check{h})^{|R^+|(2g-2+s)}\sum_{\gamma\in P^{\mathrm{reg}}}\frac{N_{\mathbf{a}}(2i\pi\gamma)}{\prod_{\alpha>0}(2i\pi(\alpha|\gamma))^{2g-2+s}}.$$

By virtue of the normalization in the basic invariant form, if α is a long root we have $\langle H_\alpha,\gamma\rangle=(\alpha|\gamma)$; otherwise $\langle H_\alpha,\gamma\rangle=\epsilon_G(\alpha|\gamma)$. Using also that the dimension of G is $r+2|R^+|$, and that $|R^+|=|\Phi(G)|$, we obtain

$$\lim_{k\to\infty}\frac{1}{(k\ell)^m}V(G,k\underline{\lambda},g,k\ell)$$
$$=(fq)^{g-1}\frac{|Z(G)|}{|W|}(-1)^{|\Phi(G)|(g-1)}\epsilon_G^{p(2g-2+s)}W(\Phi(G),P,g,s)(\mathbf{a})$$

as claimed. $\square$

Remark 3.2. *In the case of one marking, the Verlinde formula reduces to*

$$V(G,\lambda,g,\ell)=(fq)^{g-1}(\ell+\check{h})^{r(g-1)}\sum_{t\in T_\ell^{reg}}\frac{e^{\lambda}(t)}{D(t)\Delta(t)^{g-1}},$$

where $D(t) = \prod_{\alpha>0}(1 - e^{-\alpha}(t))$.

Following the same line of arguments as in the proof of Proposition 3.1 we get that for $s = 1$ *with* $\lambda = \ell\hat{a}$ *lying in the root lattice,*

$$\mathrm{vol}(G,g)(a) = (fq)^{g-1}|Z|\epsilon_G^{p(2g-1)}(-1)^{(g-1)|\Phi|} \sum_{\gamma \in P^{reg}} \frac{e^{\langle a, 2i\pi\gamma\rangle}}{\prod_{H_\alpha \in \Phi}(2i\pi\langle H_\alpha, \gamma\rangle)^{2g-1}}. \tag{3.2.1}$$

That is,

$$\mathrm{vol}(G,g)(a) = (fq)^{g-1}|Z|\epsilon_G^{p(2g-1)}(-1)^{(g-1)|\Phi|}\mathcal{B}(\Phi_{2g-1}, \check{Q})(a).$$

3.1. Volume of the moduli space as a function of the volume of T *and* G

Let us recall the formula for the symplectic volume of the moduli space $\mathcal{M}(G, g, s, \mathcal{C})$ for a set of s regular conjugacy classes $\mathcal{C} = (\mathcal{C}_j)$ in G as given by E. Witten ([16] equation 4.1.14),

$$\mathrm{vol}(\mathcal{M}(G,g,s,\mathcal{C})) = \frac{|Z(G)|}{(2\pi)^{2m}}\mathrm{vol}(G)^{2g-2}\mathrm{vol}(G/T)^s \sum_{\lambda \in P^+} \frac{\prod_{j=1}^s[\chi_{V_\lambda}(\mathcal{C}_j)\sqrt{\Delta(\mathcal{C}_j)}]}{\dim V_\lambda{}^{2g-2+s}} \tag{3.2.2}$$

where $2m$ is the real dimension of $\mathcal{M}(G, g, s, \mathcal{C})$, and $P^+ = P \cap \mathfrak{h}_+^*$ is the set of dominant weights parametrizing irreducible representations of G. Above $\mathrm{vol}(G)$, $\mathrm{vol}(G/T)$ are Riemannian volumes of G and G/T which we now express following Bourbaki (Ch. IX, pages 396-411):

Choose a $\mathfrak{g}$-invariant scalar product on $\mathfrak{g}$. This determines a Lebesgue measure μ on $\mathfrak{g}$, via identification of $\mathfrak{g}$ with $\mathbb{R}^n$ by an orthonormal basis. Similarly let τ be the Lesbegue measure on $\mathfrak{t}$ corresponding to the restriction of the scalar product on $\mathfrak{t}$. We can construct from μ and τ Haar measures μ_G and μ_T on G and T respectively.

Since we aim to compare the volume formula in Proposition 3.1 with that of Witten in Equation (3.2.2), we choose the normalized Killing form as the $\mathfrak{g}$-invariant scalar product in the above construction, as this was our choice in the previous section. Then, for this choice, with respect to μ_G and μ_T constructed as above, we get that

$$\mathrm{vol}(G) = (fq)^{1/2}(2\pi)^{|R^+|+r}\frac{1}{\prod_{\alpha>0}(\alpha|\rho)}, \quad \mathrm{vol}(G/T) = \frac{(2\pi)^{|R^+|}}{\prod_{\alpha>0}(\alpha|\rho)}.$$

Observe that from formula (3.1.1), $\Delta(t)$ takes positive values on a regular element t. Then, for $\mathcal{C}_j$ the adjoint orbit of $\exp(a_j)$, we may write

$$\chi_{V_\lambda}(\mathcal{C}_j) = \frac{\sum_{w \in W} \varepsilon(w)e^{2i\pi\langle w(\lambda+\rho), a_j\rangle}}{i^{|R^+|}\sqrt{\Delta(\mathcal{C}_j)}}.$$

Let $d(\gamma) = \prod_{\alpha>0} \frac{\langle \gamma, H_\alpha \rangle}{\langle \rho, H_\alpha \rangle}$; it computes the dimension of the irreducible representation $V_{\gamma-\rho}$. Then we have

$$\sum_{\lambda \in P^+} \frac{\prod_{j=1}^s \chi_{V_\lambda}(\mathcal{C}_j)\sqrt{\Delta(\mathcal{C}_j)}}{\dim V_\lambda{}^{2g-2+s}} = \sum_{\lambda \in P^+} \frac{\prod_{j=1}^s N_{a_j}(2i\pi(\lambda+\rho))}{i^{s|R^+|} d(\lambda+\rho)^{2g-2+s}}.$$

Observe that the summand above is invariant under the Weyl group (both the numerator and the denominator are anti-invariant by factor $(\mathrm{sign}(w))^s$ for a Weyl group element w). Using this invariance, we get

$$\sum_{\lambda \in P^+} \frac{\prod_{j=1}^s \chi_{V_\lambda}(\mathcal{C}_j)\sqrt{\Delta(\mathcal{C}_j)}}{\dim V_\lambda{}^{2g-2+s}} = \frac{1}{|W|} \sum_{\gamma \in P^{reg}} \frac{\prod_{j=1}^s N_{a_j}(2i\pi\gamma)}{i^{s|R^+|d(\gamma)^{2g-2+s}}}.$$

Inserting the expressions for the volume of G and G/T into Equation (3.2.2), all (2π) factors cancel, and combining the terms we get

$$\begin{aligned}
\mathrm{vol}(\mathcal{M}(G,g,s,\mathcal{C})) &= \frac{|Z|}{|W|}(fq)^{g-1} \frac{(\prod_{\alpha>0} \langle \rho, H_\alpha \rangle)^{2g-2+s}}{(\prod_{\alpha>0} (\alpha|\rho))^{2g-2+s}} (-1)^{(g-1)|R^+|} W(\Phi(G),P,g)(a) \\
&= \frac{|Z|}{|W|}(fq)^{g-1} \epsilon_G^{p(2g-2+s)} (-1)^{(g-1)|R^+|} W(\Phi(G),P,g)(a),
\end{aligned}$$

which is precisely the formula that we obtained in Proposition 3.1.

4. Various examples of volume calculations

Example 4.1. We now compute the volume of the moduli space of SU(3) bundles on a Riemann surface of genus one.

Simple roots are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, and fundamental weights are $\omega_1 = \frac{2e_1 - e_2 - e_3}{3}$, $\omega_2 = \frac{e_1 + e_2 - 2e_3}{3}$. The positive coroots are

$$\Phi(\mathrm{SU}(3)) = \{H_{\alpha_1} = e^1 - e^2, H_{\alpha_2} = e^2 - e^3, H_{\alpha_1+\alpha_2} = e^1 - e^3\}$$

and $P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Let $\gamma = n_1\omega_1 + n_2\omega_2$. Then $\gamma \in P^{\mathrm{reg}}$ if and only if $n_1 \neq 0$, $n_2 \neq 0$ and $n_1 + n_2 \neq 0$. Now consider $a = a_1 H_{\alpha_1} + a_2 H_{\alpha_2} = a_1(e^1 - e^3) + (a_2 - a_1)(e^2 - e^3) \in \mathfrak{h}_\mathbb{R}$. Suppose that a is a regular element in $\mathfrak{A}$, in other words, $2a_1 - a_2 > 0$, $2a_2 - a_1 > 0$ (in particular $a_1 > 0$ and $a_2 > 0$) and $\theta(a_1 H_{\alpha_1} + a_2 H_{\alpha_2}) = a_1 + a_2 < 1$.

We compute the volume using Formula (3.2.1). We have $s = 1$, $p = 0$, $q = 1$, $f = 3$ and $|Z(SU(3))| = 3$; hence, for $g = 1$, $2^{p(2g-1)}(fq)^{g-1}|Z(G)| = 3$. Then,

$$\begin{aligned}
\mathrm{vol}(\mathrm{SU}(3),1)(a) &= 3 \sum_{n_1 \neq 0, n_2 \neq 0, n_1+n_2 \neq 0} \frac{e^{2i\pi(n_1 a_1 + n_2 a_2)}}{(2i\pi n_1)(2i\pi n_2)(2i\pi(n_1+n_2))} \\
&= \begin{cases} -\frac{1}{2}(1 + a_1 - 2a_2)(a_1 - 1 + a_2)(2a_1 - a_2) & a_1 \leq a_2 \\ -1/2(a_1 - 2a_2)(a_1 - 1 + a_2)(2a_1 - 1 - a_2) & a_1 \geq a_2 \end{cases}
\end{aligned}$$

Example 4.2. With the notation of Example 4.1, we make similar computations for SU(3) when genus $g = 2$.

In this case, $s = 1$, $p = 0$, $q = 1$, $f = 3$, $|Z(\mathrm{SU}(3))| = 3$, hence, for $g = 2$, $2^{s(2g-1)}(fq)^{g-1}|Z(G)|(-1)^{(g-1)|\Phi|} = -3^2$. Then,

$$\begin{aligned}\mathrm{vol}(\mathrm{SU}(3),2)(a) &= -9 \sum_{n_1\neq 0, n_2\neq 0, n_1+n_2\neq 0} \frac{e^{2i\pi(n_1a_1+n_2a_2)}}{(2i\pi n_1)^3(2i\pi n_2)^3(2i\pi(n_1+n_2))^3} \\ &= \begin{cases} 1/40320(a_2-2a_1)(a_2-1+a_1)(-1+2a_2-a_1)P_1 & a_1\leq a_2 \\ 1/40320(a_2+1-2a_1)(2a_2-a_1)(a_2-1+a_1)P_2 & a_1 \geq a_2\end{cases}\end{aligned}$$

where the polynomials P_1 and P_2 above are too long to be included in here.

Example 4.3. We now compute the volume of the moduli space of Spin(5) bundles on a Riemann surface of genus $g = 1$ with one marking.

Positive roots are $\{\alpha_1+\alpha_2 = e_1, \alpha_2 = e_2, \theta = e_1+e_2, \alpha_1 = e_1 - e_2\}$, with associated coroots $H_{e_1} = 2e^1$, $H_{e_2} = 2e^2$, $H_{e_1-e_2} = e^1 - e^2$, $H_{e_1+e_2} = e^1 + e^2$.

Let $a = a_1H_{\alpha_1} + a_2H_{\alpha_2}$ be a regular element in $\mathfrak{A}$; in other words, $a_1 > a_2$, $2a_2 > a_1$, $2a_2 < 1$. We can express a as $a = t_1e^1 + t_2e^2$ (with $t_1 = a_1$ and $t_2 = 2a_2 - a_1$), t_1 and t_2 satisfy $t_1 > t_2, t_2 > 0, t_1 + t_2 < 1$. We calculate the volume for B_2 and genus $g = 1$ employing the Formula (3.2.1). In this case, $s = 1$, $p = 2$, $q = 2$, $f = 2$, $|Z(\mathrm{Spin}(5))| = 2$. Hence, for $g = 1$, $2^{s(2g-1)}(fq)^{g-1}|Z(\mathrm{Spin}(5))|(-1)^{(g-1)|\Phi|} = 8$. We get,

$$\begin{aligned}\mathrm{vol}(B_2, g=1)(a) &= \frac{1}{2}t_2(t_1-1)(t_1-1+t_2)(t_1-t_2) \\ &= (2a_2-a_1)(a_1-1)(-1+2a_2)(a_1-a_2).\end{aligned}$$

Example 4.4. With the notation of Example 4.3, we compute the volume of the moduli space of Spin(5) bundles on a Riemann surface of genus one and two markings.

Let $\mathbf{a} = \{a_1, a_2\}$, where a_1 and a_2 are regular elements in $\mathfrak{A}$. Write $a_1 = t_1e^1 + t_2e^2$ and $a_2 = u_1e^1 + u_2e^2$. Then the function $\mathrm{vol}(B_2, 1)(\mathbf{a})$ is a piecewise polynomial function of t_1, t_2, u_1, u_2. For example, choose $v_1 = \frac{1}{2}e^1 + \frac{1}{5}e^2$, $v_2 = \frac{1}{7}e^1 + \frac{1}{9}e^2$ and consider $\tau(\mathbf{v}) \subset \mathfrak{A}\times\mathfrak{A}$, the open set determined by the condition that $a_1 + w(a_2)$ is in the same tope as $v_1 + w(v_2)$ for each element w in the Weyl group of B_2. Then for $\mathbf{a} \in \tau(\mathbf{v})$, we have

$$\begin{aligned}&\mathrm{vol}(B_2,1)(\mathbf{a}) = 4W(\Phi(B_2), P, 1, 2)(\mathbf{a}) \\ &= -\frac{1}{60}u_2{u_1}^5 + \frac{1}{6}t_2u_2t_1{u_1}^3 - \frac{1}{6}t_2{u_2}^3t_1u_1 + \frac{1}{6}{u_2}^3u_1{t_2}^2 - \frac{1}{6}u_2{t_2}^2{u_1}^3 - \frac{1}{12}{u_2}^3u_1{t_2}^3 \\ &+ \frac{1}{12}u_2{t_2}^3{u_1}^3 - \frac{1}{60}{u_2}^5t_1u_1 + \frac{1}{12}t_2u_2{t_1}^3{u_1}^3 - \frac{1}{4}t_2u_2{t_1}^2{u_1}^3 + \frac{1}{4}t_2{u_2}^3{t_1}^2u_1 + \frac{1}{6}{t_2}^2u_2t_1{u_1}^3 \\ &- \frac{1}{6}{t_2}^2{u_2}^3t_1u_1 + \frac{1}{60}u_2t_1{u_1}^5 - \frac{1}{12}{t_2}^3u_2t_1{u_1}^3 + \frac{1}{12}{t_2}^3{u_2}^3t_1u_1 - \frac{1}{12}t_2{u_2}^3{t_1}^3u_1 + \frac{1}{60}{u_2}^5u_1.\end{aligned}$$

5. More examples

In this section we compute some instances of volumes of moduli spaces of flat G-connections for classical simple Lie groups G using the formula given in Proposition 3.1. We remark that in this case ϵ_G is always 2.

Table 1: Value tables

G-type	q	f	p	$\lvert Z(G)\rvert$	$\lvert W\rvert$
A_r	1	$r+1$	0	$r+1$	$(r+1)!$
B_r	2	2	r	2	$2^r r!$
C_r	2^{r-1}	2	$r(r-1)$	2	$2^r r!$
D_r	1	4	0	4	$2^{r-1} r!$

We will denote by c_{vol} the factor

$$c_{\mathrm{vol}} := 2^{p(2g-2+s)}(fq)^{g-1}|Z(G)|(-1)^{(g-1)|\Phi(G)|}|W|^{-1}.$$

For convenience, we list values of the parameters in c_{vol} for each type of classical simple Lie group in Table 1.

5.1. Tables of volumes of moduli spaces

In the case of no marking ($s=0$) we denote the volume simply by $\mathrm{vol}(G,g)$. We list some values of $\mathrm{vol}(G,g)$ for classical simple Lie groups in Tables 2 and 3. We also list some values of the factor c_{vol} that we need in Section 5.2 to compare our computations with other numerical results in literature.

Computations are very quick for rank less or equal to 4 (and relatively small genus). Beyond rank 5, computations cannot be made within a time limit of half-hour with our method.

Here are some explicit running times regarding the volume tables:
Type A, rank 4 and genus 4 running time less then 1 sec.
Type A, rank 5 and genus 2 running time less then 1 sec.
Type B, rank 4 and genus 3 running time less then 1 sec.
Type B, rank 5 and genus 3 running time 5 hours.
Type C and rank 4 and genus 4 running time less then 1 sec.
Type C, rank 5 and genus 2 less then 1 sec.
Type D rank 5 and genus 2 running time 2 sec.
Type D, rank 4 and genus 4 running time half an hour.

All experiments were done with Maple 15 on a MacPro with a Quad-Core intel Xeon machine running at 2.93GHz.

G	g	vol(G,g)	c_{vol}
A_2	2	$\frac{1}{20160}$	$-\frac{3}{2}$
	3	$\frac{19}{41513472000}$	$\frac{9}{2}$
	4	$\frac{1031}{189225711747072000}$	$-\frac{27}{2}$
	5	$\frac{32293}{487445433460457472000000}$	$\frac{81}{2}$
	6	$\frac{27739097}{343594397705440269686538240000000}$	$-\frac{243}{2}$
	7	$\frac{29835840687589}{30319572290041089305610122050928640000000000}$	$\frac{729}{2}$
	8	$\frac{71810708985991}{59867814661023533299285522596881655398400000000000}$	$-\frac{2187}{2}$
	9	$\frac{22113713266984288663}{151246026314426013297816671756013269041872371712000000000000}$	$\frac{6561}{2}$
	10	$\frac{72520622051158753648014443}{406913205782775738093852803149608268676912309073384308736000000000000}$	$-\frac{19683}{2}$
A_3	2	$\frac{23}{653837184000}$	$\frac{2}{3}$
	3	$\frac{14081}{6481469910963304857600000}$	$\frac{8}{3}$
	4	$\frac{17634884778757}{11155091763154851836756404076509396992000000000}$	$\frac{32}{3}$
	5	$\frac{4257463829989959473}{3672156139411416383106635929141342829221732291706880000000000000}$	$\frac{128}{3}$
A_4	2	$\frac{1}{2730366140350464000}$	$\frac{5}{24}$
	3	$\frac{3998447009863}{18546080754338014660769482402427552723160268800000000}$	$\frac{25}{24}$
	4	$\frac{638972940367595659107077701677}{453620782564535886669774366873202293870244220323999062110716772391392567623680000000000000}$	$\frac{125}{24}$
A_5	2	$\frac{46511}{12663171527434001139297366835200000000}$	$-\frac{1}{20}$
D_2	2	$\frac{1}{36}$	4
	3	$\frac{1}{32400}$	16
	4	$\frac{1}{14288400}$	64
	5	$\frac{1}{5715360000}$	256
	6	$\frac{1}{2240649734400}$	1024
	7	$\frac{477481}{4174834601377440000000}$	4096
	8	$\frac{1}{340802824602240000}$	16384
	9	$\frac{13082689}{17373991677092354304000000000}$	65536
	10	$\frac{1924313689}{99574446563452076311839744000000}$	262144
D_3	2	$\frac{23}{653837184000}$	$\frac{2}{3}$
	3	$\frac{14081}{6481469910963304857600000}$	$\frac{8}{3}$
	4	$\frac{17634884778757}{11155091763154851836756404076509396992000000000}$	$\frac{32}{3}$
	5	$\frac{4257463829989959473}{3672156139411416383106635929141342829221732291706880000000000000}$	$\frac{128}{3}$
D_4	2	$\frac{68227}{1084047447508315948449792000000}$	$\frac{1}{12}$
	3	$\frac{3727283292300079}{4163527227475565552987044342329298418787470879468908707840000000000}$	$\frac{1}{3}$
	4	$\frac{1}{10^{18}}\frac{53725509445331487981115971039438961324463}{40159528749725537591038966810533732773060819287951411386355456046700908651220161332069154224393420 8}$	$\frac{4}{3}$
D_5	2	$\frac{24136975469}{949559334622106350631397499004290699425805762560000000000}$	$\frac{1}{120}$

Table 2: Witten volumes with $s = 0$ type A, D

G	g	$vol(G,g)$	c_{vol}
B_2	2	$\frac{1}{604800}$	16
	3	$\frac{4799}{444609285120000}$	1024
	4	$\frac{830287}{10586400854573397934080000}$	65536
	5	$\frac{15426208283}{26882357138658593827179724800000000}$	4194304
	6	$\frac{116091757826572267}{2764256255594966032075350680497658265600000000000}$	268435456
	7	$\frac{890103918575782649595999577}{289587313199907652270725517206183101789728520798208000000000000}$	17179869184
	8	$\frac{1189513900446338789485454541173}{52877369206125057710964488249573327181156553092261268684800000000000000}$	1099511627776
	9	$\frac{63380838431615435525579282051142l021}{38496369428311724563658733143905642747465680662764490448883809842626560000000000000000}$	70368744177664
	10	$\frac{372724427371281986645902844259956399824270911281 87}{3093222235039827696612775181342421259437768404686950682623785816645627805246413560872960000000000000000000}$	4503599627370496
B_3	2	$\frac{19}{4302395130249216000}$	$\frac{-32}{3}$
	3	$\frac{82197390067823}{913825117237645462467084621947649801584640000000000}$	$\frac{8192}{3}$
	4	$\frac{494277616807840302658 8037}{259372284831214187236355852194766861663874462695850253729180680192000000000000}$	$\frac{-2097152}{3}$
	5	$\frac{354908282467357148490713188145792722330626811}{10^{17}878489691400246134803977398980425404410705421923937124076525601021582376745441029139012748658606 08}$	$\frac{536870912}{3}$
B_4	2	$\frac{4999}{1099339586242016873556223477363507200000}$	$\frac{16}{3}$
	3	$\frac{29052511570551212254147642843 3}{29110138980741515852319493982928220072033854458512007601770585360017103402900354236416000000000000000}$	$\frac{16384}{3}$
B_5	2	$\frac{673653727654631}{114503199208433739395536475504707052302608351142027798548106444800000000000000}$	$-\frac{32}{15}$
C_2	2	$\frac{1}{604800}$	16
	3	$\frac{479}{444609285120000}$	1024
	4	$\frac{830287}{10586400854573397934080000}$	65536
	5	$\frac{15426208283}{26882357138658593827179724800000000}$	65536
	6	$\frac{116091757826572267}{2764256255594966032075350680497658265600000000000}$	268435456
	7	$\frac{890103918575782649595999577}{289587313199907652270725517206183101789728520798208000000000000}$	17179869184
	8	$\frac{1189513900446338789485454541173}{52877369206125057710964488249573327181156553092261268684800000000000000}$	1099511627776
	9	$\frac{633808384316154355255792820511421021}{38496369428311724563658733143905642747465680662764490448883809842626560000000000000000}$	70368744177664
	10	$\frac{37272442737128198664590284425995639982427091128187}{3093222235039827696612775181342421259437768404686950682623785816645627805246413560872960000000000000000000}$	4503599627370496
C_3	2	$\frac{19}{33612461955072000}$	$\frac{-4096}{3}$
	3	$\frac{328842292593389}{2231018352630970367351280815301879398400000000}$	$\frac{134217728}{3}$
	4	$\frac{790851606011875419185575 93}{197885349144908284939846688991368760424708910137825205787033600000000000000}$	$\frac{-4398046511104}{3}$
	5	$\frac{227141383379980233539822639608462691914466356 33}{20944826397901681299304423307905802831905017421816280462182178521670874994884515503382986752000000000000000000}$	$\frac{1441151880758558 72}{3}$
C_4	2	$\frac{2668859}{223661211393639500642135310336 0000000}$	$\frac{4194304}{3}$
	3	$\frac{297521088216082657235423644965697}{43377487332733744162797173832249969351342103568482410314337291359926914279014400000000000000000}$	$\frac{1125899906842624}{3}$
C_5	2	$\frac{5395475362754527}{1066394142885075317654167073822645957605991057511677952000000000000000}$	$\frac{-274877906944/15}{15}$

Table 3: Witten volumes with $s = 0$ type B, C

5.2. Comparison results

In this section we compare some of our computations of $\mathrm{vol}(G,g)$ with that of Komori-Matsumoto-Tsumura ([6],[7],[8],[9]). The setting is as follows.

Consider a simple, connected, compact Lie group G of rank r. Here we do not assume that G is simply-connected. Let L be the weight lattice of G. Let P be the weight lattice of the simply connected group covering G and let Q be its root lattice. Then, $Q \subset L \subset P$. We denote, as before, by P^+ the monoid of dominant weights. Let $L^+ = L \cap P^+$.

Let $\mathbf{s} = [s_\alpha]$ be a sequence of real variables indexed by positive roots R^+. For $v \in \mathfrak{h}_\mathbb{R}$, Komori-Matsumoto-Tsumura introduced

$$\zeta(\mathbf{s}, v, G) = \sum_{\gamma \in \rho + L^+} e^{2i\pi\langle v, \gamma\rangle} \prod_{\alpha \in R^+} \frac{1}{\langle \gamma, H_\alpha\rangle^{s_\alpha}}.$$

If G is simply connected, then $L = P$, and we may denote $\zeta(\mathbf{s}, v, G)$ by $\zeta(\mathbf{s}, v, \mathfrak{g})$, or for the Lie algebra $\mathfrak{g}$ of type X_r by $\zeta(\mathbf{s}, v, X_r)$ as in [8].

Example 5.1. Consider the simply connected group $G = \mathrm{SU}(4)$; its positive roots are $[e_1 - e_2, e_2 - e_3, e_3 - e_4, e_1 - e_3, e_2 - e_4, e_1 - e_4]$.

The monoid of dominant weights is freely generated by fundamental weights ω_1, ω_2 and ω_3 that are dual to simple coroots $e^1 - e^2$, $e^2 - e^3$ and $e^3 - e^4$ respectively. Then, if we order the exponents $\mathbf{s} = [s_i]$ with respect to the order of the roots as given above,

$$\begin{aligned}\zeta(\mathbf{s}, v, \mathrm{SU}(4)) &= \zeta(\mathbf{s}, v, A_3)\\ &= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\sum_{m_3=1}^{\infty} \frac{e^{2i\pi\langle v, m_1\omega_1 + m_2\omega_2 + m_3\omega_3\rangle}}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_1+m_2+m_3)^{s_6}}.\end{aligned}$$

The series $\zeta(\mathbf{s}, v, G)$ converges when the exponents s_α are sufficiently large. Let $S = \sum s_\alpha$. Suppose s_α are the same for all short roots, respectively for all long roots, and both are equal to positive even integers (that are not necessarily the same positive even integers). Then $(2\pi)^{-S}\zeta(\mathbf{s}, 0, G)$ is rational. Indeed, using the invariance of the sum under the Weyl group W, $(2\pi)^{-S}\zeta(\mathbf{s}, 0, G)$ is proportional to a Bernoulli series (with repetition of coroots in Φ matching the exponent data) which is obtained by summing over all the regular elements of the full lattice L. More precisely,

$$\frac{\zeta(\mathbf{s}, 0, G)}{(2\pi)^S} = |W|^{-1} i^S \sum_{\gamma \in L_{reg}} \frac{1}{\prod_{\alpha \in R^+}(2i\pi\langle\gamma, H_\alpha\rangle)^{s_\alpha}}, \tag{5.1.1}$$

where the series on the right hand side is a multiple Bernoulli series which has (in the case that it converges absolutely) rational value.

If all s_α are equal to an even integer $2k$, we denote the sequence $\mathbf{s} = [s_\alpha]$ by $\mathbf{s}_{2k}$. Then, for exponents $\mathbf{s}_{2k}$, and G simply connected, we may compute $\zeta(\mathbf{s}_{2k}, 0, G)$ using the Witten volume formula for $g = k+1$,

$$\begin{aligned}\zeta(\mathbf{s}_{2k}, 0, G) &= |W|^{-1}(2\pi)^{2k|R^+|}(-1)^{k|R^+|} W(\Phi(G), L, k+1)(0)\\ &= |W|^{-1}(2\pi)^{2k|R^+|}(-1)^{k|R^+|}\frac{1}{c_{\mathrm{vol}}}\mathrm{vol}(G, k+1)(0).\end{aligned} \tag{5.1.2}$$

Thus we can use the values of the volume listed in the tables of the previous section to compute some instances of the series $\zeta(\mathbf{s}_{2k}, 0, G)$. We now demonstrate some computations of $\zeta(\mathbf{s}_{2k}, 0, G)$.

5.2.1. Examples of type A_r

Let $n = r + 1$. We consider the simply connected group $G = \mathrm{SU}(n)$. If we write $N = |R^+| = \frac{n(n-1)}{2}$, then

$$\zeta(\mathbf{s}_{2k}, 0, A_r) = (-1)^{kN}(2\pi)^{2kN}\frac{1}{n!}\frac{1}{c_{\mathrm{vol}}}\mathrm{vol}(\mathrm{SU}(n), k+1)(0),$$

where $c_{\mathrm{vol}} = n^{k+1}(-1)^{k\frac{n(n-1)}{2}}\dfrac{1}{n!}$. Thus we obtain the values of $\zeta(\mathbf{s}_{2k}, 0, A_r)$ for $n = 3, 4, 5, 6$ using Table 2.

For instance, if $n = 3$ (that is $r = 2$), and $k = 1$, then we have $N = 3$, $\mathrm{vol}(\mathrm{SU}(3), 2)(0) = \frac{1}{20160}$ and $c_{\mathrm{vol}} = -3/2$, and we obtain

$$\zeta(\mathbf{s}_2, 0, A_2) = (2\pi)^6(-1)^3\frac{1}{3!}\frac{1}{9(-1)^3\frac{1}{3!}}\frac{1}{20160} = \pi^6\frac{1}{2835}$$

as in [9] equation (7.11).

We give one other example whose parameters are not contained in the tables. Consider $n = 4$, $k = 5$. Then, $N = 6$ and

$$\zeta(\mathbf{s}_{2k}, 0, A_3) = (2\pi)^{60}\times \frac{13936140662907425134123100095846}{58203152419058513584890890509712229288124323632762771449711578369140625}$$

5.2.2. Examples of type B_r, C_r and D_r

For root systems of type B_r and C_r, the number of positive roots is $N = r^2$. For example, for B_r when all $s_\alpha = 2k$,

$$\zeta(\mathbf{s}_{2k}, 0, B_r) = \frac{1}{r!2^r}(2\pi)^{2kN}(-1)^{kN}\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_B, g^B_{\mathbf{s}_{2k}})(0).$$

Explicitly for C_2, positive roots are $[e_1 - e_2, 2e_2, 2e_1, e_1 + e_2]$. We order the exponents with respect to the order given in the list of positive roots. Then,

$$\zeta([s_1, s_2, s_3, s_4], 0, C_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+2n)^{s_4}},$$

In the particular case of $\mathbf{s_2} = [2, 2, 2, 2]$, using Table 3,

$$\zeta(\mathbf{s_2}, 0, C_2) = \frac{1}{8}(-1)^4(2\pi)^8\frac{1}{16}\frac{1}{604800} = \frac{1}{302400}\pi^8,$$

which is the equation (7.23) of [9].

We also give an example of D_4 with all exponents equal to 6 (that is $k = 3$ and $\mathbf{s_6} = [6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6]$).

$$\zeta(\mathbf{s_6}, 0, D_4) = \pi^{72} \times \frac{5372550944533148798111597103943896132463}{217705241582232507678568106534510431311303415213232182911994028438087168146370880000000000000000000}$$

It is also possible to compute $\zeta(\mathbf{s}, 0, G)$ when the exponents in the list $\mathbf{s} = [s_1, s_2, s_3, s_4]$ are distinct positive even integers for short and long roots. We conclude with one example of this kind.

Consider the list of exponents $[2, 4, 4, 2]$ corresponding to the list of positive roots $[e_1 - e_2, 2e_2, 2e_1, e_1 + e_2]$ of C_2. Then,

$$\zeta([2, 4, 4, 2], 0, C_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 n^4 (m + n)^4 (m + 2n)^2} = \pi^{12} \frac{53}{6810804000},$$

which coincides with equation (4.30) of [9].

5.3. Some multiple zeta values

Let k be a positive integer. Consider the multiple zeta series

$$\zeta_r(2k, 2k, \ldots, 2k) := \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{1}{m_1^{2k}} \frac{1}{(m_1 + m_2)^{2k}} \cdots \frac{1}{(m_1 + m_2 + \cdots + m_r)^{2k}}.$$

Following [6], we want to demonstrate how the above series can be computed using the Bernoulli series $\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(0)$ for the root system of type C_r, where the exponents $\mathbf{s} = [s_\alpha]$ are taken to be 0 for long positive roots, and $2k$ for short positive roots. Using the invariance of the sum under the Weyl group, which is of order $2^r r!$ for C_r,

$$\mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(0) = 2^r r! \sum_{\gamma \in (P_C^+)_{reg}} \frac{1}{\prod_{\alpha > 0} (2i\pi \langle H_\alpha, \gamma \rangle)^{s_\alpha}}.$$

A dominant integral regular weight $\gamma \in (P_C^+)_{reg}$ is of the form $\gamma = \sum_{i=1}^r m_i \omega_i$ with $m_i \geq 1$ (where ω_i denote fundamental weights). Recall that the root system of type C_r admits r long roots $\{2e_i\}_{1 \leq i \leq r}$, with corresponding (short) coroots $\{H_{2e_i} = e^i\}_{1 \leq i \leq r}$. If we express $H_{2e_i} = e^i = (e^i - e^{i+1}) + (e^{i+1} - e^{i+2}) + \cdots + e^r$, then $\langle H_{2e_i}, \gamma \rangle = m_i + m_{i+1} + \cdots + m_r$. Thus,

$$\zeta_r(2k, 2k, \ldots, 2k) = (-1)^{kr} (2\pi)^{2kr} \frac{1}{2^r r!} \mathcal{B}(\mathcal{H}_r^{BC}, \check{Q}_C, g_{\mathbf{s}}^C)(0).$$

For example, $\zeta_2(4, 4) = \dfrac{\pi^8}{113400}$, $\zeta_5(4, 4, 4, 4, 4) = \dfrac{\pi^{20}}{548828480360160000}$ and $\zeta_5(6, 6, 6, 6, 6) = \dfrac{\pi^{30}}{1347828286825972065254765625}$.

6. Appendix: Szenes formula

Let $\mathcal{H}$ be an arrangement of hyperplanes compatible with a lattice Λ. Let $g \in \mathcal{R}_{\mathcal{H}}$. Consider

$$\mathcal{B}(\mathcal{H}, \Lambda, g)(v) = \sum_{\gamma \in \Gamma_{reg}(\mathcal{H})} g(2i\pi\gamma) e^{2i\pi\langle v,\gamma\rangle}.$$

This function (a generalized function on V) coincide with a polynomial function $\mathcal{B}(\mathcal{H}, \Lambda, g, \tau)$ on a tope τ (see Proposition 1.25). The piecewise polynomial function $P(\mathcal{H}, \Lambda, g)$ has been defined in Definition 1.14. Following Szenes [12], we prove the following formula.

Theorem 6.1. *(Szenes) Let $g \in \mathcal{R}_{\mathcal{H}}$. On $V_{reg}(\mathcal{H}, \Lambda)$ we have the equality*

$$\mathcal{B}(\mathcal{H}, \Lambda, g) = P(\mathcal{H}, \Lambda, g).$$

We recall that, for $f \in S_{\mathcal{H}}$, $Z^{\Lambda}(v)(f)(z) = \sum_{\gamma\in\Gamma} f(2i\pi\gamma - z)e^{\langle v, 2i\pi\gamma - z\rangle}$, and $P(\mathcal{H}, \Lambda, g)(v)$ is the trace on $S_{\mathcal{H}}$ of the operator $A(v, g) : \mathcal{S}_{\mathcal{H}} \to \mathcal{S}_{\mathcal{H}}$ defined by

$$f(z) \mapsto \mathbf{R}(e^{\langle z,v\rangle} g(z)(Z^{\Lambda}(v)f)(z)). \tag{6.1.1}$$

Here $\mathbf{R} : R_{\mathcal{H}} \to S_{\mathcal{H}}$ is the total residue.

We first consider the one dimensional case where $V = \mathbb{R}$, and $\Lambda = \mathbb{Z}$. Here $\mathcal{H} = \{0\}$, with equation $z = 0$. The topes are the intervals $]-n, n+1[$, and the space $\mathcal{S}_{\mathcal{H}}$ is one dimensional with basis $f_{\sigma} = \frac{1}{z}$.

Let $\tau =]0, 1[$. Assume $v \in \tau$ so that $[v] = 0$. If we consider $g(z) = \frac{1}{z^k}$, the formula to be proven is

$$\sum_{n\neq 0} \frac{e^{2i\pi nv}}{(2i\pi n)^k} = \mathrm{Res}_{z=0}(\frac{1}{z^k} e^{zv} (Z^{\Lambda}(\tau) f_{\sigma})(z)). \tag{6.1.2}$$

As $Z^{\Lambda}(\tau)(f_{\sigma})(z) = \frac{1}{1-e^z}$ (see Example 1.10), we have to verify that

$$\sum_{n\neq 0} \frac{e^{2i\pi nv}}{(2i\pi n)^k} = \mathrm{Res}_{z=0}(\frac{1}{z^k} e^{zv} \frac{1}{1-e^z}).$$

The poles of the function $\frac{1}{1-e^z}$ consist of the elements $2i\pi n$, with $n \in \mathbb{Z}$. When $k \geq 0$, the equality above follows from the residue theorem in one variable. If $k < 0$, both sides vanish (the left hand side gives a generalized function supported on $\mathbb{Z}$, the right hand side has no poles).

Szenes formula generalizes this result in higher dimensions, which we aim to demonstrate below.

Proof. Our proof is a slightly modified version of Szenes's proof where we use the total residue as opposed to iterated residue.

We first remark that using the comparison formula (1.14.1) and those of Lemma 1.23 over commensurable lattices, it suffices to prove the equality for any lattice Λ (compatible with $\mathcal{H}$) of our choice. We will prove Theorem 6.1 by the standard 'deletion-contraction' argument on arrangement of hyperplanes.

Choose a set Φ^{eq} of equations for $\mathcal{H}$. For $\phi \in \Phi^{eq}$, we consider the following two arrangements:

- $\mathcal{H}' = \mathcal{H} \setminus H_\phi$.
- $\mathcal{H}_0 = \{H \cap H_\phi, H \in \mathcal{H}'\}$, the trace of the arrangement $\mathcal{H}'$ on H_ϕ.

Consider the vector space $V_0 := V/\mathbb{R}\phi$, let $p : V \to V_0$ be the projection. The dual space U_0 of V_0 is the hyperplane H_ϕ. We now compare the spaces $\mathcal{S}_{\mathcal{H}}$, $\mathcal{S}_{\mathcal{H}_0}$ and $\mathcal{S}_{\mathcal{H}'}$.

Definition 6.2. *We say that a function $f \in \mathcal{M}_{\mathcal{H}}$ has at most a simple pole along the hyperplane $\phi = 0$ if $\phi f \in \mathcal{M}_{\mathcal{H}'}$. In this case, we define $res_\phi f \in \mathcal{M}_{\mathcal{H}_0}$ by $res_\phi f = (\phi f)|_{H_\phi}$*

In other words, the meromorphic function f has at most a simple pole on H_ϕ if the denominator of f contains the factor ϕ at most once. Then we multiply f by ϕ, eliminating ϕ from the denominator of f, and we can restrict ϕf to $\phi = 0$. This operation kills the functions f having no poles of $\phi = 0$.

If $f = \frac{1}{\phi} f'$ with $f' \in \mathcal{M}_{\mathcal{H}'}$, then

$$res_\phi \mathbf{R} f = \mathbf{R} res_\phi f. \tag{6.2.1}$$

This is easy to verify using for example a decomposition of f' with denominator on a set of independent hyperplanes (see Lemma 1.2).

The map res_ϕ is well defined on $\mathcal{S}_{\mathcal{H}}$, as elements in $\mathcal{S}_{\mathcal{H}}$ have at most a simple pole on $\phi = 0$. It is easy to prove that we have the exact sequence

$$0 \longrightarrow \mathcal{S}_{\mathcal{H}'} \xrightarrow{\ i\ } \mathcal{S}_{\mathcal{H}} \xrightarrow{res_\phi} \mathcal{S}_{\mathcal{H}_0} \longrightarrow 0. \tag{6.2.2}$$

Let $v \in V_{reg}(\Lambda, \mathcal{H})$. Its projection $v_0 = p(v)$ belongs to $V_{reg}(\Lambda_0, \mathcal{H}_0)$.

Lemma 6.3. *Let $v \in V_{reg}(\Lambda, \mathcal{H})$ and $f \in \mathcal{S}_{\mathcal{H}}$. Then*

$$res_\phi Z^\Lambda(v)(f) = -Z^{\Lambda_0}(v_0)(res_\phi f),$$

with $v_0 = p(v)$.

Proof. We have $Z^\Lambda(v)(f)(z) = \sum_{\gamma \in \Gamma} f(2i\pi\gamma - z) e^{\langle v, 2i\pi\gamma - z\rangle}$. If γ is such that $\langle \phi, \gamma \rangle \neq 0$, then the term $f(2i\pi\gamma - z)$ has no pole on $\phi = 0$. Thus we obtain, for $z \in H_\phi$,

$$\begin{aligned} res_\phi Z^\Lambda(v)(f)(z) &= \sum_{\gamma\in\Gamma, \langle\gamma,\phi\rangle=0} (\phi(z) f(2i\pi\gamma - z))|_{H_\phi} e^{\langle v_0, 2i\pi\gamma - z\rangle} \\ &= -\sum_{\gamma\in\Gamma, \langle\gamma,\phi\rangle=0} \phi(2\pi\gamma - z) f(2i\pi\gamma - z)|_{H_\phi} e^{\langle v_0, 2i\pi\gamma - z\rangle}. \end{aligned}$$

□

Lemma 6.4. *Let $g \in \mathcal{R}_{\mathcal{H}'}$. Then the following diagram is commutative.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}_{\mathcal{H}'} & \longrightarrow & \mathcal{S}_{\mathcal{H}} & \longrightarrow & \mathcal{S}_{\mathcal{H}_0} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle A(v,g)} & & \Big\downarrow{\scriptstyle A(v,g)} & & \Big\downarrow{\scriptstyle -A(v_0,g_0)} & & \\ 0 & \longrightarrow & \mathcal{S}_{\mathcal{H}'} & \longrightarrow & \mathcal{S}_{\mathcal{H}} & \longrightarrow & \mathcal{S}_{\mathcal{H}_0} & \longrightarrow & 0 \end{array} \tag{6.4.1}$$

Proof. Let $g \in \mathcal{R}_{\mathcal{H}'}$, and let g_0 be its restriction to H_ϕ. Then the operator $A(v,g)$ leaves $\mathcal{S}_{\mathcal{H}'}$ stable. If F has at most a simple pole on $\phi = 0$, then gF also has at most a simple pole on $\phi = 0$, as g has no pole on $\phi = 0$. Thus maps in the above diagram are well defined. Its commutativity follows from Lemma 6.3. □

We are now ready to prove Theorem 6.1 by induction on the number of hyperplanes in $\mathcal{H}$. If there are less than r hyperplanes, then $\mathcal{S}_{\mathcal{H}} = \{0\}$, the generalized function $\mathcal{B}(\mathcal{H},\Lambda,g)$ is supported on affine walls, so both sides of the equation of Theorem 6.1 vanish.

Assume that $\mathcal{H}$ consists of r independent hyperplanes. Changing the lattice Λ, we may assume that Λ is the lattice generated by the equations ϕ_k of the hyperplanes. Then, the theorem follows from Formula (6.1.2) in the one dimensional case.

Assume that $\mathcal{H}$ has more than r hyperplanes. Then by Lemma 1.2, we can write a function in $\mathcal{R}_{\mathcal{H}}$ as a sum of functions g whose poles lie on an independent subset of hyperplanes of $\mathcal{H}$, thus in number less or equal to r. Thus $\mathcal{R}_{\mathcal{H}}$ is linearly generated by functions g such that some equation $\phi \in \Phi^{eq}$ is not a pole of g. We consider such a couple (g,ϕ) and the arrangements $\mathcal{H}'$ and $\mathcal{H}_0$ associated to ϕ by deletion and contraction. The function g is in $\mathcal{R}_{\mathcal{H}'}$.

Let $g_0 \in \mathcal{R}_{\mathcal{H}_0}$ be the restriction of g to H_ϕ. Thus $\mathcal{B}(\mathcal{H}_0,\Lambda_0,g_0)$ is a generalized function on $H_\phi^* = V/\mathbb{R}\phi$ and $p^*\mathcal{B}(\mathcal{H}_0,\Lambda_0,g_0)$ is a function on V (constant in the direction ϕ).

We have the following recurrence relation for the function (eventually generalized) $\mathcal{B}(\mathcal{H},\Lambda,g)$ associated to an element $g \in R_{\mathcal{H}'}$.

Proposition 6.5. *If $g \in \mathcal{R}_{\mathcal{H}'}$, then*

$$\mathcal{B}(\mathcal{H},\Lambda,g) = \mathcal{B}(\mathcal{H}',\Lambda,g) - p^*\mathcal{B}(\mathcal{H}_0,\Lambda_0,g_0).$$

This is clear. Indeed the set $\Gamma_{reg}(\mathcal{H}')$ is larger than $\Gamma_{reg}(\mathcal{H})$ as it may contain also elements γ with $\langle\gamma,\phi\rangle = 0$. This additional summation gives rise to the term $\mathcal{B}(\mathcal{H}_0,\Lambda_0,g_0)$.

Let $v \in V_{reg}(\mathcal{H},\Lambda)$. As $P(\mathcal{H},\Lambda,g)(v)$ is the trace of the operator $A(v,g)$ defined in (6.1.1), the commutativity of the diagram (6.4.1) above implies that

$$P(\mathcal{H},\Lambda,g)(v) = P(\mathcal{H}',\Lambda,g)(v) - P(\mathcal{H}_0,\Lambda_0,g_0)(v_0).$$

Comparing with Proposition 6.5, we see by induction that Szenes formula holds. □

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