On the Free Resolution Induced by a Pommaret Basis

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Abstract

We combine the theory of Pommaret bases with a (slight generalisation of a) recent construction by Sköldberg based on discrete Morse theory. This combination allows us the explicit determination of a (generally non-minimal) free resolution for a graded polynomial module with the computation of only one Pommaret basis. If only the Betti numbers are needed, one can considerably simplify the computations by determining only the constant part of the differential. For the special case of a quasi-stable monomial ideal, we show that the induced resolution is a mapping cone resolution. We present an implementation within the CoCoALib and test it with some common benchmark ideals.

Key words: Free resolutions; syzygies; Betti numbers; Pommaret bases; algebraic discrete Morse theory; mapping cones

1 Introduction

Free resolutions are of fundamental importance in commutative algebra and algebraic geometry. In particular, the minimal free resolution of a homogeneous polynomial ideal encodes much relevant information about the ideal and is related to many important invariants like the Betti numbers. However, the explicit determination of the minimal free resolution (or in fact of any free

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resolution) is a computationally demanding task and only for certain special cases closed-form expressions for a resolution are known. Most algorithms presented in the literature follow either a vertical or a horizontal strategy, i. e. they construct the resolution either one homological degree after the other or they proceed according to the symmetric degree. Discussions can be found in (Kreuzer and Robbiano, 2005) or (La Scala and Stillman, 1998).

Seiler (2009b) showed that the (revlex) Pommaret basis of a polynomial ideal or submodule induces a free resolution. More precisely, via the involutive form of the Schreyer theorem, one can read off the Pommaret basis without any further computations the *shape* of a free resolution (leading to sharp upper bounds for the Betti numbers). However, a closed form of the *differential* could be derived by Seiler (2009b) only for quasi-stable monomial ideals. In general, the induced resolution is not minimal. One obtains a minimal resolution (in generic coordinates), if and only if the module is componentwise linear.

The free resolution induced by the Pommaret basis is highly structured consisting of "linear layers." Despite the fact that it is generally not minimal, this structure allowed Seiler (2009b) to obtain important homological invariants like the Castelnuovo-Mumford regularity or the projective dimension from the Pommaret basis without further computations. This article presents some new results from our ongoing analysis of this resolution with emphasis on "topological" methods for its construction.

Sköldberg (2006) and independently Jöllenbeck and Welker (2009) developed an algebraic version of the *discrete Morse theory* of Forman (1998) and started to apply it to the construction of resolutions. More recently, Sköldberg (2011) derived the minimal free resolution for modules with *initially linear (minimal) syzygies* from a two-sided Koszul complex. His definition of initially linear syzygies contains a minimality condition which restricts the applicability of his results to componentwise linear modules, but which is only needed for ensuring that the final resolution is the minimal one. Our first main result is that, after dropping this condition, one can apply his construction to the Pommaret basis of a submodule of a free module and obtains then a resolution isomorphic to the one presented in (Seiler, 2009b). In particular, we show that his construction yields then a Pommaret basis of each syzygy module. Thus combining Sköldberg's work with Pommaret bases makes it fully effective and applicable to arbitrary finitely presented polynomial modules, as the Pommaret bases lead automatically to initially linear syzygies whereas in Sköldberg's work even the existence of such a presentation remains an open question.

In algebraic geometry it often suffices to obtain the (bigraded) Betti numbers; the whole resolution is not really needed. Our second main result consists of showing that it is possible to extract directly the constant part of Sköldberg's differential without determining the remainder of the resolution. Then the determination of the Betti numbers is reduced to linear algebra over the ground field. This approach yields an algorithm which seems to be faster than any other proposed in the literature so far.

For arbitrary *monomial* ideals, a free resolution in closed form was e. g. provided by Taylor (1960). However, the problem of giving a closed form minimal resolution for arbitrary monomial ideals is wide open and different approaches to it have produced many interesting results, see e. g. (Miller and Sturmfels, 2004). One line of research in this area studies the minimal free resolution of particular families of monomial ideals. A seminal result in this respect is the minimal resolution of stable ideals found by Eliahou and Kervaire (1990).

Pommaret bases only exist in generic coordinates. For polynomial ideals, this is not a fundamental problem. However, in the case of monomial ideals, the required coordinate transformations generally destroy the monomiality. This observation leads to the class of *quasi-stable ideals* as those monomial ideals possessing a monomial Pommaret basis. In the literature, these ideals have been called ideals of nested type by Bermejo and Gimenez (2006), ideals of Borel type by Herzog et al. (2003) or weakly stable ideals by Caviglia and Sbarra (2005). Stable ideals are now those monomial ideals where already the minimal basis is the Pommaret basis (Mall, 1998). In this case, the resolution induced by the Pommaret basis is the Eliahou-Kervaire resolution, i. e. the minimal one. For arbitrary quasi-stable ideals, the induced resolution is very similar to the Eliahou-Kervaire resolution, but no longer minimal.

Another approach to the construction of—generally non-minimal—resolutions that can be used for any monomial ideal consists of iterated *mapping cones* and has been studied by various authors like Charalambous and Evans (1995) or Herzog and Takayama (2002). In particular, it was shown that both the Eliahou-Kervaire and the Taylor resolution can be obtained this way. Our third main result is that this is also the case for the resolutions of quasi-stable ideals induced by their Pommaret bases. The proof is based on the recent observation (Hashemi et al., 2012) that the notion of linear quotients introduced by Herzog and Takayama (2002) is closely related to Pommaret bases and the construction of a contracting homotopy for polynomial resolutions via Gröbner bases (Seiler, 2002).

Section 2 briefly reviews Pommaret bases and describes the free resolution induced by them. The following section gives a brief survey over algebraic discrete Morse theory and presents Sköldberg's construction of a resolution for modules with initially linear syzygies. In Section 4, we combine this construction with the theory of Pommaret bases and show that the two resolutions are isomorphic. The next section discusses a special case where the differential becomes much simpler and which includes in particular monomial ideals.

Section 6 describes our implementation in CoCoA and discusses the efficient determination of Betti numbers. In Section 7, we specialise to quasi-stable monomial ideals and show that here the Pommaret basis induces an iterated mapping cone resolution. Finally, some conclusions are given.

2 Pommaret Bases and the Induced Resolution

Involutive bases are Gröbner bases with additional combinatorial properties. They were introduced by Gerdt and Blinkov (1998a,b) who combined Gröbner bases with ideas from the algebraic theory of partial differential equations (Janet, 1929; Riquier, 1910)—see also related earlier works by Amasaki (1990) and Wu (1991). A survey over their basic theory can be found in (Seiler, 2009a) or (Seiler, 2010, Chapts. 3/4). Pommaret bases represent a special case which has turned out to be particularly useful in the context of algebraic geometry, as the (revlex) Pommaret basis of an ideal reflects many of its algebraic and homological properties—see (Seiler, 2009b) or (Seiler, 2010, Chapt. 5).

Throughout this work, $\mathbb{k}$ denotes a sufficiently large (preferably infinite) field of arbitrary characteristic and $\mathcal{P} = \mathbb{k}[x_1, \dots, x_n] = \mathbb{k}[\mathcal{X}]$ the polynomial ring in n variables over $\mathbb{k}$ together with the standard grading. For notational simplicity, we will present much of the theory for a homogeneous ideal $0 \neq \mathcal{I} \lhd \mathcal{P}$ but everything extends straightforwardly to a graded submodule $0 \neq \mathcal{U} \subseteq \mathcal{P}^t$. The homogeneous maximal ideal is denoted by $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$. In contrast to the standard conventions, we define the reverse lexicographic order for two terms of the same degree by $x^\mu \prec_{\text{revlex}} x^\nu$, if the first non-vanishing entry of $\mu - \nu$ is positive (this corresponds to reverting the ordering of the variables compared with the standard definition and thus is equivalent). If not explicitly stated otherwise, we always use revlex and in the module case its TOP lift.

Given an exponent vector $\mu = [\mu_1, \dots, \mu_n] \neq 0$ (or the term x^μ or a polynomial $f \in \mathcal{P}$ with $\operatorname{lt} f = x^\mu$ for some fixed term order), we call $\min\{i \mid \mu_i \neq 0\}$ the *class* of μ (or x^μ or f), denoted by $\operatorname{cls}\mu$ (or $\operatorname{cls} x^\mu$ or $\operatorname{cls} f$). Then the *multiplicative variables* of x^μ or f are $\mathcal{X}_P(x^\mu) = \mathcal{X}_P(f) = \{x_1, \dots, x_{\operatorname{cls}\mu}\}$; the remaining variables are the *non-multiplicative ones* $\overline{\mathcal{X}}_P(f) = \mathcal{X} \setminus \mathcal{X}_P(f)$. We say that x^μ is an *involutive divisor* of another term x^ν, if $x^\mu \mid x^\nu$ and $x^{\nu-\mu} \in \mathbb{k}[x_1, \dots, x_{\operatorname{cls}\mu}]$. Given a finite set $\mathcal{F} \subset \mathcal{P}$, we write $\deg \mathcal{F}$ for the maximal degree and $\operatorname{cls}\mathcal{F}$ for the minimal class of an element of $\mathcal{F}$.

Definition 2.1. Assume first that the finite set $\mathcal{H} \subset \mathcal{P}$ consists only of terms. $\mathcal{H}$ is a *Pommaret basis* of the monomial ideal $\mathcal{I} = \langle \mathcal{H} \rangle$, if as a $\mathbb{k}$-linear space

$$\bigoplus_{h \in \mathcal{H}} \mathbb{k}[\mathcal{X}_P(h)] \cdot h = \mathcal{I} \tag{1}$$

(in this case each term $x^\nu \in \mathcal{I}$ has a unique involutive divisor $x^\mu \in \mathcal{H}$). A finite polynomial set $\mathcal{H}$ is a *Pommaret basis* of the polynomial ideal $\mathcal{I}$ for the term order $\prec$, if all elements of $\mathcal{H}$ possess distinct leading terms and these terms form a Pommaret basis of the leading ideal $\operatorname{lt}\mathcal{I}$.

As the simple example $\mathcal{I} = \langle x_1x_2\rangle \lhd \Bbbk[x_1, x_2]$ demonstrates, not every ideal possesses a finite Pommaret basis. One can show that this is solely a problem of the used coordinate system: in generic coordinates, every ideal has a finite Pommaret basis. A deterministic approach for constructing "good" coordinates can be found in (Hausdorf et al., 2006). In the sequel, we always assume that such coordinates have been chosen. A monic, involutively autoreduced Pommaret basis is unique (Gerdt and Blinkov, 1998b).

It is well-known that Pommaret bases can be characterised similarly to Gröbner bases (Apel, 1998; Gerdt and Blinkov, 1998a). However, *involutive standard representations*, i. e. standard representation where each coefficient contains only multiplicative variables for the corresponding generator, are unique. The S-polynomials in the theory of Gröbner bases are replaced by products of the generators with one of their *non*-multiplicative variables.

Proposition 2.2. (Seiler, 2009a, Thm. 5.4) The finite set $\mathcal{H} \subset \mathcal{I}$ is a Pommaret basis of the ideal $\mathcal{I} \lhd \mathcal{P}$ for the term order $\prec$, if and only if every polynomial $0 \neq f \in \mathcal{I}$ possesses a unique involutive standard representation $f = \sum_{h\in\mathcal{H}} P_h h$ where each non-zero coefficient $P_h \in \Bbbk[\mathcal{X}_P(h)]$ satisfies $\operatorname{lt}(P_h h) \preceq \operatorname{lt}(f)$.

Proposition 2.3. (Seiler, 2009a, Cor. 7.3) Let $\mathcal{H} \subset \mathcal{P}$ be a finite set of polynomials and $\prec$ a term order such that no leading term in $\operatorname{lt}\mathcal{H}$ is an involutive divisor of another one. The set $\mathcal{H}$ is a Pommaret basis of the ideal $\langle\mathcal{H}\rangle$ with respect to $\prec$, if and only if for every $h \in \mathcal{H}$ and every non-multiplicative index $\operatorname{cls} h < j \leq n$ the product $x_j h$ possesses an involutive standard representation with respect to $\mathcal{H}$.

The classical Schreyer Theorem (Schreyer, 1980) describes how every Gröbner basis induces a Gröbner basis of its syzygy module for a suitable chosen term order. If $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\} \subset \mathcal{P}^t$ is a finite subset and $\prec$ a term order on $\mathcal{P}^t$, then the *Schreyer order* $\prec_\mathcal{H}$ is the term order on $\mathcal{P}^s$ defined by $x^\mu\mathbf{e}_\alpha \prec_\mathcal{H} x^\nu\mathbf{e}_\beta$, if $\operatorname{lt}(x^\mu\mathbf{h}_\alpha) \prec \operatorname{lt}(x^\nu\mathbf{h}_\beta)$ or if these leading terms are equal and $\beta < \alpha$.

Obviously, the Schreyer order $\prec_\mathcal{H}$ depends on the ordering of the elements of the set $\mathcal{H}$. For the involutive version of the Schreyer Theorem, we must order the elements of the Pommaret basis in a suitable manner. For this purpose, we recall some notions from (Seiler, 2009b). We associate a directed graph with each Pommaret basis $\mathcal{H}$. Its vertices are given by the elements in $\mathcal{H}$. If $x_j \in \overline{\mathcal{X}}_{\mathcal{H},\prec}(\mathbf{h})$ for some generator $\mathbf{h} \in \mathcal{H}$, then, by definition of a Pommaret basis, $\mathcal{H}$ contains a unique generator $\bar{\mathbf{h}}$ such that $\operatorname{lt}\bar{\mathbf{h}}$ is an involutive divisor of

$\operatorname{lt}(x_j\mathbf{h})$. In this case we include a directed edge from $\mathbf{h}$ to $\bar{\mathbf{h}}$. The thus defined graph is called the *P-graph* of the Pommaret basis $\mathcal{H}$. We order the elements of $\mathcal{H}$ as follows: whenever the P-graph of $\mathcal{H}$ contains a path from $\mathbf{h}_\alpha$ to $\mathbf{h}_\beta$, then we must have $\alpha < \beta$. Any ordering satisfying this condition is called a *P-ordering*. An explicit P-ordering can be described as follows: we require that if either $\operatorname{cls}\mathbf{h}_\alpha < \operatorname{cls}\mathbf{h}_\beta$ or $\operatorname{cls}\mathbf{h}_\alpha = \operatorname{cls}\mathbf{h}_\beta = k$ and and the last non-vanishing entry of $\operatorname{lt}\mathbf{h}_\alpha - \operatorname{lt}\mathbf{h}_\beta$ is negative, then we must have $\alpha < \beta$. Thus we sort the generators $\mathbf{h}_\alpha$ first by their class and within each class lexicographically (according to our "reverse" conventions).

Assume that $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ is a Pommaret basis of $\mathcal{U}$. According to Proposition 2.3, for every non-multiplicative variable x_k of a generator $\mathbf{h}_\alpha$ we have an involutive standard representation $x_k\mathbf{h}_\alpha = \sum_{\beta=1}^{s} P_\beta^{(\alpha;k)}\mathbf{h}_\beta$ and thus a syzygy

$$\mathbf{S}_{\alpha;k} = x_k\mathbf{e}_\alpha - \sum_{\beta=1}^{s} P_\beta^{(\alpha;k)}\mathbf{e}_\beta \tag{2}$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ denotes the standard basis of $\mathcal{P}^s$. Let $\mathcal{H}_{\mathrm{Syz}}$ be the set of all these syzygies.

Lemma 2.4. (Seiler, 2009b, Lemma 5.7) If the Pommaret basis $\mathcal{H}$ is P-ordered, then for all admissable values of α and k we find with respect to the Schreyer order $\prec_{\mathcal{H}}$ that

$$\operatorname{lt}\mathbf{S}_{\alpha;k} = x_k\mathbf{e}_\alpha\,. \tag{3}$$

Theorem 2.5. (Seiler, 2009b, Thm. 5.10) Let $\mathcal{H}$ be a P-ordered Pommaret basis of the polynomial submodule $\mathcal{U} \subseteq \mathcal{P}^t$. Then $\mathcal{H}_{\mathrm{Syz}}$ is a Pommaret basis of $\operatorname{Syz}(\mathcal{H})$ with respect to the Schreyer order $\prec_{\mathcal{H}}$.

Like the classical Schreyer Theorem, we can iterate Theorem 2.5 and obtain then a free resolution of the submodule $\mathcal{U}$.[1] However, in constrast to the classical situation, the involutive version yields the full shape of the arising resolution without any further computations. We present here a bigraded version of this result which is obtained by a trivial extension of the arguments in (Seiler, 2009b). It provides sharp upper bounds for the Betti numbers of $\mathcal{U}$.

Theorem 2.6. (Seiler, 2009b, Thm. 6.1) Let $\mathcal{H}$ be the Pommaret basis of the polynomial submodule $\mathcal{U} \subseteq \mathcal{P}^t$. If we denote by $\beta_{0,j}^{(k)}$ the number of generators $\mathbf{h} \in \mathcal{H}$ such that $\deg\mathbf{h} = j$ and $\operatorname{cls}\operatorname{lt}\mathbf{h} = k$ and by $d = \min\{k \mid \exists j : \beta_{0,j}^{(k)} > 0\}$ the minimal class of a generator, then $\mathcal{U}$ possesses a finite free resolution

$$0 \longrightarrow \bigoplus \mathcal{P}[-j]^{r_{n-d,j}} \longrightarrow \cdots \longrightarrow \bigoplus \mathcal{P}[-j]^{r_{1,j}} \longrightarrow \bigoplus \mathcal{P}[-j]^{r_{0,j}} \longrightarrow \mathcal{U} \longrightarrow 0 \tag{4}$$

[1] Related results were obtained earlier by Amasaki (1990) who uses the terminology *Weierstraß basis* instead of Pommaret basis.

of length $n-d$ where the ranks of the free modules are given by

$$r_{i,j} = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_{0,j-i}^{(k)} \,. \tag{5}$$

For the proof of this result, one shows that the Pommaret basis $\mathcal{H}_j$ of the jth syzygy module $\operatorname{Syz}^j(\mathcal{H})$ with respect to the Schreyer order $\prec_{\mathcal{H}_{j-1}}$ consists of the syzygies $\mathbf{S}_{\alpha;\mathbf{k}}$ with an ordered integer sequence $\mathbf{k} = (k_1, \dots, k_j)$ where $\operatorname{cls} h_\alpha < k_1 < \cdots < k_j \leq n$. These syzygies are defined recursively. We denote for any $1 \leq i \leq j$ by $\mathbf{k}_i$ the sequence obtained by eliminating k_i from $\mathbf{k}$. Now $\mathbf{S}_{\alpha;\mathbf{k}}$ arises from the involutive standard representation of $x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j}$:

$$x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j} = \sum_{\beta=1}^{p} \sum_{\boldsymbol{\ell}} P_{\beta;\boldsymbol{\ell}}^{(\alpha;\mathbf{k})} \mathbf{S}_{\beta;\boldsymbol{\ell}} \,. \tag{6}$$

Here the second sum is over all ordered integer sequences $\boldsymbol{\ell}$ of length $j-1$ satisfying $\operatorname{cls} \mathbf{h}_\beta < \ell_1 < \cdots < \ell_{j-1} \leq n$. Lemma 2.4 implies that

$$\operatorname{lt} \mathbf{S}_{\alpha;\mathbf{k}} = x_{k_j}\mathbf{e}_{\alpha;\mathbf{k}_j} \tag{7}$$

—leading by a simple combinatorial computation to the ranks $r_{i,j}$—and that the coefficient $P_{\beta;\boldsymbol{\ell}}^{(\alpha;\mathbf{k})}$ lies in $\Bbbk[x_1, \dots, x_{\ell_{j-1}}]$.

One can furthermore show that the free resolution (4) is of minimal length (i. e. $\operatorname{pd}\mathcal{U} = n-d$) (Seiler, 2009b, Thm. 8.11) and that $\operatorname{reg}\mathcal{U} = \deg \mathcal{H}$ (Seiler, 2009b, Thm. 9.2). Nevertheless, it is generally not minimal. Assuming that we are in generic coordinates (more precisely, in *componentwise δ-regular* coordinates (Hashemi et al., 2012, Def. 18)), one can show (Seiler, 2009b, Thm. 9.12) that it is minimal, if and only if $\mathcal{U}$ is componentwise linear (see (Herzog and Hibi, 1999) for a definition of this notion). We also note that it is minimal, if and only if the first syzygies $\mathbf{S}_{\alpha;k}$ do not contain constants (Seiler, 2009b, Lemma 8.1). These observations lead to a simple effective criterion for componentwise linearity.

For later use, we give an alternative description of the complex underlying the resolution (4). Let $\mathcal{W} = \bigoplus_{\alpha=1}^{s} \mathcal{P}\mathbf{w}_\alpha$ and $\mathcal{V} = \bigoplus_{i=1}^{n} \mathcal{P}\mathbf{v}_i$ be two free $\mathcal{P}$-modules whose ranks are given by the size of the Pommaret basis $\mathcal{H}$ and by the number of variables in $\mathcal{P}$, respectively. Then we set $\mathcal{C}_i = \mathcal{W} \otimes_{\mathcal{P}} \Lambda_i\mathcal{V}$ where $\Lambda_\bullet$ denotes the exterior product. A $\mathcal{P}$-linear basis of $\mathcal{C}_i$ is provided by the elements $\mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$ where $\mathbf{v}_\mathbf{k} = \mathbf{v}_{k_1} \wedge \cdots \wedge \mathbf{v}_{k_i}$ for an ordered sequence $\mathbf{k} = (k_1, \dots, k_i)$ with $1 \leq k_1 < \cdots < k_i \leq n$. Then the free subcomplex $\mathcal{S}_\bullet \subset \mathcal{C}_\bullet$ generated by all elements $\mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$ with $\operatorname{cls} h_\alpha < k_1$ corresponds to (4) upon the identification $\mathbf{e}_{\alpha;\mathbf{k}} \leftrightarrow \mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$. The differential comes from (6),

$$d_{\mathcal{S}}(\mathbf{w}_\alpha \otimes \mathbf{v}_{\mathbf{k},k_{j+1}}) = x_{k_{j+1}}\mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k} - \sum_{\beta,\boldsymbol{\ell}} P_{\beta;\boldsymbol{\ell}}^{(\alpha;\mathbf{k},k_{j+1})} \mathbf{w}_\beta \otimes \mathbf{v}_{\boldsymbol{\ell}} \,, \tag{8}$$

and thus requires the explicit determination of all the higher syzygies (6).

3 Algebraic Discrete Morse Theory and Sköldberg's Resolution

Algebraic discrete Morse theory was mainly developed by Sköldberg (2006) and by Jöllenbeck and Welker (2009). It provides techniques to reduce a large complex to a smaller one with the same homology. We briefly recall the main concepts; for further details and proofs we refer to the above cited works.

We consider a finite chain complex $C_\bullet$ of $\mathcal{P}$-modules

$$0 \longrightarrow C_p \longrightarrow C_{p-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0 \tag{9}$$

where each module $C_m = \bigoplus_{a\in I_m} K_a$ is written as a direct sum of $\Bbbk$-linear spaces with disjoint index sets I_m (Sköldberg calls this a *based complex*). To such a complex, we associate a directed graph $\Gamma_{C_\bullet}$. The set of vertices is $V = \sqcup_m I_m$ and the graph contains the edge $a \to b$ if and only if $a \in I_{m+1}$, $b \in I_m$ and $d_{b,a} = \pi_b(d_C|_{K_a}) \neq 0$ where d_C is the differential in $C_\bullet$ and $\pi_b : C_m = \bigoplus_{c\in I_m} K_c \to K_b$ for $b \in I_m$ the canonical projection.

A *partial matching* on a directed graph $D = (V, E)$ with vertices V and edges E is a subset $A \subseteq E$ of edges such that any vertex is incident to at most one edge in A. For such a partial matching A, we define a new directed graph $D^A = (V, E^A)$ by reversing all the arrows contained in A: thus the graph D^A has the same vertices as D and it contains the edge $a \to b$ if and only if $(b \to a) \in A \vee \big((a \to b) \in E \wedge (a \to b) \notin A\big)$. We define $A^+ \subseteq V$ to be the subset of those vertices that are the targets of the reversed arrows in A and $A^- \subseteq V$ as their sources; finally $A^0 \subseteq V$ contains all vertices which are not incident to any reversed arrow.

Definition 3.1. A *Morse matching* on the directed graph $\Gamma_{C_\bullet}$ is a partial matching A satisfying the following conditions:

- For every edge $a \to b$, the map $d_{b,a}$ is an isomorphism.
- Every index set I_m possesses a well-founded partial order $\prec$ such that for any $a, c \in I_m$ for which there is a path $a \to b \to c$ in $\Gamma^A_{C_\bullet}$ we have $c \prec a$. We say that such an order $\prec$ *respects the Morse matching* A.

The goal is to reduce the complex $C_\bullet$ to a smaller complex with the same homology using a Morse matching A. For the definition of the differential in the smaller complex, we will use *reduction paths* in $\Gamma^A_{C_\bullet}$. An *elementary reduction path* is a "zig-zag" path $\alpha_0 \to \beta \to \alpha_1$ of length two in $\Gamma^A_{C_\bullet}$ with

$\alpha_0, \alpha_1 \in I_m$ that also satisfies

$$\beta \in I_{m-1} \Longleftrightarrow \alpha_0 \in A^0 \cup A^+ \quad \text{and} \quad \beta \in I_{m+1} \Longleftrightarrow \alpha_0 \in A^- .$$

Note that there are also paths $\alpha_0 \to \beta \to \alpha_1$ of length two in the graph $\Gamma^A_{C_\bullet}$ with $\alpha_0, \alpha_1 \in I_m$ which are *not* elementary reduction paths: a path with $\beta \in I_{m-1}$ and $\alpha_0 \in A^-$ is considered by Sköldberg not as an elementary reduction path; we will see later that it would not make any difference to include them. For the elementary reduction path $\alpha_0 \to \beta \to \alpha_1$, we define the corresponding *elementary reduction* as the map

$$\rho_{\alpha_1,\alpha_0} = \begin{cases} -d^{-1}_{\beta,\alpha_1} \circ d_{\beta,\alpha_0} & \text{if } \beta \in I_{m-1} , \\ -d_{\alpha_1,\beta} \circ d^{-1}_{\alpha_0,\beta} & \text{if } \beta \in I_{m+1} . \end{cases}$$

A *(general) reduction path* p is a composition of elementary reduction paths

$$p = \alpha_0 \to \beta_1 \to \alpha_1 \to \cdots \to \beta_q \to \alpha_q$$

where $q \geq 0$. For two indices $\alpha, \alpha^* \in I_m$, there may exist several reduction paths from α to α^*; we write $[\alpha \rightsquigarrow \alpha^*]$ for the set of all such paths. For a general reduction path p, the *reduction* ρ_p is given by

$$\rho_p = \rho_{\alpha_q,\alpha_{q-1}} \circ \rho_{\alpha_{q-1},\alpha_{q-2}} \circ \cdots \circ \rho_{\alpha_1,\alpha_0} .$$

Definition 3.2. A graded polynomial module $\mathcal{M}$ has *initially linear syzygies*, if $\mathcal{M}$ possesses a finite presentation

$$0 \longrightarrow \ker \eta \longrightarrow \mathcal{W} = \bigoplus_{\alpha=1}^{s} \mathcal{P}\mathbf{w}_\alpha \xrightarrow{\ \eta\ } \mathcal{M} \longrightarrow 0 \tag{10}$$

such that with respect to some term order $\prec$ on the free module $\mathcal{W}$ the leading module $\operatorname{lt}\ker\eta$ is generated by terms of the form $x_j\mathbf{w}_\alpha$. We say that $\mathcal{M}$ has *initially linear minimal syzygies*, if the presentation is minimal in the sense that $\ker\eta \subseteq \mathfrak{m}^s$.

These notions go back to Sköldberg (2011) who, however, does not consider the non-minimal case. In his work "initially linear syzygies" always means initially linear *minimal* syzygies. His construction begins with the following two-sided Koszul complex $(\mathcal{F}, d_\mathcal{F})$ defining a free resolution of the module $\mathcal{M}$. Let $\mathcal{V}$ be a $\Bbbk$-linear space with basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ (with n still the number of variables) and set $\mathcal{F}_j = \mathcal{P} \otimes_\Bbbk \Lambda_j\mathcal{V} \otimes_\Bbbk \mathcal{M}$ which obviously yields a free $\mathcal{P}$-module. Choosing a $\Bbbk$-linear basis $\{m_a \mid a \in A\}$ of $\mathcal{M}$, a $\mathcal{P}$-linear basis of $\mathcal{F}_j$ is given by the elements $1 \otimes v_\mathbf{k} \otimes m_a$ with ordered sequences $\mathbf{k}$ of length j. The differential is now defined by

$$d_\mathcal{F}(1 \otimes \mathbf{v_k} \otimes m_a) = \sum_{i=1}^{j} (-1)^{i+1} \Big(x_{k_i} \otimes \mathbf{v}_{\mathbf{k}_i} \otimes m_a - 1 \otimes \mathbf{v}_{\mathbf{k}_i} \otimes x_{k_i} m_a \Big) \tag{11}$$

where $\mathbf{k}_i$ denotes the sequence $\mathbf{k}$ without the element k_i. Here it should be noted that the second term on the right hand side is not yet expressed in the chosen $\Bbbk$-linear basis of $\mathcal{M}$. For notational simplicity, we will drop in the sequel the tensor sign $\otimes$ and leading factors 1 when writing elements of $\mathcal{F}_\bullet$.

Under the assumption that the module $\mathcal{M}$ has initially linear syzygies via a presentation (10), Sköldberg (2011) constructs a Morse matching leading to a smaller resolution $(\mathcal{G}, d_\mathcal{G})$. He calls the variables

$$\operatorname{crit}(\mathbf{w}_\alpha) = \{x_j \mid x_j \mathbf{w}_\alpha \in \operatorname{lt}\ker\eta\}\,; \tag{12}$$

critical for the generator $\mathbf{w}_\alpha$; the remaining *non-critical* ones are contained in the set $\operatorname{ncrit}(\mathbf{w}_\alpha)$. A $\Bbbk$-linear basis of $\mathcal{M}$ is then given by all elements $x^\mu \mathbf{h}_\alpha$ with $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$ and $x^\mu \in \Bbbk[\operatorname{ncrit}(\mathbf{w}_\alpha)]$.

For each module element $m \in \mathcal{M}$, consider the following set of vertices in the graph $\Gamma_{F_\bullet}$:

$$V_m = \Big\{\mathbf{v}_I x^\mu \mathbf{h}_\alpha \mid x^I x^\mu \mathbf{h}_\alpha = m \wedge x^\mu \in \Bbbk[\operatorname{ncrit}\mathbf{w}_\alpha]\Big\}\,. \tag{13}$$

Then V_m is not empty, if and only if m is the product of some generator $\mathbf{h}_\alpha$ with a monomial. Furthermore, we define

$$\begin{aligned} A_m = \Big\{&\mathbf{v}_I x^\mu \mathbf{h}_\alpha \to \mathbf{v}_{I\setminus i} x_i x^\mu \mathbf{h}_\alpha \in \Gamma_\mathcal{F}|_{V_m} \mid \\ &i = \min\{I \cap \operatorname{ncrit}(\mathbf{w}_\alpha)\} \wedge i \leq \operatorname{cls}(x^\mu)\Big\}\,. \end{aligned} \tag{14}$$

Sköldberg (2011) gives a slightly different definition for the sets A_m; however, we think that our definition is more precise. By (Sköldberg, 2011, Lemma 2), the union $A = \bigcup_{m\in\mathcal{M}} A_m$ is a Morse matching on the graph $\Gamma_{\mathcal{F}_\bullet}$. A vertex $\mathbf{v}_\mathbf{k}\mathbf{h}_\alpha$ is not contained in A, if and only if $\mathbf{k} \subseteq \operatorname{crit}(\mathbf{w}_\alpha)$; in particular, all vertices of the form $\mathbf{v}_I x^\mu \mathbf{h}_\alpha$ with $\mu \neq 0$ appear in this Morse matching. We now define $\mathcal{G}_j \subseteq \mathcal{F}_j$ as the free submodule generated by those vertices $\mathbf{v}_\mathbf{k}\mathbf{h}_\alpha$ where the ordered sequences $\mathbf{k}$ are of length j and such that every entry k_i is critical for $\mathbf{w}_\alpha$. In particular, $\mathcal{W} \cong \mathcal{G}_0$ with an isomorphism induced by $\mathbf{w}_\alpha \mapsto \mathbf{v}_\emptyset \mathbf{h}_\alpha$.

Sköldberg (2011) gives two descriptions of the differential $d_\mathcal{G}$ in the reduced complex, a recursive one and an explicit one. For our purposes, the explicit one is better suited. It is based on reduction paths in the associated Morse graph and expresses the differential as a triple sum. If we assume that after expanding the right hand side of (11) in the chosen $\Bbbk$-linear basis of $\mathcal{M}$ the differential of the complex $\mathcal{F}_\bullet$ can be expressed as

$$d_\mathcal{F}(\mathbf{v}_\mathbf{k}\mathbf{h}_\alpha) = \sum_{\mathbf{m},\mu,\gamma} Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma} \mathbf{v}_\mathbf{m}(x^\mu \mathbf{h}_\gamma)\,, \tag{15}$$

then $d_{\mathcal{G}}$ is defined by

$$d_{\mathcal{G}}(\mathbf{v_k}\mathbf{h}_\alpha) = \sum_{\boldsymbol{\ell},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p} \rho_p\Big(Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v_m}(x^\mu \mathbf{h}_\gamma)\Big) \tag{16}$$

where the first sum ranges over all ordered sequences $\boldsymbol{\ell}$ which consists entirely of critical indices for $\mathbf{w}_\beta$ and the second sum may be restricted to all values such that a polynomial multiple of $\mathbf{v_m}(x^\mu\mathbf{h}_\gamma)$ effectively appears in $d_{\mathcal{F}}(\mathbf{v_k}\mathbf{h}_\alpha)$ and the third sum ranges over all reduction paths p going from $\mathbf{v_m}(x^\mu\mathbf{h}_\gamma)$ to $\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta$. Finally, ρ_p is the reduction associated with the reduction path p satisfying $\rho_p\Big(\mathbf{v_m}(x^\mu\mathbf{h}_\gamma)\Big) = c_p\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta$ for some polynomial $c_p \in \mathcal{P}$. In Appendix A, we show for a concrete ideal how the evaluation of (16) works in practice.

Sköldberg's main result is that $(\mathcal{G}, d_{\mathcal{G}})$ is the *minimal* free resolution of $\mathcal{M}$, if one starts with initially linear *minimal* syzygies. However, independent of this minimality assumption, his construction always yields a free resolution. We will show in the next section that for a submodule of a free module, his resolution is isomorphic to the resolution induced by a Pommaret basis.

4 Sköldberg's Construction and Pommaret Bases

We now combine Sköldberg's construction with Pommaret bases. Assume that the considered graded module is presented as $\mathcal{M} = \mathcal{P}^t/\mathcal{U}$ for a graded submodule $\mathcal{U} \subseteq \mathcal{P}^t$. Obviously, a free resolution of $\mathcal{U}$ immediately yields one of $\mathcal{M}$. Therefore we will restrict from now to the construction of resolutions for such submodules $\mathcal{U} \subseteq \mathcal{P}^t$ given by a Pommaret basis $\mathcal{H} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$. Unless stated otherwise, we will always assume that any Pommaret basis is enumerated according to a P-ordering. Furthermore, we extend the notion of class to positive homological degrees by setting $\operatorname{cls}(\mathbf{v_k}\mathbf{h}_\alpha) = \max \mathbf{k}$ for elements $\mathbf{v_k}\mathbf{h}_\alpha \in \mathcal{G}_{|\mathbf{k}|}$ with $\mathbf{k} \neq \emptyset$. As an immediate consequence of Lemma 2.4, we obtain the following trivial assertion.

Lemma 4.1. The submodule $\mathcal{U} \subseteq \mathcal{P}^t$ has initially linear syzygies [2] for the Schreyer order $\prec_{\mathcal{H}}$ and $\operatorname{crit}(\mathbf{w}_\alpha) = \overline{\mathcal{X}}_P(\mathbf{h}_\alpha)$, i. e. the critical variables of the generator $\mathbf{w}_\alpha$ are the non-multiplicative variables of $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$.

Sköldberg (2011, Cor. 4) proves that a module with initially linear minimal syzygies is always componentwise linear. It now follows from the above mentioned results of (Seiler, 2009b) on componentwise linearity that the converse is also true: modulo a coordinate transformation any componentwise linear module has initially linear minimal syzygies.

[2] Note that we apply here Definition 3.2 directly to $\mathcal{U}$ and not to $\mathcal{M} = \mathcal{P}^t/\mathcal{U}$, i. e. in (10) one must replace $\mathcal{M}$ by $\mathcal{U}$.

Corollary 4.2. If the graded polynomial module $\mathcal{M}$ is componentwise linear, then it can be presented as $\mathcal{M} = \mathcal{P}^t/\mathcal{U}$ such that the submodule $\mathcal{U}$ has initially linear minimal syzygies in componentwise δ-regular coordinates.

For later use, we will now distinguish three types of elementary reduction paths p in the graph $\Gamma^A_{\mathcal{F}_\bullet}$.

Type 0: In this case p is a path $\alpha_0 \to \beta \to \alpha_1$ with $\alpha_0, \alpha_1 \in I_m$ and $\beta \in I_{m-1}$. We will later see that these paths are irrelevant for the construction of the differential $d_{\mathcal{G}}$ of the reduced complex $\mathcal{G}_\bullet$.

All other elementary reduction paths are of the form

$$\mathbf{v}_{\mathbf{k}}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}(\frac{x^\mu}{x_i}\mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}(x^\nu \mathbf{h}_\beta)\,.$$

Here $\mathbf{k} \cup i$ is the ordered sequence which arises when i is inserted into $\mathbf{k}$; likewise $\mathbf{k} \setminus i$ stands for the removal of an index $i \in \mathbf{k}$.

Type 1: Here $\boldsymbol{\ell} = (\mathbf{k} \cup i)\backslash j$, $x^\nu = \frac{x^\mu}{x_i}$ and $\beta = \alpha$. Note that $i = j$ is allowed. We define $\epsilon(i; \mathbf{k}) = (-1)^{|\{j \in \mathbf{k} | j > i\}|}$. Then the corresponding reduction is

$$\rho(\mathbf{v}_{\mathbf{k}} x^\mu \mathbf{h}_\alpha) = \epsilon(i; \mathbf{k} \cup i)\epsilon(j; \mathbf{k} \cup i) x_j \mathbf{v}_{(\mathbf{k}\cup i)\backslash j}\Big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\Big)\,.$$

Type 2: Now $\boldsymbol{\ell} = (\mathbf{k} \cup i) \setminus j$ and $x^\nu \mathbf{h}_\beta$ appears in the involutive standard representation of $\frac{x^\mu x_j}{x_i}\mathbf{h}_\alpha$ with the coefficient $\lambda_{j,i,\alpha,\mu,\nu,\beta} \in \Bbbk$. In this case, by construction of the Morse matching, we have $i \neq j$. The reduction is

$$\rho(\mathbf{v}_{\mathbf{k}} x^\mu \mathbf{h}_\alpha) = -\epsilon(i; \mathbf{k} \cup i)\epsilon(j; \mathbf{k} \cup i)\lambda_{j,i,\alpha,\mu,\nu,\beta} \mathbf{v}_{(\mathbf{k}\cup i)\backslash j}(x^\nu \mathbf{h}_\beta)\,.$$

These reductions follow from the differential (11): the summands appearing there are either of the form $x_{k_i}\mathbf{v}_{\mathbf{k}_i} m_a$ or of the form $\mathbf{v}_{\mathbf{k}_i}(x_{k_i} m_a)$. For each of these summands, we have a directed edge in the graph $\Gamma^A_{\mathcal{F}_\bullet}$. Thus for an elementary reduction path

$$\mathbf{v}_{\mathbf{k}}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}\Big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\Big) \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}(x^\nu \mathbf{h}_\beta)\,,$$

the second edge can originate from summands of either form. For the first form we then have an elementary reduction path of type 1 and for the second form we have type 2.

For completeness, we note the following simple result which shows that the free resolution $\mathcal{G}$ indeed extends the presentation (10) and hence yields essentially the same first syzygies as the Pommaret basis.

Lemma 4.3. For an[3] $i \in \operatorname{crit}(\mathbf{h}_\alpha)$ let $x_i\mathbf{h}_\alpha = \sum_{\beta=1}^{s} P_\beta^{(\alpha;i)}\mathbf{h}_\beta$ be the involutive standard representation. Then we have $d_\mathcal{G}(\mathbf{v}_i\mathbf{h}_\alpha) = x_i\mathbf{v}_\emptyset\mathbf{h}_\alpha - \sum_{\beta=1}^{s} P_\beta^{(\alpha;i)}\mathbf{v}_\emptyset\mathbf{h}_\beta$.

Proof. Looking at the different types of reduction paths, we immediately see that in the differential (16) we can only have concatenations of elementary reduction paths of type 1 which are of the form

$$\mathbf{v}_\emptyset(x^\mu\mathbf{h}_\alpha) \longrightarrow \mathbf{v}_i\Big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\Big) \longrightarrow \mathbf{v}_\emptyset\Big(\frac{x^\mu}{x_i}\mathbf{h}_\beta\Big)\,.$$

The corresponding reduction is $\rho(\mathbf{v}_\emptyset x^\mu\mathbf{h}_\alpha) = x_i\mathbf{v}_\emptyset(\frac{x^\mu}{x_i}\mathbf{h}_\alpha)$. As $d_\mathcal{F}(\mathbf{v}_i\mathbf{h}_\alpha) = x_i\mathbf{v}_\emptyset\mathbf{h}_\alpha - \sum_{\beta=1}^{s}\mathbf{v}_\emptyset P_\beta^{(\alpha;i)}\mathbf{h}_\beta$, the reduction paths move the variables in a way that gives us the correct reduced differential $d_\mathcal{G}$. □

Our next result states that if one starts at a vertex $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha)$ with $i \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and follows through all possible reduction paths in the graph, one will never get to a point where one must calculate an involutive standard representation. If there are no critical (i. e. non-multiplicative) variables present at the starting point, then this will not change throughout any reduction path. In order to generalise this lemma to higher homological degrees, one must simply replace the conditions $i \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$ by ordered sequences $\mathbf{k}, \boldsymbol{\ell}$ with $\mathbf{k} \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $\boldsymbol{\ell} \subseteq \operatorname{ncrit}(\mathbf{h}_\beta)$.

Lemma 4.4. Assume that $i \cup \operatorname{supp}(x^\mu) \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$. Then for any reduction path $p = \mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_j(x^\nu\mathbf{h}_\beta)$ we have $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$ and $\beta = \alpha$. In particular, in this situation there is no reduction path $p = \mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_k\mathbf{h}_\beta$ with $k \in \operatorname{crit}(\mathbf{h}_\beta)$.

Proof. Assume first that p is an elementary reduction path. We distinguish two cases depending on the position of the starting point of p.

Case 1: Here $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \in A^0 \cup A^+$. Then the elementary reduction path must be of type 0 and p is either of the form $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \mathbf{v}_\emptyset(x_ix^\mu\mathbf{h}_\alpha) \to \mathbf{v}_{\operatorname{cls}(x_ix^\mu)}(\frac{x_ix^\mu}{x_{\operatorname{cls}(x_ix^\mu)}}\mathbf{h}_\alpha)$ or $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to x_i\mathbf{v}_\emptyset(x^\mu\mathbf{h}_\alpha) \to x_i\mathbf{v}_{\operatorname{cls}(x^\mu)}(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}}\mathbf{h}_\alpha)$. Since by assumption $i \cup \operatorname{supp}(x^\mu) \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$, we also have $\operatorname{cls}(x_ix^\mu) \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $\operatorname{cls}(x^\mu) \in \operatorname{ncrit}(\mathbf{h}_\alpha)$, resp., as claimed.

Case 2: If $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \in A^-$, then p can be either of type 1 or type 2.

Type 1: If p is of the form $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \mathbf{v}_{i,\operatorname{cls}(x^\mu)}(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}}\mathbf{h}_\alpha) \to \mathbf{v}_i(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}}\mathbf{h}_\alpha)$, then the statement is obvious. If, however, p is of the form $\mathbf{v}_i(x^\mu\mathbf{h}_\alpha) \to \mathbf{v}_{i,\operatorname{cls}(x^\mu)}(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}}\mathbf{h}_\alpha) \to \mathbf{v}_{\operatorname{cls}(x^\mu)}(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}}\mathbf{h}_\alpha)$, then the assumption $\operatorname{supp}(x^\mu) \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$ entails that also $\operatorname{cls}(x^\mu) \in \operatorname{ncrit}(\mathbf{h}_\alpha)$.

[3] For notational simplicity, we will often identify sets X of variables with sets of the corresponding indices and thus simply write $i \in X$ instead of $x_i \in X$.

Type 2: Here the path p is of the form $\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \mathbf{v}_{i,\mathrm{cls}(x^\mu)}(\frac{x^\mu}{x_{\mathrm{cls}(x^\mu)}}\mathbf{h}_\alpha) \to \mathbf{v}_{\mathrm{cls}(x^\mu)}(\frac{x_i x^\mu}{x_{\mathrm{cls}(x^\mu)}}\mathbf{h}_\alpha)$. As above $\mathrm{cls}(x^\mu) \in \mathrm{ncrit}\,(\mathbf{h}_\alpha)$ and by assumption $i \in \mathrm{ncrit}\,(\mathbf{h}_\alpha)$. Thus $\frac{x_i x^\mu}{x_{\mathrm{cls}(x^\mu)}}\mathbf{h}_\alpha$ is already an involutive standard representation. (Which means that, while in general a reduction path of type 2 might introduce a $\mathbf{h}_\beta$ for $\beta \neq \alpha$, this does not happen under the additional assumptions of this lemma).

For arbitrary reduction paths p, the claim now follows by an induction over the length of p. □

Now we can show the above claim that reduction paths of type 0 are irrelevant. Implicitly, this statement is already contained in (Sköldberg, 2006, Lemma 5).

Lemma 4.5. In the differential (16), no reduction path appearing in the third sum contains an elementary reduction path of type 0; i. e. all reduction paths appearing in the third sum are concatenations of elementary reduction paths of type 1 or 2.

Proof. Let p be a reduction path appearing in the sum in (16) ending at the vertex $\mathbf{v}_j\mathbf{h}_\gamma$ and $\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \mathbf{v}_\emptyset(x^\nu \mathbf{h}_\beta) \to \mathbf{v}_{\mathrm{cls}(x^\nu)}(\frac{x^\nu}{\mathrm{cls}(x^\nu)}\mathbf{h}_\beta)$ an elementary reduction path of type 0 appearing in p. As in the proof of Lemma 4.4, we then have $\mathrm{cls}(x^\nu) \in \mathrm{ncrit}\,(\mathbf{h}_\beta)$ and Lemma 4.4 implies that $j \in \mathrm{ncrit}(\mathbf{h}_\gamma)$. On the other hand, for any such reduction path appearing in (16), we must have $j \in \mathrm{crit}\,(\mathbf{h}_\gamma)$. □

The next results use Schreyer orders on the components of the complex $\mathcal{G}$. We define $\mathcal{H}_0$ as the Pommaret basis of $d_\mathcal{G}(\mathcal{G}_1) \subseteq \mathcal{G}_0$ with respect to the Schreyer order $\prec_\mathcal{H}$ induced by the term order $\prec$ on $\mathcal{P}^t$ and $\mathcal{H}_i$ as the Pommaret basis of $d_\mathcal{G}(\mathcal{G}_{i+1}) \subseteq \mathcal{G}_i$ for the Schreyer order $\prec_{\mathcal{H}_{i-1}}$. For the next lemma, we further remark that in order to apply there these Schreyer orders, we need to have both $x^{\kappa+\mu}\mathbf{v}_j\mathbf{h}_\beta \in \mathcal{G}_1$ and $x^\nu \mathbf{v}_i \mathbf{h}_\alpha \in \mathcal{G}_1$, i.e. $j \in \mathrm{crit}\,(\mathbf{h}_\beta)$ and $i \in \mathrm{crit}\,(\mathbf{h}_\alpha)$. Indeed, by Lemma 4.4, $j \in \mathrm{ncrit}\,(\mathbf{h}_\beta)$ and $i \in \mathrm{ncrit}\,(\mathbf{h}_\alpha)$, respectively, implies that the reduction path in question cannot be part of a longer reduction path that ends in a $\mathbf{v}_k\mathbf{h}_\gamma$ with $k \in \mathrm{crit}\,(\mathbf{h}_\gamma)$ which would be necessary for the reduction path to appear in (16).

Lemma 4.6. Let $p = \mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_j(x^\nu \mathbf{h}_\beta)$ be a reduction path that appears in the differential (16) (possibly as part of a longer path). If $\rho_p\big(\mathbf{v}_i(x^\mu \mathbf{h}_\alpha)\big) = x^\kappa \mathbf{v}_j(x^\nu \mathbf{h}_\beta)$, then $\mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^{\kappa+\nu}\mathbf{v}_j\mathbf{h}_\beta) \preceq_{\mathcal{H}_1} \mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^\mu \mathbf{v}_i \mathbf{h}_\alpha)$.

Proof. Again we prove the assertion only for an elementary reduction path p and the general case follows by induction over the path length. If p is of type 1, then we have either $\rho_p\big(\mathbf{v}_i(x^\mu \mathbf{h}_\alpha)\big) = x_k \mathbf{v}_i(\frac{x^\mu}{x_k}\mathbf{h}_\alpha)$, where the claim is obvious, or $\rho_p\Big(\mathbf{v}_i(x^\mu\mathbf{h}_\alpha)\Big) = x_i\mathbf{v}_k(\frac{x^\mu}{x_k}\mathbf{h}_\alpha)$ for an index $k \in \operatorname{supp} x^\mu$, so $k \in \operatorname{ncrit}\mathbf{h}_\alpha$. But by the same argument as in the proof of Lemma 4.5, the last case cannot occur.

If p is of type 2, there exists an index $j \in \operatorname{supp} x^\mu$ (implying $j \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and thus $j \leq \operatorname{cls}(\mathbf{h}_\alpha)$), a multi index ν and a scalar $\lambda \in \Bbbk$ such that $\rho_p\Big(\mathbf{v}_i(x^\mu\mathbf{h}_\alpha)\Big) = \lambda\mathbf{v}_j(x^\nu\mathbf{h}_\gamma)$ where $x^\nu\mathbf{h}_\gamma$ appears in the involutive standard representation of $\frac{x^\mu x_i}{x_j}\mathbf{h}_\alpha$ with a non-vanishing coefficient. Lemma 4.4 implies $j \in \operatorname{crit}(\mathbf{h}_\gamma)$. By construction, $\operatorname{lt}_\prec(\frac{x_i x^\mu}{x_j}\mathbf{h}_\alpha) \succeq \operatorname{lt}_\prec(x^\nu\mathbf{h}_\gamma)$.

Here we must distinguish between equality and strict inequality. In the first case, $\operatorname{lt}_\prec(\frac{x_i x^\mu}{x_j}\mathbf{h}_\alpha) = \operatorname{lt}_\prec(x^\nu\mathbf{h}_\gamma)$ and hence $\operatorname{cls}(\mathbf{h}_\alpha) \leq \operatorname{cls}(\mathbf{h}_\gamma)$. But then $j \in \operatorname{ncrit}(\mathbf{h}_\alpha) \subseteq \operatorname{ncrit}(\mathbf{h}_\gamma)$. Analogously to Lemma 4.5, such reduction paths do not appear in the differential (16). If in the second case the strict inequality $\operatorname{lt}_\prec(\frac{x_i x^\mu}{x_j}\mathbf{h}_\alpha) \succ \operatorname{lt}_\prec(x^\nu\mathbf{h}_\gamma)$ holds, then also $\operatorname{lt}_\prec(x^\mu x_i\mathbf{h}_\alpha) \succ \operatorname{lt}_\prec(x^\nu x_j\mathbf{h}_\gamma)$ and now, by definition of the Schreyer order, $\operatorname{lt}_{\prec_{\mathcal{H}_1}}(x^\mu\mathbf{v}_i\mathbf{h}_\alpha) \succ_{\mathcal{H}_1} \operatorname{lt}_{\prec_{\mathcal{H}_1}}(x^\nu\mathbf{v}_j\mathbf{h}_\gamma)$ which proves the claim. □

For notational simplicity, we formulate the two decisive corollaries only for the special case of second syzygies, but they remain valid in any homological degree. The first one already indicates the great similarity between Sköldberg's resolution and the one induced by a Pommaret basis, as a comparison with Lemma 2.4 shows that there is a one-to-one correspondence between the leading terms of the syzygies contained in the two resolutions.

Corollary 4.7. If $i < j$, then $\operatorname{lt}_{\prec_{\mathcal{H}_1}}\Big(d_{\mathcal{G}}(\mathbf{v}_{i,j}\mathbf{h}_\alpha)\Big) = x_j\mathbf{v}_i\mathbf{h}_\alpha$.

Proof. As described in Section 2, we assume that the elements of the given Pommaret basis are numbered according to a P-order. Consider now the differential $d_{\mathcal{G}}$. We first compare the terms $x_i\mathbf{v}_j\mathbf{h}_\alpha$ and $x_j\mathbf{v}_i\mathbf{h}_\alpha$. Lemma 4.4 (or the minimality of these terms with respect to any order respecting the used Morse matching) entails that there are no reduction paths $[\mathbf{v}_j\mathbf{h}_\alpha \rightsquigarrow \mathbf{v}_k\mathbf{h}_\delta]$ with $k \in \operatorname{crit}(\mathbf{h}_\delta)$ (except trivial reduction paths of length 0), since $\mathbf{v}_j\mathbf{h}_\alpha \in A^0$; the same argument applies to $\mathbf{v}_i\mathbf{h}_\alpha$. By definition of the Schreyer order, we have $x_i\mathbf{v}_j\mathbf{h}_\alpha \prec_{\mathcal{H}_1} x_j\mathbf{v}_i\mathbf{h}_\alpha$.

Now consider any other term in the sum. We will prove $x_j\mathbf{v}_i\mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa\mathbf{v}_i\mathbf{h}_\beta$, where $x^\kappa\mathbf{h}_\beta$ effectively appears in the involutive standard representation of $x_j\mathbf{h}_\alpha$. The claim then follows from applying Lemma 4.6 with $x_j\mathbf{v}_i\mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa\mathbf{v}_i\mathbf{h}_\beta \succeq_{\mathcal{H}_1} \operatorname{lt}_{\prec_{\mathcal{H}_1}}\Big(\rho_p(\mathbf{v}_i x^\kappa\mathbf{h}_\beta)\Big)$.

We always have $\operatorname{lt}_\prec(x_j x_i\mathbf{h}_\alpha) \succeq \operatorname{lt}_\prec(x^\kappa x_i\mathbf{h}_\beta)$. If this is a strict inequality, then $x_j\mathbf{v}_i\mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa\mathbf{v}_i\mathbf{h}_\beta$ follows at once by definition of the Schreyer order. So now assume $\operatorname{lt}_\prec(x_j x_i\mathbf{h}_\alpha) = \operatorname{lt}_\prec(x^\kappa x_i\mathbf{h}_\beta)$. By construction, $x^\kappa \in \Bbbk[x_1, \ldots, x_{\operatorname{cls}(\mathbf{h}_\beta)}]$.

Again by definition of the Schreyer order, the claim follows, if we can prove $\mathrm{lt}_{\prec_{\mathcal{H}_0}}(x_j x_i \mathbf{v}_\emptyset \mathbf{h}_\alpha) \succ_{\mathcal{H}_0} \mathrm{lt}_{\prec_{\mathcal{H}_0}}(x^\kappa x_i \mathbf{v}_\emptyset \mathbf{h}_\beta)$. Since $j \in \mathrm{crit}\,(\mathbf{h}_\alpha)$ and $\mathrm{lt}_\prec(x_j \mathbf{h}_\alpha)$ is involutively divisible by $\mathrm{lt}_\prec(\mathbf{h}_\beta)$, we have $\alpha < \beta$, by definition of a P-ordering. As we have here $\mathrm{lt}_\prec(x_j \mathbf{h}_\alpha) = \mathrm{lt}_\prec(x^\kappa \mathbf{h}_\beta)$, this implies the estimate $\mathrm{lt}_{\prec_{\mathcal{H}_0}}(x_j x_i \mathbf{v}_\emptyset \mathbf{h}_\alpha) \succ_{\mathcal{H}_0} \mathrm{lt}_{\prec_{\mathcal{H}_0}}(x^\kappa x_i \mathbf{v}_\emptyset \mathbf{h}_\beta)$ and therefore $\mathrm{lt}_{\prec_{\mathcal{H}_1}}(x_j \mathbf{v}_i \mathbf{h}_\alpha) \succ_{\mathcal{H}_1} \mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^\kappa \mathbf{v}_i \mathbf{h}_\beta)$. □

Corollary 4.8. The set $\left\{ d_{\mathcal{G}}(v_{\mathbf{k}} \otimes \mathbf{h}_\alpha) \mid |\mathbf{k}| = 2; \mathbf{k} \subseteq \mathrm{crit}\,(\mathbf{w}_\alpha) \right\}$ is a Pommaret basis with respect to the term order $\prec_{\mathcal{H}_0}$.

Based on these two corollaries, it is now comparatively straightforward to prove our main result by explicitly constructing an isomorphism between the two resolutions we consider.

Theorem 4.9. Assume the situation of Lemma 4.1, i. e. we have a submodule $\mathcal{U} \subseteq \mathcal{P}^t$ and the presentation comes from a P-ordered Pommaret basis $\mathcal{H}$ of $\mathcal{U}$. Then the resolution $(\mathcal{G}, d_{\mathcal{G}})$ is isomorphic to the resolution induced by $\mathcal{H}$.

Proof. First, we recall the alternative description of the resolution induced by a Pommaret basis given at the end of Section 2, and especially the definition of the differential in the complex $\mathcal{S}_\bullet$ as in (8). There, we start with $\mathcal{W} = \bigoplus_{\alpha=1}^{s} \mathcal{P}\mathbf{w}_\alpha$ and $\mathcal{V} = \bigoplus_{i=1}^{n} \mathcal{P}\mathbf{v}_i$. Then we consider the modules $\mathcal{C}_i = \mathcal{W} \otimes_{\mathcal{P}} \Lambda_i \mathcal{V}$. By identifying $\mathbf{w}_\alpha = \mathbf{w}_\alpha \otimes_{\mathcal{P}} \mathbf{v}_\emptyset \in \mathcal{C}_0$ with $m_\alpha = d_{\mathcal{S}}(\mathbf{w}_\alpha) \in \mathcal{U}$, we get the isomorphism

$$\mathcal{C}_i = \left(\bigoplus_{\alpha=1}^{s} \mathcal{P}\mathbf{w}_\alpha \right) \otimes_{\mathcal{P}} \Lambda_i \left(\bigoplus_{j=1}^{n} \mathcal{P}\mathbf{v}_j \right) \cong \mathcal{P} \otimes_{\Bbbk} \left(\bigoplus_{\alpha=1}^{s} \Bbbk m_\alpha \right) \otimes_{\Bbbk} \Lambda_i \left(\bigoplus_{j=1}^{n} \Bbbk \mathbf{v}_j \right) \subseteq \mathcal{F}_i$$

and we can view $\mathcal{C}_i$ as a submodule of $\mathcal{F}_i$. In the same way, we see that for the submodules $\mathcal{S}_i \subseteq \mathcal{C}_i$ and $\mathcal{G}_i \subseteq \mathcal{F}_i$ we have isomorphisms $\mathcal{S}_i \cong \mathcal{G}_i$. Using these isomorphisms, we will identify $\mathcal{S}_i$ and $\mathcal{G}_i$ in the rest of this proof.

We write the two resolutions as rows in a diagram denoting the components of $d_{\mathcal{S}}$ by d_i and those of $d_{\mathcal{G}}$ by d_i^*:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathcal{G}_2 & \xrightarrow{d_1} & \mathcal{G}_1 & \xrightarrow{d_0} & \mathcal{G}_0 & \longrightarrow & \mathcal{U} & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \| & & \\ \cdots & \longrightarrow & \mathcal{G}_2 & \xrightarrow{d_1^*} & \mathcal{G}_1 & \xrightarrow{d_0^*} & \mathcal{G}_0 & \longrightarrow & \mathcal{U} & \longrightarrow & 0 \end{array} \tag{17}$$

Let $\{\mathbf{e}_{i,1}, \dots, \mathbf{e}_{i,r_i}\}$ be the basis of the free module $\mathcal{G}_i$. By Corollary 4.8, the vectors $\mathbf{h}_{i-1,\alpha} = d_i(\mathbf{e}_{i,\alpha})$ define a Pommaret basis $\mathcal{H}_i$ of $\mathrm{im}\, d_i$. Analogously, we obtain a Pommaret basis $\mathcal{H}_i^*$ of $\mathrm{im}\, d_i^*$. Here we set $\mathcal{H}_{-1} = \mathcal{H}_{-1}^* = \mathcal{H}$, the given Pommaret basis of $\mathcal{U}$, and define the term orders $\prec_i$ on $\mathcal{G}_i$ recursively as the Schreyer orders $\prec_i = \prec_{\mathcal{H}_{i-1}}$. Because of Corollary 4.7, we always have $\mathrm{lt}\, \mathbf{h}_{i,\alpha} = \mathrm{lt}\, \mathbf{h}_{i,\alpha}^*$ and hence also $\prec_i = \prec_{\mathcal{H}_{i-1}^*}$.

Assume now that an automorphism $\varphi_0 : \mathcal{G}_0 \to \mathcal{G}_0$ is given which satisfies $\varphi_0(\operatorname{im} d_0) = \operatorname{im} d_0^*$ and which preserves the term order $\prec_0$ in the sense that we have $\operatorname{lt}_{\prec_0}\big(\varphi_0(\mathbf{f})\big) = \operatorname{lt}_{\prec_0}(\mathbf{f})$ for all vectors $0 \neq \mathbf{f} \in \mathcal{G}_0$. Obviously, the identity is such an automorphism. We now show that φ_0 can be lifted to automorphisms $\varphi_i : \mathcal{G}_i \to \mathcal{G}_i$ preserving the term orders $\prec_i$ such that the diagram (17) commutes.

If $\varphi_0(\mathbf{h}_{0,\alpha}) = \sum_{\beta=1}^{r_1} P_\alpha^\beta \mathbf{h}_{0,\beta}^*$ is an involutive standard representation with respect to the Pommaret basis $\mathcal{H}_0^*$, then we set $\varphi_1(\mathbf{e}_{1,\alpha}) = \sum_{\beta=1}^{r_1} P_\alpha^\beta \mathbf{e}_{1,\beta}$ and extend $\mathcal{P}$-linearly. It is trivial that for this choice of φ_1 the rightmost square in the diagram (17) becomes commutative.

We temporarily renumber the elements of the Pommaret bases $\mathcal{H}_0$ and $\mathcal{H}_0^*$ so that now $\operatorname{lt}\mathbf{h}_\alpha \prec_0 \operatorname{lt}\mathbf{h}_\beta$, if and only if $\alpha < \beta$. By definition of an involutive standard representation, the matrix (P_α^β) is then an upper triangular matrix for this ordering and since φ_0 preserves the term order $\prec_0$ the elements P_α^α on the diagonal are non vanishing constants. This fact trivially implies that φ_1 is an automorphism.

Finally, we must show that φ_1 preserves the term order $\prec_1$. Obviously, it suffices to check this for terms. By definition, $\varphi_1(x^\kappa \mathbf{e}_{1,\alpha}) = x^\kappa \sum_{\beta=1}^{r_1} P_\alpha^\beta \mathbf{e}_{1,\beta}$. Using the definition of a Schreyer order and the fact that the coefficients P_α^β come from an involutive standard representation, we find that the equality

$$\operatorname{lt}_{\prec_1}\varphi_1(x^\kappa \mathbf{e}_{1,\alpha}) = \max_{\prec_1}\Big\{x^\kappa \operatorname{lt}_{\prec_1}(P_\alpha^\beta \mathbf{e}_{1,\beta}) \mid \beta = 1,\dots,r_1\Big\}$$

is equivalent to the equality

$$\max_{\prec_0}\Big\{x^\kappa \operatorname{lt}_{\prec_0}(P_\alpha^\beta \mathbf{h}_{0,\beta}^*) \mid \beta = 1,\dots,r_1\Big\} = x^\kappa \operatorname{lt}_{\prec_0}(\mathbf{h}_{0,\alpha}^*)\,.$$

Using again the definition of a Schreyer order, we may now conclude that $\operatorname{lt}_{\prec_1}\varphi_1(x^\kappa \mathbf{e}_{1,\alpha}) = \operatorname{lt}_{\prec_1}(x^\kappa \mathbf{e}_{1,\alpha})$ as required.

Since φ_1 is an automorphism and the rows in the diagram (17) are exact, we have $\varphi_1(\operatorname{im} d_1) = \operatorname{im} d_1^*$. Thus we can iterate the construction and obtain automorphisms $\varphi_i : \mathcal{G}_i \to \mathcal{G}_i$ for all values of i. Because of Lemma 4.3, choosing the identity for φ_0 then proves our assertion. □

5 A Simple Special Case

Whenever a reduction path p contains an elementary reduction of type 2 which is reversed in the Morse matching, then in the situation of Lemma 4.1 a factor $P_\bullet^{(\bullet;\bullet)}$ (cf. (2)) appears in the the differential $d_\mathcal{G}$ in the coefficient c_p associated with the reduction ρ_p. A special case, in which one obtains a much simpler expression for the differential $d_{\mathcal{G}}$ (Sköldberg, 2011, Thm. 2), arises when *no* appearing reduction path contains an elementary reduction of type 2. Sköldberg provides a simple sufficient condition for being in this special case, namely when the module $\mathcal{M}$ is *crit-monotone*.

Translated into the situation of Lemma 4.1, a submodule $\mathcal{U}$ is crit-monotone when on the right hand side of (2) only those generators $\mathbf{e}_\beta$ effectively appear which satisfy $\operatorname{cls}\mathbf{h}_\beta \geq \operatorname{cls}\mathbf{h}_\alpha$. It is very rare that submodules satisfy this condition. There exists one notable exception: it is always satisfied for *monomial* submodules possessing a Pommaret basis. In fact, the arising resolution is a simple generalisation of the closed-form resolution obtained in (Seiler, 2009b) for such monomial submodules (see Theorem 7.1 below). We provide now an independent proof of this result which is also much simpler than the proof given in (Seiler, 2009b) for the monomial case.

Theorem 5.1. Assume that in the syzygies (2) the coefficients $P_\beta^{(\alpha;k)}$ are non-zero only, if $\operatorname{cls}\mathbf{h}_\beta \geq \operatorname{cls}\mathbf{h}_\alpha$. Then the Pommaret basis $\mathcal{H}_j$ of the jth syzygy module consists of the syzygies [4]

$$\mathbf{S}_{\alpha;\mathbf{k}} = \sum_{\ell=1}^{j}(-1)^{j-\ell}\Big(x_{k_\ell}\mathbf{e}_{\alpha,\mathbf{k}_\ell} - \sum_{\beta=1}^{s}\big[(\mathbf{k}_\ell)_1 > \operatorname{cls}\mathbf{h}_\beta\big]P_\beta^{(\alpha,k_\ell)}\mathbf{e}_{\beta;\mathbf{k}_\ell}\Big) \tag{18}$$

where again $\mathbf{k} = (k_1, \dots, k_j)$ is an integer sequence with $\operatorname{cls}\mathbf{h}_\alpha < k_1 < \cdots < k_j$.

Proof. It suffices to consider the second syzygy module, as the assertion is then also true for all higher syzygy modules by iteration. Since we have the right leading terms (with respect to the corresponding Schreyer order), it furthermore suffices to prove that (18) is indeed a syzygy. Thus, with the short hand $[k,\beta] = [k > \operatorname{cls}\mathbf{h}_\beta]$, there remains to show that for $\operatorname{cls}\mathbf{h}_\alpha < k_1 < k_2$

$$x_{k_2}\mathbf{S}_{\alpha;k_1} - \sum_{\beta=1}^{s}[k_1,\beta]P_\beta^{(\alpha;k_2)}\mathbf{S}_{\beta;k_1} = x_{k_1}\mathbf{S}_{\alpha;k_2} - \sum_{\beta=1}^{s}[k_2,\beta]P_\beta^{(\alpha;k_1)}\mathbf{S}_{\beta;k_2}\,. \tag{19}$$

In order to see that this equation always holds after substitution of the first syzygies, we compare two different ways to determine a standard representation of $x_{k_1}x_{k_2}\mathbf{h}_\alpha$: one time we first take the involutive standard representation of $x_{k_2}\mathbf{h}_\alpha$ and then multiply it by x_{k_1}; the second time we revert the role of x_{k_1}

[4] The Kronecker-Iversion symbol $[\cdot]$ is 1, if the contained condition is true, and 0 otherwise.

and x_{k_2}. The first operation yields

$$x_{k_1}x_{k_2}\mathbf{h}_\alpha = \sum_{\gamma=1}^{s}\Bigl([\operatorname{cls}\mathbf{h}_\gamma \geq k_1]x_{k_1}P_\gamma^{(\alpha,k_2)} + \sum_{\beta=1}^{s}[\operatorname{cls}\mathbf{h}_\beta < k_1]P_\beta^{(\alpha,k_2)}P_\gamma^{(\beta,k_1)}\Bigr)\mathbf{h}_\gamma\,; \tag{20}$$

the second one yields the same result with k_1 and k_2 swapped. Due to our assumption on the coefficients $P_\beta^{(\alpha,k)}$, we obtain in both cases the *involutive* standard representation which is unique by Proposition 2.2. Hence both ways must lead to identical coefficients for each generator $\mathbf{h}_\gamma$. Substituting the obtained equalities into (19) yields the desired result. □

Example 5.2. The condition in Theorem 5.1—and similar in (Sköldberg, 2011, Thm. 2)—is sufficient but not necessary for the conclusion. The homogeneous ideal $\mathcal{I} \lhd \Bbbk[x,y,z]$ generated by the Pommaret basis

$$h_1 = x^2y\,,\quad h_2 = x^2z\,,\quad h_3 = y^2 + xz\,,\quad h_4 = yz - xz\,,\quad h_5 = z^2 + xy \tag{21}$$

provides a concrete instance where the assumptions of Theorem 5.1 are *not* satisfied, but the conclusion is nevertheless correct. The Pommaret basis of the first syzygy module consists of

$$\mathbf{S}_{1;3} = z\mathbf{e}_1 - x\mathbf{e}_2 - x^2\mathbf{e}_4\,, \tag{22a}$$
$$\mathbf{S}_{2;3} = x\mathbf{e}_1 + z\mathbf{e}_2 - x^2\mathbf{e}_5\,, \tag{22b}$$
$$\mathbf{S}_{3;3} = \mathbf{e}_1 - \mathbf{e}_2 + z\mathbf{e}_3 - (x+y)\mathbf{e}_4 - x\mathbf{e}_5\,, \tag{22c}$$
$$\mathbf{S}_{4;3} = -\mathbf{e}_1 - \mathbf{e}_2 + x\mathbf{e}_3 + z\mathbf{e}_4 + (x-y)\mathbf{e}_5\,, \tag{22d}$$
$$\mathbf{S}_{1;2} = y\mathbf{e}_1 + x\mathbf{e}_2 - x^2\mathbf{e}_3\,, \tag{22e}$$
$$\mathbf{S}_{2;2} = (y-x)\mathbf{e}_2 - x^2\mathbf{e}_4\,. \tag{22f}$$

Obviously, the syzygies $\mathbf{S}_{3;3}$ and $\mathbf{S}_{4;3}$ come from generators of class 2 but contain basis vectors corresponding to generators of class 1. The Pommaret basis of the second syzygy module comprises

$$\mathbf{S}_{1;2,3} = z\mathbf{e}_{1;2} - x\mathbf{e}_{2;2} - y\mathbf{e}_{1;3} - x\mathbf{e}_{2;3} + x^2\mathbf{e}_{3;3}\,, \tag{23a}$$
$$\mathbf{S}_{2;2,3} = z\mathbf{e}_{2;2} + x\mathbf{e}_{1;2} - (y-x)\mathbf{e}_{2;3} + x^2\mathbf{e}_{4;3} \tag{23b}$$

which is exactly (18).

6 Implementation in CoCoALib

We now describe an implementation of the above results in the computer algebra library CoCoALib (Abbott and Bigatti, 2013). [5] In contrast to the remainder of this article, our implementation is based on the standard conventions for the reverse lexicographic order in order to be consistent with CoCoALib. This also implies that a number of things like the definitions of multiplicative or critical variables must be adapted. The change of convention means that everywhere the ordering of the variables must be reverted: $x_1, \dots, x_n$ becomes $x_n, \dots, x_1$. Our implementation is currently restricted to ideals.

We want to construct the reduced complex $\mathcal{G}_\bullet$. We first need the two-sided Koszul complex $\mathcal{F}_\bullet$. Because of the form of the reduced differential (16), we only have to determine the differential $d_\mathcal{F}$ for basis elements of the form $\mathbf{v_k h}_\alpha$ where $\mathbf{k} \subseteq \operatorname{crit} \mathbf{h}_\alpha$, which is in principle straightforward with (11).

The only problem which may occur is that we obtain in the right hand summands a term $\mathbf{v}_{\mathbf{k}_i}(x_{k_i}\mathbf{h}_\alpha)$ where $k_i \notin \operatorname{ncrit} \mathbf{h}_\alpha$. In this case, we determine the involutive standard representation of $x_{k_i}\mathbf{h}_\alpha$ and split the coefficients into monomials:

$$x_{k_i}\mathbf{h}_\alpha = \sum_{\beta=1}^{s}\sum_{\mu} Q^{\mathbf{k},\alpha}_{\mathbf{k}_i,\mu,\beta} x^\mu \mathbf{h}_\beta$$

with $\mu \subseteq \mathcal{X}_P(\mathbf{h}_\beta)$ and scalars $Q^{\mathbf{k},\alpha}_{\mathbf{k}_i,\mu,\beta} \in \Bbbk$. So we can replace $\mathbf{v}_{\mathbf{k}_i}(x_{k_i}\mathbf{h}_\alpha)$ by basis elements of the right form. Thus we arrive at the following, easily computable, explicit form of the differential $d_\mathcal{F}$ (we assume that $\mathbf{k}$ is of length j)

$$d_\mathcal{F}(\mathbf{v_k h}_\alpha) = \sum_{i=1}^{j}(-1)^{i+1}\Big(x_{k_i}\mathbf{v}_{\mathbf{k}_i}\mathbf{h}_\alpha - \sum_{\beta=1}^{s}\sum_{\mu} Q^{\mathbf{k},\alpha}_{\mathbf{k}_i,\mu,\beta}\mathbf{v}_{\mathbf{k}_i}(x^\mu \mathbf{h}_\beta)\Big)\,. \tag{24}$$

Now we perform the Morse reduction. Assume that we have an edge $\mathbf{v_k h}_\alpha \to \mathbf{v_m}(x^\mu \mathbf{h}_\beta)$ in the graph $\Gamma_{\mathcal{F}_\bullet}$ with $|\mathbf{k}| = |\mathbf{m}| + 1$ and $\mu \subseteq \operatorname{ncrit} \mathbf{h}_\beta$, e.g. $\mathbf{v_k h}_\alpha$ maps to $Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\beta}\mathbf{v_m}(x^\mu \mathbf{h}_\beta)$. By (16), the element $Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\beta}\mathbf{v_m}(x^\mu \mathbf{h}_\beta)$ must reduce to

$$\sum_{\substack{\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\gamma \\ \boldsymbol{\ell} \subseteq \operatorname{ncrit} \mathbf{h}_\gamma}} \; \sum_{p \in [\mathbf{v_m}(x^\mu \mathbf{h}_\beta) \rightsquigarrow \mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\gamma]} \rho_p\Big(Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\beta}\mathbf{v_m}(x^\mu \mathbf{h}_\beta)\Big)\,.$$

We see that the result of the reduction does not really depend on the starting point $\mathbf{v_k h}_\alpha$ of the path; it solely requires the knowledge of its image. It is furthermore possible to combine some reduction paths. Assume that two reduction paths with starting points coming from the same preimage pass at

[5] Our implementation is part of the official distribution of the CoCoALib and thus freely available.

some later point through the same vertex $\mathbf{v_k}(x^\mu \mathbf{h}_\beta)$. Due to the fact that an elementary reduction path only depends on its starting point, we can combine the two paths from this point on and thus compute their remaining parts simultaneously.

We introduce now a partial order on the set of basis elements $\mathbf{v_k}(x^\mu \mathbf{h}_\beta)$, as this considerably simplifies the determination of the reduction $d_\mathcal{G}(\mathbf{v_k h}_\alpha)$ from $d_\mathcal{F}(\mathbf{v_k h}_\alpha)$. We first compute $d_\mathcal{F}(\mathbf{v_k h}_\alpha)$ which is a $\mathcal{P}$-linear combination of terms of the form $\mathbf{v_m}(x^\mu \mathbf{h}_\beta)$. Then we pick the greatest term according to our ordering and perform an elementary reduction which leads to a new sum. This process is iterated, until there are only elements of the form $\mathbf{v_m h}_\beta$ with $\mathbf{m} \subseteq \operatorname{crit} \mathbf{h}_\beta$ left which then form the differential $d_\mathcal{G}(\mathbf{v_k h}_\alpha)$. Our next result explicitly describes a useful partial order for this purpose.

Proposition 6.1. Let $\mathbf{v_k}(x^\mu \mathbf{h}_\alpha)$ and $\mathbf{v_m}(x^\nu \mathbf{h}_\beta)$ be two basis elements such that $\operatorname{supp} x^\mu \subseteq \operatorname{ncrit} \mathbf{h}_\alpha$ and $\operatorname{supp} x^\nu \subseteq \operatorname{ncrit} \mathbf{h}_\beta$ and such that there is an elementary reduction path $\mathbf{v_k} x^\mu \mathbf{h}_\alpha \to \mathbf{v_m} x^\nu \mathbf{h}_\beta$. If we set $\mathbf{v_k} x^\mu \mathbf{h}_\alpha > \mathbf{v_m} x^\nu \mathbf{h}_\beta$ whenever one of the conditions [6]

$$|\mathbf{k}| > |\mathbf{m}|\,, \tag{25}$$

$$|\mathbf{k}| = |\mathbf{m}| \quad \wedge \quad x_\mathbf{k} x^\mu \operatorname{lt} \mathbf{h}_\alpha \succ x_\mathbf{m} x^\nu \operatorname{lt} \mathbf{h}_\beta\,, \tag{26}$$

$$|\mathbf{k}| = |\mathbf{m}| \quad \wedge \quad x_\mathbf{k} x^\mu \operatorname{lt} \mathbf{h}_\alpha = x_\mathbf{m} x^\nu \operatorname{lt} \mathbf{h}_\beta \quad \wedge \quad x^\mu \operatorname{lt} \mathbf{h}_\alpha \prec x^\nu \operatorname{lt} \mathbf{h}_\beta \tag{27}$$

is satisfied, then this defines a total order on the set which contains the basis elements of all the modules $\mathcal{F}_m$ for $m \geq 0$.

Proof. Along every elementary reduction path we have $|\mathbf{k}| = |\mathbf{m}|$. Assume that the path is an elementary reduction of type 1. Then $\mathbf{m} = (\mathbf{k} \cup b) \setminus a$ and $x^\nu = x^\mu / x_b$ for some $a, b \in \{1, \dots, n\}$. Hence $\deg x_\mathbf{k} x^\mu = \deg x_\mathbf{m} x^\nu + 1$ entailing $x_\mathbf{k} x^\mu \succ x_\mathbf{m} x^\nu$ which leads to $\mathbf{v_k}(x^\mu \mathbf{h}_\alpha) < \mathbf{v_m}(x^\nu \mathbf{h}_\beta)$.

Now we consider elementary reductions of type 2. There are two possibilities: either $x_\mathbf{k} x^\mu \operatorname{lt} \mathbf{h}_\alpha \succ x_\mathbf{m} x^\nu \operatorname{lt} \mathbf{h}_\beta$ or for at least one reduction of type 2, $x_\mathbf{k} x^\mu \operatorname{lt} \mathbf{h}_\alpha = x_\mathbf{m} x^\nu \operatorname{lt} \mathbf{h}_\beta$. In the first case the statement is obvious. For the second one, we note that

$$x^\nu \operatorname{lt} \mathbf{h}_\beta = \frac{x_a x^\mu}{x_b} \operatorname{lt} \mathbf{h}_\alpha\,,$$

where $a \in \mathbf{k}$ and $b = \min \operatorname{supp} x^\mu$ and $b < \min \mathbf{k}$, according to the construction of a Morse matching (14). Therefore $a > b$ and hence we can deduce with the reverse lexicographic order that

$$x^\mu \operatorname{lt} \mathbf{h}_\alpha < \frac{x_a x^\mu}{x_b} \operatorname{lt} \mathbf{h}_\alpha\,,$$

[6] $x_\mathbf{k}$ denotes here the product of the x_{k_i} with $k_i \in \mathbf{k}$.

which completes the proof. □

Remark 6.2. The proof does not use the condition (25). This condition is trivially satisfied. But with the help of condition (25), we are able to store the complete complex $\mathcal{F}_\bullet$ in only one list. Then we can start the Morse reductions at the greatest element, lying in the last module of the free resolution, and proceed until the first module is reached.

Based on these results, we are now able to formulate a novel algorithm for computing a free resolution for an ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ given a Pommaret basis $\mathcal{H}$ of it. For the sake of completeness, we first discuss the determination of the relevant part of the two-sided Koszul complex $\mathcal{F}_\bullet$. It is a straightforward application of (24). Consider the map $d_\mathcal{F}$ restricted to the free submodule $\mathcal{P} \cdot \mathbf{v_k h}_\alpha$. To calculate the differential $d_\mathcal{F}(\mathbf{v_k h}_\alpha)$, we at worst have to calculate $d_\mathcal{F}$ for all basis elements in $\mathcal{F}_{|\mathbf{k}|}$ that are smaller than $\mathbf{v_k h}_\alpha$ with respect to the partial order introduced in 6.1. As there are only finitely many smaller basis elements in $\mathcal{F}_{|\mathbf{k}|}$, we conclude that the computation terminates.

Usually the differentials are represented as a list of pairs (preimage, image)—or (key, value) in computer science terminology—sorted according to the preimages. However, we sort the list first according to the images and only the pairs with the same image value (we cannot expect our maps to be injective) are then sorted by their preimage values. In the sequel, $M_{\mathbf{v_n}(x^\xi \mathbf{h}_\gamma)}$ denotes a set of pairs $\{\mathbf{v_m}(x^\nu \mathbf{h}_\beta), p\}$ where the first component is the preimage and the polynomial p in the second component defines the image $0 \neq p\mathbf{v_n}(x^\xi \mathbf{h}_\gamma)$.

Throughout the algorithm, basis elements are always sorted according to the order defined in Proposition 6.1. We first compute the partial differential of the complex $\mathcal{F}_\bullet$. Then we reduce this differential. Among the finitely many sets $M_{\mathbf{v_n}(x^\xi \mathbf{h}_\gamma)}$, we start with the greatest element, e. g. the greatest element $\mathbf{v_k}(x^\mu \mathbf{h}_\alpha) \in \mathcal{F}_m$ where m is the largest index such that $\mathcal{F}_m \neq 0$. If $\mathbf{k} \subseteq \overline{\mathcal{X}}_P(\mathbf{h}_\alpha)$ and $x^\mu = 1$, then this element is also a generator of the complex $\mathcal{G}_\bullet$ and we do not need to perform a reduction. If this is not the case, then we check whether there are possible elementary reductions of type 1 or 2 and perform them all. If there is no elementary reduction possible, then we remove this element. It may happen that for both the starting and the final point of a reduction path a map exists such that each point has the same preimage under "its" map. In this case we simply add the reduced map to the second map and thus combine all subsequent reduction steps.

The output consists of two sets G and M representing the complex $\mathcal{G}_\bullet$ which defines a free resolution of the ideal $\mathcal{I}$. The first set contains the basis elements of free submodules $\mathcal{G}_m$ and the second stores the differentials in the form of sets $M_{\mathbf{v_n}(x^\xi \mathbf{h}_\gamma)}$ as introduced above. The algorithm works for any ideal possessing a Pommaret basis, even for non-homogeneous ideals. In the homogeneous case, the obtained resolution can subsequently be minimised. Currently, we use for this purpose the simple linear algebra algorithm described in Cox et al. (1998).

Although the current implementation is still in a fairly early stage, it contains already a number of optimisations. As often involutive standard representations of the same element are needed, these are stored after they have been computed once. Furthermore, we check for every element $\mathbf{v_k}(x^\mu \mathbf{h}_\alpha)$ whether there is a possible reduction path even before creating it. The minimisation procedure has not yet been optimised. Much more details on the implementation and the used optimisations can be found in (Albert, 2013).

For benchmarking the implementation, we used a number of standard examples[7] given in (Yanovich et al., 2001). As most of these ideals are not homogeneous, we homogenised them by adding a new smallest variable, also we used always $\Bbbk = \mathbb{Z}/101\mathbb{Z}$ as the base field. In addition, we used the following two examples where we always worked over the base field $\Bbbk = \mathbb{Z}/101\mathbb{Z}$.

Example 6.3. In the polynomial ring $\mathcal{P} = \Bbbk[x_1, \dots, x_n]$ we consider the ideal $\mathcal{I} = \langle x_1^2, \dots, x_n^2, (x_1 + \cdots + x_n)^2 \rangle$ which possesses a homogeneous Pommaret basis.

Example 6.4. Let $6 \le g \in \mathbb{N}$ and $\mathcal{P} = \Bbbk[x_0, \dots, x_{g-4}]$. Then we generate the ideal $\mathcal{I}$ by the following binomials with $2 \le i \le j \le g-2$

$$\begin{aligned} (i+j-1)x_{i-2}x_{j-2} - (i \cdot j)x_{i+j-3}x_{g-4}\,, &\qquad i+j \le g-1\,, \\ (2g-i-j-1)x_{i-2}x_{j-2} - (g-i)\cdot(g-j)x_{i+j-g}x_{g-4}\,, &\qquad i+j > g-1\,. \end{aligned}$$

This example was deduced from (La Scala and Stillman, 1998, Ex. 6.4). As their example leads to some negative indices, we modified it a bit so that we only get indices between 0 and $g-4$. This ideal also possesses a homogeneous Pommaret basis.

For the benchmarks presented in Table 1, we used a MacBook Pro with 2.53 GHz Intel Core 2 Duo processor and 4GB DDR3 main memory. Besides the CoCoALib, we used Macaulay2 (Grayson and Stillman, 2013) and Singular (Decker et al., 2012) for comparison. For both systems we used the standard resolution command `res` which uses La Scala's method (La Scala and Stillman, 1998) to compute the resolutions. The running times are given

[7] For the benchmarks presented here, we specifically chose examples possessing Pommaret bases in the given coordinates. In our experience, between 70% and 80% of the classical benchmark examples for Gröbner bases computations satisfy this condition. This is to a considerable extent due to the fact that many of these examples are zero-dimensional and such ideals always possess a Pommaret basis. We currently work on an implementation realising an efficient deterministic algorithm for constructing Pommaret bases for arbitrary ideals by making an appropriate linear change of coordinates which is as sparse as possible. A description of this work will appear elsewhere.

	CoCoALib	**Red.**	**Minim.**	**Macaulay2**	**Singular**
Ex. 6.3 ($n = 7$)	17.777	2.383	15.394	4.527	593.780
Ex. 6.3 ($n = 8$)	928.853	22.338	906.515	483.931	*
Ex. 6.4 ($g = 11$)	3.273	0.584	2.689	0.499	0.440
Ex. 6.4 ($g = 12$)	30.426	1.833	28.593	6.828	4.710
Reimer 5	12.927	7.970	4.957	8.756	0.880
Noon 5	6.189	4.064	2.125	0.252	0.130
Noon 6	691.004	130.652	560.352	30.831	7.420
Redeco 7	25.918	9.793	16.125	7.703	0.440
Redeco 8	1 270.735	168.005	1 102.730	862.337	19.650
Eco 7	31.914	15.727	16.187	138.369	23.490
Eco 8	3 024.956	365.836	2 659.120	*	2 370.020
Katsura 6	133.124	49.585	84.539	161.985	26.670
Katsura 7	6 288.220	1 146.260	5 141.960	*	2 439.710
Cyclic 6	14.940	4.358	10.582	2.823	1 413.320

Table 1
Timings for computing minimal free resolutions of some classical examples

in seconds and we aborted computations after two hours. A * marks when we run out of time or memory. The timings obtained with our implementation are split into two parts: *Red* gives the time need to compute the complex $\mathcal{G}_\bullet$, i. e. the non-minimal resolution, and *Minim* the time required by the subsequent minimisation.

Obviously, our current implementation is still generally slower than either Macaulay2 or Singular, although one can see that—at least compared to Macaulay2—its relative performance improves the larger the examples get. The main reasons are the size of the complex $\mathcal{G}_\bullet$ (see Table 2 below) and the still very naive minimisation process. In most examples, we need much more time to minimise $\mathcal{G}_\bullet$ than to construct it. There are a number of quite obvious possible optimisations which should bring considerably savings. We hope to implement these in the near future.

The real power of our approach becomes apparent when one considers the problem of determining the Betti numbers. Often knowledge of the Betti numbers is sufficient and one does not really need the whole resolution. To our knowlege, however, all current implementations compute Betti numbers via the minimal resolution. This also becomes evident by the fact that for both

Macaulay2 and Singular the timings for the minimal resolution and for the Betti numbers, respectively, hardly differ.

In our approach, it is easily possible to modify the above presented algorithms so that they determine only the constant part of the complex $\mathcal{G}_\bullet$. If we perform an elementary reduction of type 2, then the degree of the map does not change. For an elementary reduction of type 1, the degree increases by one. Thus if we start with the constant part of the two-sided Koszul complex $\mathcal{F}_\bullet$ and only apply reductions of type 2, then we obtain the constant part of $\mathcal{G}_\bullet$. It follows from the explicit form (24) of the differential $d_\mathcal{F}$ that the left summand yields always elements of degree one and the right summand elements of degree zero. Hence, by simply skipping the left summands, we directly obtain the constant part of $\mathcal{G}_\bullet$.

The Betti numbers of $\mathcal{I}$ are now easily computed. Because of the above proven isomorphy between the complex $\mathcal{G}_\bullet$ and the resolution induced by the Pommaret basis, the bigraded ranks $r_{i,j}$ of the components of the complex $\mathcal{G}_\bullet$ can be directly determined with (5). Then, as described above, we construct (degreewise) the constant part of the matrices of $d_\mathcal{G}$. Subtracting their ranks from the corresponding $r_{i,j}$ yields the Betti numbers $b_{i,j}$. We emphasise that computing these ranks requires only linear algebra over the base field $\Bbbk$ and not over the polynomial ring $\mathcal{P}$ as for a minimisation.

The benchmarks presented in Table 2 show that our approach is generally much faster than the standard methods requiring the minimal free resolution (often by orders of magnitude) and it allows for the treatment of considerably larger examples. In particular, it scales much better when examples are getting larger. If one compares for instance the timings for Eco 7 and Eco 8 or for Katsura 6 and Katsura 7, respectively, then they increase in our approach by a factor of about 13 whereas Singular needs roughly 100 times longer.

In Table 2 we also exhibit some data about the sizes of the resolutions. The considered benchmarks are too large to write down their full Betti diagrams. Therefore, as a simple measure for their sizes we just include the total sum of all Betti numbers (column "Σ-Betti"). For comparison, we also show the corresponding values of the resolution induced by the Pommaret basis (column "Σ-PBetti"). One can see that for some examples the non-minimal resolution is up to 100 times larger, whereas for other examples the factor is as low as 2. In some families of examples the factor seems to remain about constant when moving to larger members, in other families the factor grows considerably. To some extent these observations can be explained from the combinatorial structure of the differential (16), but we have not yet performed a detailed analysis.

	Σ-PBetti	Σ-Betti	CoCoALib	Macaulay2	Singular
Ex. 6.3 ($n = 8$)	12066	3770	11.118	483.931	*
Ex. 6.3 ($n = 9$)	44322	7540	223.083	*	*
Ex. 6.4 ($g = 14$)	20482	9328	23.619	765.979	710.830
Ex. 6.4 ($g = 15$)	45058	19670	175.845	*	*
Ex. 6.4 ($g = 16$)	98306	41466	1 460.890	*	*
Reimer 6	5302	64	57.673	1 908.11	283.900
Noon 6	9558	322	15.010	30.831	7.420
Noon 7	56666	770	372.791	*	*
Redeco 8	6828	256	15.352	862.337	19.650
Redeco 9	27308	512	248.092	*	*
Redeco 10	109228	1024	6 981.240	*	*
Eco 7	1828	656	2.285	138.369	23.490
Eco 8	6916	1248	31.204	*	2 370.020
Eco 9	28292	6188	406.463	*	*
Katsura 6	1812	128	3.883	161.985	26.670
Katsura 7	6900	256	49.162	*	2 439.710
Katsura 8	27252	512	1 169.300	*	*
Cyclic 6	1060	320	1.158	2.823	1 413.320
Cyclic 7	10356	1688	107.591	2 453.73	*

Table 2
Timings for computing Betti numbers of some classical examples

7 The Monomial Case

We call a monomial ideal $\mathcal{I}$ *quasi-stable*, if it possesses a finite monomial Pommaret basis. Such ideals can be characterised by a combinatorial condition generalising the definition of stable ideals.[8] Another explanation of the terminology is given by the fact that for any degree $q \geq \operatorname{reg}\mathcal{I}$ the truncation $\mathcal{I}_{\geq q}$ is stable.

Let $\mathcal{H} = \{h_1, \ldots, h_s\}$ be the Pommaret basis of $\mathcal{I}$. For any generator h_α

[8] See (Seiler, 2009b, Prop. 4.4) for a number of equivalent characterisations of these ideals independent of Pommaret bases.

and any non-multiplicative variable $x_k \in \overline{\mathcal{X}}_P(h_\alpha)$ there exists a unique index $\Delta(\alpha, k)$ and a unique term $t_{\alpha;k} \in \Bbbk[\mathcal{X}_P(h_{\Delta(\alpha,k)})]$ such that $x_k h_\alpha = t_{\alpha;k} h_{\Delta(\alpha,k)}$. With these notations one can now give a simplified version of Theorem 5.1 which entails that a weighted version of the P-graph with the terms $t_{\alpha;k}$ as weight on the edge from h_α to $h_{\Delta(\alpha,k)}$ contains all necessary information about the resolution.

Theorem 7.1. (Seiler, 2009b, Thm. 7.2) Let $\mathcal{I}$ be a quasi-stable monomial ideal with Pommaret basis $\mathcal{H} = \{h_1, \dots, h_s\}$. A Pommaret basis $\mathcal{H}_j$ of the jth syzygy module $\mathrm{Syz}^j(\mathcal{H})$ with respect to the Schreyer order $\prec_{\mathcal{H}_{j-1}}$ is given by

$$\mathbf{S}_{\alpha;\mathbf{k}} = \sum_{\ell=1}^{j} (-1)^{j-\ell} \Big(x_{k_\ell} \mathbf{e}_{\alpha;\mathbf{k}_\ell} - t_{\alpha,k_\ell} \mathbf{e}_{\Delta(\alpha,k_\ell);\mathbf{k}_\ell} \Big) . \tag{28}$$

Herzog and Takayama (2002) introduced the notion of *linear quotients*. In our "reverse" conventions, a monomial ideal $\mathcal{I}$ has linear quotients with respect to an ordered basis $\{h_1, \dots, h_r\}$, if the colon ideals $\langle h_{k+1}, \dots, h_r \rangle : h_k$ are all generated by a subset $\mathcal{X}_k \subseteq \mathcal{X}$. The following result exhibits the relation of this notion to Pommaret bases.

Proposition 7.2. (Hashemi et al., 2012, Prop. 26) Let $\mathcal{H} = \{h_1, \dots, h_s\}$ be a P-ordered monomial Pommaret basis of the quasi-stable monomial ideal $\mathcal{I} \lhd \mathcal{P}$. Then $\mathcal{I}$ possesses linear quotients with respect to the basis $\mathcal{H}$ and

$$\langle h_{\alpha+1}, \dots, h_s \rangle : h_\alpha = \langle \overline{\mathcal{X}}_P(h_\alpha) \rangle \qquad \alpha = 1, \dots s-1 . \tag{29}$$

Conversely, assume that $\mathcal{H} = \{h_1, \dots, h_s\}$ is a monomial generating set of the monomial ideal $\mathcal{I} \lhd \mathcal{P}$ such that (29) is satisfied and $\operatorname{cls} h_s = n$. Then $\mathcal{I}$ is quasi-stable and $\mathcal{H}$ its Pommaret basis.

Let the ordered set $\{m_1, \dots, m_s\}$ generate an ideal $\mathcal{I} \lhd \mathcal{P}$ and consider the following short exact sequence

$$0 \to \mathcal{P}/\tilde{\mathcal{I}} \xrightarrow{\cdot m_1} \mathcal{P}/\mathcal{I}' \longrightarrow \mathcal{P}/\mathcal{I} \to 0 \tag{30}$$

where $\tilde{\mathcal{I}} = \mathcal{I} : m_1$ and $\mathcal{I}' = \langle m_2, \dots, m_s \rangle$. A free resolution of $\mathcal{I}$ can now always be obtained as a mapping cone of resolutions of $\tilde{\mathcal{I}}$ and $\mathcal{I}'$. We first note the following simple consequences of Proposition 7.2 and the definition of a P-ordering.

Lemma 7.3. Let $\mathcal{I} \lhd \mathcal{P}$ be a quasi-stable monomial ideal and $\mathcal{H} = \{h_1, \dots, h_s\}$ a P-ordered Pommaret basis of it. Then

(1) $\tilde{\mathcal{I}} = \langle x_k, \dots, x_n \rangle$, where $k = \operatorname{cls} h_1$, and
(2) $\mathcal{I}'$ is again a quasi-stable ideal with Pommaret basis $\{h_2, \dots, h_s\}$.

Since $\tilde{I}$ is generated by a subset of the variables, it is minimally resolved by a Koszul complex. By adding one element of $\mathcal{H}$ in each iteration, we can construct a resolution of $\mathcal{I}$ as an iterated mapping cone.

Theorem 7.4. Let $\mathcal{H} = \{h_1, \dots, h_s\}$ be a P-ordered Pommaret basis of the quasi-stable monomial ideal $\mathcal{I} \lhd \mathcal{P}$ and the complex $(\mathcal{C}, d)$ the iterated mapping cone produced by the short exact sequences

$$\mathcal{S}_\alpha : \ 0 \to \mathcal{P}/\tilde{\mathcal{I}}_\alpha \xrightarrow{\cdot h_\alpha} \mathcal{P}/\mathcal{I}'_\alpha \longrightarrow \mathcal{P}/\mathcal{I}_\alpha \to 0 \tag{31}$$

with ideals $\mathcal{I}_\alpha = \langle h_\alpha, \dots, h_s \rangle$, $\mathcal{I}'_i = \langle h_{\alpha+1}, \dots, h_s \rangle$ and $\tilde{\mathcal{I}}_\alpha = \langle x_{k_\alpha}, \cdots, x_n \rangle$ where $k_\alpha = \operatorname{cls} h_\alpha$. Then $(\mathcal{C}, d)$ is isomorphic to the resolution of $\mathcal{I}$ induced by the Pommaret basis $\mathcal{H}$.

Proof. It is clear that we obtain a resolution, so it only remains to prove that it is the same as the one induced by the Pommaret basis $\mathcal{H}$. The proof proceeds by induction on s, the number of generators in $\mathcal{H}$.

For $s = 2$, the result is easily proven. In this case, $h_2 = x_n^a$ for some $a \in \mathbb{N}$ and $h_1 = x_{n-1}^b x_n^{a-1}$ for some $b \in \mathbb{N}$ and this ideal is stable. Now the assertion follows from the already mentioned fact that in this case the Eliahou-Kervaire resolution coincides with the resolution induced by the Pommaret basis.

Assume that the result is true for all $s - 1 \geq 2$. We are now in the situation of the sequence (30). We have the following diagram where ϕ is some chain complex morphism that lifts the map given by multiplication with h_α:

$$\begin{array}{ccccccccc}
\dashrightarrow & \tilde{\mathcal{F}}_2 & \xrightarrow{\tilde{d}_2} & \tilde{\mathcal{F}}_1 & \xrightarrow{\tilde{d}_1} & \mathcal{P} & \longrightarrow & \mathcal{P}/\tilde{\mathcal{I}} & \to 0 \\
 & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \cdot h_\alpha & & \downarrow \cdot h_\alpha & \\
\dashrightarrow & \mathcal{F}'_2 & \xrightarrow{d'_2} & \mathcal{F}'_1 & \xrightarrow{d'_1} & \mathcal{P} & \longrightarrow & \mathcal{P}/\mathcal{I}' & \to 0
\end{array}$$

Each module in the mapping cone is given by $\mathcal{C}_j = \mathcal{F}'_j \oplus \tilde{\mathcal{F}}_{j-1}$ and the differential is given by $d_j = \begin{pmatrix} d'_j & \phi_{j-1} \\ 0 & -\tilde{d}_{j-1} \end{pmatrix}$. Given a generator $\mathbf{e}_\alpha \otimes \mathbf{k}$ of the jth module of the mapping cone complex, its differential is given by $d(\mathbf{e}_\alpha \otimes \mathbf{k}) = \phi_{j-1} - \tilde{d}_j(\mathbf{e}_\alpha \otimes \mathbf{k})$. We make here explicit use of the correspondence $\mathbf{e}_\alpha \otimes \mathbf{k} \leftrightarrow \mathbf{e}_{\alpha;\mathbf{k}}$. On the other hand, by Theorem 7.1, the differential of the resolution induced by the Pommaret basis $\mathcal{H}$ is given by

$$\begin{aligned}\delta(\mathbf{e}_\alpha \otimes \mathbf{k}) &= \sum_{\ell=1}^{j}(-1)^{\sigma(\ell,\mathbf{k})} x_{k_\ell}\mathbf{e}_\alpha \otimes \mathbf{k}_\ell - \sum_{\ell=1}^{j}(-1)^{\sigma(\ell,\mathbf{k})} t_{\alpha,k_\ell}\mathbf{e}_{\Delta(\alpha,k_\ell)} \otimes \mathbf{k}_\ell \\ &= A + B\,,\end{aligned}$$

where $\sigma(l, \mathbf{k})$ is the place of l in $\mathbf{k}$. One easily sees that $\tilde{d}_j(\mathbf{e}_\alpha \otimes \mathbf{k}) = -A$, so we only have to prove that $\phi_j(\mathbf{e}_\alpha \otimes \mathbf{k}) = B$.

Let $\psi_j : \mathcal{F}'_{j-1} \mapsto \mathcal{F}'_j$ be a contracting homotopy for d'_j. Then $\phi_j = \psi_j\phi_{j-1}\tilde{d}_j$. According to (Seiler, 2002, Thm. 5.2), a contracting homotopy ψ_j consists essentially of computing the involutive normal form with respect to the Pommaret basis of $\mathcal{F}'_i$ which is given by the jth syzygies induced by the Pommaret basis $\mathcal{H}$ of I.

Now we proceed by induction on j. For $j = 1$ and $j = 2$, we obtain as a corollary to (Seiler, 2009b, Lemma 7.1) by direct computation that

$$\phi_j\Bigl(\sum_{\ell=1}^{j}(-1)^{\sigma(l,\mathbf{k})} x_{k_\ell}\mathbf{e}_\alpha \otimes \mathbf{k}_\ell\Bigr) = B\,.$$

For the general case, observe that $\tilde{d}_j(\mathbf{e}_\alpha \otimes \mathbf{k}) = \sum_{\ell=1}^{j}(-1)^{\sigma(\ell,\mathbf{k})} x_{k_\ell}\mathbf{e}_\alpha \otimes \mathbf{k}_\ell$. Assuming that $\phi_h\bigl(\sum_{\ell=1}^{h}(-1)^{\sigma(\ell,\mathbf{k}_\ell)} x_{k_\ell}\mathbf{e}_\alpha \otimes \mathbf{k}_\ell\bigr) = B$ for all $h < j$ and computing B explicitly, we find that

$$\begin{aligned}&\phi_{j-1}\tilde{d}_j(\mathbf{e}_\alpha \otimes \mathbf{k}) = \\ &\qquad \sum_{\ell=1}^{j}(-1)^{\sigma(\ell,\mathbf{k})} x_{k_\ell}\Bigl(\sum_{\substack{k\in\mathbf{k}_\ell \\ \operatorname{cls}\mathbf{k}_{k\ell}>\operatorname{cls} w_{\Delta(1,k)}}}(-1)^{\sigma(k,\mathbf{k}_\ell)+1} t_{1,k}\mathbf{e}_{\Delta(1,k)} \otimes \mathbf{k}_{\ell k}\Bigr).\end{aligned} \tag{32}$$

Using normal form computations, we have that

$$\psi_j\phi_{j-1}\tilde{d}_j(\mathbf{e}_\alpha \otimes \mathbf{k}) = \sum_{\ell=1}^{j}(-1)^{\sigma(\ell,\mathbf{k})} \sum_{\substack{k\in\mathbf{k}_\ell \\ \operatorname{cls}\mathbf{k}_{k\ell}>\operatorname{cls} w_{\Delta(1,k)} \\ \operatorname{cls} x_{k_\ell}>\operatorname{cls}\mathbf{k}_{\ell k}}}(-1)^{\sigma(k,\mathbf{k}_\ell)+1} t_{1,k}\mathbf{e}_{\Delta(1,k)} \otimes \mathbf{k}_k\,. \tag{33}$$

Finally, in order to see that the expression in equation (33) equals B, observe that $t_{1,k}$ is always multiplicative for $\mathbf{e}_{\Delta(1,k)} \otimes \mathbf{k}_k$ and hence the right hand side of equation (33) is involutively autoreduced. □

8 Conclusions

In the first part, we combined the construction of a free resolution for polynomial modules due to Sköldberg (2011) with the theory of Pommaret bases as given in (Seiler, 2009b). This combination makes Sköldberg's approach via algebraic discrete Morse theory fully algorithmic, as Pommaret bases provide us with a systematic method for generating presentations with initially linear syzygies: the existence of such presentations can be seen as one of the core ideas underlying the theory of Pommaret bases.

A distinctive feature of our algorithm compared to other approaches for constructing resolutions is the fact that only *one* Pommaret basis (namely for the given submodule $\mathcal{U}$) is needed whereas usually several Gröbner bases (typically the number is given by $\operatorname{pd}\mathcal{U}$) have to be computed. Although the evaluation of the closed formula for the differential might appear very complicated, it requires mainly very cheap operations. The only polynomial operations are some involutive normal forms of products $x^{\mu}\mathbf{h}_{\alpha}$ for generators $\mathbf{h}_{\alpha}$ in the Pommaret basis of $\mathcal{U}$. As we observed that the same product often appears many times, significant savings could be achieved by simply storing these.

The main reasons for the not yet satisfactory timings are our currently still very naive implementation of the minimisation process and the sometimes extreme difference in the sizes of the non-minimal and the minimal resolution, respectively. Here one can expect on one side substantial improvements just by optimisations of the code. On the other hand, it is obvious that a decisive factor is simply the size of the complex $\mathcal{G}_{\bullet}$. While this size is completely determined by the Pommaret basis of $\mathcal{U}$, it can be affected by linear coordinate transformations. In a forthcoming work, we will present a deterministic approach to construct generally fairly sparse coordinate transformations such that the leading module $\operatorname{lt}\mathcal{U}$ is not only quasi-stable but stable (or even strongly stable). Then the complex $\mathcal{G}_{\bullet}$ should be considerably smaller; more precisely, it will have the same size as the minimal resolution of $\operatorname{lt}\mathcal{U}$. Finally, our current approach first determines the full complex $\mathcal{G}_{\bullet}$ and then minimises it. Probably very significant gains in efficiency are possible by interweaving the construction of this complex with the minimisation process (as it is done by classical algorithms to compute minimal resolutions).

The full power of our approach becomes already now apparent, if one does not need the whole resolution but only the Betti numbers. To our knowledge we presented here the first method to compute them which does not require the minimal resolution. As Table 2 demonstrates, for all examples save one our implementation was the fastest to compute the Betti numbers, often even by orders of magnitude. The main reasons are of course that, due to the knowledge of a closed form of the differential, we can construct directly only the constant part of the resolution and the subsequent "minimisations" require only linear algebra over the base field $\Bbbk$ and not over the polynomial ring $\mathcal{P}$. Obviously, similar techniques should exist for tackling other questions where only certain parts of the minimal resolution are relevant like an analysis of the linear part of the resolution or its linear strand.

In the second part of this work, we specialised to a particular class of monomial ideals, the quasi-stable ideals. We proved that the resolution induced by the Pommaret basis is also obtainable via iterated mapping cones. Thus we showed that this interesting property of the Eliahou-Kervaire resolution remains valid for our generalisation of it to quasi-stable ideals. An interesting question here is whether further nice properties of the Eliahou-Kervaire resolution like the fact that it is a cellular resolution (Mermin, 2010) also extend to the resolution induced by a Pommaret basis.

A A Detailed Example

We elaborate on some of the notions introduced in Section 3. Our starting point is the ideal $\mathcal{I} = \langle x_1^2, x_0^2, x_2 + x_0 \rangle \trianglelefteq \mathcal{P} = \Bbbk[x_0, x_1, x_2]$ over the base field $\Bbbk = \mathbb{Z}/2\mathbb{Z}$. A Pommaret basis $\mathcal{H}$ for $\mathcal{I}$ is given by

$$h_1 = x_0^2,\ h_2 = x_0^2 x_1,\ h_3 = x_1^2,\ h_4 = x_2 + x_0\,.$$

We start with the two-sided Koszul complex $(\mathcal{F}, d_{\mathcal{F}})$ induced by the initially linear syzygies associated with the Pommaret basis $\mathcal{H}$, see Lemma 4.1. This complex has the shape

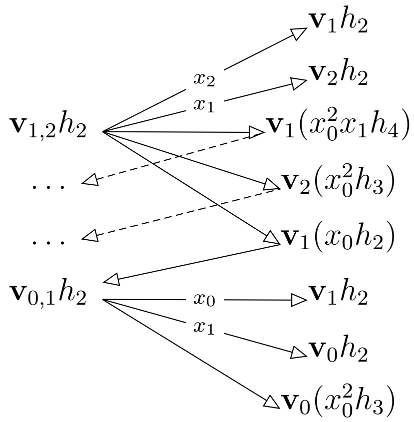
$$0 \longrightarrow \mathcal{F}_3 = \mathcal{P} \otimes_{\Bbbk} \Lambda_3 \mathcal{V} \otimes_{\Bbbk} I \longrightarrow \cdots \longrightarrow \mathcal{F}_0 = \mathcal{P} \otimes_{\Bbbk} \Lambda_0 \mathcal{V} \otimes_{\Bbbk} I = \mathcal{P} \otimes_{\Bbbk} I$$

with the differential given by (11).

In order to construct the resolution $(\mathcal{G}, d_{\mathcal{G}})$, more precisely its differential defined by (16), we study the graph $\Gamma_{\mathcal{F}}$. As this graph is huge, we will only show how to find $d_{\mathcal{G}}(\mathbf{v}_{1,2}h_2)$. The argument $\mathbf{v}_{1,2}h_2$ is indeed an element of the module G_2 as $1, 2 \in \operatorname{crit}(h_2)$. According to (15), the first step is to calculate the image $d_{\mathcal{F}}(\mathbf{v}_{1,2}h_2) = \sum_{\mathbf{m},\mu,\gamma} Q^{(1,2);2}_{\mathbf{m},\mu,\gamma} \mathbf{v}_{\mathbf{m}}(x^\mu h_\gamma)$. For each non-vanishing coefficient $Q^{(1,2);2}_{\mathbf{m},\mu,\gamma}$, the graph $\Gamma_{\mathcal{F}}$ contains the edge $\mathbf{v}_{1,2} \otimes h_2 \to \mathbf{v}_{\mathbf{m}}(x^\mu h_\gamma)$. By the definition (11) of the differential $d_{\mathcal{F}}$, we have

$$d_{\mathcal{F}}(\mathbf{v}_{1,2}h_2) = x_1 \mathbf{v}_2 h_2 + \mathbf{v}_2 x_0^2 h_3 + x_2 \mathbf{v}_1 h_2 + \mathbf{v}_1 x_0^2 x_1 h_4 + \mathbf{v}_1 x_0 h_2$$

So we get the top half of the following graph (which we want to be a subgraph of $\Gamma^A_{\mathcal{F}_\bullet}$, see the subsequent discussion):



(A.1)

According to (16), we search for reduction paths originating in the vertices on the right side of the graph and therefore we must look at the graph $\Gamma^A_{\mathcal{F}_\bullet}$, where A is the Morse matching defined in (14) and the paragraph following it. It immediately follows from (14) that no path (in $\Gamma_{\mathcal{F}_\bullet}$) ending in $\mathbf{v}_2 h_2$ or $\mathbf{v}_1 h_2$, respectively, is contained in the Morse matching, so there are no reduction paths originating in either of these two vertices.

For the other vertices, we look at $\mathbf{v}_1(x_0 h_2)$ and note that there are also reduction paths originating in the remaining vertices (indicated by the dashed arrows). The edge $\mathbf{v}_{0,1} h_2 \to \mathbf{v}_1(x_0 h_2)$ is contained in the Morse matching A or equivalently the graph $\Gamma^A_{\mathcal{F}_\bullet}$ contains the edge $\mathbf{v}_1 x_0 h_2 \to \mathbf{v}_{0,1} h_2$. Now we need to find the edges originating in $\mathbf{v}_{0,1} h_2$, i. e. to calculate $d_{\mathcal{F}}(\mathbf{v}_{0,1} h_2)$. Again, by (11), we have

$$d_{\mathcal{F}}(\mathbf{v}_{0,1} h_2) = x_0 \mathbf{v}_1 h_2 + \mathbf{v}_1(x_0 h_2) + x_1 \mathbf{v}_0 h_2 + \mathbf{v}_0(x_0^2 h_3)\,.$$

This gives us the bottom half of the above graph.

Then we should reiterate this process for the three new vertices we just constructed. But this time, (14) tells us that no edges ending in either of these three vertices are contained in the Morse matching. So we have found all reduction paths originating in $\mathbf{v}_1(x_0 h_2)$.

In order to find the associated reduction maps, we "collect the coefficients" along each reduction path. Note that in general, we would have to pay a little more attention to signs and coefficients, which we have avoided here by working in $\mathbb{Z}/2\mathbb{Z}$. For longer paths, all coefficients along the path have to be multiplied. Here, all paths are elementary reduction paths, so we can also look at their types.

The paths $\mathbf{v}_1(x_0 h_2) \to \mathbf{v}_{0,1} h_2 \to \mathbf{v}_1 h_2$ and $\mathbf{v}_1(x_0 h_2) \to \mathbf{v}_{0,1} h_2 \to \mathbf{v}_0 h_2$, respectively, are of type 1. Essentially the only things that happen are variables/indices being moved around towards components more to the left within the tensor product. This is due to the fact that those paths are coming from left summands in (11) of the differential in the two-sided Koszul complex.

The right summands in (11) yield the path $\mathbf{v}_1(x_0h_2) \to \mathbf{v}_{0,1}h_2 \to \mathbf{v}_0(x_0^2h_3)$, which accordingly is of type 2. Here we take the product of an element of the Pommaret basis with a critical variable, x_1h_2. The vertex comes from the involutive standard representation of this product.

However, for the differential $d_{\mathcal{G}}(\mathbf{v}_{1,2}h_2)$, only one of these reduction paths is relevant, as the restrictions of the sums in (16) require that only those reduction paths appear that end in a vertex $\mathbf{v}_{\boldsymbol{\ell}}h_\beta$ where $\boldsymbol{\ell} \subseteq \operatorname{crit}(h_\beta)$. Of the reduction paths in our example, only the path ending in $\mathbf{v}_1h_2$ satisfies this condition. The final result is then

$$d_{\mathcal{G}}(\mathbf{v}_{1,2}h_2) = (x_2 + x_0)\mathbf{v}_1h_2 + x_1\mathbf{v}_2h_2 + x_0^2\mathbf{v}_2h_3$$

and we have explicitly demonstrated the construction of all terms in it except the last one which stems from one of the dashed pathes in (A.1).

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