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# Term-ordering free involutive bases

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## Abstract

In this paper, we consider a monomial ideal  $J \triangleleft P := A[x_1, \dots, x_n]$ , over a commutative ring  $A$ , and we face the problem of the characterization for the family  $\mathcal{Mf}(J)$  of all homogeneous ideals  $I \triangleleft P$  such that the  $A$ -module  $P/I$  is free with basis given by the set of terms in the Gröbner escalier  $\mathbf{N}(J)$  of  $J$ . This family is in general wider than that of the ideals having  $J$  as initial ideal w.r.t. any term-ordering, hence more suited to a computational approach to the study of Hilbert schemes.

For this purpose, we exploit and enhance the concepts of multiplicative variables, complete sets and involutive bases introduced by Riquier (1893, 1899, 1910) and in Janet (1920, 1924, 1927) and we generalize the construction of  $J$ -marked bases and term-ordering free reduction process introduced and deeply studied in Bertone et al. (2013); Cioffi et al. (2011) for the special case of a strongly stable monomial ideal  $J$ .

Here, we introduce and characterize for every monomial ideal  $J$  a particular complete set of generators  $\mathcal{F}(J)$ , called stably complete, that allows an explicit description of the family  $\mathcal{Mf}(J)$ . We obtain stronger results if  $J$  is quasi stable, proving that  $\mathcal{F}(J)$  is a Pommaret basis and  $\mathcal{Mf}(J)$  has a natural structure of affine scheme.

The final section presents a detailed analysis of the origin and the historical evolution of the main notions we refer to.

*Keywords:* Involutive bases, quasi-stable ideals

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## 1. Introduction.

Let  $P := A[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over a commutative ring  $A$ . The problem we address is the following.

**Problem 1.1.** *Given any monomial ideal  $J \triangleleft P$  find a characterization for the family  $\mathcal{Mf}(J)$  of all homogeneous ideals  $I \triangleleft P$  such that the  $A$ -module  $P/I$  is free with basis given by the set of terms in the Gröbner escalier  $\mathbf{N}(J)$  of  $J$ .*

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The most relevant examples of ideals that belong to this family are those such that  $J$  is their initial ideal w.r.t. some term-ordering, but in general they form a proper subset of  $\mathcal{Mf}(J)$ . Therefore, we must overcome the Gröbner framework.

A computational description of the whole family  $\mathcal{Mf}(J)$  is obtained in Bertone et al. (2013); Cioffi et al. (2011) for  $J$  strongly stable. These families are optimal for many applications, for instance for an effective study of Hilbert schemes (see Bertone-Lella-Roggero (2013)). However, the strong stability of the monomial ideal  $J$  is a rather limiting condition.

In the present paper we give an overall view on what can be said about the above question for an *arbitrary* monomial ideal  $J$ , enhancing some ideas introduced by Riquier (1893, 1899, 1910) and Janet (1920, 1924, 1927).

The ideas we mainly deal with are those of *multiplicative variable* and *complete system*, leading to the so called *Janet-Riquier decomposition* for terms. These concepts date back to the late nineteenth century and the first decades of the twentieth. In a historical note at the end of the paper we present a detailed overview of their appearances, evolution and applications.

In Janet's theory the ideals  $I$  are generated by those we call now *involution bases*, a set which contains as a subset those that are also Gröbner bases. Indeed, Janet develops his ideas assuming to be in *generic coordinates* and, using the (deg)-revlex ordering, the monomial ideal  $J$  he gets is strongly stable.

From a computational point of view, a general change of coordinates is remarkably heavy. For this reason, we consider interesting enhancing the theory and obtaining analogous results not assuming this hypothesis, hence not requiring to perform a linear change of coordinates.

Indeed, Janet's ideas permit to go beyond this context and to recover results and techniques of both Gröbner basis theory and  $J$ -marked basis theory. In fact we do not need to impose a term-ordering on the given polynomial ring.

We identify two essential features that are key points for most computations in both the above frameworks:

- I)  $I$  is generated by a set of polynomials, marked on the terms of a suitable generating set of the monomial ideal  $J$ ;
- II) there is a reduction process w.r.t. these marked polynomials, that is used to rewrite each element of  $P/I$  as an element of the free  $A$ -module  $\langle \mathbf{N}(J) \rangle$

Janet's notions of multiplicative variable and complete system allow to construct such marked set of generators for  $I$  and to define an efficient reduction process.

First of all, we examine and compare two different definitions of multiplicative variable that are presented in Janet (1920, 1924) and in Janet (1927) and that are equivalent in general coordinates. We underline similarities and differences and introduce the notion of *stably complete* set of terms, when both conditions hold. We show that every monomial ideal  $J$  has only one stably complete set of generators (possibly made of infinitely many terms) that we call *star set* and denote by  $\mathcal{F}(J)$ .

Furthermore, we define a reduction procedure with respect to a homogeneous set of polynomials marked on a stably complete system  $\mathcal{F}(J)$  and prove its noetherianity. As a consequence we are able to give a first, general answer to Problem 1.1.

Of course, the most interesting cases are those of ideals  $J$  such that their generating stably complete set  $U$  is finite. We prove that they are the *quasi stable* ideals and that  $\mathcal{F}(J)$  is their Pommaret basis. Among them, those such that  $\mathcal{F}(J)$  coincides with the monomial basis are exactly the *stable* ones.

For the class of quasi stable ideals  $J$  we give a more complete and effective answer to Problem 1.1. Indeed, we prove that our description of  $\mathcal{M}(J)$  is natural, in the sense that it defines a representable functor from the category of (commutative) rings to the category of sets. Finally we give an effective procedure computing equations for the scheme that represents this functor.

After introducing all the notation (section 2), we introduce the Riquier-Janet decomposition for semigroup ideals and order ideals (section 3).

More precisely, we recall the notion of multiplicative variables (Janet (1920, 1924, 1927)) and complete system, pointing out that in the cited papers Janet uses two *non equivalent* definitions of multiplicative variables and completeness.

For our purpose, we then introduce the notion of *stable completeness* as a junction between the two notions. Given a complete system of terms  $U$ , we define a decomposition for terms in  $J = (U)$ , called *star decomposition*, in analogy with the star product introduced in Bertone et al. (2013); Cioffi et al. (2011) for strongly stable ideals.

In section 4 we define a very special stably complete set, i.e. the *star set*  $\mathcal{F}(J)$ , introducing the *stable ideals*, for which  $\mathcal{F}(J)$  is the minimal generating set of  $J$  and the *quasi stable ideals* for which the finiteness condition is respected.

In section 5, we define  $U$ -marked polynomials, bases and families, for a complete set  $U$ , again generalizing the definitions given for the generating set of a strongly stable ideal. Then, we define a noetherian reduction process for homogeneous polynomial w.r.t. the elements of a  $U$ -marked set. In section 6 we associate to marked families a representable functor and give a procedure computing equations for the scheme that defines it.

Finally, section 7 contains the historical note.

## 2. Notation.

Consider the polynomial ring  $P := A[x_1, \dots, x_n] = \bigoplus_{d \in \mathbb{N}} P_d$  in  $n$  variables and coefficients in the base ring  $A$ .

When an order on the variables<sup>2</sup> comes into play, we consider  $x_1 < x_2 < \dots < x_n$ . In case of  $n = 2, 3$  we will usually set  $P = A[x, y]$ ,  $x < y$  and  $P = A[x, y, z]$ ,  $x < y < z$ .

The symbol  $<_{Lex}$  will denote the lexicographic term-ordering according to this order on the set of variables.

The *set of terms* of  $P$  is

$$\mathcal{T} := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

For every polynomial  $f \in P$ ,  $deg(f)$  is its usual degree and  $deg_i(f)$  is its degree with respect to the variable  $x_i$ .

For each  $p \in \mathbb{N}$ , and for all  $W \subseteq P$ ,

$$W_p := \{f \in W : f \text{ homogeneous and } deg(f) = p\};$$

in particular:

$$\mathcal{T}_p := \{\tau \in \mathcal{T} : deg(\tau) = p\}, \quad |\mathcal{T}_p| = dim_k(P_p) = \binom{p+n-1}{n-1}.$$

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<sup>2</sup>Even if the “standard” variable ordering is actually  $x_1 > x_2 > \dots > x_n$ , we revert it, following the same notation of Bertone et al. (2013); Cioffi et al. (2011); Seiler (2009a,b); Hashemi et al. (2012). Moreover, we point out that the procedures we will explain in the following sections will be greatly similar the ones implemented in Ceria (2012a,b), for which the more intuitive ordering  $x_1 < x_2 < \dots < x_n$  has been chosen.

If  $f \in P$ , we denote by  $\text{Supp}(f)$  the *support* of  $f$ , i.e. the set of all the terms in  $\mathcal{T}$ , appearing in  $f$  with non-zero coefficient.

Fixing a polynomial  $f \in P$  and a term-ordering  $<$ , we call *leading term* of  $f$  the maximal element in  $\text{Supp}(f)$  w.r.t.  $<$  and we denote it  $\mathbb{T}(f)$ . Its coefficient is the *leading coefficient* of  $f$ . Given a term  $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{T}$ , we set

$$\max(\tau) = \max\{x_i : x_i \mid \tau\} \quad , \quad \min(\tau) = \min\{x_i : x_i \mid \tau\}$$

the maximal and the minimal variable appearing in  $\tau$  with non-zero exponent.

**Definition 2.1.** Given a term  $\tau \in \mathcal{T}$  and a variable  $x_j \mid \tau$ , the term  $\frac{\tau}{x_j}$  is the  $j$ -th *predecessor* of  $\tau$ .

**Definition 2.2.** Let  $F = \{\tau_1, \dots, \tau_s\} \subseteq \mathcal{T}$  be an ordered subset of terms, generating an ideal  $J = (F)$ . The module

$$\text{Syz}(F) = \{(g_1, \dots, g_s) \in P^s, \sum_{i=1}^s g_i \tau_i = 0\}$$

is the *syzygy module* of  $F$ .

We denote an element in  $\text{Syz}(F)$  as  $(g_1, \dots, g_s)$  and we call it *syzygy among  $F$* .

**Definition 2.3.** A set  $N \subset \mathcal{T}$  is called *order ideal* if

$$\forall s \in \mathcal{T}, t \in N : s \mid t \Rightarrow s \in N.$$

Observe that  $N$  is an order ideal if and only if the complementary set  $I := \mathcal{T} \setminus N$  is a *semigroup ideal*, i.e.  $\forall t \in \mathcal{T}, \tau \in I \Rightarrow t\tau \in I$ .

If  $I$  is either a monomial ideal or a semigroup ideal, we will denote by  $N(I)$  the order ideal  $N := \mathcal{T} \setminus I$  and by  $G(I)$  its *monomial basis*, namely the minimal set of terms generating  $I$ .

**Definition 2.4** (Reeves et al. (1993)). A (*monic*) *marked polynomial* is a polynomial  $f \in P$  together with a fixed term  $\tau$  of  $\text{Supp}(f)$ , called *head term of  $f$*  and denoted by  $\text{Ht}(f)$  and such that its coefficient is equal to  $1_A$ .

We can extend to marked polynomials the notion of  $S$ -polynomial.

### 3. Riquier-Janet decomposition.

In this section we loosely base on the paper Janet (1920), where Janet defines the notion of *multiplicative variable* for a term  $\tau$  with respect to a given set  $U \subseteq \mathcal{T}$ .

For completeness' sake, we recall Janet's decomposition for terms in the semigroup ideal generated by  $U$  into disjoint classes.

Each of them contains:

1. a term  $\tau \in U$ ;
2. the set of monomials obtained multiplying  $\tau$  by products of multiplicative variables, that we call *cone of  $\tau$*  (Gerdt et al. (1998a); Gerdt (2005); Seiler (2009a,b); Hashemi et al. (2012)) and denote  $C_J(\{\tau\})$ .

**Remark 3.1.** The decomposition by Janet and Riquier we present here has been generalized by Stanley (1978). The generalized decomposition has been employed to study Stanley depth, being more suitable than the original one. For more details, we refer to Herzog (2013).

The main difference with respect to Janet's papers is that we remove the finiteness condition on  $U$ , showing that it is not necessary for our purposes.

**Definition 3.2.** (Janet, 1920, pp .75-9), Let  $U \subset \mathcal{T}$  be a set of terms and  $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  be an element of  $U$ . A variable  $x_j$  is called *multiplicative* for  $\tau$  with respect to  $U$  if there is no term in  $U$  of the form  $\tau' = x_1^{\beta_1} \cdots x_j^{\beta_j} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$  with  $\beta_j > \alpha_j$ .

Since Janet used two different and non equivalent notions of multiplicative variables, recent notation (Gerdt, 2000, p.3) labels this concept as *Janet-multiplicative* (or *J-multiplicative*); moreover if a variable is not *J-multiplicative* for  $\tau \in U$  it is called *J-nonmultiplicative* for  $\tau$ . Following Gerdt et al. (1998a); Gerdt (2000), will denote by  $M_J(\tau, U)$  the set of *J-multiplicative* variables for  $\tau$  with respect to  $U$  and  $NM_J(\tau, U)$  the set of *J-nonmultiplicative* ones.

In what follows, we will refer to Janet-multiplicative variables simply as multiplicative variables.

If a term  $\sigma \in \mathcal{T}$  is such that  $\sigma = \tau\theta$ , where  $\theta$  is a power product of *J-multiplicative* variables<sup>3</sup> for  $\tau$  w.r.t. a set  $U \ni \tau$ , then  $\tau$  is called *involution divisor* of  $\sigma$ .

**Definition 3.3** (Gerdt et al. (1998a); Gerdt (2005)). With the previous notation, the *J-cone* of  $\tau$  with respect to  $U$  is the set

$$C_J(\{\tau\}) := \{\tau x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid \text{where } \lambda_j \neq 0 \text{ only if } x_j \in M_J(\tau, U)\}.$$

**Example 3.4.** Consider the set  $U = \{x_1^3, x_2^3, x_1^4 x_2 x_3, x_3^2\} \subseteq k[x_1, x_2, x_3]$ .

Let  $\tau = x_1^3$ , so  $\alpha_1 = 3$ ,  $\alpha_2 = \alpha_3 = 0$ . The variable  $x_1$  is multiplicative for  $\tau$  w.r.t  $U$  since there are no terms  $\tau' = x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \in U$  satisfying both conditions:

- $\beta_1 > 3$ ;
- $\beta_2 = \beta_3 = 0$ .

On the other hand,  $x_2$  is not multiplicative for  $\tau$  since  $\tau'' = x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} = x_2^3 \in U$ :

$$\gamma_2 = 3 > 0 = \alpha_2, \gamma_3 = \alpha_3 = 0.$$

Similarly,  $x_3$  is not multiplicative since  $x_3^2 \in U$ .

In conclusion, we have  $M_J(\tau, U) = \{x_1\}$ .

**Remark 3.5.** Observe that, by definition of multiplicative variable, the only element in  $C_J(\{\tau\}) \cap U$  is  $\tau$  itself.

Indeed, if  $\tau \in U$  and also  $\tau\sigma \in U$  for a non constant term  $\sigma$ , then  $\max(\sigma)$  cannot be multiplicative for  $\tau$ , hence  $\tau\sigma \notin C_J(\{\tau\})$ .

In paper Janet (1920), Janet defines multiplicative variables as in Definition 3.2 and he provides both a decomposition for the semigroup ideal  $T(U)$  generated by a finite set of terms  $U$  and a decomposition for the complementary set  $N(U)$ .

On the other hand, in Janet (1924, 1927), he defines multiplicative variables in the following way.

<sup>3</sup>Such a term is called *multiplicative* by Gerdt et al. (1998a).

**Definition 3.6.** A variable  $x_j$  is *multiplicative* for  $\tau \in \mathcal{T}$  if and only if  $x_j \leq \min(\tau)$ .

In order to avoid confusion, here we change terminology w.r.t. the one employed by Janet and we follow the one by Gerdt (2000).

**Definition 3.7** (Gerdt (2000)). A variable  $x_j$  is *Pommaret-multiplicative* or *P-multiplicative* for  $\tau \in \mathcal{T}$  if and only if  $x_j \leq \min(\tau)$ .

The *P-cone* of a term  $\tau$  is the set

$$C_P(\{\tau\}) := \{\tau x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid \text{where } \lambda_j \neq 0 \text{ only if } x_j \text{ is P-multiplicative for } \tau\}.$$

These two definitions of multiplicative variables appear to be very different. First of all, in the first formulation, the set of multiplicative variables for a term in  $U$  depends on the whole set  $U$ , while in the second it is completely independent on the set  $U$ . Indeed, the two notions are not equivalent for a general set  $U$ , as shown by the following examples.

**Example 3.8.** In  $k[x_1, x_2, x_3]$  consider the ideal  $I = (x_1^2 x_2, x_1 x_2^2)$  and let  $U$  be its monomial basis. Then,  $M_J(x_1^2 x_2, U) = \{x_1, x_3\}$  and  $M_J(x_1 x_2^2, U) = \{x_1, x_2, x_3\}$ , whereas only  $x_1$  is *P-multiplicative*.

**Example 3.9.** Taken the set  $U = \{x_1^2 x_2, x_1 x_2^2\} \subseteq k[x_1, x_2]$ , we get  $M_J(x_1 x_2^2, U) = \{x_1, x_2\}$ , while of course  $x_1 \leq \min(x_1 x_2^2)$  but  $x_2 > \min(x_1 x_2^2)$ .

However, they are equivalent in Janet setting, that is if  $U$  is the generating set of the generic initial ideal of a homogeneous ideal  $I$ .

We refer to Gerdt (2000) for a deeper and more detailed description of *J-multiplicative* and *P-multiplicative* variables.

More generally, we will see that they turn out to be equivalent also if  $U$  is the monomial basis  $\mathbf{G}(J)$  of a strongly stable ideal  $J$  and if  $U$  is the special set of generators of any monomial ideal  $J$  that we will introduce in section 4 and denote by  $\mathcal{F}(J)$ .

We will see that stronger results can be proved when a set  $U$  is such that *the two definitions* of multiplicative variables *coincide*.

The following definition will be a key point in this paper.

**Definition 3.10.** (Janet, 1920, ppg.75-9) A set of terms  $U \subset \mathcal{T}$  is called *complete* (or *Janet-involutive*) if for every  $\tau \in U$  and  $x_j \notin M_J(\tau, U)$ , there exists  $\tau' \in U$  such that  $x_j \tau \in C_J(\{\tau'\})$ .

Moreover,  $U$  is *stably complete* if it is complete and for every  $\tau \in U$  it holds  $M_J(\tau, U) = \{x_i \mid x_i \leq \min(\tau)\}$ .

If a set  $U$  is stably complete and finite, then it is the *Pommaret basis* of  $J = (U)$  and we denote it by  $\mathcal{H}(J)$ .

**Remark 3.11.** If  $U = \{\tau\} \subseteq P$  is a singleton, it is complete, with  $M_J(\tau, U) = \{x_1, \dots, x_n\}$ .

Let us examine some examples.

**Example 3.12.** In  $k[x_1, x_2, x_3]$  consider the ideal  $I = (x_1^2, x_1 x_2, x_3)$ .

Both  $U_0 = \{x_1^2, x_1 x_2, x_3\}$  and each generating set of  $I$  with the shape

$U_i = \{x_1^2, x_1 x_2, x_3, x_2 x_3, \dots, x_2^i x_3\}$  are complete systems of terms. In fact, for  $U_0$ :

- $M_J(x_1^2, U_0) = \{x_1\}$ ,  $x_1^2 x_2 \in C_J(\{x_1 x_2\})$ ,  $x_1^2 x_3 \in C_J(\{x_3\})$ ;
- $M_J(x_1 x_2, U_0) = \{x_1, x_2\}$ ,  $x_1 x_2 x_3 \in C_J(\{x_3\})$ ;

$$- M_J(x_3, U_0) = \{x_1, x_2, x_3\}.$$

For  $U_i, i \geq 1$ :

$$- M_J(x_1^2, U_i) = \{x_1\}, x_1^2 x_2 \in C_J(\{x_1 x_2\}), x_1^2 x_3 \in C_J(\{x_3\});$$

$$- M_J(x_1 x_2, U_i) = \{x_1, x_2\}, x_1 x_2 x_3 \in C_J(\{x_2 x_3\});$$

$$- M_J(x_3, U_i) = \{x_1, x_3\}, x_2 x_3 \in C_J(\{x_2 x_3\});$$

$$- M_J(x_2^j x_3, U_i) = \{x_1, x_3\}, x_2^{j+1} x_3 \in C_J(\{x_2^{j+1} x_3\}), 0 \leq j < i;$$

$$- M_J(x_2^i x_3, U_i) = \{x_1, x_2, x_3\}.$$

**Example 3.13.** Consider the ideal  $J = (xy) \triangleleft k[x, y]$ .

The monomial basis  $U_0 = \mathbb{G}(J) = \{xy\}$  is a complete system with  $M_J(xy, U_0) = \{x, y\}$ .

Also the set  $U = \{x^h y \mid h \geq 1\} \subseteq k[x, y], x < y$ , is a complete system, again according to the first definition. It generates the same ideal  $(xy)$ , but has infinitely many elements. Anyway, it is not stably complete. In fact, for each  $x^h y \in U, M_J(x^h y, U) = \{y\}$ , since no terms of the form  $x^l y^e$  with  $e > 1$  belong to  $U$ ; on the other hand  $x \notin M_J(x^h y, U)$  since  $x^{h+1} y \in U$ .

**Example 3.14.** Let  $U$  be the set of terms  $\{x, y^2\}$  in  $k[x, y]$ , with  $x < y$ .

The multiplicative variables for every term in  $U$  are those lower than or equal to its minimal one:  $M_J(x, U) = \{x\} M_J(y^2, U) = \{x, y\}$ .

However,  $U$  is not complete since  $yx$  does not belong to the J-cone of any term in  $U$ .

The following example shows that a complete generating set of terms can lose completeness when the ideal is enlarged.

**Example 3.15.** Let  $U = \{x^2, xy\} \subset k[x, y]$  and  $J = (U)$ . It is a complete system, but it is not stably complete, since  $y$  is multiplicative for  $xy$ , although  $\min(xy) = x$ .

Adding to  $U$  a term in  $\mathbb{N}(J)$ , we get a new set  $U_0$  and  $J_0 = (U_0)$ , whose Riquier-Janet decomposition clearly changes. For example, if  $U_0 = \{x^2, xy, y^2\}$  we get a stably complete system.

On the other hand, if  $U_0 = \{x^2, xy, y^3\}$  the system is not complete anymore, since  $xy^2$  does not belong to the J-cone of any term in the set.

The following technical lemma will be very useful throughout the paper. As a first application, we will prove that a system of terms  $U$  (possibly infinite) is *complete* if and only if the J-cones of the elements in  $U$  form a partition of the semigroup ideal generated by  $U$ .

**Lemma 3.16.** (Janet, 1931, pg.23) Let  $\tau, \tau'$  be elements of a set of terms  $U$  and  $x_j$  be a variable such that  $x_j \notin M_J(\tau, U)$  and  $x_j \tau \in C_J(\{\tau'\})$ . Then

$$\tau <_{Lex} \tau'. \text{ If, moreover, } x_j \leq \min(\tau), \text{ then } x_j \tau = \tau' \in U.$$

*Proof.* First of all, we observe that  $\tau \neq \tau'$ , since  $x_j \notin M_J(\tau, U)$ . By definition of J-cone, we have that  $\tau x_j = \tau' \sigma'$ , where  $\sigma'$  is a product of multiplicative variables for  $\tau'$ . Let us assume by contradiction that  $\tau >_{Lex} \tau'$  and let  $x_i$  be the maximal variable such that  $\deg_i(\tau) > \deg_i(\tau')$ . Then,  $x_i | \sigma'$ , hence  $x_i \in M_J(\tau', U)$ , but this is impossible by definition of multiplicative variable, since also  $\tau$  is in  $U$ .

Now let us assume that  $x_j \leq \min(\tau)$  and  $\sigma' \neq 1$ . If  $x_j | \sigma'$ , then  $\tau = \frac{\sigma'}{x_j} \tau' \in U \cap C_J(\{\tau'\})$ , which is not possible by Remark 3.5. If, on the contrary,  $x_j \nmid \sigma'$  we get a contradiction with the previous assertion, since in this case  $\tau' \leq_{Lex} \frac{\tau' \sigma'}{\max(\sigma')} <_{Lex} \frac{\tau' \sigma'}{x_j} = \tau$ .  $\square$



**Theorem 3.17.** *Let  $U$  be a set of terms (possibly infinite).*

*If  $\tau, \tau' \in U$  and  $\tau \neq \tau'$ , then  $C_J(\{\tau\}) \cap C_J(\{\tau'\}) = \emptyset$ .*

*If, moreover,  $U$  is complete and  $\mathbb{T}(U)$  is the semigroup ideal it generates, then  $\forall \gamma \in \mathbb{T}(U)$ ,  $\exists \tau \in U$  such that  $\gamma \in C_J(\{\tau\})$ . Hence, the  $J$ -cones of the elements in  $U$  give a partition of  $\mathbb{T}(U)$ .*

*Proof.* To prove the first assertion, let us assume by contradiction that  $\tau\sigma = \tau'\sigma' \in C_J(\{\tau\}) \cap C_J(\{\tau'\}) \neq \emptyset$  and let  $\tau >_{lex} \tau'$ . If  $x_i$  is the maximal variable such that  $deg_i(\tau) > deg_i(\tau')$ , then  $x_i | \sigma'$ . By definition of  $J$ -cone,  $x_i \in M_J(\tau', U)$ , but this is impossible by definition of multiplicative variable, since also  $\tau$  is in  $U$ .

Now we assume that  $U$  is complete and prove the second fact. We argue by contradiction. Suppose  $\mathbb{T}(U) \not\supseteq \mathcal{O} := \bigcup_{\sigma \in U} C_J(\{\sigma\})$  and take any term  $\gamma$  in  $\mathbb{T}(U) \setminus \mathcal{O}$ . As  $U$  generates  $\mathbb{T}(U)$ , there are terms in  $U$  that divide  $\gamma$ : let  $\tau$  be the one which is maximal with respect to  $<_{lex}$ . If  $\gamma = \tau\sigma$ , the term  $\sigma$  contains at least a variable  $x_i \in NM_J(\tau, U)$ , since  $\tau\sigma \notin C_J(\{\tau\})$ . Then  $\gamma = \tau x_i \eta$  and  $\tau x_i \notin C_J(\{\tau\})$ .

By the completeness of  $U$ , we have  $\tau x_i \in \mathcal{O}$ , namely there is a term  $\tau' \in U$  such that  $\tau x_i = \tau' \sigma' \in C_J(\{\tau'\})$ . By Lemma 3.16 i),  $\tau' >_{lex} \tau$ , and this is not possible since  $\tau' | \gamma = \tau x_i \eta = \tau' \sigma' \eta$ .  $\square$

Thanks to the previous result, if  $U$  is a complete system, each term in  $\mathbb{T}(U)$  can be written in a unique way as a product of

1. an element  $\tau \in U$ ;
2. a term  $x^\eta = x_i^{\eta_i} \cdots x_j^{\eta_j}$ , with  $x_i, \dots, x_j \in M_J(\tau, U)$ .

This fact suggests the following

**Definition 3.18.** Let  $U$  be a complete system of terms. The *star decomposition* of every term  $\gamma \in (U)$  with respect to  $U$ , is the unique couple of terms  $(\tau, \eta)$ , with  $\tau \in U$ , such that  $\gamma = \tau\eta$  and  $\gamma \in C_J(\{\tau\})$ . If  $(\tau, \eta)$  is the star decomposition of  $\gamma$  with respect to  $U$ , we will write  $\gamma = \tau *_U \eta$ .

The star decomposition is the term ordering free version of the decomposition of terms defined by Eliahou et al. (1990).

**Remark 3.19.** From the statements examined above, it follows the well-known explicit formula for the Hilbert function of  $P/(U)$ :

$$H(P/(U))(k) = \binom{k+n}{n} - \sum_{\tau \in U, deg(\tau) \leq k} \binom{k - deg(\tau) + s_\tau - 1}{s_\tau - 1},$$

where  $s_\tau$  is the number of multiplicative variables for  $\tau$  w.r.t  $U$  and we set equal to 0 every binomial with a negative numerator or a negative denominator. Thus, this formula makes sense even if  $|U| = \infty$ , since for every  $k$  there are only finitely many non-zero summands.

If  $U$  is a finite set of terms and  $r$  is the maximal degree of its elements, this formula gives the value of the Hilbert polynomial for every  $k \geq r$ . See also Apel (1998), in which involutivity is applied to the computation of affine Hilbert functions.

The following lemma will be very useful for the reduction process we will define in section 5.

**Lemma 3.20.** *Let  $U$  be a stably complete system of terms and let  $\gamma$  be a term such that  $\gamma = \tau *_{\mathcal{U}} \eta$  and also  $\gamma = \sigma \eta'$  with  $\tau \nmid \sigma$ .*

*Then  $\eta' >_{Lex} \eta$ .*

*Proof.* By definition of stable completeness,  $\min(\tau) \geq \max(\eta)$ .

We can suppose  $GCD(\eta, \eta') = 1$ , dividing, if necessary, by the common factors. Since  $\tau \nmid \sigma$ , then  $\eta' \neq 1$ .

If  $x_j \mid \eta'$  then  $x_j \mid \tau$ , so  $x_j \geq \max(\eta)$  but, since  $x_j \nmid \eta$ ,  $x_j > \max(\eta)$  and so  $\eta' >_{Lex} \eta$ .  $\square$

#### 4. Star set and quasi stable ideals

We introduce here a special set of terms. We will prove that it is a complete system with many interesting properties in common with the minimal monomial basis of strongly stable ideals.

**Definition 4.1.** Given a monomial ideal  $J \triangleleft P$  we define the *star set* as

$$\mathcal{F}(J) := \{x^\alpha \in \mathcal{T} \setminus \mathbf{N}(J) \mid \frac{x^\alpha}{\min(x^\alpha)} \in \mathbf{N}(J)\}.$$

For completeness' sake, we give a direct proof of the following theorem.

**Theorem 4.2** (Gerdt et al. (1998a)). *For every monomial ideal  $J$ , the star set  $\mathcal{F}(J)$  is the unique stably complete system of generators of  $J$ . Hence, if  $U$  is stably complete,  $U = \mathcal{F}(U)$ .*

*Proof.* Let  $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{F}(J)$ ,  $x_j = \min(\tau)$ ,  $x_i \leq x_j$ . We prove that  $x_i \in M_J(\tau, \mathcal{F}(J))$ .

Indeed, if  $x_i \in NM_J(\tau, \mathcal{F}(J))$ , by definition 3.2, there is a term in  $\mathcal{F}(J)$  of the form  $\tau' = x_1^{\beta_1} \cdots x_i^{\beta_i} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n}$  with  $\beta_i > \alpha_i$ .

Since  $x_i \leq x_j = \min(\tau)$ , we have  $\tau \mid \frac{\tau'}{\min(\tau')}$  and this contradicts the fact that  $\tau' \in \mathcal{F}(J)$ , so it must be  $x_i \in M_J(\tau, \mathcal{F}(J))$ .

Let  $x_i > x_k := \min(\tau)$  and set  $\sigma_0 := \tau x_i$ ,  $\sigma_r := \frac{\sigma_{r-1}}{\min(\sigma_{r-1})}$  for  $r = 1 \dots, \alpha_k + \cdots + \alpha_{i-1}$  and note that  $x_i^{\alpha_i} \cdots x_n^{\alpha_n} \notin J$ , since it divides  $\frac{\tau}{\min(\tau)}$ , while  $\sigma_r \in J$ , since it is a multiple of  $\tau$ . Then, in the sequence of terms  $\sigma_i$ ,  $0 \leq i \leq \alpha_k + \cdots + \alpha_{i-1}$ , we find an element  $\sigma_j$  that belongs to  $J$ , while the following one does not.

Then  $\sigma_j \in \mathcal{F}(J)$ , so that  $x_i \tau \in C_J(\{\sigma_j\})$  and  $x_i \in NM_J(\tau, \mathcal{F}(J))$ .

Take  $\tau = x_k^{\alpha_k} \cdots x_n^{\alpha_n} \in \mathcal{F}(J)$ , and a variable  $x_i \in NM_J(\tau, \mathcal{F}(J))$ . By the previous result  $x_i > x_k = \min(\tau)$ . By definition of nonmultiplicative variable, there is a term  $\sigma' = x_i^t x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \in \mathcal{F}(J)$ , for some integer  $t > \alpha_i$ . Let us consider the minimum one.

If  $t = \alpha_i + 1$ , then  $x_i \tau = x_k^{\alpha_k} \cdots x_i^t \cdots x_n^{\alpha_n} \in C_J(\{\sigma'\})$ .

If, on the contrary,  $t > \alpha_i + 1$ , then  $\sigma'' = x_i^{\alpha_i+1} \cdots x_n^{\alpha_n} \in \mathbf{N}(J)$  by definition. Let us consider, as in the previous proof, the sequence of terms  $\sigma_0 := \tau x_i \in J$ ,  $\sigma_r := \frac{\sigma_{r-1}}{\min(\sigma_{r-1})}$  for  $r = 1 \dots, \sum_{j=k}^{i-1} \alpha_j$ . Since the last one is  $\sigma''$ , we can find in this sequence a suitable  $\sigma_j \in I$  such that  $\sigma_{j+1} \in \mathbf{N}(J)$ , that is  $\sigma_j \in \mathcal{F}(J)$  and  $x_i \tau \in C_J(\{\sigma_j\})$ .

In order to prove that every stably complete set of terms  $U$ , with  $J = (U)$  is exactly  $\mathcal{F}(J)$ , we first notice that clearly  $\mathbf{G}(J) \subseteq U$  and  $\mathbf{G}(J) \subseteq \mathcal{F}(J)$ .

Moreover, it is sufficient to prove that  $\mathcal{F}(J) \subseteq U$ . Let  $\sigma \in \mathcal{F}(J)$ , i.e.  $\frac{\sigma}{\min(\sigma)} = \omega \in \mathbf{N}(J)$ . Then, there exists  $\tau \in U$  such that  $\sigma \in C_J(\{\tau\})$  and so  $\sigma = \tau \eta$ , with either  $\eta = 1$  or  $\max(\eta) \leq \min(\tau)$ .

This implies that either  $\tau = \sigma$  or  $\tau \mid \omega$ , but the second alternative is impossible since both  $\tau \in U$  and  $\omega \in \mathbf{N}(J)$ .  $\square$

From now on, multiplicative and  $P$ -multiplicative variables will coincide and we simply call them multiplicative. In the same way, since also  $J$ -cones and  $P$ -cones turn out to coincide, we will call them cones.

**Remark 4.3.** i. For an arbitrary monomial ideal  $J$  the set  $\mathcal{F}(J)$  can be infinite. For example, if  $J = (x) \triangleleft k[x, y]$ ,  $x < y$ , then  $\mathcal{F}(J) = \{xy^n \mid n \in \mathbb{N}\}$ .

ii. Not all the complete systems turn out to be of the form of a star set.

For example, the complete system  $U = \{x^h y, h \geq 1\} \subseteq k[x, y]$  of Example 3.13 is not the star set of the ideal  $J := (U)$ .

Indeed,  $\mathbf{N}(J) = \{x^m, m \geq 0\} \cup \{y^l, l > 0\}$  and all the terms of the form  $xy^k$ ,  $k > 1$ , do not belong to  $U$ , even if  $\frac{xy^k}{\min(xy^k)} = y^k \in \mathbf{N}(U)$ .

Moreover, for  $h > 1$ ,  $\frac{x^h y}{x} = x^{h-1} y \in U$ , so  $x^h y \notin \mathcal{F}(J)$ .

Better results hold if the monomial ideal  $J$  satisfies one of the following conditions, weaker than the strongly stable property (see section 6).

**Definition 4.4.** A monomial ideal  $J$  is called *stable* if it holds

$$\tau \in J, x_j > \min(\tau) \implies \frac{x_j \tau}{\min(\tau)} \in J$$

A monomial ideal  $J$  is called *quasi stable* if it holds

$$\tau \in J, x_j > \min(\tau) \implies \exists t \geq 0 : \frac{x_j^t \tau}{\min(\tau)} \in J.$$

We will show that this notion of quasi stable ideal coincides with the one in Seiler (2009b), by proving that  $J$  actually has a Pommaret basis.

**Remark 4.5.** In order to verify whether the conditions above are satisfied for a given ideal  $J$  it is sufficient to check the terms in the basis  $\mathbf{G}(J)$ .

The following proposition characterizes stable ideals, connecting the definition to the notion of star set (see also Mall (1998)).

**Proposition 4.6.** *Let  $J$  be a monomial ideal. Then TFAE:*

i)  $J$  is stable

ii)  $\mathcal{F}(J) = \mathbf{G}(J)$

*Proof.* i)  $\implies$  ii) The inclusion  $\mathbf{G}(J) \subseteq \mathcal{F}(J)$  is true for every monomial ideal by definition of star set. We prove now that  $\gamma \notin \mathcal{F}(J)$  for every term  $\gamma \in J \setminus \mathbf{G}(J)$ .

By hypothesis,  $\exists \tau \in \mathbf{G}(J)$ , such that  $\gamma = \tau \sigma$  and  $\sigma \neq 1$ .

Let  $x_k := \min(\gamma)$ . If  $x_k \mid \sigma$ , then  $\frac{\gamma}{\min(\gamma)} = \tau \frac{\sigma}{x_k} \in J$ , so that  $\gamma \notin \mathcal{F}(J)$ .

If, on the other hand,  $x_k \nmid \sigma$  and  $x_j$  is any variable dividing  $\sigma$ , then  $x_j > x_k$  and  $x_k = \min(\tau)$ . By the stability of  $J$  we have  $\frac{x_j \tau}{x_k} \in J$ , hence  $\frac{\gamma}{x_k} = \frac{\tau \sigma}{x_j x_k} \in J$ , hence again  $\gamma \notin \mathcal{F}(J)$ .

ii)  $\implies$  i) If ii) holds, then  $\mathbf{G}(J)$  is the only stably complete system generating  $J$ . By remark 4.5, we can check the stability on the terms  $x^\alpha \in \mathbf{G}(J)$ . Let  $x_j > x_k := \min(x^\alpha)$ . By hypothesis there exists  $x^\beta \in \mathbf{G}(J)$  such that  $x_j x^\alpha \in C_J(\{x^\beta\})$ , and, since  $x^\alpha \in \mathbf{G}(J)$ , of course  $x^\alpha x_j \notin \mathbf{G}(J)$ . Hence  $x^\beta \mid \frac{x_j x^\alpha}{x_k}$  and so  $\frac{x^\alpha x_j}{x_k} \in J$ .  $\square$

**Proposition 4.7.** *Let  $J$  be a monomial ideal. Then TFAE:*

i)  $J$  is quasi stable

ii)  $|\mathcal{F}(J)| < \infty$

iii)  $\mathcal{F}(J) = \mathcal{H}(J)$  is the Pommaret basis of  $J$ .

*Proof.* i)  $\Rightarrow$  ii) Let  $a$  be the maximum of the degrees of elements in  $\mathbf{G}(J)$  and let  $t$  be such that  $\frac{x_j^t x^\alpha}{\min(x^\alpha)} \in J$  for every  $x^\alpha \in \mathbf{G}(J)$  and  $x_j > \min(x^\alpha)$ . We prove that  $\mathcal{F}(J)$  is contained in  $P_{<d}$  where  $d := a + tn$ . Let  $x^\alpha x^\eta \in J_{\geq d}$  with  $x^\alpha \in \mathbf{G}(J)$  and  $x_k$  be  $\min(x^\alpha x^\eta)$ . If  $x_k | x^\eta$ , then obviously  $\frac{x^\alpha x^\eta}{x_k} = x^\alpha \frac{x^\eta}{x_k} \in J$ , so  $x^\alpha x^\eta \notin \mathcal{F}(J)$ . If, on the other hand,  $x_k \nmid x^\eta$ , then  $x_k = \min(x^\alpha)$ . Moreover, every variable dividing  $x^\eta$  is higher than  $x_k$  and at least one of them, let us call it  $x_j$ , appears in  $x^\eta$  with exponent  $\geq t$ , as  $\deg(x^\eta) \geq nt$ . Then  $\frac{x_j^t x^\alpha}{x_k} \in J$ , hence  $\frac{x^\alpha x^\eta}{x_k} = \frac{x_j^t x^\alpha}{x_k} \cdot \frac{x^\eta}{x_j^t} \in J$  and  $x^\alpha x^\eta \notin \mathcal{F}(J)$ .

ii)  $\Rightarrow$  iii) By ii)  $\mathcal{F}(J)$  is finite, and by 4.2 is stably complete, so it is clearly the Pommaret basis of  $J$ .

iii)  $\Rightarrow$  i) By remark 4.5, we check the quasi stability on the terms  $x^\alpha \in \mathbf{G}(J)$ . Let  $x_j > x_k := \min(x^\alpha)$ . By the hypothesis on the finiteness of  $\mathcal{F}(J)$ , there exists  $m \gg 0$  such that  $x^\alpha x_j^m \notin \mathcal{F}(J)$ . Moreover, being  $\mathcal{F}(J)$  a stably complete system, there exists  $x^\beta \in \mathcal{F}(J)$  such that  $x_j^m x^\alpha \in C_J(\{x^\beta\})$  and  $x^\beta | \frac{x_j^m x^\alpha}{x_k}$ . Therefore,  $\frac{x_j^m x^\alpha}{x_k} \in J$ , namely  $J$  is quasi stable.  $\square$

Quasi-stable ideals are known in literature with many different names, such as *ideals of nested type* or *weakly stable ideals*. For more details on this class of monomial ideals, see also Bermejo et al. (2006); Caviglia et al. (2005).

**Example 4.8.** In  $k[x, y, z]$  with  $x < y < z$ :

- considered  $J = (z, y^2)$ , we get  $U = \mathcal{F}(J) = \mathbf{G}(J) = \{z, y^2\}$ , since  $J$  is stable;
- taken the ideal  $J' = (z^2, y)$ , we get  $U = \mathcal{F}(J) = \{z^2, yz, y\} \supset \mathbf{G}(J)$ .  
In fact,  $J$  is quasi stable, but it is not stable;
- given  $J = (y)$ , the star set is  $U = \mathcal{F}(J) = \{z^k y \mid k \geq 0\}$ , and it holds  $|\mathcal{F}(J)| = \infty$ , since  $J$  is not stable.

## 5. $U$ -marked sets and reduction process.

In this section, we generalize the notions of  $J$ -marked polynomial,  $J$ -marked basis and  $J$ -marked family given in Bertone et al. (2013); Cioffi et al. (2011) for  $J$  strongly stable. In those papers, the involved polynomials are marked on the monomial basis of the given monomial ideal  $J$ . Here, we give the analogous definitions for any monomial ideal, provided that the involved polynomials are marked on a complete generating system in the sense of definition 3.10. After determining the setting, we extend to it the reduction process of the quoted papers. At the end, we will see that such a generalized procedure does not need to be noetherian for every complete system of terms. We will need to add some hypotheses on the given complete system in order to overcome this problem.

We point out that, as in Bertone et al. (2013); Cioffi et al. (2011), we do not introduce any term-ordering and this represents an important difference w.r.t. Janet's papers.

Moreover, we consider polynomials with coefficients in a ring, not necessarily in a field.

**Definition 5.1.** Let  $U$  be a complete system of terms and  $J$  be the ideal it generates.

- A  $U$ -marked set is a set  $\mathcal{G}$ , not necessarily finite, containing,  $\forall x^\alpha \in U$ , a homogeneous (monic) marked polynomial  $f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma$ , with  $\text{Ht}(f_\alpha) = x^\alpha$  and  $\text{Supp}(f_\alpha - x^\alpha) \subset \mathbf{N}(J)$ , so that  $|\text{Supp}(f) \cap J| = 1$ .
- A  $U$ -marked basis  $\mathcal{G}$  is a  $U$ -marked set such that  $\mathbf{N}(J)$  is a basis of  $P/(\mathcal{G})$  as  $A$ -module, i.e.  $P = (\mathcal{G}) \oplus \langle \mathbf{N}(J) \rangle$  as an  $A$ -module.
- The  $U$ -marked family  $\mathcal{Mf}(U)$  is the set of all homogeneous ideals  $I$  that are generated by a  $U$ -marked basis.

**Remark 5.2.** Observe that the above definition of marked family  $\mathcal{Mf}(U)$  is consistent with that given in the Introduction of  $\mathcal{Mf}(J)$  for a monomial ideal  $J$ . Indeed, if  $I \in \mathcal{Mf}(U)$ , then  $I \in \mathcal{Mf}(J)$  with  $J = (U)$ . On the other hand, for every given  $J$  there are complete systems  $U$  that generate it, for instance  $U = \mathcal{F}(J)$  and  $\mathcal{Mf}(J) = \mathcal{Mf}(U)$ . In fact, if  $I \in \mathcal{Mf}(J)$ , every polynomial  $h$  can be uniquely written as a sum  $f + g$  with  $f \in I$  and  $g \in \langle \mathbf{N}(J) \rangle$ ; especially for every  $x^\alpha \in U$ , we have

$$x^\alpha = f_\alpha + g_\alpha, \quad f_\alpha \in I \text{ and } g_\alpha \in \langle \mathbf{N}(J) \rangle. \quad (1)$$

Then  $I$  contains the  $U$ -marked basis

$$\mathcal{G} = \{f_\alpha = x^\alpha - g_\alpha, \quad x^\alpha \in U\}.$$

Furthermore  $\mathcal{G}$  is a  $U$ -marked basis since  $(\mathcal{G}) \subseteq I$  and  $P = (\mathcal{G}) + \langle \mathbf{N}(J) \rangle = I \oplus \langle \mathbf{N}(J) \rangle$ .

The only difference between the two notations  $\mathcal{Mf}(J)$  and  $\mathcal{Mf}(U)$  with  $U$  a complete system generating  $J$ , is that using the second one we present every ideal of the family by means of a special set of generators depending on  $U$ . Note that, by the definition itself of  $\mathcal{Mf}(J)$ , we can assert that for every ideal  $I \in \mathcal{Mf}(J)$  the  $U$ -marked basis generating it is *unique*.

We define now a reduction procedure for terms and polynomials, with respect to an homogeneous set  $\mathcal{G}$  of polynomials, marked on a complete system of terms  $U$ .

The usual reduction process with respect to  $\mathcal{G}$  consists of substituting each term  $x^\alpha x^\eta$ , multiple of an head term  $x^\alpha = \text{Ht}(f_\alpha)$ , with the polynomial  $(x^\alpha - f_\alpha)x^\eta = g_\alpha x^\eta$ .

We add an extra condition to the standard procedure, namely that this substitution can be performed only in the case  $x^\alpha x^\eta = x^\alpha *_U x^\eta$ .

**Definition 5.3.** Let  $U$  be a complete system and  $\mathcal{G}$  a  $U$ -marked set. We will denote by  $\xrightarrow{\mathcal{G}}$  the transitive closure of the relation  $h \xrightarrow{\mathcal{G}} h - c f_\alpha x^\eta$ , where  $x^\alpha x^\eta = x^\alpha *_U x^\eta$  is a term that appears in  $h$  with a non-zero coefficient  $c$ . We will say that  $\xrightarrow{\mathcal{G}}$  is noetherian if the length  $r$  of any sequence  $h = h_0 \xrightarrow{\mathcal{G}} h_1 \xrightarrow{\mathcal{G}} \dots \xrightarrow{\mathcal{G}} h_r$  is bounded by an integer number  $m = m(h)$ . This is equivalent to say that if we continue rewriting terms in this way we always obtain, after a finite number of reductions, a polynomial whose support is contained in  $\mathbf{N}(J)$ .

We will write  $h \xrightarrow{\mathcal{G}}_* g$  if  $h \xrightarrow{\mathcal{G}} g$  and  $\text{Supp}(g) \subset \mathbf{N}(J)$ .

**Remark 5.4.** The relation  $\xrightarrow{\mathcal{G}}$  defined in 5.3 generalizes to a term-ordering free context, the concept of *involutive polynomial reduction* (and normal form) by Gerdt et al. (1998a).

In general, the relation  $\xrightarrow{\mathcal{G}}$  is not noetherian, namely there are sequences of reduction of infinite length.

**Example 5.5.** Let  $U := \{xz, yz, y^2\}$  a set of terms in  $k[x, y, z]$  with  $x < y < z$ . We find the following sets of multiplicative variables:

- $M_J(xz, U) = \{x, z\}$
- $M_J(y^2, U) = \{x, y\}$
- $M_J(yz, U) = \{x, y, z\}$

and check that  $U$  is complete.

Let  $\mathcal{G}$  the  $U$ -marked set  $\{f_{xz} = xz - xy, f_{yz} = yz - z^2, f_{y^2} = y^2\}$ . Then we have the infinite sequence of reductions:

$$xz^2 = xz *_{U} z \xrightarrow{G} xz^2 - f_{xz}z = xyz = yz *_{U} x \xrightarrow{\mathcal{G}} xyz - f_{yz}x = xz^2$$

However, the reduction  $\xrightarrow{\mathcal{G}}$  is always noetherian if  $\mathcal{G}$  is marked on a stably complete system. In order to prove this fact we will use the following special subset of the ideal  $(\mathcal{G})$ .

**Definition 5.6.** Let  $\mathcal{G}$  be a  $U$ -marked set on a complete system of terms  $U$  and let  $J := (U)$ . For each degree  $s$ , we will denote by  $\mathcal{G}^{(s)}$  the set of homogeneous polynomial

$$\mathcal{G}^{(s)} := \{f_{\alpha}x^{\eta} \mid x^{\alpha} *_{U} x^{\eta} \in (U)_s\}$$

marked on the terms of  $J_s$  in the natural way  $\text{Ht}(f_{\alpha}x^{\eta}) = x^{\alpha}x^{\eta}$ .

**Remark 5.7.** Observe that if  $\mathcal{G}$  is a  $U$ -marked set on a stably complete system of terms  $U$ , for every homogeneous polynomial  $g$  of degree  $s$ ,  $g \xrightarrow{\mathcal{G}} h$  implies that  $g - h = \sum_{i=1}^m c_i f_{\alpha_i} x^{\eta_i} \in \langle \mathcal{G}^{(s)} \rangle$ .

It is worth noticing as a direct consequence of Lemma 3.20 that if  $f_{\alpha}x^{\eta} \in \mathcal{G}$ , then every term in  $\text{Supp}(x^{\alpha}x^{\eta} - f_{\alpha}x^{\eta})$  either belongs to  $\mathbf{N}((U))$  or is of the type  $x^{\alpha'} *_{U} x^{\eta'}$  with  $x^{\eta'} <_{\text{Lex}} x^{\eta}$ .

**Lemma 5.8.** Let  $\mathcal{G}$  be a  $U$ -marked set on the stably complete system of terms  $U = \mathcal{F}(J)$ .

1. Every term in  $\text{Supp}(x^{\beta}x^{\epsilon} - f_{\beta}x^{\epsilon})$  either belongs to  $\mathbf{N}((U))$  or is of the type  $x^{\alpha} *_{U} x^{\eta}$  with  $x^{\eta} <_{\text{Lex}} x^{\epsilon}$ .
2. If  $f_{\beta} \in \mathcal{G}$ , then all the polynomials  $f_{\alpha_i}x^{\eta_i} \in \mathcal{G}^{(s)}$  used in the reduction of  $x^{\beta}x^{\epsilon}$  (except  $f_{\beta}x^{\epsilon}$  if it belongs to  $\mathcal{G}^{(s)}$ ) are such that  $x^{\epsilon} >_{\text{Lex}} x^{\eta_i}$ .
3. If  $g = \sum_{i=1}^m c_i f_{\alpha_i} x^{\eta_i}$ , with  $c_i \in k - \{0\}$  and  $f_{\alpha_i} x^{\eta_i} \in \mathcal{G}^{(s)}$  pairwise different, then  $g \neq 0$  and its support contains some term of the ideal  $J$ .

*Proof.* (1) is a direct consequence of Lemma 3.20.

(2) Assume that the statement holds for every term  $x^{\beta'}x^{\epsilon'}$ , with  $x^{\epsilon'} <_{\text{Lex}} x^{\epsilon}$ . At a first step of reduction of  $x^{\beta}x^{\epsilon}$  we use the polynomial  $f_{\alpha}x^{\eta}$  where  $x^{\beta}x^{\epsilon} = x^{\alpha} *_{U} x^{\eta}$ , so that  $x^{\eta} \leq_{\text{Lex}} x^{\epsilon}$ ; moreover every term in the support of the obtained polynomial either belongs to  $\mathbf{N}((U))$  or is of the type  $x^{\alpha'} *_{U} x^{\eta'}$  with  $x^{\eta'} <_{\text{Lex}} x^{\eta}$  (Remark 5.7). Then we conclude since we assumed the property holds for all those terms.

(3) We assume that the summands in  $g$  are ordered so that  $x^{\eta_1} \geq_{\text{Lex}} x^{\eta_i}$  for every  $i = 1, \dots, m$  and show that  $x^{\eta_1 + \alpha_1}$  belongs to the support of  $g$ .

The term  $x^{\alpha_1 + \eta_1}$  cannot appear as the head of  $f_{\alpha_i}x^{\eta_i}$  for some  $i \neq 1$  because the star decomposition of a term is unique. Moreover it cannot appear in  $f_{\alpha_i}x^{\eta_i} - x^{\alpha_i + \eta_i}$  since  $x^{\alpha_1 + \eta_1} = x^{\beta}x^{\eta_1}$ , with  $x^{\beta} \in \mathbf{N}(J)$  would imply  $x^{\eta_1} >_{\text{Lex}} x^{\eta_1}$  (see Lemma 3.20), against the assumption.  $\square$

**Theorem 5.9.** Let  $\mathcal{G}$  be a  $U$ -marked set on a stably complete system of terms  $U$  and let  $J$  be the ideal generated by  $U$ .

Then the reduction process  $\xrightarrow{\mathcal{G}}$  is noetherian and, for every integer  $s$ ,  $P_s = \langle \mathcal{G}^{(s)} \rangle \oplus \langle \mathbf{N}(J)_s \rangle$ .  
Indeed, for every  $h \in P_s$

$$h = f + g \text{ with } f \in \langle \mathcal{G}^{(s)} \rangle \text{ and } g \in \langle \mathbf{N}(J)_s \rangle \iff h \xrightarrow{\mathcal{G}}_* g \text{ and } f = h - g$$

*Proof.* Let  $\mathcal{G} = \{f_\alpha \mid x^\alpha \in U\}$ .

We observe that we have  $\langle \mathcal{G}^{(s)} \rangle \cap \langle \mathbf{N}(J)_s \rangle = \{0\}$  by Lemma 5.8.

In order to prove that the module  $\langle \mathcal{G}^{(s)} \rangle + \langle \mathbf{N}(J)_s \rangle$  coincides with  $P_s$  it is sufficient to show that it contains all the terms in  $J_s \setminus U$ , being obvious for those in  $U$ , for which  $x^\alpha = f_\alpha + g_\alpha$  (see (1)).

Let  $\tau$  be a term in  $J_s$ .

If  $\tau = x^\alpha *_{\mathcal{U}} x^\eta$ , we may assume of having already proved the statement for all the terms  $\tau' = x^{\alpha'} *_{\mathcal{U}} x^{\eta'}$  with  $x^{\eta'} <_{\text{Lex}} x^\eta$ .

We have  $x^\alpha x^\eta = f_\alpha x^\eta + (x^\alpha - f_\alpha)x^\eta$  where  $\text{Supp}(x^\alpha - f_\alpha) \subset \mathbf{N}(J)$ . If  $x^\beta$  is any term in this support, then either  $x^{\beta+\eta} \in \mathbf{N}(J)$  or  $x^{\beta+\eta} = x^{\alpha'} *_{\mathcal{U}} x^{\eta'}$  with  $x^{\eta'} <_{\text{Lex}} x^\eta$  by Lemma 5.8.

This allows us to conclude  $P_s = \langle \mathcal{G}^{(s)} \rangle + \langle \mathbf{N}(J)_s \rangle$ .

Finally, in order to prove that  $\xrightarrow{\mathcal{G}}$  is noetherian it is sufficient to observe that every step of reduction substitutes a term of  $J$  of the type  $x^\alpha *_{\mathcal{U}} x^\eta$  with  $x^\alpha x^\eta - f_\alpha x^\eta$ . Indeed, by remark 5.7, each  $\tau \in \text{Supp}(x^\alpha x^\eta - f_\alpha x^\eta) \setminus \mathbf{N}(\langle U \rangle)$  has the form  $x^{\alpha'} *_{\mathcal{U}} x^{\eta'}$ ,  $x^{\eta'} <_{\text{Lex}} x^\eta$  and this permits to conclude by descent on Lex.  $\square$

**Remark 5.10.** With the previous notation, if  $\mathcal{G}$  is a  $U$ -marked set, but it is not a  $U$ -marked basis, then  $\exists f_\alpha, f_\beta \in \mathcal{G}$ , such that

$$S(f_\alpha, f_\beta) = x^\eta f_\alpha - x^\gamma f_\beta \xrightarrow{\mathcal{G}}_* h \neq 0.$$

Considering  $x^\eta f_\alpha$ , we point out that one can perform two different reduction processes on it, such that both of them terminate. They lead to different outcomes:

1. the reduction  $x^\eta f_\alpha \xrightarrow{f_\alpha} 0$ , w.r.t. the polynomial  $f_\alpha$ , which is clearly different from our reduction procedure;
2. the reduction process described above

$$x^\eta f_\alpha \xrightarrow{\mathcal{G}} x^\eta f_\alpha - x^\gamma f_\beta \xrightarrow{\mathcal{G}}_* h \neq 0.$$

On the other hand, if  $\mathcal{G}$  is a  $U$ -marked basis,  $\forall f \in P$ ,  $\exists! h \in \langle \mathbf{N}(J) \rangle$ , such that  $f - h \in \langle \mathcal{G} \rangle$ . This implies that any reduction process, applied to  $f$ , either gives  $h$  as outcome or it does not terminate.

As a straightforward consequence of the previous result, we obtain the following

**Corollary 5.11.** If  $U$  is a stably complete system and  $\mathcal{G}$  is a  $U$ -marked set, the following are equivalent:

- $\mathcal{G}$  is a  $U$ -marked basis

- for every  $s$ :  $\langle \mathcal{G}^{(s)} \rangle = (\mathcal{G})_s$ ,
- for every  $h \in (\mathcal{G})$ :  $h \xrightarrow{\mathcal{G}}_* 0$
- if  $h - g \in (\mathcal{G})$  and  $\text{Supp}(g) \subset \mathbf{N}(J)$ , then  $h \xrightarrow{\mathcal{G}}_* g$ .

**Remark 5.12.** We point out that if  $\mathcal{G}$  is a  $U$ -marked set, but not a  $U$ -marked basis, then there are polynomials in the ideal  $(\mathcal{G})$  whose support is contained in  $\mathbf{N}((U))$ . Hence, we do not have a "normal form" of a polynomial  $h$  modulo  $(\mathcal{G})$ , since, in general, there are several polynomials  $g'$  such that  $\text{Supp}(g') \subset \mathbf{N}(J)$  and  $h - g' \in (\mathcal{G})$ . However, the reduction process  $h \xrightarrow{\mathcal{G}}_* g$  with respect to a  $\mathcal{F}(J)$ -marked set  $\mathcal{G}$  gives a unique reduced polynomial  $g$  for every polynomial  $h$ .

Using the reduction process, we can now answer Problem 1.1 and characterize the ideals  $I$  that belong to the marked family  $\mathcal{Mf}(J)$ .

**Theorem 5.13.** *Let  $\mathcal{G}$  be a  $\mathcal{F}(J)$ -marked set. Then:*

$$(\mathcal{G}) \in \mathcal{Mf}(J) \iff \forall f_\beta \in \mathcal{G}, \forall x_i > \min(x^\beta) : f_\beta x_i \xrightarrow{\mathcal{G}}_* 0$$

*Proof.* Since " $\implies$ " is a straightforward consequence of Corollary 5.11, we only prove " $\impliedby$ ". More precisely, we prove that  $(\mathcal{G})_m = \langle \mathcal{G}^{(m)} \rangle$ , showing that if  $f_\beta \in \mathcal{G}$  and  $\deg(x^{\beta+\epsilon}) = m$ , then  $f_\beta x^\epsilon$  is either an element of  $\langle \mathcal{G}^{(m)} \rangle$  itself or a linear combination of polynomials in  $\langle \mathcal{G}^{(m)} \rangle$ .

If this were not true, we can choose an element  $f_\beta x^\epsilon \notin \langle \mathcal{G}^{(m)} \rangle$  with  $x^\epsilon$  minimal with respect to  $<_{Lex}$ . As  $f_\beta x^\epsilon \notin \langle \mathcal{G}^{(m)} \rangle$ , at least one variable  $x_i$  appearing in  $x^\epsilon$  with nonzero exponent is nonmultiplicative for  $x^\beta$ . Let  $x^\epsilon = x_i x^{\epsilon'}$ . By hypothesis  $f_\beta x_i \xrightarrow{\mathcal{G}}_* 0$ , so that  $f_\beta x_i$  is a linear combination  $\sum c_i f_{\alpha_i} x^{\eta_i}$  of polynomials in  $\langle \mathcal{G}^{(|\beta|+1)} \rangle$ . By Lemma 5.8 we have  $x^{\eta_i} <_{Lex} x_i$ . Now  $f_\beta x^\epsilon = (f_\beta x_i) x^{\epsilon'} = (\sum c_i f_{\alpha_i} x^{\eta_i}) x^{\epsilon'} = \sum c_i f_{\alpha_i} x^{\eta_i + \epsilon'}$ , where  $x^{\eta_i + \epsilon'} <_{Lex} x_i x^{\epsilon'} = x^\epsilon$ . Now we get a contradiction, since  $f_{\alpha_i} x^{\eta_i + \epsilon'} \in \langle \mathcal{G}^{(m)} \rangle$  by the minimality of  $x^\epsilon$ .  $\square$

We point out that the above Theorem 5.13 is a term-ordering free generalization of the concept of *local involutivity* in the polynomial case, defined by Gerdt et al. (1998a).

**Example 5.14.** Let  $J$  be the monomial ideal  $(x^3, xy, y^3)$  in  $k[x, y]$  with  $x < y$ . Its star set is  $\mathcal{F}(J) = \{x^3, xy, xy^2, y^3\}$ . Using the criterion given in Theorem 5.13, we can easily check that the  $\mathcal{F}(J)$ -marked set  $\mathcal{G} := \{\mathbf{f}_1 := \mathbf{x}^3, \mathbf{f}_2 := \mathbf{xy} - x^2 - y^2, \mathbf{f}_3 := \mathbf{xy}^2, \mathbf{f}_4 := \mathbf{y}^3\}$  (in bold the head terms) is a  $\mathcal{F}(J)$ -market basis:

- $yf_1 = xf_1 + x^2f_2 + xf_3 \xrightarrow{\mathcal{G}}_* 0$ ,
- $yf_2 = f_1 - xf_2 - f_4 \xrightarrow{\mathcal{G}}_* 0$
- $yf_3 = xf_4 \xrightarrow{\mathcal{G}}_* 0$ .

This is a simple example of a marked basis which is not a Gröbner basis. In fact, it is obvious that  $\text{Ht}(f_2) = xy$  cannot be the leading term of  $f_2$  with respect to any term-ordering and, more generally, that  $J$  cannot be the initial ideal of the ideal  $(\mathcal{G})$ , even though  $(\mathcal{G}) \oplus \mathbf{N}(J) = k[x, y]$ .

A wider family of ideals of this type are presented in (Cioffi et al., 2011, Example 3.18 and Appendix).



**Remark 5.15.** Observe that we can perform the first step of reduction of the polynomial  $f_\beta x_i$  rewriting the head  $x^\beta x_i$  throughout  $f_\alpha x^\eta$  with  $x^\beta x_i = x^\alpha x^\eta \in C_J(\{x^\alpha\})$ . In this way we obtain  $f_\beta x_i \xrightarrow{\mathcal{G}} f_\beta x_i - f_\alpha x^\eta$ , namely the  $S$ -polynomial

$$S(f_\beta, f_\alpha) := \frac{\text{lcm}(x^\beta, x^\alpha)}{x^\beta} f_\beta - \frac{\text{lcm}(x^\beta, x^\alpha)}{x^\alpha} f_\alpha.$$

Therefore we could reformulate the criterion given by Theorem 5.13 as follows:

$$(\mathcal{G}) \in \mathcal{Mf}(J) \iff \forall f_\alpha, f_\beta \in \mathcal{G} : S(f_\alpha, f_\beta) \xrightarrow{\mathcal{G}}_* 0.$$

However Theorem 5.13 shows that it is sufficient to check a special subset of the  $S$ -polynomials. If  $J$  is quasi stable and so  $\mathcal{F}(J)$  is a finite set, this subset corresponds to the basis for the first syzygies of the terms in  $\mathcal{F}(J)$ , namely the one considered in Seiler (2009a). It is obvious that the maximal degree of these special  $S$ -polynomials cannot exceed  $1 + \max\{\deg(x^\alpha) \mid x^\alpha \in \mathcal{F}(J)\}$ .

Indeed, if  $J$  is quasi stable,  $\text{reg}(J) = \max\{\deg(\tau), \tau \in \mathcal{F}(J)\}$  as proved in Janet (1920); Seiler (2009b); Hashemi et al. (2012).

**Remark 5.16.** If  $J$  is a quasi stable monomial ideal and  $\mathcal{G}$  is a  $\mathcal{F}(J)$ -marked set, then there are only a finite number of reduction to perform in order to decide if a  $\mathcal{F}(J)$ -marked set  $\mathcal{G}$  is a basis. We will use this algorithm in order to endow the marked family  $\mathcal{Mf}(J)$  of a structure of affine scheme

If the considered monomial ideal is not quasi stable, then the (unique) stably complete generating set is *infinite*. Actually this does not necessarily exclude we can exploit it even from a computational point of view.

## 6. Marked families, schemes and functors

In this section we follow Bertone et al. (2013); Cioffi et al. (2011) and show how it is possible to associate a scheme to each marked family  $\mathcal{Mf}(J)$ . Due to the naturality of this construction, we can mimic that of Lella et al. (2014), and define marked families as functors.

Our results are very similar, but more general, than those of Bertone et al. (2013); Cioffi et al. (2011); Lella et al. (2014); in fact in those papers the ideal  $J$  is assumed to be strongly stable. Recall that a monomial ideal  $J$  is called *strongly stable* if for every term  $\tau \in J$  and pair of variables  $x_i, x_j$  such that  $x_i | \tau$  and  $x_i < x_j$ , then also  $\frac{\tau x_j}{x_i}$  belongs to  $J$ .

Obviously, a strongly stable ideal is also stable, so that  $\mathcal{F}(J) = \mathbf{G}(J)$ . If  $J$  is strongly stable, the notions of  $\mathbf{G}(J)$ -marked sets,  $\mathbf{G}(J)$ -marked bases and  $\mathbf{G}(J)$ -marked family introduced in the previous sections exactly correspond to those of  $J$ -marked sets,  $J$ -marked bases,  $J$ -marked family considered in Bertone et al. (2013); Cioffi et al. (2011) and the reduction procedure  $\xrightarrow{\mathcal{G}}$  with respect to a  $\mathbf{G}(J)$ -marked set  $\mathcal{G}$  introduced in definition 5.3 coincides with the one used in those papers.

Moreover, for such an ideal  $J$ , the scheme structure that we will define is the same obtained in Bertone et al. (2013); Cioffi et al. (2011) and used in Bertone-Lella-Roggero (2013); Lella et al. (2014) for a local study of Hilbert schemes. Indeed, for every monomial ideal  $J$ , if  $I \in \mathcal{Mf}(J)$ , then the ideals  $I$  and  $J$  share the same Hilbert polynomial (and also the same Hilbert function), so that they correspond to points in the same Hilbert scheme.

The scheme we associate to  $\mathcal{Mf}(J)$  only depends on the monomial ideal  $J$ , but the way we use in order to define it needs a set of generators  $U$  complete, finite and such that for every  $U$ -marked set  $\mathcal{G}$  the reduction procedure  $\xrightarrow{\mathcal{G}}$  is noetherian.

**Then, in the following  $J$  will be a quasi stable monomial ideal and  $U$  will be its finite star-set  $\mathcal{F}(J)$ , that is its Pommaret basis  $\mathcal{H}(J)$ .**

Let  $\{x_1^\alpha, \dots, x_s^\alpha\}$  be the monomials in  $U$  and consider the polynomial ring  $B := A[C]$ , where  $C$  is a compact notation for the set of variables  $C_{i,\beta}$   $i = 1, \dots, s$  and  $x^\beta \in \mathbf{N}(J)_{|\alpha_i|}$ . We also define the  $U$ -marked set in  $B[x_1, \dots, x_n]$

$$\mathcal{G} := \{f_{\alpha_i} := x^{\alpha_i} + \sum C_{i,\beta} x^\beta \mid x^\beta \in \mathbf{N}(J)_{|\alpha_i|}, \text{Ht}(f_{\alpha_i}) = x^{\alpha_i}\}.$$

Clearly, every  $U$ -marked set can be obtained specializing  $\mathcal{G}$ , namely as  $\phi(\mathcal{G})$  for a suitable morphism of  $A$ -algebras  $\phi : A[C] \rightarrow A$ . Moreover, by the uniqueness of the  $U$ -marked basis generating each ideal in  $\mathcal{Mf}(J)$ , we can assert that for every ideal  $I \in \mathcal{Mf}(J)$  there exists a unique specialization  $\phi$  such that  $(\phi(\mathcal{G})) = I$ .

We use Theorem 5.13 in order to construct a set of polynomials  $\mathcal{R}$  that will define the scheme we associate to  $U$ . If  $g$  is a polynomial in  $B[x_1, \dots, x_n]$ , we denote with  $\text{coeff}_x(g)$  the set of coefficients of  $g$  with respect to the variables  $x_1, \dots, x_n$ ; hence  $\text{coeff}_x(g) \subset B = A[C]$  is a set of polynomials in the variables  $C$ . For every  $x^{\alpha_i} \in U$  and  $x_j > \min(x^{\alpha_i})$ , let  $g_{\alpha_i, j} \in B[x_1, \dots, x_n]$  be such that  $f_{\alpha_i, x_j} \xrightarrow{\mathcal{G}}_* g_{\alpha_i, j}$ .

**Definition 6.1.** Let  $U$  be a stably complete system in  $\mathcal{T}$ ,  $A$  be any ring, and  $\mathcal{R}$  be the union of  $\text{coeff}_x(g_{\alpha_i, j})$  for every  $x^{\alpha_i} \in U$  and  $x_j > \min(x^{\alpha_i})$ .

We will call  $U$ -marked scheme over the ring  $A$ , and denote with  $\mathbf{Mf}_U(A)$  the affine scheme  $\text{Spec}(A[C]/(\mathcal{R}))$ .

**Remark 6.2.** Every  $U$ -marked set in  $A[x_1, \dots, x_n]$  is a  $U$ -marked basis if and only if the coefficients of the terms in the tails satisfy the conditions given by  $\mathcal{R}$ .

In particular, if  $A = k$  is an algebraically closed field, then the closed points of  $\mathbf{Mf}_U(A)$  correspond to the ideals in the marked family  $\mathcal{Mf}(J)$  where  $J$  is the ideal in  $k[x_1, \dots, x_n]$  generated by  $U$ .

**Remark 6.3.** The above construction of  $\mathcal{R}$  is in fact *independent* from the fixed commutative ring  $A$ , in the sense that it is preserved by extension of scalars. We can first choose  $\mathbb{Z}$  as the coefficient ring and then apply the standard map  $\mathbb{Z} \rightarrow A$ .

More formally, for every stably complete set of terms  $U$  we can define a functor between the category of (commutative) rings to the category of sets

$$\mathbf{Mf}_U : \underline{\text{CRng}} \rightarrow \underline{\text{Set}}$$

that associates to any ring  $A$  the set  $\mathbf{Mf}_U(A) := \mathcal{Mf}(UA[x_1, \dots, x_n])$  and to any morphism  $\phi : A \rightarrow B$  the map

$$\begin{aligned} \mathbf{Mf}_U(\phi) : \mathbf{Mf}_U(A) &\longrightarrow \mathbf{Mf}_U(B) \\ \mathbf{I} &\longmapsto \mathbf{I} \otimes_A B. \end{aligned}$$

Moreover, again following Lella et al. (2014), it is possible to prove that  $\mathbf{Mf}_U$  is a representable functor represented by the scheme  $\mathbf{Mf}_U(\mathbb{Z}) = \text{Spec}(\mathbb{Z}[C]/(\mathcal{R}))$ .

## 7. Historical notes.

Through the trivial interpretation of derivatives

$$\frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{\alpha_1 + \alpha_2 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}},$$

in terms of the corresponding term  $\tau = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \in \mathcal{T}$ , Riquier (1893, 1899, 1910) was able to algebraically transform the problem of solving differential partial equations in terms of ideal membership.

After introducing the concept (but not the notion) of S-polynomials he proved that if the normal form (in terms of Gauss-Buchberger reduction) of each S-polynomial among the elements of the basis  $\mathcal{G}$  goes to zero then

- the given basis  $\mathcal{G}$  generates the related ideal and the related problem could be solvable;
- a solution of the PDE is completely determined (and computed) as series in terms of initial conditions which can be described and formulated in terms of a Hironaka-Galligo-like decomposition (Hironaka (1977); Galligo (1979), but more general) of the related *escalier*  $N$ ;

if the normal form computation produces conflicts among the data then the PED has no solution (for instance the problem  $\frac{\partial u}{\partial y} = f$ ,  $\frac{\partial u}{\partial x} = g$  has no solution unless  $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$ ; if no conflict arose and not all normal forms are 0, then, exactly as in Buchberger Algorithm, the non-zero normal forms are included in the basis and the procedure is repeated.

For instance, the system (Riquier, 1910, pp.188-9)

$$\frac{\partial^3 u}{\partial y^3} = A(x, y, z), \quad \frac{\partial^2 u}{\partial x \partial z} = B(x, y, z), \quad \frac{\partial^3 u}{\partial x^2 \partial y} = C(x, y, z),$$

must satisfy the integrability conditions

$$\frac{\partial^2 A}{\partial x \partial z} = \frac{\partial^3 B}{\partial y^3}, \quad \frac{\partial^2 A}{\partial x^2} = \frac{\partial^2 C}{\partial y^2}, \quad \frac{\partial^2 B}{\partial x \partial y} = \frac{\partial C}{\partial z},$$

in which case the initial conditions have the shape

$$\left\{ \begin{array}{l} u = \phi_0(z) \\ \frac{\partial u}{\partial y} = \phi_1(z) \\ \frac{\partial^2 u}{\partial y^2} = \phi_2(z) \end{array} \right\} x - x_0 = y - y_0 = 0,$$

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial x} = \alpha_0 \\ \frac{\partial^2 u}{\partial x \partial y} = \alpha_1 \\ \frac{\partial^3 u}{\partial x \partial y^2} = \alpha_2 \end{array} \right\} x - x_0 = y - y_0 = z - z_0 = 0,$$

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial x^2} = \psi(x) \end{array} \right\} y - y_0 = z - z_0 = 0.$$

Their 'solution' cannot be considered complete; a fully effective theory was achieved by Thomas (1937, 1962) (see Gerdt (2008)) and, more recently, by Boulier et al. (1995) in the

framework of Ritt's differential algebra (Ritt (1932, 1950)). For a general overview of this subject see Hubert (2003).

In his theory, Riquier was assuming that the set  $\mathcal{T}$  of the terms was ordered by a term-ordering; he was mainly using (Riquier, 1910, p.67) the deglex ordering induced by  $x_1 > x_2 > \dots > x_n$ , but he gave a large class of term-orderings to which his theory was applicable; actually (but he never stated that) his characterization is the classical one of all term-orderings (Erdős (1956); Robbiano (1985)). He was however forced to restrict himself to degree-compatible term-orderings in order to be granted convergency.

In his gaussian reduction, Riquier, as Buchberger, considered as head term of each "marked" polynomial its maximal term.

In his considerations on generic initial ideal, Delassus (1896), followed by Robinson (1913) used (deg)-rev-lex induced by  $x_1 < x_2 < \dots < x_n$  and the minimal term as head term of each "marked" polynomial.

In order to "harmonize" the two notations, Janet (1920, 1931) applied deglex induced by  $x_1 < x_2 < \dots < x_n$  and chose the maximal term as head term, but expressed all terms as (!)  $x_n^{\alpha_n} x_{n-1}^{\alpha_{n-1}} \dots x_1^{\alpha_1}$ , while in Janet (1924) went back to use deglex induced by  $x_1 > x_2 > \dots > x_n$ .

What is worst, in Janet (1927), Janet not only applied deglex induced by  $x_1 < x_2, \dots < x_n$  but presented all results within his notation; so, in his presentation of Delassus's result, the head term is again, *à la* Buchberger, the maximal one.

This is not helpful, as regards his reformulation of the previous results on generic initial ideals and stability; thus while, for Robinson (1913, 1917) and Gunther (1913a,b) a generic initial ideal  $\epsilon(I)$  satisfies

$$\mu \in \epsilon(I), x_h \mid \mu, i < h \implies x_i \frac{\mu}{x_h} \in \epsilon(I),$$

according to Janet (1927) the formula is

$$\mu \in \epsilon(I), x_h \mid \mu, i > h \implies x_i \frac{\mu}{x_h} \in \epsilon(I).$$

Under the suggestion of Hadamard (Pommaret (1978)), Janet dedicated his doctoral thesis (Janet (1920)) to a reformulation of Riquier's results in terms of Hilbert's results (Hilbert (1890)).

In particular, given a finite set of monomials  $U$ , he associates to each term  $\tau \in U$ , as functions of its relation with the other elements of  $U$ , a set of variables which he labels *multiplicative* (Definition 3.2) and a subset of terms in  $(U)$  which he called his *class* and which we labelled as its *cone* and considered  $U$  *complete* (Definition 3.10) when the disjoint cones of  $U$  cover  $(U)$ .

He then gave (Janet, 1920, p.80) a *procédé régulier pour obtenir un système complet base d'un module donné* which *ne pourra se prolonger indéfiniment*; it simply consisted to enlarge  $U$  with the elements  $xt \notin \cup_{\tau \in U} C_J(\{\tau\}), t \in U, x$  nonmultiplicative for  $t$ .

Janet can now formulate (Janet, 1931, p.75) Riquier's procedure; we can assume to have a finite basis  $\mathcal{G} \subset P$ ; denoting  $U = \{\mathbb{T}(f) : f \in \mathcal{G}\}$ ,

- we enlarge  $U$  in order to made it complete and at the same time
- we similarly enlarge  $\mathcal{G}$ , adding  $xg$  to  $\mathcal{G}$  when we add  $x\mathbb{T}(g) \notin \cup_{\tau \in U} C_J(\{\tau\})$ ;
- we then perform Riquier's test, which, for a complete system, consists in computing the normal form of each element  $xg, g \in \mathcal{G}, x$  nonmultiplicative for  $\mathbb{T}(g)$ .

Janet (Janet, 1920, p.112-3) further remarks (in connection with Hilbert's syzygy theory) that the reduction-to-zero of all such elements give a basis  $S$  of the syzygy module of  $\mathcal{G}$ . Actually he repeatedly applied the same procedure to  $S$ , thus computing a resolution of  $\mathcal{G}$  and anticipating Schreyer's Algorithm (Schreyer (1991)).

Next, Janet (1924) moved his interest in extending the study to the homogeneous case, adapting his approach on one side to the solution of partial differential equation given by Cartan (1901); Cartan (1904, 1920) via his characters (the values  $\sigma$  defined below) and test (equation (2) below) and on the other side to the introduction by Delassus (1896) of the concept of generic initial ideal and the precise description of it given by Robinson (1913, 1917) and Gunther (1913a,b); he thus discussed the notion of *système de forms (de même ordre) en involution*. The notion, as he explains, is independent from the variable chosen and allows to assign to the system a series of values  $\sigma_i^{(p)}$ ,  $1 \leq i \leq n$ ,  $p \in \mathbb{N}$  which (Janet, 1931, p.87) *sont évidemment invariables lorsqu'on fait un changement linéaire et homogène des variables indépendantes* which, under the assumption of generality, allow to describe the structure of the *generic escalier* of the considered ideal.

The procedure, given a finite set  $\mathcal{G}$  of forms, repeatedly produces *à la Macaulay* a linear basis  $\mathcal{B}_p$  of  $(\mathcal{G})_p$  by performing linear algebra on the set  $\{x_i g : g \in \mathcal{B}_{p-1}, 1 \leq i \leq n\}$ ; termination is granted when the formula (2) below is satisfied.

Given a homogeneous ideal  $I \subset k[x_1, x_2, \dots, x_n]$ , where the variables are assumed to be generic, so that  $N(I)$  is stable, Janet defined (Janet, 1924, pp.30-2),(Janet, 1927, p.30),(Janet, 1931, pp.90-1),(Pommaret, 1978, p.93, p.99) multiplicative variables according 3.7, introduced values  $\sigma_i^{(p)}(I)$  (or  $\sigma_i^{(p)}$  for short when no confusion is possible) for every  $1 \leq i \leq n$ , and  $p \in \mathbb{N}$ , which can be described as

$$\sigma_i^{(p)} := \#\{\tau \in N(I), \deg(\tau) = p, \min(\tau) = i\}$$

and, fixing a value  $p$  and denoting  $\sigma_i := \sigma_i^{(p)}$ , and  $\sigma'_i := \sigma_i^{(p+1)}$  proved

**Proposition 7.1** (Janet). *It holds,*

1.  $\sigma'_1 + \sigma'_2 + \dots + \sigma'_n \leq \sigma_1 + 2\sigma_2 + \dots + n\sigma_n$ ;
2.  $\sum_{i=1}^n \sigma'_i = \sum_{i=1}^n i\sigma_i \implies \sigma'_j = \sum_{i=j}^n \sigma_i$  for each  $j$ .
3.  $\sum_{i=1}^n \sigma'_i = \sum_{i=1}^n i\sigma_i \implies \sum_{i=1}^n \sigma_i^{(P+1)} = \sum_{i=1}^n i\sigma_i^{(P)}$  for each  $P > p$ .

He can then state

**Definition 7.2.** (Janet, 1931, pp.90-1) A finite set  $E \subset \mathcal{P}$  of forms of degree at most  $p$  generating the ideal  $I \subset \mathcal{P}$ , is said to be *involution*<sup>4</sup> if, with the present notation, it satisfies the formula

$$\sum_{i=1}^n \sigma_i^{(p+1)} = \sum_{i=1}^n i\sigma_i^{(p)}. \quad (2)$$

□

Thus, once the iterated Macaulay-like procedure satisfies (2) at degree  $\bar{p}$  then it successfully terminates and the *finite* bases produced by it is involutive; Janet is therefore able to present the ideal  $\{\tau \in T(I), \deg(\tau) \geq \bar{p}\}$  by explicitly producing (Janet (1931)) the decomposition

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<sup>4</sup>en involution.

$$\{\tau \in \mathbb{T}(I), \deg(\tau) \geq \bar{p}\} = \sqcup_{\tau \in U} C_J(\{\tau\})$$

where  $U$  is the stably complete set  $U = \{\tau \in \mathbb{T}(I), \deg(\tau) \geq \bar{p}\}$  and to express its Hilbert polynomial as

$${}^h H_I(t) = \sum_{h=1}^{n-1} \binom{t-p+h-1}{h-1} \sigma_h^{(p)}(I).$$

The degree  $\bar{p}$  is greater or equal than Castelnuovo-Mumford regularity, and the equality holds if we consider the minimal such  $\bar{p}$ . This was first noted by Malgrange (2003).

In our context, the characterization of  $\sigma_i^{(p)}$  and definition 7.2 lead to the following

**Proposition 7.3.** *With the previous notation, if  $J$  is a quasi stable monomial ideal, then*

$$\sum_{i=1}^n \sigma_i^{(p+1)}(J) = \sum_{i=1}^n i \sigma_i^{(p)}(J).$$

*The same equality holds if  $I$  is a homogeneous ideal generated by a  $J$ -marked basis  $\mathcal{G}$  with  $J$  quasi stable.*

*Therefore  $\mathcal{G}$  is an involutive basis.*

*Proof.* For the first statement we observe that each  $x^\beta \in \mathbb{N}(J)$  is such that  $\frac{x^\beta}{\min(x^\beta)} \in \mathbb{N}(J)$ .

Then, we can stratify  $\mathbb{N}(J)_{p+1}$  w.r.t. the terms  $x^\gamma := \frac{x^\beta}{\min(x^\beta)}$ : for each  $x^\gamma \in \mathbb{N}(J_p)$  there are  $i$  terms  $x^\beta = x_j x^\gamma, j \leq i$  s.t.  $x_i = \min(x^\gamma)$ .

If  $I$  is the homogeneous ideal generated by a  $J$ -marked basis  $\mathcal{G}$ , then by Corollary 5.11 we get  $\langle \mathcal{G}^{(t)} \rangle = (\mathcal{G})_t$  for each  $t$ .

On the other hand, by theorem 5.9,  $P_t = \langle \mathcal{G}^{(t)} \rangle \oplus \langle \mathbb{N}(J)_t \rangle$ , then  $\mathbb{N}(J)$  is a basis for  $P/I$ . Since the equality holds for  $\mathbb{N}(J)$ , it holds for the basis of  $P/I$  as well.  $\square$

Note that if  $I$  is generated by a  $J$ -marked set  $\mathcal{G}$  which is not a  $J$ -marked basis, only the inequality  $\sum_{i=1}^n \sigma_i^{(p+1)} \leq \sum_{i=1}^n i \sigma_i^{(p)}$  holds true, since we have the inclusion  $\langle \mathcal{G}^{(t)} \rangle \subseteq (\mathcal{G})_t$ .

The iterated Macaulay-like procedure gives also a fine decomposition of  $\mathbb{N}(I)_{\geq \bar{p}-1}$  as follows:

- he partitions the set  $\mathbb{N}(I)_{\bar{p}-1}$  as  $\mathbb{N}_{\bar{p}-1} = \sqcup_{i=0}^{n-1} N_i$  associating to
  - $N_0$  the monomials  $\tau \in \mathbb{N}_{\bar{p}-1}(I)$  for which  $x_1 \tau \in \mathbb{T}(I)$ ;
  - while each of the  $\sigma_1$  elements  $\tau = \frac{v}{x_1} \in \mathbb{N}(I)_{\bar{p}-1} \setminus N_0$ ,  $\text{class}(v) = 1$ , is inserted in  $N_i$  if it is one of the  $\sigma_i$  elements which can be expressed as  $\tau = \frac{v_i}{x_i}$ ,  $\text{class}(v_i) = i$  but is not one of the  $\sigma_{i+1}$  elements which can be expressed as  $\tau = \frac{v_{i+1}}{x_{i+1}}$ ,  $\text{class}(v_{i+1}) = i + 1$ .
- he then associate to each  $\tau \in N_i$   $M_J(\tau) = \{x_j, 1 \leq j \leq i\}$  as multiplicative variables and  $C_J(\{\tau\}) := \{\tau \omega, \omega = x_1^{\omega_1} \cdots x_i^{\omega_i}\}$  as its cone
- and states

$$\{\tau \in \mathbb{N}(I), \deg(\tau) \geq \bar{p} - 1\} = \sqcup_{i=0}^{n-1} \sqcup_{\tau \in N_i} C_J(\{\tau\}).$$

Riquier's and Janet's results were introduced to the Computational Algebra commutative at the MEGA-90 Symposium in 1990 by a survey by Pommaret et al. (1990) of his theory and, two years later, through a paper by Schwarz (1992) (which had previously discussed Riquier–Janet in Schwarz (1984)) where he remarked:

The concept of a Gröbner base and algorithmic methods for constructing it for a given system of multivariate polynomials has been established as an extremely important tool in commutative algebra. It seems to be less well known that similar ideas have been applied for investigating partial differential equations (pde's) around the turn of the century in the pioneering work of the French mathematicians Riquier and Janet. [...] [T]heir theory [...] is basically a critical-pair/completion procedure. All basic concepts like a term-ordering, reductions and formation of critical pairs are already there.

This prompted V. Gerdt to suggest his coworkers Zharkov and Blinkov to investigate whether the results by Janet and Pommaret were translatable from pde's to polynomial rings in order to produce an effective alternative approach to Buchberger's Algorithm; the conclusion of this investigation (Zarkov et al. (1996); Zarkov (1996)) was successful — the proposed algorithm was able to give a solution with a speed-up of 20 w.r.t. degrevlex Buchberger's algorithm on classical test-suites and caused sensation in the community.

An algorithm based on Janet's notion (Janet (1920)) of completeness is reported in (Gerdt et al. (1998a,b); Gerdt (2005))

Involutiveness is the argument of the *Habilitation* thesis (2002) of Seiler (2002, 2009a,b); an improved version has recently appeared as Seiler (2010). Finiteness is a required condition for the notion of Pommaret bases (Hashemi et al. (2012)).

Janet-Riquier theory influenced also the work of Reid (1991) ; one of the first implementations is due to Topunov (1989); Apel (1995, 1991) introduced his own version of involutive bases which is markedly different from the approach of Gerdt and his group. An adaptation of Riquier–Janet Theory as a tool for computing Gröbner bases was also proposed by Wu (1991).

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