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Abstract

Given a real algebraic surface S in $\mathbb{RP}^3$, we propose a procedure to determine the topology of S and to compute non-trivial topological invariants for the pair $(\mathbb{RP}^3, S)$ under the hypothesis that the real singularities of S are isolated. In particular, starting from an implicit equation of the surface, we compute the number of connected components of S, their Euler characteristics and the labelled 2-adjacency graph of the surface.

Keywords: Real algebraic surfaces, topology computation, adjacency graph

Dedicated to the memory of Domenico Luminati

1. Introduction

Given a real algebraic surface S in $\mathbb{RP}^3$ defined by a square-free polynomial equation, the problem of recognizing the topology of the surface can be addressed at two different levels: either considering S only as an abstract topological space, or taking into account also its embedding in $\mathbb{RP}^3$ and looking at the topology of the pair $(\mathbb{RP}^3, S)$.

We say that two surfaces S, S' are ambient-homeomorphic if there exists a homeomorphism $\varphi : \mathbb{RP}^3 \to \mathbb{RP}^3$ such that $\varphi(S) = S'$; in this case we also say that the pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are homeomorphic. At present there is no classification of the pairs $(\mathbb{RP}^3, S)$ up to homeomorphism even in the non-singular case and deciding whether two pairs are homeomorphic is a very hard problem, even for simple classes of surfaces such as tori. Hence a useful contribution in this direction is to be able to compute topological invariants of the pair $(\mathbb{RP}^3, S)$.

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A classical approach to an algorithmic determination of the topology of a surface is via Cylindrical Algebraic Decomposition ([7], see also [2]), which provides a cellular decomposition of the pair $(\mathbb{RP}^3, S)$.

Other algorithms for studying the topology of affine surfaces of degree d in $\mathbb{R}^3$ with arbitrary singularities have been presented in [3], [1], [9] and [6]. The first three compute triangulations of the surface using $\mathcal{O}(d^7)$ cells, while the latter one computes a curvilinear wireframe model. [3] handles surfaces not in general position, while the other three put conditions on the coordinate frame. All of these methods perform much better than CAD, e.g. [1] and [9] show that their method requires $\mathcal{O}(d^7)$ points to compute the triangulation while CAD would require $\mathcal{O}(d^{13})$.

It is not easy to decide whether two pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are homeomorphic or not, even after computing a combinatorial description of each pair. This is why our aim is a different one: we want to find a small discrete set of data that are algorithmically computable and sufficient to determine the topology of the surface, that are non-trivial topological invariants for the pair $(\mathbb{RP}^3, S)$ and, most important, easy to compare and hence easy to be used as a test to exclude topological equivalence.

In [13], developing the ideas in [15], [14] and [12], this goal is achieved in the case of a non-singular surface; the authors describe a method to compute not only the number of its connected components and the Euler characteristic of each of them but also the "labelled adjacency graph" of the surface. This graph, though not sufficient to determine the pair $(\mathbb{RP}^3, S)$, gives information both on the mutual disposition of the connected components and on their contractibility.

In this paper we address the same questions when the surface contains at most isolated singularities.

In our approach the basic topological information is that in a small 3-dimensional ball D, centered at an isolated singular point, $S \cap D$ is homeomorphic to the cone over the curve C obtained as the intersection of S with the boundary of D (see [17]). Then, up to homeomorphism, the portion of S inside the ball can be seen as the space obtained taking the union of as many 2-dimensional disks as the connected components of C, choosing a point in each of these disks and collapsing these points to a single point. In this way we see the isolated singularity as the result of two successive operations: first the glueing of a 2-cell (i.e. a subset homeomorphic to a closed 2-dimensional disk) along each connected component of C and then the collapsing of a set containing a point in each of the attached 2-cells.

Applying this procedure to all of the singularities, we obtain a compact topological surface T without boundary such that S, except its isolated points, is homeomorphic to the topological quotient $T/\mathcal{R}(T)$ where $\mathcal{R}(T)$ is the equivalence relation that collapses suitable finite families of points of T.

In order to determine the topology of T, it is sufficient to compute the number of its connected components and to find a topological model for each of them. By the topological classification theorem any compact connected orientable topological surface is homeomorphic to the connected sum of a sphere and g tori, i.e. it is homeomorphic to a sphere with g handles. The number g, called the *genus*, is a topological invariant that determines a compact connected orientable surface up to homeomorphism. We can equivalently determine any orientable connected surface by computing its *Euler characteristic* χ, since it turns out that $\chi = 2 - 2g$ (which in particular is always an even integer). Moreover (see [18], 1.3.A) any compact connected non-orientable surface contained in $\mathbb{RP}^3$ is homeomorphic to the connected sum of a projective plane and a compact connected orientable surface of genus g; the Euler characteristic is then $\chi = 1 - 2g$ (i.e. an odd integer).

Thus we topologically determine S by computing the Euler characteristics of the connected components of T, the equivalence relation that, through a collapsing process, produces the singularities of S and the number of isolated points on the surface.

Then, considering how S is embedded in $\mathbb{RP}^3$, we define the *labelled 2-adjacency graph* of S containing information about the mutual disposition of the connected components of the surface and we show that it is a topological invariant of the pair $(\mathbb{RP}^3, S)$. We also describe an algorithm to compute the previous labelled graph and to recognize the position of the isolated points of S in the connected components of $\mathbb{RP}^3 \setminus S$.

This paper is a natural evolution of the articles [14], [12] and [13], which dealt with non-singular surfaces. Here we use some of those techniques for reducing the problem to the compact affine case; for this latter case we develop a new procedure able to investigate the topological nature of the isolated singularities.

The main definitions, the necessary theoretical background and the list $D(S)$ of data invariant up to homeomorphism of the pair $(\mathbb{RP}^3, S)$ are presented in Section 2. In order to deal with isolated singularities we use a generalization of classical Morse theory to singular spaces, first introduced by Lazzeri ([16]), that we briefly recall in Section 3, where we prove also some related results necessary for the algorithm. In Section 4 we describe an algorithm to compute $D(S)$ when the surface is contained in an affine chart of $\mathbb{RP}^3$. When S is not affine, it is possible to construct a suitable compact algebraic surface $\widehat{S}$ in $\mathbb{R}^3$ and to recover $D(S)$ from $D(\widehat{S})$, which can be computed by means of the affine-case algorithm. This reduction procedure and the general-case algorithm are presented in Section 5 which also contains some examples. Finally in Section 6 we present algorithms to perform some auxiliary computations concerning the critical and singular points and tests for connectedness. We also give a rough complexity analysis showing that, after the auxiliary computations, the total number of points required by our algorithm to compute all our topological invariants is $\mathcal{O}(d^7)$.

A preliminary version of this paper was written in collaboration with our dear friend "Mimmo" with whom we collaborated for a long time and who passed away too early. With him we had many stimulating and enriching discussions concerning many of the topics presented in this paper. This is why we wish to dedicate this paper to him.

2. Remarks on the topology of surfaces with isolated singularities

Let S be the real projective algebraic surface in $\mathbb{RP}^3$ defined by the equation $F(x,y,z,t)=0$, where F is a square-free homogeneous polynomial of degree d with rational coefficients (or more generally with coefficients in a computable subfield of $\mathbb{R}$). A point $P \in S$ is a singular point of the surface if it annihilates all the first partial derivatives of F; we will denote by $Sing\, S$ the algebraic set consisting of the singular points of S.

In this paper we will consider only the case when $Sing\, S$ contains only finitely many points. Note that we make no assumption on the singular locus $Sing\, S_{\mathbb{C}}$ of the complex projective surface $S_{\mathbb{C}}$ in $\mathbb{CP}^3$ defined by the equation $F=0$; since F is square-free, $Sing\, S_{\mathbb{C}}$ cannot have dimension 2, but it can be a complex curve.

A point $P \in S$ is called an *isolated point* of S if there exists an open neighborhood U of P in $\mathbb{RP}^3$ such that $S \cap U = \{P\}$; the set $Isol(S)$ of all isolated points of S is contained in $Sing\, S$.

Let us present an explicit way to see $S \setminus Isol(S)$, up to homeomorphism, as the quotient space $T/\mathcal{R}(T)$ where T is a compact topological surface (i.e. a 2-dimensional topological manifold) and $\mathcal{R}(T)$ is an equivalence relation on T which collapses finitely many subsets, each consisting of finitely many points.

For simplicity, consider first the case when S is contained in an affine chart of $\mathbb{RP}^3$ and defined in affine coordinates by the square-free polynomial equation $f(x,y,z)=0$. We will use the following notations:

Notation 2.1. 1. *If $Q \in \mathbb{R}^3$, ϵ is a real positive number and $d(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R}^3$, denote by:*
— $B(Q,\epsilon)=\{X \in \mathbb{R}^3 \mid d(X,Q)<\epsilon\}$
— $D(Q,\epsilon)=\{X \in \mathbb{R}^3 \mid d(X,Q)\le\epsilon\}$
— $S(Q,\epsilon)=\{X \in \mathbb{R}^3 \mid d(X,Q)=\epsilon\}$,
respectively the open ball, the closed ball and the sphere of radius ϵ centered at Q. Moreover denote by $C(Q,\epsilon)$ the curve $S \cap S(Q,\epsilon)$.

2. *If $A \subseteq \mathbb{R}^3$ and P is a point in $\mathbb{R}^3$, by* cone over A with vertex P *we will mean the union of all segments joining P with any point of A. By convention the cone over the empty set with vertex P is the set $\{P\}$.*
3. *If U is a topological space and W is a subspace, let $G(U,W)$ denote the* adjacency graph of the pair (U,W)*, that is the graph whose vertices are the connected components of $U \setminus W$ and where two distinct vertices Ω_1, Ω_2 are joined by an edge if and only if the topological closures of Ω_1 and Ω_2 are not disjoint.*

The following result, proved by Milnor also in the complex case, gives the important information that locally at an isolated singularity S is topologically a cone:

Theorem 2.2. *([17], Proposition 2.10) Let Q be an isolated singular point of $S \subseteq \mathbb{R}^3$. Then there exists $r>0$ such that for all positive $\epsilon \le r$*

i) $C(Q,\epsilon)$ is a non-singular curve (possibly empty) *ii)* $S \cap D(Q,\epsilon)$ *is homeomorphic to the cone over* $C(Q,\epsilon)$ *with vertex* Q.

Recall that the *critical points* of a polynomial function $g\colon S \to \mathbb{R}$ are the points of $S = V(f)$ where the rank of the matrix $\begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix}$ is less than or equal to 1; in particular all the points of $Sing\, S$ are critical. If P is critical for g, the real number $g(P)$ is called a *critical value*. Any non critical value is called a *regular value*. Recall also (see [17], Corollary 2.8) that the restriction of g to $S \setminus Sing\, S$ can have at most finitely many critical values; since in our hypotheses S has finitely many singular points, any polynomial function g defined on S can have at most finitely many critical values.

If $Q \in S$, any $r > 0$ such that $D(Q,r) \setminus \{Q\}$ contains no singular points of S and no critical points of the restriction to S of the function $X \to d(X,Q)^2$ satisfies the thesis of Theorem 2.2. For any $\epsilon \leq r$, we will call ϵ a *Milnor radius* at Q and $S(Q,\epsilon)$ a *Milnor sphere* at Q. In this case, though the non-singular curve $C(Q,\epsilon)$ is not plane, we will call its connected components *ovals*. In particular, if $Q \notin Sing\, S$ then $C(Q,\epsilon)$ consists of a single oval; if Q is an isolated point then $C(Q,\epsilon)$ is empty.

Remark 2.3. *According to the literature (see [10]), an* algebraic neighborhood *of a point* Q *in a compact algebraic set* $M \subseteq \mathbb{R}^n$ *is the set* $\alpha^{-1}([0,\epsilon])$ *where* $\alpha\colon M \to \mathbb{R}$ *is a non-negative proper polynomial (or rational or even semialgebraic) function such that* $\alpha^{-1}(0) = \{Q\}$ *and* $\epsilon > 0$ *is a real number such that* $(0,\epsilon]$ *does not contain any critical value of* α *(any such function is called a* rug function*). The Uniqueness Theorem proved by Durfee in [10] shows that any two algebraic neighborhoods* V_1, V_2 *of an isolated singular point* Q *in* M *are homeomorphic by means of a homeomorphism* $h\colon M \to M$ *such that* $h(Q) = Q$ *and* $h|_{V_1 \setminus \{Q\}}$ *is a diffeomorphism between* $V_1 \setminus \{Q\}$ *and* $V_2 \setminus \{Q\}$. *In particular the notion of algebraic neighborhood of* Q *in* M *is independent of the embedding of* M *in* $\mathbb{R}^n$*; moreover the* link *(i.e. the boundary of an algebraic neighborhood of* Q *in* M*) is a topological invariant which is independent of the embedding of* M *in its ambient space.*

Thus, coming back to our situation and using as α *the polynomial function* $X \to d(X,Q)^2$ *, the set* $S \cap D(Q,\epsilon)$ *is an algebraic neighborhood of* Q *in the surface* S *and* $C(Q,\epsilon)$ *is the link of* Q *in* S.

It is possible to associate to S a topological surface $T \subset \mathbb{RP}^3$ from which we can obtain again S, except its isolated points if any, by means of a suitable quotient. This can be done by removing from S the portion contained in Milnor spheres centered at the singular points, and then suitably attaching, inside the previous Milnor spheres, non-intersecting 2-cells along the ovals of the links. The next proposition describes the main properties of the surface obtained by means of this procedure:

Proposition 2.4. *For any* $Q \in Sing\, S$, *let* $\epsilon_Q > 0$ *be a Milnor radius at* Q *and assume that* $D(Q,\epsilon_Q) \cap D(Q',\epsilon_{Q'}) = \emptyset$ *whenever* Q, Q' *are distinct singular points. Then, for each* $Q \in Sing\, S$, *we can attach to* $S \setminus \left(\bigcup_{Q \in Sing\, S} D(Q,\epsilon_Q)\right)$

a 2-cell along each oval of the non-singular curve $C(Q, \epsilon_Q)$ in such a way that the embedded topological surface without boundary T so obtained satisfies the following properties:

1. $T \cap S(Q, \epsilon_Q) = C(Q, \epsilon_Q)$ *for each* $Q \in Sing\, S$,
2. $S \setminus \left(\bigcup_{Q \in Sing\, S} D(Q, \epsilon_Q)\right) = T \setminus \left(\bigcup_{Q \in Sing\, S} D(Q, \epsilon_Q)\right)$
3. *for each* $Q \in Sing\, S$ *the adjacency graph* $G(D(Q, \epsilon_Q), T \cap D(Q, \epsilon_Q))$ *is isomorphic to the adjacency graph* $G(S(Q, \epsilon_Q), C(Q, \epsilon_Q))$
4. *if for any* $Q \in Sing\, S$ *we denote by* $Z(Q)$ *the set obtained by choosing a point in each* 2*-cell attached to the ovals of* $C(Q, \epsilon_Q)$ *and by* $\mathcal{R}(T)$ *the equivalence relation on* T *that collapses to a point each of the sets* $Z(Q)$*, then* S *is homeomorphic to the disjoint union of* $T/\mathcal{R}(T)$ *and* $Isol(S)$.

Example 2.5. The surface represented in Figure 1 has three isolated singularities N_1, N_2, R_1 and R_1 is an isolated point. The topological surface T constructed as explained above has three connected components T^1, T^2, T^3 and all of them are spheres. The singularity N_1 can be obtained collapsing one point of T^1 and two points of T^2 to a single point; similarly N_2 can be obtained collapsing three points chosen respectively in T^1, T^2 and T^3. □

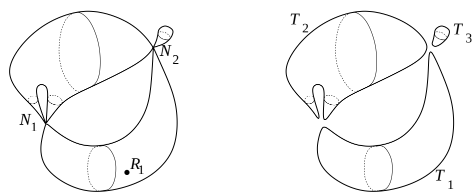

Figure 1: *A surface with three singular points (left-hand side) and the topological surface T associated to it (right-hand side).*

Remark 2.6. *The choice of the sets $Z(Q)$ in Proposition 2.4 is not unique. However, since the topological type of the space obtained by collapsing finitely many points in a compact connected topological surface does not depend on the choice of these points but only on their number, in order to determine $\mathcal{R}(T)$ it will be sufficient to compute the number of points it collapses in each connected component of T at each singular point.*

Observe also that, if Q is a singular non-isolated point which does not disconnect S locally, then $S \cap D(Q, \epsilon_Q)$ is homeomorphic to a 2-cell, $Z(Q)$ contains a unique point and the action of $\mathcal{R}(T)$ is trivial at Q. In other words the effect of $\mathcal{R}(T)$ is interesting only at the disconnecting singular points.

Remark 2.7. *If Q is a non-singular point of the surface and ϵ_Q is a Milnor radius at Q, then $C(Q, \epsilon_Q)$ consists of a single oval and $S \cap D(Q, \epsilon_Q)$ is homeomorphic to a 2-dimensional disk. Thus, if we modify S also inside the Milnor ball $D(Q, \epsilon_Q)$ as described in Proposition 2.4, the 2-cell that we attach along $C(Q, \epsilon_Q)$ is homeomorphic to $S \cap D(Q, \epsilon_Q)$. In this case the set $Z(Q)$ of points* *to be glued (defined as in the statement of Proposition 2.4) contains a unique point, so that collapsing $Z(Q)$ to a point has a trivial effect. This shows that, when we modify locally S according to Proposition 2.4 to get T, it is possible to enlarge the set of the singularities of S, including in that set also finitely many non-singular points: all the topological surfaces we obtain are the same up to homeomorphism.*

In the general case when S is a projective surface contained in $\mathbb{RP}^3$, it is possible to see $S \setminus Isol(S)$ as a quotient of a topological surface using affine charts. Namely, let N be a singular point of S, H a hyperplane not passing through N and let $g\colon U_H = \mathbb{RP}^3 \setminus H \to \mathbb{R}^3$ be a chart. Then, according to Proposition 2.4 we can remove from $g(S \cap U_H) \subseteq \mathbb{R}^3$ the portion $g(S \cap U_h) \cap D(g(N), \epsilon)$ and attach along the link finitely many 2-cells inside $D(g(N), \epsilon)$. If $Y_{g(N)}$ is the union of these 2-cells, we locally modify $S \subseteq \mathbb{RP}^3$ at N attaching to $S \setminus g^{-1}(D(g(N), \epsilon))$ the 2-cells $g^{-1}(Y_{g(N)})$ along the link of S at N. By Durfee's results ([10]), the surface T so obtained after performing these local modifications at all the singular points does not depend on the choice of the charts.

We can show that any topological surface T and equivalence relation $\mathcal{R}(T)$ associated to S according to Proposition 2.4 (see also Remark 2.6 and Remark 2.7), together with the number of isolated points of S, uniquely characterize the surface S up to homeomorphism in the sense we are going to make precise.

Definition 2.8.

1. *We say that an equivalence relation $\mathcal{R}_U$ on a compact topological surface U is* generically $1-1$ *if the projection to the quotient $\pi_U : U \to U/\mathcal{R}_U$ has finite fibers and π_U is bijective except on a finite subset of U.*
2. *Let $\varphi\colon U \to W$ be a homeomorphism between compact topological surfaces. Two generically $1-1$ equivalence relations $\mathcal{R}_U, \mathcal{R}_W$ respectively on U and W are called* φ-compatible *if for each $x \in U$ the cardinality of the equivalence class $[x]_{\mathcal{R}_U}$ coincides with the cardinality of the equivalence class $[\varphi(x)]_{\mathcal{R}_W}$.*

Proposition 2.9. *Let U, W be compact topological surfaces and let $\mathcal{R}_U, \mathcal{R}_W$ be generically $1-1$ equivalence relations respectively on U and W. Then $U/\mathcal{R}_U$ and $W/\mathcal{R}_W$ are homeomorphic if and only if there exists a homeomorphism $\varphi\colon U \to W$ such that $\mathcal{R}_U$ and $\mathcal{R}_W$ are* φ-compatible.

Proof. Denote by $\pi_U : U \to U/\mathcal{R}_U$ and $\pi_W : W \to W/\mathcal{R}_W$ the projections to the quotient. Let $U_1 = \{x \in U \mid [x]_{\mathcal{R}_U} = \{x\}\}$ and $W_1 = \{x \in W \mid [x]_{\mathcal{R}_W} = \{x\}\}$. By hypothesis both $U \setminus U_1$ and $W \setminus W_1$ are finite sets and both $\pi_U|_{U_1}$ and $\pi_W|_{W_1}$ are injective.

Assuming that $\psi\colon U/\mathcal{R}_U \to W/\mathcal{R}_W$ is a homeomorphism, we want to define a homeomorphism $\varphi\colon U \to W$.

Let $x \in U_1$. Since U_1 is open in U, there exists an open neighborhood V_x of x contained in U_1 and homeomorphic to a disk. The restriction $\pi_U|_{V_x}$ is a homeomorphism, hence also $U/\mathcal{R}_U$ is locally homeomorphic to a disk near

$\pi_U(x)$. By hypothesis $W/\mathcal{R}_W$ is then locally homeomorphic to a disk near $\psi(\pi_U(x))$ and the fiber of π_W above $\psi(\pi_U(x))$ consists of a single point. Thus the map $\varphi_1 \colon U_1 \to W_1$ given by $\varphi_1 = (\pi_W|W_1)^{-1} \circ \psi \circ \pi_U|_{U_1}$ is well defined and a homeomorphism between U_1 and W_1.

We want to extend the map φ_1 to the whole U. Let $x \in U \setminus U_1$ and assume that $[x]_{\mathcal{R}_U} = \{x_1, \dots x_m\}$ so that $\pi_U(x_1) = \dots = \pi_U(x_m) = P_x$. Locally at each x_i the surface T is homeomorphic to a disk D_i and φ_1 is defined on $D_i \setminus \{x_i\}$. Up to shrinking the disks D_i if necessary, locally at P_x the space $U/\mathcal{R}_U$ is the union of m disks modulo the glueing of a point from each disk, hence it is locally at P_x homeomorphic to the cone over m disjoint ovals. Since modifying a space by means of a homeomorphism preserves the number of local connected components, also $W/\mathcal{R}_W$ is homeomorphic to the cone over m disjoint ovals at $\psi(P_x)$. Hence there exist $y_1, \dots, y_m \in W$ such that $\{y_1, \dots, y_m\}$ is an equivalence class w.r.t. $\mathcal{R}_W$ and $\pi_W(y_1) = \dots = \pi_W(y_m) = \psi(P_x)$. Since φ_1 is already defined on U_1, by continuity for each i there exists a unique $j_i \in \{1, \dots, m\}$ such that, setting $\varphi(x_i) = y_{j_i}$, φ is a bijective continuous extension of φ_1 and hence a homeomorphism.

Conversely if $\varphi \colon U \to W$ is a homeomorphism and $\mathcal{R}_U$ and $\mathcal{R}_W$ are φ-*compatible*, then a bijection $\psi \colon U/\mathcal{R}_U \to W/\mathcal{R}_W$ such that $\psi \circ \pi_U = \pi_W \circ \varphi$ is well defined. By the universal property of quotient spaces, ψ is continuous and hence a homeomorphism. □

Henceforth we will always denote by T and $\mathcal{R}(T)$ a topological surface and an equivalence relation associated to S according to Proposition 2.4; observe that T can be considered as a sort of topological desingularization of $S \setminus Isol(S)$.

As well known the surface T is topologically determined by the Euler characteristics of its connected components. Thus, as a consequence of the previous results, we can determine S topologically by computing the list $\chi(T) = [\chi(T^1), \dots, \chi(T^r)]$ of the Euler characteristics of the connected components $T^1, \dots, T^r$ of T, recognizing $\mathcal{R}(T)$ and computing the number of the isolated points of S.

So far our topological investigation has not taken into account the way in which the surface is embedded in $\mathbb{RP}^3$, which can occur in different ways with possible phenomena of self-knotting of a connected component or linking of distinct components. In particular it gives no information about the mutual disposition of the connected components of S and the connected components (or *regions*) of $\mathbb{RP}^3 \setminus S$. We introduce now the so-called *labelled 2-adjacency graph* of the surface S and we show that it is a topological invariant of the pair $(\mathbb{RP}^3, S)$ (see [13] where similar data were introduced and computed in the case of a non-singular surface).

2.1. The 2-adjacency graph

If S is singular, it may occur that the closures of two regions of the complement of S share only finitely many points; think for instance of the surface consisting of two spheres tangent at a common point. This justifies the following definition:

Definition 2.10. *The* 2-adjacency graph $G(S)$ *is the graph whose vertices are the regions of* $\mathbb{RP}^3 \setminus S$ *and where two distinct vertices are joined by an edge if and only if the closures of the two regions of* $\mathbb{RP}^3 \setminus S$ *meet in a 2-dimensional subset of* S.

Remark 2.11. *Recall (see for instance [18]) that a connected topological surface can be situated in* $\mathbb{RP}^3$ *in two different ways: either it disconnects* $\mathbb{RP}^3$ *into two connected components and it is the common boundary of those regions (and then it is called* two-sided*), or it does not disconnect* $\mathbb{RP}^3$ *(and then it is called* one-sided*). When* $S \subseteq \mathbb{RP}^3$ *is a non-singular algebraic surface, the nature of its connected components with respect to the previous property is determined by the degree of the surface: if it is even, all the components are two-sided; if it is odd,* S *contains exactly one component which is one-sided and all the others are two-sided. For odd-degree surfaces the one-sided component is the only one having an odd Euler characteristic. Since two adjacent regions of* $\mathbb{RP}^3 \setminus S$ *share in their boundaries a connected component of* S*, the edges of the standard adjacency graph* $G(\mathbb{RP}^3, S)$ *are in 1-1 correspondence with the connected components of the surface, with the only exception that, when* S *has odd degree, the unique one-sided component of* S *is not represented in the graph. Thus the 2-adjacency graph* $G(S)$ *just defined coincides with the ordinary adjacency graph in the non-singular case.*

When S *is singular, in general there is not a bijective correspondence between the set of the edges of* $G(S)$ *and the set of the connected components of the surface, even if* S *has even degree. However if* S *contains only isolated singularities and* $S_1, \ldots, S_n$ *are the connected components of* $S \setminus Sing\, S$*, if the degree of* S *is even then for all* j $\overline{S_j}$ *disconnects* $\mathbb{RP}^3$ *into two connected regions; if instead the degree of* S *is odd, then there exists a unique* i_0 *such that* $\mathbb{RP}^3 \setminus \overline{S_{i_0}}$ *is connected, while for any* $j \neq i_0$ *the set* $\overline{S_j}$ *disconnects* $\mathbb{RP}^3$ *into two connected regions. Thus the situation of the non-singular case still holds, up to replacing the connected components of* S *by the closures of the connected components of* $S \setminus Sing\, S$*. By an abuse of terminology we will call* S_{i_0} *the one-sided component of* $S \setminus Sing\, S$*, we will denote* $\overline{S_{i_0}}$ *by* Γ *and call it the one-sided component of* S*. As a consequence, there is a 1-1 correspondence between the closures of the connected components of* $S \setminus Sing\, S$ *and the edges of* $G(S)$*, except for odd-degree surfaces when the one-sided component is not represented in* $G(S)$*.*

Remark 2.12.

1. *The graph* $G(S)$ *is a topological invariant of the pair* $(\mathbb{RP}^3, S)$.
2. *The isolated points of* S*, if any, are not represented in* $G(S)$.
3. *The 2-adjacency graph* $G(S)$ *is isomorphic to the adjacency graph* $G(T)$.

2.2. Labels and roots

In order to give information about the mutual disposition of connected components of the surface, we have to precise the intuitive idea by which, if A and B are connected surfaces, we say that A envelopes B if B is contained in the interior part of A.

Even in the case of a topological surface it may occur that a connected component disconnects $\mathbb{RP}^3$ in two regions (i.e. it is two-sided) which are homologically equivalent, so that it is not possible to distinguish one of them from the other (think for instance of a one-sheeted hyperboloid). This phenomenon is strictly related to the so-called property of "contractibility"; we refer to Viro ([18]) for a detailed survey of this notion and for a proof of the results we are going to recall.

Adopting the terminology used by Viro, we say that a subset $A \subset \mathbb{RP}^3$ is *contractible in* $\mathbb{RP}^3$ (or briefly *contractible*) if any loop in A is homotopically trivial as a loop in $\mathbb{RP}^3$, *non-contractible* otherwise. Equivalently A is contractible if the homomorphism $\pi_1(A) \to \pi_1(\mathbb{RP}^3)$ induced by the inclusion is trivial.

Classical general results ensure that the one-sided component of a topological surface V, if present, is always non-contractible, while the other components can be either contractible or non-contractible. For instance any component contained in an affine chart of $\mathbb{RP}^3$ is contractible, while a one-sheeted hyperboloid is not. Moreover a contractible connected component W of V disconnects $\mathbb{RP}^3$ into two connected regions, only one of which is contractible and called the *interior part* of W. Hence for such components it is possible to define a natural partial order relation.

Coming back to our situation and taking into account the previous remarks, we can define a partial order relation in the set of the closures of the connected components of $S \setminus Sing\, S$ as follows:

Definition 2.13. *Let S_i and S_j be distinct connected components of $S \setminus Sing\, S$. We say that $\overline{S_i}$ is inside $\overline{S_j}$ if the following two conditions hold:*

1. *$\overline{S_j}$ disconnects $\mathbb{RP}^3$ into two regions and one of these is contractible,*
2. *S_i is contained in the contractible component of $\mathbb{RP}^3 \setminus \overline{S_j}$.*

The partial order relation among the closures $\overline{S_i}$ of the connected components of $S \setminus Sing\, S$ introduced in Definition 2.13 coincides with the one among the corresponding components of T.

We can now endow $G(S)$ with labels by means of a function

$$Contr(S)\colon \{\text{vertices of } G(S)\} \to \{c, nc\}$$

that marks each vertex of $G(S)$ (i.e. each region of $\mathbb{RP}^3 \setminus S$) as *contractible* or *non-contractible*. In a similar way we can define the function

$$Contr(T)\colon \{\text{vertices of } G(T)\} \to \{c, nc\}$$

that marks each vertex of $G(T)$ (i.e. each region of $\mathbb{RP}^3 \setminus T$) as contractible or non-contractible.

Remark 2.14. *For a topological surface, such as T, it turns out that a two-sided component is non-contractible if and only if the two regions of its complement are both non-contractible. Thus the knowledge of the function $Contr(T)$ allows us also to know which components of T are contractible and which components are not. Note that in the singular case it is no longer true that the labels on the two vertices of an edge are sufficient to determine the contractibility of the edge. For instance, if S is a cone, $G(S)$ has two vertices, one marked c, the other nc and still the only edge of $G(S)$ (i.e. the cone itself) is non-contractible.*

It is possible to reconstruct the partial order relation introduced in Definition 2.13 through $G(S)$ fixing on it a set of roots as follows.

Denote by $G_{nc}(S)$ the subgraph of $G(S)$ formed by the non-contractible vertices and by the edges having both vertices marked *nc*; instead denote by $\overline{G_c(S)}$ the subgraph formed by all the contractible vertices, all the edges where at least one vertex is contractible and all the vertices of these edges. One can see that, if S has odd degree, all the regions of $\mathbb{RP}^3 \setminus S$ are contractible and hence $G_{nc}(S) = \emptyset$. Instead, when the degree of S is even, the closure $\overline{S_j}$ of each connected component of $S \setminus Sing\, S$ disconnects $\mathbb{RP}^3$ into two connected regions and at least one of them is non-contractible (possibly both, as in the case of a one-sheeted hyperboloid); in particular $G_{nc}(S)$ is not empty.

The way we fix roots in $G(S)$ depends on the degree of S.

If S has even degree, $G_{nc}(S)$ is connected and each connected component of $\overline{G_c(S)}$ is a tree that contains exactly one vertex marked *nc*: we choose these vertices as a *set of roots* of $G(S)$. In this way the order induced on each connected component of $\overline{G_c(S)}$ by the only root contained in it coincides with the partial order described in Definition 2.13.

If S has odd degree, $G_{nc}(S)$ is empty; however we are able to choose a root in $G(S)$ also in this case: we will call *root* of $G(S)$ the unique region of $\mathbb{RP}^3 \setminus S$ which is adherent to the one-sided component Γ.

Note that, while for even-degree surfaces the information about which vertices are the roots of $G(S)$ is obtained from the labels, for odd-degree surfaces this notion is independent of the labels on $G(S)$ because in this case each vertex is marked c. However this piece of information is quite important since sometimes it is the only one that allows us to realize that two pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are not homeomorphic: if, for instance, S consists of a projective plane and two topological spheres, only the knowledge of the root allows us to recognize whether the two spheres are mutually external or one of them encircles the other one.

If $Roots(S)$ denotes the set of roots of S defined as above, we will call the triple $(G(S), Contr(S), Roots(S))$ the *labelled 2-adjacency graph* of S.

The previous definition of $G_{nc}(S)$ and $\overline{G_c(S)}$ can be used also for the graph $G(T)$; this allows us to define a set of roots in $G(T)$, denoted $Roots(T)$, independently of the notion of degree used above: if $G_{nc}(T) \neq \emptyset$, we choose as roots in $G(T)$ the common vertices to $G_{nc}(T)$ and $\overline{G_c(T)}$; if $G_{nc}(T) = \emptyset$, we choose as unique root the region of $\mathbb{RP}^3 \setminus T$ external to all the connected components of T.

The triple $(G(T), Contr(T), Roots(T))$ will be called the *labelled adjacency graph* of T.

With the previous definitions we have that

Proposition 2.15. *(i) The labelled 2-adjacency graph of S is an invariant of the pair $(\mathbb{RP}^3, S)$ up to homeomorphism.*

(ii) The labelled 2-adjacency graph of S is isomorphic to the labelled adjacency graph of T.

The isolated points of S do not appear at all in $G(S)$. While their number is sufficient for the topological characterization of S, in order to take into account the embedding of S in $\mathbb{RP}^3$ we need to know in which regions of the complement they lie. For that it will be sufficient to compute a list $IsReg(T) = [isReg(\Sigma_1), \ldots, isReg(\Sigma_k)]$ where $\Sigma_1, \ldots, \Sigma_k$ are the regions of $\mathbb{R}^3 \setminus T$ and $isReg(\Sigma_i)$ is the number of isolated points contained in Σ_i.

We will collect all the mentioned data concerning the surface in a single list of data

$$D(S) = [\chi(T), G(T), Contr(T), Roots(T), \mathcal{R}(T), IsReg(T)].$$

By the previous considerations we have that

1. $D(S)$ is an invariant up to homeomorphism of the pair $(\mathbb{RP}^3, S)$,
2. $D(S)$ completely determines the topological type of S,
3. though not sufficient to determine the pair $(\mathbb{RP}^3, S)$ (see for instance the example shown in Figure 2), the set $D(S)$ gives useful information on the surface up to ambient-homeomorphism. For instance the surfaces S and S' represented in Figure 3 are homeomorphic but the pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are not homeomorphic since the labelled adjacency graphs of T and T' are not isomorphic (because of the position of the labels c and nc).

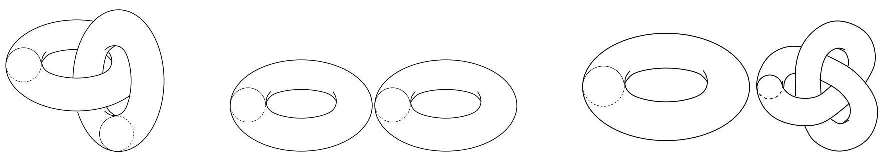

Figure 2: *Surfaces pairwise not ambient-homeomorphic but sharing the same $D(S)$*

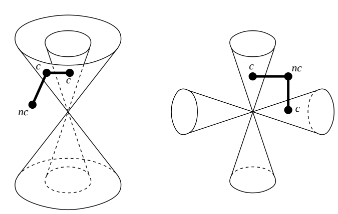

Figure 3: *Two surfaces that are homeomorphic, but not ambient-homeomorphic.*

The next sections will be devoted to show that all the data in $D(S)$ can be computed starting from an equation of S, even though T is not algebraic.

3. Passing through a critical point - Theoretical results

In this section we describe our general strategy to compute the topology of a real algebraic surface S having only isolated singularities and we present the mathematical theoretical results on which the algorithm bases its correctness. We will present the algorithm and discuss all the computational aspects in the next sections.

In the present section we assume that S is a compact affine surface in $\mathbb{R}^3$ defined by a square-free polynomial equation $f(x,y,z) = 0$ (as we will see, the general case can be reduced to this one).

Denote by $p : S \to \mathbb{R}$ the projection defined by $p(x,y,z) = z$. Thus a point $P \in S$ is critical for p if it annihilates the first partial derivatives f_x and f_y. As recalled in Section 2, since in our hypotheses S has finitely many singular points, the projection p can have at most finitely many critical values.

We assume that (x,y,z) is a *good frame*, i.e. that the fiber $p^{-1}(\gamma)$ over any critical value γ contains only one critical point in S, which turns out to be the only singular point of the section curve $S \cap \{z = \gamma\}$. In Section 6 we will show that, up to a generic linear change of coordinates, we can assume that (x,y,z) is a good frame.

Since S is compact, let τ be a positive real number such that $S \subset \mathbb{R}^2 \times (-\tau,\tau)$.

For any $W \subseteq \mathbb{R}^3$ and for any $U \subseteq \mathbb{R}$ let $W_U = W \cap p^{-1}(U)$. Thus, if $a < b$, we have that $S_{[a,b]} = S \cap \{a \le z \le b\}$. For simplicity we will denote by S_a the level curve $S_{\{a\}} = S \cap \{z = a\}$ and by $S_{\le a}$ the level surface $S_{[-\infty,a]} = S \cap \{z \le a\}$ of S having S_a as its boundary.

Denote by $Q_1, \ldots, Q_m$ the points in S which are critical for p.

We will need to work at each point Q_i in a sufficiently small Milnor ball where both the surface and its plane horizontal section at Q_i have a conical structure. To this purpose, using the Theorem of semialgebraic triviality for the function $X \to d(X,Q)$ and adapting suitably the proof of Theorem 9.3.6 in [4], we have:

Proposition 3.1. *Let $Q = (\alpha,\beta,\gamma)$ be an isolated singular point of S. Let r be a Milnor radius at Q both for the surface S and for the plane curve $S \cap \{z = \gamma\}$. Then for all $0 < \epsilon \le r$ there exists a homeomorphism $\phi : D(Q,\epsilon) \to D(Q,\epsilon)$ such that:*

1. *$\phi(S \cap D(Q,\epsilon))$ coincides with the cone over $C(Q,\epsilon)$ with vertex Q;*
2. *$\phi(D(Q,\epsilon) \cap \{z = \gamma\}) = D(Q,\epsilon) \cap \{z = \gamma\}$;*
3. *$\phi(x) = x$ for all $x \in S(Q,\epsilon)$;*
4. *$\phi(S(Q,\epsilon')) = S(Q,\epsilon')$ for all $\epsilon' \le \epsilon$.*

As an immediate consequence, one also has

Corollary 3.2. *In the hypotheses of Proposition 3.1, $\phi(S \cap D(Q,\epsilon) \cap \{z = \gamma\})$ coincides with the cone over $C(Q,\epsilon) \cap \{z = \gamma\}$ with vertex Q.*

Henceforth denote by T a topological surface that we obtain by modifying S as described in Proposition 2.4 (see also Remark 2.6 and Remark 2.7) inside all the Milnor balls $D(Q_i,\epsilon_i)$ centered at the critical points Q_i, for sufficiently small ϵ_i. As explained in Section 2, if S contains s isolated points, the surface S is homeomorphic to the disjoint union of s points and the quotient $T/\mathcal{R}(T)$ where $\mathcal{R}(T)$ is a generically 1-1 equivalence relation.

We choose to represent the equivalent relation $\mathcal{R}(T)$ by means of an $r \times m$ matrix $Rel(T)$, where r denotes the number of connected components of T, including the data relative to all the critical points:

Notation 3.3. 1. *For any connected component Y of T, we denote by $rel(Y)$ the vector of length m whose i-th element $rel(Y)(i)$ is the number of ovals in which Y meets the Milnor sphere $S(Q_i, \epsilon_i)$ centered at Q_i (which coincides with the number of points in $Y \cap Z(Q_i)$.*

2. *If $T^1, \ldots, T^r$ are the connected components of T, we denote by $Rel(T)$ the $r \times m$ matrix whose h-th row is the vector $rel(T^h)$.*

Remark 3.4. 1. *It is possible to extract from $Rel(T)$ the data sufficient to recover the equivalence relation $\mathcal{R}(T)$ such that $S \setminus Isol(S)$ is homeomorphic to $T/\mathcal{R}(T)$ using only the columns corresponding to the singular non-isolated points.*

2. *The notation introduced in Notation 3.3 can be used also for any level surface $T_{\leq b}$ with respect to its connected components: if $Y^1, \ldots, Y^g$ are the connected components of $T_{\leq b}$, then $rel(Y^j)$ is a vector of length m whose i-th element $rel(Y^j)(i)$ is the number of ovals in which Y^j meets the Milnor sphere $S(Q_i, \epsilon)$ centered at Q_i, while $Rel(T_{\leq b})$ is the $g \times m$ matrix having as rows $rel(Y^1), \ldots, rel(Y^g)$.*

Thus our aim is to compute the list of data

$$D(S) = [\chi(T), G(T), Contr(T), Roots(T), Rel(T), IsReg(T)].$$

In the affine case the computation of the function $Contr(T)$ and of the set $Roots(T)$ is straightforward: since T is contained in $\mathbb{R}^3$, all its components are contractible and all the regions of $\mathbb{RP}^3 \setminus T$ are contractible except the only one, say U_0, external to all the components of T. It is therefore sufficient to recognize U_0 as the only unbounded region of $\mathbb{R}^3 \setminus T$, choose it as the only root of $G(T)$, mark it as non-contractible and mark as contractible all other vertices in $G(T)$.

In order to compute $D(S)$ it is sufficient to be able to study the level surface $T_{\leq b}$ assuming the knowledge of $T_{\leq a}$ when the strip $\mathbb{R}^2 \times [a, b]$ contains only one critical point Q for p in its interior part. Since $T_{\leq b} = T_{\leq a} \cup T_{[a,b]}$, our strategy to compute $D(S_{\leq b})$ from $D(S_{\leq a})$ will consist in combining the data concerning the level surface $T_{\leq a}$ contained in $D(S_{\leq a})$ with information concerning the surface $T_{[a,b]}$ in the strip $\mathbb{R}^2 \times [a, b]$.

Therefore for the rest of this section we assume that:

• the strip $\mathbb{R}^2 \times [a, b]$ contains only one critical point $Q = (\alpha, \beta, \gamma)$ for p which is not isolated in S, with $a < \gamma < b$, (in particular a and b are regular values),

• ϵ is a positive number which is a Milnor radius both for S and for the curve S_γ (which in particular guarantees that each oval of the curve $C(Q, \epsilon) = S \cap S(Q, \epsilon)$ is transversal to $\{z = \gamma\}$); we assume also that $D(Q, \epsilon)$ is contained in the open strip $\mathbb{R}^2 \times (a, b)$.

The tool we intend to use in order to compute the topology of the connected components of $T_{\leq b}$ starting from the topology of $T_{\leq a}$ is the following theorem, which is an easy consequence of results due to Lazzeri ([16]) concerning more general "spaces with isolated singularities". Recall that, if Z is a topological space, the space $Z\times[0,1]/Z\times\{1\}$, obtained by collapsing to a point the subspace $Z\times\{1\}$, is called *the cone over* Z; we will denote it by $Cone(Z)$. Conventionally the cone over the empty set consists of a point. If $W\subseteq Z$, we will denote by $C(Z,W)$ the space obtained from the disjoint union of Z and $Cone(W)$ by identifying each $w\in W$ with $w\times\{0\}\in Cone(W)$.

Theorem 3.5. *([16]) Let X be a compact semialgebraic subset of $\mathbb{R}^3$ of dimension 2. Let $Q=(\alpha,\beta,\gamma)\in X$ be a critical point for the projection $p:X\to\mathbb{R}$ defined by $p(x,y,z)=z$. Let $a<\gamma<b$ and assume that*

(i) the boundary of $X_{[a,b]}$ is contained in $\{z=a\}\cup\{z=b\}$,
(ii) the strip $\mathbb{R}^2\times[a,b]$ does not contain any critical point for p except Q (in particular $X_{(a,b)}\setminus\{Q\}$ is a 2-dimensional differentiable manifold and Q is the only singular point for the level curve X_γ).

Then there exists $r>0$ such that

1. *for any $\epsilon\in(0,r)$ there exists $\eta(\epsilon)>0$ such that for any $\eta\in(0,\eta(\epsilon)]$ the plane set $V^\epsilon_{\gamma-\eta}=X_{\gamma-\eta}\cap D(Q,\epsilon)$ is a smooth compact curve (with boundary $X_{\gamma-\eta}\cap S(Q,\epsilon)$) whose diffeomorphism type depends neither on ϵ nor on η (the diffeomorphism class will be called the* vanishing manifold *of X at Q and denoted simply by $VM(Q)$),*
2. *$X_{\le\gamma}$ is homeomorphic to $X_{\le\gamma-\eta}/V^\epsilon_{\gamma-\eta}$ for any $\eta\in(0,\eta(\epsilon))$,*
3. *if $K\subseteq X_{\gamma-\eta}$ is any deformation retract of $V^\epsilon_{\gamma-\eta}$, then $X_{\le\gamma}$ is homotopically equivalent to $C(X_{\le\gamma-\eta},K)$,*
4. *$X_{\le\gamma}$ is a deformation retract of $X_{\le b}$,*
5. *$(X_\gamma\setminus B(Q,\epsilon))\cup(X\cap S(Q,\epsilon))$ is a deformation retract of $X_{[a,b]}\setminus B(Q,\epsilon)$.*

As outlined in [16], the previous theorem can be proved by making use of the integral curves of a vector field suitably constructed starting from the gradient field of the projection p on $X_{[a,b]}\setminus\{Q\}$ and adapted with respect to the Milnor ball $D(Q,\epsilon)$.

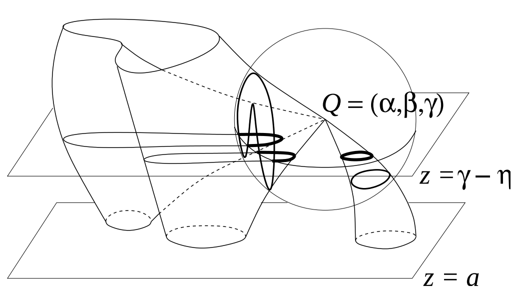

Figure 4: *Sections of a surface with a Milnor sphere centered at a singular point Q and with a horizontal plane $\{z=\gamma-\eta\}$ passing below the singularity.*

Remark 3.6. *When the strip $\mathbb{R}^2 \times [a, b]$ does not contain any critical point for p on the algebraic surface S except Q, the vanishing manifold $VM(Q)$ is the union of h ovals and k arcs, with $h \geq 0$ and $k \geq 0$. Since an arc of a curve is homotopically equivalent to a point, a deformation retract K of $VM(Q)$ consists of h ovals and k points, so that by Theorem 3.5 (3) and (4) $S_{\leq b}$ is homotopically equivalent to the space obtained by suitably attaching to $S_{\leq \gamma-\eta}$ the cone over h ovals and k points. Since $S_{\leq a}$ is a deformation retract of $S_{\leq \gamma-\eta}$ (by means of a deformation which is the identity on $S_{\leq a}$), $S_{\leq b}$ is also homotopically equivalent to the space obtained by suitably attaching to $S_{\leq a}$ the cone over h ovals and k points. In particular, if the vanishing manifold consists of k arcs, then $\chi(S_{\leq b}) = \chi(S_{\leq a}) + (1 - k)$.*

Theorem 3.5 cannot be directly applied to study the topology of T because T is only a topological manifold. It might be applied to S, but S is not homotopically equivalent to T. Proposition 3.16 will show that it is possible to compute the Euler characteristics of the connected components W of $T_{[a,b]}$ applying Theorem 3.5 to a suitable semialgebraic set homeomorphic to W. In this spirit observe that, even though T is not algebraic, however outside the union of the Milnor balls centered at the critical points of S, the surfaces S and T coincide (see Proposition 2.4 (2)), hence T is here described by an algebraic equation. Instead, inside each Milnor ball $D(Q, \epsilon)$ the local behavior of S and T is different, for instance because $S \cap D(Q, \epsilon)$ is connected, while $T \cap D(Q, \epsilon)$ has as many connected components as the curve $C(Q, \epsilon)$.

3.1. Computation of $\chi(T_{\leq b})$.

Observe that (see Figure 5):

- each connected component of $T_{\leq a}$ is contained in a single connected component of $T_{\leq b}$, but several connected components of $T_{\leq a}$ may be contained in the same connected component of $T_{\leq b}$,

- a connected component of $T_{\leq b}$ can intersect the strip $\{a \leq z \leq b\}$ in several distinct connected components of $T_{[a,b]}$.

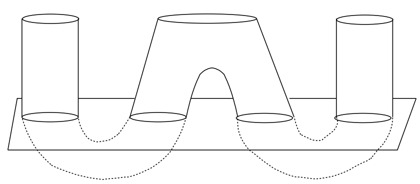

Figure 5: *A connected component Y of $T_{\leq b}$ for which both $Y \cap T_{\leq a}$ and $Y \cap T_{[a,b]}$ are not connected.*

Recall that:

Lemma 3.7. *If U_1, U_2 are compact topological surfaces with boundary such that U_1 intersects U_2 in a union of ovals, then $\chi(U_1 \cup U_2) = \chi(U_1) + \chi(U_2)$.*

As a consequence, the next result, assuming we know $T_{\leq a}$, reduces the computation of the topology of $T_{\leq b}$ to the investigation of the connected components of $T_{[a,b]}$:

Proposition 3.8. *Let Y be a connected component of $T_{\leq b}$. Then the Euler characteristic $\chi(Y)$ is the sum of the Euler characteristics of all the connected components of $Y \cap T_{\leq a}$ and of the Euler characteristics of all the connected components of $T_{[a,b]}$ contained in Y.*

Therefore we now focus to the first task of recognizing the connected components of $T_{[a,b]}$ and computing the Euler characteristic of each of them.

Let W be a connected component of $T_{[a,b]}$. If $W \cap D(Q,\epsilon) = \emptyset$, then W is topologically a cylinder. Otherwise W intersects the Milnor sphere $S(Q,\epsilon)$ in finitely many ovals of the curve $C(Q,\epsilon)$, say $\omega_1, \ldots, \omega_q$ with $q \geq 1$. As already pointed out, we cannot directly apply Theorem 3.5 to the topological surface W obtained by attaching a 2-cell along each oval ω_i to the semialgebraic set $E = W \setminus B(Q,\epsilon)$. However the 2-cell attached along ω_i is homeomorphic to the cone over ω_i with vertex Q and, by the conical structure, it is also homeomorphic to the closure of the connected component of $(S \cap D(Q,\epsilon)) \setminus \{Q\}$ containing ω_i.

In order to be able to give precise statements, it will be useful to introduce a suitable notation to indicate, for each oval ω of $C(Q,\epsilon)$, the "external" component of $S_{[a,b]}$ and the "internal" component of $S \setminus \{Q\}$ inside the Milnor ball $D(Q,\epsilon)$ containing ω in its boundary:

Notation 3.9. *For any oval ω of $C(Q,\epsilon)$*

1. *$E(\omega)$ denotes the connected component of $S_{[a,b]} \setminus B(Q,\epsilon)$ ($= T_{[a,b]} \setminus B(Q,\epsilon)$) containing ω,*
2. *$I(\omega)$ denotes the connected component of $(S \cap D(Q,\epsilon)) \setminus \{Q\}$ containing ω.*

Remark 3.10. *It may occur that $E(\omega_1) = E(\omega_2)$ for distinct ovals ω_1, ω_2 of $C(Q,\epsilon)$: for instance in the case of the "pinched torus", represented in Figure 6 and obtained by collapsing a circle in a torus, we have $C(Q,\epsilon) = \omega_1 \cup \omega_2$ and $E(\omega_1) = E(\omega_2)$. In particular this example shows that $S_{[a,b]} \setminus B(Q,\epsilon)$ may not have as many connected components as the curve $C(Q,\epsilon)$.*

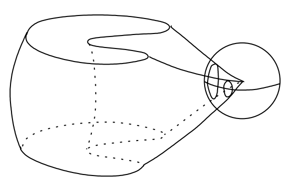

Figure 6: *Sections of a pinched torus with a Milnor sphere centered at a singular point and with two planes $\{z = a\}$ and $\{z = b\}$.*

In order to compute the Euler characteristic of any connected component W of $T_{[a,b]}$, if $W \cap S(Q,\epsilon) = \omega_1 \cup \ldots \cup \omega_q$, we will make use of the vanishing manifolds of $\overline{I(\omega_1)}, \ldots, \overline{I(\omega_q)}$. We will see that, for each oval ω of $C(Q,\epsilon)$, the nature of the vanishing manifold of $\overline{I(\omega)}$ at Q and also of $E(\omega) \cap (\{z=a\} \cup \{z=b\})$ is strictly related to the position of the oval ω with respect to the circle $S(Q,\epsilon) \cap \{z=\gamma\}$ that we will call the "equator" and denote by $Eq(Q,\epsilon)$. This is the reason why we introduce the following terminology:

Definition 3.11. *If $Q = (\alpha, \beta, \gamma)$, a connected subset X of $S(Q,\epsilon)$ is called*

1. *of type $(+)$ if $X \subseteq \{z > \gamma\}$*
2. *of type $(-)$ if $X \subseteq \{z < \gamma\}$*
3. *of type $(+-)$ if $X \cap \{z = \gamma\} \neq \emptyset$.*

Observe that, by Proposition 3.1, the type of an oval ω of $C(Q,\epsilon)$ does not change if we reduce the sphere radius, i.e.

Proposition 3.12. *Let ω be an oval of $C(Q,\epsilon)$. For any positive $\epsilon' \leq \epsilon$, let $\omega(\epsilon') = I(\omega) \cap S(Q,\epsilon')$. Then, for all $\epsilon' \leq \epsilon$ we have:*

1. *If ω is an oval of type $(+)$ (resp. $(-)$), then $\omega(\epsilon')$ is an oval of type $(+)$ (resp. $(-)$).*
2. *If ω is an oval of type $(+-)$, then $\omega(\epsilon')$ is an oval of type $(+-)$ and $\omega(\epsilon') \cap \{z=\gamma\}$ contains as many points as $\omega \cap \{z = \gamma\}$.*

The following result describes the vanishing manifold of $\overline{I(\omega)}$ and the topological type of $E(\omega)$ for an oval ω of type $(-)$:

Proposition 3.13. *Let ω be an oval of $C(Q,\epsilon)$ of type $(-)$. Then*

1. *there exists $\eta_0 > 0$ such that for all positive $\eta \leq \eta_0$ the set $\overline{I(\omega)} \cap \{z = \gamma - \eta\}$ consists of exactly one oval;*
2. *$E(\omega) \cap \{z=a\}$ consists of a single oval ω', the boundary of $E(\omega)$ is equal to $\omega \cup \omega'$ (in particular $E(\omega) \cap \{z=b\} = \emptyset$) and $E(\omega)$ is homeomorphic to a connected cylinder.*

Proof. (1) Since ϵ is a Milnor radius both for S and for the curve S_γ, $I(\omega) \cap \{z = \gamma\} = \emptyset$ and $\overline{I(\omega)} \cap \{z=\gamma\} = \{Q\}$.

Let $\eta_0 > 0$ be such that $\omega \subset \{z < \gamma - \eta_0\}$. By connectivity, for any $c \in (\gamma - \eta_0, \gamma)$ the set $I(\omega)_c$ is not empty. Since $\omega \cap \{z = c\} = \emptyset$ and the plane $\{z=c\}$ is transversal to $I(\omega)$, then $I(\omega)_c$ is a non-singular curve without boundary, i.e. consisting only of ovals.

Let Y be the connected component of $S_{[a,\gamma)}$ containing ω. By Thom's first isotopy Lemma there exist an oval $\omega' \subseteq S_a$ such that Y is homeomorphic to $\omega' \times [a,\gamma)$. In particular for any $c \in [a,\gamma)$ Y_c is connected and $Y_a = \omega'$.

Since $I(\omega)$ is a connected set containing ω, then $I(\omega) \subseteq Y$. In particular for any $c \in (\gamma - \eta_0, \gamma)$ we have that $I(\omega)_c \subseteq Y_c$ and hence $I(\omega)_c$ consists of just one oval.

More precisely $Y \cap S(Q,\epsilon) = \omega$, because if there exists a point $R \in (Y \cap S(Q,\epsilon)) \backslash \omega$ and σ is the oval of $C(Q,\epsilon)$ containing R, then Y contains $I(\sigma) \cap \{z <$

$\gamma\}$. Then for any $c \in (\gamma - \eta_0, \gamma)$ the set $Y_c \setminus I(\omega)_c$ contains points in $I(\sigma)_c$, while Y_c is connected.

Consider the connected component W of $E(\omega)_{[a,\gamma)}$ containing ω. Since $I(\omega) \cup W$ is a connected set containing ω, then $I(\omega) \cup W \subseteq Y$. It follows that $W \cap S(Q, \epsilon) = \omega$ and hence $Y = I(\omega) \cup W$.

Since $\overline{Y} \cap \{z = \gamma\} = \{Q\}$, then $\overline{W} \cap \{z = \gamma\} = \emptyset$ so that W coincides with the connected component of $S_{[a,b]} \setminus B(Q, \epsilon)$ containing ω, i.e. $W = E(\omega)$.

As a consequence $E(\omega) \cap \{z = a\} = Y \cap \{z = a\} = \omega'$ and the boundary of $E(\omega)$ is equal to $\omega \cup \omega'$. Moreover $\overline{Y} = Y \cup \{Q\}$ is topologically a disk; also $\overline{I(\omega)} = I(\omega) \cup \{Q\}$ is topologically a disk; hence $E(\omega) = \overline{Y} \setminus \overline{I(\omega)}$ is a connected cylinder. □

In a similar way one can prove that:

Proposition 3.14. *Let ω be an oval of $C(Q, \epsilon)$ of type $(+)$. Then*

1. *there exists $\eta_0 > 0$ such that for all positive $\eta \leq \eta_0$ we have that $\overline{I(\omega)} \cap \{z = \gamma - \eta\} = \emptyset$;*
2. *$E(\omega) \cap \{z = b\}$ consists of a single oval ω', the boundary of $E(\omega)$ is equal to $\omega \cup \omega'$ (in particular $E(\omega) \cap \{z = a\} = \emptyset$) and $E(\omega)$ is homeomorphic to a connected cylinder.*

For the ovals of type $(+-)$ the situation is more complicated (see for instance Figure 7 and Figure 6). Nevertheless the next proposition guarantees that also for any oval ω of $C(Q, \epsilon)$ of type $(+-)$ the nature of the vanishing manifold of $\overline{I(\omega)}$ can be derived from information concerning the algebraic curve $C(Q, \epsilon)$.

Proposition 3.15. *Let ω be an oval of $C(Q, \epsilon)$ of type $(+-)$ and assume that $\omega \cap \{z \leq \gamma\}$ is the union of k arcs. Then there exists $\eta_0 > 0$ such that, for all positive $\eta \leq \eta_0$, $\overline{I(\omega)} \cap \{z = \gamma - \eta\}$ is also the union of k arcs (in particular, in order to count the number of arcs in the vanishing manifold of $\overline{I(\omega)}$, it suffices to count the number of points in ω on the equator $Eq(Q, \epsilon)$).*

Proof. Since ω is transversal to $\{z = \gamma\}$, there exists $\eta_0 > 0$ such that $\omega \cap \{z \leq \gamma - \eta\}$ is the union of k arcs for all positive $\eta \leq \eta_0$. Furthermore, for $\eta \leq \eta_0$ by a transversality argument $I(\omega) \cap \{z = \gamma - \eta\}$ is a smooth curve with a boundary equal to $\omega \cap \{z = \gamma - \eta\}$ and hence it contains exactly k arcs .

We have to prove that this curve cannot contain any oval. Suppose that there exists $\eta' \leq \eta_0$ such that $I(\omega) \cap \{z = \gamma - \eta'\}$ contains an oval σ. Since $\overline{I(\omega)}$ is homeomorphic to a disk, then σ disconnects $\overline{I(\omega)}$. Let $I(\omega)'$ be the connected component of $\overline{I(\omega)} \setminus \sigma$ not containing ω. It is clear that $I(\omega)' \cap S(Q, \epsilon) = \emptyset$.

Let $m = \min_{\overline{I(\omega)'}} z$ and $M = \max_{\overline{I(\omega)'}} z$. Since the function $p(x, y, z) = z$ cannot be constant on $\overline{I(\omega)'}$, then $M \neq m$, and since z is constant on the boundary of $\overline{I(\omega)'}$ then at least one of them must be achieved in an interior point R of $\overline{I(\omega)'}$.

If $Q \notin I(\omega)'$, such a point R should be a critical point for the function p on S different from Q, while no such point exists in the strip we are working in.

Suppose instead that $Q \in I(\omega)'$. If the maximum M were achieved in a point different from Q, we would have a contradiction as before. So we can suppose that $I(\omega)' \setminus \{Q\} \subseteq \{z < \gamma\}$. The set $I(\omega)'$ is open in $\overline{I(\omega)}$, hence for ϵ' small enough $D(Q,\epsilon') \cap \overline{I(\omega)} \subseteq I(\omega)'$. Then $D(Q,\epsilon') \cap \overline{I(\omega)} = D(Q,\epsilon') \cap I(\omega)'$ and thus $S(Q,\epsilon') \cap \overline{I(\omega)} = S(Q,\epsilon') \cap I(\omega)'$. Since $I(\omega)' \setminus \{Q\} \subseteq \{z < \gamma\}$, we have $S(Q,\epsilon') \cap \overline{I(\omega)} \subseteq S(Q,\epsilon') \cap \{z < \gamma\}$ which means that $\omega(\epsilon') = I(\omega) \cap S(Q,\epsilon')$ should be an oval of type $(-)$, contradicting Proposition 3.12. □

As a consequence of all the previous results we obtain a complete description of the possible types of connected components of $T_{[a,b]}$ and of the Euler characteristic of each of them:

Theorem 3.16. *Let W be a connected component of $T_{[a,b]}$. Then*

1. *if $W \cap C(Q,\epsilon) = \emptyset$, then W is a cylinder intersecting both $\{z = a\}$ and $\{z = b\}$ in a single oval, and $\chi(W) = 0$,*
   (in this case we say that W is a component of $T_{[a,b]}$ of type Cylinder)
2. *if $W \cap C(Q,\epsilon)$ contains an oval ω of type $(-)$, then $W \cap C(Q,\epsilon) = \omega$; moreover W intersects $\{z = a\}$ in a single oval, does not intersect $\{z = b\}$ and $\chi(W) = 1$,*
   (in this case we say that W is a component of $T_{[a,b]}$ of type $(-)$)
3. *if $W \cap C(Q,\epsilon)$ contains an oval ω of type $(+)$, then $W \cap C(Q,\epsilon) = \omega$; moreover W intersects $\{z = b\}$ in a single oval, does not intersect $\{z = a\}$ and $\chi(W) = 1$,*
   (in this case we say that W is a component of $T_{[a,b]}$ of type $(+)$)
4. *if $W \cap C(Q,\epsilon)$ contains an oval ω of type $(+-)$, then all the other ovals of $W \cap C(Q,\epsilon)$, say $\omega_1 = \omega, \omega_2, \dots, \omega_q$, are of type $(+-)$; if k_i denotes the number of arcs of $\omega_i \cap \{z \le \gamma\}$ (also called the negative arcs of ω_i), then $\chi(W) = \sum_{i=1}^{q}(1 - k_i)$; moreover W intersects both $\{z = a\}$ and $\{z = b\}$ in at least one oval,*
   (in this case we say that W is a component of $T_{[a,b]}$ of type $(+-)$)

Proof. (1) follows from the fact that the restriction of p to W has no critical points.

(2) (resp. (3)) W is homeomorphic to $E(\omega) \cup \overline{I(\omega)}$, where $E(\omega)$ is a topological cylinder by Proposition 3.13 (resp. Proposition 3.14) and $\overline{I(\omega)}$ is a 2-cell. Since $E(\omega) \cap \overline{I(\omega)} = \omega$, by Lemma 3.7 we have $\chi(W) = \chi(E(\omega)) + \chi(\overline{I(\omega)}) = 1$.

(4) The exterior part $E(\omega)$ cannot intersect $C(Q,\epsilon)$ in ovals different from ω of type $(+)$ or $(-)$ by Propositions 3.13 and 3.14. Let $\widehat{W} = (W \setminus B(Q,\epsilon)) \cup \overline{I(\omega_1)} \cup \ldots \cup \overline{I(\omega_q)}$. By Theorem 3.5 we get that $\chi(\widehat{W}) = 1 - \sum_{i=1}^{q} k_i$. Since $\widehat{W} \cap D(Q,\epsilon)$ is contractible, then $\chi(\widehat{W} \cap D(Q,\epsilon)) = 1$. Thus $\chi(W \setminus B(Q,\epsilon)) = \chi(\widehat{W} \setminus B(Q,\epsilon)) = (1 - \sum_{i=1}^{q} k_i) - 1 = -\sum_{i=1}^{q} k_i$. Since W is obtained attaching q 2-cells to $W \setminus B(Q,\epsilon)$, we have that $\chi(W) = \chi(W \setminus B(Q,\epsilon)) + q = -\sum_{i=1}^{q} k_i + q = \sum_{i=1}^{q}(1 - k_i)$. □

By Proposition 3.13 and Proposition 3.14 there is a $1-1$ correspondence between the ovals ω of $C(Q,\epsilon)$ of type $(-)$ or $(+)$ and the connected components of $T_{[a,b]}$ containing them, which is not true for ovals of type $(+-)$. Moreover:

1. if ω is of type $(+)$, attaching a 2-cell along ω to $S \setminus B(Q,\epsilon)$, there appears in $T_{[a,b]}$ (and in $T_{\leq b}$) a new connected component homeomorphic to a disk and with boundary a single oval $\omega' \subset C_b$;
2. if ω is of type $(-)$, the attachment of a 2-cell along ω is topologically equivalent to the attachment of a 2-cell along an oval ω' of S_a; thus it contributes increasing by 1 the Euler characteristic of the connected component Y' of $T_{\leq a}$ containing ω'. Note that the connected component of $T_{\leq b}$ containing Y' may intersect $S(Q,\epsilon)$ also in other ovals of type $(-)$ or $(+-)$, thus its topology passing from level a to level b will be simultaneously influenced by the attachment of 2-cells along all these ovals.

Therefore, in order to reconstruct correctly $\chi(T_{\leq b})$, it is necessary

• to recognize the connected components of $T_{[a,b]}$ on the basis of the characterization given by Theorem 3.16,

• to determine all the attachment relations among the connected components of $T_{\leq a}$ and the connected components of $T_{[a,b]}$,

• to compute the Euler characteristic of any connected component Y of $T_{\leq b}$ using Proposition 3.8 and the Euler characteristic of the components of $T_{\leq a}$ contained in Y.

We will see in Section 4 how all these tasks can be performed effectively.

3.2. Computation of $G(T_{\leq b})$.

The ability to determine the attachment relations among the connected components of $T_{\leq a}$ and the connected components of $T_{[a,b]}$ is also important for computing $G(T_{\leq b}) = G(\{z \leq b\}, T_{\leq b})$ starting from $G(T_{\leq a})$.

Recall that we can choose as a root in the tree $G(T_{\leq a})$ the vertex corresponding to the unique unbounded region of $\{z \leq a\} \setminus T_{\leq a}$. This choice makes $G(T_{\leq a})$ an ordered tree; in particular if v_1, v_2 are the two vertices of an edge e and v_1 is the vertex which is nearer to the root, we can call v_1 the *first vertex* of e and v_2 its *second vertex*. We can apply to $G(T_{\leq a})$ the following terminology:

Definition 3.17. *Let G be a rooted tree, r its root and let e be an edge of G.*

(i) Unless r is a vertex of e, we call parent *of e (denoted* **parent**(e)*) the unique edge of G having as second vertex the first vertex of e.*

(ii) We call children *of e (denoted* **children**(e)*) the set of all the edges of G having as first vertex the second vertex of e.*

Let Y be a connected component of $T_{\leq b}$ such that $Y \cap \{z \leq a\} = \emptyset$; then Y coincides with a connected component of $T_{[a,b]}$ of type $(+)$ and therefore we will call it a component of $T_{\leq b}$ of type $(+)$. Let $\beta = Y \cap \{z = b\}$. Each other connected component of $T_{\leq b}$ which meets $\{z = b\}$ in an oval contained in β is necessarily of type $(+)$. Therefore removing from $G(T_{\leq b})$ all the edges corresponding to the connected components of $T_{\leq b}$ of type $(+)$ yields a connected tree $G'(T_{\leq b})$. We will compute $G(T_{\leq b})$ by first computing $G'(T_{\leq b})$ and then glueing to it the edges corresponding to the connected components of type $(+)$.

If Y is a connected component of $T_{\leq b}$ that meets $\{z = a\}$, and we regard it as $Y = (Y \cap T_{\leq a}) \cup (Y \cap T_{[a,b]})$, then all the distinct edges of $G(T_{\leq a})$ corresponding to the connected components of $Y \cap T_{\leq a}$ glue into a single edge in $G(T_{\leq b})$. If we have recognized which edges of $G(T_{\leq a})$ glue, we can compute the subgraph $G'(T_{\leq b})$ as a quotient of $G(T_{\leq a})$ by means of the equivalence relation given by the edge glueing.

However observe that two connected components of $T_{\leq a}$ that glue in $T_{\leq b}$ do not necessarily lie in the boundary of one region of $\{z \leq a\} \setminus T_{\leq a}$ (see Figure 7). So the corresponding two edges in $G(T_{\leq a})$ glue into a single edge of $G(T_{\leq b})$ but they do not necessarily have a vertex in common.

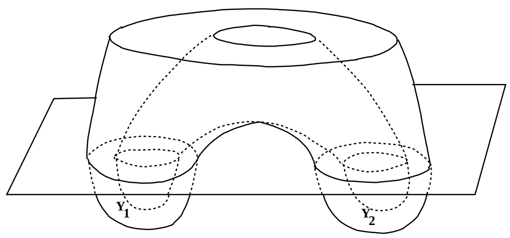

Figure 7: *Example of two connected components Y_1, Y_2 of $T_{\leq a}$ which lie in the boundaries of different regions of $\{z \leq a\} \setminus T_{\leq a}$ but which glue in $T_{\leq b}$.*

In general identifying two edges without a common vertex is ambiguous, since it is not clear how the two pairs of vertices have to be identified. Though, using the fact that $G'(T_{\leq b})$ is a tree, we can see that $G'(T_{\leq b})$ can be recursively obtained from $G(T_{\leq a})$ by applying finitely many identifications between pairs of edges sharing a common vertex:

Definition 3.18. *A fold is a graph transformation which identifies 2 vertices of distance 2 and the path between them with a single edge, i.e. the 2 edges along the path are folded at the common vertex.*

Lemma 3.19. *Applying a fold to a tree produces a tree.*

Proof. Recall that a connected graph is a tree if and only if the number of vertices is one more than the number of edges. The graph obtained applying a fold to a tree is obviously connected. Moreover a fold reduces by one both the number of vertices and edges, so the number of vertices remains one more than the number of edges; hence the resulting graph is a tree. □

Proposition 3.20. *Any reduction of a tree modulo an equivalence relation on edges and vertices which produces a tree can be achieved by applying an arbitrary sequence of folds compatible with the equivalence relation until there are no more folds possible.*

Proof. Since the result of applying a maximum compatible sequence of folds is a tree, we can assume we begin with the folded tree and a reduced equivalence relation such that there are no remaining compatible folds. We want to show that the equivalence relation is trivial. If not let $v_1 \neq v_2$ be two equivalent vertices of minimum distance. Since there are no possible folds, their distance is greater than 2. Let w_1 and w_2 be the first and last vertices encountered on the unique path from v_1 to v_2. Since the length of the path is greater than 2, w_1 and w_2 are distinct. By minimality of the path between v_1 and v_2 there are no pairs of nodes along the path from v_1 to v_2 which are equivalent, in particular w_1 and w_2 are inequivalent and there are no nodes along the path from w_1 to w_2 which are equivalent to v_1. If we now identify v_1 and v_2, then we obtain two distinct paths from w_1 to w_2 which contradicts the hypothesis that the result is a tree, thus the resulting equivalence relation is trivial. □

The previous proposition will be used in a situation where we have determined the equivalence relation only on the edges (components). However the fact that the associated equivalence relation on the vertices exists (although we do not compute it) and the fact that the merging can be performed by a sequence of folds, is enough to uniquely determine the merged graph.

After computing $G'(T_{\leq b})$ we need to attach to it the edges corresponding to the connected components of $T_{\leq b}$ of type $(+)$ (i.e. the "new" components of $T_{\leq b}$).

Let β_1 be an oval of S_b which bounds a connected component Y_1 of $T_{\leq b}$ of type $(+)$ and maximal with respect to this property. We want to update $G'(T_{\leq b})$ by attaching to it the edge Y_1 in a vertex v that we need to detect.

There are two possible cases:

(i) β_1 is contained in the interior part of an oval β of S_b :
in this case β bounds a connected component of $T_{\leq b}$ (necessarily not of type $(+)$) that we have already reconstructed as an edge e_β of $G'(T_{\leq b})$; then we attach an edge (corresponding to Y_1) to the second vertex v of e_β.

(ii) β_1 is not contained in the interior part of any oval of S_b:
in this case the exterior part of β_1 is unbounded and lies in the boundary of the unbounded region of $\{z \leq b\} \setminus T_{\leq b}$; then we attach an edge (corresponding to Y_1) to the root of $G'(T_{\leq b})$ (which is the required vertex v).

If $G_{\beta_1}(S_b)$ is the subtree of $G(S_b)$ consisting of the ovals contained in β_1, then the connected components of $T_{[a,b]}$ bounded by these ovals form a subtree isomorphic to $G_{\beta_1}(S_b)$. Hence $G'(T_{\leq b})$ can be updated by attaching to it a tree isomorphic to $G_{\beta_1}(S_b)$ in the vertex v detected before.

3.3. Computation of $Rel(T_{\leq b})$.

Assume that $Q = Q_i$, i.e. it is the i-th point in the list of the critical points $Q_1, \ldots, Q_m$.

Recall that, if $Y^1, \ldots, Y^g$ are the connected components of $T_{\leq b}$, then (see Remark 3.4 (2)) $Rel(T_{\leq b}) = [rel(Y^1), \ldots, rel(Y^g)]$, where, for any $j = 1, \ldots, g$, $rel(Y^j)$ is a vector of length m whose i-th element $rel(Y^j)(i)$ is the number of ovals in which Y^j meets the Milnor sphere $S(Q_i, \epsilon)$ centered at Q_i.

If Y is a connected component of $T_{\leq b}$ such that $Y \cap \{z \leq a\} = \emptyset$, then Y is a connected component of $T_{[a,b]}$ of type $(+)$, it meets $S(Q_i, \epsilon)$ in one oval (see Theorem 3.16) and so $rel(Y)$ is the vector whose elements are zero except $rel(Y)(i) = 1$.

If Y is a connected component of $T_{\leq b}$ such that $Y \cap \{z \leq a\} \neq \emptyset$ and $Y \cap S(Q_i, \epsilon) = \emptyset$, then $rel(Y)$ coincides with the vector $rel(Y \cap T_{\leq a})$ as appearing in $Rel(T_{\leq a})$.

It remains to consider the case when $Y \cap \{z \leq a\} \neq \emptyset$ and $Y \cap S(Q_i, \epsilon) \neq \emptyset$. In this case Y intersects $S(Q_i, \epsilon)$ in some ovals, say n_Y, of type either $(+-)$ or $(-)$, that we detected during the reconstruction process described above. If $T_a^1, \ldots, T_a^c$ are the connected components of $T_{\leq a}$ contained in Y, then $rel(Y) = rel(T_a^1) + \ldots + rel(T_a^c) + n_Y e_i$, where e_i is the vector of length m whose i-th element is 1 and all the other elements are zero.

If Q is an isolated point, since $T \cap S(Q, \epsilon) = \emptyset$, then $Rel(T_{\leq b}) = Rel(T_{\leq a})$.

3.4. Computation of $IsReg(T_{\leq b})$.

We compute the list $IsReg(T)$ by recursively computing the list

$$IsReg(T_{\leq b}) = [isReg(\Omega_1), \ldots, isReg(\Omega_h)]$$

where $\Omega_1, \ldots, \Omega_h$ are the regions of $\{z \leq b\} \setminus T_{\leq b}$ and $isReg(\Omega_i)$ is the number of isolated points contained in Ω_i. There are two possible situations:

Case 1: the strip $\mathbb{R}^2 \times (a, b)$ does not contain any isolated point.

In this case, for any region Ω of $\{z \leq b\} \setminus T_{\leq b}$, if $\Omega \cap \{z \leq a\} = \Sigma_1 \cup \ldots \cup \Sigma_j$, we recover from the list $IsReg(T_{\leq a})$ the elements $isReg(\Sigma_1), \ldots, isReg(\Sigma_j)$ and we insert in $IsReg(T_{\leq b})$ the element $isReg(\Omega) = isReg(\Sigma_1) + \ldots + isReg(\Sigma_j)$.

Case 2: the strip $\mathbb{R}^2 \times (a, b)$ contains an isolated point $Q = (\alpha, \beta, \gamma)$ (which is revealed by the fact that $S(Q, \epsilon) \cap S = \emptyset$).

In this case, denote by Ω the region of $\{z \leq b\} \setminus T_{\leq b}$ containing Q. Since $\Omega \cap \{a \leq z \leq b\}$ is a cylinder, we can detect it by means of the plane horizontal section S_γ passing through Q having equation $f(x, y, \gamma) = 0$. If Q is contained in the unbounded region of $\{z = \gamma\} \setminus S_\gamma$, then it is contained in the unbounded region of $\{z \leq b\} \setminus T_{\leq b}$. Otherwise, after choosing a point M on the innermost oval of S_γ containing Q in its interior part, it suffices to lift M up to the level $z = b$, to recognize $\Omega \cap \{z = b\}$ and hence Ω. The techniques to do that will be described in detail in the next section. After detecting Ω, if $\Omega \cap \{z \leq a\} = \Sigma$, we insert in $IsReg(T_{\leq b})$ the element $isReg(\Omega) = isReg(\Sigma) + 1$.

4. The algorithm in the compact affine case

In this section we describe a constructive procedure to compute the list of data $D(S) = [\chi(T), G(T), Contr(T), Roots(T), Rel(T), IsReg(T)]$ when the real algebraic surface $S \subset \mathbb{R}^3$ is compact, defined by a square-free polynomial equation $f(x, y, z) = 0$. We assume that:

1. S has finitely many critical points $Q_1, \ldots, Q_m$ w.r.t. the projection $p \colon S \to \mathbb{R}$, $p(x, y, z) = z$ (and in particular finitely many singular points); let $Crit(S)$ denote the critical locus,

2. if Q_i and Q_j are distinct critical points and $i < j$ then $p(Q_i) < p(Q_j)$,
3. $m+1$ rational numbers $a_1, \ldots, a_{m+1}$ are chosen such that each a_i is not a critical value and Q_i is the only critical point such that $p(Q_i) \in (a_i, a_{i+1})$
4. for each critical point Q_i, a real $\epsilon_i > 0$ is known, which is a Milnor radius at Q_i both for the surface and for the plane curve obtained intersecting S with the horizontal plane through Q_i; if $Q_i \in \mathbb{R}^2 \times (a_i, a_{i+1})$, we require also that $D(Q_i, \epsilon_i) \subset \mathbb{R}^2 \times (a_i, a_{i+1})$.

Since S has finitely many singular points, up to a generic linear change of coordinates, conditions (1) and (2) are satisfied as shown in Section 6, where a method to compute a Milnor radius ϵ for the critical point Q will also be described.

In order to compute the data $D(S)$ we construct a rooted tree, that, by abuse of notation, we call again $G(T)$, with edges representing connected components and vertices representing regions, labelled with all the required information. To describe the edges of $G(T)$ we introduce the following data type:
compRegion=[**ccComp**, **position**], where:

1. **ccComp** $= [points, \chi, rel]$ is a record representing a connected component cc with:
   - $points(cc)$ a set of points on cc
   - $\chi(cc)$ the Euler characteristic of cc
   - $rel(cc)$ a vector of length $m = \#Crit(S)$, such that $rel(cc)(i)$= number of ovals in $cc \cap S(Q_i, \epsilon_i)$
2. **position** $= [isolPts, parent, children]$ is a record where *parent* and *children* are respectively the parent of cc and the children of cc, and $isolPts$ are the isolated points lying in the region of $\mathbb{R}^3 \setminus T$ corresponding to the common vertex of cc and $children(cc)$, (i.e. the second vertex of cc).

In this representation the first field **ccComp** contains all the information relative to the connected components of T, while the second field **position** contains the data necessary both to recover the structure of the tree (i.e. the relation of an edge with the other edges of the tree) and to recognize the isolated points in the associated region. We omit $Contr(T)$ since in this case it is completely determined by the knowledge of the root (see Section 3). Notice that the vertices (i.e. the regions of $\mathbb{R}^3 \setminus T$) are only implicitly represented. Finally we insert a conventional element $rootG := [\textbf{null}, [isolPts, \textbf{null}, children]]$, with **ccComp** = **null** and $parent$ = **null** so that its second vertex represents the unbounded region of $\mathbb{R}^3 \setminus T$.

For simplicity of notation, let $a = a_i$ and $b = a_{i+1}$. As described in the previous section, the algorithm iteratively constructs $G(T)$ merging the information (already) computed for $T_{\leq a}$ and the new information for $T_{[a,b]}$ in order to determine $T_{\leq b}$, where $\mathbb{R}^2 \times [a, b]$ contains exactly one critical point Q of S. This construction is done with the functions **ccStrip**, **lift** and **mergeComponents**.

As described in Theorem 3.16 the topological structure of the connected components of $T_{[a,b]}$ are determined by their intersections with the planes $\{z = a\}$, $\{z = b\}$ and with the equator $Eq(Q, \epsilon)$ of the Milnor sphere of radius ϵ centered at Q. If we know $\mathcal{A} \subset S \cap \{z = a\}$, $\mathcal{B} \subset S \cap \{z = b\}$ and $\mathcal{E} \subset S \cap Eq(Q, \epsilon)$ finite sets of points (containing at least one point in each oval of the respective curves), we distinguish the connected components of $T_{[a,b]}$ partitioning the set $\mathcal{A} \cup \mathcal{B} \cup \mathcal{E}$ into maximal subsets of points belonging to the same connected component. This action is performed by applying a function **connComp** which, given a semialgebraic set SAL and a finite set of points $\mathcal{P}ts \subset SAL$, computes a partition of $\mathcal{P}ts$, say **connComp**$(SAL, \mathcal{P}ts) = [Z_1, \ldots, Z_h]$, having the property that, for any points $P_i, P_j \in Pts$ there exists $k \in \{1, \ldots, h\}$ such that $P_i, P_j \in Z_k$ if and only if P_i and P_j belong to the same connected component of SAL. A possible description of **connComp** is given is Section 6.

In order to compute $\mathcal{A}$ and $\mathcal{B}$ we consider the affine compact non-singular algebraic curves S_a and S_b. All their connected components are ovals whose containment relations are described by rooted adjacency graphs $G(S_a)$ and $G(S_b)$, where the root corresponds to the unbounded region of $\{z = a\} \setminus S_a$, (resp.$\{z = b\} \setminus S_b$) and each edge represents an oval. It is possible to study these curves using an algorithm that returns their adjacency graphs (see Section 6 and, for additional references, [2]). More precisely we assume to have a function **planeGraph**(g), which, given a polynomial $g(x, y) \in \mathbb{Q}[x, y]$ such that $V(g)$ is compact and non-singular, returns a description of the topology of $(\mathbb{R}^2, \{g = 0\})$ as a list of **plane-oval** $=[points, parent, children]$, a record where:

1. $points(\omega)$ is a non-empty list of points on the oval represented by ω
2. $parent(\omega)$ is the oval ω' such that $\omega' \supsetneq \omega$ and ω' is contained in all the ovals properly containing ω or **null** if no oval contains properly ω
3. $children(\omega)$ is the list of the set of ovals having ω as parent.

and a function **findOval**(g, Q) which finds a point on the innermost oval of $V(g)$ containing Q in its interior or **null**.

Regarding the set $\mathcal{E}$, it is a subset of $S \cap Eq(Q, \epsilon)$ containing exactly one point for each oval of $C(Q, \epsilon)$ intersecting the equator. It can be computed applying the function **connComp** to the semialgebraic set $\{f = 0, \|X - Q\|^2 = \epsilon^2\}$ and to the set $\mathcal{P}ts = S \cap Eq(Q, \epsilon)$ and then choosing a single element in each set of the partition. .

We describe now the function **ccStrip**, which identifies the connected components of T in a strip $\mathbb{R}^2 \times [a, b]$. In it and in the following algorithms, $\mathbf{e}_i \in \mathbb{N}^m$ denotes the unit vector of length $m = \#Crit(S)$ with 1 in the i-th position.

Algorithm ccStrip$(f, \mathcal{A}, \mathcal{B}, i, Q, a, b)$
where

- $f \in \mathbb{Q}[x, y, z]$
- $\mathcal{A}$ is a set of points on S_a and $\mathcal{B}$ is a set of points on S_b (at least one for each oval)
- $i \in \{1, ..., m\}$
- $Q = (\alpha, \beta, \gamma) = Crit(S)(i)$ is the i-th element of $Crit(S)$
- $a, b \in \mathbb{Q}$ such that $a < \gamma < b$

Output: a list of **ccComp**=[$points, \chi, rel$], each element representing a connected component of $T_{[a,b]}$.

Begin

– compute the Milnor radius at Q
$\epsilon :=$ **milnorRadius**(f, Q, a, b)
– analysis of the Milnor sphere; find the points on $Eq(Q, \epsilon)$
$EqPts := \mathbf{V}(f, (x-\alpha)^2 + (y-\beta)^2 - \epsilon^2, z - \gamma)$
if $\#EqPts > 0$ then
  – partition points on the equator into subsets of points
  – belonging to the same oval
  $\{Eq_1, .., Eq_h\} :=$ **connComp**$(\{f = 0, \|X - Q\|^2 = \epsilon^2\}, EqPts)$
  $\mathcal{E} := \{\mathbf{first}(Eq_1), ..., \mathbf{first}(Eq_h)\}$
  $numArcs := [\#Eq_i/2$ for i in $1..h]$
else $\mathcal{E} := \emptyset$
– partition the points of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{E}$ into subsets of points
– belonging to the same connected component of $T_{[a,b]}$
$\mathcal{Z}:=$ **connComp**$(\{f = 0, \|X - Q\|^2 \geq \epsilon^2, a \leq z \leq b\}, \mathcal{A} \cup \mathcal{B} \cup \mathcal{E})$
– attach to any element of $\mathcal{Z}$ its Euler characteristic and its rel value
$ccSt :=$ emptyList
for ζ in $\mathcal{Z}$ repeat
  $[E_1, .., E_s] := \zeta \cap \mathcal{E}$
  if $s > 0$ then
    – the component is of type $(+-)$
    $\chi := \sum_{i=1}^{s}(1 - k_i)$ where $k_i = numArcs(position(E_i, \mathcal{E}))$
    $new\_cc := [\zeta \cap (\mathcal{A} \cup \mathcal{B}), \chi, s\mathbf{e}_i]$
  if $s = 0$ then
    – components without intersection with the equator
    $(na, nb) := (\#(\zeta \cap \mathcal{A}), \#(\zeta \cap \mathcal{B}))$
    – component of type Cylinder
    if $(na, nb) = (1, 1)$ then $new\_cc := [\zeta, 0, 0\mathbf{e}_i]$
    – component of type $(-)$
    if $(na, nb) = (1, 0)$ then $new\_cc := [\zeta, 1, \mathbf{e}_i]$
    – component of type $(+)$
    if $(na, nb) = (0, 1)$ then $new\_cc := [\zeta, 1, \mathbf{e}_i]$
  $ccSt :=$ **insert**$(new\_cc, ccSt)$
return $ccSt$

End

After computing the connected components of $T_{[a,b]}$ we need to determine their attachment relations with the connected components of $T_{\leq a}$ in order to "lift" all the information to level $z = b$. As we already remarked, different components of $T_{[a,b]}$ intersecting the plane $\{z = a\}$ may belong to the same connected component of $T_{\leq b}$. The identification of these components and the computation of the corresponding values of χ and rel is done in the function **lift**, applying **mergeComponents**.

At first we limit our attention to the components cc_{st} of $T_{[a,b]}$ that intersect the plane $\{z = a\}$, postponing the analysis of the components of type (+).

If cc_{st} intersects $\{z = a\}$ in precisely one oval belonging to a component cc_a of $T_{\leq a}$, then cc_{st} is connected to cc_a and we add the points of cc_{st} at level $z = b$ (maybe none, if cc_{st} is of type (−)) to the list of points identifying cc_a and we accordingly update the fields χ and rel.

If cc_{st} intersects $\{z = a\}$ in several ovals, possibly contained in different components of $T_{\leq a}$, (i.e. cc_{st} is of type (+−)), we add the points of cc_{st} to all the components of $T_{\leq a}$ connected to cc_{st} but we only modify the fields χ and rel of one of them. This corresponds to attaching the component in the strip to only one of the components of $T_{\leq a}$ it meets at level $z = a$.

In this way all the edges of the original tree $G(T_{\leq a})$ intersecting components of $T_{[a,b]}$ of type Cylinder or of type (+−) contain points both on $\{z = a\}$ and on $\{z = b\}$ and the edges of $G(T_{\leq a})$ that are contained in a single connected component of $G(T_{\leq b})$ can be detected being precisely the edges sharing a common point on $\{z = b\}$. The merging of these components into a unique edge of $G(T_{\leq b})$ is done by **mergeComponents**.

Before returning the final tree $G(T_{\leq b})$, **lift** analyzes the components of $T_{[a,b]}$ that do not intersect the plane $\{z = a\}$ (i.e. the components of type (+)). In this case new elements need to be inserted in the tree; the information on the ovals of S_b and their mutual positions are used to determine the *parent* and the *children* of these new edges. This identification is done processing the inclusion relations of the ovals of S_b starting from the outermost oval of each nest.

Algorithm lift$(f, a, b, \mathcal{G}, ccS, curveG_b)$
where

- $f \in \mathbb{Q}[x, y, z]$
- $a, b \in \mathbb{Q}$, with $a < b$
- $\mathcal{G}$ is the list of **compRegion** of $G(T_{\leq a})$
- ccS is the list of **ccComp** of $T_{[a,b]}$
- $curveG_b$ is the (rooted) tree of the plane curve $\{f(x, y, b) = 0\}$

Output: the list of **compRegion** of $G(T_{\leq b})$

Begin

$ccPlus := \emptyset$
– iterative step
for cc_{st} in ccS repeat
    – select the components of $T_{\leq a}$ which intersect cc_{st}
    $cc_a := [g \ \ in \ \ \mathcal{G} \mid points(\mathbf{ccComp}(g)) \cap points(cc_{st}) \neq \emptyset]$
    if $(\#cc_a) > 0$ then
        – cc_{st} is not of type (+), temporarily we collect all the points,
        – update χ and the number of intersections

– with the Milnor sphere
for cc in cc_a repeat
$points(\textbf{ccComp}(cc)) := points(\textbf{ccComp}(cc)) \cup points(cc_{st})$
$cc_1 := \textbf{ccComp}(\textbf{first}(cc_a))$
$\chi(cc_1) := \chi(cc_1) + \chi(cc_{st})$
$rel(cc_1) := rel(cc_1) + rel(cc_{st})$
else
– collect the components of type (+) (to be processed later)
$ccPlus :=$**insert**$(cc_{st}, ccPlus)$
– identify the components of $T_{[a,b]}$ which are contained in the same
– connected component of $T_{\leq b}$
$\mathcal{G} :=$ **mergeComponents**$(\mathcal{G})$
– consider the components of $T_{[a,b]}$ of type (+)
– the entries of $curveG_b$ are sorted with respect to containment
– partial order
for ov *in* $curveG_b$ repeat
for cc_1 in $ccPlus$ repeat
if $points(ov) \cap points(cc_1) \neq \emptyset$ then
– insert the new component in the tree
if $parent(ov) =$ **null** then $new\_parent :=$ **root**$(\mathcal{G})$
else $new\_parent :=$
find$(cc \in \mathcal{G} \mid points(\textbf{ccComp}(cc)) \ni points(parent(ov)))$
$new\_cc := cc_1$
$children(\textbf{position}(new\_parent)) :=$
insert$(new\_cc, children(\textbf{position}(new\_parent)))$
$\mathcal{G} :=$ **insert**$(new\_cc, \mathcal{G})$
return $\mathcal{G}$

End

In the previous function we called **mergeComponents** to complete the construction of the connected components of $T_{\leq b}$ in the case when components of $T_{[a,b]}$ of type $(+-)$ attach to multiple components of $T_{\leq a}$. As already remarked, the components to be identified correspond to edges sharing a common point on $\{z = b\}$. We perform a sequence of foldings in the adjacency graph looking for pairs of edges having a common point and a common vertex. We identify them and update accordingly all the data on the new merged edge. This procedure is then repeated on the updated graph. According to Proposition 3.20, eventually any pair of edges that must be identified will have a common point and a common vertex, so that the algorithm will be able to identify them.

Algorithm mergeComponents$(\mathcal{G})$
where $\mathcal{G}$ is a list of **compRegion**.

Output: updated $\mathcal{G}$

Begin

– search for edges with a common vertex and with points in common
for (cr_1, cr_2) in $\mathcal{G} \times \mathcal{G} \mid cr_1 \neq cr_2$ repeat
    $c_1 :=$ **ccComp**(cr_1)
    $c_2 :=$ **ccComp**(cr_2)
    if $points(c_1) \cap points(c_2) \neq \emptyset$ then
        $r_1 :=$ **position**(cr_1)
        $r_2 :=$ **position**(cr_2)
        if $cr_1 = parent(r_2)$ then
            – the first edge is the parent of the other one
            $points(c_1) := points(c_1) \cup points(c_2)$
            $\chi(c_1) := \chi(c_1) + \chi(c_2)$
            $rel(c_1) := rel(c_1) + rel(c_2)$
            $children(r_1) :=$ **remove**$(cr_2, children(r_1))$
            $reg\_par :=$ **position**$(parent(r_1))$
            $children(reg\_par) := children(r_2) \cup children(reg\_par)$
            $isolPts(reg\_par) := isolPts(r_2) \cup isolPts(reg\_par)$
            return **mergeComponents** (**remove**$(cr_2, \mathcal{G})$)

        if $parent(r_1) = parent(r_2)$ then
            – the edges have a common parent
            $points(c_1) := points(c_1) \cup points(c_2)$
            $\chi(c_1) := \chi(c_1) + \chi(c_2)$
            $Rel(c_1) := Rel(c_1) + Rel(c_2)$
            $children(r_1) := children(r_1) \cup children(r_2)$
            $isolPts(r_1) := isolPts(r_1) \cup isolPts(r_2)$
            $reg\_par :=$ **position**$(parent(r_1))$
            $children(reg\_par) :=$ **remove**$(cr_2, children(reg\_par))$
            return **mergeComponents** (**remove**$(cr_2, \mathcal{G})$)
return $\mathcal{G}$

End

The last operation to describe is how to deal with the isolated singular points. This is done with the function **isolatedPt** which is used when the critical point $Q = (\alpha, \beta, \gamma)$ is an isolated point. In this case, as seen in Section 3, essentially we only need to find the region to which Q belongs, which is determined by the position of the oval containing Q in the graph of the plane section S_γ, if there exists one.

Algorithm isolatedPt$(f, Q, b, \mathcal{B}, \mathcal{G})$ where

- $f \in \mathbb{Q}[x, y, z]$
- $Q = (\alpha, \beta, \gamma)$ is an isolated singular point
- $\mathcal{B}$ is a list of points on $S \cap \{z = b\}$ (one for each oval)
- $\mathcal{G}$ is a list of **compRegion**.

Output: the element of $\mathcal{G}$ identifying the region containing Q

Begin

–find the innermost oval of S_γ containing the isolated point Q or **null**
$ov_{iso} :=$ **findOval**$(f(x, y, \gamma), Q)$
– Q belongs to the unlimited region
if $ov_{iso} =$ **null** then return **root**$(\mathcal{G})$
else

$iso\_pts := \mathcal{B} \cup ov_{iso}$
$Z :=$ **connComp**$(\{f = 0, \gamma \leq z \leq b\}, iso\_pts)$
$\zeta_1 := [\ \zeta \in Z \mid ov_{iso} \in \zeta]$
return $[cc\ \ in\ \ \mathcal{G} \mid points($**ccComp**$(cc)) \cap \zeta_1 \neq \emptyset]$

End

We can now describe the complete algorithm:

Algorithm affineSurface(f)

where $f \in \mathbb{Q}[x, y, z]$ is such that $S = V(f)$ satisfies the hypotheses required in this section
Output: A list of **compRegion** describing the tree $G(T)$

Begin

– compute the critical points in S and sort them by ascending z-coordinates
$Crit$:= **criticalPoints**(f)
– compute a list $la = [a_1, .., a_{m+1}]$ of rationals separating the z-coordinates
– of the critical points
la:= $\mathbf{separate_z}(Crit)$
– initialize the adjacency graph of T
$rootG := [$**null**$, [\emptyset,$ **null**$, \emptyset]]$
– $\mathcal{G}$ is a list of compRegion representing at each step $T_{\leq a_i}$
$\mathcal{G} := [rootG]$
– iterative step
for i in $1..\#Crit$ repeat

– consider the i-th critical point Q
$Q := Crit(i)$
– study the curve $S_{a_{i+1}}$
$curveG :=$ **planeGraph**$(f(x, y, a_{i+1}))$
– extract the points on $T_{\leq a_i} \cap \{z = a_i\}$ from $\mathcal{G}$,
– there is at least one point for each connected component of $\mathcal{G}$
$\mathcal{A} := \cup_{cc \in \mathcal{G}}\{points($**ccComp**$(cc)) \cap \{z = a_i\}\}$
– extract the points from $curveG$, there is one point for each oval
$\mathcal{B} := \{points(ov)$ for ov in $curveG\}$
– analyze the connected components of $T_{[a_i, a_{i+1}]}$
$ccSt :=$ **ccStrip**$(f, \mathcal{A}, \mathcal{B}, i, Q, a_i, a_{i+1})$
– lift the information to level a_{i+1}
$\mathcal{G} :=$ **lift**$(f, a_i, a_{i+1}, Q, \mathcal{G}, ccSt, curveG)$
if $rel($**ccComp**$(cc)).i = 0$ forall $cc \in \mathcal{G}$ then

– none of the components intersects the Milnor ball,
– Q is an isolated singular point
$isolCC :=$ **isolatedPt**$(f, Q, a_{i+1}, \mathcal{B}, \mathcal{G})$
$isolPts($**position**$(isolCC)) :=$
insert$(Q, isolPts($**position**$(isolCC)))$

return $\mathcal{G}$

End

Example 4.1. Let $f = f_1 f_2$ where

$$f_1(x,y,z) = z^4 + (2x^2 + 2y^2 - 26)z^2 + x^4 + (2y^2 + 10)x^2 + y^4 - 26y^2 + 25$$
$$f_2(x,y,z) = 16z^4 + 8z^2(y^2 + 4x^2 - 10) + 16x^4 + 8(y^2 + 8)x^2 + y^4 - 20y^2 + 64.$$

The equation $f(x,y,z) = 0$ defines a surface S which is the union of two tori, one inside the other and tangent at the points $(0,0,-1)$ and $(0,0,1)$. In particular we have that $Sing\,S = \{(0,0,-1),(0,0,1)\}$, while the critical values w.r.t. the projection $p(x,y,z) = z$ are $-5,-2,-1,1,2,5$.

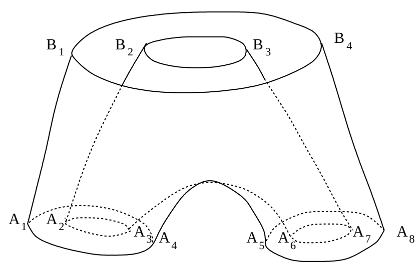

Figure 8: *Components of the surface of Example 4.1 in the strip containing the singular point* $(0,0,1)$.

The overall algorithm chooses as separating rationals the values $a_1 = -6, a_2 = -3.5, a_3 = -1.5, a_4 = 0, a_5 = 1.5, a_6 = 3.5, a_7 = 6$ which correspond to the boundaries of the six strips to be studied.

Figure 8 corresponds to the components in the fourth strip (i.e. $i = 4$ in **ccStrip**) between $z = 0$ and $z = 1.5$.

The output of the third step of the algorithm marks the four ovals in the curve S_0 with 8 points $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ as represented in Figure 8, while the function **planeGraph** marks the two ovals of $S_{1,5}$ with the points B_1, B_2, B_3, B_4.

The function **ccStrip** determines that there are 8 points on the equator of the Milnor sphere which are contained in two nested ovals, 4 points per oval. Thus $\mathcal{E} = [E_1, E_2]$ and $numArcs = [2,2]$.

The result of the second call to **connComp** is

$$\mathcal{Z} = [\zeta_1, \zeta_2] = [[A_1, A_4, A_5, A_8, E_1, B_1, B_4], [A_2, A_3, A_6, A_7, E_2, B_2, B_3]]$$

The two components ζ_1, ζ_2 in the strip are both of type $(+-)$ and for each we have that $\chi = 1 - 2 = -1$ and $s = \#(\zeta_1 \cap \mathcal{E}) = \#(\zeta_2 \cap \mathcal{E}) = 1$. Since $i = 4$, we have $s\mathbf{e}_i = [0, 0, 0, 1, 0, 0]$. Thus the two components returned by **ccStrip** are:

$$[[A_1, A_4, A_5, A_8, B_1, B_4], -1, [0, 0, 0, 1, 0, 0]]$$

$$[[A_2, A_3, A_6, A_7, B_2, B_3], -1, [0, 0, 0, 1, 0, 0]]$$

Here we show the information returned by the function **lift** for the adjacency graph associated to the various strips

$T_{\leq -6}$: $\chi = []$, $rel = []$, $G = \bullet$
$T_{\leq -3.5}$: $\chi = [1]$, $rel = [[1, 0, 0, 0, 0, 0]]$, $G = \bullet\!\!-\!\!\bullet$
$T_{\leq -1.5}$: $\chi = [1, 1]$, $rel = [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0]]$, $G = \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$
$T_{\leq 0}$: $\chi = [0, 0]$, $rel = [[1, 0, 1, 0, 0, 0], [0, 1, 1, 0, 0, 0]]$, $G = \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$
$T_{\leq 1.5}$: $\chi = [-1, -1]$, $rel = [[1, 0, 1, 1, 0, 0], [0, 1, 1, 1, 0, 0]]$, $G = \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$
$T_{\leq 3.5}$: $\chi = [-1, 0]$, $rel = [[1, 0, 1, 1, 0, 0], [0, 1, 1, 1, 1, 0]]$, $G = \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$
$T_{\leq 6}$: $\chi = [0, 0]$, $rel = [[1, 0, 1, 1, 0, 1], [0, 1, 1, 1, 1, 0]]$, $G = \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$

5. The general case and examples

If the real algebraic surface $S = \{F(x, y, z, t) = 0\}$ does not intersect in real points the plane $\{t = 0\}$ of $\mathbb{RP}^3$, then it is contained in the affine chart $\{t \neq 0\}$ where it is described by the affine equation $f(x, y, z) = F(x, y, z, 1) = 0$; thus we can compute $D(S)$ by using the Algorithm **affineSurface** described in Section 4.

If S intersects the plane $\{t = 0\}$, in this section we show that we can achieve the same goal by constructing an affine real algebraic surface $\widehat{S} \subset \mathbb{R}^3$, computing $D(\widehat{S})$ by means of the Algorithm **affineSurface** and then recovering $D(S)$ from $D(\widehat{S})$.

The strategy of using $\widehat{S}$ was already used in [12] and [13] respectively to compute the topological type and the labelled adjacency graph in the case of a non-singular algebraic surface. Here we will show that the previous construction can be helpful even when S has isolated singularities and can be used to compute also the matrix $Rel(T)$ and the list $IsReg(T)$ containing the needed information about the singularities of S. At first we briefly recall the essential features of the construction of $\widehat{S}$ and some of its properties that can be found in detail in the two papers mentioned above; then we focus our attention on how to use $D(\widehat{S})$ to recover $D(S)$.

Denote by $\mathbf{S^3}$ the unit sphere in $\mathbb{R}^4$ centered at the origin and by $\pi : \mathbf{S^3} \to \mathbb{RP}^3$ the map that associates to any point (x, y, z, t) of $\mathbf{S^3}$ the point having homogeneous coordinates $[x, y, z, t]$ in $\mathbb{RP}^3$; then each fiber contains two antipodal points on the sphere and $\mathbf{S^3}$ turns out to be a 2-sheeted covering space of $\mathbb{RP}^3$. If S is defined by the homogeneous equation $F(x, y, z, t) = 0$ and we lift S through π, the surface

$$\widetilde{S} = \pi^{-1}(S) = \{(x, y, z, t) \in \mathbb{R}^4 \mid F(x, y, z, t) = 0\} \cap \mathbf{S^3}$$

is invariant with respect to the antipodal map $ap : \mathbf{S^3} \to \mathbf{S^3}$ defined by $ap(v) = -v$. If $(0,0,0,1) \notin \widetilde{S}$ (which we can assume up to an affine translation of S) and if $\varphi : \mathbf{S^3} \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$ denotes the stereographic projection given by $\varphi(x,y,z,t) = (\frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t})$, then the image $\widehat{S} = \varphi(\widetilde{S})$ is a compact algebraic surface in $\mathbb{R}^3$, homeomorphic to $\widetilde{S}$ and defined implicitly by the polynomial equation $F(2X, \|X\|^2 - 1) = 0$ where $X = (x,y,z)$ and $\|X\|^2 = x^2 + y^2 + z^2$. Furthermore, if $inv = \varphi \circ ap \circ \varphi^{-1} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ denotes the involution $inv(X) = -\frac{X}{\|X\|^2}$ corresponding to ap via the stereographic projection, then $\widehat{S}$ is invariant with respect to inv.

In [12] and [13] it was shown that, when S is non-singular, the ability to recognize the action of inv on the set of the connected components of $\widehat{S}$ and on the set of the regions of $\mathbb{R}^3 \setminus \widehat{S}$, together with the topology of $\widehat{S}$ and the adjacency graph $G(\widehat{S})$, is sufficient to compute $\chi(S)$ and the labelled adjacency graph of S.

In the case we are examining, when S has at most isolated singularities, in order to compute $D(S)$ we make use of the topological surface T obtained from S applying the construction described in Proposition 2.4 inside the Milnor balls at the singularities of S that are not isolated points. Also the topological surface $\varphi(\pi^{-1}(T))$ is invariant with respect to inv; more precisely inv induces an involution on the set $\mathcal{F}$ of the connected components of $\varphi(\pi^{-1}(T))$ and on the set $\mathcal{R}$ of the regions of $\mathbb{R}^3 \setminus \varphi(\pi^{-1}(T))$.

Hence we can split $\mathcal{F}$ as the union $\mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{\widehat{Y} \in \mathcal{F} \mid inv(\widehat{Y}) = \widehat{Y}\} \quad \text{and} \quad \mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$$

and split $\mathcal{R}$ as the union $\mathcal{R}_1 \cup \mathcal{R}_2$, where

$$\mathcal{R}_1 = \{\widehat{\Sigma} \in \mathcal{R} \mid inv(\widehat{\Sigma}) = \widehat{\Sigma}\} \quad \text{and} \quad \mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1.$$

Our procedure to derive the needed data relative to T (and hence to S) from the data on $\varphi(\pi^{-1}(T))$ is based on the following characterization:

Proposition 5.1. *Let Y be a connected component of T and Σ a region of $\mathbb{RP}^3 \setminus T$. Then*

1. *$\varphi(\pi^{-1}(Y))$ is either a connected component of $\varphi(\pi^{-1}(T))$ (so that it belongs to $\mathcal{F}_1$) or it is the union of two distinct connected components $\widehat{Y}_1$ and $\widehat{Y}_2$ of $\varphi(\pi^{-1}(T))$ transformed each into the other by inv. Moreover in the first case $\chi(\varphi(\pi^{-1}(Y))) = 2\,\chi(Y)$, in the second case $\chi(\widehat{Y}_1) = \chi(\widehat{Y}_2) = \chi(Y)$.*
2. *Y is non-contractible if and only if $\varphi(\pi^{-1}(Y)) \in \mathcal{F}_1$,*
3. *Σ is non-contractible if and only if $\varphi(\pi^{-1}(\Sigma)) \in \mathcal{R}_1$.*

The previous results were proved in the mentioned papers [12] and [13] for the components and regions of a non-singular algebraic surface; since the proof uses only the fact that $(\mathbf{S^3}, \pi, \mathbb{RP}^3)$ is a double covering, it holds also for singular surfaces and even for topological 2-manifolds contained in $\mathbb{RP}^3$.

Since S has only isolated singularities, also the singularities of $\widehat{S}$ are isolated, so that we can compute $D(\widehat{S})$ by means of the Algorithm **affineSurface**; we obtain these data from the study of the topological 2-manifold $\widehat{T}$ associated to $\widehat{S}$ by modifying this latter surface inside Milnor balls centered at its singularities which are not isolated points, according to Proposition 2.4 .

Note that, by Durfee's result recalled in Section 2, $\widehat{T}$ is homeomorphic to $\varphi(\pi^{-1}(T))$ and also the pairs $(\mathbb{R}^3, \widehat{T})$ and $(\mathbb{R}^3, \varphi(\pi^{-1}(T)))$ are homeomorphic, so that $G(\varphi(\pi^{-1}(T))) = G(\widehat{T})$. The fact that the pairs $(\mathbb{R}^3, \widehat{T})$ and $(\mathbb{R}^3, \varphi(\pi^{-1}(T)))$ are homeomorphic is very important because it allows us to recover the data in $D(S)$ by means of a procedure based on the properties of $\varphi(\pi^{-1}(T))$ with respect to inv, but using the data relative to the surface $\widehat{T}$ computed by the Algorithm **affineSurface**.

The previous considerations allow us to attribute to $\widehat{T}$ the properties of pairing among components of $\varphi(\pi^{-1}(T))$. For instance, by an abuse of terminology we will say that two components of $\widehat{T}$ are paired through inv if the corresponding components of $\varphi(\pi^{-1}(T))$ are exchanged by inv; accordingly we will denote such a pair of components by $\widehat{Y}$ and $inv(\widehat{Y})$.

Let us now see how we can identify the components in $\mathcal{F}_1$ and recognize the pairing among the components of $\mathcal{F}_2$. We obtain a procedure to do that by combining the previous observations with the method to recognize the connected components used in Section 4, where any component was identified by means of a list of points lying on the planes $\{z = a_i\}$ that it meets. We can slightly modify that procedure with the aim of using it also to recognize the pairing in $\mathcal{F}$.

Let $Q_1, \dots, Q_m$ be the critical points of the projection p on $\widehat{S}$. Up to a translation of the surface S, we can assume that 0 is not a critical value for p, so that we can choose 0 as one of the regular levels $a_1, \dots, a_{m+1}$ used by the Algorithm **affineSurface**. We can also assume that the a_is are chosen in such a way that a_i is different from the z-coordinate of $inv(Q_j)$ for each $i = 1, \dots, m+1$ and for each $j = 1, \dots, m$.

Since $inv(X) = -\frac{X}{\|X\|^2}$, the plane $\{z = 0\}$ is invariant through inv and each point with a negative z-coordinate is transformed by inv into a point with a positive z-coordinate and viceversa.

Consider first the strips $\{a_{i-1} \leq z \leq a_i\}$ with $a_i \leq 0$. Whenever a new component $\widehat{Y}$ of $\widehat{T}$ appears in the strip, a point in $\widehat{Y} \cap \{z = a_i\}$ is chosen to identify the component; assume to mark such a point and denote by $\{V_1, \dots, V_g\}$ the set of points so found. By the assumption made above, for any V_i the point $inv(V_i)$ cannot be a critical point for p.

Consider now the strips $\{a_{i-1} \leq z \leq a_i\}$ with $a_i > 0$. For each such strip consider the set $\mathcal{M}_i$ of all the points V_j such that $inv(V_j) \in \{a_{i-1} \leq z \leq a_i\}$ and choose a Milnor radius ϵ at the unique critical point Q contained in the strip such that $D(Q, \epsilon)$ does not contain any of the $inv(V_j)$s. Enlarge the set of points used by the algorithm in the strip to identify the components of $\widehat{T} \cap \{a_{i-1} \leq z \leq a_i\}$ by adding to it the points $inv(V_j)$ for $V_j \in \mathcal{M}_i$. When the algorithm stops, any component of $\widehat{T}$ is identified by a list of points containing at least one point in $\{V_1, \ldots, V_g, inv(V_1), \ldots, inv(V_g)\}$. Comparing the lists of points identifying the different connected components we can detect the pairing: if two points $V_j, inv(V_j)$ lie on the same connected component, this component belongs to $\mathcal{F}_1$; if they belong to different components, these two components are exchanged by inv and so they belong to $\mathcal{F}_2$.

The following result shows how the knowledge of the action of inv on the set $\mathcal{F}$ and on the set of the singular points of $\widehat{S}$ is sufficient to derive from $D(\widehat{S})$ the data $\chi(T), Rel(T)$ and $IsReg(T)$.

Each row of the matrix $Rel(\widehat{T})$ computed by the Algorithm **affineSurface** contains the data concerning the behavior of a connected component $\widehat{Y}$ of $\widehat{T}$ near all the critical points for p on $\widehat{S}$.

If $N_1, \ldots, N_t$ are the singular points of S which are not isolated points, then $\widehat{S}$ has $2t$ singular non-isolated points that we can order as $\widehat{N}_1, \ldots, \widehat{N}_t$, $inv(\widehat{N}_1), \ldots, inv(\widehat{N}_t)$. We can extract from each row of $Rel(\widehat{T})$ the information concerning only the singular non-isolated points of $\widehat{S}$, so that each row is a vector of length $2t$. In the following proposition for any connected component $\widehat{Y}$ of $\widehat{T}$ we denote by $rel(\widehat{Y})$ the vector of length $2t$ so extracted regarding the behavior of $\widehat{Y}$ only at the singular non-isolated points.

Proposition 5.2. *Let Y be a connected component of T and let Σ be a region of $\mathbb{RP}^3 \setminus T$. For any singular non-isolated point N of S, let $\varphi(\pi^{-1}(N)) = \{\widehat{N}, inv(\widehat{N})\}$.*

1. *If $\varphi(\pi^{-1}(Y)) = \widehat{Y}$, with $\widehat{Y}$ connected, then*

$$\chi(Y) = \frac{\chi(\widehat{Y})}{2} \quad \text{and} \quad rel(Y)(N) = rel(\widehat{Y})(\widehat{N}) = rel(\widehat{Y})(inv(\widehat{N})).$$

2. *If $\varphi(\pi^{-1}(Y)$ is the union of the two distinct connected components $\widehat{Y}$ and $inv(\widehat{Y})$ of $\varphi(\pi^{-1}(T))$, then $\chi(Y) = \chi(\widehat{Y}) = \chi(inv(\widehat{Y}))$; moreover*
   (a) $rel(Y)(N) = rel(\widehat{Y})(\widehat{N}) + rel(inv(\widehat{Y}))(\widehat{N})$
   (b) $rel(Y)(N) = rel(\widehat{Y})(\widehat{N}) + rel(\widehat{Y})(inv(\widehat{N}))$.
3. *If Σ contains n isolated points of $\widehat{S}$, then $\varphi(\pi^{-1}(\Sigma))$ contains $2n$ points if it is connected.*
4. *If Σ contains n isolated points of $\widehat{S}$ and $\varphi(\pi^{-1}(\Sigma))$ is the union of the two distinct regions Ω, Ω' of $\mathbb{R}^3 \setminus \varphi(\pi^{-1}(T))$, then both Ω and Ω' contain n isolated points.*

Proof. The results concerning $\chi(Y)$ follow from Proposition 5.1 (1).

(3) and (4) are easy consequences of the pairing induced by inv.

As for rel, it suffices to observe that by construction Y intersects a Milnor sphere at N in as many ovals as $\widehat{Y} \cup inv(\widehat{Y})$ intersects a Milnor sphere at $\widehat{N}$. Moreover, because of the pairing induced by inv, $\widehat{Y}$ intersects a Milnor sphere at $\widehat{N}$ in as many ovals as $inv(\widehat{Y})$ intersects a Milnor sphere at $inv(\widehat{N})$, i.e. $rel(\widehat{Y})(\widehat{N}) = rel(inv(\widehat{Y}))(inv(\widehat{N}))$. □

We can now see that the knowledge of the pairing among the connected components and also among the singular points of $\widehat{S}$ induced by inv is sufficient to compute all the data in $D(S)$. It is convenient to distinguish two cases according to the parity of the degree d of S.

Case 1: d is even.

By Section 2 $G_{nc}(T)$ is non-empty and connected; moreover each connected component of $\overline{G_c(T)}$ is a tree containing exactly one vertex v lying in $G_{nc}(T)$.

The pairing among components in $\mathcal{F}$ can be seen as an equivalence relation in the set of the connected components of $\varphi(\pi^{-1}(T))$ and hence on the edges of $G(\varphi(\pi^{-1}(T))) = G(\widehat{T})$, which determines the edges of $G(T)$ as equivalent classes either of the form $[\widehat{Y}]$ or of the form $[\widehat{Y}, inv(\widehat{Y})]$. In other words the edges of $G(T)$ can be obtained by means of the corresponding identification of pairs of edges in $G(\widehat{T})$. Since $G(T)$ is a tree, as seen in Section 3 $G(T)$ can be obtained by means of a finite sequence of folds between pairs of edges sharing a common vertex.

We can also think of $\chi(Y)$ and $rel(Y)$ as labels attached to each edge Y of $G(T)$ and of $isReg(\Sigma)$ as a label attached to each vertex Σ of $G(T)$.

The result of Proposition 5.2 shows how it is possible to compute the labelled graph $G(T)$, and hence $D(S)$, from the labelled graph $G(\widehat{T})$ by means of the edge identification process recalled above, proceeding as follows:

(a) We perform on $G(\widehat{T})$ the folds that realize the identification process induced by inv; in particular, whenever we identify two paired edges e, e'
  - we attach to the new edge $[e, e']$ as χ-label the value $\chi(e) + \chi(e')$ and as rel-label the vector $rel(e) + rel(e')$;
  - if v, v' are the two distinct vertices identified by the fold, we attach to the new vertex $[v, v']$ as $isReg$-label the value $isReg(v) + isReg(v')$ and mark it c.

(b) At the end of step (a), we divide by 2 the χ-label of each edge, we divide by 2 the $isReg$-label of each vertex and we remove from all the rel-vectors the duplicates associated to paired singular points.

(c) We mark nc all the vertices not marked c so far.

(d) We choose the roots according to the process described in Subsection 2.2.

The labelled graph so obtained and endowed with the chosen roots is isomorphic to the rooted labelled graph $G(T)$.

Case 2: d is odd.

In this case (see Section 2) T contains a non-contractible component Γ, not represented in $G(T)$ since it does not disconnect $\mathbb{RP}^3$, while the component $\varphi(\pi^{-1}(\Gamma))$, lying in $\mathcal{F}_1$, is represented in $G(\widehat{T})$ as an edge e_0. Since in this case all the other components of T are contractible, $\mathcal{F}_1$ contains only one element, and each component of T different from Γ has a preimage in $\mathbb{R}^3$ formed by two components lying in $\mathcal{F}_2$ and exchanged by inv. Moreover the preimage of each region in $\mathbb{RP}^3 \setminus T$ consists of two connected regions of $\mathbb{R}^3 \setminus \widehat{T}$ exchanged by inv (in other words $\mathcal{R}_1 = \emptyset$); in particular, if Σ_0 is the region in $\mathbb{RP}^3 \setminus \Gamma$ external to all the two-sided components of T, then the two regions in $\varphi(\pi^{-1}(\Sigma_0))$ correspond to the vertices v_1, v_2 of e_0.

Thus the graph $G(\widehat{T})$, apart from the "special" edge e_0, detectable as the unique element in $\mathcal{F}_1$, having no counterpart in $G(T)$, is such that $G(\widehat{T}) \setminus \{e_0\}$ consists of two connected and isomorphic subgraphs $\widehat{G_1}$ and $\widehat{G_2}$.

As a consequence, we might compute $D(S)$ proceeding as in the even-degree case after collapsing to a vertex the edge e_0. However, taking advantage of the theoretical result that the two subgraphs $\widehat{G_1}$ and $\widehat{G_2}$ are isomorphic (and each of them is isomorphic to $G(T)$), we can compute the labelled graph $G(T)$, and hence $D(S)$, as follows:

(a) We detect in $G(\widehat{T})$ the edge e_0 as the unique element in $\mathcal{F}_1$.

(b) We choose one of the two connected components of $G(\widehat{T}) \setminus \{e_0\}$, say for instance $\widehat{G_1}$.

(c) We leave unchanged the χ-labels on all edges and the label $isReg(v)$ on all vertices.

(d) We modify the rel-label of each edge of $\widehat{G_1}$, according to Proposition 5.2 (2)(b), producing a vector of lenght t obtained by summing the entries of position i and $i+t$ for all $i = 1, \dots, t$.

(e) We mark c all the vertices.

(f) If v_1 denotes the vertex of e_0 contained in $\widehat{G_1}$, we choose v_1 as a root.

(g) Since the edge e_0 does not appear in $\widehat{G_1}$, apart we insert among the data as $\chi(e_0)$ half of the χ-value relative to e_0 that appeared in $D(\widehat{S})$; moreover we compute the datum $rel(e_0)$ using Proposition 5.2 (1).

The labelled graph so obtained and endowed with the chosen roots is isomorphic to the rooted labelled graph $G(T)$.

Example 5.3. Consider the surface represented in the left-hand side of Figure 9 consisting of a cone and two isolated points.

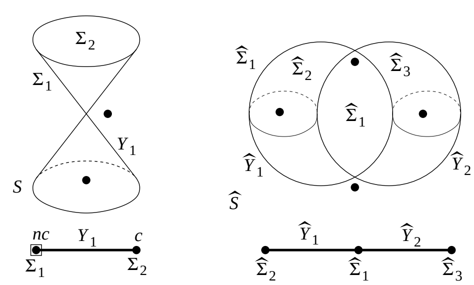

Figure 9: *An even degree non-affine surface.*

The right-hand side represents the doubled surface $\widehat{S}$ and the adjacency graph $G(\widehat{T})$, where $\widehat{T}$ is the union of two spheres. Note that the symbols used in the figure, according to the notation used in this section, indicate the components and regions of T and $\widehat{T}$, even if these surfaces are not represented in the figure.

Using the method described above we get the following data concerning $\widehat{T}$:
Regions: $inv(\widehat{\Sigma}_1) = \widehat{\Sigma}_1$ and $inv(\widehat{\Sigma}_2) = \widehat{\Sigma}_3$; hence, labelling by means of the index i each region $\widehat{\Sigma}_i$, we have $\mathcal{R}_1 = \{1\}$ and $\mathcal{R}_2 = \{2, 3\}$.
Components: $inv(\widehat{Y}_1) = \widehat{Y}_2$; hence, again labeling by means of the index i each connected component $\widehat{Y}_i$, we get $\mathcal{F}_1 = \emptyset$ and $\mathcal{F}_2 = \{1, 2\}$. Moreover $\chi(\widehat{T}) = [2, 2]$
Singularities: $rel(\widehat{Y}_1) = [1, 1]$, $rel(\widehat{Y}_2) = [1, 1]$, $isReg(\widehat{\Sigma}_1) = 2, isReg(\widehat{\Sigma}_2) = 1, isReg(\widehat{\Sigma}_3) = 1$.

The procedure described in this section yields $\chi(T) = [2]$, $rel(Y_1) = [2]$, $isReg(\Sigma_1) = isReg(\Sigma_2) = 1$, so we recognize that T is a sphere and that S is the union of two isolated points and the space obtained collapsings two points in the sphere T. The labelled 2-adjacency graph of S is represented in Figure 9 below S.

Example 5.4. The surface S represented in the left-hand side of Figure 10 contains a cone and a plane passing through the vertex of the cone, thus there is only one singular point in S which is not an isolated point.

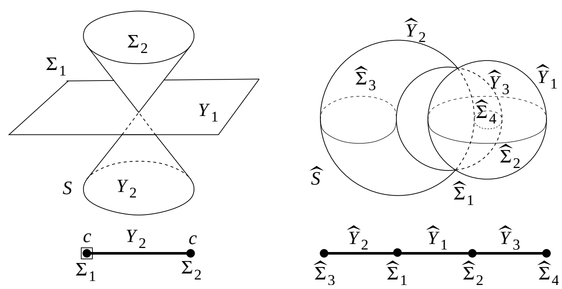

Figure 10: *An odd degree non-affine surface.*

Proceeding as in the previous example, using the notations appearing in the figure and the same way of labeling, we compute:
Regions: $inv(\widehat{\Sigma}_3) = \widehat{\Sigma}_4$ and $inv(\widehat{\Sigma}_1) = \widehat{\Sigma}_2$; hence $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_2 = \{1, 2, 3, 4\}$.
Components: $inv(\widehat{Y}_1) = \widehat{Y}_1, inv(\widehat{Y}_2) = \widehat{Y}_3$; hence $\mathcal{F}_1 = \{1\}$ and $\mathcal{F}_2 = \{2, 3\}$. Moreover $\chi(\widehat{T}) = [2, 2, 2]$.
Singularities: $rel(\widehat{Y}_1) = rel(\widehat{Y}_2) = rel(\widehat{Y}_3) = [1, 1]$, $isReg(\widehat{\Sigma}_i) = 0$ for $i = 1, 2, 3, 4$.

Using the procedure described in this section we get $\chi(T) = [1, 2]$, $rel(Y_1) = [1]$, $rel(Y_2) = [2]$, $isReg(\Sigma_1) = isReg(\Sigma_2) = 0$, i.e. T is the disjoint union of a projective plane and a sphere and S is obtained from it collapsing a point in the plane with two points in the sphere.

Example 5.5. Also the surface in Figure 11 contains a projective plane and only one singular point.

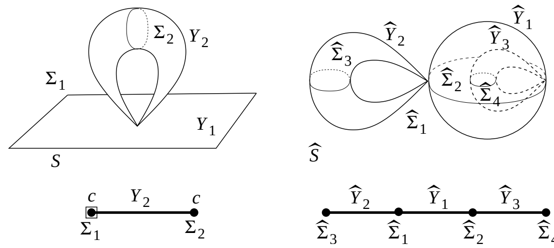

Figure 11: *Another odd degree non-affine surface.*

The usual procedure yields:
Regions: $inv(\widehat{\Sigma}_3) = \widehat{\Sigma}_4$ and $inv(\widehat{\Sigma}_1) = \widehat{\Sigma}_2$; hence $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_2 = \{1,2,3,4\}$.
Components: $inv(\widehat{Y}_1) = \widehat{Y}_1, inv(\widehat{Y}_2) = \widehat{Y}_3$; hence $\mathcal{F}_1 = \{1\}$ and $\mathcal{F}_2 = \{2,3\}$. Moreover $\chi(\widehat{T}) = [2,2,2]$.
Singularities: $rel(\widehat{Y}_1) = [1,1]$, $rel(\widehat{Y}_2) = [2,0]$, $rel(\widehat{Y}_3) = [0,2]$, $isReg(\widehat{\Sigma}_i) = 0$ for $i = 1,2,3,4$.

Using the procedure described in this section we get $\chi(T) = [1,2]$, $rel(Y_1) = [1]$, $rel(Y_2) = [2]$, $isReg(\Sigma_1) = isReg(\Sigma_2) = 0$, i.e. T is the disjoint union of a projective plane and a sphere and S is obtained from it collapsing a point in the plane with two points in the sphere.

Remark 5.6. *Observe that the output $D(S)$ obtained in Example 5.5 coincides with the one of the surface S of Example 5.4, in spite of the fact that the two surfaces cannot be mapped each into the other by means of a homeomorphism of $\mathbb{RP}^3$. However also in this case there is an invariant that distinguishes the two surfaces: the lists $\widehat{l}_1, \widehat{l}_2$. This shows that the method of "doubling" S into $\widehat{S}$ is not only a useful technical device to compute $D(S)$ but also provides new additional invariants by homeomorphism.*

6. Computational remarks and complexity of the algorithm

6.1. Good frame test and genericity

In Section 3 we assumed that (x, y, z) is a *good frame* for the surface S, i.e. that the fiber over any critical value of the projection p contains only one critical point in S. When S is non-singular, it is well-known that we can assume to be in this situation, up to a generic linear change of coordinates. A proof of this result can be found for instance in [2]. We show here how the same result can be obtained also in our singular situation by modifying the proof of [2] taking into account the presence of the isolated singularities.

Denote by $\mathbf{S^2}$ the unit sphere in $\mathbb{R}^3$ centered at the origin. For any $v \in \mathbf{S^2}$ let $\phi_v : S \to \mathbb{R}$ be the map defined by $\phi_v(X) = \langle v, X \rangle$ for any $X \in S$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^3$. If $v = (0,0,1)$, then $\phi_v(x,y,z) = z = p(x,y,z)$. A point $X \in S$ is critical for ϕ_v if and only if the gradient $\nabla f(X)$ is parallel to v.

Let $G\colon S\setminus Sing\ S \to \mathbf{S^2}$ denote the Gauss map defined by $G(X) = \frac{\nabla f(X)}{\|\nabla f(X)\|}$ and recall the following classical result (for a proof see for instance [2]):

Lemma 6.1. *Let $X_0 \in S\setminus Sing\ S$ be a critical point for ϕ_v. Then X_0 is non-degenerate for ϕ_v if and only if X_0 is not a critical point for G.*

Proposition 6.2. *Up to a linear change of coordinates in $\mathbb{R}^3$ we can assume that the projection $p(x, y, z) = z$ has finitely many critical points on S and that the critical values are distinct.*

Proof. By Lemma 6.1 the points $v \in \mathbf{S^2}$ such that ϕ_v is not a Morse function on $S\setminus Sing\ S$ are exactly the critical values of the Gauss map G, which by the semialgebraic version of Sard's Lemma (see [2], Theorem 5.57) form a semialgebraic set of dimension at most 1 in $\mathbf{S^2}$. Hence there exists a dense open subset A in $\mathbf{S^2}$, invariant w.r.t. the antipodal map $ap\colon \mathbf{S^2} \to \mathbf{S^2}$, $ap(X) = -X$, such that $\phi_v|_{S\setminus Sing\ S}$ is a Morse function for all $v \in A$.

Let $Z = \{(X, v) \in \mathbb{R}^3 \times \mathbf{S^2} \mid X \in S, \nabla f(X) \text{ is parallel to } v\}$. The set Z is a closed algebraic set of dimension 2 such that, for each $X \in S\setminus Sing\ S$, $Z \cap (\{X\} \times \mathbf{S^2})$ consists exactly of the two points $(X, \frac{\nabla f(X)}{\|\nabla f(X)\|})$ and $(X, -\frac{\nabla f(X)}{\|\nabla f(X)\|})$.

Denote by W the topological closure of $\{(X, v) \in Z \mid X \notin Sing\ S\}$, which is a semialgebraic set. If $Sing\ S = \{P_1, \ldots, P_t\}$, for each $i = 1, \ldots, t$ denote by Γ_i the subset of $\mathbf{S^2}$ such that $W \cap (\{P_i\} \times \mathbf{S^2}) = \{P_i\} \times \Gamma_i$. Since $\{P_i\} \times \Gamma_i$ is contained in the singular locus of Z, then $\dim \Gamma_i \leq 1$ for each i. Hence $\Gamma = \bigcup_{i=1}^t \Gamma_i$ is a semialgebraic curve in $\mathbf{S^2}$.

Observe that, if L is a compact subset of $\mathbf{S^2}$ such that $L \cap \Gamma_i = \emptyset$, then there exists an open neighborhood U_i of P_i in S such that v is not parallel to $\nabla f(X)$ for all $v \in L$ and all $X \in U_i \setminus \{P_i\}$.

Up to a generic linear change of coordinates we can assume that the points $(0, 0, \pm 1)$ belong to $A \setminus \Gamma$.

Since $(0, 0, \pm 1) \in A$, the projection $p(x, y, z) = z$ has only finitely many critical points in $S\setminus Sing\ S$, which are non-degenerate, say $Q_1, \ldots, Q_m$. Up to a generic linear change of coordinates in the (x, y)-plane, we can also assume that all the critical points for p, i.e. $Q_1, \ldots, Q_m, P_1, \ldots, P_t$, either have distinct z-coordinates or have distinct y-coordinates.

By Lemma 6.1 for each $j = 1, \ldots, m$ there exists an open neighborhood V_j of Q_j such that $G|_{V_j}\colon V_j \to G(V_j)$ is a diffeomorphism. Let $V = \bigcup_{j=1}^m V_j$; let also $B = \bigcap_{j=1}^m (G(V_j) \cup ap(G(V_j)))$, which is an open subset of $\mathbf{S^2}$ containing $(0, 0, 1)$ and $(0, 0, -1)$. Since $(0, 0, \pm 1) \notin \Gamma$, up to shrinking the V_js we can assume that $\overline{B} \cap \Gamma = \emptyset$.

As a consequence of the observation made above, for any $i = 1, \ldots, t$ there exists an open neighborhood U_i of the singular point P_i such that v is not parallel to $\nabla f(X)$ for all $v \in \overline{B}$ and all $X \in U_i \setminus \{P_i\}$. Let $U = \bigcup_{i=1}^t U_i$.

Since $S\setminus U$ is compact, $G|_{S\setminus U}$ is a proper map. Moreover, since V is an open neighborhood of $(G|_{S\setminus U})^{-1}(0, 0, \pm 1)$, then there exists an open neighborhood $B' \subseteq B$ of $(0, 0, \pm 1)$ such that $(G|_{S\setminus U})^{-1}(B') \subseteq V$. In other words for any $v' \in B'$ the points $X \in S\setminus U$ where $\nabla f(X)$ is parallel to v' lie in V. Moreover, since $B' \subseteq B$, $\nabla f(X)$ is not parallel to v' for all $X \in U \setminus Sing\ S$. Hence the only non-singular points $X \in S$ where $\nabla f(X)$ is parallel to v' lie in V; in particular $\phi_{v'}$ has as many critical points as $\phi_{(0,0,1)} = p$ had.

For $\epsilon \in \mathbb{R}$ consider the linear change of coordinates $\Phi\colon \mathbb{R}^3_{(X,Y,Z)} \to \mathbb{R}^3_{(x,y,z)}$ given by

$$\Phi(X, Y, Z) = (X, Y + \epsilon Z, Z - \epsilon Y).$$

With respect to the coordinates (X, Y, Z) the surface S is defined by $V(g)$ where $g = f \circ \Phi$. A point $Q \in S$ is critical for $p'(X, Y, Z) = Z$ if and only if $g_X(Q) = g_Y(Q) = 0$, which happens if and only if $\nabla f(\Phi(Q))$ is parallel to $v_\epsilon = \frac{(0,\epsilon,1)}{\|(0,\epsilon,1)\|} \in \mathbf{S^2}$. In this case the critical value w.r.t. p' is $Z = \frac{z+\epsilon y}{1+\epsilon^2}$.

Whenever v_ϵ lies in B', v_ϵ is a normal direction to S in exactly one point in each of the neighborhoods V_j. In particular the total number of critical points for p' coincides with the total number of critical points for p. If (x_i, y_i, z_i) are the critical points for p, the critical values for p' are therefore the values $\frac{z_i+\epsilon y_i}{1+\epsilon^2}$ which are distinct for ϵ small enough. □

6.2. Computation of a Milnor radius

Let $Q = (\alpha, \beta, \gamma)$ be a point of S; we want to compute a Milnor radius for S at Q, i.e a positive real number r_0 such that, for all positive $\epsilon \leq r_0$, $S \cap D(Q, \epsilon)$ does not contain any critical point for the polynomial function $\rho(x, y, z) = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2$ except Q itself.

In order to compute an r_0 with the previous property, we can start by computing a $\delta \in \mathbb{R}^+$ such that $B(Q, \delta) \setminus \{Q\}$ does not contain any non-singular critical point for ρ.

A point $P = (x, y, z) \in S$ is critical for ρ if the rank of the matrix $M = \begin{pmatrix} f_x & f_y & f_z \\ x-\alpha & y-\beta & z-\gamma \end{pmatrix}$ is less than or equal to 1.

Let $L = (f, M_1, M_2, M_3)$ denote the ideal of $\mathbb{C}[x, y, z]$ generated by f and by the determinants M_1, M_2, M_3 of the three square submatrices of order 2 of M. Then $Sing\, S_{\mathbb{C}} \subseteq V(L)$. More precisely, if we denote by Γ the set of the non-singular points of $S_{\mathbb{C}}$ which are critical for ρ, then $V(L) = \Gamma \cup Sing\, S_{\mathbb{C}}$.

Recall that in our situation the set $Sing\, S_{\mathbb{C}}$ is not necessarily finite and hence $\rho(Sing\, S_{\mathbb{C}})$ might coincide with $\mathbb{C}$. However (see [17] Corollary 2.8) the restriction of ρ to $S_{\mathbb{C}} \setminus Sing\, S_{\mathbb{C}}$ can have at most finitely many critical values, i.e. $\rho(\Gamma)$ is finite in $\mathbb{C}$; in particular it coincides with its Zariski closure $\overline{\rho(\Gamma)}^Z$. Then $\rho(\Gamma) \subseteq \rho(\overline{\Gamma}^Z) \subseteq \overline{\rho(\Gamma)}^Z = \rho(\Gamma)$ and hence $\rho(\Gamma) = \rho(\overline{\Gamma}^Z)$.

Since $\overline{\Gamma}^Z = \overline{V(L) \setminus Sing\, S_{\mathbb{C}}}^Z$, the algebraic set $\overline{\Gamma}^Z$ is the zero-set of the ideal $(L\colon J^\infty)$, where $J = (f, f_x, f_y, f_z) \subseteq \mathbb{C}[x, y, z]$ is the ideal defining the singular locus $Sing\, S_{\mathbb{C}}$. Thus there exists a non-zero polynomial $P(t) \in \mathbb{C}[t]$ whose roots are the critical values of ρ on $S_{\mathbb{C}} \setminus Sing\, S_{\mathbb{C}}$; we can compute such a $P(t)$ by standard methods, for instance using Gröbner bases or via linear algebra, searching for a relation of linear dependence among the powers of ρ modulo $(L\colon J^\infty)$.

If we take $\delta^2 = \min\{\theta \in \mathbb{R} \mid P(\theta) = 0, \theta > 0\}$, the punctured real ball $B(Q,\delta) \setminus \{Q\}$ does not contain any real critical point of S for ρ, but it can still possibly contain some point in $Sing\,S$. We can easily avoid this: since in our hypothesis the real singular locus $Sing\,S$ is finite, it is sufficient to compute $\eta = \min\{\|Q - P_i\| \mid P_i \in Sing\,S,\ P_i \neq Q\}$and then to take $r_0 < \min(\delta, \eta)$.

If we require that r_0 is also a Milnor radius at Q for the plane curve $S_\gamma \subseteq \{z = \gamma\}$ defined by $g(x,y) = f(x,y,\gamma) = 0$, we need to compute a $\sigma \in \mathbb{R}^+$ such that the punctured plane disk $B((\alpha,\beta),\sigma) \setminus \{(\alpha,\beta)\}$ does not contain any critical point for $\rho|_{S_\gamma}$. In this case the computation is simpler, because the critical points for $\rho|_{S_\gamma}$ are the points on $V(g)$ where $\det \begin{pmatrix} g_x & g_y \\ x-\alpha & y-\beta \end{pmatrix} = 0$. After computing σ as above, it suffices to take $r_0 < \min(\delta, \eta, \sigma)$.

6.3. Recognizing connectivity relations

In the description of the algorithms in Section 4 we assumed the existence of a general function **connComp** which takes as arguments a semialgebraic set SAL in $\mathbb{R}^n$ and a finite set of points. Its job is to produce a partition of those points into maximal subsets which belong to the same connected component of SAL. Our first use of **connComp** in **ccStrip** is to decide which points on the equator of a Milnor sphere belong to the same oval. In this case the semi algebraic set SAL is a smooth one-dimensional curve in $\mathbb{R}^3$ defined by the equation of our surface S along with the equation of the Milnor sphere. In our other two uses of **connComp**, the semialgebraic set is a two-dimensional subset of S. We use Canny's concept ([5]) of a "roadmap" to reduce the problem of deciding connectedness for these two-dimensional sets to the same problem on one-dimensional sets. In our case a roadmap for S can be described by the union of a silhouette curve defined by f and f_x (which will be one-dimensional if our surface is in general position), along with suitable sections of our surface for constant values of z. The essential property of our use of a roadmap is that points are pathwise connected on the roadmap if and only if they are pathwise connected on S. The next proposition shows that to form a roadmap for our surface, in addition to the silhouette curve, it is sufficient to take z sections of S through the critical points of the silhouette curve.

Proposition 6.3. *Assume that the strip $\mathbb{R}^2 \times [a,b]$ contains only one critical point $Q = (\alpha,\beta,\gamma)$ for the projection $p(x,y,z) = z$ which is not isolated in S, with $a < \gamma < b$. Let $R = (S_\gamma \setminus B(Q,\epsilon)) \cup C(Q,\epsilon)$ and $\Sigma = V(f, f_x) \cap \{a \leq z \leq b\}$. Denote by $c_1 < \ldots < c_k$ the critical values of the restriction of p to $\Sigma \cap \{a < z < b\}$, $c_0 = a$ and $c_{k+1} = b$. Let $\Gamma = \Sigma \cup \bigcup_{i=0}^{k+1} S_{c_i}$. Then:*

1. *for any $P \in S_a \cup S_b$ there exists an arc α_P contained in $\Gamma \setminus \{Q\}$ such that $\alpha_P(0) = P$ and $\alpha_P(1) \in R$,*
2. *two points $P_1, P_2 \in R$ lie in the same connected component of $T_{[a,b]}$ if and only if they lie in the same connected component of R,*
3. *two points $P_1, P_2 \in S_a \cup S_b$ lie in the same connected component of $T_{[a,b]}$ if and only if the points $\alpha_{P_1}(1)$ and $\alpha_{P_2}(1)$ lie in the same connected component of R.*

Proof. (1) It suffices to consider the case when $P \in S_a$.

Let W be the connected component of $S_{[a,\gamma]} \setminus \{Q\}$ containing P. Then, for any $c \in [a, \gamma)$, W_c is a single oval which intersects Σ in at least two points. In particular W_a is an oval σ containing P.

Since γ is a critical value for p, there exists h with $1 \le h \le k$ such that $\gamma = c_h$. Let β_1 be a connected component of $\Sigma \cap W_{(a,c_1)}$. Since $p_{|\beta_1}$ has no critical point, for any $u_1 \in (c_0, c_1)$ the arc β_1 meets $\{z = u_1\}$ in a single point; moreover, since $\overline{W_{(c_0,c_1)}}$ is compact, $\overline{\beta_1}$ intersects $\{z = c_0\}$ in a point belonging to the oval σ and intersects $\{z = c_1\}$ in a point P_1. Thus there exists an arc joining P with P_1 and contained in $\sigma \cup \Sigma$.

Iterating the previous argument in the strips $\mathbb{R}^2 \times (c_1, c_2), \dots, \mathbb{R}^2 \times (c_{h-1}, \gamma)$, eventually we find an arc $\beta \subseteq \Gamma \cap \overline{W}$ joining P with a point of S_γ. If the final point $\beta(1)$ does not belong to $B(Q, \epsilon)$, we are done. Otherwise β necessarily meets $S(Q, \epsilon)$ in some point: in this case, if t_0 is the smallest real number in $[0, 1]$ such that $\beta(t_0) \in S(Q, \epsilon)$, it suffices to take as α_P the restriction of β to $[0, t_0]$.

(2) The points $P_1, P_2 \in R$ lie in the same connected component of $T_{[a,b]}$ if and only if they lie in the same connected component of $S_{[a,b]} \setminus B(Q, \epsilon)$. Then the result follows from the fact that R is a deformation retract of $S_{[a,b]} \setminus B(Q, \epsilon)$ by Theorem 3.5 (5).

(3) The point $P_1 \in S_a \cup S_b$ (resp. P_2) lies in the same connected component of $T_{[a,b]}$ as $\alpha_{P_1}(1)$ (resp. $\alpha_{P_2}(1)$). Then the result follows from (2). □

Remark 6.4. *For space algebraic curves it is possible to decide whether two points of the curve belong to the same connected component, using any algorithm that computes the connectivity graph of the curve. The curve R found in Proposition 6.3 is not algebraic but it is a deformation retract of $(S_\gamma \setminus \{Q\}) \cup C(Q, \epsilon)$. Thus two points $P_1, P_2 \in R$ belong to the same connected component of R if and only if they belong to the same connected component of $(S_\gamma \setminus \{Q\}) \cup C(Q, \epsilon)$, which can be decided by computing the connectivity graph of the algebraic curve $S_\gamma \cup C(Q, \epsilon)$ and by recovering from it the connectivity graph of $(S_\gamma \setminus \{Q\}) \cup C(Q, \epsilon)$. Observe that $(\Gamma \setminus \{Q\}) \cup C(Q, \epsilon)$ satisfies Canny's roadmap conditions for $S_{[a,b]} \setminus \{Q\}$ and thus for $T_{[a,b]}$.*

All of our calls to **connComp** can be reduced to the study of connectedness of points lying on curves in $\mathbb{R}^3$. There are many approaches to solving connectivity on curves (for references see for instance [2]), one in particular is due to El Kahoui ([11]). His original purpose was to study the topology of arbitrary space curves in $\mathbb{R}^3$. In order to do this he studies the connectivity between points in adjacent critical fibers. This provides us with a general technique for studying connectivity between points on a space curve. We just add the fibers containing these points of interest to the set of fibers being studied by El Kahoui's algorithm. Once we know the connectivity between points in adjacent fibers, we can compute the transitive closure of the connectivity graph to decide the connectivity of our original points. Since his algorithm is based on connectivity between adjacent fibers, if we want to restrict to paths not passing though a particular point in a fiber, we can remove that point from the connectivity graph before computing the transitive closure. This gives us a complete implementation of the **connComp** algorithm needed for our purposes. The basic construction used in El Kahoui's algorithm is to count the number of real points in a fiber and to determine the position of the critical point in the fiber. We should point out that his algorithm does require some strong general position hypotheses in order to work correctly. In addition his algorithm for studying connectedness of points on space curves can also be used to decide whether our surface has a finite number of real critical points, and if so locate them. If the critical point ideal is zero-dimensional complex, this is relatively easy; when the critical point ideal is one-dimensional complex, then we can use El Kahoui's algorithm to decide whether the real points in the critical fibers are isolated or not. Finally El Kahoui's algorithm can be adapted to computing the topology of plane curves, i.e. implementing **planeGraph**. The structure of the ovals can be computed from El Kahoui's fiber counts, and in the case of non-singular curves adjacency graphs can be obtained.

6.4. Comments on computational complexity

Assuming our compact affine surface is described by a square-free polynomial f of degree d, by [8] the number of connected components of $S = V(f)$ can be $\mathcal{O}(d^3)$. By our general position hypothesis there are at most $d(d-1)^2$ critical points, which bounds the number of top level iterations of our algorithm (calls to **lift**). The silhouette curve (f, f_x) will have degree $d(d-1)$ and our choice of El Kahoui's algorithm for analyzing this space curve requires a generic projection to a plane curve of the same degree, whose discriminant will have degree $\mathcal{O}(d^4)$. His algorithm studies the real points in the fibers above all the real roots of the discriminant, i.e. $\mathcal{O}(d^6)$ points. Although we could need $\mathcal{O}(d^3)$ calls to **connComp**, the silhouette curve is the same for all these calls and only the points to test for connectedness change. In each strip we only need to test connectedness among $\mathcal{O}(d^2)$ points, since this is a bound on the number of intersections of the silhouette curve with the planes $z = a_i$ and $z = a_{i+1}$. El Kahoui's algorithm requires us to study the fibers over the projection of each of these points; we can have $\mathcal{O}(d^2)$ points in each fiber and the total number of points to be studied in each strip is $\mathcal{O}(d^4)$. So the total number of points required for all the calls to **connComp** is $\mathcal{O}(d^6) + (d(d-1))^2\mathcal{O}(d^3) = \mathcal{O}(d^7)$. This the is same as required by the algorithm of Alberti, Mourrain and Tecourt ([1]) for the triangulation of affine real algebraic surfaces and compares very favorably with their $\mathcal{O}(d^{13})$ estimate for the number of points required to compute a Cylindrical Algebraic Decomposition of the surface.

Our algorithm requires some preliminary constructions to find the critical points and a Milnor radius at each such point; once this is done, we claim that the cost of the calls to **connComp** dominates the cost of rest of our entire algorithm. Here we show that the cost of testing connectedness between components in the strip and the cost of merging components are both dominated by $\mathcal{O}(d^7)$ operations, and thus dominated by the cost of computing the points needed by the various calls to **connComp**. For example each of our $\mathcal{O}(d^3)$ calls to **mergeComponents** requires collapsing at most $\mathcal{O}(d)$ components in each strip, since that is a bound on the number of components of type $(+-)$ in each strip (derived from the $2d$ bound on the number of intersections of the surface and the equator of a Milnor sphere). Assuming we first compute a list of the components which need to be merged, each fold requires $\mathcal{O}(d)$ comparisons and $\mathcal{O}(d^3)$ integer arithmetic operations to combine labels, and we do at most $\mathcal{O}(d)$ folds per strip, for a total cost of $\mathcal{O}(d^5)$ point comparisons and $\mathcal{O}(d^7)$ integer operations to combine the relevant data. Similarly for the problem of matching connected components, although the total number of components of our surface is bounded by $\mathcal{O}(d^3)$, the $\mathcal{O}(d^2)$ bound for the number of ovals in a section shows that at most $\mathcal{O}(d^2)$ components meet any section. Thus we can see that in each strip the cost of testing which of the $\mathcal{O}(d^2)$ components in a strip $T_{[a,b]}$ connect to one of the $\mathcal{O}(d^2)$ components from $T_{\leq a}$ which meet $z = a$ requires at most $\mathcal{O}(d^4)$ comparisons per strip, again for a overall cost of $\mathcal{O}(d^7)$. Our preliminary implementation of this algorithm also demonstrates that nearly all the time is spent in calls to **connComp**; we intend to investigate seminumerical approaches to improve this performance.

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