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On L^r hypoellipticity of solutions with compact support of the Cauchy-Riemann equation

ERIC AMAR AND SAMUELE MONGODI

1 Introduction.

In this paper, we investigate the inhomogeneous Cauchy-Riemann equation

$$\bar{\partial}u = g$$

when g has compact support and belongs to some L^r space. The question is if it is possible to find a solution u with the same properties, namely, compactly supported and in L^r .

The L^r solvability of the Cauchy-Riemann equation has been discussed by Kerzman for smoothly bounded strongly pseudoconvex domains (see [9] and [10]), by Fornaess and Sibony in \mathbb{C} with weights and in Runge domains in \mathbb{C}^2 (see [6]). Other works on the subject are [12], [8], [5], [11], [4], [1] and [16].

The problem of controlling the support of the solution is also widely discussed. In one complex variable, the existence of a compactly supported solution in \mathbb{C} is related to the vanishing of some integrals, resemblant of the *moment conditions* which appear in CR geometry:

$$\int_{\mathbb{C}} g(z) z^k dm_1(z).$$

If these integrals vanish for every $k \in \mathbb{N}$, then there exists a function u such that $\partial u / \partial \bar{z} = g$ and $\text{supp} u \Subset \{|z| < R\}$ for some R .

It is not hard to generalize this result to domains like punctured discs, as we do in Lemma 3.2.

In higher dimension, it is well known that the existence of a compactly supported solution depends on the vanishing of the cohomology with compact supports; $H_c^{p,q}(\Omega)$ vanishes, for $\Omega \subseteq \mathbb{C}^n$ Stein, if $q < n$. For smooth forms, the existence of a solution compactly supported in a sublevel of some strictly plurisubharmonic exhausting function has also been studied widely, beginning from the work of Andreotti and Grauert ([3]).

Some attempts at controlling the support of the solution were made by Landucci, in the case of smoothly bounded strictly pseudoconvex domains (see [13] and [14]).

We tackle the problem for a very special class of domains, which generalize the punctured disc: we consider the Stein open domain obtained by removing a complex hypersurface from a polydisc \mathbb{D}^n . Given $f \in \mathcal{O}(\overline{\mathbb{D}^n})$ with $Z = \{f = 0\}$, we consider the domain $\mathbb{D}^n \setminus Z$: the particular structure of

these open sets allows us to give a constructive proof of our results. We will state our results in terms of $(0, q)$ -forms, the extension to the (p, q) -forms being obvious.

First of all, for $(0, 1)$ -forms, we have the following.

Proposition 3.5. *Let $\Omega \subseteq \mathbb{C}^n$ be a Stein domain and ω a $(0, 1)$ -form with coefficients in $L_c^r(\Omega)$ such that $\bar{\partial}\omega = 0$. Then there exists a unique $f \in L_c^r(\Omega)$ such that $\bar{\partial}f = \omega$, with $\|f\|_r \leq C\|\omega\|_r$, where C depends only on Ω .*

This result leaves the question open for $q > 1$.

Let ω be a generic $(0, q)$ -form and let us write

$$\omega = \sum_{|J|=n-q} \omega_J d\hat{z}_J.$$

We are going to work with the forms satisfying the following condition

$$(*) \quad \bar{\partial}_{j_{n-q}} \cdots \bar{\partial}_{j_k} \omega_J \in L^r(\mathbb{C}^n) \quad k = 1, \dots, n-q, \quad \forall |J| = n-q.$$

In Theorems 6.2, 7.4 and 8.1, we show that, given ω a $(0, q)$ -form compactly supported in \mathbb{C}^n , with $\bar{\partial}\omega = 0$, with L^r coefficients and satisfying $(*)$, we can find a $(0, q-1)$ -form $\beta \in L_c^r(\mathbb{C}^n)$ such that $\bar{\partial}\beta = \omega$.

This result in \mathbb{C}^n easily gives the corollary

Corollary 8.4. *Let ω be a $(0, q)$ -form with compact support in $\mathbb{D}^n \setminus Z$ and satisfying conditions $(*)$, then, for any $k \in \mathbb{N}$, we can find a $(0, q-1)$ -form $\beta \in L_c^r(\mathbb{D}^n)$ such that $\bar{\partial}(f^k \beta) = \omega$. Equivalently, we can find a $(0, q-1)$ -form $\eta = f^k \beta$ such that $\eta \in L_c^r(\mathbb{D}^n)$, η is 0 on Z up to order k and $\bar{\partial}\eta = \omega$.*

Moreover, in the case of $(0, n)$ -forms, our construction allows us to obtain a slightly better result.

Theorem 4.8. *Let $f \in \mathcal{O}(\overline{\mathbb{D}^n})$ be a holomorphic function in a neighbourhood of the closed unit polydisc in \mathbb{C}^n and set $Z = f^{-1}(0)$. If ω is a $(0, n)$ -form in $L_c^r(\mathbb{D}^n \setminus Z)$, then for every $k \in \mathbb{N}$ we can find a $(0, n-1)$ -form $\eta \in L^r(\mathbb{D}^n)$ such that $f^{-k}\eta \in L^r(\mathbb{D}^n)$ and all the coefficients of η but at most one are in $L_c^r(\mathbb{D}^n \setminus Z)$; moreover, η is such that $\bar{\partial}\eta = \omega$.*

The starting point of this work was an incisive question asked by G. Tomassini and the second author to the first author.

2 Notations.

We denote by \mathbb{D} the unit disc in \mathbb{C} and by \mathbb{D}^n its n -fold product, the unit polydisc in \mathbb{C}^n . The projection from \mathbb{C}^n onto the j -th coordinate will be denoted by π_j .

The standard Lebesgue measure on \mathbb{C}^n will be dm_n and we will denote by $g *_k h$ the partial convolution in the k -th variable:

$$(g *_k h)(z_1, \dots, z_n) := \int_{\mathbb{C}} g(\cdots, z_{k-1}, \zeta, z_{k+1}, \cdots) h(\cdots, z_{k-1}, z_k - \zeta, z_{k+1}, \cdots) dm_1(\zeta).$$

If T is a distribution in \mathbb{C}^n , we set $\bar{\partial}_j T = \frac{\partial T}{\partial \bar{z}_j}$, $j = 1, \dots, n$.

Let $J = (j_1, \dots, j_q)$, $j_k = 1, \dots, n$, then we define $\hat{z}_J \in \mathbb{C}^{n-q}$ with coordinates in J deleted. For instance $\hat{z}_k = (\cdots, z_{k-1}, z_{k+1}, \cdots) \in \mathbb{C}^{n-1}$.

3 On the Cauchy transform.

Given $\varphi \in \mathcal{D}(\mathbb{C}^n)$ a smooth functions with compact support, the functions

$$\zeta \rightarrow \varphi(\cdots, z_{k-1}, \zeta, z_{k+1}, \cdots),$$

for $k = 1, \dots, n$, are still smooth and with compact support, contained in $\pi_k(\text{supp}\varphi)$. The Cauchy transform of φ in the k^{th} variable is

$$G_k(\varphi)(z) = \varphi *_{k} := \int_{\mathbb{C}} \frac{\varphi(\cdots, z_{k-1}, \zeta_k, z_{k+1}, \cdots)}{\pi(\zeta_k - z_k)} dm_1(\zeta_k)$$

and we know that [2]

Lemma 3.1. *We have, with the above notations,*

$$\bar{\partial}_k G_k(\varphi)(z) = \varphi(z) \quad \forall z \in \mathbb{C}^n$$

and

$$\|G_k(\varphi)\|_{L^r} \leq \left\| \frac{1}{\pi \zeta_1} \right\|_{L^1(\mathbb{D})} \times \|\varphi\|_{L^r}.$$

So the Cauchy transform extends as a bounded linear operator on $\varphi \in L_c^r(\mathbb{D}^n)$. Moreover $G_k(\varphi)$ is holomorphic in z_k outside of the support of φ considered as a function of z_k , \hat{z}_k being fixed.

Throughout this note, f will be a given function holomorphic in a neighbourhood of $\overline{\mathbb{D}^n}$ and $Z = Z(f)$ will denote its zero locus.

The set of directions for which there is a complex lines with that direction contained in Z is an analytic subset of $\mathbb{C}\mathbb{P}^{n-1}$ of dimension $n - 2$; therefore we can find n linearly independent complex directions not lying in it. So, after a linear change of coordinates, for every $1 \leq k \leq n$, we can find a number N_k such that, given $n - 1$ complex numbers a_j , $j \in \{1, \dots, n\} \setminus \{k\}$, with $|a_j| < 1$, the number of solutions of

$$f(\cdots, a_{k-1}, z_k, a_{k+1}, \cdots) = 0$$

as an equation in z_k , is less than $N_k + 1$.

Because these solutions are those of an analytic function, there is always a parametrization of them by measurable functions: it is an easy application of [18, Theorem 7.34]; let us denote these solutions by $\{c_{1,k}(a), \dots, c_{N_k,k}(a)\}$ where the functions $c_{j,k} = c_{j,k}(a)$ are measurable from \mathbb{C}^{n-1} to \mathbb{C} .

Let $\varphi \in L_c^r(\mathbb{D}^n \setminus Z)$ and fix $\hat{z}_k \in \mathbb{D}^{n-1}$; denote by $S_\varphi(\hat{z}_k)$ its support as a function of z_k which depends on \hat{z}_k . Then, by compactness, there exists numbers $\delta_1, \dots, \delta_n$ such that $S_\varphi(\hat{z}_k)$ has distance at least δ_k from $c_{1,k}(a), \dots, c_{N_k,k}(a)$, for every $a \in \mathbb{C}^{n-1}$, so there are numbers $r_{j,k} = r_{j,k}(\hat{z}_k) \geq \delta > 0$ such that the disc $D(c_{j,k}, r_{j,k})$ in the z_k variable is *not* in $S_\varphi(\hat{z}_k)$.

However, these discs could intersect without coinciding; suppose that the discs

$$D(c_{j_1,k}, r_{j_1,k}), \dots, D(c_{j_h,k}, r_{j_h,k})$$

form a connected component of the union of all the discs for the variable z_k , then we can suppose that $r_{j_i,k} = \delta_k$ for $i = 1, \dots, h$. If the discs

$$D(c_{j_1,k}, \delta_k/3N_k), \dots, D(c_{j_h,k}, \delta_k/3N_k)$$

are disjoint, then we are done, otherwise, let us consider a connected component of their union and let us suppose, wlog, that it coincides with the union. Obviously, the diameter of such a connected component is less than δ_k , therefore a disc centered in one of the centers with radius δ_k will enclose the whole connected component and, by the definition of δ_k will still be in the complement of S_φ . Therefore, we can set all the centers equals to one of them (it is not relevant which one) and take δ_k as a radius. The functions $c_{j,k}$ will still be measurable. The discs will be then either disjoint or coinciding and their radii will be bounded from below by $\delta_k/3N_k$; we set $\delta = \min\{\delta_1/3N_1, \dots, \delta_n/3N_n\}$.

As we already said, $\varphi *_k = G_k(\varphi)$ is holomorphic for $z_k \notin \mathbb{D}$ and for $z_k \in D(c_{j,k}, r_{j,k})$.

This will be precised in the next section with the help of the following definitions.

Let $\varphi \in L_c^r(\mathbb{D}^n)$, we define

$$[\varphi]_k(l) = \frac{1}{\pi} \int_{\mathbb{C}} \varphi(\dots, z_{k-1}, \zeta_k, z_{k+1}, \dots) \zeta_k^l dm_1(\zeta_k) ;$$

let $\varphi \in L_c^r(\mathbb{D}^n \setminus Z)$, we define

$$[\varphi, j]_k(l) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi(\dots, z_{k-1}, \zeta_k, z_{k+1}, \dots)}{(\zeta_k - c_{j,k})^{m+1}} dm_1(\zeta_k).$$

We have the following lemma linking this with $\bar{\partial}$ equation.

Lemma 3.2. *Let $\varphi \in L_c^r(\mathbb{D}^n \setminus Z)$, then the following are equivalent :*

- (i) $[\varphi]_k(l) = [\varphi, j]_k(l) = 0$ for every $l \in \mathbb{N}$ and $1 \leq j \leq N_k$
- (ii) $G_k(\varphi) \in L_c^r(\mathbb{D}^n \setminus Z) \Rightarrow \bar{\partial}_k G_k(\varphi) = \varphi$.

Proof: Without loss of generality, we can set $k = 1$; we notice that, by Lemma 3.1, $G_1(\varphi)$ is in $L^r(\mathbb{C}^n)$, so (ii) is equivalent to the compactness of its support. Moreover, we remark that $G_1(\varphi)$ has compact support in $\mathbb{D}^n \setminus Z$ if and only if for almost every $a = (a_2, \dots, a_n) \in \mathbb{C}^{n-1}$ the function $z \mapsto G_1(\varphi)(z, a_2, \dots, a_n)$ has compact support in

$$(\mathbb{D}^n \setminus Z) \cap \{z_2 = a_2, \dots, z_n = a_n\} = \mathbb{D} \setminus \{c_{1,1}(a), \dots, c_{1,N_1}(a)\} .$$

On the other hand, $[\varphi]_1(l)$ and $[\varphi, j]_1(l)$ vanish if and only if the integrals that define them vanish for almost every z_2, \dots, z_n . So, we are reduced to the 1 variable case: let then c_1, \dots, c_N be points in $\mathbb{D} \subset \mathbb{C}$ and $\phi \in L_c^r(\mathbb{D} \setminus \{c_1, \dots, c_N\})$; we set $G(z) = G_1(\varphi)(z)$.

If (ii) holds, for any $h \in \mathcal{O}(\mathbb{D} \setminus \{c_1, \dots, c_N\})$ we have

$$\int_{\mathbb{C}} \varphi(z) h(z) dm_1(z) = \int_{\mathbb{C}} \frac{\partial G(z)}{\partial \bar{z}} h(z) dm_1(z) = - \int_{\mathbb{C}} G(z) \frac{\partial h(z)}{\partial \bar{z}} dm_1(z) = 0$$

where we have used Stokes' theorem, as $G(z)$ has compact support. The last integral vanishes because h is holomorphic.

On the other hand, suppose that (i) holds and let $K = \text{supp}\varphi$. Consider $r < 1$ such that $K \Subset \mathbb{D}_r = \{|z| < r\}$ and take z with $|z| > r$; then

$$\begin{aligned} G(z) &= -\frac{1}{z\pi} \int_K \varphi(\zeta) \frac{1}{1 - \frac{\zeta}{z}} dm_1(\zeta) = -\frac{1}{\pi z} \int_K \varphi(\zeta) \sum_{l \geq 0} \frac{\zeta^l}{z^l} dm_1(\zeta) \\ &= -\frac{1}{\pi z} \sum_{l \geq 0} z^{-l} \int_K \varphi(\zeta) \zeta^l dm_1(\zeta) = -\frac{1}{\pi} \sum_{l \geq 0} z^{-l-1} [\varphi]_1(l). \end{aligned}$$

So, $G(z) = 0$ if $|z| > r$, therefore $\text{supp}G(z) \Subset \mathbb{D}$.

Moreover, fix j , $1 \leq j \leq N$; there exists $r_j > 0$ such that the closure of $D(c_j, r_j) = \{|z - c_j| < r_j\}$ does not meet $\text{supp}\varphi(z)$. So, if $|z - c_j| < r_j$, we have

$$\begin{aligned} G(z) &= \frac{1}{\pi} \int_K \varphi(\zeta) \frac{1}{(\zeta - c_j) - (z - c_j)} dm_1(\zeta) = \frac{1}{\pi} \int_K \varphi(\zeta) \frac{1}{\zeta - c_j} \frac{1}{1 - (z - c_j)/(\zeta - c_j)} dm_1(\zeta) \\ &= \frac{1}{\pi} \int_K \varphi(\zeta) \frac{1}{\zeta - c_j} \sum_{l \geq 0} \frac{(z - c_j)^l}{(\zeta - c_j)^l} dm_1(\zeta) = \frac{1}{\pi} \sum_{l \geq 0} (z - c_j)^l [\varphi, j]_1(l). \end{aligned}$$

Therefore, by hypothesis, $G(z) = 0$ if $|z - c_j| < r_j$, so $\text{supp}G(z) \Subset \mathbb{D} \setminus \{c_1, \dots, c_N\}$. ■

Moreover, we have the following relations between the Cauchy transform and the quantities defined above.

Lemma 3.3. *If g and h are L^r functions, compactly supported in \mathbb{D}^n , and $g \star_1 \frac{1}{z_1} = h \star_1 \frac{1}{z_1}$ for $z_1 \notin \mathbb{D}$, then $[g]_1(k) = [h]_1(k)$ for every k .*

Proof: If $z_1 \notin \mathbb{D}$, we have

$$\begin{aligned} g \star_1 \frac{1}{z_1} &= \int_{\mathbb{D}} g(\zeta_1, \hat{z}_1) \frac{1}{z_1 - \zeta_1} dm_1(\zeta_1) = \frac{1}{z_1} \int_{\mathbb{D}} g(\zeta_1, \hat{z}_1) \frac{1}{1 - \frac{\zeta_1}{z_1}} dm_1(\zeta_1) = \\ &= \frac{1}{z_1} \sum_{k \geq 0} z_1^{-k} \int_{\mathbb{D}} g(\zeta_1, \hat{z}_1) \zeta_1^k dm_1(\zeta_1) = \sum_{k \geq 0} [g]_1(k) z_1^{-k-1}. \end{aligned}$$

A similar expansion holds for h , so that

$$h \star_1 \frac{1}{z_1} = \sum_{k \geq 0} [h]_1(k) z_1^{-k-1}.$$

Therefore, given that $(g - h) \star_1 \frac{1}{z_1} = 0$ for $z_1 \notin \mathbb{D}$, we have $[g]_1(k) = [h]_1(k)$ for every k . ■

Lemma 3.4. *If g and h are L^r functions, compactly supported in \mathbb{D}^n , and there exists $j \geq 1$ such that $g \star_1 \frac{1}{z_1} = h \star_1 \frac{1}{z_1}$ for every $z_1 \in D(c_{j,1}(\hat{z}_1), r_{j,1}(\hat{z}_1))$, then $[g, j]_1(k) = [h, j]_1(k)$ for every k .*

We omit the proof as it can be easily obtained from the previous one.

Finally, we recall a result about the solution with compact support of the equation $\bar{\partial}f = \omega$ when ω is a $(0, 1)$ -form with compact support.

Proposition 3.5. *Let $\Omega \subseteq \mathbb{C}^n$, $n \geq 2$, be a Stein domain and ω a $(0,1)$ -form with coefficients in $L_c^r(\Omega)$ such that $\bar{\partial}\omega = 0$. Then there exists a unique $f \in L_c^r(\Omega)$ such that $\bar{\partial}f = \omega$, with $\|f\|_r \leq C\|\omega\|_r$, where C depends only on Ω .*

Proof: For the proof in the case $\Omega = \mathbb{C}^n$, see for instance [15, Chapter III]. For a generic $\Omega \subset \mathbb{C}^n$, we notice that if f_1 and f_2 are two compactly supported (distributional) solutions, then the difference $f_1 - f_2$ is $\bar{\partial}$ -closed, that is, a holomorphic function, but then $f_1 = f_2$. Moreover, by [17], $H_c^{0,1}(\Omega) = 0$, so there exists at least one distributional solution to $\bar{\partial}T = \omega$, compactly supported in Ω , on the other hand, we know that there is $f \in L_c^r(\mathbb{C}^n)$, solving $\bar{\partial}f = \omega$, given, as described in [15], by convolution with the Cauchy kernel.

Therefore we have $T = f$ and the desired estimate follows. ■

4 The coronas construction

Let φ be a function in $L_c^r(\mathbb{D}^n \setminus Z)$ and consider the Cauchy transform $G_1(\varphi)(z)$; for a.e. \hat{z}_1 , $G_1(z)$ is holomorphic in z_1 in the complement of $S(\hat{z}_1)$.

Because $\pi_1(\text{supp}\varphi)$ is compact in \mathbb{D} , there exists $D(0, r)$ containing $S(\hat{z}_1)$; let $\delta = (1 - r)/3$ and define the corona

$$C_0 = \{z_1 \in \mathbb{D} : r + \delta < |z_1| < r + 2\delta\} \Subset \mathbb{D}$$

and let $A_0 = m_1(C_0)$.

In the same way, set $\delta_j(\hat{z}_1) = r_{j,1}(\hat{z}_1)/3$ and define

$$C_j(\hat{z}_1) = \{z_1 \in \mathbb{D} : \delta_j(\hat{z}_1) \leq |z_1 - c_{j,1}| \leq 2\delta_j(\hat{z}_1)\} \Subset \mathbb{D}$$

and set $A_j(\hat{z}_1) = 1/m_1(C_j(\hat{z}_1))$.

Definition 4.1. *The outer corona component of φ is the function*

$$K_0^{(1)}(\varphi)(z) = A_0 \mathbb{1}_{C_0}(z_1) z_1 G_1(\varphi)(z)$$

and the inner coronas components of φ are the functions

$$K_j^{(1)}(\varphi)(z) = A_j(\hat{z}_1) \mathbb{1}_{C_j(\hat{z}_1)}(z_1) (z_1 - c_{j,1}) G_1(\varphi)(z).$$

Remark 4.2. *The outer and inner coronas components of φ are well defined for a.e. \hat{z}_1 , because $\varphi(\cdot, \hat{z}_1)$ is in $L^r(\mathbb{C})$ and has compact support for a.e. \hat{z}_1 . We define exactly the same way the quantities $K_j^{(k)}(\varphi)(z)$ with respect to the variables z_k .*

Lemma 4.3. *The operators $K_m^{(1)}$, $m = 0, \dots, N_1$, are linear and well defined from $L_c^r(\mathbb{C}^n)$ to $L_c^r(\mathbb{C}^n)$.*

Proof: As noted before, $K_m^{(1)}(\varphi)$ is well defined a.e. and it is obviously linear; moreover, it has compact support in \mathbb{D} by definition. We know that, by Lemma 3.1, $\|G_1(\varphi)\|_{L^r(\mathbb{C}^n)} \leq M \|\varphi\|_{L^r(\mathbb{C}^n)}$ hence we have

$$\|K_0^{(1)}(\varphi)\|_r \leq A_0 \|\mathbb{1}_{C_0} G_1(\varphi)\|_{L^r} \leq A_0 M \|\varphi\|_{L^r},$$

where $M := \left\| \frac{1}{\pi z_1} \right\|_{L^1(\mathbb{D})}$.

For $j \geq 1$, $A_j(\hat{z}_1) = 1/m_1(C_j(\hat{z}_1))$, but $m_1(C_j(\hat{z}_1)) \geq \delta > 0$ uniformly in $\hat{z}_1 \in \mathbb{D}_{n-1}^n$ hence $A_j(\hat{z}_1) \geq \delta^{-1} < \infty$, uniformly in $\hat{z}_1 \in \mathbb{D}_{n-1}^n$. So we get

$$\|K_m^{(1)}(\varphi)\|_r \leq \|A_m(\cdot)\|_{L^\infty(\mathbb{D}_{n-1}^n)} \times \|\mathbb{1}_{C_0} G_1(\varphi)\|_{L^r} \leq \delta^{-1} M \|\varphi\|_{L^r}.$$

So for fixed $\hat{z}_1 \in \mathbb{D}_{n-1}^n$ $K_m^{(1)}(\varphi)$ has compact support in z_1 and, because it operates only in z_1 and φ has compact support in \mathbb{C}^n , then $K_m^{(1)}(\varphi)$ has compact support in \mathbb{C}^n . \blacksquare

Remark 4.4. The operator $K_0^{(1)}$ is also bounded from L_c^r to L_c^r , therefore continuous. The operators $K_m^{(1)}$ for $m \geq 1$ are not.

The following results link the quantities $[\varphi]_1(k)$ and $[\varphi, c_{j,1}]_1(k)$ with the corresponding ones for $K_0^{(1)}(\varphi)$ and $K_j^{(1)}(\varphi)$.

Lemma 4.5. We have

$$K_0^{(1)}(\varphi)(z) = A_0 \mathbb{1}_{C_0}(z_1) \sum_{k \geq 0} [\varphi]_1(k) z_1^{-k},$$

$$K_j^{(1)}(\varphi)(z) = A_j \mathbb{1}_{C_j(\hat{z}_1)}(z_1) \sum_{k \geq 0} [\varphi, j]_1(k) (z_1 - c_{j,1}(\hat{z}_1))^{k+1},$$

the convergence of the series being uniform in $\overline{C_0}$ or $\overline{C_j}$.

Proof: If $|z_1| > r + \delta$ and $(\zeta_1, \hat{z}_1) \in \text{supp}\varphi$, then

$$\frac{|\zeta_1|}{|z_1|} \leq \frac{r}{r + \delta} < 1.$$

so, in particular, if $z_1 \in C_0$, then $|z_1| > |\zeta_1|$. Therefore, if $z_1 \in C_0$, we have

$$\begin{aligned} G_1(z) &= \frac{1}{\pi} \int_{\mathbb{C}} \varphi(\zeta_1, \hat{z}_1) \frac{1}{z_1 - \zeta_1} dm_1(\zeta_1) = \frac{1}{\pi z_1} \int_{\mathbb{C}} \varphi(\zeta_1, \hat{z}_1) \frac{1}{1 - \frac{\zeta_1}{z_1}} dm_1(\zeta_1) = \\ &= \frac{1}{\pi z_1} \int_{\mathbb{C}} \varphi(\zeta_1, \hat{z}_1) \sum_{k \geq 0} \frac{\zeta_1^k}{z_1^k} dm_1(\zeta_1) = \frac{1}{z_1} \sum [\varphi]_1(k) z_1^{-k} = \sum_{k \geq 0} [\varphi]_1(k) z_1^{-k-1} \end{aligned}$$

So $K_0^{(1)}(\varphi) = A_0 \mathbb{1}_{C_0}(z_1) \sum_{k \geq 0} [\varphi]_1(k) z_1^{-k}$ and the convergence is obviously uniform on $\overline{C_0}$. On the other hand, if $z_1 \in \overline{C_j(\hat{z}_1)}$ and $(\zeta_1, \hat{z}_1) \in \text{supp}\varphi$, then

$$\frac{|z_1 - c_{j,1}(\hat{z}_1)|}{|\zeta_1 - c_{j,1}(\hat{z}_1)|} \leq \frac{2}{3} < 1,$$

so we have that, for $z_1 \in C_j(\hat{z}_1)$,

$$\begin{aligned} G_1(z) &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi}{z_1 - \zeta_1} dm_1(\zeta) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi}{(z_1 - c_{j,1}(\hat{z}_1)) + (c_{j,1}(\hat{z}_1) - \zeta_1)} dm_1(\zeta_1) = \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi}{\zeta_1 - c_{j,1}(\hat{z}_1)} \frac{1}{1 - \frac{z_1 - c_{j,1}(\hat{z}_1)}{\zeta_1 - c_{j,1}(\hat{z}_1)}} dm_1(\zeta_1) = \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\varphi}{\zeta_1 - c_{j,1}(\hat{z}_1)} \sum_{k \geq 0} \frac{(z_1 - c_{j,1}(\hat{z}_1))^k}{(\zeta_1 - c_{j,1}(\hat{z}_1))^k} dm_1(\zeta_1) = \sum_{k \geq 0} [\varphi, j]_1(k) (z_1 - c_{j,1}(\hat{z}_1))^k. \end{aligned}$$

So $K_j^{(1)}(\varphi) = A_j(\hat{z}_1) \mathbf{1}_{C_j(\hat{z}_1)}(z_1) \sum_{k \geq 0} [\varphi, j]_1(k) (z_1 - c_{j,1}(\hat{z}_1))^{k+1}$ and the convergence is obviously uniform on $\overline{C_j}$. \blacksquare

We set

$$K^{(1)}(\varphi) = \sum_{m=0}^{N_1} K_m^{(1)}(\varphi).$$

Proposition 4.6. *We have $[K^{(1)}(\varphi)]_1 = [\varphi]_1$ and $[K^{(1)}(\varphi), j]_1 = [\varphi, j]_1$.*

Proof: We divide the proof in several steps.

1. $[K_0^{(1)}(\varphi)]_1(k) = [\varphi]_1(k)$ - We calculate

$$\begin{aligned} H(z) &= K_0^{(1)}(\varphi) \star_1 \frac{1}{z_1} = \left(A_0 \mathbf{1}_{C_0}(z_1) \sum_{k \geq 0} [\varphi]_1(k) z_1^{-k} \right) \star_1 \frac{1}{z_1} = \\ &= A_0 \sum_{k \geq 0} [\varphi]_1(k) (\mathbf{1}_{C_0}(z_1) z_1^{-k} \star_1 \frac{1}{z_1}) = A_0 \sum_{k \geq 0} [\varphi]_1(k) \int_{C_0} \frac{\zeta_1^{-k}}{z_1 - \zeta_1} dm_1(\zeta_1). \end{aligned}$$

If $z_1 \notin \mathbb{D}$, we know that

$$\int_{C_0} \frac{\zeta_1^{-k}}{z_1 - \zeta_1} dm_1(\zeta_1) = A_0^{-1} z_1^{-k-1}$$

so

$$H(z) = A_0 \sum_{k \geq 0} [\varphi]_1(k) (A_0^{-1} z_1^{-k-1}) = \sum_{k \geq 0} [\varphi]_1(k) z_1^{-k-1}.$$

Then we have that, if $z_1 \notin \mathbb{D}$,

$$H(z) = G_1(z)$$

so, by Lemma 3.3, $[\varphi]_1(k) = [K_0^{(1)}(\varphi)]_1(k)$.

2. $[K_j^{(1)}(\varphi)]_1(k) = 0$ for $j > 0$ - We calculate

$$H_j(z) = K_j^{(1)}(\varphi) \star_1 \frac{1}{z_1} = \left(A_j(\hat{z}_1) \mathbf{1}_{C_j(\hat{z}_1)}(z_1) \sum_{k \geq 0} [\varphi, j]_1(k) (z_1 - c_{j,1}(\hat{z}_1))^{k+1} \right) \star_1 \frac{1}{z_1} =$$

$$A_j(\hat{z}_1) \sum_{k \geq 0} [\varphi, j]_1(k) (\mathbb{1}_{C_j(\hat{z}_1)}(z_1 - c_{j,1}(\hat{z}_1))^{k+1} \star_1 \frac{1}{z_1}) =$$

$$A_j(\hat{z}_1) \sum_{k \geq 0} k \geq 0 [\varphi, j]_1(k) \int_{C_j} \frac{(\zeta_1 - c_{j,1}(\hat{z}_1))^{k+1}}{z_1 - \zeta_1} dm_1(\zeta_1).$$

If $|z_1 - c_{j,1}(\hat{z}_1)| > r_{j,1}(\hat{z}_1)$, then

$$\int_{C_j} \frac{(\zeta_1 - c_{j,1}(\hat{z}_1))^{k+1}}{z_1 - \zeta_1} dm_1(\zeta_1) = 0$$

for every $k \geq 0$. Therefore $H_j(z) = 0$, so by Lemma 3.3 $0 = [K_j^{(1)}(\varphi)]_1(k)$.

3. $[K_j^{(1)}(\varphi), j]_1(k) = [\varphi, j]_1(k)$ for $j > 0$ - By direct computation, using Lemma 4.5, we have

$$[K_j^{(1)}\varphi, j]_1(l) = A_j(\hat{z}_1) \sum_{k \geq 0} [\varphi, j]_1(k) \int_{C_j(\hat{z}_1)} (\zeta_1 - c_{j,1}(\hat{z}_1))^{k+1} (\zeta_1 - c_{j,1}(\hat{z}_1))^{-l-1} dm_1(\zeta_1) =$$

$$\sum_{k \geq 0} [\varphi, j]_1(k) \delta_{k,l} = [\varphi, j]_1(l).$$

4. $[K_m^{(1)}\varphi, j]_1(k) = 0$ if $m \neq j$ - By step 2, $H_m(z) = 0$ if $|z_1 - c_{m,1}(\hat{z}_1)| > r_{m,1}(\hat{z}_1)$, so in particular if $z_1 \in D(c_{j,1}(\hat{z}_1), r_{j,1}(\hat{z}_1))$ with $j \neq m$, we have $H(z) = 0$. By Lemma 3.4, it follows that

$$[K_m^{(1)}(\varphi), j]_1(k) = 0$$

if $m \neq j$ and $m \neq 0$.

If $m = 0$, we notice that, if $|z_1| < r$,

$$H(z) = A_0 \sum_{k \geq 0} [\varphi]_1(k) \int_{C_0} \frac{\zeta_1^{-k}}{z_1 - \zeta_1} dm_1(\zeta_1)$$

and

$$\int_{C_0} \frac{\zeta_1^{-k}}{z_1 - \zeta_1} dm_1(\zeta_1) = 0$$

for every k , as $|z_1| < r < |\zeta_1|$. So $H(z) = 0$ and by Lemma 3.3 we have that $[K_0^{(1)}(\varphi), j]_1(k) = 0$ for every k . ■

Corollary 4.7. *Let $\varphi \in L_c^r(\mathbb{D}^n \setminus Z)$, there are $\varphi_1, \dots, \varphi_n$, all in $L_c^r(\mathbb{D}^n \setminus Z)$ and such that $\varphi = \varphi_1 + \dots + \varphi_n$, $\forall i < n$, $\forall j = 1, \dots, N_i$, $[\varphi_i]_i = [\varphi_i, j]_i = 0$.*

Proof: We set $\varphi_1 := \varphi - K^{(1)}\varphi$ and we notice that $[\varphi_1]_1 = 0$, $[\varphi_1, j]_1 = 0$ for every $j = 1, \dots, N_1$.

Now, we can repeat this procedure replacing z_1 by z_2 and φ by $K^{(1)}(\varphi)$; we will apply then the operators $K_m^{(2)}$, defined with respect to the variable z_2 , with the relative coronas.

We set $\varphi_2 := K^{(1)}\varphi - K^{(2)}K^{(1)}\varphi$ with the property that $[\varphi_2]_2 = 0$, $[\varphi_2, j]_2 = 0$ for every $j = 1, \dots, N_2$.

Iterating the algorithm we set $\varphi_{n-1} := K^{(n-2)} \dots K^{(1)}\varphi - K^{(n-1)} \dots K^{(1)}\varphi$ and

$$\varphi_n := \varphi - \varphi_1 - \dots - \varphi_{n-1}.$$

By an easy recursion we have

$$\varphi_n = K^{(n-1)} \dots K^{(1)}\varphi$$

with, of course $\varphi = \varphi_1 + \dots + \varphi_n$.

So finally we find a decomposition $\varphi = \varphi_1 + \dots + \varphi_n$ such that, for $i < n$, we have $[\varphi_i]_i = 0$, $[\varphi_i, j]_i = 0$ for every $j = 1, \dots, N_i$. ■

We have a first result on solvability of the Cauchy-Riemann equation with some control on the support of the solution.

Theorem 4.8. *Let $f \in \mathcal{O}(\overline{\mathbb{D}^n})$ be a holomorphic function in a neighbourhood of the closed unit polydisc in \mathbb{C}^n and set $Z = f^{-1}(0)$. If ω is a $(0, n)$ -form in $L^r_c(\mathbb{D}^n \setminus Z)$, then for every $k \in \mathbb{N}$ we can find a $(0, n-1)$ -form $\eta \in L^r(\mathbb{D}^n)$ such that $f^{-k}\eta \in L^r(\mathbb{D}^n)$ and all the coefficients of η but at most one are in $L^r_c(\mathbb{D}^n \setminus Z)$; moreover, η is such that $\bar{\partial}\eta = \omega$.*

Proof: We write

$$\omega = \phi d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n$$

and apply the result of Corollary 4.7 to $f^{-k}\phi$. We get

$$f^{-k}\phi = \phi_1 + \dots + \phi_n$$

and $[\phi_i]_i = [\phi_i, h]_i = 0$ for $i = 1, \dots, n-1$ and $h = 1, \dots, N_i$. Therefore, by Lemma 3.2, the functions

$$F_1 = \phi_1 \star_1 \frac{1}{\pi z_1}, \quad \dots, \quad F_{n-1} = \phi_{n-1} \star_{n-1} \frac{1}{\pi z_{n-1}}$$

are compactly supported in $\mathbb{D}^n \setminus Z$. However, for

$$F_n = \phi_n \star_n \frac{1}{\pi z_n}$$

we only know that $F_n \in L^r(\mathbb{D}^n)$. We note that

$$\bar{\partial}(F_1 d\hat{z}_1 + \dots + F_n d\hat{z}_n) = f^{-k}\phi,$$

therefore we define

$$\eta = f^k(F_1 d\hat{z}_1 + \dots + F_n d\hat{z}_n)$$

and we have

$$\bar{\partial}\eta = f^k \bar{\partial}(F_1 d\hat{z}_1 + \dots + F_n d\hat{z}_n) = \phi.$$

It is easy to see that η satisfies all the requests of the theorem. ■

5 Obstructions to a solution with compact support

Let us define the two quantities which tell us when the last term in the decomposition from Corollary 4.7 verifies also

$$\forall j = 1, \dots, N_n, [\varphi_n]_n = 0, [\varphi_n, j]_n = 0.$$

We note that

$$\varphi_n = K^{(n-1)} \dots K^{(1)} \varphi$$

and, more precisely, we have

$$\varphi_n = \sum_{m_{n-1}=0}^{N_{n-1}} \dots \sum_{m_1=0}^{N_1} K_{m_{n-1}}^{(n-1)} \dots K_{m_1}^{(1)}(\varphi).$$

We set

$$M_{n-1} := \{(m_1, \dots, m_{n-1}) :: m_j \leq N_j\} \subset \mathbb{N}^{n-1};$$

$$\mu = (m_1, \dots, m_{n-1}) \in M_{n-1}, I(\mu) := \{k \leq n-1 :: m_k = 0\}, l = (l_1, \dots, l_{n-1}) \in \mathbb{N}^{n-1}$$

and

$$J_{\mu, l}^{(0)}(\varphi)(k) := \frac{1}{\pi^n} \int_{\mathbb{C}^n} \varphi(\zeta) \zeta_n^k \prod_{i \in I(\mu)} \zeta_i^{l_i} \prod_{j \notin I(\mu)} \mathbb{1}_{C_{m_j}^{(j)}(z, \zeta)}(z_j) \frac{(z_j - c_{m_j, j}(z, \zeta))^{l_j+1}}{(\zeta_j - c_{m_j, j}(z, \zeta))^{-l_j-1}} dm_n(\zeta)$$

$$J_{\mu, l}^{(j)}(\varphi)(k) := \frac{1}{\pi^n} \int_{\mathbb{C}^n} \varphi(\zeta) (\zeta_n - c_{j, n})^{-k-1} \prod_{i \in I(\mu)} \zeta_i^{l_i} \prod_{s \notin I(\mu)} \mathbb{1}_{C_{m_s}^{(s)}(z, \zeta)}(z_s) \frac{(z_s - c_{m_s, s}(z, \zeta))^{l_s+1}}{(\zeta_s - c_{m_s, s}(z, \zeta))^{-l_s-1}} dm_n(\zeta);$$

where

$$c_{h, k}(z, \zeta) = c_{h, k}(z_1, \dots, z_{k-1}, \zeta_{k+1}, \dots, \zeta_n) \quad 1 < k < n$$

$$c_{h, 1}(z, \zeta) = c_{h, 1}(\zeta_2, \dots, \zeta_n)$$

$$c_{h, n}(z, \zeta) = c_{h, n}(z_1, \dots, z_{n-1}).$$

and the same notation is used for $\mathbb{1}_{C_k^{(j)}(z, \zeta)}(z_j)$. We have the link :

Theorem 5.1. *If $J_{\mu, l}^{(0)}(\varphi) = 0$ for every $\mu \in M_{n-1}$ and $l \in \mathbb{N}^n$, then $[\varphi_n]_n = 0$; given also $j = 1, \dots, N_n$, if $J_{\mu, l}^{(j)}(\varphi) = 0$ for every $\mu \in M_{n-1}$ and $l \in \mathbb{N}^n$, then $[\varphi_n, j]_n = 0$.*

Proof: By direct calculation, using the series expansions given by Lemma 4.5, we have that

$$[K_0^{(h)}(\psi)]_{h+1}(k) = \frac{1}{\pi} A_0^{(h)} \mathbb{1}_{C_0^{(h)}(z_h)} \sum_{l \geq 0} z_h^{-l} \int_{\mathbb{C}} [\psi]_h(l) \zeta_{h+1}^k dm_1(\zeta_{h+1})$$

$$[K_0^{(h)}(\psi), m]_{h+1}(k) = \frac{1}{\pi} A_0^{(h)} \mathbb{1}_{C_0^{(h)}(z_h)} \sum_{l \geq 0} z_h^{-l} \int_{\mathbb{C}} \frac{[\psi]_h(l)}{(\zeta_{h+1} - c_{m, h+1})^{k+1}} dm_1(\zeta_{h+1})$$

$$[K_j^{(h)}(\psi)]_{h+1}(k) = \frac{1}{\pi} A_j^{(h)} \mathbb{1}_{C_j^{(h)}(z_h)} \sum_{l \geq 0} (z_h - c_{j, h})^{l+1} \int_{\mathbb{C}} [\psi, j]_h(l) \zeta_{h+1}^k dm_1(\zeta_{h+1})$$

$$[K_j^{(h)}(\psi), m]_{h+1}(k) = \frac{1}{\pi} A_j^{(h)} \mathbb{1}_{C_j^{(h)}(z_h)} \sum_{l \geq 0} (z_h - c_{j, h})^{l+1} \int_{\mathbb{C}} \frac{[\psi, j]_h(l)}{(\zeta_{h+1} - c_{m, h+1})^{k+1}} dm_1(\zeta_{h+1}).$$

Therefore, by induction, we obtain that

$$[K_{\mu_{n-1}}^{(n-1)} \cdots K_{\mu_1}^{(1)} \phi]_n(l_n) = \prod_{i=1}^{n-1} A_{\mu_i}^{(i)} \prod_{i \in I(\mu)} \mathbb{1}_{C_{\mu_i}^{(i)}}(z_i) \sum_{l' \in \mathbb{N}^{n-1}} \prod_{i \in I(\mu)} z_i^{-l'_i} J_{\mu, l' \cup \{l_n\}}^{(0)}(\phi)$$

and

$$[K_{\mu_{n-1}}^{(n-1)} \cdots K_{\mu_1}^{(1)} \phi, j]_n(l_n) = \prod_{i=1}^{n-1} A_{\mu_i}^{(i)} \prod_{i \in I(\mu)} \mathbb{1}_{C_{\mu_i}^{(i)}}(z_i) \sum_{l' \in \mathbb{N}^{n-1}} \prod_{i \in I(\mu)} z_i^{-l'_i} J_{\mu, l' \cup \{l_n\}}^{(j)}(\phi).$$

So, if $J_{\mu, l}^{(0)}(\phi) = J_{\mu, l}^{(j)}(\phi) = 0$, all the coefficients vanish, then

$$[\phi_n]_n(k) = 0 \quad [\phi_n, j]_n(k) = 0$$

as we wanted. ■

Definition 5.2. We shall say that $\varphi \in L_c^r(\mathbb{D}^n \setminus Z)$ verifies the structure conditions if $J_{\mu, l}^{(0)}(\varphi) = 0$ for every $\mu \in M_{n-1}$ and $l \in \mathbb{N}^n$, and if $J_{\mu, l}^{(j)}(\varphi) = 0$ for every $\mu \in M_{n-1}$ and $l \in \mathbb{N}^n$.

6 The polydisc - $q = n$

As for now, we don't have a way to deal with the integrals $J_{\mu, l}^{(m)}(k)$ on the domain $\mathbb{D}^n \setminus Z$, so we turn to the much easier case of the polydisc itself. We look first at the problem for $(0, n)$ -forms. Let ω be a $(0, n)$ -form with $L_c^r(\mathbb{D}^n)$ coefficients; we can find a function $\varphi \in L_c^r(\mathbb{D}^n)$ such that

$$\omega = \varphi d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

In this case, the operators $K^{(m)}$ coincide with the outer corona components $K_0^{(m)}$, so the obstructions to a solution of compact support are given by the integrals $J_{0, l}^{(0)}(k)$, where the subscript 0 stands for a multi-index of the appropriate length containing only 0s. We have the following result.

Lemma 6.1. *If there is a current T with compact support in \mathbb{D}^n such that $\bar{\partial}T = \omega$, then we have*

$$\forall l \in \mathbb{N}^{n-1}, \forall j = 1, \dots, N_n, J_{0, l}^{(0)}(\varphi) = 0,$$

i.e. φ verifies the structure conditions for the polydisc.

Proof: Let $\{\rho_\epsilon\} \subset \mathcal{D}(\mathbb{C}^n)$ be a family of functions such that $\rho_\epsilon \rightarrow \delta_0$, when $\epsilon \rightarrow 0$, in the sense of distributions, with $\text{supp} \rho_\epsilon \subset \{|z| < \epsilon\}$ and $\|\rho_\epsilon\|_1 = 1$.

We write

$$T = T_1 d\hat{z}_1 + \dots + T_n d\hat{z}_n$$

so we have

$$\varphi = \bar{\partial}_1 T_1 + \dots + \bar{\partial}_n T_n = t_1 + \dots + t_n$$

where, obviously, every t_h is compactly supported in \mathbb{D}^n .

We set $T_h^\epsilon = T_h \star \rho_\epsilon \in \mathcal{D}(\mathbb{C}^n)$; by standard theorems on convolution,

$$\text{supp}(T_h^\epsilon) \subseteq \{z \mid \text{dist}(z, \text{supp}T_h) \leq \epsilon\}$$

so, for ϵ small enough, all the regularized functions are compactly supported in \mathbb{D}^n and

$$\bar{\partial}_h T_h^\epsilon = t_h \star \rho_\epsilon = t_h^\epsilon .$$

By Lemma 3.2, we have that

$$[t_h^\epsilon]_h(k) = 0$$

for every $k \in \mathbb{N}$ and $h = 1, \dots, n$.

Moreover, we have that

$$\varphi^\epsilon = \varphi \star \rho_\epsilon = t_1^\epsilon + \dots + t_n^\epsilon$$

and $\varphi^\epsilon \rightarrow \varphi$ in L^r as $\epsilon \rightarrow 0$.

As φ and φ^ϵ are compactly supported in \mathbb{D}^n , for ϵ small enough, we can see them as continuous functionals on $L_{\text{loc}}^q(\mathbb{D}^n)$ (where $q^{-1} + r^{-1} = 1$). The convergence $\varphi^\epsilon \rightarrow \varphi$ holds also in this sense. The functions $\zeta_n^k \prod_{i=1}^n \zeta_i^{l_i}$ are in $L_{\text{loc}}^q(\mathbb{D}^n)$ for every $l \in \mathbb{N}^{n-1}$, $k \in \mathbb{N}$; therefore

$$J_{0,l}^{(0)}(\varphi^\epsilon)(k) \xrightarrow{\epsilon \rightarrow 0} J_{0,l}^{(0)}(\varphi)(k) .$$

Now, consider t_h^ϵ , with $h \leq n-1$; we know that $[t_h^\epsilon]_h(l) = 0$, for every l so we can apply Fubini and get

$$\begin{aligned} J_{0,l}^{(0)}(t_h^\epsilon)(k) &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} t_h^\epsilon(\zeta) \zeta_n^k \prod_{i=1}^n \zeta_i^{l_i} dm_n(\zeta) = \\ &= \frac{1}{\pi^n} \int_{\mathbb{C}^{n-1}} \zeta_n^k \prod_{\substack{i=1 \\ i \neq h}}^n \zeta_i^{l_i} \int_{\mathbb{C}} t_h^\epsilon(\zeta) \zeta_h^{l_h} dm_1(\zeta_h) dm_{n-1}(\hat{\zeta}_h) = \\ &= \frac{1}{\pi^n} \int_{\mathbb{C}^{n-1}} \zeta_n^k \prod_{\substack{i=1 \\ i \neq h}}^n \zeta_i^{l_i} [t_h^\epsilon]_h(l_h) dm_{n-1}(\hat{\zeta}_h) = 0 ; \end{aligned}$$

If $h = n$, it is again an application of Fubini's theorem to show that $J_{0,l}^{(0)}(t_n^\epsilon)(k) = 0$.

By additivity of the integral, it follows that $J_{0,l}^{(0)}(\varphi^\epsilon)(k) = 0$, so letting $\epsilon \rightarrow 0$ we obtain the thesis. \blacksquare

Theorem 6.2. *If ω is a $(0, n)$ -form in $L_c^r(\mathbb{D}^n)$ such that there is a $(0, n-1)$ current T , compactly supported in \mathbb{D}^n , such that $\bar{\partial}T = \omega$, then we can find a $(p, n-1)$ -form $\eta \in L_c^r(\mathbb{D}^n)$ such that $\bar{\partial}\eta = \omega$.*

Proof: By Corollary 4.7 we can write $\varphi = \varphi_1 + \dots + \varphi_n$ and, by Lemma 3.2, the convolutions

$$f_1 = \varphi_1 \star_1 \frac{1}{\pi z_1}, \dots, f_{n-1} = \varphi_{n-1} \star_{n-1} \frac{1}{\pi z_{n-1}}$$

are compactly supported and

$$\bar{\partial}_1 f_1 + \dots + \bar{\partial}_{n-1} f_{n-1} = \varphi_1 + \dots + \varphi_{n-1} = \varphi - \varphi_n .$$

Moreover, by Lemma 6.1, ϕ satisfies the structure conditions, then, by Theorem 5.1, $[\phi_n]_n(k) = 0$ for every $k \in \mathbb{N}$. So, also

$$f_n = \varphi_n \star_n \frac{1}{\pi z_n}$$

is compactly supported, always by Lemma 3.2.

We set

$$\eta = \sum_{j=1}^n (-1)^{j-1} f_j d\hat{z}_j$$

so that

$$\bar{\partial}\eta = \varphi d\bar{z}$$

and the coefficients of η belong to $L_c^r(\mathbb{D}^n)$. ■

Remark 6.3. We have that $\|f_j\|_r \leq \gamma \|\phi_j\|_r$, where γ depends only on the dimension n and on the radii of \mathbb{D}^n . We recall that $\|K_0^{(m)}\phi\|_r \leq A_0 M \|\phi\|_r$, so $\|f_j\|_r \leq (A_0 M + 1)^j \gamma \|\phi\|_r$; this means that the linear operator associating to ω the solution η is linear and bounded from L_c^r to L_c^r .

7 The polydisc - $q = n - 1$

We turn our attention to $(0, n - 1)$ -forms. Firstly, we give a refined version of Lemma 6.1.

Proposition 7.1. Suppose $\varphi \in L^r(\mathbb{C}^n)$ and T_1, \dots, T_{n-1} are distributions, compactly supported in \mathbb{D}^n , such that

$$\varphi = \bar{\partial}_1 T_1 + \dots + \bar{\partial}_{n-1} T_{n-1} .$$

Then we can find $\varphi_1, \dots, \varphi_{n-1} \in L^r(\mathbb{C}^n)$, compactly supported in P such that $\varphi = \varphi_1 + \dots + \varphi_{n-1}$ and $[\varphi_i]_i(k) = 0$, for every $k \in \mathbb{N}$.

Proof: After performing the same regularization as in the proof of Lemma 6.1, we have

$$\frac{1}{\pi^n} \int_{\mathbb{C}^n} t_h^\epsilon(\zeta) a(\zeta_n) \prod_{i=1}^{n-1} \zeta_i^{l_i} dm_n(\zeta) = 0$$

for every $a(\zeta_n)$ for which the integral is well-defined (e.g. $a \in L^1$). This is because h ranges from 1 to $n - 1$, so we can isolate the terms $[t_h^\epsilon]_h(l)$ employing only the functions which appear in the product.

Therefore, the function

$$\frac{1}{\pi^n} \int_{\mathbb{C}^{n-1}} t_h^\epsilon(\zeta) \prod_{i=1}^{n-1} \zeta_i^{l_i} dm_n(\hat{\zeta}_n)$$

vanishes for a.e. z_n and the same is true for the function

$$\frac{1}{\pi^n} \int_{\mathbb{C}^{n-1}} \varphi^\epsilon(\zeta) \prod_{i=1}^{n-1} \zeta_i^{l_i} dm_n(\hat{\zeta}_n)$$

and, letting $\epsilon \rightarrow 0$, also for

$$\frac{1}{\pi^n} \int_{\mathbb{C}^{n-1}} \varphi(\zeta) \prod_{i=1}^{n-1} \zeta_i^{l_i} dm_n(\hat{\zeta}_n).$$

By the analogue of Theorem 5.1 in the first $n - 1$ coordinates,

$$[K^{(n-2)} \dots K^{(1)} \varphi]_{n-1}(k) = 0,$$

so defining $\varphi_1, \dots, \varphi_{n-2}$ as in Corollary 4.7 and setting $\varphi_{n-1} = \varphi - \varphi_1 - \dots - \varphi_{n-2}$ we have that $[\varphi_i]_i(k) = 0$, as requested. \blacksquare

The following corollary is immediate.

Corollary 7.2. *Given ω as before and a current T , compactly supported in \mathbb{D}^n such that $\bar{\partial}T = \omega$, with*

$$T = T_1 d\hat{z}_1 + \dots + T_{n-1} d\hat{z}_{n-1},$$

we can find η with $L^r(\mathbb{C}^n)$ coefficients, compactly supported in \mathbb{D}^n , such that $\bar{\partial}\eta = \omega$ and with

$$\eta = \eta_1 d\hat{z}_1 + \dots + \eta_{n-1} d\hat{z}_{n-1}.$$

Remark 7.3. *Obviously, we can suppose that the coefficient of $d\hat{z}_k$ in T is zero and obtain that there exists a solution with coefficients in $L^r(\mathbb{C}^n)$ with compact support in \mathbb{D}^n where the coefficient of $d\hat{z}_k$ is zero.*

By induction, we can show that if there exists a solution with the coefficients of $d\hat{z}_{k_1}, \dots, d\hat{z}_{k_r}$ equal to zero, then we can produce a solution in L^r with the same vanishing coefficients.

We note that the construction of $\varphi_1, \dots, \varphi_{n-1}$ doesn't involve the n -th coordinate, so, $\bar{\partial}_n \varphi$ and $\bar{\partial}_n \varphi_j$ share the same regularity, whatever it is.

Theorem 7.4. *If ω is a $(0, n - 1)$ -form in $L_c^r(\mathbb{D}^n)$, $\bar{\partial}\omega = 0$, such that $\bar{\partial}_n \omega_n \in L^r$, then we can find a $(0, n - 2)$ -form $\beta \in L_c^r(\mathbb{D}^n)$ such that $\bar{\partial}\beta = \omega$.*

Proof: We proceed by induction on n ; the case $n = 2$ is true. If there exists a distribution T with compact support such that $\bar{\partial}T = \omega$, then, by Corollary 7.2, we have

$$\omega_n = \sum_{j=1}^{n-1} \omega_{nj}$$

with $\omega_{nj} \in L^r$ and $[\omega_{nj}]_j(k) = 0$.

We consider the following family of compactly supported $(0, n - 2)$ -forms in \mathbb{C}^{n-1} , depending on the parameter z_n :

$$\psi_{z_n} = \sum_{j=1}^{n-1} \left(\omega_j + (-1)^{n+j} \frac{\partial \omega_{nj}}{\partial \bar{z}_n} \star_j \frac{1}{\pi z_j} \right) d\hat{z}_j .$$

Note that, as ψ_{z_n} is thought as a form in \mathbb{C}^{n-1} , the notation $d\hat{z}_j$ has to be understood as the exterior product of the differentials $d\bar{z}_1, \dots, d\bar{z}_{n-1}$, with $d\bar{z}_j$ missing.

Now, we have that

$$(\bar{\partial}' \psi_{z_n}) \wedge d\bar{z}_n = \bar{\partial} \omega = 0$$

where $\bar{\partial}'$ operates in the first $n - 1$ coordinates. We note that

$$\frac{\partial}{\partial \bar{z}_j} \left(\omega_j + (-1)^{n+j} \frac{\partial \omega_{nj}}{\partial \bar{z}_n} \star_j \frac{1}{\pi z_j} \right) = \bar{\partial}_j \omega_j + (-1)^{n+j} \bar{\partial}_n \omega_{nj}$$

belongs to $L^r(\mathbb{C}^n)$ for almost all z_n . By inductive hypothesis, we can solve $\bar{\partial}' \xi_{z_n} = \psi_{z_n}$ with compact support (and the result will be in $L^r(\mathbb{C}^n)$).

We have $\bar{\partial}(\xi_{z_n} \wedge d\bar{z}_n) = \psi_{z_n} \wedge d\bar{z}_n$; we define a $(0, n - 2)$ -form in \mathbb{C}^n with

$$\gamma = \sum_{j=1}^{n-1} (-1)^{j-1} \omega_{nj} \star_j \frac{1}{\pi z_j} d\hat{z}_{jn} .$$

So we have

$$\bar{\partial} \gamma = \omega_n d\hat{z}_n + \sum_{j=1}^{n-1} (-1)^{n+j-2} \frac{\partial \omega_{nj}}{\partial \bar{z}_n} \star_j \frac{1}{\pi z_j} ;$$

therefore

$$\bar{\partial}(\gamma + \xi_{z_n} \wedge d\bar{z}_n) = \omega .$$

The form $\gamma + \xi_{z_n} \wedge d\bar{z}_n$ has compact support and belongs to $L^r(\mathbb{C}^n)$. ■

8 The polydisc - $1 < q < n - 1$

Let ω be a generic $(0, q)$ -form and let us write

$$\omega = \sum_{|J|=n-q} \omega_J d\hat{z}_J .$$

We restate here the condition $(*)$ given in the introduction

$$(*) \quad \bar{\partial}_{j_{n-q}} \cdots \bar{\partial}_{j_k} \omega_J \in L^r(\mathbb{C}^n) \quad k = 1, \dots, n - q, \quad \forall |J| = n - q . \quad (8.1)$$

Theorem 8.1. *If ω is a $(0, q)$ -form in $L_c^r(\mathbb{D}^n)$, $\bar{\partial} \omega = 0$, fulfilling condition (8.1), then we can find a $(0, q - 1)$ -form $\beta \in L_c^r(\mathbb{D}^n)$ such that $\bar{\partial} \beta = \omega$.*

Proof: Following Hörmander [7, Chapter 2], we can write

$$\omega = g \wedge d\bar{z}_n + h$$

where g, h do not contain $d\bar{z}_n$.

We can look at h as a family of $(0, q)$ -forms in \mathbb{C}^{n-1} , depending on the complex parameter z_n ; similarly, g can be understood as a family of $(0, q-1)$ -forms.

We denote by $\bar{\partial}_{\mathbb{C}^{n-1}}$ the $\bar{\partial}$ operator in the first $n-1$ variables, that is

$$\bar{\partial}_{\mathbb{C}^{n-1}}\psi = \sum_{n \notin I} \sum_{k \notin I \cup \{n\}} \bar{\partial}_k \psi_I d\bar{z}_k \wedge d\bar{z}_I.$$

If ψ doesn't contain $d\bar{z}_n$, then $\bar{\partial}'\psi = \bar{\partial}_{\mathbb{C}^{n-1}}\psi$.

We proceed by induction on the dimension and we prove the following:

I_n.1 the statement of the theorem holds in \mathbb{C}^n and β depends linearly on ω ;

I_n.2 if the coefficients of ω depend on a parameter $z_{n+1} \in \mathbb{C}$ in such a way that $\omega, \bar{\partial}\omega \in L_c^r(\mathbb{C}^{n+1})$, then also $\beta, \bar{\partial}\beta \in L_c^r(\mathbb{C}^{n+1})$, where the $\bar{\partial}$ is intended in $n+1$ variables.

We note that $I_{2.1}$ and $I_{2.2}$ hold. We assume $I_{n-1.1}$ and $I_{n-1.2}$ to hold.

Reduction. We note that $\bar{\partial}_{\mathbb{C}^{n-1}}h = 0$; therefore, h is a family of $\bar{\partial}$ -closed $(0, q)$ -forms in \mathbb{C}^{n-1} depending on the parameter z_n . Moreover, by assumption, $\bar{\partial}_n h_I \in L_c^r(\mathbb{C}^n)$. We denote by U_t the $(n-1)$ -dimensional open set $\mathbb{D}^n \cap \{z_n = t\}$ and we note that U_t is still a polydisc, hence Stein, for every t for which it is non-empty.

As a well known consequence of Serre's duality (see [17]) we have $H_c^q(U_t, \mathcal{O}) = 0$, if $2 \leq q \leq n-2$; therefore we can find a family T of $(0, q-1)$ -currents in \mathbb{C}^{n-1} such that $\bar{\partial}_{\mathbb{C}^{n-1}}T = h$ for almost every z_n . Then, by $I_{n-1.2}$, we can find a family H with $H \in L_c^r(\mathbb{D}^n)$ (and therefore $H_{z_n} \in L_c^r(U_{z_n})$ for almost every z_n) and with $\bar{\partial}H \in L_c^r(\mathbb{C}^n)$.

Moreover, as H_{z_n} depends linearly on h by $I_{n-1.1}$, if $h_{z_n} = 0$, then also $H_{z_n} = 0$. Therefore, H is compactly supported in P .

Now,

$$\bar{\partial}H = \bar{\partial}_{\mathbb{C}^{n-1}}H + \sum_I \bar{\partial}_n H_I d\bar{z}_n \wedge d\bar{z}_I = h + \sum_I \bar{\partial}_n H_I d\bar{z}_n \wedge d\bar{z}_I$$

so

$$\omega - \bar{\partial}H = g' \wedge d\bar{z}_n$$

where g' does not contain $d\bar{z}_n$. Moreover, as ω and $\bar{\partial}H$ are in $L_c^r(\mathbb{D}^n)$, also g' is. Further, we observe that

$$(\bar{\partial}_{\mathbb{C}^{n-1}}g') \wedge d\bar{z}_n = \bar{\partial}(\omega - \bar{\partial}H) = \bar{\partial}\omega = 0$$

and finally, for z_n fixed, g' is a $(0, q-2)$ -form in \mathbb{C}^{n-1} , fulfilling condition (8.1).

Solution. We have reduced ourselves to solve $\bar{\partial}G = g' \wedge d\bar{z}_n$, but as $\bar{\partial}_{\mathbb{C}^{n-1}}g' = 0$, we can, by the same argument used in the reduction, obtain a family G' of $(0, q-2)$ forms in \mathbb{C}^{n-1} such that $\bar{\partial}_{\mathbb{C}^{n-1}}G' = g'$, by $I_{n-1.2}$.

Again, by the same reasoning, $G' \in L_c^r(\mathbb{D}^n)$ and if we set $G = G' \wedge d\bar{z}_n$, we obtain a $(0, q-1)$ -form $G \in L_c^r(\mathbb{D}^n)$ such that $\bar{\partial}G = g' \wedge d\bar{z}_n$.

So, $\beta = G + H$ is the solution we looked for. This shows $I_n.1$.

To show $I_n.2$ it is enough to notice that all our operations are constructive and preserve the regularity (or summability) of an extra parameter. ■

Remark 8.2. *We have to separate the case of $(0, n-1)$ -forms from the general case because in that case Serre's duality tells us only that $H_c^{n-1}(U_t, \mathcal{O})$ is equal to the topological dual of $H^0(U_t, \Omega^{n-1})$, in general not vanishing, so the induction doesn't work there.*

Remark 8.3. *We note that, in the proof of Theorem 8.1, we never actually used the fact that our domain is the polydisc. Indeed, if we had the analogues of Theorems 6.2 and 7.4 for the domain $\mathbb{D}^n \setminus Z$ in every dimension, then we could apply the same proof to get Theorem 8.1 for $\mathbb{D}^n \setminus Z$, with exactly the same statement.*

As a corollary of the previous results, we obtain the following.

Corollary 8.4. *Let ω be a $(0, q)$ -form with compact support in $\mathbb{D}^n \setminus Z$ and satisfying conditions 8.1, then, for any $k \in \mathbb{N}$, we can find a $(0, q-1)$ -form $\beta \in L_c^r(\mathbb{D}^n)$ such that $\bar{\partial}(f^k\beta) = \omega$. Equivalently, we can find a $(0, q-1)$ -form $\eta = f^k\beta$ such that $\eta \in L_c^r(\mathbb{D}^n)$, η is 0 on Z up to order k and $\bar{\partial}\eta = \omega$.*

Proof: The $(0, q)$ -form $\phi := \omega/f^k$ is still $\bar{\partial}$ -closed and satisfies 8.1; hence we have a $(0, q-1)$ -form $\beta \in L_c^r(\mathbb{D}^n)$ such that $\bar{\partial}\beta = \phi$. So $\eta = f^k\beta$ verifies all the requirements. ■

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