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Abstract

In this paper, some results on the asymptotic behavior of stiff thin interfaces in elastostatics are recalled. A specific study of stiff interfaces in elastodynamics is presented and a numerical procedure is given.

Key words: Thin film, Elasticity, Dynamics, Asymptotic analysis, Adhesive bonding, Imperfect interface

1 Introduction

The consideration of interfaces became a major challenge in mechanical and civil engineering. For example, the more and more important use of structural bonding led to the development of new techniques of characterization and to the implementation of more and more precise models. However, considering bonding conditions, in particular in dynamics, is not easy. The purpose of this paper is to propose a methodology based on asymptotic theory allowing to obtain a family of interface laws in elastodynamics and to show how the problem can be solved numerically.
In this paper, we consider the bonding of two elastic bodies (the adherents) by a third one (the adhesive). The thickness of the adhesive is supposed to be small. Thus, its seems natural mathematically to study the limit problem i.e. when the thickness tends to zero. This methodology was employed successfully in previous papers (see Klarbring, 1991, Licht et al. 1997, Abdelmoula et al., 1998, Krasucki et al. 2000 , Zaittouni et al., 2000, Lebon et al., 1997-2011,

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Benveniste, 2006, Dumont et al., to appear) and references therein, for soft (the stiffness of the glue is small) and hard interfaces (the stiffness of the glue is of the same order as that of the adherents). The novelty of this paper is to consider "hard" interface in elastodynamics and to propose a numerical procedure able to solve the limit problem.
The paper is organized as follows. Section 2 is devoted to some generalities and recalls in elastostatics. In Section 3, a result in elastodynamics for soft interface is recalled. Section 4 is devoted to the derivation of an interface law for hard thin layers. In Section 5, a numerical scheme is proposed.

2 Theoretical results for thin stiff films: a recall in elastostatics

2.1 Generalities

In this section, the equilibrium of a mechanical system constitued by two elastic bodies glued together by a third one is considered. The two adherent and the adhesive are supposed to have mechanical characteristics of same order. However, the thickness of the glue is considered as thin in regards of the dimensions of the two adherents. In the next section, notations are given.

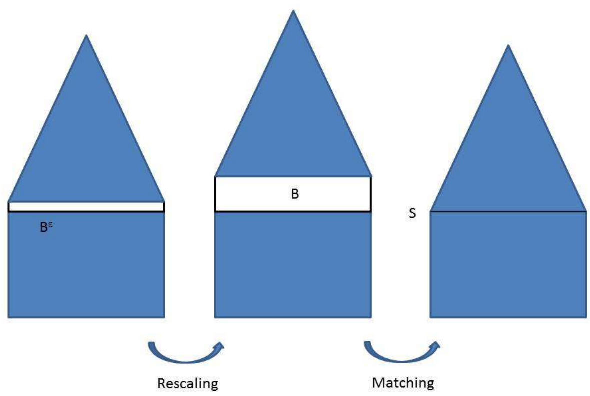

Figure 1: Schema of the procedure.

2.2 Notations

- $B^\varepsilon = \{(x_1, x_2) \in \Omega : \; |x_2| < \dfrac{\varepsilon}{2}\}$ (the glue);
- $\Omega^\varepsilon_\pm = \{(x_1, x_2) \in \Omega : \; \pm x_2 > \dfrac{\varepsilon}{2}\}$ (the adherents);
- $S^\varepsilon_\pm = \{(x_1, x_2) \in \Omega : \; x_2 = \pm\dfrac{\varepsilon}{2}\}$ (the interfaces between the glue and the adherents);
- $\Omega_\pm = \{(x_1, x_2) \in \Omega : \; \pm x_2 > \dfrac{1}{2}\}$ (the rescalled adherents);
- $B = \{(x_1, x_2) \in \Omega : \; |x_2| < \dfrac{1}{2}\}$ (the rescalled adhesive);
- $S_\pm = \{(x_1, x_2) \in \Omega : \; x_\pm = \pm\dfrac{1}{2}\}$ (the rescalled interfaces);
- $S = \{(x_1, x_2) \in \Omega : \; x_2 = 0\}$ (the interface at the limit);
- $\Omega^0_\pm = \{(x_1, x_2) \in \Omega : \; \pm x_2 > 0\}$ (the adherents at the limit).

2.3 The mechanical problem

On a part Γ_1 of $\partial\Omega$, an external load g is applied, and on a part Γ_0 of $\partial\Omega$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ a displacement u_d is imposed. Moreover, we suppose that $\Gamma_0 \cap B^\varepsilon = \emptyset$ and $\Gamma_1 \cap B^\varepsilon = \emptyset$. A body force f is applied in $\Omega^\varepsilon_\pm$. The equations of the problem are:

$$\begin{cases} \mathrm{div}\sigma^\varepsilon + f = 0 & \text{in } \Omega^\varepsilon_\pm \\ \mathrm{div}\sigma^\varepsilon = 0 & \text{in } B^\varepsilon \\ \sigma^\varepsilon n = g & \text{on } \Gamma_1 \\ u^\varepsilon = u_d & \text{on } \Gamma_0 \\ \sigma^\varepsilon = A_\pm e(u^\varepsilon) & \text{in } \Omega^\varepsilon_\pm \\ \sigma^\varepsilon = \hat{A} e(u^\varepsilon) & \text{in } B^\varepsilon \end{cases} \tag{1}$$

where σ^ε is the stress tensor, $e(u^\varepsilon)$ is the strain tensor ($e_{ij}(u^\varepsilon) = \dfrac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, 2, 3$) and $A_\pm$, $\hat{A}$ are the elasticity tensors of the deformable adherents and the adhesive, respectively.

We consider also that the interface S is a plane normal to the third direction e_3. We consider now, the limit problem i.e. the problem obtained when the thickness tends to zero.

2.4 Asymptotic analysis

The thickness of the interphase being very small, we seek the solution of the problem using asymptotic expansions with respect to the parameter ε:

$$\begin{cases} u^\varepsilon = u^0 + \varepsilon u^1 + o(\varepsilon) \\ \sigma^\varepsilon = \sigma^0 + \varepsilon \sigma^1 + o(\varepsilon) \end{cases} \tag{2}$$

We recall the results obtained in elastostatics. At order 0, we obtain

$$\begin{cases} \mathrm{div}\sigma^0 + f = 0 & \text{in } \Omega^0_\pm \\ \sigma^0 n = g & \text{on } \Gamma_1 \\ u^0 = u_d & \text{on } \Gamma_0 \\ \sigma^0 = A_\pm e(u^0) & \text{in } \Omega^0_\pm \\ \left[u^0\right] = 0 & \text{on S} \\ \left[\sigma^0 n\right] = 0 & \text{on S} \end{cases} \tag{3}$$

where [ ] is chosen to denote the jump along the surface S, i.e. $[f] = f(0^+) - f(0^-)$.

At order 1, we obtain

$$\begin{cases} \mathrm{div}\sigma^1 = 0 & \text{in } \Omega^0_\pm \\ \sigma^1 n = 0 & \text{on } \Gamma_1 \\ u^1 = u_d & \text{on } \Gamma_0 \\ \sigma^1 = A_\pm e(u^1) & \text{in } \Omega^0_\pm \\ \left[u^1\right] = D & \text{on S} \\ \left[\sigma^1 n\right] = G & \text{on S} \end{cases} \tag{4}$$

where D and G are given by

$$\begin{cases} \left[u^1_3\right] = D_3 = \dfrac{\sigma^0_{33}}{\lambda + 2\mu} - \dfrac{\lambda}{\lambda + 2\mu}\left(u^0_{1,1} + u^0_{2,2}\right) - \left\langle u^0_{3,3} \right\rangle \\ \left[u^1_\alpha\right] = D_\alpha = \dfrac{u^0_{\alpha 3}}{\mu} - u^0_{3,\alpha} - \left\langle u^0_{\alpha,3} \right\rangle, \ \alpha = 1,2 \end{cases} \tag{5}$$

with $\langle f \rangle = \dfrac{1}{2}(f(0^+) + f(0^-))$

$$\begin{cases} \left[\sigma^1_{13}\right] = G_1 & = -\dfrac{\lambda}{\lambda+2\mu}\sigma^0_{33,1} - \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}u^0_{1,11} - \dfrac{2\lambda\mu}{\lambda+2\mu}u^0_{2,21} - \mu\left(u^0_{1,22}+u^0_{2,12}\right) \\ & -\left\langle\sigma^0_{13,3}\right\rangle \\ \left[\sigma^1_{23}\right] = G_2 = & -\dfrac{\lambda}{\lambda+2\mu}\sigma^0_{33,2} - \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}u^0_{2,22} - \dfrac{2\lambda\mu}{\lambda+2\mu}u^0_{1,12} - \mu\left(u^0_{1,21}+u^0_{2,11}\right) \\ & -\left\langle\sigma^0_{23,3}\right\rangle \\ \left[\sigma^1_{33}\right] = G_3 = & -\sigma^0_{13,1} - \sigma^0_{23,2} - \left\langle\sigma^0_{33,3}\right\rangle \end{cases} \tag{6}$$

3 A recall of some theoretical results for thin soft films in elastodynamics

In the sequel, we consider that the glue is isotropic, with Lamé's coefficients equal to λ and μ in the interphase B^ε. We are interested in the dynamics of such a structure. The equations of the problem are written as follows:

$$\begin{cases} \mathrm{div}\sigma^\varepsilon + f = \rho_\pm \ddot{u}^\varepsilon & \text{in } \Omega^\varepsilon_\pm \\ \mathrm{div}\sigma^\varepsilon = \hat{\rho}\ddot{u}^\varepsilon & \text{in } \cup B^\varepsilon \\ \sigma^\varepsilon n = g & \text{on } \Gamma_1 \\ u^\varepsilon = u_d & \text{on } \Gamma_0 \\ \sigma^\varepsilon = A_\pm e(u^\varepsilon) & \text{in } \Omega^\varepsilon_\pm \\ \sigma^\varepsilon = \hat{A}e(u^\varepsilon) & \text{in } B^\varepsilon \end{cases} \tag{7}$$

where $\rho_\pm$, $\hat{\rho}$ are the densities of the deformable adherents and the adhesive, respectively. $\ddot{u}$ denotes the second derivative in time of u. We consider in this section the case of a soft interface i.e. the stiffness coefficients and the density of the thin adhesive are small that is mathematically depend on the thickness of the glue.

In this case, it is proved in (Licht et al., 2008) at order 0 that, using the Trotter theory of semi-groups,

$$\begin{cases} \mathrm{div}\sigma^0 + f = \rho_\pm \ddot{u}^0 & \text{in } \Omega^\varepsilon_\pm \\ \sigma^0 n = g & \text{on } \Gamma_1 \\ u^0 = u_d & \text{on } \Gamma_0 \\ \sigma^0 = A_\pm e(u^0) & \text{in } \Omega^\varepsilon_\pm \\ \left[\sigma^0 n\right] = 0 & \text{on S} \\ \sigma^0 n = C\left[u^0\right] & \text{on S} \end{cases} \tag{8}$$

where $C_{ij} = 0$ if $i \neq j$, $C_{11} = C_{22} = \bar{\mu}$, $C_{33} = \bar{\lambda} + 2\bar{\mu}$, $\bar{f} = \lim(f/\varepsilon; \varepsilon \to 0)$. Note that the limit case i.e. $\bar{\lambda}$ and $\bar{\mu}$ equal to ∞, we obtain $\left[u^0\right] = 0$.

4 Theoretical results for thin stiff films in elastodynamics

Let us emphasize that in this section the Lamé's coefficients of the interphase do not depend on the thickness ε of the interphase (this will be referred as the case of a stiff interface hereinafter).

At this level, the domain is rescaled using the classical procedure:

- In the glue, we define the following change of variable

$$\begin{array}{c}(x_1, x_2, x_3) \in B^\varepsilon \to (z_1, z_2, z_3) \in B, \\ \text{with } (z_1, z_2, z_3) = (x_1, x_2, \dfrac{x_3}{\varepsilon})\end{array}$$

  and we denote $\hat{u}^\varepsilon(z_1, z_2, z_3) = u^\varepsilon(x_1, x_2, x_3)$.

- In the adherent, we define the following change of variable

$$\begin{array}{c}(x_1, x_2, x_3) \in \Omega^\varepsilon_\pm \to (z_1, z_2, z_3) \in \Omega_\pm, \\ \text{with } (z_1, z_2, z_3) = (x_1, x_2, x_3 + 1/2 - \varepsilon/2)\end{array}$$

  and we denote $\bar{u}^\varepsilon(z_1, z_2, z_3) = u^\varepsilon(x_1, x_2, x_3)$. We suppose that the external forces and the prescribed displacement u_d are assumed to be independent of ε. As a consequence, we define $\bar{f}(z_1, z_2, z_3) = f(x_1, x_2, x_3)$, $\bar{g}(z_1, z_2, z_3) = g(x_1, x_2, x_3)$ and $\bar{u}_d(z_1, z_2, z_3) = u_d(x_1, x_2, x_3)$.

4.1 Internal expansions

From the equation

$$\hat{\sigma}_{ij,j} = \rho \ddot{\hat{u}}_i$$

we obtain at order -1

$$\hat{\sigma}^0_{i3,3} = 0$$

That leads to

$$[[\hat{\sigma}^0_{i3}]] = 0$$

and

$$\hat{\sigma}^0_{i\alpha,\alpha} + \hat{\sigma}^1_{i3,3} = \rho\ddot{\hat{u}}_i$$

for $i = 1, 2, 3$ and $\alpha = 1, 2$, where $[[f]] = f(z_1, z_2, 1/2^+) - f(z_1, z_2, 1/2^-)$. The constitutive equation gives

$$\begin{cases} (\lambda + 2\mu)\hat{u}^0_{3,3} = 0 \\ \mu\hat{u}^0_{\alpha,3} = 0 \end{cases} \tag{9}$$

That leads to

$$[[\hat{u}^0]] = 0$$

which generalizes the results of (Licht et al., 2008).

At order 0, the constitutive equation is written:

$$\begin{cases} (\lambda + 2\mu)\hat{u}^1_{3,3} + \lambda\left(\hat{u}^0_{1,1} + \hat{u}^0_{2,2}\right) = \hat{\sigma}^0_{33} \\ \mu\left(\hat{u}^0_{3,\alpha} + \hat{u}^1_{\alpha,3}\right) = \hat{\sigma}^0_{\alpha 3} \end{cases} \tag{10}$$

That is

$$\begin{cases} [[\hat{u}^1_3]] = \dfrac{\hat{\sigma}^0_{33}}{\lambda + 2\mu} - \dfrac{\lambda}{\lambda + 2\mu}\left(\hat{u}^0_{1,1} + \hat{u}^0_{2,2}\right) \\ [[\hat{u}^1_\alpha]] = \dfrac{\hat{\sigma}^0_{\alpha 3}}{\mu} - \hat{u}^0_{3,\alpha} \end{cases} \tag{11}$$

Note that the jump in the displacements is equal to those in the static case.

We use the three other terms of the constitutive equation

$$\begin{cases} \hat{\sigma}^0_{11} = (\lambda + 2\mu)\hat{u}^0_{1,1} + \lambda\left(\hat{u}^0_{2,2} + \hat{u}^1_{3,3}\right) \\ \hat{\sigma}^0_{22} = (\lambda + 2\mu)\hat{u}^0_{2,2} + \lambda\left(\hat{u}^0_{1,1} + \hat{u}^1_{3,3}\right) \\ \hat{\sigma}^0_{12} = \mu\left(\hat{u}^0_{1,2} + \hat{u}^0_{2,1}\right) \end{cases} \tag{12}$$

which gives us using (**??**)

$$\left\{ \begin{array}{l} \hat{\sigma}^0_{11} = \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\hat{u}^0_{1,1} + \dfrac{2\lambda\mu}{\lambda+2\mu}\hat{u}^0_{2,2} + \dfrac{\lambda}{\lambda+2\mu}\hat{\sigma}^0_{33} \\ \hat{\sigma}^0_{22} = \dfrac{2\lambda\mu}{\lambda+2\mu}\hat{u}^0_{1,1} + \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\hat{u}^0_{2,2} + \dfrac{\lambda}{\lambda+2\mu}\hat{\sigma}^0_{33} \\ \hat{\sigma}^0_{12} = \mu\left(\hat{u}^0_{1,2} + \hat{u}^0_{2,1}\right) \end{array} \right. \tag{13}$$

Introducing (**??**) in the dynamics equation, we have

$$\left\{ \begin{array}{l} \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\hat{u}^0_{1,11} + \dfrac{2\lambda\mu}{\lambda+2\mu}\hat{u}^0_{2,21} + \dfrac{\lambda}{\lambda+2\mu}\hat{\sigma}^0_{33,1} + \mu\left(\hat{u}^0_{1,22} + \hat{u}^0_{2,12}\right) + \hat{\sigma}^1_{13,3} = \rho\ddot{\hat{u}}^0_1 \\ \mu\left(\hat{u}^0_{1,21} + \hat{u}^0_{2,11}\right) + \dfrac{2\lambda\mu}{\lambda+2\mu}\hat{u}^0_{1,12} + \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\hat{u}^0_{2,22} + \dfrac{\lambda}{\lambda+2\mu}\hat{\sigma}^0_{33,2} + \hat{\sigma}^1_{23,3} = \rho\ddot{\hat{u}}^0_2 \\ \hat{\sigma}^0_{13,1} + \hat{\sigma}^0_{23,2} + \hat{\sigma}^1_{33,3} = \rho\ddot{\hat{u}}^0_3 \end{array} \right. \tag{14}$$

That is

$$\left\{ \begin{array}{l} [[\hat{\sigma}^1_{13}]] = \rho\ddot{\hat{u}}^0_1 - \dfrac{\lambda}{\lambda+2\mu}\hat{\sigma}^0_{33,1} - \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\hat{u}^0_{1,11} - \dfrac{2\lambda\mu}{\lambda+2\mu}\hat{u}^0_{2,21} - \mu\left(\hat{u}^0_{1,22} + \hat{u}^0_{2,12}\right) \\ [[\hat{\sigma}^1_{23}]] = \rho\ddot{\hat{u}}^0_2 - \dfrac{\lambda}{\lambda+2\mu}\hat{\sigma}^0_{33,2} - \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}\hat{u}^0_{2,22} - \dfrac{2\lambda\mu}{\lambda+2\mu}\hat{u}^0_{1,12} - \mu\left(\hat{u}^0_{1,21} + \hat{u}^0_{2,11}\right) \\ [[\hat{\sigma}^1_{33}]] = \rho\ddot{\hat{u}}^0_3 - \hat{\sigma}^0_{13,1} - \hat{\sigma}^0_{23,2} \end{array} \right. \tag{15}$$

Note that the difference with the static case comes from the addition of inertial terms.

4.2 Matching with external expansions

Using standard arguments (see for example (Lebon et al., 2011)), the jump $[[f]]$ along $S_\pm$ can be replaced by the jump $[f]$ along S up to a term $\langle f_{,3}\rangle$ at order one. We obtain at order 0

$$\left\{ \begin{array}{ll} \mathrm{div}\sigma^0 + f = \rho_\pm \ddot{u}^0 & \text{in } \Omega^0_\pm \\ \sigma^0 n = g & \text{on } \Gamma_1 \\ u^0 = u_d & \text{on } \Gamma_0 \\ \sigma^0 = A_\pm e(u^0) & \text{in } \Omega^0_\pm \\ \left[u^0\right] = 0 & \text{on S} \\ \left[\sigma^0 n\right] = 0 & \text{on S} \end{array} \right. \tag{16}$$

At order 1, we obtain

$$\begin{cases} \mathrm{div}\sigma^1 = \rho_\pm \ddot{u}^1 & \text{in } \Omega^0_\pm \\ \sigma^1 n = 0 & \text{on } \Gamma_1 \\ u^1 = 0 & \text{on } \Gamma_0 \\ \sigma^1 = A_\pm e(u^1) & \text{in } \Omega^0_\pm \\ \left[u^1\right] = D & \text{on S} \\ \left[\sigma^1 n\right] = G + G^\rho & \text{on S} \end{cases} \tag{17}$$

where G^ρ is given by

$$\begin{cases} G^\rho_1 = \rho\ddot{\hat{u}}^0_1 \\ G^\rho_2 = \rho\ddot{\hat{u}}^0_2 \\ G^\rho_3 = \rho\ddot{\hat{u}}^0_3 \end{cases} \tag{18}$$

and D and G are given in eqs. (5) and (6) respectively.

5 A numerical procedure

In this paragraph, we focus on the numerical method developed to solve the problem at order 1, the problem at order 0 being very classical. The generic problem associated to this problem can be written (without exponent 1)

$$\begin{cases} \mathrm{div}\sigma(u) = \rho_\pm \ddot{u} & \text{in } \Omega^0_\pm \\ \sigma(u) n = 0 & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_0 \\ \sigma = A_\pm e(u) & \text{in } \Omega^0_\pm \\ [u] = D & \text{on } S \\ [\sigma(u) n] = G + G^\rho & \text{on } S \end{cases} \tag{19}$$

Note that D, G and G^ρ are given functions, provided by the solutions u^0 and σ^0 of problem at order 0.

In the following, we will denote the restriction of u on Ω^0_+ (resp. Ω^0_-) by u^+ (resp. u^-).

Without loss of generality, an explicit time stepping is introduced i.e. the term $\rho_\pm \ddot{u}$ is given. We chose η to denote this term. The indices in time are omitted.

The weak symmetrical formulation of the problem is given by

$$\int_{\Omega_+^0\cup\Omega_-^0} A_\pm e(u^\pm)\cdot e(v^\pm)dx + \int_S (< Ae(u)n > \cdot[v] + [u]\cdot < Ae(v)n >)\, dS =$$
$$\int_{\Omega_+^0\cup\Omega_-^0} \eta v^\pm dx - \int_S (G+G^\rho)\cdot < v > dS + \int_S D\cdot < Ae(v)n > dS \tag{20}$$

for all $v \in \{H^1(\Omega):\ \gamma(v)=0 \text{ on } \partial\Omega\backslash\Gamma\}$.

This formulation, which is known as the Nitsche's method (Nitsche, 1974) is not stable. It is then necessary to add a stabilization term such as $\dfrac{\beta}{h}\displaystyle\int_S [u]\cdot[v]dS$, where h is the size of the smallest element of the finite element discretization of $\Omega_\pm^0$ considered, and $\beta > 0$ is a fixed number that must be sufficiently large to ensure the stability of the method (see (Dumont et al, 2006, Stenberg, 1995) for the complete study of this method and for a priori and a posteriori error estimates in the case $D = 0$). Note that this weak formulation is equivalent to the initial strong formulation.

Unfortunately, this method does not work properly to solve the problem (**??**) as soon as $D \neq 0$. To overcome this difficulty, we split the problem (**??**) into two parts. More precisely, we write $u^\pm = w^\pm + z^\pm$ where the unknowns $z^\pm$ and $w^\pm$ solve the problems

$$\begin{cases} \text{div } \sigma(z^\pm) = \eta & \text{in } \Omega_\pm^0 \\ \sigma(z^\pm)n = 0 & \text{on } \Gamma_1 \\ z^\pm = 0 & \text{on } \Gamma_0 \\ \sigma(z^\pm) = A^\pm e(z^\pm) & \text{in } \Omega_\pm^0 \\ z^\pm = \pm\frac{1}{2}D & \text{on } S \end{cases} \tag{21}$$

$$\begin{cases} \text{div } \sigma(w^\pm) = 0 & \text{in } \Omega_\pm^0 \\ \sigma(w^\pm)n = 0 & \text{on } \Gamma_1 \\ w^\pm = 0 & \text{on } \Gamma_0 \\ \sigma(w^\pm) = A^\pm e(w^\pm) & \text{in } \Omega_\pm^0 \\ [w] = 0 & \text{on } S \\ [\sigma(w)n] = G + G^\rho - [\sigma(z)n] & \text{on } S \end{cases} \tag{22}$$

since $[w] = w^+ - w^- = [u] - z^+ + z^- = (1 - \frac{1}{2} - \frac{1}{2})D = 0$. The two first problems defined in the left of both in Ω_+^0 and Ω_-^0 are standard and can be solved simultaneously using a standard finite element method. The problem in (**??**) is solved using the Nitsche's method developed above.

6 Conclusions

In this paper, the asymptotic analysis of a thin elastic layer bonded with two elastic adherent being a stiffness of the same order as that of the adherents was studied in dynamics. It is shown that at order zero, the thin layer inertial terms do not intervene. A problem of elastodynamics with perfect gluing is obtained, extending the results obtained in (Licht et al, 2008). At order one, the inertial terms of the thin layer only intervene in the jump in the stress vector along the interface. The jump in the displacement is not modified.
In a second part of the paper, a numerical method to solve the problem at order 1 is proposed. This method is closed to the method proposed by the authors for the elastostatics case (Dumont et al., to appear).
In the future, we intend to implement the numerical schema proposed in the paper and to extend the methodology to non linear constitutive equations.

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