Pointwise estimates and existence of solutions of porous medium and p-Laplace evolution equations with absorption and measure data

Marie-Françoise Bidaut-Véron, Quoc-Hung Nguyen

Pointwise estimates and existence of solutions of porous medium and p-Laplace evolution equations with absorption and measure data

Marie-Françoise Bidaut-Véron* Quoc-Hung Nguyen†

Abstract

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 2)$. We obtain a necessary and a sufficient condition, expressed in terms of capacities, for existence of a solution to the porous medium equation with absorption

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu \ \text{ in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma, \end{cases}$$

where σ and μ are bounded Radon measures, $q > \max(m,1)$, $m > \frac{N-2}{N}$. We also obtain a sufficient condition for existence of a solution to the p-Laplace evolution equation

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu \ \text{ in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma. \end{cases}$$

where $q > p-1$ and $p > 2$.

keywords: Sobolev-Besov capacities; Bessel capacities; Radon measures; renormalized solutions.
MSC: 35K92; 35K55; 35K15

Contents

*Laboratoire de Mathématiques et Physique Théorique, Faculté des Sciences, Université François Rabelais, Tours, France. E-mail: veronmf@univ-tours.fr

†Laboratoire de Mathématiques et Physique Théorique, Faculté des Sciences, Université François Rabelais, Tours, France. E-mail: Hung.Nguyen-Quoc@lmpt.univ-tours.fr

1 Introduction and main results

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$ and $T > 0$, and $\Omega_T = \Omega \times (0,T)$. In this paper we study the existence of solutions to the following two types of evolution problems: the porous medium problem with absorption

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \mu \ \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma, \end{cases} \tag{1.1}$$

where $m > \frac{N-2}{N}$ and $q > \max(1,m)$, and the p-Laplace evolution problem with absorption

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu \ \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma, \end{cases} \tag{1.2}$$

where $q > p-1 > 1$, and μ and σ are bounded Radon measures respectively on Ω_T and Ω. In the sequel, for any bounded domain O of $\mathbb{R}^l (l \geqq 1)$, we denote by $\mathcal{M}_b(O)$ the set of bounded Radon measures in O, and by $\mathcal{M}_b^+(O)$ its positive cone. For any $\nu \in \mathcal{M}_b(O)$, we denote by ν^+ and ν^- respectively its positive and negative part.

When $m = 1, p = 2$ and $q > 1$ the problem has been studied by Brezis and Friedman [12] with $\mu = 0$. It is shown that in the subcritical case $q < 1 + 2/N$, the problem can be solved for any $\sigma \in \mathcal{M}_b(\Omega)$, and it has no solution when $q \geq 1 + 2/N$ and σ is a Dirac mass. The general case has been solved by Baras and Pierre [5] and their results are expressed in terms of capacities. For $s > 1, \alpha > 0$, the capacity $\mathrm{Cap}_{\mathbf{G}_\alpha,s}$ of a Borel set $E \subset \mathbb{R}^N$, defined by

$$\mathrm{Cap}_{\mathbf{G}_\alpha,s}(E) = \inf\{||g||^s_{L^s(\mathbb{R}^N)} : g \in L^s_+(\mathbb{R}^N), \mathbf{G}_\alpha * g \geq 1 \text{ on } E\},$$

where $\mathbf{G}_\alpha$ is the Bessel kernel of order α and the capacity $\mathrm{Cap}_{2,1,s}$ of a compact set $K \subset \mathbb{R}^{N+1}$ is defined by

$$\mathrm{Cap}_{2,1,s}(K) = \inf\left\{||\varphi||^s_{W^{2,1}_s(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K\right\},$$

where

$$||\varphi||_{W^{2,1}_s(\mathbb{R}^{N+1})} = ||\varphi||_{L^s(\mathbb{R}^{N+1})} + ||\varphi_t||_{L^s(\mathbb{R}^{N+1})} + ||\,|\nabla\varphi|\,||_{L^s(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\dots,N} ||\varphi_{x_i x_j}||_{L^s(\mathbb{R}^{N+1})}.$$

The capacity $\mathrm{Cap}_{2,1,s}$ is extended to Borel sets by the usual method. Note the relation between the two capacities:

$$C^{-1}\mathrm{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}(E) \leq \mathrm{Cap}_{2,1,s}(E \times \{0\}) \leq C\mathrm{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}(E)$$

for any Borel set $E \subset \mathbb{R}^N$, see [34, Corollary 4.21]. In particular, for any $\omega \in \mathcal{M}_b(\mathbb{R}^N)$ and $a \in \mathbb{R}$, the measure $\omega \otimes \delta_{\{t=a\}}$ in $\mathbb{R}^{N+1}$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,s}$ (in $\mathbb{R}^{N+1}$) if and only if ω is absolutely continuous with respect to the capacity $\mathrm{Cap}_{\mathbf{G}_{2-\frac{2}{s}},s}$ (in $\mathbb{R}^N$).

From [5], the problem

$$\begin{cases} u_t - \Delta u + |u|^{q-1}u = \mu \ \text{ in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma, \end{cases}$$

has a solution if and only if the measures μ and σ are absolutely continuous with respect to the capacities $\mathrm{Cap}_{2,1,q'}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω respectively, where $q' = \frac{q}{q-1}$.

In Section 2 we study problem (1.1).

For $m > 1$, Chasseigne [14] has extended the results of [12] for $\mu = 0$ in the new subcritical range $m < q < m + \frac{2}{N}$. The supercritical case $q \geq m + \frac{2}{N}$ with $\mu = 0$ and σ is positive is studied in [13]. He has essentially proved that if problem (1.1) has a solution, then $\sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}$, defined for anycompact set $K \subset \mathbb{R}^{N+1}$ by

$$\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}(K) = \inf\left\{ ||\varphi||^{\frac{q}{q-m}}_{W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^N), \varphi \geq 1 \text{ in a neighborhood of } E \right\},$$

where

$$||\varphi||_{W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})} = ||\varphi||_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})} + ||\varphi_t||_{L^{q'}(\mathbb{R}^{N+1})} + ||\,|\nabla\varphi|\,||_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\dots,N} ||\varphi_{x_ix_j}||_{L^{\frac{q}{q-m}}(\mathbb{R}^{N+1})}.$$

In this Section, we first give *necessary conditions* on the measures μ and σ for existence, which cover the results mentioned above.

Theorem 1.1 *Let $m > \frac{N-2}{N}$ and $q > \max(1,m)$ and $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$. If problem (1.1) has a very weak solution then μ and $\sigma \otimes \delta_{\{t=0\}}$ are absolutely continuous with respect to the capacity $Cap_{2,1,\frac{q}{q-m},\frac{q}{q-1}}$.*

Remark 1.2 *It is easy to see that the capacity $Cap_{2,1,\frac{q}{q-m},\frac{q}{q-1}}$ is absolutely continuous with respect to the capacity $Cap_{2,1,\frac{q}{q-\max\{m,1\}}}$. Therefore μ and $\sigma \otimes \delta_{\{t=0\}}$ are absolutely continuous with respect to the capacities $Cap_{2,1,\frac{q}{q-\max\{m,1\}}}$.In particular σ is absolutely continuous with respect to the capacity $Cap_{\mathbf{G}_{\frac{2\max\{m,1\}}{q}},\frac{q}{q-\max\{m,1\}}}$.*

The main result of this Section is the following *sufficient condition* for existence, where we use the notion of R-truncated Riesz parabolic potential $\mathbb{I}_2$ on $\mathbb{R}^{N+1}$ of a measure $\mu \in \mathcal{M}_b^+(\Omega_T)$, defined by

$$\mathbb{I}_2^R[\mu](x,t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho} \quad \text{for any } \ (x,t) \in \mathbb{R}^{N+1},$$

with $R \in (0,\infty]$, and $\tilde{Q}_\rho(x,t) = B_\rho(x) \times (t-\rho^2, t+\rho^2)$.

Theorem 1.3 *Let $m > \frac{N-2}{N}$, $q > \max(1, m)$, $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$.*

i. *If $m > 1$ and μ and σ are absolutely continuous with respect to the capacities $Cap_{2,1,q'}$ in Ω_T and $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω, then there exists a very weak solution u of (1.1), satisfying for a.e.$(x,t) \in \Omega_T$*

$$|u(x,t)| \le C\left(\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\sigma|(\Omega)+|\mu|(\Omega_T)+1+\mathbb{I}_2^{2d}[|\sigma|\otimes\delta_{\{t=0\}}+|\mu|](x,t)\right),$$

where $C = C(N,m) > 0$ and

$$m_1 = \frac{(N+2)(2mN+1)}{m(mN+2)(1+2N)}, \qquad d = diam(\Omega) + T^{1/2}.$$

ii. *If $\frac{N-2}{N} < m \le 1$, and μ and σ are absolutely continuous with respect to the capacities $Cap_{2,1,\frac{2q}{2(q-1)+N(1-m)}}$ in Ω_T and $Cap_{\mathbf{G}_{\frac{2-N(1-m)}{q}},\frac{2q}{2(q-1)+N(1-m)}}$ in Ω, there exists a very weak solution u of (1.1), such that for a.e.$(x,t) \in \Omega_T$*

$$|u(x,t)| \le C\left(\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\mathbb{I}_2^{2d}[|\sigma|\otimes\delta_{\{t=0\}}+|\mu|](x,t)\right)^{\frac{2}{2-N(1-m)}}\right),$$

where $C = C(N,m) > 0$ and

$$m_2 = \frac{2N(N+2)(m+1)}{(2+Nm)(2-N(1-m))(2+N(1+m))}.$$

.

Remark 1.4 *These estimates are not homogeneous in u. In particular if $\mu \equiv 0$, u satisfies the decay estimates, for a.e. $(x,t) \in \Omega_T$,*

i. *if $m > 1$,*

$$|u(x,t)| \le C\left(\left(\frac{|\sigma|(\Omega)}{d^N}\right)^{m_1} + |\sigma|(\Omega) + 1 + \frac{|\sigma|(\Omega)}{Nt^{N/2}}\right),$$

ii. *if $m < 1$,*

$$|u(x,t)| \le C\left(\left(\frac{|\sigma|(\Omega)}{d_1^N}\right)^{m_2} + 1 + \left(\frac{|\sigma|(\Omega)}{Nt^{N/2}}\right)^{\frac{2}{2-N(m-1)}}\right).$$

We also give other types of *sufficient* conditions for *measures which are good in time*, that means such that

$$\sigma \in L^1(\Omega) \quad \text{ and } |\mu| \le f + \omega \otimes F, \quad \text{ where } f \in L^1_+(\Omega_T), F \in L^{1,+}((0,T)), \tag{1.3}$$

see Theorem 2.10. The proof is based on estimates for the stationary problem in terms of elliptic Riesz potential.

In Section 3, we consider problem (1.2). Let us recall some former results about it.

For $q > p-1 > 0$, Pettitta, Ponce and Porretta [36] have proved that it admits a (unique renormalized) solution provided $\sigma \in L^1(\Omega)$ and $\mu \in \mathcal{M}_b(\Omega_T)$ is a *diffuse* measure, i.e. absolutely continuous with respect to C_p-capacity in Ω_T, defined on a compact set $K \subset \Omega_T$ by

$$C_p(K, \Omega_T) = \inf \{||\varphi||_W : \varphi \in C_c^\infty(\Omega_T)\varphi \geq 1 \text{ on } K\}, \tag{1.4}$$

where

$$W = \{z : z \in L^p(0,T,W_0^{1,p}(\Omega) \cap L^2(\Omega)), z_t \in L^{p'}(0,T,W^{-1,p'}(\Omega) + L^2(\Omega))\}.$$

In the recent work [7], we have proved a stability result for the p-Laplace parabolic equation, see Theorem 3.5, for $p > \frac{2N+1}{N+1}$. As a first consequence, in the new subcritical range

$$q < p - 1 + \frac{p}{N},$$

problem (1.2) admits a renormalized solution for any measures $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in L^1(\Omega)$. Moreover, we have obtained sufficient conditions for existence, for measures that have a *good behavior in time*, of the form (1.3). It is shown that (1.2) has a renormalized solution if $\omega \in \mathcal{M}_b^+(\Omega)$ is absolutely continuous with respect to $\mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}$. The proof is based on estimates of [8] for the stationary problem which involve Wolff potentials.

Here we give *new sufficient conditions when* $p > 2$. The next Theorem is our second main result:

Theorem 1.5 *Let $q > p-1 > 1$ and $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$. If μ and σ are absolutely continuous with respect to the capacities $Cap_{2,1,q'}$ in Ω_T and $Cap_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω, then there exists a distribution solution of problem (1.2) which satisfies the pointwise estimate*

$$|u(x,t)| \leq C\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|\right](x,t)\right) \tag{1.5}$$

for a.e $(x,t) \in \Omega_T$ with $C = C(N,p)$ and

$$m_3 = \frac{(N+p)(\lambda+1)(p-1)}{((p-1)N+p)(1+\lambda(p-1))}, \qquad \lambda = \min\{1/(p-1), 1/N\}, \qquad D = diam(\Omega) + T^{1/p}. \tag{1.6}$$

Moreover, if $\sigma \in L^1(\Omega)$, u is a renormalized solution.

2 Porous medium equation

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s,k\}, -k\}$. The solutions of (1.1) are considered in a weak sense:

Definition 2.1 *Let $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$ and $g \in C(\mathbb{R})$.*

i. *A function u is a weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + g(u) = \mu \quad in\ \Omega_T, \\ u = 0 \quad on\ \partial\Omega \times (0,T), \\ u(0) = \sigma \quad in\ \ \Omega. \end{cases} \tag{2.1}$$

if $u \in C([0,T]\,; L^2(\Omega))$, $|u|^m \in L^2((0,T); H_0^1(\Omega))$ *and* $g(u) \in L^1(\Omega_T)$, *and for any* $\varphi \in C_c^{2,1}(\Omega \times [0,T))$,

$$-\int_{\Omega_T} u\varphi_t dxdt + \int_{\Omega_T} \nabla(|u|^{m-1}u).\nabla\varphi dxdt + \int_{\Omega_T} g(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_{\Omega} \varphi(0) d\sigma.$$

ii. *A function* u *is a very weak solution of (2.1) if* $u \in L^{\max\{m,1\}}(\Omega_T)$ *and* $g(u) \in L^1(\Omega_T)$, *and for any* $\varphi \in C_c^{2,1}(\Omega \times [0,T))$,

$$-\int_{\Omega_T} u\varphi_t dxdt - \int_{\Omega_T} |u|^{m-1}u\Delta\varphi dxdt + \int_{\Omega_T} g(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_{\Omega} \varphi(0) d\sigma.$$

First we give a priori estimates for the problem without perturbation term:

Proposition 2.2 *Let* $u \in L^\infty(\Omega_T)$ *with* $|u|^m \in L^2((0,T); H_0^1(\Omega))$ *be a weak solution to problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) = \mu \quad in\ \Omega_T, \\ u = 0 \quad on\ \partial\Omega \times (0,T), \\ u(0) = \sigma \quad in\ \ \Omega, \end{cases} \tag{2.2}$$

with $\sigma \in C_b(\Omega)$ *and* $\mu \in C_b(\Omega_T)$. *Then,*

$$||u||_{L^\infty((0,T);L^1(\Omega))} \le |\sigma|(\Omega) + |\mu|(\Omega_T), \tag{2.3}$$

$$||u||_{L^{m+2/N,\infty}(\Omega_T)} \le C_1(|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{N+2}{mN+2}}, \tag{2.4}$$

$$|||\nabla(|u|^{m-1}u)|||_{L^{\frac{mN+2}{mN+1},\infty}(\Omega_T)} \le C_2(|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{m(N+1)+1}{mN+2}}, \tag{2.5}$$

where $C_1 = C_1(N,m), C_2 = C_2(N,m)$.

Proof of Proposition 2.2. For any $\tau \in (0,T)$, and $k > 0$ we have

$$\int_{\Omega_\tau} (H_k(u))_t dxdt + \int_{\Omega_\tau} |\nabla T_k(|u|^{m-1}u)|^2 dxdt = \int_{\Omega_\tau} T_k(|u|^{m-1}u) d\mu(x,t),$$

where $H(a) = \int_0^a T_k(|y|^{m-1}y)dy$. This leads to

$$\begin{aligned} &\int_{\Omega_T} |\nabla T_k(|u|^{m-1}u)|^2 dxdt \le k(|\sigma|(\Omega) + |\mu|(\Omega_T)) \quad \text{and} \\ &\int_{\Omega} (H_k(u))(\tau)dx \le k(|\sigma|(\Omega) + |\mu|(\Omega_T)),\ \forall \tau \in (0,T). \end{aligned} \tag{2.6}$$

Since $H_k(a) \ge k(|a| - k)$ for any a and $k > 0$, we find

$$\int_{\Omega} (|u|(\tau) - k)dx \le |\sigma|(\Omega) + |\mu|(\Omega_T),\ \forall \tau \in (0,T).$$

Letting $k \to 0$, we get (2.3).

Next we prove (2.4). By the Gagliardo-Nirenberg embedding theorem, there holds

$$\int_{\Omega_T} |T_k(|u|^{m-1}u)|^{\frac{2(N+1)}{N}} dxdt \le C_1 ||T_k(|u|^{m-1}u)||^{2/N}_{L^\infty((0,T);L^1(\Omega))} \int_{\Omega_T} |\nabla T_k(|u|^{m-1}u)|^2 dxdt$$
$$\le C_1 k^{\frac{2(m-1)}{mN}} ||u||^{2/N}_{L^\infty((0,T);L^1(\Omega))} \int_{\Omega_T} |\nabla T_k(|u|^{m-1}u)|^2 dxdt.$$

Thus, from (2.6) and (2.3) we get

$$k^{\frac{2(N+1)}{N}} |\{|u|^m > k\}| \le \int_{\Omega_T} |T_k(|u|^{m-1}u)|^{\frac{2(N+1)}{N}} dxdt \le c_1 k^{\frac{2(m-1)}{mN}+1} (|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{N+2}{N}},$$

which implies (2.4). Finally, we prove (2.5). Thanks to (2.6) and (2.4) we have for $k, k_0 > 0$

$$|\{|\nabla(|u|^{m-1}u)| > k\}| \le \frac{1}{k^2} \int_0^{k^2} |\{|\nabla(|u|^{m-1}u)| > \ell\}| d\ell$$
$$\le |\{|u|^m > k_0\}| + \frac{1}{k^2} \int_{\Omega_T} |\nabla T_{k_0}(|u|^{m-1}u)|^2 dxdt$$
$$\le C_1 k_0^{-\frac{2}{mN}-1} (|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{N+2}{N}} + k_0 k^{-2} (|\sigma|(\Omega) + |\mu|(\Omega_T)).$$

Choosing $k_0 = k^{\frac{Nm}{Nm+1}} (|\sigma|(\Omega) + |\mu|(\Omega_T))^{\frac{m}{Nm+1}}$, we get (2.5). ∎

Lemma 2.3 *Let $m > \frac{N-2}{N}$, and $g \in C_b(\mathbb{R})$ be nondecreasing with $g(0) = 0$, and $\mu \in C_b(\Omega_T)$. There exists a weak solution $u \in L^\infty(\Omega_T)$ with $|u|^m \in L^2(0,T,H_0^1(\Omega))$ of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + g(u) = \mu \quad in\ \Omega_T, \\ u = 0 \quad on\ \partial\Omega \times (0,T), \\ u(0) = 0 \quad in\ \ \Omega. \end{cases} \tag{2.7}$$

Moreover, the comparison principle holds for these solutions: if u_1, u_2 are weak solutions of (2.7) when (μ, g) is replaced by (μ_1, g_1) and (μ_2, g_2), where $\mu_1, \mu_2 \in C_b(\Omega_T)$ with $\mu_1 \ge \mu_2$ and g_1, g_2 have the same properties as g with $g_1 \le g_2$ in $\mathbb{R}$ then $u_1 \ge u_2$ in Ω_T.

As a consequence, if $\mu \ge 0$ then $u \ge 0$.

Proof of Lemma 2.3. Set $a_n(s) = m|s|^{m-1}$ if $1/n \le |s| \le n$ and $a_n(s) = m|n|^{m-1}$ if $|s| \ge n$, $a_n(s) = m(1/n)^{m-1}$ if $|s| \le 1/n$. Also $A_n(\tau) = \int_0^\tau a_n(s)ds$. Then one can find u_n being a weak solution to the following equation

$$\begin{cases} (u_n)_t - \mathrm{div}(a_n(u_n)\nabla u_n) + g(u_n) = \mu \ \text{ in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = 0 \quad \text{in } \ \Omega. \end{cases} \tag{2.8}$$

It is easy to see that $|u_n(x,t)| \le t||\mu||_{L^\infty(\Omega_T)}$ for all $(x,t) \in \Omega_T$. Thus, choosing $A_n(u_n)$ as a test function, we obtain

$$\int_{\Omega_T} |\nabla A_n(u_n)|^2 dxdt \le C_1(T, ||\mu||_{L^\infty(\Omega_T)}). \tag{2.9}$$

Now set $\Phi_n(\tau)=\int_0^\tau |A_n(s)|ds$. Choosing $|A_n(u_n)|\varphi$ as a test function in (2.8), where $\varphi\in C_c^{2,1}(\Omega_T)$, we get the relation in $\mathcal{D}'(\Omega_T)$:

$$(\Phi_n(u_n))_t-\operatorname{div}(|A_n(u_n)|\nabla A_n(u_n))+\nabla A_n(u_n).\nabla|A_n(u_n)|+|A_n(u_n)|g(u_n)=|A_n(u_n)|\mu.$$

Hence,

$$\begin{aligned}||(\Phi_n(u_n))_t||_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))}&\leq ||A_n(u_n)\nabla A_n(u_n)||_{L^2(\Omega_T)}+||\nabla A_n(u_n)||^2_{L^2(\Omega_T)}\\&\quad+||A_n(u_n)g(u_n)||_{L^1(\Omega_T)}+||A_n(u_n)\mu||_{L^1(\Omega_T)}.\end{aligned}$$

Combining this with (2.9) and the estimate $|A_n(u_n)|\leq C_2(T,||\mu||_{L^\infty(\Omega)})$, we deduce that

$$\sup_n||(\Phi_n(u_n))_t||_{L^1(\Omega_T)+L^2(0,T,H^{-1}(\Omega))}<\infty.$$

On the other hand, since $|A_n(u_n)|\leq|u_n|a_n(u_n)\leq T||\mu||_{L^\infty(\Omega)}a_n(u_n)$, there holds

$$\begin{aligned}\int_{\Omega_T}|\nabla\Phi_n(u_n)|^2dxdt&=\int_{\Omega_T}|A_n(u_n)|^2|\nabla u_n|^2dxdt\leq T||\mu||_{L^\infty(\Omega)}\int_{\Omega_T}|a_n(u_n)|^2|\nabla u_n|^2dxdt\\&\leq T||\mu||_{L^\infty(\Omega)}\int_{\Omega_T}|\nabla A_n(u_n)|^2dxdt\leq C_3(T,||\mu||_{L^\infty(\Omega)}).\end{aligned}$$

Therefore, $\Phi_n(u_n)$ is relatively compact in $L^1(\Omega_T)$. Note that

$$\Phi_n(s)=\begin{cases}\frac{m}{2}\left(\frac{1}{n}\right)^m|s|^2\operatorname{sign}(s) & \text{if } |s|\leq\frac{1}{n}\\ (m-1)\left(\frac{1}{n}\right)^m\left(|s|-\frac{1}{n}\right)\operatorname{sign}(s)+\frac{1}{m+1}\left(|s|^{m+1}-\left(\frac{1}{n}\right)^{m+1}\right)\operatorname{sign}(s) & \text{if } \frac{1}{n}\leq|s|\leq n.\end{cases}$$

So, for every $n_1,n_2\geq n$ and $|s_1|,|s_2|\leq T||\mu||_{L^\infty(\Omega)}$,

$$\frac{1}{m+1}||s_1|^ms_1-|s_2|^ms_2|\leq C_4(m,T||\mu||_{L^\infty(\Omega)})\left(\frac{1}{n}\right)^m+|\Phi_{n_1}(s_1)-\Phi_{n_2}(s_2)|.$$

Hence, for any $\varepsilon>0$,

$$\left|\left\{\frac{1}{m+1}||u_{n_1}|^mu_{n_1}-|u_{n_1}|^mu_{n_1}|>2\varepsilon\right\}\right|\leq|\left\{|\Phi_{n_1}(u_{n_1})-\Phi_{n_2}(u_{n_2})|>\varepsilon\right\}|,$$

for all $n_1,n_2\geq\left(C_4(m,T||\mu||_{L^\infty(\Omega)})/\varepsilon\right)^{1/m}$. Thus, up to a subsequence $\{u_n\}$ converges a.e in Ω_T to a function u. From (2.8) we can write

$$-\int_{\Omega_T}u_n\varphi_tdxdt-\int_{\Omega_T}A_n(u_n)\Delta\varphi dxdt+\int_{\Omega_T}g(u_n)\varphi dxdt=\int_{\Omega_T}\varphi d\mu,$$

for any $\varphi\in C_c^{2,1}(\Omega_T)$. Thanks to the dominated convergence Theorem we deduce that

$$-\int_{\Omega_T}u\varphi_tdxdt-\int_{\Omega_T}|u|^{m-1}u\Delta\varphi dxdt+\int_{\Omega_T}g(u)\varphi dxdt=\int_{\Omega_T}\varphi d\mu.$$

By Fatou's lemma and (2.9) we also get $|u|^m \in L^2((0,T);H_0^1(\Omega))$.
Furthermore, by the classic maximum principle, see [29, Theorem 9.7], if $\{\tilde{u}_n\}$ is a sequence of solutions to equations (2.8) where (g,μ) is replaced by (h,ν) such that $\nu \in C_b(\Omega_T)$ with $\nu \geq \mu$ and h has the same properties as g satisfying $h \leq g$ in $\mathbb{R}$, then, $u_n \leq \tilde{u}_n$. As $n \to \infty$, we get $u \leq \tilde{u}$. This achieves the proof. ■

We first consider the subcritical case.

Theorem 2.4 *Let $m > \frac{N-2}{N}$ and $0 < q < m + \frac{2}{N}$. Then problem (1.2) has a very weak solution for any $\mu \in \mathcal{M}_b(\Omega_T)$ and $\sigma \in \mathcal{M}_b(\Omega)$.*

Proof. We can reduce to the case $\sigma = 0$ by considering $\mu = \sigma \otimes \delta_{\{t=0\}}$, see proof of Theorem 1.3. Let $\{\mu_n\}$ be approximation of μ by smooth data, and $\{u_n\}$ be a sequence of solutions of

$$\begin{cases} (u_n)_t - \Delta(|u_n|^{m-1}u_n) + T_n(|u_n|^{q-1}u_n) = \mu_n \ \text{ in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = 0 \quad \text{ in } \ \Omega, \end{cases}$$

From Proposition 2.2, $\{u_n\}$ is bounded in $L^\infty((0,T);L^1(\Omega))$ and in $L^{m+2/N,\infty}(\Omega_T)$. By equi-integrability, we get easily that, up to a subsequence, $\{u_n\}$ converges to a very weak solution of (1.1). ■

Next we show the necessary conditions given at Theorem 1.1.

Proof of Theorem 1.1. As in [5, Proof of Proposition 3.1], it is enough to claim that for any compact $K \subset \Omega \times [0,T)$ such that $\mu^-(K) = 0$, $(\sigma^- \otimes \delta_{\{t=0\}})(K) = 0$ and $\mathrm{Cap}_{2,1,\frac{q}{q-m},q'}(K) = 0$ then $\mu^+(K) = 0$ and $(\sigma^+ \otimes \delta_{\{t=0\}})(K) = 0$. Let $\varepsilon > 0$ and choose an open set O such that $(|\mu| + |\sigma| \otimes \delta_{\{t=0\}})(O\backslash K) < \varepsilon$ and $K \subset O \subset \Omega \times (-T,T)$. One can find a sequence $\{\varphi_n\} \subset C_c^\infty(O)$ which satisfies $0 \leq \varphi_n \leq 1$, $\varphi_n|_K = 1$ and $\varphi_n \to 0$ in $W^{2,1}_{\frac{q}{q-m},q'}(\mathbb{R}^{N+1})$ and almost everywhere in O (see [5, Proposition 2.2]). We get

$$\begin{aligned} \int_{\Omega_T} \varphi_n d\mu + \int_\Omega \varphi_n(0) d\sigma &= -\int_{\Omega_T} u(\varphi_n)_t dxdt - \int_{\Omega_T} |u|^{m-1}u\Delta\varphi_n dxdt + \int_{\Omega_T} |u|^{q-1}u\varphi_n dxdt \\ &\leq (||u||_{L^q(\Omega_T)} + ||u||^m_{L^q(\Omega_T)})||\varphi_n||_{W^{2,1}_{\frac{q}{q-m},\frac{q}{q-1}}(\mathbb{R}^{N+1})} + \int_{\Omega_T} |u|^q\varphi_n dxdt. \end{aligned}$$

Note that

$$\begin{aligned} \int_{\Omega_T} \varphi_n d\mu + \int_\Omega \varphi_n(0) d\sigma &\geq \mu^+(K) + (\sigma^+ \otimes \delta_{\{t=0\}})(K) - (|\mu| + |\sigma| \otimes \delta_{\{t=0\}})(O\backslash K) \\ &\geq \mu^+(K) + (\sigma^+ \otimes \delta_{\{t=0\}})(K) - \varepsilon. \end{aligned}$$

This implies

$$\mu^+(K) + (\sigma^+ \otimes \delta_{\{t=0\}})(K) \leq (||u||_{L^q(\Omega_T)} + ||u||^m_{L^q(\Omega_T)})||\varphi_n||_{W^{2,1}_{\frac{q}{q-m},\frac{q}{q-1}}(\mathbb{R}^{N+1})} + \int_{\Omega_T} |u|^q\varphi_n dxdt + \varepsilon.$$

Letting the limit we get $\mu^+(K) + (\sigma^+ \otimes \delta_{\{t=0\}})(K) \leq \varepsilon$. Therefore, $\mu^+(K) = (\sigma^+ \otimes \delta_{\{t=0\}})(K) = 0$. ■

Next we look for sufficient conditions of existence in the supercritical case $q \geq m + 2/N$. The crucial result used to establish Theorem 1.3 is the following a priori estimates, due to of Liskevich and Skrypnik [31] for $m \geq 1$ and Bogelein, Duzaar and Gianazza [11] for $m \leq 1$.

Theorem 2.5 *Let $m > \frac{N-2}{N}$ and $\mu \in C_b^+(\Omega_T)^+$. Let $u \in L^{\infty,+}(\Omega_T)$ with $u^m \in L^2(0,T,H^1_{loc}(\Omega))$ be a weak solution to equation*

$$u_t - \Delta(u^m) = \mu \quad in\ \Omega_T.$$

Then there exists $C = C(N,m)$ such that, for almost all $(y,\tau) \in \Omega_T$ and any cylinder $\tilde{Q}_r(y,\tau) \subset\subset \Omega_T$, there holds

i. *if $m > 1$*

$$u(y,\tau) \le C\left(\left(\frac{1}{r^{N+2}}\int_{\tilde{Q}_r(y,\tau)} |u|^{m+\frac{1}{2N}}dxdt\right)^{\frac{2N}{1+2N}} + ||u||_{L^\infty((\tau-r^2,\tau+r^2);L^1(B_r(y)))} + 1 + \mathbb{I}_2^{2r}[\mu](y,\tau)\right),$$

ii. *if $m \le 1$,*

$$u(y,\tau) \le C\left(\left(\frac{1}{r^{N+2}}\int_{\tilde{Q}_r(y,s)} |u|^{\frac{2(1+mN)}{N(1+m)}}dxdt\right)^{\frac{2N(m+1)}{(2-N(1-m))(2+N(1+m))}} + 1 + \left(\mathbb{I}_2^{2r}[\mu](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right).$$

As a consequence we get a new a priori estimate for the porous medium equation:

Corollary 2.6 *Let $m > \frac{N-2}{N}$ and $\mu \in C_b(\Omega_T)$. Let $u \in L^\infty(\Omega_T)$ with $|u|^m \in L^2(0,T,H^1_0(\Omega))$ be the weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) = \mu \quad in\ \Omega_T, \\ u = 0 \quad on\ \partial\Omega \times (0,T), \\ u(0) = 0 \quad in\ \ \Omega. \end{cases}$$

Then there exists $C = C(N,m)$ such that, for a.e. $(y,\tau) \in \Omega_T$,

i. *if $m > 1$,*

$$|u(y,\tau)| \le C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\mu|(\Omega_T) + 1 + \mathbb{I}_2^{2d}[|\mu|](y,\tau)\right), \tag{2.10}$$

ii. *if $m \le 1$,*

$$|u(y,\tau)| \le C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\mathbb{I}_2^{2d_1}[|\mu|](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right), \tag{2.11}$$

where m_1, m_2 and d are defined in Theorem 1.3.

Proof. Let $x_0 \in \Omega$, and $Q = B_{2d}(x_0) \times (-(2d)^2, (2d)^2)$. Consider the function $U \in C^+(Q)$, with $U^m \in L^p((-(2d)^2,(2d)^2); H^1_0(B_{2d}(x_0)))$ such that U is weak solution of

$$\begin{cases} U_t - \Delta(U^m) = \chi_{\Omega_T}|\mu| & \text{in } B_{2d}(x_0) \times (-(2d)^2,(2d)^2), \\ U = 0 & \text{on } \partial B_{2d}(x_0) \times (-(2d)^2,(2d)^2), \\ U(-(2d)^2) = 0 & \text{in } \ B_{2d}(x_0). \end{cases} \tag{2.12}$$

From Theorem 2.5, we get, for a.e $(y,\tau) \in \Omega_T$,

$$U(y,\tau) \le c_1\left(\left(\frac{1}{d^{N+2}}\int_{\tilde{Q}_d(y,\tau)} |U|^{m+\frac{1}{2N}}dxdt\right)^{\frac{2N}{1+2N}} + ||U||_{L^\infty((\tau-d^2,\tau+d^2);L^1(B_d(y)))} + 1 + \mathbb{I}_2^{2d}[|\mu|](y,\tau)\right)$$

if $m > 1$ and

$$u(y,\tau) \le C\left(\left(\frac{1}{d^{N+2}}\int_{\tilde{Q}_d(y,s)} |u|^{\frac{2(1+mN)}{N(1+m)}}\,dxdt\right)^{\frac{2N(m+1)}{(2-N(1-m))(2+N(1+m))}} + 1 + \left(\mathbb{I}_2^{2r}[\mu](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right)$$

if $m \le 1$. By Proposition 2.2, we have

$$\begin{aligned} &||U||_{L^\infty((\tau-d^2,\tau+d^2);L^1(B_d(y)))} \le |\mu|(\Omega_T),\\ &|\{|U| > \ell\}| \le c_2(|\mu|(\Omega_T))^{\frac{2+N}{N}}\ell^{-\frac{2}{N}-m}, \qquad \forall \ell > 0. \end{aligned}$$

Thus, for any $\ell_0 > 0$,

$$\begin{aligned} \int_Q U^{m+\frac{1}{2N}}dxdt &= (m+\frac{1}{2N})\int_0^\infty \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell\\ &= (m+\frac{1}{2N})\int_0^{\ell_0} \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell + (m+\frac{1}{2N})\int_{\ell_0}^\infty \ell^{m+\frac{1}{2N}-1}|\{U>\ell\}|d\ell\\ &\le c_3 d^{N+2}\ell_0^{m+\frac{1}{2N}} + c_4\ell_0^{\frac{1}{2N}-\frac{2}{N}}(|\mu|(\Omega_T))^{\frac{2+N}{N}}. \end{aligned}$$

Choosing $\ell_0 = \left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{\frac{N+2}{mN+2}}$, we get

$$\int_Q U^{(\lambda+1)(p-1)}dxdt \le c_5 d^{N+2}\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{\frac{(N+2)(2mN+1)}{2mN(mN+2)}}.$$

Thus, for a.e $(y,\tau) \in \Omega_T$,

$$U(y,\tau) \le c_6\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\mu|(\Omega_T) + 1 + \mathbb{I}_2^{2d}[|\mu|](y,\tau)\right)$$

if $m > 1$. Similarly, we also obtain for a.e $(y,\tau) \in \Omega_T$,

$$U(y,\tau) \le c_7\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\mathbb{I}_2^{2d_1}[|\mu|](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right).$$

if $m \le 1$. By the comparison principle we get $|u| \le U$ in Ω_T, and (2.10)-(2.11) follow. ∎

Lemma 2.7 *Let $m > \frac{N-2}{N}$, and $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function, such that $g \in C_b(\mathbb{R})$, $g(0) = 0$, and let $\mu \in \mathcal{M}_b(\Omega_T)$. There exists a very weak solution $u \in L^{\frac{2}{N}+m,\infty}(\Omega_T)$ of equation (2.7) which satisfies (2.10)-(2.11) and*

$$\int_{\Omega_T} |g(u)|dxdt \le |\mu|(\Omega_T).$$

Moreover, the comparison principle holds for these solutions: if u_1, u_2 are very weak solutions of (2.7) when (μ, g) is replaced by (μ_1, g_1) and (μ_2, g_2), where $\mu_1, \mu_2 \in \mathcal{M}_b(\Omega_T)$ with $\mu_1 \ge \mu_2$ and g_1, g_2 have the same properties as g with $g_1 \le g_2$ in $\mathbb{R}$ then $u_1 \ge u_2$ in Ω_T.

Proof. Let $\{\mu_n\}$ be a sequence in $C_c^\infty(\Omega_T)$ converging to μ in $\mathcal{M}_b(\Omega_T)$, such that $|\mu_n| \leq \varphi_n * |\mu|$ and $|\mu_n|(\Omega_T) \leq |\mu|(\Omega_T)$ for any $n \in \mathbb{N}$ where $\{\varphi_n\}$ be a sequence of mollifiers in $\mathbb{R}^{N+1}$. By Lemma 2.3 there exists a very weak solution u_n of problem

$$\begin{cases} (u_n)_t - \Delta(|u_n|^{m-1}u_n) + g(u_n) = \mu_n \ \text{ in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = 0 \quad \text{ in } \ \Omega, \end{cases}$$

which satisfies for a.e $(y,\tau) \in \Omega_T$,

$$|u_n(y,\tau)| \leq C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\mu|(\Omega_T) + 1 + \varphi_n * \mathbb{I}_2^{2d}[|\mu|](y,\tau)\right) \quad \text{if } m > 1,$$

$$|u_n(y,\tau)| \leq C\left(\left(\frac{|\mu|(\Omega_T)}{d^N}\right)^{m_2} + 1 + \left(\varphi_n * \mathbb{I}_2^{2d_1}[|\mu|](y,\tau)\right)^{\frac{2}{2-N(1-m)}}\right) \quad \text{if } m \leq 1,$$

and

$$\int_{\Omega_T} |\nabla T_k(|u_n|^{m-1}u_n)|^2 dxdt \leq k|\mu|(\Omega_T), \qquad \forall k > 0, \tag{2.13}$$

$$|\{|u_n| > \ell\}| \leq C_1 \ell^{-\frac{2}{N}-m}|\mu|(\Omega_T)^{\frac{N+2}{N}}, \qquad \forall \ell > 0, \tag{2.14}$$

$$\int_{\Omega_T} |g(u_n)|dxdt \leq |\mu|(\Omega_T).$$

For $l > 0$, we consider $S_l \in C_c^2(\mathbb{R})$ such that

$$S_l(a) = |a|^m a, \quad \text{for } |a| \leqq l, \quad \text{and} \quad S_l(a) = (2l)^{m+1}\text{sign}(a), \quad \text{for } |a| \geq 2l.$$

Then we find the relation in $\mathcal{D}'(\Omega_T)$:

$$(S_l(u_n))_t - \text{div}\left(S_l'(u_n)\nabla(|u_n|^{m-1}u_n)\right) + m|u_n|^{m-1}|\nabla u_n|^2 S_l''(u_n) + g(u_n)S_l'(u_n) = S_l'(u_n)\mu_n.$$

It leads to

$$\begin{aligned} ||(S_l(u_n))_t||_{L^1(\Omega_T)+L^2(0,T,H^{-1}(\Omega))} \leq\ & ||S_l'(u_n)\nabla(|u_n|^{m-1}u_n)||_{L^2(\Omega_T)} + m||\,|u_n|^{m-1}|\nabla u_n|^2 S_l''(u_n)||_{L^1(\Omega_T)} \\ & + ||g(u_n)S_l'(u_n)||_{L^1(\Omega_T)} + ||S_l'(u_n)\mu_n||_{L^1(\Omega_T)}. \end{aligned}$$

Since $|S_l'(u_n)| \leq C_2\chi_{[-2l,2l]}(u_n)$ and $|S_l''(u_n)| \leq C_3|u_n|^{m-1}\chi_{[-2l,2l]}(u_n)$, we obtain

$$||(S_l(u_n))_t||_{L^1(\Omega_T)+L^2(0,T,H^{-1}(\Omega))} \leq C_4\left(||\nabla T_{(2l)^m}(|u_n|^{m-1}u_n)||_{L^2(\Omega_T)} + ||g||_{L^\infty(\mathbb{R})}|\Omega_T| + |\mu_n|(\Omega_T)\right).$$

So from (2.13) we deduce that $\{(S_l(u_n))_t\}$ is bounded in $L^1(\Omega_T) + L^2((0,T);H^{-1}(\Omega))$ and for any $n \in \mathbb{N}$,

$$||(S_l(u_n))_t||_{L^1(\Omega_T)+L^2((0,T);H^{-1}(\Omega))} \leq C_4\left((2l)^{m/2}(|\mu|(\Omega_T))^{1/2} + ||g||_{L^\infty(\mathbb{R})}|\Omega_T| + |\mu|(\Omega_T)\right).$$

Moreover, $\{S_l(u_n)\}$ is bounded in $L^2(0,T,H_0^1(\Omega))$. Hence, $\{S_l(u_n)\}$ is relatively compact in $L^1(\Omega_T)$ for any $l > 0$. Thanks to (2.14) we find

$$\begin{aligned} |\{||u_{n_1}|^m u_{n_1} - |u_{n_1}|^m u_{n_1}| > \ell\}| &\leq |\{|u_{n_1}| > l\}| + |\{|u_{n_2}| > l\}| + |\{|S_l(u_{n_1}) - S_l(u_{n_2})| > \ell\}| \\ &\leq 2C_2 l^{-\frac{2}{N}-m}|\mu|(\Omega_T)^{\frac{N+2}{N}} + |\{|S_l(u_{n_1}) - S_l(u_{n_2})| > \ell\}|. \end{aligned}$$

Thus, up to a subsequence $\{u_n\}$ converges a.e in Ω_T to a function u. Consequently, $u \in L^{\frac{2}{N}+m,\infty}(\Omega_T)$ is a very weak solution of equation (2.7) and satisfies (2.10)-(2.11). The other conclusions follow in the same way. ∎

Remark 2.8 *If supp$(\mu) \subset \overline{\Omega}\times[a,T]$ for $a>0$, then the solution u in Lemma 2.7 satisfies $u=0$ in $\Omega\times[0,a)$.*

Now we recall the important property of Radon measures which was proved in [6] and [34].

Proposition 2.9 *Let $s>1$ and $\mu \in \mathcal{M}_b^+(\Omega_T)$. If μ is absolutely continuous with respect to $Cap_{2,1,s'}$ in Ω_T, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathcal{M}_b^+(\Omega_T)$, with compact support in Ω_T which converges to μ weakly in $\mathcal{M}_b(\Omega_T)$ and satisfies $\mathbb{I}_2^R[\mu_n] \in L^s_{loc}(\mathbb{R}^{N+1})$ for all $R>0$.*

Next we prove Theorem 1.3 in several steps of approximation:

Proof of Theorem 1.3. First suppose $m>1$. Assume that μ,σ are absolutely continuous with respect to the capacities $\mathrm{Cap}_{2,1,q'}$ in Ω_T and $\mathrm{Cap}_{\mathbf{G}_{\frac{2}{q}},q'}$ in Ω. Then $\sigma^+\otimes\delta_{\{t=0\}}+\mu^+, \sigma^-\otimes\delta_{\{t=0\}}+\mu^-$ are absolutely continuous with respect to the capacities $\mathrm{Cap}_{2,1,q'}$ in $\Omega\times(-T,T)$. Applying Proposition 2.9 to $\sigma^+\otimes\delta_{\{t=0\}}+\mu^+, \sigma^-\otimes\delta_{\{t=0\}}+\mu^-$, there exist two nondecreasing sequences $\{\upsilon_{1,n}\}$ and $\{\upsilon_{2,n}\}$ of positive bounded measures with compact support in $\Omega\times(-T,T)$ which converge respectively to $\sigma^+\otimes\delta_{\{t=0\}}+\mu^+$ and $\sigma^-\otimes\delta_{\{t=0\}}+\mu^-$ in $\mathcal{M}_b(\Omega\times(-T,T))$ and such that $\mathbb{I}_2^{2d_1}[\upsilon_{1,n}], \mathbb{I}_2^{2d_1}[\upsilon_{2,n}] \in L^q(\Omega\times(-T,T))$ for all $n\in\mathbb{N}$. By Lemma 2.7, there exists a sequence $\{u_{n_1,n_2,k_1,k_2}\}$ of of weak solution of the problems

$$\begin{cases} (u_{n_1,n_2,k_1,k_2})_t - \Delta(|u_{n_1,n_2,k_1,k_2}|^{m-1}u_{n_1,n_2,k_1,k_2}) + T_{k_1}((u^+_{n_1,n_2,k_1,k_2})^q) \\ \qquad\qquad - T_{k_2}((u^-_{n_1,n_2,k_1,k_2})^q) = \upsilon_{1,n_1} - \upsilon_{2,n_2} \ \text{ in } \Omega\times(-T,T), \\ u_{n_1,n_2,k_1,k_2} = 0 \quad \text{on } \partial\Omega\times(-T,T), \\ u_{n_1,n_2,k_1,k_2}(-T) = 0 \quad \text{in } \ \Omega, \end{cases}$$

which satisfy

$$|u_{n_1,n_2,k_1,k_2}| \le C\left(\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{d^N}\right)^{m_1} + |\sigma|(\Omega)+|\mu|(\Omega_T)+1+\mathbb{I}_2^{2d}[\upsilon_{1,n_1}]\right), \tag{2.15}$$

and

$$\int_{\Omega_T} T_{k_1}((u^+_{n_1,n_2,k_1,k_2})^q)dxdt + \int_{\Omega_T} T_{k_2}((u^-_{n_1,n_2,k_1,k_2})^q)dxdt \le |\mu|(\Omega_T).$$

Moreover, for any $n_1\in\mathbb{N}, k_2>0$, $\{u_{n_1,n_2,k_1,k_2}\}_{n_2,k_1}$ is non-increasing and for any $n_2\in\mathbb{N}, k_1>0$, $\{u_{n_1,n_2,k_1,k_2}\}_{n_1,k_2}$ is non-decreasing. Therefore, thanks to the fact that $\mathbb{I}_2^{2d_1}[\upsilon_{1,n}]$, $\mathbb{I}_2^{2d_1}[\upsilon_{2,n}] \in L^q(\Omega\times(-T,T))$ and from (2.15) and the dominated convergence Theorem, we deduce that $u_{n_1,n_2} = \lim_{k_1\to\infty}\lim_{k_2\to\infty} u_{n_1,n_2,k_1,k_2}$ is a very weak solution of

$$\begin{cases} (u_{n_1,n_2})_t - \Delta(|u_{n_1,n_2}|^{m-1}u_{n_1,n_2}) + |u_{n_1,n_2}|^{q-1}u_{n_1,n_2} = \upsilon_{1,n_1} - \upsilon_{2,n_2} \ \text{ in } \Omega\times(-T,T), \\ u_{n_1,n_2} = 0 \quad \text{on } \partial\Omega\times(-T,T), \\ u_{n_1,n_2}(-T) = 0 \quad \text{in } \ \Omega. \end{cases}$$

And (2.15) is true when u_{n_1,n_2,k_1,k_2} is replaced by u_{n_1,n_2}. Note that $\{u_{n_1,n_2}\}_{n_1}$ is non-increasing, $\{u_{n_1,n_2}\}_{n_2}$ is non-decreasing and

$$\int_{\Omega_T} |u_{n_1,n_2}|^q dxdt \leq |\mu|(\Omega_T) \quad \forall\ n_1, n_2 \in \mathbb{N}.$$

From the monotone convergence Theorem we obtain that $u = \lim_{n_2\to\infty} \lim_{n_1\to\infty} u_{n_1,n_2}$ is a very weak solution of

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \sigma \otimes \delta_{\{t=0\}} + \chi_{\Omega_T}\mu \ \text{ in } \Omega \times (-T,T), \\ u = 0 \quad \text{on } \partial\Omega \times (-T,T), \\ u(-T) = 0 \quad \text{in } \ \Omega. \end{cases}$$

which $u = 0$ in $\Omega \times (-T, 0)$ and u satisfies (**??**). Clearly, u is a very weak solution of equation (1.1).

Next suppose $m \leq 1$. The proof is similar, with the new capacitary assumptions and (**??**) is replaced by (**??**). ∎

Next, from an idea of [7, Theorem 2.3], we obtain an existence result *for measures which present a good behaviour in time*:

Theorem 2.10 *Let $m > \frac{N-2}{N}$, $q > \max(1, m)$ and $f \in L^1(\Omega_T)$, $\mu \in \mathcal{M}_b(\Omega_T)$, such that*

$$|\mu| \leq \omega \otimes F, \qquad \text{for some } \omega \in \mathcal{M}_b^+(\Omega) \text{ and } F \in L^{1,+}((0,T)).$$

If ω is absolutely continuous with respect to the capacities $Cap_{\mathbf{G}_2,\frac{q}{q-m}}$ in Ω, then there exists a very weak solution to problem

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = f + \mu \ \ \text{in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = 0. \end{cases} \tag{2.16}$$

Proof. For $R \in (0, \infty]$, we define the R-truncated Riesz elliptic potential of a measure $\nu \in \mathcal{M}_b^+(\Omega)$ by

$$\mathbf{I}_2^R[\nu](x) = \int_0^R \frac{\nu(B_\rho(x))}{\rho^{N-2}} \frac{d\rho}{\rho} \quad \forall x \in \Omega.$$

By [8, Theorem 2.6],there exists sequence $\{\omega_n\} \subset \mathcal{M}_b^+(\Omega)$ with compact support in Ω which converges to ω in $\mathcal{M}_b(\Omega)$ and such that $\mathbf{I}_2^{2\text{diam}(\Omega)}[\omega_n] \in L^{q/m}(\Omega)$ for any $n \in \mathbb{N}$. We can write

$$f + \mu = \mu_1 - \mu_2, \qquad \mu_1 = f^+ + \mu^+, \qquad \mu_2 = f^- + \mu^-,$$

and $\mu^+, \mu^- \leqq \omega \otimes F$. We set

$$\mu_{1,n} = T_n(f^+) + \inf\{\mu^+, \omega_n \otimes T_n(F)\}, \qquad \mu_{2,n} = T_n(f^-) + \inf\{\mu^-, \omega_n \otimes T_n(F)\}.$$

Then $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ are nondecreasing sequences converging to μ_1, μ_2 respectively in $\mathcal{M}_b(\Omega_T)$ and $\mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}$, with $\tilde{\omega}_n = n(\chi_\Omega + \omega_n)$ and $\mathbf{I}_2^{2\text{diam}(\Omega)}[\tilde{\omega}_n] \in L^{q/m}(\Omega)$. As in the proof of Theorem 1.3, there exists a sequence of weak solution $\{u_{n_1,n_2,k_1,k_2}\}$ of equations

$$\begin{cases} (u_{n_1,n_2,k_1,k_2})_t - \Delta(|u_{n_1,n_2,k_1,k_2}|^{m-1}u_{n_1,n_2,k_1,k_2}) + T_{k_1}((u^+_{n_1,n_2,k_1,k_2})^q) \\ \qquad\qquad - T_{k_2}((u^-_{n_1,n_2,k_1,k_2})^q) = \mu_{1,n_1} - \mu_{2,n_2} \text{ in } \Omega \times (-T,T), \\ u_{n_1,n_2,k_1,k_2} = 0 \quad \text{on } \partial\Omega \times (-T,T), \\ u_{n_1,n_2,k_1,k_2}(-T) = 0 \quad \text{in } \ \Omega. \end{cases}$$

Using the comparison principle as in [7], we can assume that

$$-v_{n_2} \leq |u_{n_1,n_2,k_1,k_2}|^{m-1}u_{n_1,n_2,k_1,k_2} \leq v_{n_1},$$

where for any $n \in \mathbb{N}$, v_n is a nonnegative weak solution of

$$\begin{cases} -\Delta v_n = \tilde{\omega}_n \ \text{ in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{cases}$$

such that

$$v_n \leq c_1 \mathbf{I}_2^{2\text{diam}(\Omega)}[\tilde{\omega}_n] \ \ \forall\ n \in \mathbb{N}.$$

Hence, utilizing the arguments in the proof of Theorem 1.3, it is easy to obtain the result as desired. ∎

It is easy to show that $\omega \otimes \chi_{[0,T]}$ is absolutely continuous with respect to the capacities $\text{Cap}_{2,1,\frac{q}{q-m},q'}$ in Ω_T if any only if ω is absolutely continuous with respect to the capacities $\text{Cap}_{\mathbf{G}_2,\frac{q}{q-m}}$ in Ω. Consequently, we obtain the following:

Corollary 2.11 *Let $m > \frac{N-2}{N}$, $q > \max(1,m)$ and $\omega \in \mathcal{M}_b(\Omega)$. Then, ω is absolutely continuous with respect to the capacities $Cap_{\mathbf{G}_2,\frac{q}{q-m}}$ in Ω if and only if there exists a very weak solution of problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) + |u|^{q-1}u = \omega \otimes \chi_{[0,T]} \ \ \textit{in } \Omega_T, \\ u = 0 \quad \textit{on } \partial\Omega \times (0,T), \\ u(0) = 0 \quad \textit{in } \ \Omega. \end{cases} \tag{2.17}$$

3 $p-$Laplacian evolution equation

Here we consider solutions in the week sense of distributions, or in the renormalized sense,.

3.1 Distribution solutions

Definition 3.1 *Let $\mu \in \mathcal{M}_b(\Omega_T)$, $\sigma \in \mathcal{M}_b(\Omega)$ and $B \in C(\mathbb{R})$. A measurable function u is a distribution solution to problem (3.1) if $u \in L^s(0,T,W_0^{1,s}(\Omega))$ for any $s \in \left[1, p - \frac{N}{N+1}\right)$, and $B(u) \in L^1(\Omega_T)$, such that*

$$-\int_{\Omega_T} u\varphi_t dxdt + \int_{\Omega_T} |\nabla u|^{p-2}\nabla u.\nabla\varphi dxdt + \int_{\Omega_T} B(u)\varphi dxdt = \int_{\Omega_T} \varphi d\mu + \int_{\Omega} \varphi(0)d\sigma,$$

for every $\varphi \in C_c^1(\Omega \times [0,T))$.

Remark 3.2 *Let $\sigma' \in \mathcal{M}_b(\Omega)$ and $a' \in (0,T)$, set $\omega = \mu + \sigma' \otimes \delta_{\{t=a'\}}$. Let u is a distribution solution to problem (3.1) with data ω and $\sigma = 0$, such that $supp(\mu) \subset \overline{\Omega} \times [a',T]$, and $u = 0, B(u) = 0$ in $\Omega \times (0,a')$. Then $\tilde{u} := u|_{\Omega\times[a',T)}$ is a distribution solution to problem (3.1) in $\Omega \times (a',T)$ with data μ and σ'.*

3.2 Renormalized solutions

The notion of renormalized solution is stronger. It was first introduced by Blanchard and Murat [10] to obtain uniqueness results for the p-Laplace evolution problem for L^1 data μ and σ, and developed by Petitta [35] for measure data μ. It requires a decomposition of the measure μ, that we recall now.

Let $\mathcal{M}_0(\Omega_T)$ be the space of Radon measures in Ω_T which are absolutely continuous with respect to the C_p-capacity, defined at (1.4), and $\mathcal{M}_s(\Omega_T)$ be the space of measures in Ω_T with support on a set of zero C_p-capacity. Classically, any $\mu \in \mathcal{M}_b(\Omega_T)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathcal{M}_0(\Omega_T) \cap \mathcal{M}_b(\Omega_T)$ and $\mu_s \in \mathcal{M}_s(\Omega_T)$. In turn μ_0 can be decomposed under the form

$$\mu_0 = f - \text{div } g + h_t,$$

where $f \in L^1(\Omega_T)$, $g \in (L^{p'}(\Omega_T))^N$ and $h \in L^p(0,T;W_0^{1,p}(\Omega))$, see [20]; and we say that (f,g,h) is a decomposition of μ_0. We say that a sequence of $\{\mu_n\}$ in $\mathcal{M}_b(\Omega_T)$ converges to $\mu \in \mathcal{M}_b(\Omega_T)$ in the *narrow topology* of measures if

$$\lim_{n\to\infty} \int_{\Omega_T} \varphi d\mu_n = \int_{\Omega_T} \varphi d\mu \quad \forall \varphi \in C(\Omega_T) \cap L^\infty(\Omega_T).$$

We recall that if u is a measurable function defined and finite a.e. in Ω_T, such that $T_k(u) \in L^p(0,T,W_0^{1,p}(\Omega))$ for any $k > 0$, there exists a measurable function $v : \Omega_T \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\le k} v$ a.e. in Ω_T and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$.

Definition 3.3 *Let $p > \frac{2N+1}{N+1}$ and $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(\Omega_T)$, $\sigma \in L^1(\Omega)$ and $B \in C(\mathbb{R})$. A measurable function u is a renormalized solution of*

$$\begin{cases} u_t - \Delta_p u + B(u) = \mu \text{ in } \Omega_T, \\ u = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

if there exists a decomposition (f,g,h) of μ_0 such that

$$\begin{aligned} v = u - h \in L^s((0,T);W_0^{1,s}(\Omega)) \cap L^\infty((0,T);L^1(\Omega)),\ \forall s \in \left[1, p - \frac{N}{N+1}\right), \\ T_k(v) \in L^p((0,T);W_0^{1,p}(\Omega))\ \forall k > 0, B(u) \in L^1(\Omega_T), \end{aligned} \tag{3.2}$$

and:

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$\begin{aligned} &-\int_\Omega S(\sigma)\varphi(0)dx - \int_{\Omega_T} \varphi_t S(v)dxdt + \int_{\Omega_T} S'(v)|\nabla u|^{p-2}\nabla u \nabla \varphi dxdt \\ &+ \int_{\Omega_T} S''(v)\varphi|\nabla u|^{p-2}\nabla u \nabla v dxdt + \int_{\Omega_T} S'(v)\varphi B(u)dxdt = \int_{\Omega_T} (fS'(v)\varphi + g.\nabla(S'(v)\varphi)dxdt \end{aligned} \tag{3.3}$$

for any $\varphi \in L^p((0,T);W_0^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ such that $\varphi_t \in L^{p'}((0,T);W^{-1,p'}(\Omega)) + L^1(\Omega_T)$ and $\varphi(.,T) = 0$;

(ii) for any $\phi \in C(\overline{\Omega_T})$,

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{m\le v<2m\}} \phi|\nabla u|^{p-2}\nabla u\nabla v dxdt = \int_{\Omega_T} \phi d\mu_s^+ \quad \textit{and} \tag{3.4}$$

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{-m\ge v>-2m\}} \phi|\nabla u|^{p-2}\nabla u\nabla v dxdt = \int_{\Omega_T} \phi d\mu_s^-. \tag{3.5}$$

We first mention a convergence result of [7].

Proposition 3.4 *Let* $\{\mu_n\}$ *be bounded in* $\mathcal{M}_b(\Omega_T)$ *and* $\{\sigma_n\}$ *be bounded in* $L^1(\Omega)$, *and* $B \equiv 0$. *Let* u_n *be a renormalized solution of* (3.1) *with data* $\mu_n = \mu_{n,0} + \mu_{n,s}$ *relative to a decomposition* (f_n, g_n, h_n) *of* $\mu_{n,0}$ *and initial data* σ_n. *If* $\{f_n\}$ *is bounded in* $L^1(\Omega_T)$, $\{g_n\}$ *bounded in* $(L^{p'}(\Omega_T))^N$ *and* $\{h_n\}$ *convergent in* $L^p(0,T,W_0^{1,p}(\Omega))$, *then, up to a subsequence,* $\{u_n\}$ *converges to a function* u *in* $L^1(\Omega_T)$. *Moreover, if* $\{\mu_n\}$ *is bounded in* $L^1(\Omega_T)$ *then* $\{u_n\}$ *is convergent in* $L^s(0,T,W_0^{1,s}(\Omega))$ *for any* $s \in \left[1, p-\frac{N}{N+1}\right)$.

Next we recall the fundamental stability result of [7].

Theorem 3.5 *Suppose that* $p > \frac{2N+1}{N+1}$ *and* $B \equiv 0$. *Let* $\sigma \in L^1(\Omega)$ *and*

$$\mu = f - divg + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(\Omega_T),$$

with $f \in L^1(\Omega_T), g \in (L^{p'}(\Omega_T))^N$, $h \in L^p((0,T);W_0^{1,p}(\Omega))$ *and* $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(\Omega_T)$. *Let* $\sigma_n \in L^1(\Omega)$ *and*

$$\mu_n = f_n - divg_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(\Omega_T),$$

with $f_n \in L^1(\Omega_T), g_n \in (L^{p'}(\Omega_T))^N, h_n \in L^p((0,T);W_0^{1,p}(\Omega))$, *and* $\rho_n, \eta_n \in \mathcal{M}_b^+(\Omega_T)$, *such that*

$$\rho_n = \rho_n^1 - div\ \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - div\ \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(\Omega_T), \rho_n^2, \eta_n^2 \in (L^{p'}(\Omega_T))^N$ *and* $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(\Omega_T)$.

Assume that $\{\mu_n\}$ *is bounded in* $\mathcal{M}_b(\Omega_T)$, $\{\sigma_n\}, \{f_n\}, \{g_n\}, \{h_n\}$ *converge to* σ, f, g, h *in* $L^1(\Omega)$, *weakly in* $L^1(\Omega_T)$, *in* $(L^{p'}(\Omega_T))^N$, *in* $L^p(0,T,W_0^{1,p}(\Omega))$ *respectively and* $\{\rho_n\}, \{\eta_n\}$ *converge to* μ_s^+, μ_s^- *in the narrow topology of measures; and* $\left\{\rho_n^1\right\}, \left\{\eta_n^1\right\}$ *are bounded in* $L^1(\Omega_T)$, *and* $\left\{\rho_n^2\right\}, \left\{\eta_n^2\right\}$ *bounded in* $(L^{p'}(\Omega_T))^N$. *Let* $\{u_n\}$ *be a sequence of renormalized solutions of*

$$\begin{cases} (u_n)_t - \Delta_p u_n = \mu_n \text{ in } \Omega_T, \\ u_n = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n \quad \text{in } \Omega, \end{cases} \tag{3.6}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ *of* $\mu_{n,0}$. *Let* $v_n = u_n - h_n$.

Then up to a subsequence, $\{u_n\}$ *converges a.e. in* Ω_T *to a renormalized solution* u *of* (3.1), *and* $\{v_n\}$ *converges a.e. in* Ω_T *to* $v = u - h$. *Moreover,* $\{\nabla v_n\}$ *converge to* ∇v *a.e in* Ω_T, *and* $\{T_k(v_n)\}$ *converges to* $T_k(v)$ *strongly in* $L^p(0,T,W_0^{1,p}(\Omega))$ *for any* $k > 0$.

In order to apply this Theorem, we need some the following properties concerning approximate measures of $\mu \in \mathcal{M}_b^+(\Omega_T)$, see also [7].

Proposition 3.6 *Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b^+(\Omega_T)$, $\mu_0 \in \mathcal{M}_0(\Omega_T) \cap \mathcal{M}_b^+(\Omega_T)$ and $\mu_s \in \mathcal{M}_s(\Omega_T)$. Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of mollifiers in $\mathbb{R}^N, \mathbb{R}$ respectively. There exists a sequence of measures $\mu_{n,0} = (f_n, g_n, h_n)$, such that $f_n, g_n, h_n, \mu_{n,s} \in C_c^\infty(\Omega_T)$ and strongly converge to f, g, h in $L^1(\Omega_T), (L^{p'}(\Omega_T))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, $\mu_{n,s}$ converges to $\mu_s \in \mathcal{M}_s^+(\Omega_T)$, and $\mu_n = \mu_{n,0} + \mu_{n,s}$ converges to μ, in the narrow topology, and satisfying $0 \le \mu_n \le (\varphi_{1,n}\varphi_{2,n}) * \mu$, and*

$$||f_n||_{L^1(\Omega_T)} + \|g_n\|_{(L^{p'}(\Omega_T))^N} + ||h_n||_{L^p(0,T,W_0^{1,p}(\Omega))} + \mu_{n,s}(\Omega_T) \le 2\mu(\Omega_T) \quad \text{for any} \quad n \in \mathbb{N}.$$

Proposition 3.7 *Let $\mu = \mu_0 + \mu_s$, $\mu_n = \mu_{n,0} + \mu_{n,s} \in \mathcal{M}_b^+(\Omega_T)$ with $\mu_0, \mu_{n,0} \in \mathcal{M}_0(\Omega_T) \cap \mathcal{M}_b^+(\Omega_T)$ and $\mu_{n,s}, \mu_s \in \mathcal{M}_s^+(\Omega_T)$ such that $\{\mu_n\}$ is nondecreasing and converges to μ in $\mathcal{M}_b(\Omega_T)$. Then, $\{\mu_{n,s}\}$ is nondecreasing and converging to μ_s in $\mathcal{M}_b(\Omega_T)$; and there exist decompositions (f, g, h) of μ_0, (f_n, g_n, h_n) of $\mu_{n,0}$ such that $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(\Omega_T), (L^{p'}(\Omega_T))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, satisfying*

$$||f_n||_{L^1(\Omega_T)} + \|g_n\|_{(L^{p'}(\Omega_T))^N} + ||h_n||_{L^p((0,T);W_0^{1,p}(\Omega))} + \mu_{n,s}(\Omega_T) \le 2\mu(\Omega_T) \quad \text{for any} \quad n \in \mathbb{N}.$$

3.3 Proof of Theorem 1.5

Here the crucial point is a result of Liskevich, Skrypnik and Sobol [30] for the p-Laplace evolution problem without absorption:

Theorem 3.8 *Let $p > 2$, and $\mu \in \mathcal{M}_b(\Omega_T)$. If $u \in C([0,T]; L^2_{loc}(\Omega)) \cap L^p_{loc}(0,T, W^{1,p}_{loc}(\Omega))$ is a distribution solution to equation*

$$u_t - \Delta_p u = \mu \quad \text{in } \Omega_T,$$

then there exists $C = C(N,p)$ such that, for every Lebesgue point $(x,t) \in \Omega_T$ of u and any $\rho > 0$ such that $Q_{\rho,\rho^p}(x,t) := B_\rho(x) \times (t - \rho^p, t + \rho^p) \subset \Omega_T$ one has

$$|u(x,t)| \le C \left(1 + \left(\frac{1}{\rho^{N+p}} \int_{Q_{\rho,\rho^p}(x,t)} |u|^{(\lambda+1)(p-1)} \right)^{\frac{1}{1+\lambda(p-1)}} + \mathbf{P}_p^\rho[\mu](x,t) \right), \tag{3.7}$$

where $\lambda = \min\{1/(p-1), 1/N\}$ and

$$\mathbf{P}_p^\rho[\mu](x,t) = \sum_{i=0}^{\infty} D_p(\rho_i)(x,t),$$

$$D_p(\rho_i)(x,t) = \inf_{\tau > 0} \left\{ (p-2)\tau^{-\frac{1}{p-2}} + \frac{1}{2(p-1)^{p-1}} \frac{|\mu|(Q_{\rho_i,\tau\rho_i^p}(x,t))}{\rho_i^N} \right\},$$

with $\rho_i = 2^{-i}\rho$, $Q_{\rho,\tau\rho^p}(x,t) = B_\rho(x) \times (t - \tau\rho^p, t + \tau\rho^p)$.

As a consequence, we deduce the following estimate:

Proposition 3.9 *If u is a distribution solution of problem*

$$\begin{cases} u_t - \Delta_p u = \mu \text{ in } \Omega_T, \\ u = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u(0) = 0 \quad \text{in } \Omega, \end{cases}$$

with data $\mu \in C_b(\Omega_T)$. Then there exists $C = C(N,p)$ such that for a.e. $(x,t) \in \Omega_T$,

$$|u(x,t)| \le C\left(1 + D + \left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}[|\mu|](x,t)\right), \tag{3.8}$$

where m_3 and D are defined at (1.6).

Proof. Let $x_0 \in \Omega$ and $Q = B_{2D}(x_0) \times (-(2D)^p, (2D)^p)$. Let $U \in C(Q) \cap L^p((-(2D)^p,(2D)^p); W_0^{1,p}(B_{2D}(x_0)))$ the distribution solution of

$$\begin{cases} U_t - \Delta_p U = \chi_{\Omega_T}|\mu| \qquad \text{in } Q, \\ u = 0 \qquad\qquad\quad \text{on } \partial B_{2D}(x_0) \times (-(2D)^p, (2D)^p), \\ u(-(2D)^p) = 0 \quad \text{in } \ B_{2D}(x_0), \end{cases} \tag{3.9}$$

where for $x_0 \in \Omega$. Thus, by Theorem 3.8 we have, for any $(x,t) \in \Omega_T$,

$$U(x,t) \le c_1\left(1 + \left(\frac{1}{D^{N+p}}\int_{Q_{D,D^p}(x,t)} |U|^{(\lambda+1)(p-1)}\right)^{\frac{1}{1+\lambda(p-1)}} + \mathbf{P}_p^D[\mu](x,t)\right), \tag{3.10}$$

where $Q_{D,D^p}(x,t) = B_D(x) \times (t - D^p, t + D^p)$.

According to Proposition 4.8 and Remark 4.9 of [7], there exists a constant $C_2 > 0$ such that

$$|\{|U| > \ell\}| \le c_2(|\mu|(\Omega_T))^{\frac{p+N}{N}} \ell^{-p+1-\frac{p}{N}} \qquad \forall \ell > 0.$$

Thus, for any $\ell_0 > 0$,

$$\begin{aligned} \int_Q |U|^{(\lambda+1)(p-1)} dxdt &= (\lambda+1)(p-1)\int_0^\infty \ell^{(\lambda+1)(p-1)-1}|\{|U| > \ell\}|d\ell \\ &= (\lambda+1)(p-1)\int_0^{\ell_0} \ell^{(\lambda+1)(p-1)-1}|\{|U| > \ell\}|d\ell + (\lambda+1)(p-1)\int_{\ell_0}^\infty \ell^{(\lambda+1)(p-1)-1}|\{|U| > \ell\}|d\ell \\ &\le c_3 D^{N+p}\ell_0^{(\lambda+1)(p-1)} + c_4\ell_0^{(\lambda+1)(p-1)-p+1-\frac{p}{N}}(|\mu|(\Omega_T))^{\frac{p+N}{N}}. \end{aligned}$$

Choosing $\ell_0 = \left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{\frac{N+p}{(p-1)N+p}}$, we get

$$\int_Q |U|^{(\lambda+1)(p-1)} dxdt \le c_5 D^{N+p}\left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{\frac{(N+p)(\lambda+1)(p-1)}{(p-1)N+p}}. \tag{3.11}$$

Next we show that

$$\mathbf{P}_p^{d_2}[\mu](x,t) \le (p-2)D + c_6\mathbb{I}_2^{2D}[|\mu|](x,t). \tag{3.12}$$

Indeed, we have

$$D_p(\rho_i)(x,t) \le (p-2)\rho_i + \frac{1}{2(p-1)^{p-1}} \frac{|\mu|(\tilde{Q}_{\rho_i}(x,t))}{\rho_i^N},$$

where $\rho_i = 2^{-i}D$. Thus,

$$\begin{aligned}\mathbf{P}_p^D[\mu](x,t) &\le (p-2)D + \frac{1}{2(p-1)^{p-1}} \sum_{i=0}^{\infty} \frac{|\mu|(\tilde{Q}_{\rho_i}(x,t))}{\rho_i^N} \\ &\le (p-2)D + C_5 \int_0^{2D} \frac{|\mu|(\tilde{Q}_{\rho}(x,t))}{\rho^N} \frac{d\rho}{\rho}.\end{aligned}$$

So from (3.11), (3.12) and (3.10) we get, for any $(x,t) \in \Omega_T$,

$$|U(x,t)| \le C \left(1 + D + \left(\frac{|\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}[|\mu|](x,t)\right).$$

By the comparison principle we get $|u| \le U$ in Ω_T, thus (3.8) follows. ∎

Proposition 3.10 *Let $p > 2$, and $\mu \in \mathcal{M}_b(\Omega_T)$, $\sigma \in \mathcal{M}_b(\Omega)$. There exists a distribution solution u of problem*

$$\begin{cases} u_t - \Delta_p u = \mu \quad \text{in } \Omega_T, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma. \end{cases} \tag{3.13}$$

which satisfies for any $(x,t) \in \Omega_T$

$$|u(x,t)| \le C \left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|\right](x,t)\right), \tag{3.14}$$

where $C = C(N,p)$. Moreover, if $\sigma \in L^1(\Omega)$, u is a renormalized solution.

Proof. Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of standard mollifiers in $\mathbb{R}^N$ and $\mathbb{R}$. Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(\Omega_T)$, with $\mu_0 \in \mathcal{M}_0(\Omega_T), \mu_s \in \mathcal{M}_s(\Omega_T)$. By Lemma 3.6, there exist sequences of nonnegative measures $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$ and $\mu_{n,s,i}$ such that $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(\Omega_T)$ and strongly converge to some f_i, g_i, h_i in $L^1(\Omega_T), (L^{p'}(\Omega_T))^N$ and $L^p((0,T); W_0^{1,p}(\Omega))$ respectively, and $\mu_{n,1}, \mu_{n,2}, \mu_{n,s,1}, \mu_{n,s,2} \in C_c^\infty(\Omega_T)$ converge to $\mu^+, \mu^-, \mu_s^+, \mu_s^-$ in the narrow topology, with $\mu_{n,i} = \mu_{n,0,i} + \mu_{n,s,i}$, for $i = 1, 2$, and satisfying

$$\mu_0^+ = (f_1, g_1, h_1), \mu_0^- = (f_2, g_2, h_2) \text{ and } 0 \le \mu_{n,1} \le (\varphi_{1,n}\varphi_{2,n}) * \mu^+, 0 \le \mu_{n,2} \le (\varphi_{1,n}\varphi_{2,n}) * \mu^-.$$

Let $\sigma_{1,n}, \sigma_{2,n} \in C_c^\infty(\Omega)$, converging to σ^+ and σ^- in the narrow topology, and in $L^1(\Omega)$ if $\sigma \in L^1(\Omega)$, such that

$$0 \le \sigma_{1,n} \le \varphi_{1,n} * \sigma^+, 0 \le \sigma_{2,n} \le \varphi_{1,n} * \sigma^-.$$

Set $\mu_n = \mu_{n,1} - \mu_{n,2}$ and $\sigma_n = \sigma_{1,n} - \sigma_{2,n}$.
Let u_n be solution of the approximate problem

$$\begin{cases} (u_n)_t - \Delta_p u_n = \mu_n \quad \text{in } \Omega_T, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = \sigma_n \quad \text{on } \Omega. \end{cases} \tag{3.15}$$

Let $g_{n,m}(x,t)=\sigma_n(x)\int_{-T}^t\varphi_{2,m}(s)ds$. As in proof of Theorem 2.1 in [34], by Theorem 3.5, there exists a sequence $\{u_{n,m}\}_m$ of solutions of the problem

$$\begin{cases}(u_{n,m})_t-\Delta_p u_{n,1,m}=(g_{n,m})_t+\chi_{\Omega_T}\mu_n \ \text{ in } \Omega\times(-T,T),\\ u_{n,1,m}=0 \quad \text{on } \partial\Omega\times(-T,T),\\ u_{n,m}(-T)=0 \ \text{ on } \Omega,\end{cases}\tag{3.16}$$

which converges to u_n in $\Omega\times(0,T)$. By Proposition 3.9, there holds, for any $(x,t)\in\Omega_T$,

$$|u_{n,m}(x,t)|\le C\left(1+D+\left(\frac{|\mu_n|(\Omega_T)+(|\sigma_n|\otimes\varphi_{2,m})(\Omega\times(-T,T))}{D^N}\right)^{m_3}+\mathbb{I}_2^{2D}[|\mu_n|+|\sigma_n|\otimes\varphi_{2,m}](x,t)\right).$$

Therefore

$$\begin{aligned}|u_{n,m}(x,t)|\le C&\left(1+D+\left(\frac{|\mu_n|(\Omega_T)+(|\sigma_n|\otimes\varphi_{2,m})(\Omega\times(-T,T))}{D^N}\right)^{m_3}\right)\\&+C(\varphi_{1,n}\varphi_{2,m})*\mathbb{I}_2^{2D}[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}](x,t).\end{aligned}$$

Letting $m\to\infty$, we get

$$|u_n(x,t)|\le C\left(1+D+\left(\frac{|\mu_n|(\Omega_T)+|\sigma_n|(\Omega)}{D^N}\right)^{m_3}\right)+c_1(\varphi_{1,n})*(\mathbb{I}_2^{2D}[|\mu|+|\sigma|\otimes\delta_{\{t=0\}}](.,t))(x).$$

Therefore, by Proposition 3.4 and Theorem 3.5 , up to a subsequence, $\{u_n\}$ converges to a distribution solution u of (3.13) (a renormalized solution if $\sigma\in L^1(\Omega)$), and satisfying (3.14). ∎

Proof of Theorem 1.5. Step 1. First, assume that $\sigma\in L^1(\Omega)$. Because μ is absolutely continuous with respect to the capacity $\mathrm{Cap}_{2,1,q'}$, so are μ^+ and μ^-. Applying Proposition 2.9 to μ^+,μ^-, there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω_T which converge to μ^+ and μ^- in $\mathcal{M}_b(\Omega_T)$ respectively and such that $\mathbb{I}_2^{2D}[\mu_{1,n}],\mathbb{I}_2^{2D}[\mu_{2,n}]\in L^q(\Omega_T)$ for all $n\in\mathbb{N}$.
For $i=1,2$, set $\tilde\mu_{i,1}=\mu_{i,1}$ and $\tilde\mu_{i,j}=\mu_{i,j}-\mu_{i,j-1}\ge 0$, so $\mu_{i,n}=\sum_{j=1}^n\tilde\mu_{i,j}$. We write

$$\mu_{i,n}=\mu_{i,n,0}+\mu_{i,n,s},\tilde\mu_{i,j}=\tilde\mu_{i,j,0}+\tilde\mu_{i,j,s},\quad\text{with } \mu_{i,n,0},\tilde\mu_{i,n,0}\in\mathcal{M}_0(\Omega_T),\mu_{i,n,s},\tilde\mu_{i,n,s}\in\mathcal{M}_s(\Omega_T).$$

Let $\{\varphi_m\}$ be a sequence of mollifiers in $\mathbb{R}^{N+1}$. As in the proof of Proposition 3.10, for any $j\in\mathbb{N}$ and $i=1,2$, there exist sequences of nonnegative measures $\tilde\mu_{m,i,j,0}=(f_{m,i,j},g_{m,i,j},h_{m,i,j})$ and $\tilde\mu_{m,i,j,s}$ such that $f_{m,i,j},g_{m,i,j},h_{m,i,j}\in C_c^\infty(\Omega_T)$ strongly converge to some $f_{i,j},g_{i,j},h_{i,j}$ in $L^1(\Omega_T),(L^{p'}(\Omega_T))^N$ and $L^p(0,T,W_0^{1,p}(\Omega))$ respectively; and $\tilde\mu_{m,i,j},\tilde\mu_{m,i,j,s}\in C_c^\infty(\Omega_T)$ converge to $\tilde\mu_{i,j},\tilde\mu_{i,j,s}$ in the narrow topology with $\tilde\mu_{m,i,j}=\tilde\mu_{m,i,j,0}+\tilde\mu_{m,i,j,s}$, which satisfy $\tilde\mu_{i,j,0}=(f_{i,j},g_{i,j},h_{i,j})$, and

$$0\le\tilde\mu_{m,i,j}\le\varphi_m*\tilde\mu_{i,j},\tilde\mu_{m,i,j}(\Omega_T)\le\tilde\mu_{i,j}(\Omega_T),$$

$$||f_{m,i,j}||_{L^1(\Omega_T)}+\|g_{m,i,j}\|_{(L^{p'}(\Omega_T))^N}+||h_{m,i,j}||_{L^p(0,T,W_0^{1,p}(\Omega))}+\mu_{m,i,j,s}(\Omega_T)\le 2\tilde\mu_{i,j}(\Omega_T).\tag{3.17}$$

Note that, for any $n,m\in\mathbb{N}$,

$$\sum_{j=1}^n(\tilde\mu_{m,1,j}+\tilde\mu_{m,2,j})\le\varphi_m*(\mu_{1,n}+\mu_{2,n})\text{ and }\sum_{j=1}^n(\tilde\mu_{m,1,j}(\Omega_T)+\tilde\mu_{m,2,j}(\Omega_T))\le|\mu|(\Omega_T).$$

For any $n,k,m\in\mathbb{N}$, let $u_{n,k,m},v_{n,k,m}\in W$ be solutions of problems

$$\begin{cases}(u_{n,k,m})_t-\Delta_p u_{n,k,m}+T_k(|u_{n,k,m}|^{q-1}u_{n,k,m})=\sum_{j=1}^n(\tilde\mu_{m,1,j}-\tilde\mu_{m,2,j}) \text{ in } \Omega_T,\\ u_{n,k,m}=0 \qquad\qquad \text{on } \partial\Omega\times(0,T),\\ u_{n,k,m}(0)=T_n(\sigma^+)-T_n(\sigma^-) \quad \text{on } \Omega,\end{cases}\tag{3.18}$$

and

$$\begin{cases}(v_{n,k,m})_t-\Delta_p v_{n,k,m}+T_k(v^q_{n,k,m})=\sum_{j=1}^n(\tilde\mu_{m,1,j}+\tilde\mu_{m,2,j}) \text{ in } \Omega_T,\\ v_{n,k,m}=0 \qquad\qquad \text{on } \partial\Omega\times(0,T),\\ v_{n,k,m}(0)=T_n(|\sigma|) \quad \text{on } \Omega.\end{cases}\tag{3.19}$$

By the comparison principle and Proposition 2.9 we have for any m,k the sequences $\{v_{n,k,m}\}_n$ is increasing and

$$\begin{aligned}|u_{n,k,m}|\le v_{n,k,m}\le c_1\left(1+D+\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{D^N}\right)^{m_3}+\mathbb{I}_2^{2D}\left[T_n(|\sigma|)\otimes\delta_{\{t=0\}}\right]\right)\\ +c_1\varphi_m*\mathbb{I}_2^{2D}\left[\mu_{1,n}+\mu_{2,n}\right].\end{aligned}$$

Moreover,

$$\int_{\Omega_T}T_k(v^q_{n,k,m})dxdt\le|\mu|(\Omega_T)+|\sigma|(\Omega).$$

As in [7, Proof of Lemma 6.4], thanks to Proposition 3.4 and Theorem 3.5, up to subsequences, $\{u_{n,k,m}\}_m$ converge to a renormalized solutions $u_{n,k}$ of problem

$$\begin{cases}(u_{n,k})_t-\Delta_p u_{n,k}+T_k(|u_{n,k}|^{q-1}u_{n,k})=\mu_{1,n}-\mu_{2,n} \text{ in } \Omega_T,\\ u_{n,k}=0 \qquad\qquad \text{on } \partial\Omega\times(0,T),\\ u_{n,k}(0)=T_n(\sigma^+)-T_n(\sigma^-) \quad \text{on } \Omega,\end{cases}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j}-\sum_{j=1}^n f_{2,j},\sum_{j=1}^n g_{1,j}-\sum_{j=1}^n g_{2,j},\sum_{j=1}^n h_{1,j}-\sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0}-\mu_{2,n,0}$; and $\{v_{n,k,m}\}_m$ converge to a solution $v_{n,k}$ of

$$\begin{cases}(v_{n,k})_t-\Delta_p v_{n,k}+T_k(v^q_{n,k})=\mu_{1,n}+\mu_{2,n} \text{ in } \Omega_T,\\ v_{n,k}=0 \qquad\qquad \text{on } \partial\Omega\times(0,T),\\ v_{n,k}(0)=T_n(|\sigma|) \quad \text{on } \Omega.\end{cases}$$

relative to the decomposition $(\sum_{j=1}^n f_{1,j}+\sum_{j=1}^n f_{2,j},\sum_{j=1}^n g_{1,j}+\sum_{j=1}^n g_{2,j},\sum_{j=1}^n h_{1,j}+\sum_{j=1}^n h_{2,j})$ of $\mu_{1,n,0}+\mu_{2,n,0}$. And there holds

$$|u_{n,k}|\le v_{n,k}\le C\left(1+D+\left(\frac{|\sigma|(\Omega)+|\mu|(\Omega_T)}{D^N}\right)^{m_3}+\mathbb{I}_2^{2D}\left[T_n(|\sigma|)\otimes\delta_{\{t=0\}}\right]\right)+C\mathbb{I}_2^{2D}\left[\mu_{1,n}+\mu_{2,n}\right].$$

Observe that $\mathbb{I}_2^{2D}[\mu_{1,n}+\mu_{2,n}]\in L^q(\Omega_T)$ for any $n\in\mathbb{N}$. Then, as in [7, Proof of Lemma 6.5], thanks to Proposition 3.4 and Theorem 3.5, up to a subsequence, $\{u_{n,k}\}_k$ $\{v_{n,k}\}_k$ converge to renormalized solutions u_n,v_n of problems

$$\begin{cases}(u_n)_t-\Delta_p u_n+|u_n|^{q-1}u_n=\mu_{1,n}-\mu_{2,n} \text{ in } \Omega_T,\\ u_n=0 \qquad\qquad \text{on } \partial\Omega\times(0,T),\\ u_n(0)=T_n(\sigma^+)-T_n(\sigma^-) \quad \text{in } \Omega,\end{cases}\tag{3.20}$$

$$\begin{cases} (v_n)_t - \Delta_p v_n + v_n^q = \mu_{1,n} + \mu_{2,n} \ \text{ in } \ \Omega_T, \\ v_n = 0 \qquad\qquad\quad \text{on } \partial\Omega \times (0,T), \\ v_n(0) = T_n(|\sigma|) \quad \text{in } \ \Omega, \end{cases} \tag{3.21}$$

which still satisfy

$$|u_n| \le v_n \le C\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[T_n(|\sigma|) \otimes \delta_{\{t=0\}}\right]\right) + C\mathbb{I}_2^{2D}\left[\mu_{1,n} + \mu_{2,n}\right].$$

and the sequence $\{v_n\}_n$ is increasing and

$$\int_{\Omega_T} v_n^q dxdt \le |\mu|(\Omega_T) + |\sigma|(\Omega).$$

Note that from (3.17) we have

$$||f_{i,j}||_{L^1(\Omega_T)} + \|g_{i,j}\|_{(L^{p'}(\Omega_T))^N} + ||h_{i,j}||_{L^p(0,T,W_0^{1,p}(\Omega))} \le 2\tilde{\mu}_{i,j}(\Omega_T),$$

which implies

$$||\sum_{j=1}^n f_{i,j}||_{L^1(\Omega_T)} + ||\sum_{j=1}^n g_{i,j}||_{(L^{p'}(\Omega_T))^N} + ||\sum_{j=1}^n h_{i,j}||_{L^p(0,T,W_0^{1,p}(\Omega))} \le 2\tilde{\mu}_{i,n}(\Omega_T) \le 2|\mu|(\Omega_T).$$

Finally, as in [7, Proof of Theorem 6.3], from Proposition 3.4, Theorem 3.5 and the monotone convergence Theorem, up to subsequences $\{u_n\}_n$, $\{v_n\}_n$ converge to a renormalized solutions u, v of problem

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \mu \ \text{ in } \ \Omega_T, \\ u = 0 \qquad\qquad\quad \text{on } \partial\Omega \times (0,T), \\ u(0) = \sigma \quad \text{in } \ \Omega, \end{cases}$$

relative to the decomposition $(\sum_{j=1}^\infty f_{1,j} - \sum_{j=1}^\infty f_{2,j}, \sum_{j=1}^\infty g_{1,j} - \sum_{j=1}^\infty g_{2,j}, \sum_{j=1}^\infty h_{1,j} - \sum_{j=1}^\infty h_{2,j})$ of μ_0. And

$$\begin{cases} v_t - \Delta_p v + |v|^{q-1}v = |\mu| \ \text{ in } \ \Omega_T, \\ v = 0 \qquad\qquad\quad \text{on } \partial\Omega \times (0,T), \\ v(0) = |\sigma| \quad \text{in } \ \Omega, \end{cases}$$

relative to the decomposition $(\sum_{j=1}^\infty f_{1,j} + \sum_{j=1}^\infty f_{2,j}, \sum_{j=1}^\infty g_{1,j} + \sum_{j=1}^\infty g_{2,j}, \sum_{j=1}^\infty h_{1,j} + \sum_{j=1}^\infty h_{2,j})$ of $|\mu_0|$ respectively; and

$$|u| \le v \le C\left(1 + D + \left(\frac{|\sigma|(\Omega) + |\mu|(\Omega_T)}{D^N}\right)^{m_3} + \mathbb{I}_2^{2D}\left[|\sigma| \otimes \delta_{\{t=0\}} + |\mu|\right]\right)$$

Remark that, if $\sigma \equiv 0$ and $\text{supp}(\mu) \subset \overline{\Omega} \times [a,T]$, $a > 0$, then $u = v = 0 \text{in } \Omega \times (0,a)$, since $u_{n,k} = v_{n,k} = 0$ in $\Omega \times (0,a)$.

Step 2. We consider any $\sigma \in \mathcal{M}_b(\Omega)$ such that σ is absolutely continuous with respect to the capacity $\text{Cap}_{\mathbf{G}_{\frac{2}{q},q'}}$ in Ω. So, $\mu + \sigma \otimes \delta_{\{t=0\}}$ is absolutely continuous with respect to the capacity $\text{Cap}_{2,1,q'}$ in $\Omega \times (-T,T)$. As above, we verify that there exists a renormalized solution u of

$$\begin{cases} u_t - \Delta_p u + |u|^{q-1}u = \chi_{\Omega_T}\mu + \sigma \otimes \delta_{\{t=0\}} \ \text{ in } \ \Omega \times (-T,T) \\ u = 0 \qquad\qquad\quad \text{on } \partial\Omega \times (-T,T), \\ u(-T) = 0 \quad \text{on } \ \Omega, \end{cases}$$

satisfying $u = 0$ in $\Omega \times (-T, 0)$ and (1.5). Finally, from Remark 3.2 we get the result. This completes the proof of the Theorem. ∎

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J., **42**, no. 4, 765-778 (1975).

[2] D. R. Adams, L.I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wisenschaften 31, Springer-Verlag (1999).

[3] D.R. Adams, R.J. Bagby, *Translation-dilation invariant estimates for Riesz potentials*, Indiana Univ. Math. J., **23**, 1051-1067 (1974).

[4] R.J. Bagby, *Lebesgue spaces of parabolic potentials*, Ill. J. Math., **15**, 610-634 (1971).

[5] P. Baras and M. Pierre, *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18**, 111-149 (1984).

[6] P. Baras, M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin., **3**, 185-212 (1985).

[7] M. F. Bidaut-Véron, Q.H. Nguyen, *Stability properties for quasilinear parabolic equations with measure data and applications*. Submitted.

[8] M. F. Bidaut-Véron, Q.H Nguyen, L. Véron, *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math.Pures et Appl.,**102**, 315-337 (2014).

[9] L. Boccardo , A. Dall'Aglio, T. Gallouet and L. Orsina, *Nonlinear parabolic equations with measure data*, J. Funct. Anal. **147**, 237-258 (1997).

[10] D. Blanchard, F. Murat, *Renormalized solutions of nonlinear parabolic equation with L^1 data: existence and uniqueness*, Proc. Roy. Soc. Edinburgh, **127A**, 1153-1179 (1997).

[11] V. Bogelein, F. Duzaar and U. Gianazza, Inst. Mittag-Leffler, Report 31 (2013-2014).

[12] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions,* J.Math.Pures Appl., **62**, 73-97 (1983).

[13] E. Chasseigne, *Contribution à la Théorie des traces pour les équations quasilinéaires paraboliques,* Thesis, Univ. Tours, France (2000).

[14] E. Chasseigne, *Initial trace for a porous medium equation: I. The strong absorption case,* Ann. Mat. Pura Appl. IV, **179**, 413-458 (2001).

[15] E. Chasseigne and J.L. Vazquez, *Sets of admissible initialdata for porous-medium equation with absorption,* E.J.D.E., conf.**08**, 53-83 (2001).

[16] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Sc. Norm. Sup. Pisa, **28**, 741-808 (1999).

[17] A. Dall'Aglio and L. Orsina, *Existence results for some nonlinear parabolic equations with nonregular data*, Diff. Int. Equ., **5**, 1335-1354 (1992).

[18] P. Daskalopoulou and C. Kenig, *Degenerate diffusions. Initial value problems and local regularity theory*, EMS Tracts in Mathematics **1**, European Mathematical Society Zurich (2007).

[19] E. DiBenedetto : *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York (1993).

[20] J. Droniou , A. Porretta and A. Prignet, *Parabolic capacity and soft measures for nonlinear equations*, Potential Anal. **19**, 99-161 (2003).

[21] F. Duzaar, G. Mingione, *Gradient estimate via non-linear potentials*, the American Journal of Mathematics, **133**, no. 4, 1093-1149 (2011).

[22] D. Feyel, A. de la Pradelle: *Topologies fines et compactifications associées à certains espaces de Dirichlet*, Ann. Inst. Fourier Grenoble, **27**, 121-146 (1977).

[23] J. Heinonen, T. Kilpelainen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations.* Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, N.Y. (2006).

[24] T. Kilpelainen, P. Koskela: *Global integrability of the gradients of solutions to partial differential equations.* Nonlinear Anal., **23**, no. 7, 899-909 (1994).

[25] T. Kuusi, G. Mingione: *Riesz potentials and nonlinear parabolic equations,* to appear in Arch. for Rat. Mech. Anal. (2013)

[26] T. Kuusi, G. Mingione: *the Wolff gradient bound for degenerate parabolic equations* to appear in J. European Math. Soc.

[27] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. UralÃ-Ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. **23**, Amer. Math. Soc., Providence (1968).

[28] R. Landes, *On the existence of weak solutions for quasilinear parabolic initial boundary-value problems,* Proc. Royal Soc. Edinburg Sect A, **89**, 217-237 (1981).

[29] G.M. Lieberman: *Second Order Parabolic Differential Equations*, World Scientific press, River Edge (1996).

[30] Liskevich, Vitali; Skrypnik, Igor I.; Sobol, Zeev *Estimates of solutions for the parabolic p-Laplacian equation with measure via parabolic nonlinear potentials.* Commun. Pure Appl. Anal., **12**, 1731-1744 (2013).

[31] Liskevich, Vitali; Skrypnik, Igor I. *Pointwise estimates for solutions to the porous medium equation with measure as a forcing term.* Isreal J. of mathematics, **194**, 259-275 (2013).

[32] G. Mingione: *Nonlinear measure data problems* Milan J. Math., **79** 429-496 (2011).

[33] Q.H. Nguyen , L. Véron *Quasilinear and Hessian type equations with exponential reaction and measure data*, to appear in Arch. for Rat. Mech. Anal. (2014).

[34] Q.H. Nguyen, *Potential estimates and quasilinear equations with measure data*, arXiv:1405.2587v1.

[35] F. Petitta, *Renormalized solutions of nonlinear parabolic equations with general measure data,* Ann. Math. Pura Appl., **187**, 563-604 (2008).

[36] F. Petitta, A. Ponce and A. Porretta, *Diffuse measures and nonlinear parabolic equations*, J. Evol. Equ., **11**, 861-905 (2011).

[37] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailiov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin and New York (1995).

[38] J.L. Vazquez, *Smoothing and decay estimates for nonlinear diffusion equations. Equations if porous medium type*, Oxford lecture series in mathematics ans its applications, 33, Oxford University Press, Oxford (2006).

[39] J.L.Vazquez, *The porous medium equation. Mathematical theory* Oxford Mathematical monographs, the clarendon, Oxford University Press, Oxford (2007).