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# HIGH LEVEL QUANTILE ESTIMATIONS OF SUMS OF RISKS

A. CUBEROS, E. MASIELLO, AND V. MAUME-DESCHAMPS

ABSTRACT. The estimation of high level quantile or of the expectation over a high quantile (Value at Risk - VaR - or Tail Value at Risk - TVaR - in risk management contexts) is crucial for the insurance industry. We propose a new method to estimate high level quantiles of sums of risks. It is based on the estimation of the ratio between the VaR of the sum and the VaR of the maximum. We use results on consistently varying functions. We compare the efficiency of our method with classical ones, on several models.

## 1. INTRODUCTION

Because of regulatory rules (such as Solvency 2 in Europe) or for internal risk management purposes, the estimation of high level quantile of sums of risks is of major interest.

Consider an insurance company that has a portfolio of  $d \geq 2$  (possibly) dependent risks which is represented as a random vector  $\mathbf{X} = (X_1, \dots, X_d)$  with cumulative distribution function (c.d.f.)  $\mathbf{F}(x_1, \dots, x_d)$ . We assume that all the risks are almost surely positive but we do not assume that they are identically distributed. Let  $S$  denote the aggregated risk

$$S = X_1 + \dots + X_d,$$

we are interested here in the computation of the Value-at-Risk and the Tail Value-at-Risk of the sum,

$$\text{VaR}_p(S) = F_S^{\leftarrow}(p) \quad \text{and} \quad \text{TVaR}_p(S) = \frac{1}{1-p} \int_p^1 \text{VaR}_u(S) \, du,$$

for confidence levels  $p \in ]0, 1[$  near to 1, where  $F_S$  is the c.d.f. of  $S$  and  $F^{\leftarrow}$  is its generalized inverse. Even when the distribution function  $\mathbf{F}$  is known, good estimations for  $\text{VaR}_p(S)$  are not trivial since they require precise calculation of  $F_S$ , which is given by the integral

$$F_S(x) = \int_{\{x_1 + \dots + x_d \leq x\}} d\mathbf{F}(x_1, \dots, x_d).$$

This integral is more difficult to approximate when  $d$  is large and generally it is more efficient to apply Monte Carlo methods to estimate it. However, when  $p$  is near to 1, the number of simulations required to give precise estimations is also large so new methods are always well received. Classical Extreme Value Theory (EVT) allows to get some estimation of the VaR ([14, 23]). Nevertheless, using EVT based methods require estimations of

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*Key words and phrases.* consistently varying functions, value at risk estimation, risk aggregation.

the EVT parameters, which is known to be not an easy task. Recently, in [7, 15, 8], some bounds on the VaR are obtained for some specific models, see also [16] where theoretical results on the asymptotic behavior of the ratio

$$\frac{\text{VaR}_p(S)}{\sum_{i=1}^d \text{VaR}_p(X_i)}$$

are given. In [5], an algorithm to compute the distribution function of  $S$  is also proposed, it may be used to estimate  $\text{VaR}_p(S)$  for small dimension ( $d < 4$ ) and gives ranges in dimension 4 or 5. We shall compare our method to the EVT driven ones, especially for very high level quantiles.

Let us denote by  $M$  the maximum risk on the portfolio of the company,  $M = \max\{X_1, \dots, X_d\}$ . The c.d.f of  $M$ , that we denote  $F_M$ , is given by

$$F_M(x) = \mathbf{F}(x, \dots, x).$$

Contrary to the sum, the c.d.f of the maximum is directly determined by the c.d.f.  $\mathbf{F}$  of the portfolio, numerical integration or Monte Carlo methods are not necessary. This means also that  $\text{VaR}_p(M)$  can be easily calculated for any level of confidence  $p$  given, at most a simple numerical inversion is needed.

In this paper we give some conditions on  $\mathbf{X}$  under which the value-at-risk and the tail value-at-risk of the sum and maximum are asymptotically equivalent in the sense that there exists some  $\Delta > 0$  such that

$$\text{VaR}_{1-p}(S) \sim \text{VaR}_{1-\Delta^{-1}p}(M) \quad \text{and} \quad \text{TVaR}_{1-p}(S) \sim \text{TVaR}_{1-\Delta^{-1}p}(M),$$

when  $p \rightarrow 0$  and where  $a(t) \sim b(t)$  when  $t \rightarrow l$ , for  $l \in [-\infty, \infty]$  means through this paper that  $\lim_{t \rightarrow l} \frac{a(t)}{b(t)} = 1$ . This result is interesting because it allows to estimate the VaR (or TVaR) of the sum by using the VaR (or TVaR) of the maximum, which is easy to calculate, and the estimation of  $\Delta$ .

For random vectors with common marginals (Frechet, Gumbel, Weibull) with an Archimedean copula dependence structure [2] and [3] get an asymptotic approximation of the tail of  $S$ . These results are generalized in [4] to more general dependence structures. In [6] the same results are obtained in the multivariate regularly varying framework. We consider a more general framework (not common marginals, consistely varying tails) and propose estimations. Finally, we would like to mention [19] which is related to our work, in an independent framework and for Pareto marginals.

The paper is organized as follows. In Section 2 we recall definitions and classical results on regularly and consistently varying functions. Section 3 contains our main results. In Section 4 we give classes of random vectors satisfying our hypothesis. Section 5 is devoted to the estimation of  $\Delta$ . In Section 6 we give explicit expressions of the VaR on some specific models (introduced in [20, 24]) and also considered in [12] where the expression of the VaR is derived. In Section 7 we compare our method with classical

ones on several models. We would like to emphasize that our method is tractable, even in high dimension (dimension 10 tested). Section 8 is an appendix which contains useful results on consistently varying functions.

## 2. PRELIMINARIES

In this section, we will first recall the definition of regularly and consistently varying functions and then give some results on generalized inverse functions and consistently varying functions.

**Definition 1.** Let  $f$  be a positive measurable function on  $\mathbb{R}_+$ .

- We say that  $f$  is regularly varying at infinity if there exists a real  $\rho$  such that

$$\lim_{x \rightarrow \infty} \frac{f(tx)}{f(x)} = t^\rho,$$

for any  $t > 0$ . This will be denoted by  $f \in \mathcal{RV}_\infty(\rho)$ . Similarly we say that  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is regularly varying at  $a \geq 0$  if  $f(a + 1/x)$  is regularly varying at infinity. This will be denoted by  $f \in \mathcal{RV}_a(\rho)$ . If  $\rho = 0$  then  $f$  is said to be slowly varying.

- We say that  $f$  is consistently varying at infinity, denoted as  $f \in \mathcal{C}_\infty$ , if

$$\lim_{t \downarrow 1} \liminf_{x \rightarrow \infty} \frac{f(tx)}{f(x)} = 1.$$

Similarly to the regularly varying definition,  $f$  is said to be consistently varying at  $a > 0$ , denoted by  $f \in \mathcal{C}_a$ , if  $g(x) = f(a + 1/x)$  is consistently varying at infinity.

Notice that every regularly varying function is consistently varying. Examples of consistently varying functions which are not regularly can be found in [11].

**Definition 2.** Let  $f$  be a non-decreasing function and  $h$  a non-increasing function. The generalized inverses of  $f$  and  $h$  are defined respectively as

$$f^-(t) = \inf\{s : f(s) \geq t\} \quad \text{and} \quad h^-(t) = \inf\{s : h(s) \leq t\}$$

Remark that if  $h$  is positive we have then  $h^-(t) = (1/h)^-(1/t)$ .

Let us recall some well known facts on the generalized inverse functions.

**Proposition 2.1.** *If  $f$  and  $h$  are two right-continuous functions, respectively non-decreasing and non-increasing, then for any  $x$  and  $y$  the following is satisfied:*

- (i)  $f^-(f(x)) \leq x$ ,  $f(f^-(x)) \geq x$  and  $f^-(y) \leq x \Leftrightarrow y \leq f(x)$
- (ii)  $h^-(h(x)) \leq x$ ,  $h(h^-(x)) \leq x$  and  $h^-(y) \leq x \Leftrightarrow y \geq h(x)$

**Proposition 2.2.** *If  $f_t, t \geq 0$  are non-decreasing functions on  $\mathbb{R}$  and  $f_t \rightarrow f_0$ , then  $f_t^- \rightarrow f_0^-$  in the sense that*

$$f_t^-(x) \rightarrow f_0^-(x)$$

for all  $x$  on the continuity points of  $f_0^-$ . The same is true for non-increasing functions changing  $^-$  by  $^+$  above.

*Proof.* For the proof on non-decreasing function see for example [21] Chapter 2. Non-increasing case follows using that  $f_n^-(x) = (\frac{1}{f_n})^-(\frac{1}{x})$ .  $\square$

**2.1. Some results on Consistently Varying functions.** It is well known that the sum of regularly varying functions is regularly varying, if  $f_i \in \mathcal{RV}_\infty(\rho_i)$ ,  $i = 1, 2$ , then  $f_1 + f_2 \in \mathcal{RV}_\infty(\rho)$  with  $\rho = \max\{\rho_1, \rho_2\}$  (see for example [9]). Below we prove that consistently varying functions also satisfy this closure property.

**Proposition 2.3.** *Let  $f_1$  and  $f_2$  be two consistently varying and non-increasing functions. Then  $f_1 + f_2$  is consistently varying.*

*Proof.* Set  $\epsilon > 0$ , and define for  $t > 1$

$$L(t) := \liminf_{x \rightarrow \infty} \frac{f_1(tx) + f_2(tx)}{f_1(x) + f_2(x)} = \liminf_{x \rightarrow \infty} \left( \frac{f_1(tx)/f_1(x)}{1 + f_2(x)/f_1(x)} + \frac{f_2(tx)/f_2(x)}{f_1(x)/f_2(x) + 1} \right).$$

Then, as  $f_1$  and  $f_2$  are consistently varying, there exist reals  $s > 1$  and  $N > 0$  such that  $f_i(sx)/f_i(x) > 1 - \epsilon$  for all  $x > N$ , and  $i = 1, 2$ . Then

$$L(s) \geq (1 - \epsilon) \liminf_{x \rightarrow \infty} \left( \frac{1}{1 + f_2(x)/f_1(x)} + \frac{1}{1 + f_1(x)/f_2(x)} \right) = (1 - \epsilon).$$

As  $L$  is non-increasing, then also  $L(s') \geq 1 - \epsilon$  for all  $1 < s' < s$ . Finally as  $L(t) \leq 1$  for all  $t > 1$  we had shown

$$\lim_{t \downarrow 1} L(t) = 1.$$

□

**Proposition 2.4.** *Let  $f$  and  $g$  be non-increasing and consistently varying functions such that  $f(\infty) = 0 = g(\infty)$ . Then  $g \circ (1/f)$  is consistently varying.*

*Proof.* Set  $\epsilon > 0$  and choose  $s' > 1$  and  $s > 1$  such that

$$\liminf_{x \rightarrow \infty} \frac{g(s'x)}{g(x)} > 1 - \epsilon \quad \text{and} \quad \liminf_{x \rightarrow \infty} \frac{f(sx)}{f(x)} > 1/s'.$$

Then

$$\liminf_{x \rightarrow \infty} \frac{g(1/f(sx))}{g(1/f(x))} \geq \liminf_{x \rightarrow \infty} \frac{g(s'/f(x))}{g(1/f(x))} > 1 - \epsilon,$$

which proves the proposition. □

**Proposition 2.5.** *Let  $f$  be a non-increasing measurable function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . If  $f \in \mathcal{C}_\infty$  then the function  $x \mapsto f^\leftarrow(1/x)$  is consistently varying.*

*Proof.* Set  $H_x(t) = f(tx)/f(x)$  and  $H(t) = \liminf_{x \rightarrow \infty} H_x(t)$ . We will first show that  $\liminf_{x \rightarrow \infty} H_x^\leftarrow(\omega) \leq H^\leftarrow(\omega)$  for all  $0 < \omega < 1$ . Take  $\omega \in (0, 1)$ , for each  $x > 0$  let us denote by  $I_x(\omega)$  the value

$$I_x(\omega) = \inf\{H_s^\leftarrow(\omega) : s \geq x\}.$$

Then  $H_x^\leftarrow(\omega) \geq I_x(\omega)$  and by Proposition 2.1 (ii)  $\omega \leq H_x(I_x(\omega))$ , for all  $x > 0$ . We have then that  $\omega \leq \inf\{H_s(I_s(\omega)) : s \geq x\}$  for any  $x > 0$ . Notice now that as  $I_x(\omega)$  is non-decreasing on  $x$  and that each  $H_x$  is a non-increasing function then

$$\omega \leq \inf\{H_s(I_s(\omega)) : s \geq x\} \leq \inf\{H_s(I_x(\omega)) : s \geq x\} \leq H(I_x(\omega))$$

for all  $x > 0$ . By Proposition 2.1 (ii) we have

$$I_x(\omega) \leq H^\leftarrow(\omega)$$

for all  $x > 0$ . Thus by taking limits we find

$$\liminf_{x \rightarrow \infty} H_x^-(\omega) \leq H^-(\omega).$$

Now, we prove the announced result. For each  $x > 0$  and  $0 < \omega < 1$  we have  $H_x^-(\omega) = f^-(\omega f(x))/x$ . Proposition 2.1 (ii) gives,  $f \circ f^-(1/x) \leq 1/x$  for all  $x > 0$  and then for  $0 < \omega < 1$  we have  $f^-(\omega/x) \leq f^-(\omega f \circ f^-(1/x))$ . Now, we have

$$1 \leq \liminf_{x \rightarrow \infty} \frac{f^-(\omega/x)}{f^-(1/x)} \leq \liminf_{x \rightarrow \infty} \frac{f^-(\omega f \circ f^-(1/x))}{f^-(1/x)} = \liminf_{x \rightarrow \infty} H_x^-(\omega) \leq H^-(\omega).$$

As  $f \in \mathcal{C}_\infty$  implies  $H(t) \uparrow 1$  when  $t \downarrow 1$ , and thus, as  $H$  is non-increasing, then  $H^-(\omega) \downarrow 1$  when  $\omega \uparrow 1$  and

$$\lim_{\omega \uparrow 1} \liminf_{x \rightarrow \infty} \frac{f^-(\omega/x)}{f^-(1/x)} = 1,$$

which proves the Proposition.  $\square$

Below, we prove results, that are classical for regularly varying functions, for consistently varying functions.

**Proposition 2.6.** *Let  $f$  be a non-increasing function such that  $f(\infty) = 0$ . If  $f \in \mathcal{C}_\infty$  then  $f^- \circ f(x) \sim x$  and  $f \circ f^-(1/x) \sim 1/x$  when  $x \rightarrow \infty$ .*

*Proof.* Proposition 2.1 (ii) gives  $f^-(f(x)) \leq x$  for all  $x > 0$ . By definition of the generalized inverses it follows that

$$0 \leq x - f^- \circ f(x) \leq \lim_{s \uparrow 1} f^-(sf(x)) - \lim_{t \downarrow 1} f^-(tf(x))$$

where the last expression represents the size of the possible jump of  $f^-$  at  $f(x)$ . Take  $\epsilon > 0$ , by Proposition 2.4 the mapping  $x \mapsto f^-(1/x)$  is consistently varying so we can choose  $N > 0$ ,  $0 < s < 1$  and  $t > 1$  such that for all  $x > N$

$$\frac{f^-(s/x)}{f^-(1/x)} \leq 1 + \frac{\epsilon}{2} \quad \text{and} \quad \frac{f^-(t/x)}{f^-(1/x)} \geq 1 - \frac{\epsilon}{2}.$$

Then

$$0 \leq 1 - \frac{f^- \circ f(x)}{x} \leq \frac{f^-(sf(x)) - f^-(tf(x))}{f^- \circ f(x)} \leq \epsilon$$

where the last inequality holds for all  $x > f^-(1/N)$ . As  $\epsilon$  was arbitrary it had been proven that  $f^- \circ f(x) \sim x$ . The proof that  $f \circ f^-(1/x) \sim 1/x$  is similar.  $\square$

**Proposition 2.7.** *Let  $f$  be a non-increasing and consistently varying function and suppose that  $a(t)$  and  $b(t)$  are two positive sequences such that  $a(t) \sim b(t)$ .*

(i) *Suppose  $a(\infty) = \infty$ . Then*

$$f(a(t)) \sim f(b(t)).$$

(ii) *Suppose  $a(\infty) = 0$ . Then*

$$f^-(a(t)) \sim f^-(b(t)).$$

*Proof.* (i): Set  $\epsilon > 0$ , and let  $T$  be such that  $1 - \epsilon \leq a(t)/b(t) \leq 1 + \epsilon$  for any  $t \geq T$ . Then as  $f$  is non-increasing, for  $t \geq T$  we have

$$\frac{f((1 + \epsilon)b(t))}{f(b(t))} \leq \frac{f(a(t))}{f(b(t))} \leq \frac{f((1 - \epsilon)b(t))}{f(b(t))}.$$

Applying limits in the equation above we get,

$$\liminf_{t \rightarrow \infty} \frac{f((1 + \epsilon)b(t))}{f(b(t))} \leq \liminf_{t \rightarrow \infty} \frac{f(a(t))}{f(b(t))}$$

and

$$\limsup_{t \rightarrow \infty} \frac{f(a(t))}{f(b(t))} \leq \limsup_{t \rightarrow \infty} \frac{f((1 - \epsilon)b(t))}{f(b(t))}.$$

As  $f \in \mathcal{C}_\infty$  then

$$\lim_{\epsilon \downarrow 0} \liminf_{t \rightarrow \infty} \frac{f((1 + \epsilon)b(t))}{f(b(t))} = 1.$$

Similarly, as

$$\limsup_{t \rightarrow \infty} \frac{f((1 - \epsilon)b(t))}{f(b(t))} = \liminf_{t \rightarrow \infty} \frac{f(b(t))}{f((1 - \epsilon)b(t))}$$

then  $f \in \mathcal{C}_\infty$  implies

$$\lim_{\epsilon \downarrow 0} \limsup_{t \rightarrow \infty} \frac{f((1 - \epsilon)b(t))}{f(b(t))} = 1.$$

Thus we had proved

$$\lim_{t \rightarrow \infty} \frac{f(a(t))}{f(b(t))} = 1.$$

(ii): The proof follows from (i) and Proposition 2.5. □

### 3. ASYMPTOTIC RESULTS ON THE VaR AND THE TVAR OF THE SUM AND THE MAXIMUM

Our main results are now stated: they link the value-at-risk of the sum and the maximum in case where  $\bar{F}_M$  is consistently varying. The results still hold for the TVaR. Recall that we do not assume that the marginal distributions are neither identically distributed nor independent.

**Theorem 3.1.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a vector of positive random variables (r.v.s). Suppose that  $\bar{F}_M$  is consistently varying and that  $\delta(x) := \frac{\bar{F}_S(x)}{\bar{F}_M(x)} \rightarrow \Delta$  as  $x \rightarrow \infty$ , for some  $1 \leq \Delta < \infty$ . Then*

$$\text{VaR}_{1-p}(S) \sim \text{VaR}_{1-\Delta^{-1}p}(M) \text{ as } p \rightarrow 0.$$

*The same result applies for the TVaR if  $\text{TVaR}_p(M)$  exists for all  $p$ . If moreover  $\bar{F}_M$  is  $\rho$ -varying then  $\Delta \leq d^{-\rho}$ .*

*Proof.* By Proposition 2.6 and by Proposition 2.7(ii) we have

$$t \sim \bar{F}_M^\vee \circ \bar{F}_M(t) \sim \bar{F}_M^\vee(\Delta^{-1}\bar{F}_S(t)).$$

Again, by combining Proposition 2.6 and Proposition 2.7 we have as  $p \rightarrow 0$

$$\bar{F}_S^\vee(p) \sim \bar{F}_M^\vee(\Delta^{-1}\bar{F}_S \circ \bar{F}_S^\vee(p)) \sim \bar{F}_M^\vee(\Delta^{-1}p)$$

After rewriting the last equation in terms of the VaR function the result follows. Now we prove that the result is still valid when changing the VaR measure by the TVaR above, assuming the last exists. As we have that

$$\text{VaR}_{1-u}(S) \sim \text{VaR}_{1-\Delta^{-1}u}(M)$$

when  $u \rightarrow 0$  then there exists a function  $\epsilon(p)$ ,  $\epsilon(p) \downarrow 0$  as  $p \downarrow 0$  such that for any  $0 < u \leq p$

$$(1 - \epsilon(p)) \text{VaR}_{1-\Delta^{-1}u}(M) \leq \text{VaR}_{1-u}(S) \leq (1 + \epsilon(p)) \text{VaR}_{1-\Delta^{-1}u}(M).$$

After integrating over  $u$  from 0 to  $p$  each side of the inequality below we get,

$$(1 - \epsilon(p)) \text{TVaR}_{1-\Delta^{-1}p}(M) \leq \text{TVaR}_{1-p}(S) \leq (1 + \epsilon(p)) \text{TVaR}_{1-\Delta^{-1}p}(M),$$

and then as  $\epsilon(p) \downarrow 0$  it is clear from the last inequality that

$$\text{TVaR}_{1-p}(S) \sim \text{TVaR}_{1-\Delta^{-1}p}(M)$$

when  $p \rightarrow 0$ . Remark that as we assume that marginal risks are almost surely positive we always have that

$$\{\max\{X_1, \dots, X_d\} > t\} \subset \{X_1 + \dots + X_d > t\} \subset \{\max\{X_1, \dots, X_d\} > t/d\}$$

and thus  $\delta(t) \leq \bar{F}_M(t/d)/\bar{F}_M(t)$ . So that if  $\bar{F}_M$  is regularly varying with index  $\rho$  then  $\Delta \leq d^{-\rho}$ . □

In Section 4, we give classes of random vectors such that the tail of the maximum is consistently varying. In Section 5 we provide a method to estimate  $\Delta$ . The hypothesis of the convergence of  $\delta(x)$  stating this result can be relaxed if we assume regularity of the maximum. Before explaining that point, we prove the following lemma.

**Lemma 3.2.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a vector of positive r.v.s. If both  $\bar{F}_M$  and  $\bar{F}_S$  are regularly varying at infinity then they have the same variation index.*

*Proof.* Suppose that  $\bar{F}_M \in \mathcal{RV}_\infty(-\rho_M)$  and that  $\bar{F}_S \in \mathcal{RV}_\infty(-\rho_S)$ , for some positive values  $\rho_S$  and  $\rho_M$ . As before, because  $\mathbf{X}$  has positive components,  $\bar{F}_M(t) \leq \bar{F}_S(t) \leq \bar{F}_M(t/d)$  for any  $t > 0$  and then

$$1 \leq \frac{\bar{F}_S(t)}{\bar{F}_M(t)} \leq \frac{\bar{F}_M(t/d)}{\bar{F}_M(t)}$$

for any  $t > 0$ . Regular variation of  $\bar{F}_M$  implies then that

$$(3.1) \quad 1 \leq \frac{\bar{F}_S(t)}{\bar{F}_M(t)} \leq d^{\rho_M} + \epsilon$$

for some positive values  $\epsilon$ ,  $T$  and any  $t > T$ . Now let  $L_M$  and  $L_S$  be the slowly varying functions that satisfy

$$\bar{F}_M(t) = t^{-\rho_M} L_M(t) \quad \text{and} \quad \bar{F}_S(t) = t^{-\rho_S} L_S(t),$$

then

$$\frac{\bar{F}_S(t)}{\bar{F}_M(t)} = t^\rho \frac{L_S(t)}{L_M(t)},$$



with  $\rho = \rho_M - \rho_S$ . Rewriting last equation as

$$\frac{\overline{F}_S(t)}{\overline{F}_M(t)} = \frac{t^{\rho/2} L_S(t)}{t^{-\rho/2} L_M(t)},$$

allows us to check, by Proposition 1.3.6 in [9], that as  $t \rightarrow \infty$  either  $\frac{\overline{F}_S(t)}{\overline{F}_M(t)} \rightarrow \infty$  if  $\rho > 0$  or  $\frac{\overline{F}_S(t)}{\overline{F}_M(t)} \rightarrow 0$  if  $\rho < 0$ . As both possibilities contradict inequality (3.1) we must have  $\rho = 0$  and thus  $\rho_M = \rho_S$ .  $\square$

**Theorem 3.3.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a vector of positive r.v.s. If the function  $\delta(x) = \frac{\overline{F}_S(x)}{\overline{F}_M(x)}$  is continuous but not convergent and  $\overline{F}_M$  and  $\overline{F}_S$  are both regularly varying with negative indexes then*

$$\text{VaR}_{1-\beta p_n}(S) \sim \text{VaR}_{1-\Delta^{-1}\beta p_n}(M)$$

for some sequence  $p_n \rightarrow 0$ , some  $\Delta \geq 1$  and any  $\beta > 0$ . The same result applies for the TVaR if  $\text{TVaR}_p(M)$  exists for any  $p$ .

*Proof.* We have  $\overline{F}_M(t) \leq \overline{F}_S(t) \leq \overline{F}_M(t/d)$  for any  $t > 0$ , so  $\overline{F}_M \in RV_\rho$ ,  $\rho < 0$ , implies that eventually  $1 \leq \delta(t) \leq d^{-\rho}$ . So if  $\delta(t)$  is not convergent there exist two strictly increasing and unbounded real sequences  $(u_n)_{n>0}$ ,  $(v_n)_{n>0}$  such that  $b(u_n) \rightarrow \Delta_1$  and  $b(v_n) \rightarrow \Delta_2$  as  $n \rightarrow \infty$ , for some  $1 \leq \Delta_1 < \Delta_2 \leq d^{-\rho}$ . Without loss of generality we may suppose also that  $\Delta(v_n) > \Delta$  and  $\Delta(u_n) < \Delta$  for all  $n > 0$  with  $\Delta = (\Delta_1 + \Delta_2)/2$ . As both sequences  $v_n$  and  $u_n$  are unbounded we can construct two natural sequences  $(j_i)_{i>0}$  and  $(k_i)_{i>0}$  such that  $u_{k_i} < v_{j_i} < u_{k_{i+1}}$  for all  $i = 0, 1, \dots$ . Continuity of  $\delta(t)$  allows us then to pick up a strictly increasing and unbounded real sequence  $(t_n)_{n>0}$ , such that  $u_{k_n} < t_n < v_{j_n}$  and such that  $\Delta(t_n) = \Delta$ . Then

$$\overline{F}_S(t_n) = \Delta \overline{F}_M(t_n),$$

for all  $n > 0$ . As both  $\overline{F}_S$  and  $\overline{F}_M$  are regularly varying with same index, for any  $s > 0$

$$\overline{F}_S(st_n) \sim \Delta \overline{F}_M(st_n)$$

and then after applying Proposition 8.2 (ii) we get

$$\overline{F}_S^{\leftarrow}(\beta p_n) \sim \Delta^{-\rho^{-1}} \overline{F}_M^{\leftarrow}(\beta p_n) \sim \overline{F}_M^{\leftarrow}(\Delta^{-1} \beta p_n),$$

where  $p_n = 1/t_n$  and  $\beta = 1/s$ . Rewriting last equation with the VaR notation finishes the proof. The TVaR case follows exactly as in the proof of Theorem 3.1.  $\square$

In theory this theorem allows us to approximate the  $\text{VaR}_{1-p}(S)$  at any level  $p$ , by taking  $\beta = p/p_n$ , for  $n$  big. However in practice it will not be usable as the sequence  $p_n$  is in general not known.

#### 4. ON THE REGULAR VARIATION OF THE TAILS OF THE MAX AND THE SUM

In this section we explore in which cases  $\overline{F}_M$  is regularly varying. We also exhibit some classes of random vectors for which the limit  $\Delta$  exists.

**4.1. Multivariate regular framework.** Alink et al., ([2], [3] and [4]) studied the asymptotic behaviour of the tail of the sum when the marginals of the vector  $\mathbf{X} = (X_1, \dots, X_d)$  are identically distributed as one of the three extreme value families: Gumbel, Fréchet or Weibull and when the dependence within the vector is given by an archimedean copula. Then Barbe et al. ([6]) generalized these results under the framework of the multivariate regular variation distributions. Their main contribution is the explicit calculation of the limit

$$\lim_{t \rightarrow \infty} \frac{\overline{F}_S(t)}{\overline{F}_1(t)},$$

where  $F_1$  is the common distribution function of the marginal risks  $X_1, \dots, X_d$ . Let us recall the definition of multivariate regularly varying random vectors.

**Definition 3** (Multivariate Regular Variation). A random vector  $\mathbf{X}$  is said to be multivariate regularly varying of index  $\beta > 0$  if there exists a probability measure  $\mu$  on  $\Gamma_d = \left\{ \frac{\mathbf{x}}{\|\mathbf{x}\|} : \mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\} \right\}$ , a function  $b : (0, \infty) \rightarrow (0, \infty)$  and a scalar  $\gamma$ , such that for all  $x > 0$  and all  $A \subset \Gamma_d$ ,

$$\lim_{t \rightarrow \infty} t \Pr \left( \|\mathbf{X}\| > xb(t), \frac{\mathbf{X}}{\|\mathbf{X}\|} \in A \right) = \frac{\gamma}{x^\beta} \mu(A),$$

where  $\gamma = \gamma(\|\cdot\|, b)$  depends on both the norm  $\|\cdot\|$  used and the function  $b$ .

As shown in Barbe et al. (2006) ([6]), using the  $L^1$  norm,  $\|\mathbf{X}\|_1 = |X_1| + \dots + |X_d|$ , and  $b(t) = F_1^{-1}(1 - 1/t)$  in Definition 3 gives

$$\lim_{t \rightarrow \infty} \frac{\overline{F}_S(t)}{\overline{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{\overline{F}_S(b(t))}{\overline{F}_1(b(t))} = \lim_{t \rightarrow \infty} t \Pr(\|\mathbf{X}\|_1 > b(t)) = \gamma(L^1, b).$$

Similarly, we can use the  $L^\infty$  norm,  $\|\mathbf{X}\|_\infty = \max\{|X_1|, \dots, |X_d|\}$  to get

$$\lim_{t \rightarrow \infty} \frac{\overline{F}_M(t)}{\overline{F}_1(t)} = \lim_{t \rightarrow \infty} t \Pr(\|\mathbf{X}\|_\infty > b(t)) = \gamma(L^\infty, b).$$

We conclude then that when  $\mathbf{X}$  is multivariate regularly varying

$$\lim_{t \rightarrow \infty} \frac{\overline{F}_S(t)}{\overline{F}_M(t)} = \frac{\gamma(L^1, b)}{\gamma(L^\infty, b)} = \Delta.$$

So that, when  $\mathbf{X}$  is multivariate regularly varying Theorem 3.1 applies.

We are also interested in random vectors whose coordinates are not identically distributed. Results for identically distributed marginals will not lead to results for no identically distributed marginals. This is the purpose of the next section.

**4.2. Random vectors whose maximum survival's function is regularly varying.** In this section, the results presented in Section 8 are used to prove, in the case of independence and for random vectors with an Archimedean copula dependence structure, that the maximal survival function is regularly varying.

Let us remark that considering a random vector with regularly varying marginals does not imply that  $\overline{F}_M$  is regularly varying. Nevertheless, some bounds may be obtained. Denote by  $\overline{F}_i$  the survival function of  $X_i$  and

assume that  $\bar{F}_i \in \mathcal{RV}_\infty(-\rho_i)$  for  $i = 1, \dots, d$ .  
Fréchet bounds imply that for any  $t > 0$

$$1 - \min_i \{F_i(t)\} \leq \bar{F}_M(t) \leq 1 - \left(1 - \sum_{i=1}^d \bar{F}_i(t)\right)_+,$$

where  $(\cdot)_+$  stands for the positive part function, and so when  $t$  is such that  $\bar{F}_i(t) \leq d^{-1}$  for all  $i = 1, \dots, d$

$$\max_i \{\bar{F}_i(t)\} \leq \bar{F}_M(t) \leq \sum_{i=1}^d \bar{F}_i(t) \leq d \max_i \{\bar{F}_i(t)\}.$$

So

$$\frac{\max_i \{\bar{F}_i(tx)\}}{d \max_i \{\bar{F}_i(t)\}} \leq \frac{\bar{F}_M(tx)}{\bar{F}_M(x)} \leq \frac{d \max_i \{\bar{F}_i(tx)\}}{\max_i \{\bar{F}_i(t)\}}.$$

Then if there exists  $j$  such that eventually  $\bar{F}_j(x) \geq \bar{F}_i(x)$  for any  $i$  and that for such a  $j$ ,  $\bar{F}_j \in \mathcal{RV}_\infty(-\rho)$ , then for  $\bar{F}_{M*}(x) = \liminf_{t \rightarrow \infty} \frac{\bar{F}_M(tx)}{\bar{F}_M(x)}$  and

$$\bar{F}_M^*(x) = \limsup_{t \rightarrow \infty} \frac{\bar{F}_M(tx)}{\bar{F}_M(x)},$$

$$d^{-1}x^{-\rho_j} \leq \bar{F}_{M*}(x) \leq \bar{F}_M^*(x) \leq dx^{-\rho_j}$$

$$\bar{F}_1(t) \leq \bar{F}_M(t) \leq \sum_{i=1}^d \bar{F}_i(t).$$

Now as (see e.g. [9])  $\sum_{i=1}^d \bar{F}_i(t)$  is in  $\mathcal{RV}_\infty(-\rho_*)$  with  $\rho_* = \min\{\rho_i : i = 1, \dots, d\} = \rho_j$ , then there exists  $g$  and  $h$  in  $\mathcal{RV}_\infty(-\rho)$  such that

$$g(t) \leq \bar{F}_M(t) \leq h(t).$$

Under some additional hypothesis, we now get that  $\bar{F}_M$  is regularly varying.

**Proposition 4.1.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a vector of positive r.v.s. If  $\bar{F}_j(x)/\bar{F}_1(x) \rightarrow 0$  when  $x \rightarrow \infty$  for all  $2 \leq j \leq d$ , then  $\bar{F}_M(x) \sim \bar{F}_1(x)$ . Then if  $\bar{F}_1 \in \mathcal{RV}_\infty(\rho)$ ,  $\rho < 0$  so does  $\bar{F}_M$ .*

*Proof.* It follows directly from the remark that

$$\bar{F}_M(t) \geq \bar{F}_1(t) \text{ and } \bar{F}_M(t) \leq \sum_{i=1}^d \bar{F}_i(t).$$

□

We now consider the case where the dependence structure between components of the random vector is given by an Archimedean copula. We study conditions under which the tail of the maximum is regularly or consistently varying.

4.2.1. *Study of the maximum of a vector with Archimedean Copula.***Definition 4.** (Archimedean Copulas) $C$  is an archimedean copula with generator  $\psi$  if

$$C(u_1, \dots, u_d) = \psi^{\leftarrow} \{ \psi(u_1) + \dots + \psi(u_d) \},$$

for all  $u_i \in [0, 1]$  and  $i = 1, \dots, d$ , where the generator  $\psi$  is a function  $\psi : [0, 1] \rightarrow [0, \infty]$  that satisfies:

- (i)  $\psi$  is strictly decreasing with  $\psi(1) = 0$ ;
- (ii) The first  $k$  derivatives of  $\psi^{\leftarrow}$  exist and are continuous;
- (iii) For  $k = 0, 1, \dots, d$ ,

$$(-1)^k \frac{d^k \psi^{\leftarrow}(t)}{dt^k} \geq 0, \text{ for all } t > 0.$$

**Proposition 4.2.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector with regularly varying tails,  $\bar{F}_i \in \mathcal{RV}_\infty(\rho_i)$ ,  $i = 1, \dots, d$ . Suppose that the copula  $C$  of  $\mathbf{X}$  is Archimedean with generator  $\psi$  and  $\psi \in \mathcal{RV}_1(-\theta)$  with  $\theta > 0$ .**Then the tail of the maximum is regularly varying,  $\bar{F}_M \in \mathcal{RV}_\infty(\rho)$ , with index  $\rho = \max_i \{\rho_i\}$ .**Proof.* By definition

$$\bar{F}_M(x) = 1 - \psi^{-1} \left( \sum_{i=1}^d \psi(F_i(x)) \right).$$

Lemma 8.4 gives

$$\psi \circ F_i \in \mathcal{RV}_\infty(\rho_i \theta)$$

for each  $i = 1, \dots, d$ , and then

$$\phi := \sum_{i=1}^d (\psi \circ F_i) \in \mathcal{RV}_\infty(\rho \theta)$$

with  $\rho = \max\{\rho_i : i = 1, \dots, d\}$ . Lemma 8.3 (ii) implies that

$$1 - \psi^{-1} \in \mathcal{RV}_0(\theta^{-1})$$

and applying again Lemma 8.4 gives

$$\bar{F}_M \equiv (1 - \psi^{-1}) \circ \phi \in \mathcal{RV}_\infty(\rho)$$

as required. □**Corollary 4.3.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a vector of positive r.v.s. If the marginals are independent and their survival functions  $\bar{F}_i \in \mathcal{RV}_\infty(\rho_i)$ ,  $\rho_i < 0$ ,  $i = 1, \dots, d$ , then*

$$\bar{F}_M \in \mathcal{RV}_\infty(\rho)$$

*with  $\rho = \max_i \{\rho_i\}$ .**Proof.* It suffices to notice that the independent copula is an Archimedean copula with generator  $\psi(t) = -\ln(t)$ . □

**Proposition 4.4.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector with consistently varying marginal tails. Suppose that the copula  $C$  of  $\mathbf{X}$  is Archimedean with generator  $\psi$  and that  $x \mapsto \psi(1 - 1/x)$  is consistently varying. Then the tail of the maximum is consistently varying.*

*Proof.* The proof is exactly the same as before, but instead of Lemmas 8.3 and 8.4, we use here Propositions 2.5 and 2.4. The closure by sums of the consistently varying functions stated in Proposition 2.3 is also required.  $\square$

In the case of survival Archimedean copulas, we derive the regular variation property for the tail of maximum  $M$  and the minimum  $m$  in the following proposition.

**Proposition 4.5.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector with regularly varying marginal tails,  $\bar{F}_i \in \mathcal{RV}_\infty(\rho_i)$ ,  $i = 1, \dots, d$ . Suppose that the copula  $C$  of  $\mathbf{X}$  is the survival copula of an Archimedean copula  $\tilde{C}$  with generator  $\psi$  with  $\psi \in \mathcal{RV}_0(-\theta)$ ,  $\theta > 0$ . Then*

- (i) *The tail of the minimum of  $\mathbf{X}$ ,  $m := \min\{X_1, \dots, X_d\}$ , is regularly varying,  $\bar{F}_m \in \mathcal{RV}_\infty(\rho^*)$ , with index  $\rho^* = \min_i\{\rho_i\}$ .*
- (ii) *If it exists a distribution  $F$  such that*

$$(4.1) \quad \lim_{x \rightarrow \infty} \frac{\bar{F}_i(x)}{\bar{F}(x)} = a_i,$$

*where  $0 \leq a_i \leq 1$  for all  $i = 1, \dots, d$  and at least one is non-zero, then the tail of the maximum is regularly varying,  $\bar{F}_M \in \mathcal{RV}_\infty(\rho)$ , with index  $\rho = \max_i\{\rho_i\}$ .*

*Proof.* (i) Notice first that

$$\bar{F}_m(x) = \Pr(X_1 > x, \dots, X_d > x).$$

The survival copula dependence of  $\mathbf{X}$  implies

$$\bar{F}_m(x) = \tilde{C}(\bar{F}_1(x), \dots, \bar{F}_d(x)),$$

and then by definition,

$$\bar{F}_m(x) = \psi^{-1} \left( \sum_{i=1}^d \psi(\bar{F}_i(x)) \right).$$

Lemma 8.4 gives, for each  $i = 1, \dots, d$ ,  $\psi \circ \bar{F}_i \in \mathcal{RV}_\infty(-\rho_i \times \theta)$  and thus

$$\sum_{i=1}^d \psi(\bar{F}_i(x)) \in \mathcal{RV}_\infty(-\rho^* \times \theta)$$

with  $\rho^* = \min\{\rho_i, i = 1, \dots, d\}$ . Combining Lemmas 8.3 and 8.4 as in the proof of Proposition 4.2 allows us to conclude that  $\bar{F}_m \in \mathcal{RV}_\infty(\rho^*)$ .

(ii) For  $i = 1, \dots, d$  set  $A_i = \{X_i \leq x\}$ , and  $A = \bigcap_{i=1}^d A_i$ . Then, by the inclusion-exclusion principle, we have that

$$(4.2) \quad \bar{F}_M(x) = \Pr(A^c) = \sum_{k=1}^d (-1)^{k+1} \sum_{i_1, \dots, i_k \in \{1, \dots, d\}} \Pr(A_{i_1}^c, \dots, A_{i_k}^c).$$

Notice that by (i), for each  $k = 1, \dots, d$  and each set  $I = \{i_1, \dots, i_k\} \subset \{1, \dots, d\}$  each addend in the sum above, in absolute value, is regularly varying. That is because if  $m_I = \min\{X_{i_1}, \dots, X_{i_k}\}$  then

$$\overline{F}_{m_I}(x) = \Pr(A_{i_1}^c, \dots, A_{i_k}^c) \in \mathcal{RV}_\infty(\rho^*)$$

with  $\rho^* = \min\{\rho_{i_1}, \dots, \rho_{i_k}\}$ . However as not all elements in the sum are positive we can just conclude that  $\overline{F}_M$  is the difference of two regularly varying functions, and even if the difference is known to be positive we cannot conclude directly on its regularity (see for example [17]). Nevertheless if it exists a distribution  $F$  such that

$$\lim_{x \rightarrow \infty} \frac{\overline{F}_i(x)}{\overline{F}(x)} = a_i$$

whence  $0 \leq a_i \leq 1$  for all  $i = 1, \dots, d$  with at least one  $a_i$  non-zero, then the regularity of  $\overline{F}_M$  can be proved. In this case for  $k = 1, \dots, d$  and each set  $I = \{i_1, \dots, i_k\} \subset \{1, \dots, d\}$  then

$$\sum_{i \in I} \psi(\overline{F}_i(x)) \sim \sum_{i \in I} \psi(a_i \overline{F}(x)) \sim \psi(\overline{F}(x)) \sum_{i \in I} a_i^{-\theta}$$

and by Lemma 8.3 and Proposition 2.7

$$(4.3) \quad \overline{F}_{m_I}(x) = \psi^{-1} \left( \sum_{i \in I} \psi(\overline{F}_i(x)) \right) \sim \left( \sum_{i \in I} a_i^{-\theta} \right)^{-1/\theta} \overline{F}(x).$$

Then combining equations (4.2) and (4.3) gives

$$\lim_{x \rightarrow \infty} \frac{\overline{F}_M(x)}{\overline{F}(x)} = \sum_{k=1}^d (-1)^{k+1} \sum_{i_1, \dots, i_k \in \{1, \dots, d\}} \left( \sum_{j=1}^k a_{i_j}^{-\theta} \right)^{-1/\theta}.$$

Remark that for any index  $i$  and any  $x \in \mathbb{R}$ ,  $\overline{F}_M(x) \geq \overline{F}_i(x)$ , so that

$$\frac{\overline{F}_M(x)}{\overline{F}(x)} \geq \frac{\overline{F}_i(x)}{\overline{F}(x)} \xrightarrow{x \rightarrow \infty} a_i.$$

Since we assume that at least for one index  $i$ ,  $a_i > 0$ , we conclude that  $\overline{F}_M \in \mathcal{RV}_\infty(\rho)$  where  $\rho$  is the index of regularity of  $\overline{F}$ . Let us check that the index of  $\overline{F}$  should be  $\rho = \max_i \{\rho_i\}$ . By hypothesis it exists  $i$  such that  $a_i \neq 0$ , and thus (4.1) implies  $\overline{F} \in \mathcal{RV}_{\rho_i}$ . If it exists  $j \neq i$  such that  $\rho_j > \rho_i$  then  $\overline{F}_j(x)/\overline{F}(x) \rightarrow \infty$  as  $x \rightarrow \infty$  which contradicts that  $0 \leq a_j \leq 1$ . This shows that  $\rho = \max_i \{\rho_i\}$ .  $\square$

**4.3. On the convergence of  $\delta(x) = \overline{F}_S(x)/\overline{F}_M(x)$ .** We finish this section with some remarks on the convergence of  $\delta(x) = \overline{F}_S(x)/\overline{F}_M(x)$ . As before (see the proof of Lemma 3.2), we have

$$\overline{F}_M(t) \leq \overline{F}_S(t) \leq \overline{F}_M(t/d)$$

so

$$1 \leq \liminf_{t \rightarrow \infty} \frac{\overline{F}_S(t)}{\overline{F}_M(t)} \leq \limsup_{t \rightarrow \infty} \frac{\overline{F}_S(t)}{\overline{F}_M(t)} \leq \limsup_{t \rightarrow \infty} \frac{\overline{F}_M(t/d)}{\overline{F}_M(t)}.$$

When  $\overline{F}_M$  is regularly varying with index  $\rho < 0$  we have then that the quotient  $\overline{F}_S(t)/\overline{F}_M(t)$  is eventually bounded, i.e., for any  $\epsilon > 0$ , there exists  $T$  such that

$$1 \leq \frac{\overline{F}_S(t)}{\overline{F}_M(t)} \leq d^{-\rho} + \epsilon$$

for any  $t > T$ . Then, in this case, the monotonicity of  $F_S(x)/F_M(x)$  would be sufficient to prove the convergence of  $\delta(x)$ .

Now, if we assume also that  $\overline{F}_S$  is regularly varying, by Lemma 3.2 we must have that the index of variation is the same as in  $\overline{F}_M$  and then it can be easily shown that  $L(t) = \frac{\overline{F}_S(t)}{\overline{F}_M(t)}$  is a slowly varying function. However this doesn't allow us to conclude anything on the convergence because a slowly varying function, even bounded, doesn't converge necessarily, see for example [22].

In the simple case where one of the components of  $\mathbf{X}$  has a strictly heavier tail than the others then the convergence holds.

**Proposition 4.6.** *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a vector of positive regularly varying r.v.s with indexes  $\rho_i < 0$ ,  $i = 1, \dots, d$ . If  $\rho_1 > \rho_i$ , for any  $1 < i \leq d$  then*

$$\frac{\overline{F}_S(x)}{\overline{F}_M(x)} \rightarrow 1,$$

when  $x \rightarrow \infty$ .

*Proof.* By Proposition 4.1 it follows that  $\overline{F}_M(x) \sim \overline{F}_1(x)$  as  $x \rightarrow \infty$ . Similarly it is easy to show that  $\overline{F}_S(x) \sim \overline{F}_1(x)$  and thus the proposition follows.  $\square$

## 5. ESTIMATION OF THE LIMIT $\Delta$

In this section, we show how to estimate the limit  $\Delta$  using samples of  $\mathbf{X}$ . We shall use this estimation to approximate  $VaR_p(S)$ , for different values of  $p$  close to 1 using Theorem 3.1.

Recall that  $\delta$  is the real valued function defined by  $\delta(t) = \overline{F}_S(t)/\overline{F}_M(t)$  and continue to denote by  $\Delta$  its limit at infinity if it exists. If a sample of  $\mathbf{X}$  is available, the function  $\delta$  can be estimated using the empirical cumulative distribution function (e.c.d.f.) of  $S$ , i.e. by

$$\widehat{\delta}(t) = \frac{1 - \widehat{F}_S(t)}{1 - F_M(t)}.$$

In some cases it can be detected graphically that the function  $\widehat{\delta}$  becomes stable after some point and the limit  $\Delta$  can be inferred. More specifically, let  $(S_1, \dots, S_n)$  be an i.i.d. sample of  $S$ . According to Donsker's Theorem, the empirical process

$$\sqrt{n}(\widehat{F}_S(t) - F_S(t))$$

converges in distribution to a Gaussian process with zero mean and covariance given by

$$F_S(t_1) - F_S(t_1)F_S(t_2)$$

for  $t_1 \leq t_2$ . Thus given any sequence  $0 < t_1 < \dots < t_k$  the vector

$$\sqrt{n} \left( \widehat{\delta}(t_1) - \delta(t_1), \dots, \widehat{\delta}(t_k) - \delta(t_k) \right)$$

converges in law to a centred Gaussian vector with covariances given by

$$\frac{F_S(t_i) - F_S(t_i)F_S(t_j)}{(1 - F_M(t_i))(1 - F_M(t_j))} = \frac{\delta(t_j)}{1 - F_M(t_i)} - \delta(t_i)\delta(t_j)$$

for any  $i \leq j$ . Then

$$\sqrt{n} \left( \frac{1}{k} \sum_{i=1}^k \widehat{\delta}(t_i) - \frac{1}{k} \sum_{i=1}^k \delta(t_i) \right)$$

converges to a normal distribution with zero mean and variance

$$(5.1) \quad \frac{1}{k^2} \sum_{1 \leq i \leq j \leq k} \left\{ \frac{\delta(t_j)}{1 - F_M(t_i)} - \delta(t_i)\delta(t_j) \right\}.$$

If we assume that the values  $t_i$  are large enough, the approximation  $\delta(t_i) \approx \Delta$  holds for each  $i = 1, \dots, k$ , then the variance (5.1) can be approximated by

$$\frac{\Delta}{k^2} \sum_{i=1}^k \left\{ \frac{i}{1 - F_M(t_i)} \right\} - \frac{\Delta^2(k+1)}{2k}.$$

In practice we should plot points  $(S_{(i)}, \widehat{\delta}(S_{(i)}))$  where  $S_{(1)} < \dots < S_{(n)}$  is the ordered sample of  $S$ . Then we choose a threshold in such a way that that the approximation  $\delta(S_{(n-i)}) \approx \Delta$  holds for any  $0 \leq i \leq k$ . We propose then to estimate  $\Delta$  by

$$\widehat{\Delta} = \frac{1}{k} \sum_{i=1}^k \widehat{\delta}(S_{(n-i)}).$$

The behavior of  $\widehat{\delta}(x)$  for the model described in Section 6 may be seen on Figure 1.

## 6. SOME EXPLICIT CALCULATIONS

In this section we will consider a simplified model that will allow us to give explicit formulas for  $F_S$  and  $F_M$ . Besides it will allow us to understand the scope and the limitations of our  $\Delta$  estimation. The model is described by the following compound process: let  $\Lambda$  be a positive random variable and let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector such that

$$(6.1) \quad \Pr(X_1 > x_1, \dots, X_d > x_d \mid \Lambda = \lambda) = \prod_{i=1}^d e^{-\lambda x_i},$$

for each  $x_1, \dots, x_d \geq 0$ .

That means that conditionally to  $\Lambda$  the marginals of  $\mathbf{X}$  are independent and exponentially distributed. The final distribution of  $\mathbf{X}$  will not have, in general, independent marginals and they will not be exponential either. Actually the dependence structure of  $\mathbf{X}$  and its marginal distributions will depend on the distribution of  $\Lambda$ .



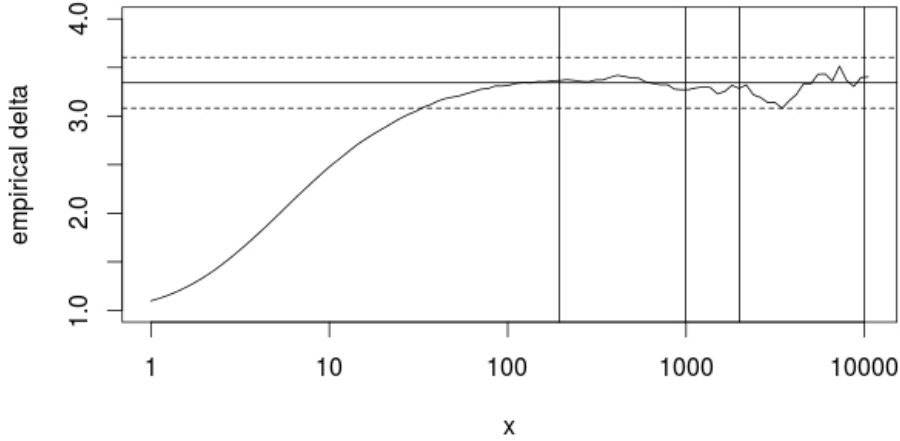


FIGURE 1. Shape of the  $\hat{\delta}$  function of the Pareto-Clayton model with parameters  $\alpha = 1, \beta = 1$  and  $d = 10$  (see Section 6) based on sample of size  $10^4$ . Vertical lines are displayed at the empirical VaR of the sum at confidence levels 95%, 99%, 99.5%, 99.9%. The estimation  $\hat{\Delta}$  with its estimated 95% confidence interval is represented by the horizontal lines.

Some particular  $\Lambda$  distributions define some well-known models in which the explicit calculation of  $F_S$  and  $F_M$  is possible. For example when  $\Lambda$  is Gamma distributed then the marginals of  $\mathbf{X}$  are of Pareto type with dependence given by a survival Clayton copula. When  $\Lambda$  is Levy distributed the marginals will be Weibull distributed with a Gumbel survival copula. These models have been studied in [20, 24] and used in [1] where explicit formulas for ruin probabilities have been derived. In [10, 18] explicit results for the minimum of some risk indicators are obtained for this kind of models. We also would like to mention that the computation of the VaR for this model is given in [12].

When  $\Lambda$  is Gamma( $\alpha, \beta$ ) distributed with density

$$f_{\Lambda}(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

then the marginal distributions of  $X_i$  are Pareto( $\alpha, \beta$ ) with tail given by

$$\bar{F}_i(x) = \left(1 - \frac{x}{\beta}\right)^{-\alpha}$$

and the dependence structure is described by a survival Clayton copula with parameter  $1/\alpha$ . Through this paper we will refer to this model as a Pareto-Clayton vector with parameters  $(\alpha, \beta)$ . This model is a particular Multivariate Pareto of type II with location parameters  $\mu_i = 0$  and scale parameters  $\sigma_i = \beta$  for  $i = 1, \dots, d$ , see [24].

In the Pareto-Clayton model the exact distribution function of  $S = \sum_{i=1}^d X_i$  can be calculated. Conditionally to  $\Lambda = \lambda$ ,  $\sum_{i=1}^d X_i$  is Gamma( $1/\lambda, d$ ) distributed, distribution also known as the Erlang distribution. Then, as here we are assuming that  $\Lambda$  is Gamma( $\alpha, \beta$ ) distributed, the total distribution of  $S$  is the result of compounding two Gamma distributions, more precisely

$$S \sim \text{Gamma}(1/\Lambda, d) \text{ where } \Lambda \sim \text{Gamma}(\alpha, \beta).$$

It is well known that the result of this compound distribution is the so-called Beta prime distribution (see [13]). The c.d.f of  $S$  can be expressed in terms of  $F_\beta$ , the c.d.f of the Beta( $d\beta, \alpha$ ) distribution, as

$$F_S(x) = F_\beta\left(\frac{x}{1+x}\right).$$

Naturally, the inverse of  $F_S$  can also be expressed in function of the inverse of the beta distribution

$$F_S^{\leftarrow}(p) = \frac{F_\beta^{\leftarrow}(p)}{1 - F_\beta^{\leftarrow}(p)}.$$

In this example, the  $\delta$  function is explicitly calculated, see Figure 2. Moreover, computer algebra softwares allow us to calculate explicitly the limit  $\Delta$  for specified parameters.

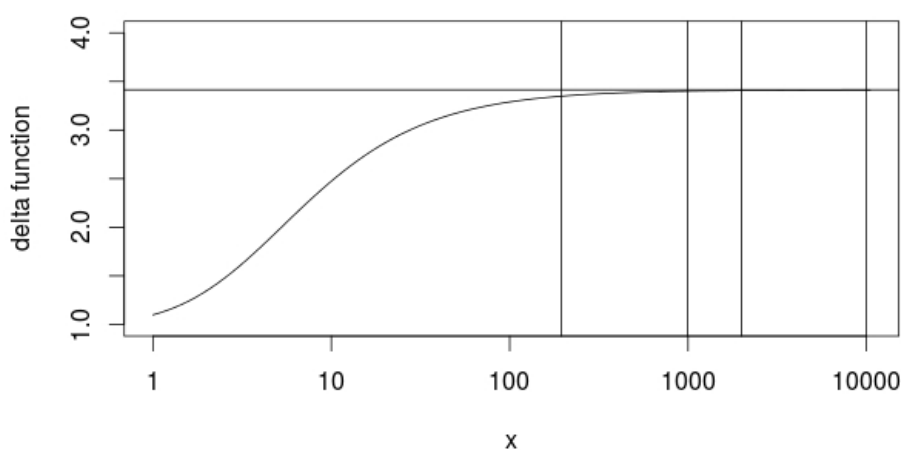


FIGURE 2. Shape of the  $\delta$  function of the Pareto-Clayton model, with parameters  $\alpha = 1, \beta = 1$  and  $d = 10$ . Vertical lines are displayed at the VaR of the sum at confidence levels 95%, 99%, 99.5%, 99.9%. The limit  $\Delta \approx 3.4142$  is represented by the horizontal line.

In order to see how fast the function  $\delta$  converges to its limit  $\Delta$  we plot the function  $p \mapsto \delta(\text{VaR}_p(S))$  for different parameters  $\alpha$  and different dimensions  $d$  (see Figure 3). The plots show that  $\delta(x)$  is already very close

to  $\Delta$  when  $x = \text{VaR}_{0.95}(S)$ , for  $\alpha \leq 2.5$ . The lower the value  $\alpha$ , the flatter the tail of  $\delta$  and thus the limit  $\Delta$  is attained rapidly. Remark that the lower the levels of  $\alpha$ , the heavier tails of the Pareto Marginals. Finally, this plot confirms the intuition that for heavier marginals the tail of the sum is better approximated by the tail of the maximum.

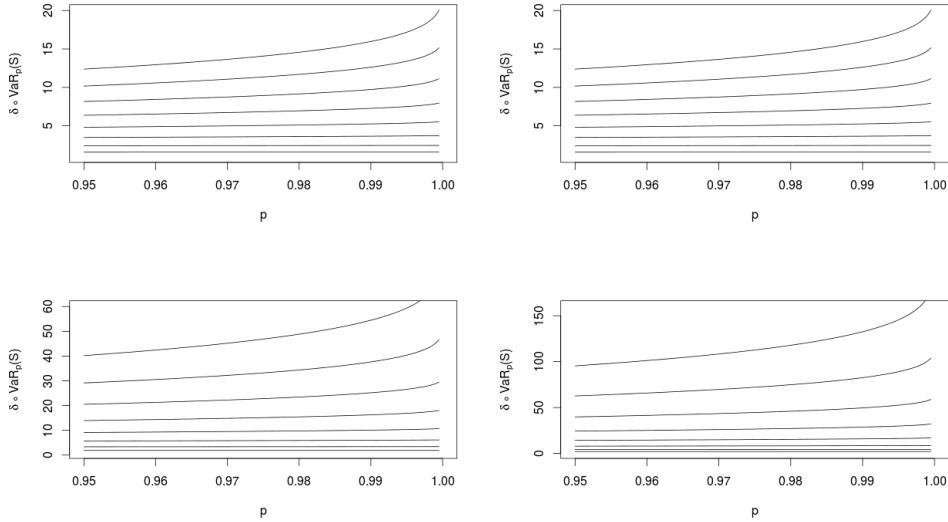


FIGURE 3. Four plots of the  $p \mapsto \delta(\text{VaR}_p(S))$  function of the Pareto-Clayton model for dimensions  $d = 2, 6, 10$  and  $14$  (from top-left to bottom-right) are represented. For each dimension the curves with  $\alpha = 0.5, 1, 1.5, 2, 2.5, 3, 3.5$  and  $4$  are plotted and they can be seen from bottom to top at each chart.

## 7. SOME NUMERICAL EXAMPLES

In this section we show how the ideas presented in the above section can help to estimate in practice the VaR and the TVaR of a sum at confidence levels close to 1. We compare our results with the empirical quantiles of the sample used to estimate  $\Delta$  and with three different quantile estimation methods from the Extreme Value Theory.

We first consider the Pareto-Clayton model presented in Section 6 (dimension 2 and 10), where exact values for the Value-at-Risk are computable. Then we test our method with a different model where exact values are not known. We compare the estimation done via the  $\Delta$ -limit estimation with other common quantile estimation methods:

- (1) The direct Monte Carlo quantile estimation (MC).
- (2) The quantile estimation from a GPD fitted distribution where parameters are estimated using maximum likelihood method (GPD 1).
- (3) The quantile estimation from a GPD fitted distribution where parameters are estimated using the moment method (GPD 2).

- (4) The high quantile estimate based on a method by Weissman [23] (Weiss.).

In order to study the performance of our estimator and to compare with the main competitors, we consider the root-mean-squared error (RMSE) loss function. When  $n$  estimations have been performed, it is defined by

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^n \left( \widehat{\text{VaR}}_p(S^i) - \text{VaR}_p(S) \right)^2},$$

where  $\widehat{\text{VaR}}_p(S^i)$  represents the estimate of  $\text{VaR}_p(S)$  for any of the different methods presented above, on the  $i$ th sample.

**7.1. Pareto-Clayton Model dimension 2.** Here we consider the model presented in Section 6. We first consider  $d = 2$  and  $\alpha = 1$  which corresponds to a model with Pareto marginals with  $\alpha = 1$  and dependence given by a Survival Clayton copula with parameter  $\theta = 1$ .

In Table 1, the exact VaR at different confidence levels (from 95% to 99,999%) is presented. In Table 2 and Table 3 we present the RMSE criterion in percentage of the real value based on 1000 simulations at different confidence levels. At each simulation a sample of size  $10^4$  in Table 2 and size  $10^5$  in Table 3 is used to estimate the VaR. Clearly, in term of RMSE, our method performs better than classical methods at each confidence level, even for very high levels. When increasing the size of the sample ( $10^5$  instead of  $10^4$ ) classical methods improve but our method remains the most performant.

| VaR<br>95% | VaR<br>99% | VaR<br>99.5% | VaR<br>99.9% | VaR<br>99.95% | VaR<br>99.99% | VaR<br>99.995% | VaR<br>99.999% |
|------------|------------|--------------|--------------|---------------|---------------|----------------|----------------|
| 194.5      | 994.5      | 1994.5       | 9994.5       | 19994.5       | 99994.5       | 199994.5       | 999994.5       |

TABLE 1. Exact Value-at-Risk at different confidence levels on the Pareto-Clayton Model in dimension  $d = 2$  with  $\alpha = 1$ .

| Method | VaR<br>95% | VaR<br>99% | VaR<br>99.5% | VaR<br>99.9% | VaR<br>99.95% | VaR<br>99.99% | VaR<br>99.995% | VaR<br>99.999% |
|--------|------------|------------|--------------|--------------|---------------|---------------|----------------|----------------|
| New    | 1.9%       | 1.7%       | 1.7%         | 1.7%         | 1.7%          | 1.7%          | 1.7%           | 1.7%           |
| MC     | 4.4%       | 10.3%      | 14.1%        | 38.2%        | 76.2%         | 8985.7%       | -              | -              |
| GPD 1  | 11.3%      | 8.5%       | 11.8%        | 23.8%        | 30.2%         | 47.9%         | 57.1%          | 83.6%          |
| GPD 2  | 4.4%       | 11.1%      | 15.1%        | 25.1%        | 29.9%         | 42.2%         | 48.1%          | 63.8%          |
| Weiss. | 4.4%       | 11.2%      | 15.1%        | 25.0%        | 29.6%         | 41.8%         | 47.7%          | 63.2%          |

TABLE 2. RMSE in percentage of the real value based on 1000 simulations. At each simulation a sample of size  $10^4$  is used to estimate the VaR.

**7.2. Pareto-Clayton Model dimension 10.** Here we consider the model presented in Section 6. We consider the case  $d = 10$  and  $\alpha = 1$  which corresponds to a model with Pareto marginals with  $\alpha = 1$  and dependence given by a Survival Clayton copula with parameter  $\theta = 1$ . Results are presented in Tables 4, 5 and 6. We mention that even in dimension 10, the estimation remains efficient for high level quantiles.

| Method | VaR<br>95% | VaR<br>99% | VaR<br>99.5% | VaR<br>99.9% | VaR<br>99.95% | VaR<br>99.99% | VaR<br>99.995% | VaR<br>99.999% |
|--------|------------|------------|--------------|--------------|---------------|---------------|----------------|----------------|
| New    | 0.7%       | 0.5%       | 0.6%         | 0.6%         | 0.6%          | 0.6%          | 0.6%           | 0.6%           |
| MC     | 1.4%       | 3.1%       | 4.4%         | 9.7%         | 14.4%         | 37.6%         | 76.9%          | 2763.3%        |
| GPD 1  | 5.2%       | 2.6%       | 3.6%         | 7.2%         | 8.9%          | 13.2%         | 15.2%          | 19.8%          |
| GPD 2  | 1.4%       | 3.7%       | 4.7%         | 7.7%         | 9.1%          | 12.7%         | 14.4%          | 18.4%          |
| Weiss. | 1.4%       | 3.9%       | 4.9%         | 7.7%         | 9.0%          | 12.5%         | 14.1%          | 18.1%          |

TABLE 3. RMSE in percentage of the real value based on 1000 simulations. At each simulation a sample of size  $10^5$  is used to estimate the VaR.

| VaR<br>95% | VaR<br>99% | VaR<br>99.5% | VaR<br>99.9% | VaR<br>99.95% | VaR<br>99.99% | VaR<br>99.995% | VaR<br>99.999% |
|------------|------------|--------------|--------------|---------------|---------------|----------------|----------------|
| 194.5      | 994.5      | 1994.5       | 9994.5       | 19994.5       | 99994.5       | 199994.5       | 999994.5       |

TABLE 4. Exact Value-at-Risk at different confidence levels on the Pareto-Clayton Model in dimension  $d = 10$  with  $\alpha = 1$ .

| Method     | VaR<br>95% | VaR<br>99% | VaR<br>99.5% | VaR<br>99.9% | VaR<br>99.95% | VaR<br>99.99% | VaR<br>99.995% | VaR<br>99.999% |
|------------|------------|------------|--------------|--------------|---------------|---------------|----------------|----------------|
| New Method | 8.4%       | 7.8%       | 7.7%         | 7.7%         | 7.7%          | 7.7%          | 7.7%           | 7.7%           |
| MC         | 4.5%       | 10.1%      | 14.5%        | 43.6%        | 85.5%         | 3785.6%       | -              | -              |
| GPD 1      | 10.7%      | 8.5%       | 12.1%        | 25.0%        | 32.1%         | 51.7%         | 62.0%          | 91.3%          |
| GPD 2      | 4.5%       | 11.3%      | 15.6%        | 26.5%        | 31.8%         | 45.7%         | 52.5%          | 71.0%          |
| Weiss.     | 4.5%       | 11.4%      | 15.5%        | 26.1%        | 31.2%         | 44.8%         | 51.5%          | 69.7%          |

TABLE 5. RMSE in percentage of the real value based on 1000 simulations. At each simulation a sample of size  $10^4$  is used to estimate the VaR.

| Method | VaR<br>95% | VaR<br>99% | VaR<br>99.5% | VaR<br>99.9% | VaR<br>99.95% | VaR<br>99.99% | VaR<br>99.995% | VaR<br>99.999% |
|--------|------------|------------|--------------|--------------|---------------|---------------|----------------|----------------|
| New    | 2.6%       | 2.2%       | 2.2%         | 2.3%         | 2.3%          | 2.3%          | 2.3%           | 2.3%           |
| MC     | 1.4%       | 3.2%       | 4.6%         | 10.1%        | 14.8%         | 39.5%         | 71.9%          | 9779.5%        |
| GPD 1  | 4.3%       | 2.7%       | 3.8%         | 7.4%         | 9.2%          | 13.6%         | 15.6%          | 20.3%          |
| GPD 2  | 1.4%       | 3.6%       | 4.8%         | 7.8%         | 9.2%          | 12.6%         | 14.2%          | 17.9%          |
| Weiss. | 1.4%       | 4.1%       | 5.2%         | 7.9%         | 9.1%          | 12.3%         | 13.8%          | 17.4%          |

TABLE 6. RMSE in percentage of the real value based on 1000 simulations. At each simulation a sample of size  $10^5$  is used to estimate the VaR.

### 7.3. Model with different Pareto marginals and Gumbel copula.

We apply here our method to a model where the exact value of  $\text{VaR}_p(S)$  is not known. The model is composed of 10 marginals  $\text{Pareto}(\alpha_i, \beta_i)$  distributed with parameters  $\alpha_i = 1$  and  $\beta_i = 5 * i$  for  $i = 1, \dots, 10$ . The dependence structure is described by a Gumbel copula of parameter 1.5. Results are presented in Table 7. It is notable that our method is very stable with respect to others.

| Method  | VaR<br>95% | VaR<br>99% | VaR<br>99.5% | VaR<br>99.9% | VaR<br>99.95% | VaR<br>99.99% | VaR<br>99.995% | VaR<br>99.999% |
|---------|------------|------------|--------------|--------------|---------------|---------------|----------------|----------------|
| New     | 1.3%       | 2.0%       | 1.6%         | 1.6%         | 1.8%          | 1.5%          | 1.3%           | 1.6%           |
| MC      | 1.3%       | 3.2%       | 4.4%         | 10.6%        | 15.6%         | 40.1%         | 80.7%          | 6009.1%        |
| GPD 1 1 | 3.0%       | 2.7%       | 3.8%         | 7.7%         | 9.6%          | 14.3%         | 16.3%          | 21.0%          |
| GPD 2   | 1.3%       | 3.6%       | 5.1%         | 8.2%         | 9.6%          | 12.9%         | 14.3%          | 17.6%          |
| Weiss.  | 1.3%       | 3.7%       | 5.0%         | 8.2%         | 9.7%          | 13.0%         | 14.4%          | 17.7%          |

TABLE 7. Standard deviation of the estimations, in percentage of the mean estimated value based on 1000 simulations. At each simulation a sample of size  $10^5$  is used to estimate the VaR.

## 8. APPENDIX: SOME GENERAL PROPERTIES ON REGULARLY VARYING FUNCTIONS AND THEIR INVERSES

The results given in this section are used in Section 4.2.

**Proposition 8.1.** (i) *Suppose  $f$  is non-decreasing, and  $f \in \mathcal{RV}_\infty(\rho)$ ,  $\rho > 0$ . Then*

$$f^\leftarrow \in \mathcal{RV}_\infty(\rho^{-1}) \quad \text{and} \quad f \circ f^\leftarrow(x) \sim f^\leftarrow \circ f(x) \sim x.$$

(ii) *Suppose  $f$  is non-increasing, and  $f \in \mathcal{RV}_\infty(\rho)$ ,  $\rho < 0$ . Then*

$$f^\leftarrow \in \mathcal{RV}_0(-\rho^{-1}), \quad f \circ f^\leftarrow(1/x) \sim 1/x \quad \text{and} \quad f^\leftarrow \circ f(x) \sim x.$$

*Proof.* (i): For the proof see [21] Chapter 2. (ii): Now suppose  $f$  is non-increasing and that  $f \in \mathcal{RV}_\infty(\rho)$ ,  $\rho < 0$ . Then  $(1/f) \in \mathcal{RV}_\infty(-\rho)$ , and by the non-decreasing case  $(1/f)^\leftarrow \in \mathcal{RV}_\infty(-1/\rho)$  which is equivalent to say that  $f^\leftarrow(1/x) \in \mathcal{RV}_\infty(-1/\rho)$  and this by definition implies  $f^\leftarrow \in \mathcal{RV}_0(-1/\rho)$ . To show that  $f \circ f^\leftarrow(1/x) \sim 1/x$  simply note that this is equivalent to show that  $(1/f) \circ (1/f)^\leftarrow(x) \sim x$ .  $\square$

**Proposition 8.2.** (i) *Suppose  $f_1$  and  $f_2$  are non-decreasing,  $\rho$ -varying at infinity,  $\rho > 0$ . Then for  $0 < c < \infty$*

$$f_1(x) \sim cf_2(x) \quad \text{iff} \quad f_1^\leftarrow(x) \sim c^{-\rho^{-1}} f_2^\leftarrow(x).$$

(ii) *Suppose  $f_1$  and  $f_2$  are non-increasing,  $\rho$ -varying at infinity,  $\rho < 0$ . Then for  $0 < c < \infty$*

$$(8.1) \quad f_1(x) \sim cf_2(x) \quad \text{iff} \quad f_1^\leftarrow(1/x) \sim c^{-\rho^{-1}} f_2^\leftarrow(1/x).$$

*Proof.* The proof for the non-decreasing functions can be found on [21] Chapter 2. The non-increasing case then follows immediately by noticing that  $f_1 \sim cf_2$  iff  $(1/f_1) \sim (1/c)(1/f_2)$  iff  $(1/f_1)^\leftarrow \sim (1/c)^{\rho^{-1}}(1/f_2)^\leftarrow$  iff  $f_1^\leftarrow(1/x) \sim c^{-\rho^{-1}} f_2^\leftarrow(1/x)$ .  $\square$

**Lemma 8.3.** *Let  $\psi : [0, 1] \rightarrow \mathbb{R}_+$  be a non-increasing function with  $\psi(0) = +\infty$  and  $\psi(1) = 0$ .*

- (i) *If  $\psi \in \mathcal{RV}_0(-\theta_0)$  for  $\theta_0 > 0$  then*

$$\psi^\leftarrow \in \mathcal{RV}_\infty(-\theta_0^{-1})$$

- (ii) If  $\psi \in \mathcal{RV}_1(-\theta_1)$  for  $\theta_1 > 0$  then

$$(1 - \psi^-) \in \mathcal{RV}_0(-\theta_1^{-1})$$

*Proof.* i) Set  $f(x) = \psi(1/x)$ , then  $f \in \mathcal{RV}_\infty(-\theta_0)$ . By Proposition 8.1 (ii)  $f^- \in \mathcal{RV}_\infty(\theta_0^{-1})$ , and as  $f^- = 1/\psi^-$  then  $\psi^- \in \mathcal{RV}_\infty(-\theta_0^{-1})$ .

ii) Set  $f(x) = \psi(1 - 1/x)$ , then  $f \in \mathcal{RV}_\infty(-\theta_1)$ . By Proposition 8.1 (ii)  $f^- \in \mathcal{RV}_0(\theta_1^{-1})$  and as  $f^- = 1/(1 - \psi^-)$  then  $(1 - \psi^-) \in \mathcal{RV}_0(-\theta_1^{-1})$   $\square$

**Lemma 8.4.** *If  $f \in \mathcal{RV}_\infty(\rho)$  with  $f(t) \downarrow 0$  as  $t \rightarrow \infty$ ,  $\psi_0 \in \mathcal{RV}_0(\theta_0)$  and  $\psi_1 \in \mathcal{RV}_1(\theta_1)$  then*

$$\psi_0 \circ f \in \mathcal{RV}_\infty(\rho \times \theta_0) \quad \text{and} \quad \psi_1 \circ (1 - f) \in \mathcal{RV}_\infty(-\rho \times \theta_1).$$

*Proof.* Notice that as  $f(tx) \sim t^\rho f(x)$  then

$$\psi \circ f(tx) \sim \psi(t^\rho f(x)) \sim t^{\rho \times \theta_0} \psi \circ f(x).$$

Notice that the first  $\sim$  is consequence from the Proposition 2.7. To prove the other equation let  $h(x) = \psi_1(1 - 1/x)$ , then by definition  $h \in \mathcal{RV}_\infty(\theta_1)$ . The proof follows by noticing that  $\psi_1 \circ (1 - f) = h \circ (1/f)$ .  $\square$

## 9. CONCLUSION

In this paper we give some conditions under which the tail distribution of a sum can be approximated using the tail of the maximum of a vector. We show how, basically whenever the maximum is consistently varying, the VaR or the TVaR on high levels for the sum can be approximated, by estimating first a limit  $\Delta$ . The models in which our results can be applied include those where marginals are consistently varying and such that dependence is given by an Archimedean copula or survival copula. We don't require the marginals to be identically distributed. Our method gives a good approximation for the VaR and the TVaR when the convergence of  $\delta(x)$  to  $\Delta$  is fast enough. This generally happens when at least one of the marginal risks is strongly heavy tailed, with tail index  $\leq 2$ . We should remark also that this kind of models are also those where Monte Carlo approximations are less efficient.

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UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, LABORATOIRE SAF EA 2429, SCOR SE  
E-mail address: [acuberos@scor.com](mailto:acuberos@scor.com)

UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, INSTITUT CAMILLE JORDAN ICJ UMR  
5208 CNRS  
E-mail address: [esterina.masiello@univ-lyon1.fr](mailto:esterina.masiello@univ-lyon1.fr)

UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, INSTITUT CAMILLE JORDAN ICJ UMR  
5208 CNRS  
E-mail address: [veronique.maume@univ-lyon1.fr](mailto:veronique.maume@univ-lyon1.fr)