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ON A CHARACTERISTIC OF THE FIRST EIGENVALUE OF THE DIRAC OPERATOR ON COMPACT SPIN SYMMETRIC SPACES WITH A KÄHLER OR QUATERNION-KÄHLER STRUCTURE

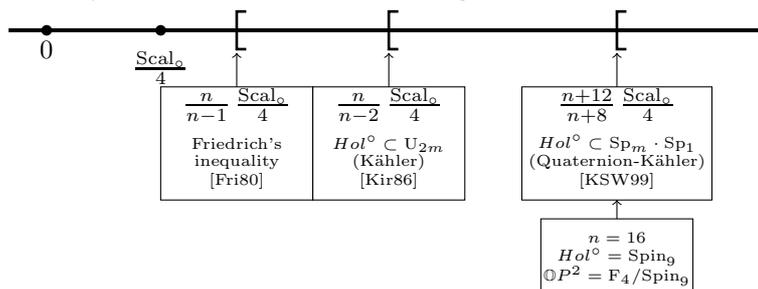
JEAN-LOUIS MILHORAT

ABSTRACT. It is shown that on a compact spin symmetric space with a Kähler or Quaternion-Kähler structure, the first eigenvalue of the Dirac operator is linked to a “lowest” action of the holonomy, given by the fiberwise action on spinors of the canonical forms characterized by this holonomy. The result is also verified for the symmetric space F_4/Spin_9 , proving that it is valid for all the “possible” holonomies in the Berger’s list occurring in that context. The proof is based on a characterization of the first eigenvalue of the Dirac operator given in [Mil05] and [Mil06]. By the way, we review an incorrect statement in the proof of the first lemma in [Mil05].

1. INTRODUCTION

Let (M^n, g) be a spin compact Riemannian manifold with positive scalar curvature, more precisely such that $\text{Scal}_o := \min_{m \in M} \text{Scal}(m) > 0$. Under this assumption, the only groups G in Berger’s list such that the restricted holonomy group of M verifies $\text{Hol}^\circ \subset G$ are (cf. for instance [Bes87]) $G = U_m$, $n = 2m$, (M is then Kähler), $G = \text{Sp}_m \cdot \text{Sp}_1$, $n = 4m$, (M is then Quaternion-Kähler) or $G = \text{Spin}_9$, $n = 16$, and then M is isometric to the Cayley plane $\mathbb{O}P^2 = F_4/\text{Spin}_9$, [Ale68], [BG72].

Assuming $n = 4m$ in order to compare all the possible cases, there exist sharp lower bounds for the square of the eigenvalues of the Dirac operator whose dependence on the holonomy is summarized in the following illustration:



The study of limiting manifolds, that are manifolds for which there exists a spinor-field Ψ such that

$$(1.1) \quad \mathcal{D}^2 \Psi = \lambda^2 \Psi,$$

where \mathcal{D} is the Dirac operator, and where λ^2 is one of the bounds quoted above is due to C. Bär in the general case, [Bär93], to A. Moroianu in the case of Kähler manifolds, [Mor95], [Mor99], and to W. Kramer, U. Semmelmann and G. Weingart in the case of Quaternion-Kähler manifolds, [KSW98].

The study of limiting manifolds in the Kähler and Quaternion-Kähler cases involves a special condition for spinor fields Ψ verifying (1.1), which is linked to the decomposition of the spinor space Σ into irreducible components under the action of the holonomy group.

The spinor bundle $\Sigma(M)$ of a spin Kähler manifold (M, g, J) of complex dimension m decomposes into a sum of parallel subbundles $\Sigma(M) = \bigoplus_{r=0}^m \Sigma_r(M)$, each section of the bundle $\Sigma_r(M)$ being an eigenvector for the eigenvalue $i(m-2r)$, for the fiberwise action of the Kähler form Ω on spinors, [Kir86].

It is then a characteristic of limiting Kähler manifolds that in the space of spinor fields Ψ verifying (1.1), there always exists an element such that $\Omega \cdot \Psi = 0$ if m is even, or $\Omega \cdot \Psi = \pm i$, if m is odd. We also may formulate this property as :

- In the space of spinor fields Ψ verifying (1.1), there always exists an element such that

$$(1.2) \quad \frac{\|\Omega \cdot \Psi\|^2}{\|\Psi\|^2} \quad \text{is minimal.}$$

In order to illustrate how this property corresponds to a “lower action” of the “Kähler holonomy”, recall that the above decomposition of the spinor bundle corresponds to the decomposition of the spinor space Σ_{2m} under the action of the groups $U_1 \times SU_m$ when m is even, or $S(U_1 \times U_m)$ when m is odd, actions given by the commutative diagrams

$$\begin{array}{ccc} U_1 \times SU_m & \longrightarrow & Spin_{2m} \\ & \searrow (m \text{ even}) & \downarrow \\ & & U_m \subset SO_{2m} \end{array} \qquad \begin{array}{ccc} S(U_1 \times U_m) & \longrightarrow & Spin_{2m} \\ & \searrow (m \text{ odd}) & \downarrow \\ & & U_m \subset SO_{2m} \end{array}$$

This decomposition may be easily expressed in the case m even, [Sal93],

$$(1.3) \quad \Sigma_{2m} = \bigoplus_{r=0}^m L^{-\frac{1}{2}(m-2r)} \otimes \Lambda^r E,$$

where $L^k := L^{\otimes k}$ is the k -symmetric power of the standard representation L of U_1 , and E is the standard representation of the group SU_m .

The action of the Kähler form corresponds to twice the action of the Lie algebra of U_1 on Σ_{2m} , hence is equal to $i(m-2r)$ id on each component of the sum (1.3). So, limiting Kähler manifolds of even complex dimension are characterized by the existence of a spinor field verifying (1.1), which is also a section of the bundle corresponding to the component $L^0 \otimes \Lambda^{m/2}$ in (1.3), this component being, roughly speaking, the one with minimal “ U_1 holonomy” in (1.3). There is an analogous description in the case m odd.

There exists an analogous criterion for Quaternion-Kähler limiting manifolds. The spinor bundle $\Sigma(M)$ of a spin Quaternion-Kähler manifold (M, g) of dimension $4m$ decomposes into a sum of parallel subbundles $\Sigma(M) = \bigoplus_{r=0}^m \Sigma_r(M)$, each section of the bundle $\Sigma_r(M)$ being an eigenvector for the eigenvalue $6m-4r(r+2)$, for the

fiberwise action of the fundamental 4-form Ω on spinors, [HM95a]. This decomposition corresponds to the decomposition of the spinor space Σ_{4m} into irreducible components under the action of the group $\mathrm{Sp}_m \times \mathrm{Sp}_1$ given by the commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}_m \times \mathrm{Sp}_1 & \xrightarrow{\quad} & \mathrm{Spin}_{4m} \\ & \searrow & \downarrow \\ & & \mathrm{Sp}_m \cdot \mathrm{Sp}_1 \subset \mathrm{SO}_{4m} \end{array}$$

One gets [BS83], [Wan89], [HM95a],

$$(1.4) \quad \Sigma_{4m} = \bigoplus_{r=0}^m S^r H \otimes \Lambda_{\circ}^{m-r} E,$$

where $S^k H$ is the k -symmetric power of the standard representation H of Sp_1 in the space $\mathbb{H} \simeq \mathbb{C}^2$, E is the standard representation of the group Sp_m in the space $\mathbb{H}^m \simeq \mathbb{C}^{2m}$, and $\Lambda_{\circ}^k E$ is the irreducible hermitian complement of $\omega \wedge \Lambda_{\circ}^{k-2} E$ in $\Lambda^k E$, ω being the standard symplectic form on E .

Quaternion-Kähler limiting manifolds are characterized by the fact that, among the spinor fields Ψ verifying (1.1), there always exists a section of the bundle corresponding to the space $S^0 H \otimes \Lambda_{\circ}^m E$.

This space may be characterized by the fact that it is the only component in the decomposition (1.4) on which the Casimir operator of the subgroup Sp_1 acts trivially (roughly speaking, one could say that this is the component with minimal “ Sp_1 holonomy” in (1.4)). From a geometrical point of view, the invariant given by this action of the Casimir operator, gives raise to the operator $\Omega - 6m \mathrm{id}$, where Ω is the 4-fundamental form Ω acting on spinor fields by Clifford multiplication. Hence, we may re-formulate the characterization of Quaternion-Kähler limiting manifolds as:

- In the space of spinor fields Ψ verifying (1.1), there always exists an element such that $(\Omega - 6m \mathrm{id}) \cdot \Psi = 0$.
- In the space of spinor fields Ψ verifying (1.1), there always exists an element such that

$$(1.5) \quad \frac{\|(\Omega - 6m \mathrm{id}) \cdot \Psi\|^2}{\|\Psi\|^2} \quad \text{is minimal.}$$

Now, the following example makes think that the above criteria are not characteristic of limiting manifolds. Consider the grassmannian $\mathrm{Gr}_2(\mathbb{C}^{m+2}) = \frac{\mathrm{SU}_{m+2}}{\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_m)}$, endowed with its canonical metric induced by the Killing form of SU_{m+2} sign-changed, which is both Kähler and Quaternion-Kähler (cf. 14.53 in [Bes87] for details). It is shown in [Mil98] that there exists a spinorfield Ψ such that (1.1) is verified for the first eigenvalue λ and (1.5) is verified for the fundamental “Quaternion-Kähler” 4-form. Actually, it may also be checked that (1.2) is also verified for the Kähler form.

Hence it seems natural to conjecture that the above property is not a characteristic of limiting manifolds and the aim of this paper is to prove that the conjecture is true for compact spin symmetric spaces.

Proposition 1.1. *Any spin compact simply connected irreducible symmetric space G/K of “type I”, endowed with a Kähler or Quaternion-Kähler structure, verifies the following criterion.*

Among the spinorfields Ψ verifying $\mathcal{D}^2\Psi = \lambda^2\Psi$, where λ is the first eigenvalue of the Dirac operator, there exists at least one such that

$$\frac{\|\Omega \cdot \Psi\|^2}{\|\Psi\|^2}, \quad \text{respectively} \quad \frac{\|(\Omega - 6m \text{id}) \cdot \Psi\|^2}{\|\Psi\|^2}, \quad \text{is minimal,}$$

where Ω is respectively the Kähler form or the fundamental “Quaternion-Kähler” 4-form of the manifold under consideration.

There is an analogous result for the Cayley plane $\mathbb{O}P^2 = F_4/\text{Spin}_9$, Ω being in this case the canonical 8-form on manifolds with holonomy Spin_9 .

2. PRELIMINARIES FOR THE PROOF

2.1. Spectrum of the Dirac operator on spin compact irreducible symmetric spaces. We consider a spin compact simply connected irreducible symmetric space G/K of “type I”, where G is a simple compact and simply-connected Lie group and K is the connected subgroup formed by the fixed elements of an involution σ of G . This involution induces the Cartan decomposition of the Lie algebra \mathfrak{g} of G into

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where \mathfrak{k} is the Lie algebra of K and \mathfrak{p} is the vector space $\{X \in \mathfrak{g}; \sigma_* \cdot X = -X\}$. The symmetric space G/K is endowed with the Riemannian metric induced by the restriction to \mathfrak{p} of the Killing form of G sign-changed.

The spin condition implies that the homomorphism

$$\alpha : h \in K \longmapsto \text{Ad}_G(h)|_{\mathfrak{p}} \in \text{SO}(\mathfrak{p})$$

lifts to a homomorphism $\tilde{\alpha} : H \longrightarrow \text{Spin}(\mathfrak{p})$ such that $\xi \circ \tilde{\alpha} = \alpha$ where ξ is the two-fold covering $\text{Spin}(\mathfrak{p}) \rightarrow \text{SO}(\mathfrak{p})$, [CG88].

Then the group K inherits a spin representation given by

$$\tilde{\rho}_K : K \xrightarrow{\tilde{\alpha}} \text{Spin}(\mathfrak{p}) \xrightarrow{\rho} \text{GL}_{\mathbb{C}}(\Sigma),$$

where ρ is the spinor representation in the complex spinor space Σ .

The Dirac operator has a real discrete spectrum, symmetric with respect to the origin. A real number λ belongs to the spectrum if and only if there exists an irreducible representation $\gamma : G \rightarrow \text{GL}_{\mathbb{C}}(V_\gamma)$ whose restriction $\text{Res}_K^G(\gamma)$ to the subgroup K , contains in its decomposition into irreducible parts, a representation equivalent to some irreducible component of the decomposition of the spin representation $\tilde{\rho}_K$ of K . Then

$$(2.1) \quad \lambda^2 = c_\gamma + n/16,$$

where c_γ is the Casimir eigenvalue of the irreducible representation γ (which only depends on the equivalence class of γ) and where $n = \dim(G/K)$, $n/16$ being $\text{Scal}/8$ for the choice of the metric (cf. [BHM+] or [Gin09] for details).

2.2. A characterization of the decomposition of the spin representation of K .

We henceforward assume that G and K have same rank and consider a fixed common maximal torus T . Hence G/K has even dimension $n = 2m$.

It is shown in [Mil05] that the lowest eigenvalue λ_{min} of the Dirac operator verifies

$$(2.2) \quad \lambda_{min}^2 = 2 \min_{1 \leq i \leq N} \|\beta_i\|^2 + \frac{n}{8},$$

where the β_i , $1 \leq i \leq N$, are the dominant weights (relative to the choice of T) occurring in the decomposition of the spin representation of K , and where the norm $\|\cdot\|$ on the space of weights is induced by the Killing form of G sign-changed.

The proof of proposition (1.1) is based on a characterization of those dominant weights β_i for which $\|\beta_i\|^2$ is minimum, a characterization which appears implicitly in [Mil06]. It is based on a lemma of R. Parthasaraty in [Par71] (cf. lemma 2.2), which gives the following characterization of dominant weights occurring in the decomposition of the spin representation of K .

Let Φ be the set of non-zero roots of the group G with respect to T . According to a classical terminology, a root θ is called compact if the corresponding root space is contained in $\mathfrak{k}_{\mathbb{C}}$ (that is, θ is a root of K with respect to T) and non compact if the root space is contained in $\mathfrak{p}_{\mathbb{C}}$. Let Φ_G^+ be the set of positive roots of G , Φ_K^+ be the set of positive roots of K , and $\Phi_{\mathfrak{p}}^+$ be the set of positive non compact roots with respect to a fixed lexicographic ordering in Φ . The half-sums of the positive roots of G and K are respectively denoted δ_G and δ_K and the half-sum of non compact positive roots is denoted by $\delta_{\mathfrak{p}}$. The Weyl group of G is denoted W_G . The space of weights is endowed with the W_G -invariant scalar product \langle, \rangle induced by the Killing form of G sign-changed.

To introduce the result of Parthasaraty first note that the common torus T may be chosen in such a way that the weights of the spin representation of K are

$$(2.3) \quad \frac{1}{2} (\pm\alpha_1 \pm \alpha_2 \cdots \pm \alpha_m),$$

where $\alpha_1, \dots, \alpha_m$ is an enumeration of the non compact positive roots, the weights of the half-spin representations Σ^{\pm} corresponding to an even (resp. odd) number of negative signs.

Thus weights of the spin representation of K have the form

$$(2.4) \quad \delta_{\mathfrak{p}} - \sum_{i \in I} \alpha_i, \quad I \subset \{1, \dots, m\}.$$

Lemma 2.1 (R. Parathasaraty, [Par71]). *Let*

$$(2.5) \quad W := \{w \in W_G ; w \cdot \Phi_G^+ \supset \Phi_K^+\}.$$

The spin representation of K decomposes into irreducible components as

$$(2.6) \quad \widetilde{\rho}_K = \bigoplus_{w \in W} \widetilde{\rho}_{K|_w},$$

where $\widetilde{\rho}_{K|_w}$ has for dominant weight

$$(2.7) \quad \beta_w := w \cdot \delta_G - \delta_K.$$

2.3. A characterization of highest weights of the spin representation of K with minimal norm.

Proposition 2.2. *Let*

$$I_- := \{i \in \{1, \dots, m\}, \langle \delta_K, \alpha_i \rangle < 0\},$$

and

$$I_0 := \{i \in \{1, \dots, m\}, \langle \delta_K, \alpha_i \rangle = 0\}.$$

Then, for any subset $I \subset I_0$

$$(2.8) \quad \beta_I := \delta_{\mathfrak{p}} - \sum_{i \in I_-} \alpha_i - \sum_{i \in I} \alpha_i,$$

is a highest weight of the spin representation of K with minimal norm. So there are exactly $1 + \#I_0$ such highest weights.

Proof. Let I be a subset of I_0 . First β_I is a weight of the spin representation of K by (2.4). If β_I is not a highest weight, then there exists a K -positive root θ such that $\beta_I + \theta$ is a weight. So there exists a subset $J \subset \{1, \dots, m\}$ such that

$$\beta_I + \theta = \delta_{\mathfrak{p}} - \sum_{i \in J} \alpha_i.$$

Then,

$$- \sum_{i \in I_- \setminus J} \alpha_i - \sum_{i \in I \setminus J} \alpha_i + \theta = - \sum_{i \in J \setminus (I_- \cup I)} \alpha_i.$$

Now, it is well known that $\langle \theta, \delta_K \rangle > 0$, (cf. for instance § 10.2 in [Hum72]), hence $\langle - \sum_{i \in I_- \setminus J} \alpha_i - \sum_{i \in I \setminus J} \alpha_i + \theta, \delta_K \rangle > 0$, whereas $\langle - \sum_{i \in J \setminus (I_- \cup I)} \alpha_i, \delta_K \rangle \leq 0$, hence a contradiction.

By the lemma 2.1, there exists a $w_I \in W$ such that

$$\beta_I = w_I \cdot \delta_G - \delta_K = \delta_{\mathfrak{p}} - \sum_{i \in I_-} \alpha_i - \sum_{i \in I} \alpha_i.$$

Now, using the W_G -invariance of the scalar product,

$$(2.9) \quad \begin{aligned} \|\beta_I\|^2 &= \|\delta_G\|^2 + \|\delta_K\|^2 - 2\langle w_I \cdot \delta_G, \delta_K \rangle \\ &= \|\delta_G\|^2 + \|\delta_K\|^2 - 2\langle \beta_I + \delta_K, \delta_K \rangle \\ &= \|\delta_G\|^2 + \|\delta_K\|^2 - 2\langle \delta_G, \delta_K \rangle + 2 \sum_{i \in I_-} \langle \alpha_i, \delta_K \rangle \\ &= \|\delta_{\mathfrak{p}}\|^2 + 2 \sum_{i \in I_-} \langle \alpha_i, \delta_K \rangle. \end{aligned}$$

Hence all the highest weights β_I , with $I \subset I_0$ have the same norm. In order to prove that among the highest weights of the spin representation of K , they are those with lower norm, we use the same argument as in [Mil06].

Let $\theta_1, \dots, \theta_p$ be an enumeration of the K -positive roots. Let $w \in W$ (or $\in W_G$ as well). By the lemma 3.1 in [Mil06], using the expression of w in reduced form, one has

$$w \cdot \delta_G = \delta_G - \sum_{i \in I_w} \alpha_i - \sum_{j \in J_w} \theta_j,$$

where I_w is a subset of $\{1, \dots, m\}$, and J_w a subset of $\{1, \dots, p\}$.

Since $\langle \theta_j, \delta_K \rangle > 0$, $j = 1, \dots, p$, as we remark before, and since $\langle \alpha_i, \delta_K \rangle > 0$ if i does not belong to I_- or I_0 , one gets

$$\begin{aligned}
(2.10) \quad \langle w \cdot \delta_G, \delta_K \rangle &\leq \langle \delta_G - \sum_{i \in I_w \cap (I_- \cup I_0)} \alpha_i, \delta_K \rangle \\
&\leq \langle \delta_G - \sum_{i \in I_- \cup I_0} \alpha_i, \delta_K \rangle \\
&\leq \langle \delta_G - \sum_{i \in I_-} \alpha_i, \delta_K \rangle = \langle w_\emptyset \cdot \delta_G, \delta_K \rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
\|w \cdot \delta_G - \delta_K\|^2 &= \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \langle w \cdot \delta_G, \delta_K \rangle \\
&\geq \|\delta_G\|^2 + \|\delta_K\|^2 - 2 \langle w_\emptyset \cdot \delta_G, \delta_K \rangle = \|\beta_\emptyset\|^2.
\end{aligned}$$

Note that if the above inequality is an equality, then all the inequalities (2.10) are equalities, hence $J_w = \emptyset$ and $I_w = I_- \cup I$, where $I \subset I_0$, so $w \cdot \delta_G - \delta_K = \beta_I$.

As the above result is valid for any $w \in W_G$, it may be concluded that for any subset $I \subset I_0$,

$$\min_{w \in W_G} \|w \cdot \delta_G - \delta_K\|^2 = \min_{w \in W} \|w \cdot \delta_G - \delta_K\|^2 = \|\beta_I\|^2.$$

□

Now the proof of (2.2) consists in the following steps ([Mil05]):

- (1) For any $w \in W$ such that the highest weight $w \cdot \delta_G - \delta_K$ has minimal norm, $\delta_G - w^{-1} \cdot \delta_K$ is the dominant weight of an irreducible representation γ of G .
- (2) The restriction of γ to K contains in its decomposition into irreducible components a representation with dominant weight $w \cdot \delta_G - \delta_K$.
- (3) The Casimir eigenvalue for γ is given by $c_\gamma = 2 \|w \cdot \delta_G - \delta_K\|^2 + \frac{n}{16}$ (hence equal to $2 \|\delta_{\mathfrak{p}}\|^2 + 4 \sum_{i \in I_-} \langle \alpha_i, \delta_K \rangle + \frac{n}{16}$, by (2.9)).
- (4) The above Casimir eigenvalue gives the lowest eigenvalue of the Dirac operator.

In the preparation of this paper, we found a gap in the proof of the first item. We give a different proof in appendix, which is indeed based on the result of proposition 2.2.

2.4. The space of eigenvectors of the Dirac operator corresponding to the lowest eigenvalue. In order to understand the action of a form characterized by the holonomy on the eigenvectors of the Dirac operator for the lowest eigenvalue, we now review some well-known results (see [BHM+] for details).

First, recall that a spinor field Ψ on G/K may be viewed as a function

$$\Psi : G \mapsto \Sigma, \quad \forall g \in G, \forall k \in K, \quad \Psi(gk) = \tilde{\rho}(k^{-1}) \cdot \Psi(g).$$

Denoting by Σ_w the irreducible K -space of Σ with dominant weight β_w , $w \in W$, and by Π_w the projection $\Sigma \rightarrow \Sigma_w$, any spinor field Ψ decomposes into

$$(2.11) \quad \Psi = \sum_{w \in W} \Psi_w, \quad \Psi_w := \Pi_w \circ \Psi.$$

Since the restricted holonomy group of G/K is K , the spin Levi-Civita connection ∇ respects the above decomposition, hence by the Lichnerowicz-Schrödinger formula,

$$\mathcal{D}^2 = \nabla^* \nabla + \frac{\text{Scal}}{4},$$

if a spinor field Ψ is an eigenspinor of the Dirac operator \mathcal{D} for the eigenvalue λ , then each non-trivial component Ψ_w in the above decomposition is an eigenspinor of \mathcal{D}^2 for the eigenvalue λ^2 .

Now, as it was recalled above (see (2.1)), any eigenvalue λ of the Dirac operator \mathcal{D} corresponds (up to equivalence) to an irreducible complex G -representation $\gamma : G \rightarrow \text{GL}_{\mathbb{C}}(V_\gamma)$, whose restriction $\text{Res}_K^G(\gamma)$ to the subgroup K , contains in its decomposition into irreducible parts, a representation with dominant weight β_w , $w \in W$. The corresponding eigenvectors are given by a pair (v_γ, A_γ) , where $v_\gamma \in V_\gamma$ and $A_\gamma \in \text{Hom}_K(V_\gamma, \Sigma) := \{A \in \text{Hom}_{\mathbb{C}}(V_\gamma, \Sigma), \forall k \in K, A \circ \gamma(k) = \widetilde{\rho}_K(k) \circ A\}$, giving raise to the spinor field

$$\Psi_{v_\gamma, A_\gamma} : G \longrightarrow \Sigma, \quad \Psi_{v_\gamma, A_\gamma}(g) = A_\gamma(\gamma(g^{-1}) \cdot v_\gamma).$$

Moreover

$$\dim \text{Hom}_K(V_\gamma, \Sigma) = \sum_{w \in W} \text{mult.}(\text{Res}_K^G(\gamma), \widetilde{\rho}_{K|_w}),$$

where $\text{mult.}(\text{Res}_K^G(\gamma), \widetilde{\rho}_{K|_w})$ denotes the multiplicity of the irreducible representation $\widetilde{\rho}_{K|_w}$ in $\text{Res}_K^G(\gamma)$.

So, if $\text{mult.}(\text{Res}_K^G(\gamma), \widetilde{\rho}_{K|_w}) \neq 0$, then the component Ψ_w in the decomposition of Ψ given by (2.11) is a non-trivial eigenvector of \mathcal{D}^2 for λ^2 .

All that discussion applies to the irreducible G -representation γ with dominant weight $\delta_G - w^{-1} \cdot \delta_K$, where $\beta_w = w \cdot \delta_G - \delta_K$ is a highest weight of the spin representation of K with minimal norm (cf. prop. 4.1). As we recall it above, this irreducible G -representation gives raise to the lowest eigenvalue λ of the Dirac operator. Moreover by prop. 4.2, the representation $\text{Res}_K^G(\gamma)$ contains in its decomposition into irreducible components all the irreducible K -representations with dominant weights (2.8).

Hence we may conclude

Lemma 2.3. *For any subset $I \subset I_0 = \{i \in \{1, \dots, n\}, \langle \delta_K, \alpha_i \rangle = 0\}$, denoting by Σ_I the irreducible component of Σ with highest weight $\beta_I = \delta_{\mathfrak{p}} - \sum_{i \in I_-} \alpha_i - \sum_{i \in I} \alpha_i$, there exists a spinor field $\Psi_I : G \rightarrow \Sigma_I$, such that*

$$\mathcal{D}^2 \Psi_I = \lambda^2 \Psi_I,$$

where λ is the lowest eigenvalue of the Dirac operator.

Now let Ω be a parallel form on G/K . First, Ω may be viewed as a K -equivariant function

$$\Omega : G \longrightarrow \Lambda^*(\mathfrak{p}), \quad \forall g \in G, \forall k \in K, \quad \Omega(gk) = \alpha(k^{-1}) \cdot \Omega(g),$$

By the fundamental principle of holonomy, parallel forms correspond to K -invariants of $\Lambda^*(\mathfrak{p})$. Hence there exists a K -invariant Ω in $\Lambda^*(\mathfrak{p})$ such that Ω is the constant function

$$\forall g \in G, \quad \Omega(g) = \Omega.$$

Note that as Ω is K -invariant, Ω is K -equivariant since

$$\forall g \in G, \forall k \in K, \quad \Omega(gk) = \Omega = \alpha(k^{-1}) \cdot \Omega = \alpha(k^{-1}) \cdot \Omega(g).$$

The form Ω acts on a spinor field Ψ by Clifford multiplication, giving a spinor field $\Omega \cdot \Psi$ defined by the function

$$\Omega \cdot \Psi : G \longrightarrow \Sigma, \quad \Omega \cdot \Psi(g) := \Omega(g) \cdot \Psi(g) = \mathbf{\Omega} \cdot \Psi(g),$$

where $\mathbf{\Omega}$ is viewed as an element of the Clifford algebra, which acts on the spinor $\Psi(g)$ by means of the standard representation of this algebra.

Since $\mathbf{\Omega}$ is K -invariant, the Schur lemma implies that the action of Ω on spinor fields with values in some K -irreducible subspace Σ_w , $w \in W$, of Σ , is a scalar multiple of the identity, the value of the scalar depending only of $w \in W$.

In particular, the spinor fields Ψ_I , $I \subset I_0$, introduced in lemma 2.3, which are eigenspinors for the square of the Dirac operator for the lowest eigenvalue, are also eigenvectors for the action of any parallel form.

3. PROOF OF THE RESULT

3.1. The Kähler case. On a spin Kähler manifold of complex dimension m , the Kähler 2-form Ω acts fiberwise on spinors as an anti-hermitian operator with eigenvalues $i(m-2r)$, $r = 0, \dots, m$, [Kir86]. Hence

$$(3.1) \quad \min_{\Psi \neq 0} \frac{\|\Omega \cdot \Psi\|^2}{\|\Psi\|^2} = \begin{cases} 0, & \text{if } m \text{ is even} \\ 1, & \text{if } m \text{ is odd.} \end{cases}$$

We are going to prove that (3.1) is verified for one of the spinor fields Ψ_I of lemma 2.3.

First, an irreducible symmetric space G/K is Kähler if and only if K has a center $Z \simeq U_1$, [KN69].

Let $\mathfrak{p}_{\mathbb{C}}^+$, (resp. $\mathfrak{p}_{\mathbb{C}}^-$) be the space generated by root-vectors corresponding to the positive non compact roots (resp. negative non compact roots). Any element H in the Lie algebra \mathfrak{z} of the center has a K -invariant adjoint action on $\mathfrak{p}_{\mathbb{C}}^+$, (resp. $\mathfrak{p}_{\mathbb{C}}^-$), hence by the Schur lemma acts as a scalar multiple of identity. The element H is chosen such that $\text{ad}(H)|_{\mathfrak{p}_{\mathbb{C}}^+} = i \text{ id}$ (hence $\alpha_j(H) = i$, $j = 1, \dots, m$) and $\text{ad}(H)|_{\mathfrak{p}_{\mathbb{C}}^-} = -i \text{ id}$. This action defines a K -invariant homomorphism \mathbf{J} of \mathfrak{p} such that $\mathbf{J}^2 = -\text{id}$, which induces a Kähler structure on G/K . The Kähler form is then defined by the K -invariant $\mathbf{\Omega}$ corresponding to $\alpha_*(H)$ by the isomorphism $\Lambda^2(\mathfrak{p}) \simeq \mathfrak{so}(\mathfrak{p})$. Hence the action of the Kähler form on spinor fields is given by the action of $\mathbf{\Omega}$ on Σ , which corresponds to 2 times the action of H on Σ by the spinor representation of K , since viewed as a 2-form, $\mathbf{\Omega}$ is identified with an element of the Clifford algebra, whereas viewed as $\alpha_*(H) \in \mathfrak{so}(\mathfrak{p})$, it acts on spinors by the isomorphism $\xi_* : \mathfrak{spin}(\mathfrak{p}) \rightarrow \mathfrak{so}(\mathfrak{p})$, which generates a factor 2. Finally, as H belongs to the Lie algebra of the maximal torus¹ T , the Kähler form acts on spinor fields with values in Σ_w , $w \in W$, as a scalar multiple of identity, the eigenvalue being given by $2\beta_w(H)$.

Hence we only have to prove that there exists a subset $I \subset I_0 = \{i \in \{1, \dots, m\}, \langle \delta_K, \alpha_i \rangle = 0\}$, such that $\beta_I(H) = 0$ if m is even and $\beta_I(H) = \pm i/2$, if m is odd, where $\beta_I = \delta_{\mathfrak{p}} - \sum_{i \in I_-} \alpha_i - \sum_{i \in I} \alpha_i$.

Let $I_+ = \{i \in \{1, \dots, m\}, \langle \alpha_i, \delta_K \rangle > 0\}$. Then

$$\beta_I(H) = \frac{1}{2} \sum_{j \in I_+} \alpha_j(H) - \frac{1}{2} \sum_{j \in I_-} \alpha_j(H) + \frac{1}{2} \sum_{j \in I_0 \setminus I} \alpha_j(H) - \frac{1}{2} \sum_{j \in I} \alpha_j(H).$$

¹since the center of K is the intersection of the maximal tori.

Lemma 3.1. *The sets I_+ and I_- have the same number of elements.*

Proof. Let Φ_K^- be the set of negative roots of K . There exists an element w_0 in the Weyl group of K (hence in the Weyl group of G) which sends Φ_K^+ to Φ_K^- , see for instance theorem 3.1.9. in [GW09].

Note that, as H belongs to the Lie algebra \mathfrak{z} of the center Z of K , one has for any K -root θ , $\theta(H) = 0$, since for any root vector Y_θ ,

$$0 = [H, Y_\theta] = \theta(H) Y_\theta.$$

From this remark, one deduces that $w_0(\Phi_{\mathfrak{p}}^+) \subset \Phi_{\mathfrak{p}}^+$. Indeed, as H belongs to \mathfrak{z} , if $k \in K$ is some representative of w_0 , one has since $\text{Ad}(k^{-1}) \cdot H = H$, and $\alpha_j(H) = i$, $j = 1, \dots, m$,

$$w_0 \cdot \alpha_j(H) = \alpha_j(\text{Ad}(k^{-1}) \cdot H) = \alpha_j(H) = i, \quad j = 1, \dots, m.$$

Now let $j \in I_+$ so that $\langle \alpha_j, \delta_K \rangle > 0$. Then $\langle w_0 \cdot \alpha_j, w_0 \cdot \delta_K \rangle > 0$, so as $w_0 \cdot \delta_K = -\delta_K$, one gets

$$\langle w_0 \cdot \alpha_j, \delta_K \rangle < 0,$$

hence there exists $i_j \in I_-$ such that $\alpha_{i_j} = w_0 \cdot \alpha_j$. This defines a one-to-one correspondence between I_+ and I_- . □

Hence since $\alpha_j(H) = i$, $j = 1, \dots, m$, we obtain from the lemma

$$\beta_I(H) = \frac{1}{2} \sum_{j \in I_0 \setminus I} \alpha_j(H) - \frac{1}{2} \sum_{j \in I} \alpha_j(H).$$

Now if m is even, then by the result of the lemma, the set I_0 has an even number of elements. If $I_0 = \emptyset$, then $\beta_0(H) = 0$. If $I_0 \neq \emptyset$, then choosing a subset $I \subset I_0$ such that $\# I = \frac{1}{2} \# I_0$, one gets $\beta_I(H) = 0$.

If m is odd, then by the result of the lemma, the set I_0 has an odd number $2r + 1$ of elements. Choosing now a subset $I \subset I_0$ such that $\# I = r$, (resp. $r + 1$), one gets $\beta_I(H) = \frac{1}{2} i$, (resp. $-\frac{1}{2} i$), and the result is proved.

3.2. The Quaternion-Kähler case. A Quaternion-Kähler manifold is a $n = 4m$ -dimensional Riemannian manifold (M, g) whose restricted holonomy group is contained in the group $\text{Sp}_m \cdot \text{Sp}_1 = \text{Sp}_m \times_{\mathbb{Z}_2} \text{Sp}_1$, $m \geq 2$. This group is identified with a subgroup of SO_{4m} by the representation

$$(A, q) \in \text{Sp}_m \cdot \text{Sp}_1 \longmapsto \left(x \in \mathbb{H}^m \simeq \mathbb{R}^{4m} \mapsto Ax\bar{q} \right).$$

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the standard basis of imaginary quaternions. The action on the right of $-\mathbf{i}, -\mathbf{j}, -\mathbf{k}$ on \mathbb{H}^m defines three hermitian operators $\mathbf{I}, \mathbf{J}, \mathbf{K}$, verifying the same multiplication rules as the imaginary quaternions. The space \mathcal{Q} generated by $\mathbf{I}, \mathbf{J}, \mathbf{K}$ is K -invariant, hence by transport on the fibres, it defines a globally parallel subbundle $\mathcal{Q}(M)$ of the bundle $\text{End}(TM)$. By transporting the operators $\mathbf{I}, \mathbf{J}, \mathbf{K}$ on fibres with the help of a trivialization, one gets three local almost complex structures I, J, K , for which the metric g is hermitian, verifying the same multiplication rules as the imaginary quaternions. Using the metric, one obtains three local 2-forms $\Omega_I, \Omega_J, \Omega_K$. Now, the 4-form

$$\Omega = \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K,$$

is well-defined over M , parallel and non-degenerate, [Kra66], [Bon67].

On any spin Quaternion-Kähler manifold this 4-form Ω acts fiberwise on spinors as an hermitian operator with eigenvalues $6m - 4r(r + 2)$, $r = 0, \dots, m$, [HM95a]. Hence

$$(3.2) \quad \min_{\Psi \neq 0} \frac{\|(\Omega - 6m \text{id}) \cdot \Psi\|^2}{\|\Psi\|^2} = 0.$$

We are going to prove that (3.2) is verified for one of the spinor fields Ψ_I of lemma 2.3.

Compact symmetric spaces with a Quaternion-Kähler structure were classified by J. A. Wolf in [Wol65]. It is well known in the theory of representations of compact groups that any root associated to the choice of a maximal torus gives raise to a subgroup of G isomorphic to Sp_1 . J. A. Wolf has shown that compact symmetric spaces with a Quaternion-Kähler are all inner symmetric space of type I of the form G/K , where G is a simple group and $K = K_1 \text{Sp}_1$, where K_1 is the centralizer of Sp_1 in G . The subgroup Sp_1 of K in consideration here being defined by the maximal root β (for a fixed ordering of roots).

Indeed, let $H_\beta \in \mathfrak{t}$ such that for any $H \in \mathfrak{t}$, $\langle H_\beta, H \rangle = -i\beta(H)$. Then $\|H_\beta\|^2 = -i\beta(H_\beta) = \|\beta\|^2$. Let $H_\beta^\circ := 2/\|\beta\|^2 H_\beta$.

Let X_β be a root-vector for the root β . There exists a root-vector $X_{-\beta}$ for the root $-\beta$ such that $[X_\beta, X_{-\beta}] = -iH_\beta^\circ$.

Then $(H_\beta^\circ, Y_\beta := i(X_{-\beta} + X_\beta), Z_\beta := X_{-\beta} - X_\beta)$ defined a basis of a subalgebra of \mathfrak{g} isomorphic to \mathfrak{sp}_1 as

$$[H_\beta^\circ, Y_\beta] = 2Z_\beta, \quad [H_\beta^\circ, Z_\beta] = -2Y_\beta \quad \text{and} \quad [Y_\beta, Z_\beta] = 2H_\beta^\circ.$$

Now, the condition that β is the maximal root implies that $\text{ad}(H_\beta^\circ)|_{\mathfrak{p}_\mathbb{C}^+} = i \text{id}$, [Wol65], so the action of H_β°, Y_β and Z_β on \mathfrak{p} by $\alpha_* : \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{p})$, induces three hermitian operators $\mathbf{I}, \mathbf{J}, \mathbf{K}$, verifying the same multiplication rules as the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the standard basis of imaginary quaternions. The space \mathcal{Q} generated by $\mathbf{I}, \mathbf{J}, \mathbf{K}$, which is K -invariant, generates the Quaternion-Kähler structure on G/K . Identifying \mathbf{I}, \mathbf{J} and \mathbf{K} with 2-forms $\Omega_{\mathbf{I}}, \Omega_{\mathbf{J}}, \Omega_{\mathbf{K}}$, via the metric, one gets the K -invariant 4-form on \mathfrak{p}

$$\Omega = \Omega_{\mathbf{I}} \wedge \Omega_{\mathbf{I}} + \Omega_{\mathbf{J}} \wedge \Omega_{\mathbf{J}} + \Omega_{\mathbf{K}} \wedge \Omega_{\mathbf{K}},$$

which induces the Quaternion-Kähler parallel fundamental 4-form Ω on G/K .

Now, if the symmetric space has a spin structure, the 4-form Ω acts on the spinor space Σ as the operator ([HM95b])

$$\Omega = 6m \text{id} + \Omega_{\mathbf{I}} \cdot \Omega_{\mathbf{I}} + \Omega_{\mathbf{J}} \cdot \Omega_{\mathbf{J}} + \Omega_{\mathbf{K}} \cdot \Omega_{\mathbf{K}},$$

where the 2-forms $\Omega_{\mathbf{I}}, \Omega_{\mathbf{J}}, \Omega_{\mathbf{K}}$ act by Clifford multiplication. Hence ²

$$(3.3) \quad \Omega - 6m \text{id} = 4 \left(\tilde{\rho}_*(H_\beta^\circ)^2 + \tilde{\rho}_*(Y_\beta)^2 + \tilde{\rho}_*(Z_\beta)^2 \right).$$

Note that the second term in the r.h.s. of the above equation is the Casimir operator³ of the representation $\tilde{\rho}_*$ restricted to \mathfrak{sp}_1 .

²here again the presence of the scalar factor 4 is due to the use of the isomorphism $\xi_* : \mathfrak{spin}(\mathfrak{p}) \rightarrow \mathfrak{so}(\mathfrak{p})$, when the two-forms $\Omega_{\mathbf{I}}, \Omega_{\mathbf{J}}, \Omega_{\mathbf{K}}$ are identified with $H_\beta^\circ, Y_\beta, Z_\beta$, acting on spinors by the representation $\tilde{\rho}_*$ of \mathfrak{k} .

³up to some normalization.

Expressing the r.h.s. of (3.3) in the basis $(H_\beta^{\mathfrak{c}} := -iH_\beta^{\circ}, X_\beta, X_{-\beta})$ of $\mathfrak{sp}_{1,\mathbb{C}} \simeq \mathfrak{sl}_{2,\mathbb{C}}$, one gets

$$\begin{aligned} \Omega - 6m \text{ id} &= -4 \left(\tilde{\rho}_*(H_\beta^{\mathfrak{c}})^2 + 2\tilde{\rho}_*(X_\beta) \circ \tilde{\rho}_*(X_{-\beta}) + 2\tilde{\rho}_*(X_{-\beta}) \circ \tilde{\rho}_*(X_\beta) \right) \\ (3.4) \quad &= -4 \left(\tilde{\rho}_*(H_\beta^{\mathfrak{c}})^2 + 2\tilde{\rho}_*(H_\beta^{\mathfrak{c}}) + 4\tilde{\rho}_*(X_{-\beta}) \circ \tilde{\rho}_*(X_\beta) \right). \end{aligned}$$

Hence we may conclude

Lemma 3.2. *For any subset $I \subset I_0$,*

$$(\Omega - 6m \text{ id}) \cdot \Psi_I = 0 \iff \beta_I(H_\beta^{\circ}) = 0.$$

Proof. By the Schur lemma, $\Omega - 6m$ acts on the K -irreducible space Σ_I as a scalar multiple of identity. If c_I is the eigenvalue, one then has $(\Omega - 6m \text{ id}) \cdot \Psi_I = c_I \Psi_I$. To compute the eigenvalue, one applies (3.4) to a highest weight vector of Σ_I . Since the action of $\tilde{\rho}_*(X_\beta)$ is zero on such a vector, whereas $\tilde{\rho}_*(H_\beta^{\mathfrak{c}})$ acts by a non-negative integer multiple of identity on it, one has $c_I = 0$ if and only if $\tilde{\rho}_*(H_\beta^{\mathfrak{c}})$ acts trivially, hence the result. \square

Let $I_+ = \{i \in \{1, \dots, m\}, \langle \alpha_i, \delta_K \rangle > 0\}$. Then

$$(3.5) \quad \beta_I(H_\beta^{\circ}) = \frac{1}{2} \sum_{j \in I_+} \alpha_j(H_\beta^{\circ}) - \frac{1}{2} \sum_{j \in I_-} \alpha_j(H_\beta^{\circ}) + \frac{1}{2} \sum_{j \in I_0 \setminus I} \alpha_j(H_\beta^{\circ}) - \frac{1}{2} \sum_{j \in I} \alpha_j(H_\beta^{\circ}).$$

Lemma 3.3. *Apart from G_2/SO_4 , for any Quaternion-Kähler compact spin symmetric space, one has*

$$\# I_- + \# I_0 = \# I_+.$$

Proof. Note first that

$$(3.6) \quad \forall \theta \in \Phi_K, \theta \neq \pm\beta, \quad \langle \beta, \theta \rangle = 0.$$

Indeed if X_θ is a root-vector for the root θ , one has $[H_\beta, X_\theta] = 0$, since K_1 is the centralizer of Sp_1 in K . So $\theta(H_\beta) = 0$. Now, let $H_\theta \in \mathfrak{t}$ be such that for any $H \in \mathfrak{t}$ $\langle H_\theta, H \rangle = -i\theta(H)$. We then have $\langle H_\theta, H_\beta \rangle = 0$, hence $\langle \theta, \beta \rangle = 0$.

By this remark, positive non compact roots α are then characterized by the condition $\langle \alpha, \beta \rangle = \frac{1}{2} \|\beta\|^2$.

Let α_i be a positive non compact root. Then $\beta - \alpha_i$ is a positive compact root. It is a root since $\langle \alpha_i, \beta \rangle > 0$ (cf. for instance⁴ § 9.4 in [Hum72]). And it is positive since $\langle \beta - \alpha_i, \beta \rangle = \frac{1}{2} \|\beta\|^2$. Note furthermore that $\beta - \alpha_i \neq \alpha_i$, since otherwise $2\alpha_i = \beta$ should be a root, which is impossible.

Now by (3.6), $\langle \delta_K, \beta \rangle = \frac{1}{2} \|\beta\|^2$, hence

$$\langle \delta_K, \beta - \alpha_i \rangle = \frac{1}{2} \|\beta\|^2 - \langle \delta_K, \alpha_i \rangle.$$

Hence, if $j \in I_- \cup I_0$, then $\langle \delta_K, \beta - \alpha_j \rangle \geq \frac{1}{2} \|\beta\|^2$, hence $\beta - \alpha_j = \alpha_{i_j}$, with $i_j \in I_+$. We thus get an injective map $I_- \cup I_0 \rightarrow I_+$, so we may conclude $\# I_- + \# I_0 \leq \# I_+$. On the other hand, if $\langle \delta_K, \alpha_j \rangle > 0$, then as δ_K is an integral weight⁵, one has $\langle \delta_K, \alpha_j \rangle \geq \frac{1}{2} \|\alpha_j\|^2$, hence

$$\langle \delta_K, \beta - \alpha_j \rangle \leq \frac{1}{2} (\|\beta\|^2 - \|\alpha_j\|^2).$$

⁴or note that $\sigma_\beta(\alpha_i) = \alpha_i - \beta$, where σ_β the reflection across the hyperplane β^\perp .

⁵since δ_K is a difference of integral weights: $\delta_K = w \cdot \delta_G - (w \cdot \delta_G - \delta_K)$, $w \in W$.

Now as G is a simple group, the root system is irreducible and there are at most two root lengths (see for instance § 10.4. in [Hum72]).

If all the roots have same length, then $\langle \delta_K, \beta - \alpha_j \rangle \leq 0$, so $\beta - \alpha_j = \alpha_{i_j}$, where $i_j \in I_- \cup I_0$. In this case, there is an injective map $I_+ \rightarrow I_- \cup I_0$, so $\# I_+ \leq \# I_- + \# I_0$, and the result is proved.

Now by the result [Wol65], using furthermore the result [CG88], the list of spin compact Quaternion-Kähler symmetric spaces is given by

| G | K | G/K | $\dim G/K$ | Spin structure |
|-----------------------|---------------------------------------|--|-----------------|--------------------------------------|
| Sp_{m+1} | $\mathrm{Sp}_m \times \mathrm{Sp}_1$ | Quaternionic projective space $\mathbb{H}P^m$ | $4m (m \geq 1)$ | Yes (unique) |
| SU_{m+2} | $S(\mathrm{U}_m \times \mathrm{U}_2)$ | Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{m+2})$ | $4m (m \geq 1)$ | iff m even unique in that case |
| Spin_{m+4} | $\mathrm{Spin}_m \mathrm{Spin}_4$ | Grassmannian $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{m+4})$ | $4m (m \geq 3)$ | iff m even, unique in that case |
| G_2 | SO_4 | | 8 | Yes (unique) |
| F_4 | $\mathrm{Sp}_3 \mathrm{SU}_2$ | | 28 | No |
| E_6 | $\mathrm{SU}_6 \mathrm{SU}_2$ | | 40 | Yes (unique) |
| E_7 | $\mathrm{Spin}_{12} \mathrm{SU}_2$ | | 64 | Yes (unique) |
| E_8 | $E_7 \mathrm{SU}_2$ | | 112 | Yes (unique) |

Note that all of them are inner as it was noticed in [Wol65].

Now, apart from⁶ $\mathrm{Sp}(m+1)$ and G_2 , there are only one root length for the groups G in the above list, hence the result is proved for the corresponding symmetric spaces.

So it remains to prove the result for quaternionic projective spaces $\mathbb{H}P^m = \mathrm{Sp}(m+1)/\mathrm{Sp}(m) \times \mathrm{Sp}_1$, $m \geq 1$.

We consider the standard maximal torus T of Sp_{m+1} made up of diagonal matrices with entries of the form $e^{\beta \mathbf{i}} := \cos(\beta) + \sin(\beta) \mathbf{i}$, $\beta \in \mathbb{R}$. We denote by (x_0, x_1, \dots, x_m) the standard basis of \mathfrak{t}^* such that the value of x_k on a diagonal matrix with entries $(\beta_0 \mathbf{i}, \dots, \beta_m \mathbf{i})$ is β_k , $k = 0, \dots, m$. We set $\hat{x}_k = i x_k$. The scalar product on $i \mathfrak{t}^*$ induced by the Killing form sign-changed verifies $\langle \hat{x}_i, \hat{x}_j \rangle = \frac{1}{4(m+2)} \delta_{ij}$. We choose as positive roots

$$\hat{x}_i \pm \hat{x}_j, \quad 0 \leq i < j \leq m; \quad 2\hat{x}_i, \quad 0 \leq i \leq m.$$

The roots $\hat{x}_i - \hat{x}_{i+1}$, $0 \leq i \leq m-1$, and $2\hat{x}_m$ then define a basis of the root system. In order to avoid a re-ordering of roots, we consider $K = \mathrm{Sp}_1 \times \mathrm{Sp}_m$, (instead of $\mathrm{Sp}_m \times \mathrm{Sp}_1$) in such a way that the positive compact roots are

$$\hat{x}_i \pm \hat{x}_j, \quad 1 \leq i < j \leq m; \quad 2\hat{x}_i, \quad 0 \leq i \leq m.$$

⁶there are two root lengths for $G = F_4$, but the corresponding symmetric space is not spin.

Then, the positive non compact roots are

$$\widehat{x}_0 \pm \widehat{x}_k, \quad 1 \leq k \leq m.$$

The maximal root is $\beta = 2\widehat{x}_0$. Note that $\widehat{x}_0 \pm \widehat{x}_k = \beta - (\widehat{x}_0 \mp \widehat{x}_k)$, and $\langle \beta, \widehat{x}_0 \pm \widehat{x}_k \rangle = 1/2 \|\beta\|^2$.

Now $\delta_K = \widehat{x}_0 + \sum_{k=1}^m (m-k+1)\widehat{x}_k$. Hence it is easy to verify that

$$\begin{aligned} \langle \delta, \widehat{x}_0 + \widehat{x}_k \rangle &> 0, & 1 \leq k, \\ \langle \delta, \widehat{x}_0 + \widehat{x}_k \rangle &< 0, & 1 \leq k-1, \\ \langle \delta, \widehat{x}_0 + \widehat{x}_m \rangle &= 0. \end{aligned}$$

So $\# I_- + \# I_0 = \# I_+$. □

Going back to (3.5), one deduces from the above lemma, as $\text{ad}(H_\beta^\circ)|_{\mathfrak{p}_\mathbb{C}^+} = i \text{id}$,

$$\beta_{I_0}(H_\beta^\circ) = \frac{1}{2} i (\# I_+ - \# I_- - \# I_0) = 0.$$

Hence by the lemma 3.2, $(\Omega - 6m \text{id}) \cdot \Psi_{I_0} = 0$, and the result of proposition 1.1 is proved.

3.3. The case of G_2/SO_4 . The group SO_4 is identified with $\text{Sp}_1 \cdot \text{Sp}_1 \simeq \text{Sp}_1 \times_{\mathbb{Z}_2} \text{Sp}_1$. The inclusion $\text{Sp}_1 \cdot \text{Sp}_1 \subset \text{Sp}_2 \cdot \text{Sp}_1$ is not the ‘‘natural’’ one since the group acts irreducibly on $\mathbb{H}^2 = \mathbb{R}^8$ with highest weight $3\omega_1 + \omega_2$, where (ω_1, ω_2) is the standard basis of fundamental weights corresponding to the half spinors representations, see for instance §11 in [Sal89].

We use the result of [See99], where all the roots data for G and K are expressed in the basis (ω_1, ω_2) by

$$\begin{aligned} \Phi_G^+ &= \{2\omega_1, -3\omega_1 + \omega_2, -\omega_1 + \omega_2, \omega_1 + \omega_2, 2\omega_2, 3\omega_1 + \omega_2\} \\ \Phi_K^+ &= \{2\omega_1, 2\omega_2\}, \\ \Phi_{\mathfrak{p}}^+ &= \{-3\omega_1 + \omega_2, -\omega_1 + \omega_2, \omega_1 + \omega_2, 3\omega_1 + \omega_2\}. \end{aligned}$$

The scalar product on weights induced by the Killing form sign-changed is given by

$$\langle a_1\omega_1 + a_2\omega_2, b_1\omega_1 + b_2\omega_2 \rangle = \frac{1}{16} \left(\frac{1}{3} a_1 b_1 + a_2 b_2 \right).$$

The highest weights of the spin representation are obtained by means of the Parthasarathy formula, [See99]

$$4\omega_1, \quad 2\omega_2 \quad \text{and} \quad 3\omega_1 + \omega_2.$$

The half spin representation Σ_8^- is irreducible with highest weight $3\omega_1 + \omega_2$, the half spin representation Σ_8^+ decomposes into two components with respective highest weights $4\omega_1$ and $2\omega_2$. So, denoting by H the standard representation of Sp_1 , the spin representation decomposes as

$$(3.7) \quad \Sigma_8 = (\text{S}^3 H \otimes \text{S}^1 H) \oplus (\text{S}^4 H \otimes \text{S}^0 H) \oplus (\text{S}^0 H \otimes \text{S}^2 H).$$

There are two weights for which the norm is minimal: $\beta_\emptyset = 2\omega_2 = \delta_{\mathfrak{p}}$ and $\beta_{I_0} = 3\omega_1 + \omega_2 = \delta_{\mathfrak{p}} - (-3\omega_1 + \omega_2)$.

Hence the spinor field Ψ_\emptyset verifies $\mathcal{D}\Psi_\emptyset = \lambda^2 \Psi_\emptyset$, where λ is the lowest eigenvalue of the Dirac operator, and is also a section of the bundle corresponding to the component $\text{S}^0 H \otimes \text{S}^2 H$ in the decomposition (3.7), on which the action (on the first

component) of Sp_1 is trivial, thus our “holonomy criterion” is verified. However, it has to be noticed that the Quaternion-Kähler structure of G_2/SO_4 corresponds to the action of Sp_1 on the second component, and that this criterion is not verified for that action.

Indeed, it easy to see that $\theta_1 = 2\omega_1$ and $\theta_2 = -3\omega_1 + \omega_2$ are simple G -roots, and that the maximal root is $\beta := 2\omega_2 = 3\theta_1 + 2\theta_2$. Now note that

$$\begin{cases} \langle \beta, \alpha \rangle = 0 & \text{if } \alpha \in \Phi_K, \alpha \neq \pm\beta, \\ \langle \beta, \alpha \rangle = \frac{1}{2} \|\beta\|^2 & \text{if } \alpha \in \Phi_{\mathfrak{p}}^+, \end{cases}$$

hence we are in the description of Wolf spaces given above. It may be checked that for any positive non compact positive root α , $\langle \delta_K, \alpha \rangle = 0, 1/24, 1/12$ or $1/8$, so $\#I_+ = 3$, $\#I_0 = 1$ and $\#I_- = 0$, hence the result of lemma 3.3 is not verified here.

Moreover note that, by definition of H_β° , $\beta_\emptyset(H_\beta^\circ) = \frac{2i}{\|\beta\|^2} \langle \beta, \beta_\emptyset \rangle \neq 0$ and $\beta_{I_0}(H_\beta^\circ) = \frac{2i}{\|\beta\|^2} \langle \beta, \beta_{I_0} \rangle \neq 0$, so by lemma 3.2, the criterion is not verified for the action of Sp_1 on the second component.

As an additional argument, one may remark that since $\langle \beta, 4\omega_1 \rangle = 0$, $4\omega_1(H_\beta^\circ) = 0$, so by the proof of lemma 3.2, the action of $\Omega - 6m \mathrm{id}$ on spinors is zero only on the K irreducible subspace of Σ with highest weight $4\omega_1$. But, by the results of [See99], this is not a highest weight for the restriction to K of the G_2 irreducible representation giving raise to the lowest eigenvalue of the Dirac operator.

Indeed, the action of Sp_1 on the first component may also be described in terms of the action of fundamental 4-form on spinors by Clifford multiplication. This is a consequence of a more general result.

Lemma 3.4. *The eigenvalue of the Casimir operator of the spin representation of K has the same value for all irreducible components.*

Proof. By the Parthasaraty formula, any highest weight of the spin representation of K has the form $\beta_w := w \cdot \delta_G - \delta_K$, where w is an element of the Weyl group W_G of G . By the Freudenthal formula, the eigenvalue of the Casimir operator of K acting on the K -representation with highest weight β_w is given by

$$\langle \beta_w + 2\delta_K, \beta_w \rangle.$$

Hence, by the W_G -invariance of the scalar product,

$$\begin{aligned} \langle \beta_w + 2\delta_K, \beta_w \rangle &= \langle w \cdot \delta_G + \delta_K, w \cdot \delta_G - \delta_K \rangle \\ &= \|\delta_G\|^2 - \|\delta_K\|^2. \end{aligned}$$

□

Hence, since $\|\delta_G\|^2 - \|\delta_K\|^2 = \frac{1}{2}$ here, the Casimir operator \mathcal{C}_K of the spin representation of K acts on each irreducible component as $\frac{1}{2} \mathrm{id}$.

Denote by \mathcal{C}_1 (resp. \mathcal{C}_2) the Casimir operator of the restriction of the spinor representation to the first (resp. second) Sp_1 component in K , (for the scalar product given by the Killing form sign-changed of G_2). One has $\mathcal{C}_K = \mathcal{C}_1 + \mathcal{C}_2$. The eigenvalue of \mathcal{C}_1 (\mathcal{C}_2) acting on $S^k H$ is a scalar multiple of $k(k+2)$ times the identity. Using the result of the above lemma, one gets by (3.7)

$$\mathcal{C}_{1|S^k H} = \frac{1}{48} k(k+2) \mathrm{id} \quad \text{and} \quad \mathcal{C}_{2|S^k H} = \frac{1}{16} k(k+2) \mathrm{id}.$$

Hence using (3.4), one obtains

$$C_1 = \frac{1}{2} \text{id} - C_2 = \frac{1}{64} (\Omega + 20 \text{id}).$$

So for the symmetric space G_2/SO_4 , the holonomy criterion is valid for the 4-fundamental form acting (by Clifford multiplication) on spinors as $\Omega + 20 \text{id}$.

3.4. The Cayley plane F_4/Spin_9 . The above ‘‘holonomy criterion’’ is also valid for the Cayley plane F_4/Spin_9 . As it is said in [Bes87], the $\text{Spin}(9)$ holonomy is extremely special since a Riemannian manifold whose holonomy group is contained in $\text{Spin}(9)$ is either flat or (locally) isometric to the Cayley plane F_4/Spin_9 , or its non-compact dual, [Ale68], [BG72]. There is an analogy between $\text{Spin}(9)$ -structures (for 16-dimensional manifolds) and Quaternion-Kähler structures, since such a structure on a manifold M may be characterized by the existence of a 9-dimensional subbundle of the bundle $\text{End}(TM)$, with local sections I_α , $1 \leq \alpha \leq 9$, satisfying

$$I_\alpha^2 = \text{id} \quad , \quad I_\alpha^* = I_\alpha \quad , \quad I_\alpha I_\beta = -I_\beta I_\alpha \quad , \quad \alpha \neq \beta \quad ,$$

[Fri01]. There is also an analogy by the existence of a canonical 8-form Ω , which corresponds to the unique parallel 8-form on F_4/Spin_9 . This canonical 8-form was first introduced in [BG72] by means an integral formula. Explicit algebraic expressions of this form are far from being simple [BPT85], [AM96], [LGM10], [PP12]. Roughly speaking, the form is constructed by means of the Kähler 2-forms associated to the almost complex structures $J_{\alpha\beta} := I_\alpha \circ I_\beta$, [LGM10], [PP12]. To avoid an explicit expression, we will use here the fact that it may be expressed in terms of ‘‘higher Casimir operators’’, see §125 and §126 in [Žel73] or [Hom04].

Indeed, this parallel 8-form is induced by an $\text{Ad}(K)$ -invariant 8-form Ω on $\mathfrak{p} = \mathbb{R}^{16}$. From the expression of Ω given in [LGM10], [PP12], the action of Ω on spinors is (up to a shift by a scalar multiple of the identity) a sum of terms of the form $\widetilde{\rho}_{K^*}(\Omega_{\alpha_1\beta_1}) \circ \widetilde{\rho}_{K^*}(\Omega_{\alpha_2\beta_2}) \circ \widetilde{\rho}_{K^*}(\Omega_{\alpha_3\beta_3}) \circ \widetilde{\rho}_{K^*}(\Omega_{\alpha_4\beta_4})$, where the $\Omega_{\alpha\beta}$ are the Kähler 2-forms, identified with elements of the Lie algebra \mathfrak{spin}_9 , associated to the almost complex structures $J_{\alpha\beta}$. We thus may identify the action of Ω on spinors as the action of an element of the universal enveloping algebra $U(\mathfrak{spin}_{9,\mathbb{C}})$, also denoted Ω . Furthermore this element belongs to the center \mathfrak{Z} of $U(\mathfrak{spin}_{9,\mathbb{C}})$ since Ω is $\text{Ad}(K)$ -invariant.

Now it is known that the center \mathfrak{Z} is algebraically generated by a system of ‘‘higher Casimir elements’’, cf. §.125 and §.126 in [Žel73], which we briefly introduce in that context.

Let $(e_i)_{1 \leq i \leq 9}$ be the standard basis of \mathbb{R}^9 . Denote by $e_{ij} = e_i \wedge e_j$ the standard basis of $\mathfrak{spin}_9 \simeq \mathfrak{so}_9 \simeq \Lambda^2(\mathbb{R}^9)$. For any non negative integer q , consider $e_{ij}^q \in U(\mathfrak{so}_{9,\mathbb{C}})$ defined by

$$e_{ij}^q := \begin{cases} \sum_{1 \leq k_1, \dots, k_{q-1} \leq 9} e_{ik_1} e_{k_1 k_2} \cdots e_{k_{q-1} j} \quad , & q \geq 1 \quad , \\ \delta_{ij} \quad , & q = 0 \quad . \end{cases}$$

Then the trace of e_{ij}^q , $C_q := \sum_{i=1}^9 e_{ii}^q$ belongs to the center \mathfrak{Z} . For $q = 0$, $C_0 = 9$, for $q = 1$, $C_1 = 0$ and for $q = 2$, $C_2 = \sum_{ij} e_{ij} e_{ji}$ is the usual Casimir element.

It may be shown that the center \mathfrak{Z} is algebraically generated by C_2 , C_4 , C_6 and C_8 , [Žel73].

Now, the above description of the element $\Omega \in U(\mathfrak{spin}_{9,\mathbb{C}})$ shows that it is expressed only in terms of C_2 and C_4 . If we consider an irreducible component of the spin

representation, then the Schur lemma implies that $\widetilde{\rho}_{K^*}(C_2)$ and $\widetilde{\rho}_{K^*}(C_4)$ acts on it as scalar multiples of identity. Furthermore, we already note in lemma 3.4, that $\rho_*(C_2)$ acts on each irreducible component as a scalar multiple of identity with the same eigenvalue. Finally, to prove our holonomy criterion, we only have to examine the eigenvalues of $\rho_*(C_4)$ on the irreducible components of the spin representation of K . Those eigenvalues may be computed with the help of a formula given in [CGH00].

Denoting by (e_1, e_2, e_3, e_4) the standard basis of \mathbb{R}^4 , the root system of F_4 is the set of elements $x = \sum_{i=1}^4 x_i e_i$ with integer or half-integer coordinates in \mathbb{R}^4 such that $\|x\|^2 = 1$ or 2 , [Hum72],[BMP85]. Using the results in [CG88], we may consider

$$\begin{aligned}\Phi_G^+ &= \{e_i, i = 1, 2, 3, 4; e_i \pm e_j, 1 \leq i < j \leq 4; \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\} \\ \Phi_K^+ &= \{e_i, i = 1, 2, 3, 4; e_i \pm e_j, 1 \leq i < j \leq 4\}, \\ \Phi_{\mathfrak{p}}^+ &= \{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.\end{aligned}$$

Thus

$$\begin{aligned}\delta_G &= \frac{1}{2}(11e_1 + 5e_2 + 3e_3 + e_4), \\ \delta_K &= \frac{1}{2}(7e_1 + 5e_2 + 3e_3 + e_4), \\ \delta_{\mathfrak{p}} &= 2e_1.\end{aligned}$$

The scalar product induced by the Killing form sign-changed is a scalar multiple of the restriction to the set of roots of the usual scalar product on \mathbb{R}^4 . Using the ‘‘strange’’ formula of Freudenthal and De Vries, [FdV69], one obtains $\|\delta_G\|^2 = \frac{\dim \mathfrak{g}}{24} = 13/6$, hence this scalar product is given by

$$\langle \sum_{i=1}^4 x_i e_i, \sum_{i=1}^4 y_i e_i \rangle = \frac{1}{18} \sum_{i=1}^4 x_i y_i.$$

With the help of the Parthasaraty formula, it is easy to find the highest weights of the spin representation of K . The half spin representation Σ_{16}^- is irreducible with highest weight $\frac{1}{2}(3e_1 + e_2 + e_3 + e_4)$, whereas the half spin representation Σ_{16}^+ decomposes into two components with highest weights $e_1 + e_2 + e_3$ and $2e_1$.

Note that the sets I_- and I_0 have both only one element since for any positive non compact root α , one has $\langle \delta_K, \alpha \rangle < 0 \iff \alpha = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, and $\langle \delta_K, \alpha \rangle = 0 \iff \alpha = \frac{1}{2}(e_1 - e_2 - e_3 + e_4)$.

Indeed, there are two weights for which the norm is minimal

$$\beta_{\emptyset} = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4) = \delta_{\mathfrak{p}} - \frac{1}{2}(e_1 - e_2 - e_3 - e_4),$$

and

$$\beta_{I_0} = e_1 + e_2 + e_3 = \delta_{\mathfrak{p}} - \frac{1}{2}(e_1 - e_2 - e_3 - e_4) - \frac{1}{2}(e_1 - e_2 - e_3 + e_4).$$

By the result of [Mil05], the square of the first eigenvalue of the Dirac operator is then given by $2\|\beta_{\emptyset}\|^2 + 2 = 2\|\beta_{I_0}\|^2 + 2$, hence

Proposition 3.5. *On the symmetric space F_4/Spin_9 endowed with the Riemannian metric induced by the Killing form of F_4 sign-changed, the square of the first eigenvalue λ of the Dirac operator verifies*

$$\lambda^2 = \frac{7}{3} = \frac{7}{6} \frac{\text{Scal}}{4}.$$

We now verify the holonomy criterion by determining explicitly the eigenvalues of the operator $\widetilde{\rho}_{K^*}(C_4)$. This may be done with the help of a formula given in [CGH00] which applies here as follows.

For each dominant weight μ of SO_9 (relatively to the usual maximal torus), denote by $d(\mu)$ the dimension of a complex K -irreducible representation $(\rho_\mu, V(\mu))$ with highest weight μ .

Let $V(\mu_o)$ be the standard representation of SO_9 corresponding to the weight $\mu_o = (1, 0, 0, 0)$. For each highest weight $\beta_1, \beta_2, \beta_3$ of the spin representation of K where

$$\beta_1 = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4), \quad \beta_2 = e_1 + e_2 + e_3 \quad \text{and} \quad \beta_3 = 2e_1,$$

denote by Π_i the set of highest weights occurring in the decomposition into irreducible components of the tensor product $V(\beta_i) \otimes V(\mu_o)$. So

$$V(\beta_i) \otimes V(\mu_o) = \sum_{\lambda_{ij} \in \Pi_i} V(\lambda_{ij}).$$

For each highest weight λ_{ij} , let $m(\lambda_{ij})$ be the so called conformal weight given by

$$m(\lambda_{ij}) := \frac{1}{2} (9 - \|\delta_K + \lambda_{ij}\|^2 + \|\delta_K + \beta_i\|^2 - 1),$$

where $\|\cdot\|$ is the standard norm on the sets of weights of SO_9 : if $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, then $\|\mu\|^2 = \sum_{i=1}^4 \mu_i^2$.

The eigenvalue $c_{\beta_i, k}$ of the operator $(\rho_{\beta_i})_*(C_k)$ is then given by (see for instance [Hom04])

$$c_{\beta_i, k} = \frac{1}{d(\beta_i)} \sum_{\lambda_{ij} \in \Pi_i} (-m(\lambda_{ij}))^k d(\lambda_{ij}).$$

The values of $c_{\beta_i, k}$, $i = 1, 2, 3$, $k = 2, 3, 4$ are given in the following table. They were obtained with the help of the LiE Program⁷.

| | $\beta_1 = \frac{1}{2}(3, 1, 1, 1)$ | $\beta_2 = (1, 1, 1, 0)$ | $\beta_3 = (2, 0, 0, 0)$ |
|------------------|-------------------------------------|--------------------------|--------------------------|
| $c_{\beta_i, 2}$ | 36 | 36 | 36 |
| $c_{\beta_i, 3}$ | -126 | -126 | -126 |
| $c_{\beta_i, 4}$ | 1863/2 | 684 | 1404 |

The first line is not a surprise by lemma 3.4. The second line too since C_3 may be expressed in terms of C_2 . Finally, the last line shows that the lowest eigenvalue of the operator $\widetilde{\rho}_{K^*}(C_4)$ is obtained for the space with highest weight $\beta_2 = (1, 1, 1, 0)$. But $\beta_2 = \beta_{I_0}$, which as we saw it before, is linked to the first eigenvalue of the Dirac operator. Hence the ‘‘holonomy criterion’’ of prop. 1.1 is verified.

⁷<http://www-math.univ-poitiers.fr/~maavl/LiE/>.

4. APPENDIX

Recall that the highest weights of the spin representation of K are given by

$$\beta_w = w \cdot \delta_G - \delta_K, \quad w \in W,$$

where W is the subset of the Weyl group W_G defined by

$$W := \{w \in W_G; w \cdot \Phi_G^+ \supset \Phi_K^+\}.$$

In this appendix, we review the proof of the following result.

Proposition 4.1. *Let $w_0 \in W$ be such that*

$$(4.1) \quad \|\beta_{w_0}\|^2 = \min_{w \in W} \|\beta_w\|^2.$$

Then the weight

$$\beta_{w_0}^G := w_0^{-1} \cdot \beta_{w_0} = \delta_G - w_0^{-1} \cdot \delta_K,$$

is G -dominant.

We will also justify the following remark :

Proposition 4.2. *Let w_0 and $w_1 \in W$ be such that*

$$\|\beta_{w_1}\|^2 = \|\beta_{w_0}\|^2 = \min_{w \in W} \|\beta_w\|^2.$$

Then the G -dominant weights

$$\beta_{w_0}^G := w_0^{-1} \cdot \beta_{w_0} = \delta_G - w_0^{-1} \cdot \delta_K \quad \text{and} \quad \beta_{w_1}^G := w_1^{-1} \cdot \beta_{w_0} = \delta_G - w_1^{-1} \cdot \delta_K$$

verify

$$\beta_{w_0}^G = \beta_{w_1}^G.$$

4.1. Review of the proof of prop. 4.1. The main error in the proof given in [Mil05] concerns a technical assumption which is used at the end, asserting that the highest weights with same minimal length may be ordered with respect to the usual order of K -weights. As it can be seen in (2.8), this is actually not correct since two such highest weights may only differ by a *non compact* positive root⁸. Before that, there is also an imprecise statement, asserting that a certain weight of the spin representation does lie in the W_G -orbit of a highest weight, which makes the proof not satisfactory.

Hence we propose here an alternative proof we think correct.

Proof. First note that considering the sets

$$\Lambda_{w_0}^+ := \{\theta \in \Phi_G^+, w_0 \cdot \theta \in \Phi_{\mathfrak{p}}^+\} \quad \text{and} \quad \Lambda_{w_0}^- := \{\theta \in \Phi_G^+, -w_0 \cdot \theta \in \Phi_{\mathfrak{p}}^+\},$$

one has

$$\beta_{w_0} = \frac{1}{2} \sum_{\theta \in \Phi_G^+ \setminus w_0^{-1}(\Phi_K^+)} w_0 \cdot \theta = \frac{1}{2} \sum_{\theta \in \Lambda_{w_0}^+} w_0 \cdot \theta + \frac{1}{2} \sum_{\theta \in \Lambda_{w_0}^-} w_0 \cdot \theta.$$

Hence since

$$\delta_{\mathfrak{p}} = \frac{1}{2} \sum_{\theta \in \Lambda_{w_0}^+} w_0 \cdot \theta - \frac{1}{2} \sum_{\theta \in \Lambda_{w_0}^-} w_0 \cdot \theta,$$

$$(4.2) \quad \beta_{w_0} = \delta_{\mathfrak{p}} + \sum_{\theta \in \Lambda_{w_0}^-} w_0 \cdot \theta.$$

⁸whereas, if it is a root, a sum of compact roots is necessarily a compact one.

Now by the result of prop. 2.2, there exists a subset $I_{w_0} \subset I_0$ such that

$$(4.3) \quad \beta_{w_0} = \delta_{\mathfrak{p}} - \sum_{i \in I_-} \alpha_i - \sum_{i \in I_{w_0}} \alpha_i.$$

Comparing (4.2) and (4.3), we get the following alternative : if a positive G -root is such that $w_0 \cdot \theta$ is a non compact root, then either $w_0 \cdot \theta$ is negative and then $w_0 \cdot \theta = -\alpha_i$, $i \in I_- \cup I_{w_0}$, so $\langle w_0 \cdot \theta, \delta_K \rangle \geq 0$, or $w_0 \cdot \theta$ is positive and then $w_0 \cdot \theta = \alpha_i$, $i \notin I_- \cup I_{w_0}$, so $\langle w_0 \cdot \theta, \delta_K \rangle \geq 0$.

So we may conclude that

Remark 4.3. If α is a positive G -root such that $w_0 \cdot \alpha$ is a non compact root then, whatever $w_0 \cdot \alpha$ is positive or not, one always has

$$\langle \delta_K, w_0 \cdot \alpha \rangle \geq 0.$$

Let $\Pi_G = \{\theta_1, \dots, \theta_r\} \subset \Phi_G^+$ be the set of simple roots. It is sufficient to prove that $2 \frac{\langle \beta_{w_0}^G, \theta_i \rangle}{\langle \theta_i, \theta_i \rangle}$ is a non-negative integer for any simple root θ_i . First, as T is a maximal common torus of G and K , β_{w_0} , which is an integral weight for K is also an integral weight for G . Now since the Weyl group W_G permutes the weights, $\beta_{w_0}^G = w_0^{-1} \cdot \beta_{w_0}$ is also a integral weight for G , hence $2 \frac{\langle \beta_{w_0}^G, \theta_i \rangle}{\langle \theta_i, \theta_i \rangle}$ is an integer for any simple root θ_i . So we only have to prove that $\langle \beta_{w_0}^G, \theta_i \rangle = \langle \delta_G - w_0^{-1} \cdot \delta_K, \theta_i \rangle \geq 0$, or equivalently (by the W_G -invariance of the scalar product) that

$$(4.4) \quad \langle w_0 \cdot \delta_G - \delta_K, w_0 \cdot \theta_i \rangle \geq 0.$$

Let θ_i be a simple root. Suppose first that $w_0 \cdot \theta_i \in \Phi_K$.

In this case, necessarily $w_0 \cdot \theta_i \in \Phi_K^+$, otherwise since $w_0 \in W$, $-\theta_i = w_0^{-1}(-w_0 \cdot \theta_i)$ should be a positive root.

Then since $w_0 \cdot \delta_G - \delta_K$ is K -dominant, inequality (4.4) is verified in this case, since $w_0 \cdot \theta_i$ is a linear combination with non-negative integer coefficients of K -simple roots.

Suppose now that $w_0 \cdot \theta_i \notin \Phi_K$, that is $w_0 \cdot \theta_i$ is a non compact root. We are going to prove that

$$(4.5) \quad \langle w_0 \cdot \delta_G - \delta_K, w_0 \cdot \theta_i \rangle < 0,$$

is impossible. Suppose that (4.5) is true. Note first that since $2 \frac{\langle \delta_G, \theta_i \rangle}{\langle \theta_i, \theta_i \rangle} = 1$, (see for instance §10.2 in [Hum72]) and since the scalar product $\langle \cdot, \cdot \rangle$ is W_G -invariant, inequation (4.5) is equivalent to

$$(4.6) \quad \langle \delta_K, w_0 \cdot \theta_i \rangle > \frac{1}{2} \|\theta_i\|^2.$$

This implies that

$$\langle \delta_K, w_0 \cdot \theta_i \rangle > 0.$$

Now since δ_K is a linear combination with non-negative coefficients of K -simple roots, this implies that there exists a K -simple root θ' such that

$$(4.7) \quad \langle \theta', w_0 \cdot \theta_i \rangle > 0.$$

By property of roots (cf. for instance §9.4 in [Hum72]), this implies that

$$\theta' - w_0 \cdot \theta_i,$$

is a root, and moreover a non compact root by the bracket relations $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

By the definition of W , $\theta := w_0^{-1} \cdot \theta'$ is a positive G -root. By inequality (4.7) $\langle \theta, \theta_i \rangle > 0$, hence $\theta - \theta_i$ is a root, and moreover a positive root.

Now, the non compact root $\theta' - w_0 \cdot \theta_i$ being the image by w_0 of the G -positive root $\theta - \theta_i$, the above remark 4.3 applies:

$$(4.8) \quad \langle \delta_K, \theta' - w_0 \cdot \theta_i \rangle \geq 0.$$

Since θ' is a K -simple root, $\langle \delta_K, \theta' \rangle = \frac{1}{2} \|\theta'\|^2$, hence we obtain from inequality (4.6)

$$(4.9) \quad \frac{1}{2} \|\theta_i\|^2 < \langle \delta_K, w_0 \cdot \theta_i \rangle \leq \frac{1}{2} \|\theta'\|^2.$$

Now as G is a simple group, the root system is irreducible and there are at most two root lengths (see for instance § 10.4. in [Hum72]).

If all roots are of equal length, then the above inequality is impossible, and the result is proven.

If there are two distinct root lengths, then the above inequality is only possible if θ' is a long root and θ_i a short one.

We rewrite inequalities (4.9) as

$$1 < 2 \frac{\langle \delta_K, w_0 \cdot \theta_i \rangle}{\langle \theta_i, \theta_i \rangle} \leq \frac{\|\theta'\|^2}{\|\theta_i\|^2}.$$

Now $\|\theta'\|^2/\|\theta_i\|^2$ is equal to 2 or 3, see for instance § 9.4 in [Hum72].

Let us examine the case where $\|\theta'\|^2/\|\theta_i\|^2 = 2$ first.

If the inequality (4.8) is strict, one obtains

$$1 < 2 \frac{\langle \delta_K, w_0 \cdot \theta_i \rangle}{\langle \theta_i, \theta_i \rangle} < \frac{\|\theta'\|^2}{\|\theta_i\|^2},$$

hence a contradiction, since as $\delta_K = w_0 \cdot \delta_G - (w_0 \cdot \delta_G - \delta_K)$ is an integral weight, $2 \frac{\langle \delta_K, w_0 \cdot \theta_i \rangle}{\langle \theta_i, \theta_i \rangle}$ is an integer.

Thus (4.8) is an equality :

$$(4.10) \quad \langle \delta_K, \theta' - w_0 \cdot \theta_i \rangle = 0.$$

This implies that there exists $j \in I_0$ such that $\theta' - w_0 \cdot \theta_i = \pm \alpha_j$.

So, by the result of prop. 2.2

- either $\theta' - w_0 \cdot \theta_i = \alpha_j$ and $j \in I_{w_0}$ or $\theta' - w_0 \cdot \theta_i = -\alpha_j$ and $j \notin I_{w_0}$ and then

$$\mu_0 := (w_0 \cdot \delta_G - \delta_K) + (\theta' - w_0 \cdot \theta_i)$$

is a highest weight of the spin representation with minimal length,

- either $\theta' - w_0 \cdot \theta_i = \alpha_j$ and $j \notin I_{w_0}$ or $\theta' - w_0 \cdot \theta_i = -\alpha_j$ and $j \in I_{w_0}$ and then

$$\mu_0 := (w_0 \cdot \delta_G - \delta_K) - (\theta' - w_0 \cdot \theta_i)$$

is a highest weight of the spin representation with minimal length.

In the first case, one gets using $\|\mu_0\|^2 = \|w_0 \cdot \delta_G - \delta_K\|^2$, (4.10) and $\langle \delta_G, \theta_i \rangle = \frac{1}{2} \|\theta_i\|^2$,

$$\begin{aligned} 0 &= 2 \langle w_0 \cdot \delta_G - \delta_K, \theta' - w_0 \cdot \theta_i \rangle + \|\theta' - w_0 \cdot \theta_i\|^2, \\ &= 2 \langle w_0 \cdot \delta_G, \theta' \rangle - \|\theta_i\|^2 + \|\theta' - w_0 \cdot \theta_i\|^2, \\ &= 2 \langle \delta_G, w_0^{-1} \cdot \theta' \rangle - \|\theta_i\|^2 + \|\theta' - w_0 \cdot \theta_i\|^2, \end{aligned}$$

whereas in the second case, one obtains

$$0 = -2 \langle \delta_G, w_0^{-1} \cdot \theta' \rangle + \|\theta_i\|^2 + \|\theta' - w_0 \cdot \theta_i\|^2,$$

But inequality (4.7) implies as $2 \frac{\langle \theta', w_0 \cdot \theta_i \rangle}{\langle \theta', \theta' \rangle}$ is an integer

$$2 \langle \theta', w_0 \cdot \theta_i \rangle \geq \|\theta'\|^2.$$

Hence

$$\|\theta' - w_0 \cdot \theta_i\|^2 = \|\theta'\|^2 - 2 \langle \theta', w_0 \cdot \theta_i \rangle + \|\theta_i\|^2 \leq \|\theta_i\|^2.$$

This implies that $\theta' - w_0 \cdot \theta_i$ is a short root, hence that

$$\|\theta' - w_0 \cdot \theta_i\|^2 = \|\theta_i\|^2, \quad \text{and} \quad 2 \langle \theta', w_0 \cdot \theta_i \rangle = \|\theta'\|^2.$$

In the first case, this implies

$$\langle \delta_G, w_0^{-1} \cdot \theta' \rangle = 0,$$

which is impossible, since $w_0^{-1} \cdot \theta'$ is a positive root, because $\langle \delta_G, \theta_j \rangle > 0$ for any simple G -root θ_j .

In the second case, one obtains

$$\begin{aligned} 0 &= -2 \langle w_0 \cdot \delta_G, \theta' \rangle + 2 \|\theta_i\|^2 \\ &= -2 \langle w_0 \cdot \delta_G, \theta' \rangle + \|\theta'\|^2 \\ &= -2 \langle w_0 \cdot \delta_G, \theta' \rangle + 2 \langle \theta', w_0 \cdot \theta_i \rangle, \end{aligned}$$

hence

$$\langle w_0 \cdot \delta_G, \theta' \rangle = \langle w_0 \cdot \theta_i, \theta' \rangle.$$

Let σ_i be the reflection across the hyperplane θ_i^\perp . Since $\sigma_i \cdot \delta_G = \delta_G - \theta_i$ (see for instance § 10.2 in [Hum72]), we obtain

$$\langle w_0 \sigma_i \cdot \delta_G, \theta' \rangle = 0,$$

hence using the W_G -invariance of the scalar product

$$(4.11) \quad \langle \delta_G, \sigma_i w_0^{-1} \cdot \theta' \rangle = 0.$$

But $w_0^{-1} \cdot \theta'$ is a positive root which, being a long root, is different from θ_i . Hence as σ_i permutes the positive roots other than θ_i , (see for instance § 10.2 in [Hum72]), $\sigma_i w_0^{-1} \cdot \theta'$ is a positive root. Then as $\langle \delta_G, \theta_j \rangle > 0$ for any simple G -root θ_j , (4.11) is impossible.

So the result is proven in the case where $\|\theta'\|^2/\|\theta_i\|^2 = 2$.

We finally examine the case where $\|\theta'\|^2/\|\theta_i\|^2 = 3$. The only simple group for which this is possible is the group G_2 , and there is only one symmetric space of type I to be considered: G_2/SO_4 . So we may give a direct proof of proposition 4.1 in that case.

We use the result of [See99], where all the roots data for G and K are expressed in terms of the fundamental weights (ω_1, ω_2) of SO_4 , corresponding to the half spinors representations.

$$\begin{aligned} \Phi_G^+ &= \{2\omega_1, -3\omega_1 + \omega_2, -\omega_1 + \omega_2, \omega_1 + \omega_2, 2\omega_2, 3\omega_1 + \omega_2\} \\ \Phi_K^+ &= \{2\omega_1, 2\omega_2\}. \end{aligned}$$

Note that $\theta_1 = 2\omega_1$ and $\theta_2 = -3\omega_1 + \omega_2$ are simple G -roots, and that

$$\delta_G = \omega_1 + 3\omega_2 \quad \text{and} \quad \delta_K = \omega_1 + \omega_2.$$

The scalar product on $i\mathfrak{T}^*$ induced by the Killing form sign-changed is given by

$$\langle a_1 \omega_1 + a_2 \omega_2, b_1 \omega_1 + b_2 \omega_2 \rangle = \frac{1}{16} \left(\frac{1}{3} a_1 b_1 + a_2 b_2 \right).$$

The highest weights of the spin representation are ([See99]):

$$\beta_1 = 4\omega_1, \quad \beta_2 = 2\omega_2 \quad \text{and} \quad \beta_3 = 3\omega_1 + \omega_2.$$

Note that there are two highest weights for which the norm is minimal: β_2 and β_3 . One has

$$\beta_2 = \delta_G - \delta_K \quad \text{and} \quad \beta_3 = \sigma_2 \cdot \delta_G - \delta_K,$$

where σ_2 is the reflection across the hyperplane orthogonal to the simple root θ_2 . Note that

$$\sigma_2^{-1} \cdot \beta_3 = \sigma_2 \cdot \beta_3 = 2\omega_2.$$

Now

$$2 \frac{\langle 2\omega_2, \theta_1 \rangle}{\langle \theta_1, \theta_1 \rangle} = 0 \quad \text{and} \quad 2 \frac{\langle 2\omega_2, \theta_2 \rangle}{\langle \theta_2, \theta_2 \rangle} = 1,$$

hence the weight $2\omega_2 = \beta_2 = \sigma_2^{-1} \cdot \beta_3$ is G -dominant, so the result is also proven in that case. \square

4.2. Proof of prop. 4.2. By the result of prop. 2.2, there exist two distinct elements w_0 and w_1 in W such that $\|\beta_{w_1}\|^2 = \|\beta_{w_0}\|^2 = \min_{w \in W} \|\beta_w\|^2$, only if $I_0 \neq \emptyset$. Hence we suppose $I_0 \neq \emptyset$. Let $i \in I_0$ and let $I = \{i\}$.

By (2.8), $\beta_I = \beta_\emptyset - \alpha_I$.

Since $\|\beta_I\|^2 = \|\beta_\emptyset\|^2$, one obtains

$$2 \langle \beta_\emptyset, \alpha_i \rangle = \langle \alpha_i, \alpha_i \rangle,$$

hence denoting by σ_{α_i} the reflection across the hyperplane α_i^\perp ,

$$\sigma_{\alpha_i} \cdot \beta_\emptyset = \beta_\emptyset - \alpha_i = \beta_I.$$

On the other hand, as $i \in I_0$, $\langle \delta_K, \alpha_i \rangle = 0$, so

$$\sigma_{\alpha_i} \cdot \delta_K = \delta_K.$$

Hence, considering w_\emptyset and $w_I \in W$ such that $\beta_\emptyset = w_\emptyset \cdot \delta_G - \delta_K$, and $\beta_I = w_I \cdot \delta_G - \delta_K$, one gets

$$w_I \cdot \delta_G - \delta_K = \beta_I = \beta_\emptyset - \alpha_I = \sigma_{\alpha_i} \cdot \beta_\emptyset = \sigma_{\alpha_i} w_\emptyset \cdot \delta_G - \delta_K.$$

This implies

$$w_I^{-1} \sigma_{\alpha_i} w_\emptyset \cdot \delta_G = \delta_G.$$

But this is only possible if

$$w_I = \sigma_{\alpha_i} w_\emptyset,$$

(see for instance § 122 in [Žel73]). Thus

$$\beta_I^G = \delta_G - w_I^{-1} \cdot \delta_K = \delta_G - w_\emptyset^{-1} \sigma_{\alpha_i} \cdot \delta_K = \delta_G - w_\emptyset^{-1} \cdot \delta_K = \beta_\emptyset^G.$$

Repeating the argument, the result follows by induction on the cardinal of $I \subset I_0$. \square

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