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Local and global estimates of solutions of Hamilton-Jacobi parabolic equation with absorption

Marie Françoise BIDAUT-VERON

Abstract

We obtain new a priori estimates for the nonnegative solutions of the equation

$$u_t - \Delta u + |\nabla u|^q = 0$$

in $Q_{\Omega,T} = \Omega \times (0, T)$, $T \leq \infty$, where $q > 0$, and $\Omega = \mathbb{R}^N$, or Ω is a smooth bounded domain of \mathbb{R}^N and $u = 0$ on $\partial\Omega \times (0, T)$.

In case $\Omega = \mathbb{R}^N$, we show that any solution $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$ of equation (1.1) in $Q_{\mathbb{R}^N,T}$ (in particular any weak solution if $q \leq 2$), without condition as $|x| \rightarrow \infty$, satisfies the universal estimate

$$|\nabla u(\cdot, t)|^q \leq \frac{1}{q-1} \frac{u(\cdot, t)}{t}, \quad \text{in } Q_{\mathbb{R}^N,T}.$$

Moreover we prove that the growth of u is limited by $C(t+t^{-1/(q-1)})(1+|x|^q)$, where C depends on u .

We also give existence properties of solutions in $Q_{\Omega,T}$, for initial data locally integrable or even unbounded Radon measures. We give a nonuniqueness result in case $q > 2$. Finally we show that besides the local regularizing effect of the heat equation, u satisfies a second effect of type $L_{loc}^R - L_{loc}^\infty$, due to the gradient term.

Keywords Hamilton-Jacobi equation; Radon measures; initial trace; universal bounds., regularizing effects.

A.M.S. Subject Classification 35K15, 35K55, 35B33, 35B65, 35D30

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1 Introduction

Here we consider the *nonnegative* solutions of the parabolic Hamilton-Jacobi equation

$$u_t - \nu \Delta u + |\nabla u|^q = 0, \quad (1.1)$$

where $q > 1$, in $Q_{\Omega, T} = \Omega \times (0, T)$, where Ω is any domain of \mathbb{R}^N , $\nu \in (0, 1]$. We study the problem of a priori estimates of the *nonnegative* solutions, with possibly rough *unbounded* initial data

$$u(x, 0) = u_0 \in \mathcal{M}^+(\Omega), \quad (1.2)$$

where we denote by $\mathcal{M}^+(\Omega)$ the set of nonnegative Radon measures in Ω , and $\mathcal{M}_b^+(\Omega)$ the subset of bounded ones. We say that u is a solution of (1.1) if it satisfies (1.1) in $Q_{\Omega, T}$ in the weak sense of distributions, see Section 2. We say that u has a trace u_0 in $\mathcal{M}^+(\Omega)$ if $u(\cdot, t)$ converges to u_0 in the weak* topology of measures:

$$\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t) \psi dx = \int_{\Omega} \psi du_0, \quad \forall \psi \in C_c(\Omega). \quad (1.3)$$

Our purpose is to obtain a priori estimates valid for any solution in $Q_{\Omega, T} = \Omega \times (0, T)$, without assumption on the boundary of Ω , or for large $|x|$ if $\Omega = \mathbb{R}^N$.

First recall some known results. The Cauchy problem in $Q_{\mathbb{R}^N, T}$

$$(P_{\mathbb{R}^N, T}) \begin{cases} u_t - \nu \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\mathbb{R}^N, T}, \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.4)$$

is the object of a rich literature, see among them [2], [9], [5], [11], [26], [12], [13], and references therein. The first studies concern *classical* solutions, that means $u \in C^{2,1}(Q_{\mathbb{R}^N, T})$, with *smooth bounded initial data* $u_0 \in C_b^2(\mathbb{R}^N)$: there a unique global solution such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}, \quad \text{and} \quad \|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}, \quad \text{in } Q_{\mathbb{R}^N, T},$$

see [2]. Then universal a priori estimates of the gradient are obtained *for this solution*, by using the Bernstein technique, which consists in computing the equation satisfied by $|\nabla u|^2$: first from [23],

$$\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}^q \leq \frac{\|u_0\|_{L^\infty(\mathbb{R}^N)}}{t},$$

in $Q_{\mathbb{R}^N, T}$, then from [9],

$$|\nabla u(\cdot, t)|^q \leq \frac{1}{q-1} \frac{u(\cdot, t)}{t}, \quad (1.5)$$

$$\|\nabla(u^{\frac{q-1}{q}})(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}^N)}^{\frac{q-1}{q}}, \quad C = C(N, q, \nu). \quad (1.6)$$

Existence and uniqueness was extended to any $u_0 \in C_b(\mathbb{R}^N)$ in [20]; then the estimates (1.6) and (1.5) are still valid, see [5]. In case of nonnegative rough initial data $u_0 \in L^R(\mathbb{R}^N)$, $R \geq 1$, or $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$, the problem was studied in a semi-group formulation [9], [11], [26], then in the

larger class of weak solutions in [12], [13]. Recall that two critical values appear: $q = 2$, where the equation can be reduced to the heat equation, and

$$q_* = \frac{N+2}{N+1}.$$

Indeed the Cauchy problem with initial value $u_0 = \kappa\delta_0$, where δ_0 is the Dirac mass at 0 and $\kappa > 0$, has a weak solution u^κ if and only if $q < q_*$, see [9], [12]. Moreover as $\kappa \rightarrow \infty$, (u^κ) converges to a unique very singular solution Y , see [25], [10], [8], [12]. And $Y(x, t) = t^{-a/2}F(|x|/\sqrt{t})$, where

$$a = \frac{2-q}{q-1}, \quad (1.7)$$

and F is bounded and has an exponential decay at infinity.

In [13, Theorem 2.2] it is shown that for any $R \geq 1$ global regularizing L^R - L^∞ properties of two types hold for the Cauchy problem in $Q_{\mathbb{R}^N, T}$: one due to the heat operator:

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2R}}\|u_0\|_{L^R(\mathbb{R}^N)}, \quad C = C(N, R, \nu), \quad (1.8)$$

and the other due to the gradient term, independent of ν ($\nu > 0$):

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{N}{qR+N(q-1)}}\|u_0\|_{\frac{qR}{L^R(\mathbb{R}^N)}}, \quad C = C(N, q, R). \quad (1.9)$$

A great part of the results has been extended to the Dirichlet problem in a bounded domain Ω :

$$(P_{\Omega, T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega, T}, \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0, \end{cases} \quad (1.10)$$

where $u_0 \in \mathcal{M}_b^+(\Omega)$, and $u(\cdot, t)$ converges to u_0 weakly in $\mathcal{M}_b^+(\Omega)$, see [6], [26], [12], [13]. Universal estimates are given in [16], see also [12]. Note that (1.5) cannot hold, since it contradicts the Höpf Lemma.

Finally local estimates in any domain Ω were proved in [26]: for any classical solution u in $Q_{\Omega, T}$ and any ball $B(x_0, 2\eta) \subset \Omega$, there holds in $Q_{B(x_0, \eta), T}$

$$|\nabla u|(\cdot, t) \leq C(t^{-\frac{1}{q}} + \eta^{-1} + \eta^{-\frac{1}{q-1}})(1 + u(\cdot, t)), \quad C = C(N, q, \nu). \quad (1.11)$$

1.1 Main results

In Section 3 we give *local integral estimates* of the solutions *in terms of the initial data*, and a *first regularizing effect*, local version of (1.8), see Theorem 3.3.

Theorem 1.1 *Let $q > 1$. Let u be any nonnegative weak solution of equation (1.1) in $Q_{\Omega, T}$, and let $B(x_0, 2\eta) \subset \subset \Omega$ such that u has a trace $u_0 \in L_{loc}^R(\Omega)$, $R \geq 1$ and $u \in C([0, T]; L_{loc}^R(\Omega))$. Then for any $0 < t \leq \tau < T$,*

$$\sup_{x \in B(x_0, \eta/2)} u(x, t) \leq Ct^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0, \eta))}), \quad C = C(N, q, \nu, R, \eta, \tau).$$

If $R = 1$, the estimate remains true when $u_0 \in \mathcal{M}^+(\Omega)$ (with $\|u_0\|_{L^1(B(x_0, \eta))}$ replaced by $\int_{B(x_0, \eta)} du_0$).

In Section 4, we give *global estimates* of the solutions of (1.1) in $Q_{\mathbb{R}^N, T}$, and this is our main result. We show that *the universal estimate (1.5) in \mathbb{R}^N holds without assuming that the solutions are initially bounded*:

Theorem 1.2 *Let $q > 1$. Let u be any classical solution, in particular **any weak solution** if $q \leq 2$, of equation (1.1) in $Q_{\mathbb{R}^N, T}$. Then*

$$|\nabla u(\cdot, t)|^q \leq \frac{1}{q-1} \frac{u(\cdot, t)}{t}, \quad \text{in } Q_{\mathbb{R}^N, T}. \quad (1.12)$$

And we prove that *the growth of the solutions is limited, in $|x|^{q'}$ as $|x| \rightarrow \infty$ and in $t^{-1/(q-1)}$ as $t \rightarrow 0$* :

Theorem 1.3 *Let $q > 1$. Let u be any classical solution, in particular **any weak solution** if $q \leq 2$, of equation (1.1) in $Q_{\mathbb{R}^N, T}$, such that there exists a ball $B(x_0, 2\eta)$ such that u has a trace $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$. Then*

$$u(x, t) \leq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \int_{B(x_0, \eta)} du_0), \quad C = C(N, q, \eta). \quad (1.13)$$

In [14], we show that there exist solutions with precisely this type of behaviour of order $t^{-1/(q-1)} |x|^{q'}$ as $|x| \rightarrow \infty$ or $t \rightarrow 0$. Moreover we prove that the condition on the trace is always satisfied for $q < q_*$.

In Section 5 we complete the study by giving *existence results* with only *local assumptions* on u_0 , extending some results of [5] where u_0 is continuous:

Theorem 1.4 *Let $\Omega = \mathbb{R}^N$ (resp. Ω bounded).*

(i) *If $1 < q < q_*$, then for any $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$ (resp. $\mathcal{M}^+(\Omega)$), there exists a weak solution u of equation (1.1) (resp. of $(D_{\Omega, T})$) with trace u_0 .*

(ii) *If $q_* \leq q \leq 2$, then existence still holds for any nonnegative $u_0 \in L^1_{loc}(\mathbb{R}^N)$ (resp. $L^1_{loc}(\Omega)$).*

(iii) *If $q > 2$, existence holds for any nonnegative $u_0 \in L^1_{loc}(\mathbb{R}^N)$ (resp. $L^1_{loc}(\Omega)$) which is limit of a nondecreasing sequence of continuous functions.*

Our proof of (ii) (iii) is based on approximations by nonincreasing sequences, following the methods of [11], [13]. Another proof can be obtained when $u_0 \in L^1_{loc}(\mathbb{R}^N)$ and $q \leq 2$, by techniques of equiintegrability, see [22] for a connected problem.

Moreover we give a result of *nonuniqueness* of weak solutions in case $q > 2$:

Theorem 1.5 *Assume that $q > 2$, $N \geq 2$. Then the Cauchy problem $(P_{\mathbb{R}^N, \infty})$ with initial data*

$$\tilde{U}(x) = \tilde{C} |x|^{|a|} \in C(\mathbb{R}^N), \quad \tilde{C} = \frac{q-1}{q-2} \left(\frac{(N-1)q-N}{q-1} \right)^{\frac{1}{q-1}},$$

admits at least two weak solutions: the stationary solution \tilde{U} , and a radial self-similar solution of the form

$$U_{\tilde{C}}(x, t) = t^{|a|/2} f(|x|/\sqrt{t}), \quad (1.14)$$

where f is increasing on $[0, \infty)$, $f(0) > 0$, and $\lim_{\eta \rightarrow \infty} \eta^{-|a|/2} f(\eta) = \tilde{C}$.

Finally we give in Section 6a second type of regularizing effects giving a local version of (1.9).

Theorem 1.6 *Let $q > 1$, and let u be any nonnegative classical solution (resp. any weak solution if $q \leq 2$) of equation (1.1) in $Q_{\Omega,T}$, and let $B(x_0, 2\eta) \subset \Omega$. Assume that $u_0 \in L_{loc}^R(\Omega)$ for some $R \geq 1$, $R > q - 1$, and $u \in C([0, T]; L_{loc}^R(\Omega))$. Then for any $\varepsilon > 0$, and for any $\tau \in (0, T)$, then there exists $C = C(N, q, R, \eta, \varepsilon, \tau)$ such that*

$$\sup_{B_{\eta/2}} u(\cdot, t) \leq Ct^{-\frac{N}{qR+N(q-1)}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} + Ct^{\frac{1-\varepsilon}{R+1-q}} \|u_0\|_{L^R(B_\eta)}^{\frac{R}{R+1-q}}. \quad (1.15)$$

If $q < 2$, the estimates for $R = 1$ are also valid when u has a trace $u_0 \in \mathcal{M}^+(\Omega)$, with $\|u_0\|_{L^1(B_\eta)}$ replaced by $\int_{B_\eta} du_0$.

In conclusion, note that a part of our results could be extended to more general quasilinear operators, for example to the case of equation involving the p -Laplace operator

$$u_t - \nu \Delta_p u + |\nabla u|^q = 0$$

with $p > 1$, following the results of [13], [4], [21], [19].

2 Classical and weak solutions

We set $Q_{\Omega,s,\tau} = \Omega \times (s, \tau)$, for any $0 \leq s < \tau \leq \infty$, thus $Q_{\Omega,T} = Q_{\Omega,0,T}$.

Definition 2.1 *Let $q > 1$ and Ω be any domain of \mathbb{R}^N . We say that a nonnegative function u is a **classical** solution of (1.1) in $Q_{\Omega,T}$ if $u \in C^{2,1}(Q_{\Omega,T})$. We say that u is a **weak solution** (resp. weak subsolution) of (1.1) in $Q_{\Omega,T}$, if $u \in C((0, T); L_{loc}^1(Q_{\Omega,T})) \cap L_{loc}^1((0, T); W_{loc}^{1,1}(\Omega))$, $|\nabla u|^q \in L_{loc}^1(Q_{\Omega,T})$ and u satisfies (1.1) in the distribution sense:*

$$\int_0^T \int_{\Omega} (-u\varphi_t - \nu u \Delta \varphi + |\nabla u|^q \varphi) = 0, \quad \forall \varphi \in \mathcal{D}(Q_{\Omega,T}), \quad (2.1)$$

(resp.

$$\int_0^T \int_{\Omega} (-u\varphi_t - \nu u \Delta \varphi + |\nabla u|^q \varphi) \leq 0, \quad \forall \varphi \in \mathcal{D}^+(Q_{\Omega,T}).) \quad (2.2)$$

And then for any $0 < s < t < T$, and any $\varphi \in C^1((0, T), C_c^1(\Omega))$,

$$\int_{\Omega} (u\varphi)(\cdot, t) - \int_{\Omega} (u\varphi)(\cdot, s) + \int_s^t \int_{\Omega} (-u\varphi_t + \nu \nabla u \cdot \nabla \varphi + |\nabla u|^q \varphi) = 0 \quad (\text{resp. } \leq 0). \quad (2.3)$$

Remark 2.2 *Any weak subsolution u is locally bounded in $Q_{\Omega,T}$. Indeed, since u is ν -subcaloric, there holds for any ball $B(x_0, \rho) \subset \subset \Omega$ and any $\rho^2 \leq t < T$,*

$$\sup_{B(x_0, \frac{\rho}{2}) \times [t - \frac{\rho^2}{4}, t]} u \leq C(N, \nu) \rho^{-(N+2)} \int_{t - \frac{\rho^2}{2}}^t \int_{B(x_0, \rho)} u. \quad (2.4)$$

Any nonnegative function $u \in L_{loc}^1(Q_{\Omega,T})$, such that $|\nabla u|^q \in L_{loc}^1(Q_{\Omega,T})$, and u satisfies (2.1), is a weak solution and $|\nabla u| \in L_{loc}^2(Q_{\Omega,T})$, $u \in C((0, T); L_{loc}^s(Q_{\Omega,T}))$, $\forall s \geq 1$, see [12, Lemma 2.4].

Next we recall the regularity of the weak solutions of (1.1) for $q \leq 2$, see [12, Theorem 2.9], [13, Corollary 5.14]:

Theorem 2.3 *Let $1 < q \leq 2$. Let Ω be any domain in \mathbb{R}^N . Let u be any weak nonnegative solution of (1.1) in $Q_{\Omega,T}$. Then $u \in C_{loc}^{2+\gamma,1+\gamma/2}(Q_{\Omega,T})$ for some $\gamma \in (0,1)$, and for any smooth domains $\omega \subset\subset \omega' \subset\subset \Omega$, and $0 < s < \tau < T$, $\|u\|_{C^{2+\gamma,1+\gamma/2}(Q_{\omega',s,\tau})}$ is bounded in terms of $\|u\|_{L^\infty(Q_{\omega',s/2,\tau})}$. Thus for any sequence (u_n) of nonnegative weak solutions of equation (1.1) in $Q_{\Omega,T}$, uniformly locally bounded, one can extract a subsequence converging in $C_{loc}^{2,1}(Q_{\Omega,T})$ to a weak solution u of (1.1) in $Q_{\Omega,T}$.*

Remark 2.4 *Let $q > 1$. From the estimates (1.11), for any sequence of classical nonnegative solutions (u_n) of (1.1) in $Q_{\Omega,T}$, uniformly bounded in $L_{loc}^\infty(Q_{\Omega,T})$, one can extract a subsequence converging in $C_{loc}^{2,1}(Q_{\mathbb{R}^N,T})$ to a classical solution u of (1.1).*

Remark 2.5 *Let us mention some results of concerning the trace, valid for any $q > 1$, see [12, Lemma 2.14]. Let u be any nonnegative weak solution u of (1.1) in $Q_{\Omega,T}$. Then u has a trace u_0 in $\mathcal{M}^+(\Omega)$ if and only if $u \in L_{loc}^\infty([0,T]; L_{loc}^1(\Omega))$, and if and only if $|\nabla u|^q \in L_{loc}^1(\Omega \times [0,T])$. And then for any $t \in (0,T)$, and any $\varphi \in C_c^1(\Omega \times [0,T])$, and any $\zeta \in C_c^1(\Omega)$,*

$$\int_{\Omega} u(\cdot, t) \varphi dx + \int_0^t \int_{\Omega} (-u \varphi_t + \nu \nabla u \cdot \nabla \varphi + |\nabla u|^q \varphi) = \int_{\Omega} \varphi(\cdot, 0) du_0, \quad (2.5)$$

$$\int_{\Omega} u(\cdot, t) \zeta + \int_0^t \int_{\Omega} (\nu \nabla u \cdot \nabla \zeta + |\nabla u|^q \zeta) = \int_{\Omega} \zeta du_0. \quad (2.6)$$

If $u_0 \in L_{loc}^1(\Omega)$, then $u \in C([0,T]; L_{loc}^1(\Omega))$.

Finally we consider the Dirichlet problem in a smooth bounded domain Ω :

$$(D_{\Omega,T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \quad (2.7)$$

Definition 2.6 *We say that a function u is a **weak solution** of $(D_{\Omega,T})$ if it is a weak solution of equation (1.1) such that $u \in C((0,T); L^1(\Omega)) \cap L_{loc}^1((0,T); W_0^{1,1}(\Omega))$, and $|\nabla u|^q \in L_{loc}^1((0,T); L^1(\Omega))$. We say that u is a **classical solution** of $(D_{\Omega,T})$ if $u \in C^{2,1}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0,T))$.*

3 Local integral properties and first regularizing effect

3.1 Local integral properties

Lemma 3.1 *Let Ω be any domain in \mathbb{R}^N , $q > 1, R \geq 1$. Let u be any nonnegative weak subsolution of equation (1.1) in $Q_{\Omega,T}$, such that $u \in C((0,T); L_{loc}^R(\Omega))$. Let $\xi \in C^1((0,T); C_c^1(\Omega))$, with values in $[0,1]$. Let $\lambda > 1$. Then there exists $C = C(q, R, \lambda)$, such that, for any $0 < s < t \leq \tau < T$,*

$$\begin{aligned} & \int_{\Omega} u^R(\cdot, t) \xi^\lambda + \frac{1}{2} \int_s^\tau \int_{\Omega} u^{R-1} |\nabla u|^q \xi^\lambda + \nu \frac{R-1}{2} \int_s^\tau \int_{\Omega} u^{R-2} |\nabla u|^2 \xi^\lambda \\ & \leq \int_{\Omega} u^R(\cdot, s) \xi^\lambda + \lambda R \int_s^t \int_{\Omega} u^R \xi^{\lambda-1} |\xi_t| + C \int_s^t \int_{\Omega} u^{R-1} \xi^{\lambda-q'} |\nabla \xi|^{q'}. \end{aligned} \quad (3.1)$$

Proof. (i) Let $R = 1$. Taking $\varphi = \xi^\lambda$ in (2.3), we obtain, since $\nu \leq 1$,

$$\begin{aligned} \int_{\Omega} u(\cdot, t) \xi^\lambda + \int_s^t \int_{\Omega} |\nabla u|^q \xi^\lambda &\leq \int_{\Omega} u(s, \cdot) \xi^\lambda + \lambda \int_s^t \int_{\Omega} \xi^{\lambda-1} u \xi_t + \lambda \nu \int_s^t \int_{\Omega} \xi^{\lambda-1} \nabla u \cdot \nabla \xi \\ &\leq \int_{\Omega} u(\cdot, s) \xi^\lambda + \lambda \int_s^t \int_{\Omega} \xi^{\lambda-1} u |\xi_t| + \frac{1}{2} \int_s^t \int_{\Omega} |\nabla u|^q \xi^{q'} + C(q, \lambda) \int_s^t \int_{\Omega} \xi^{\lambda-q'} |\nabla \xi|^{q'}, \end{aligned}$$

hence (3.1) follows.

(ii) Next assume $R > 1$. Consider $u_{\delta, n} = ((u + \delta) * \varphi_n)$, where (φ_n) is a sequence of mollifiers, and $\delta > 0$. Then by convexity, $u_{\delta, n}$ is also a subsolution of (1.1):

$$(u_{\delta, n})_t - \nu \Delta u_{\delta, n} + |\nabla u_{\delta, n}|^q \leq 0.$$

Multiplying by $u_{\delta, n}^{R-1} \xi^\lambda$ and integrating between s and t , and going to the limit as $\delta \rightarrow 0$ and $n \rightarrow \infty$, see [13], we get with different constants $C = (N, q, R, \lambda)$, independent of ν ,

$$\begin{aligned} &\frac{1}{R} \int_{\Omega} u^R(\cdot, t) \xi^\lambda + \nu(R-1) \int_s^t \int_{\Omega} u^{R-2} |\nabla u|^2 \xi^\lambda + \int_s^t \int_{\Omega} u^{R-1} |\nabla u|^q \xi^\lambda \\ &\leq \frac{1}{R} \int_{\Omega} u^R(\cdot, s) \xi^\lambda + \lambda \int_s^t \int_{B_\rho} \xi^{\lambda-1} u^R |\xi_t| + \lambda \nu \int_s^t \int_{\Omega} u^{R-1} |\nabla u| |\nabla \xi| \xi^{\lambda-1} \\ &\leq \frac{1}{R} \int_{\Omega} u^R(\cdot, s) \xi^\lambda + \lambda \int_s^t \int_{B_\rho} \xi^{\lambda-1} u^R |\xi_t| \\ &\quad + \frac{1}{2} \int_s^t \int_{\Omega} u^{R-1} |\nabla u|^q \xi^\lambda + C(\lambda, R) \int_s^t \int_{\Omega} u^{R-1} \xi^{\lambda-q'} |\nabla \xi|^{q'}, \end{aligned}$$

and (3.1) follows again. ■

Then we give local integral estimates of $u(\cdot, t)$ in terms of the initial data:

Lemma 3.2 *Let $q > 1$. Let $\eta > 0$. Let u be any nonnegative weak solution of equation (1.1) in $Q_{\Omega, T}$, with trace $u_0 \in \mathcal{M}^+(\Omega)$, and let $B(x_0, 2\eta) \subset\subset \Omega$. Then for any $t \in (0, T)$,*

$$\int_{B(x_0, \eta)} u(x, t) \leq C(N, q) \eta^{N-q'} t + \int_{B(x_0, 2\eta)} du_0. \quad (3.2)$$

Moreover if $u_0 \in L_{loc}^R(\Omega)$ ($R > 1$), and $u \in C([0, T]; L_{loc}^R(\Omega))$, then

$$\|u(\cdot, t)\|_{L^R(B(x_0, \eta))} \leq C(N, q, R) \eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B(x_0, 2\eta))}. \quad (3.3)$$

If $u \in C(\overline{B(x_0, 2\eta)} \times [0, T])$, then

$$\|u(\cdot, t)\|_{L^\infty(B(x_0, \eta))} \leq C(N, q) \eta^{-q'} t + \|u_0\|_{L^\infty(B(x_0, 2\eta))}. \quad (3.4)$$

Proof. We can assume that $0 \in \Omega$ and $x_0 = 0$. We take $\xi \in C_c^1(\Omega)$, independent of t , with values in $[0, 1]$, and $R = 1$ in (3.1), $\lambda = q'$. Then for any $0 < s < t < T$,

$$\int_{\Omega} u(\cdot, t) \xi^{q'} + \frac{1}{2} \int_s^t \int_{\Omega} |\nabla u|^q \xi^{q'} \leq \int_{\Omega} u(\cdot, s) \xi^{q'} + C(q) \int_s^t \int_{\Omega} |\nabla \xi|^{q'} \leq \int_{\Omega} u(\cdot, s) \xi^{q'} + C(q) t \int_{\Omega} |\nabla \xi|^{q'}.$$

Hence as $s \rightarrow 0$, we get

$$\int_{\Omega} u(\cdot, t) \xi^{q'} + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla u|^q \xi^{q'} \leq C(q)t \int_{\Omega} |\nabla \xi|^{q'} + \int_{\Omega} \xi^{q'} du_0. \quad (3.5)$$

Then taking $\xi = 1$ in B_{η} with support in $B_{2\eta}$ and $|\nabla \xi| \leq C_0(N)/\eta$,

$$\int_{B_{\eta}} u(x, t) \leq C(N, q) \eta^{N-q'} t + \int_{B_{2\eta}} \xi^{q'} du_0, \quad (3.6)$$

hence we get (3.2). Next assume $u_0 \in L_{loc}^R(\Omega)$ ($R > 1$), and $u \in C([0, T]; L_{loc}^R(\Omega))$. Then from (3.1), for any $0 < s < t \leq \tau < T$, we find,

$$\begin{aligned} \int_{\Omega} u^R(\cdot, t) \xi^{\lambda} + \frac{1}{2} \int_s^{\tau} \int_{\Omega} u^{R-1} |\nabla u|^q \xi^{\lambda} &\leq \int_{\Omega} u^R(\cdot, s) \xi^{\lambda} + \int_s^t \int_{\Omega} u^{R-1} \xi^{\lambda-q'} |\nabla \xi|^{q'} \\ &\leq \int_{\Omega} u^R(\cdot, s) \xi^{\lambda} + \varepsilon \int_s^t \int_{B_{2\eta}} u^R \xi^{\lambda} + \varepsilon^{1-R} \int_s^t \int_{B_{2\eta}} \xi^{\lambda-Rq'} |\nabla \xi|^{Rq'}. \end{aligned}$$

Taking $\lambda = Rq'$, and ξ as above, we find

$$\int_{B_{2\eta}} u^R(\cdot, t) \xi^{Rq'} \leq \int_{B_{2\eta}} u^R(\cdot, s) \xi^{Rq'} + \varepsilon \int_s^t \int_{B_{2\eta}} u^R \xi^{Rq'} + \varepsilon^{1-R} C(N) C_0^{Rq'}(N) \eta^{N-Rq'} t.$$

Next we set $\varpi(t) = \sup_{\sigma \in [s, t]} \int_{B_{2\eta}} u^R(\cdot, \sigma) \xi^{Rq'}$. Then

$$\varpi(t) \leq \int_{B_{2\eta}} u^R(\cdot, s) \xi^{Rq'} + \varepsilon(t-s) \varpi(t) + \varepsilon^{1-R} C(N) C_0^{Rq'}(N) \eta^{N-Rq'} t.$$

Taking $\varepsilon = 1/2t$, we get

$$\frac{1}{2} \int_{B_{2\eta}} u^R(\cdot, t) \xi^{Rq'} \leq \int_{B_{2\eta}} u^R(\cdot, s) \xi^{Rq'} + C(N) C_0^{Rq'}(N) \eta^{N-Rq'} t^R.$$

Then going to the limit as $s \rightarrow 0$,

$$\int_{B_{\eta}} u^R(x, t) \leq C(N) C_0^{Rq'}(N) \eta^{N-Rq'} t^R + \int_{B_{2\eta}} u_0^R \xi^{Rq'}, \quad (3.7)$$

thus (3.3) follows.

If $u \in C(\overline{B_{2\rho}} \times [0, T])$, then (3.7) holds for any $R \geq 1$, implying

$$\|u(\cdot, t)\|_{L^R(B_{\eta})} \leq C^{\frac{1}{R}}(N) C_0^{q'}(N) \eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_{2\eta})},$$

and (3.3) follows as $R \rightarrow \infty$. ■

3.2 Regularizing effect of the heat operator

We first give a first regularizing effect due to the Laplace operator in $Q_{\Omega,T}$, for any domain Ω , for classical or weak solutions in terms of the initial data.

Theorem 3.3 *Let $q > 1$. Let u be any nonnegative weak subsolution of equation (1.1) in $Q_{\Omega,T}$, and let $B(x_0, 2\eta) \subset \Omega$ such that u has a trace $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$. Then for any $\tau < T$, and any $t \in (0, \tau]$,*

$$\sup_{x \in B(x_0, \eta/2)} u(x, t) \leq Ct^{-\frac{N}{2}} \left(t + \int_{B(x_0, \eta)} du_0 \right), \quad C = C(N, q, \nu, \eta, \tau). \quad (3.8)$$

Moreover if $u_0 \in L_{loc}^R(\Omega)$ ($R > 1$), and $u \in C([0, T]; L_{loc}^R(\Omega))$, then

$$\sup_{x \in B(x_0, \eta/2)} u(x, t) \leq Ct^{-\frac{N}{2R}} \left(t + \|u_0\|_{L^R(B(x_0, \eta))} \right), \quad C = C(N, q, \nu, R, \eta, \tau). \quad (3.9)$$

Proof. We still assume that $x_0 = 0 \in \Omega$. Let $\xi \in C_c^1(B_{2\eta})$ be nonnegative, radial, with values in $[0, 1]$, with $\xi = 1$ on B_η and $|\nabla \xi| \leq C_0(N)/\eta$. Since u is ν -subcaloric, from (2.4), for any $\rho \in (0, \eta)$ such that $\rho^2 \leq t < \tau$,

$$\sup_{B_{\eta/2}} u(\cdot, t) \leq C(N, \nu) \rho^{-(N+2)} \int_{t-\rho^2/4}^t \int_{B_\eta} u, \quad (3.10)$$

hence from Lemma 3.2,

$$\sup_{B_{\eta/2}} u(\cdot, t) \leq C(N, q, \nu) \rho^{-N} (\eta^{N-q'} t + \int_{B_{2\eta}} du_0).$$

Let $k_0 \in \mathbb{N}$ such that $k_0 \eta^2/2 \geq \tau$. For any $t \in (0, \tau]$, there exists $k \in \mathbb{N}$ with $k \leq k_0$ such that $t \in (k\eta^2/2, (k+1)\eta^2/2]$. Taking $\rho^2 = t/(k+1)$, we find

$$\begin{aligned} \sup_{B_{\eta/2}} u(\cdot, t) &\leq C(N, q, \nu) (k_0 + 1)^{\frac{N}{2}} t^{-\frac{N}{2}} (\eta^{N-q'} t + \int_{B_{2\eta}} du_0) \\ &\leq C(N, q, \nu) (\eta^{-N} \tau^{\frac{N}{2}} + 1) t^{-\frac{N}{2}} (\eta^{N-q'} t + \int_{B_{2\eta}} du_0). \end{aligned} \quad (3.11)$$

Thus we obtain (3.8). Next assume that $u \in C([0, T]; L_{loc}^R(B_{2\eta}))$, with $R > 1$. We still approximate u by $u_{\delta, n} = (u + \delta) * \varphi_n$, where (φ_n) is a sequence of mollifiers, and $\delta > 0$. Since u is ν -subcaloric, then $u_{\delta, n}^R$ is also ν -subcaloric. Then for any $\rho \in (0, \eta)$ such that $\rho^2 \leq t < \tau$, we have

$$\sup_{B_{\eta/2}} u_{\delta, n}^R(\cdot, t) \leq C(N, \nu) \rho^{-(N+2)} \int_{t-\rho^2/4}^t \int_{B_{\rho/2}} u_{\delta, n}^R,$$

hence as $\delta \rightarrow 0$ and $n \rightarrow \infty$, from Lemma (3.2),

$$\sup_{B_{\eta/2}} u^R(\cdot, t) \leq C(N, \nu) \rho^{-(N+2)} \int_{t-\rho^2/4}^t \int_{B_{\rho/2}} u^R \leq C(N, q, \nu, R) (\eta^{-N} \tau^{\frac{N}{2}} + 1) (\eta^{N-Rq'} t^R + \int_{B_{2\eta}} u_0^R). \quad (3.12)$$

We deduce (3.9) as above. \blacksquare

4 Global estimates in \mathbb{R}^N

We first show that the universal estimate of the gradient (1.12) implies the estimate (1.13) of the function:

Theorem 4.1 *Let $q > 1$. Let u be a classical solution of equation (1.1) in $Q_{\mathbb{R}^N, T}$. Assume that there exists a ball $B(x_0, 2\eta)$ such that u has a trace $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$. If u satisfies (1.12), then for any $t \in (0, T)$,*

$$u(x, t) \leq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \int_{B(x_0, \eta)} du_0), \quad C = C(N, q, \eta), \quad (4.1)$$

If $u_0 \in L_{loc}^R(\Omega)$, $R \geq 1$ and $u \in C([0, T]; L_{loc}^R(\Omega))$, then

$$u(x, t) \leq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + Ct^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0, \eta))}), \quad C = C(N, q, R, \nu, \eta). \quad (4.2)$$

$$u(x, t) \leq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \|u_0\|_{L^R(B(x_0, \eta))}), \quad C = C(N, q, R, \eta). \quad (4.3)$$

Proof. Estimate (1.12) is equivalent to

$$\left| \nabla(u^{\frac{1}{q'}}) \right|(\cdot, t) \leq \frac{(q-1)^{\frac{1}{q'}}}{q} t^{-\frac{1}{q}}, \quad \text{in } Q_{\mathbb{R}^N, T}. \quad (4.4)$$

Then with constants $C(q)$ only depending of q ,

$$u^{\frac{1}{q'}}(x, t) \leq u^{\frac{1}{q'}}(x_0, t) + C(q)t^{-\frac{1}{q}} |x - x_0|, \quad (4.5)$$

then

$$u(x, t) \leq C(q)(u(x_0, t) + t^{-\frac{1}{q-1}} |x - x_0|^{q'}), \quad (4.6)$$

and, from Theorem 3.3,

$$u(x_0, t) \leq C(N, q, R, \nu, \eta)t^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0, \eta))}).$$

Therefore (4.2) follows. Also, interverting x and x_0 , for any $R \geq 1$,

$$u^R(x_0, t) \leq C(q, R)(u^R(x, t) + t^{-\frac{R}{q-1}} |x - x_0|^{Rq'}).$$

Integrating on $B(x_0, \eta/2)$, we get

$$\eta^N u^R(x_0, t) \leq C(q, R) \left(\int_{B(x_0, \eta/2)} u^R(\cdot, t) + t^{-\frac{R}{q-1}} \eta^{N-Rq'} \right);$$

using Lemma 3.2, we deduce

$$u(x_0, t) \leq C(N, q, R, \eta) \left(t^{-\frac{1}{q-1}} + t + \int_{B(x_0, \eta)} du_0 \right),$$

and if $u_0 \in L_{loc}^R(\Omega)$,

$$u(x_0, t) \leq C(N, q, R, \eta) \left(t^{-\frac{1}{q-1}} + t + \|u_0\|_{L^R(B(x_0, \eta))} \right),$$

and the conclusions follow from (4.6). ■

Remark 4.2 *In particular, the estimates (4.1)-(4.3) hold for solutions with $u_0 \in C_b(\mathbb{R}^N)$, and more generally for limits a.e. of such solutions, that we can call **reachable** solutions. Inequality (4.5) was used in [5, Theorem 3.3] for obtaining local estimates of classical of bounded solutions in $Q_{\mathbb{R}^N, T}$.*

In order to prove Theorem 1.2, we first give an estimate of the type of (1.13) on a time interval $(0, \tau]$, with constants depending on τ and ν , which is not obtained from any estimate of the gradient. Our result is based on the construction of suitable supersolutions in annulus of type $Q_{B_{3\rho} \setminus \overline{B_\rho}, \infty}$, $\rho > 0$. For the construction we consider the function $t \in (0, \infty) \mapsto \psi_h(t) \in (1, \infty)$, where $h > 0$ is a parameter, solution of the problem

$$(\psi_h)_t + h(\psi_h^q - \psi_h) = 0 \quad \text{in } (0, \infty), \quad \psi_h(0) = \infty, \quad \psi_h(\infty) = 1, \quad (4.7)$$

given explicitly by $\psi_h(t) = (1 - e^{-h(q-1)t})^{-\frac{1}{q-1}}$; hence $\psi_h^q - \psi_h \geq 0$, and for any $t > 0$,

$$((q-1)ht)^{-\frac{1}{q-1}} \leq \psi_h(t) \leq 2^{\frac{1}{q-1}}(1 + ((q-1)ht)^{-\frac{1}{q-1}}). \quad (4.8)$$

since, for $x > 0$, $x(1 - x/2) \leq 1 - e^{-x} \leq x$, hence $x/2 \leq 1 - e^{-x} \leq x$, for $x \leq 1$.

Proposition 4.3 *Let $q > 1$. Then there exists a nonnegative function V defined in $Q_{B_3 \times (0, \infty)}$, such that V is a supersolution of equation (1.1) on $Q_{B_3 \setminus \overline{B_1}, \infty}$, and V converges to ∞ as $t \rightarrow 0$, uniformly on B_3 and converges to ∞ as $x \rightarrow \partial B_3$, uniformly on $(0, \tau)$ for any $\tau < \infty$. And V has the form*

$$V(x, t) = e^t \Phi(|x|) \psi_h(t) \quad \text{in } Q_{B_3, \infty} \quad (4.9)$$

for some $h = h(N, q, \nu) > 0$, where ψ_h is given by (4.7), and Φ is a suitable radial function depending on N, q, ν , such that

$$-\nu \Delta \Phi + \Phi + |\nabla \Phi|^q \geq 0 \quad \text{in } B_3. \quad (4.10)$$

Proof. We first construct Φ . Let $\sigma > 0$, such that $\sigma \geq a = (2 - q)/(q - 1)$. Let φ_1 be the first eigenfunction of the Laplacian in B_3 such that $\varphi_1(0) = 1$, associated to the first eigenvalue λ_1 , hence φ_1 is radial; let $m_1 = \min_{\overline{B_1}} \varphi_1 > 0$ and $M_1 = \min_{\overline{B_3} \setminus B_1} |\nabla \varphi_1|$. Let us take $\Phi = \Phi_K = \Phi_0 + K$, where $\Phi_0 = \gamma \varphi_1^{-\sigma}$, $K > 0$ and $\gamma > 0$ are parameters. Then

$$-\nu \Delta \Phi + \Phi + |\nabla \Phi|^q = F(\Phi_0) + K, \quad \text{with}$$

$$F(\Phi_0) = \gamma \varphi_1^{-(\sigma+2)} (\gamma^{q-1} \sigma^q \varphi_1^{(q-1)(a-\sigma)} |\varphi_1'|^q + (1 - \nu \sigma \lambda_1) \varphi_1^2 - \nu \sigma (\sigma + 1) \varphi_1'^2).$$

There holds $\lim_{r \rightarrow 3} |\varphi_1'| = c_1 > 0$ from the Höpf Lemma. Taking $\sigma > a$ we fix $\gamma = 1$, and then $\lim_{r \rightarrow 3} F(\Phi_0) = \infty$. If $q < 2$ we can also take $\sigma = a$, we get

$$F(\Phi_0) = \gamma \varphi_1^{-q} (\gamma^{q-1} a^q |\varphi_1'|^q + (1 - \nu a \lambda_1) \varphi_1^2 - a q' \varphi_1'^2),$$

hence fixing $\gamma > \gamma(N, q, \nu)$ large enough, we still get $\lim_{r \rightarrow 3} F(\Phi_0) = \infty$. Thus F has a minimum μ in B_3 . Taking $K = K(N, q, \nu) > |\mu|$ we deduce that Φ satisfies (4.10), and $\lim_{r \rightarrow 3} \Phi = \infty$.

Observe that $\Phi'^q/\Phi = \gamma^q \sigma^q / (\gamma \varphi_1^{q+\sigma(q-1)} + K \varphi_1^{q(\sigma+1)})$ is increasing, then $m_K = m_K(N, q, \nu) = \min_{[1,3]} |\Phi'|^q / \Phi = |\Phi'(1)|^q / \Phi(1) > 0$. We define V by (4.9) and compute

$$\begin{aligned} V_t - \nu \Delta V + |\nabla V|^q &= e^t (\Phi \psi_h + \Phi(\psi_h)_t - \nu \Delta \Phi) + e^{qt} |\nabla \Phi|^q \psi_h^q \\ &\geq e^t (\Phi \psi_h + \Phi \psi_t - \nu \Delta \Phi + |\nabla \Phi|^q \psi^q) = e^t (\psi^q - \psi_h) (|\nabla \Phi|^q - h \Phi). \end{aligned}$$

We take $h = h(N, q, \nu) < m_K$. Then on $B_3 \setminus B_1$ we have $|\nabla \Phi|^q - h \Phi > 0$, and $\psi^q \geq \psi_h$, then V is a supersolution on $B_3 \setminus B_1$. Moreover V is radial and increasing with respect to $|x|$, then

$$\begin{aligned} \sup_{\overline{B_2}} V(x, t) &= \sup_{\partial \overline{B_2}} V(x, t) = e^t \Phi(2) \psi_h(t) \leq 2^{\frac{1}{q-1}} e^t \Phi(2) (1 + ((q-1)ht)^{-\frac{1}{q-1}}) \\ &\leq C(N, q, \nu) e^t \Phi(2) (1 + t^{-\frac{1}{q-1}}). \end{aligned} \quad (4.11)$$

■

Theorem 4.4 *Let u be a classical solution, (in particular any weak solution if $q \leq 2$) of equation (1.1) in $Q_{\mathbb{R}^N, T}$. Assume that there exists a ball $B(x_0, 2\eta)$ such that u admits a trace $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$.*

(i) *Then for any $\tau \in (0, T)$, and $t \leq \tau$,*

$$u(x, t) \leq C(t^{-\frac{1}{q-1}} |x - x_0|^{q'} + t^{-\frac{N}{2}} (t + \int_{B(x_0, \eta)} du_0)), \quad C = C(N, q, \nu, \eta, \tau), \quad (4.12)$$

(ii) *Also if $u \in C([0, T]; L_{loc}^R(B(x_0, 2\eta)))$,*

$$u(x, t) \leq C(t^{-\frac{1}{q-1}} |x - x_0|^{q'} + t^{-\frac{N}{2R}} (t + \|u_0\|_{L^R(B(x_0, \eta))})), \quad C = C(N, q, \nu, R, \eta, \tau), \quad (4.13)$$

if $u \in C([0, T] \times B(x_0, 2\eta))$, then

$$u(x, t) \leq C(t^{-\frac{1}{q-1}} |x - x_0|^{q'} + t + \sup_{B(x_0, \eta)} u_0), \quad C = C(N, q, \nu, \eta, \tau). \quad (4.14)$$

Proof. We use the function V constructed above. We can assume $x_0 = 0$. For any $\rho > 0$, we consider the function V_ρ defined in $B_{3\rho} \times (0, \infty)$ by

$$V_\rho(x, t) = \rho^{-a} V(\rho^{-1}x, \rho^{-2}t).$$

It is a supersolution of the equation (1.1) on $B_{3\rho} \setminus \overline{B_\rho} \times (0, \infty)$, infinite on $\partial B_{3\rho} \times (0, \infty)$ and on $B_{3\rho} \times \{0\}$, and from (4.11)

$$\begin{aligned} \sup_{\overline{B_{2\rho}}} V_\rho(x, t) &= \sup_{\partial \overline{B_{2\rho}}} V_\rho(x, t) \leq C_1(N, q, \nu) \rho^{-a} e^{\frac{t}{\rho^2}} \Phi(2) (1 + \rho^{\frac{2}{q-1}} t^{-\frac{1}{q-1}}) \\ &\leq C_2(N, q, \nu) \rho^{q'} e^{\frac{t}{\rho^2}} (\rho^{-\frac{2}{q-1}} + t^{-\frac{1}{q-1}}). \end{aligned} \quad (4.15)$$

(i) First suppose that $u \in C([0, T] \times \mathbb{R}^N)$. Let $\tau \in (0, T)$, and $C(\tau) = \sup_{Q_{B_\rho, \tau}} u$. Then $w = C(\tau) + V_\rho$ is a supersolution in $Q = (B_{3\rho} \setminus \overline{B_\rho}) \times (0, \tau]$, and from the comparison principle we obtain $u \leq C(\tau) + V_\rho$ in that set. Indeed let $\epsilon > 0$ small enough. Then there exists $\tau_\epsilon < \epsilon$ and

$r_\epsilon \in (3\rho - \epsilon, 3\rho)$, such that $w(\cdot, s) \geq \max_{\overline{B_{3\rho}}} u(\cdot, \epsilon)$ for any $s \in (0, \tau_\epsilon]$, and $w(x, t) \geq \max_{\overline{B_{3\rho} \times [0, \tau]}} u$ for any $t \in (0, \tau]$ and $r_\epsilon \leq |x| < 3\rho$. We compare $u(x, t + \epsilon)$ with $w(x, t + s)$ on $[0, \tau - \epsilon] \times \overline{B_{r_\epsilon}} \setminus \overline{B_\rho}$. And for $|x| = \rho$, we have $u(x, t + \epsilon) \leq C(\tau) \leq w(x, t + s)$. Then $u(\cdot, t + \epsilon) \leq w(\cdot, t + s)$ in $\overline{B_{r_\epsilon}} \setminus \overline{B_\rho} \times (0, \tau - \epsilon]$. As $s, \epsilon \rightarrow 0$, we deduce that $u \leq w$ in Q .

Hence in $\overline{B_{2\rho}} \times (0, \tau)$, we find from (4.15)

$$u \leq C(\tau) + \sup_{\overline{B_{2\rho}}} V_\rho(x, t) \leq C(\tau) + C_2 \rho^{q'} e^{\frac{t}{\rho^2}} (\rho^{-\frac{2}{q-1}} + t^{-\frac{1}{q-1}}). \quad (4.16)$$

Making t tend to τ , this proves that

$$\sup_{Q_{B_{2\rho}, \tau}} u \leq \sup_{Q_{B_\rho, \tau}} u + C_2 \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}})$$

By induction, we get

$$\begin{aligned} \sup_{Q_{B_{2^{n+1}\rho}, \tau}} u &\leq \sup_{Q_{B_{2^n\rho}, \tau}} u + C_2 2^{nq'} \rho^{q'} e^{\frac{\tau}{4^n \rho^2}} ((2^n \rho)^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \\ &\leq \sup_{Q_{B_{2^n\rho}, \tau}} u + C_2 2^{nq'} \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}); \end{aligned}$$

$$\begin{aligned} \sup_{Q_{B_{2^{n+1}\rho}, \tau}} u &\leq \sup_{Q_{B_\rho}} u + C_2 (1 + 2^{q'} + \dots + 2^{nq'}) \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \\ &\leq \sup_{Q_{B_\rho, \tau}} u + C_2 2^{q'} (2^n \rho)^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}). \end{aligned}$$

For any $x \in \mathbb{R}^N$ such that $|x| \geq \rho$, there exists $n \in \mathbb{N}^*$ such that $x \in B_{2^{n+1}\rho} \setminus \overline{B_{2^n\rho}}$, then

$$u(x, \tau) \leq \sup_{Q_{B_\rho, \tau}} u + C_2 2^{q'} |x|^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \quad (4.17)$$

thus

$$\sup_{Q_{\mathbb{R}^N, \tau}} u \leq \sup_{Q_{B_\rho, \tau}} u + C_2 2^{q'} |x|^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}). \quad (4.18)$$

(ii) Next we consider any classical solution u in $Q_{\mathbb{R}^N, T}$ with trace u_0 in $B(x_0, 2\eta)$. We still assume $x_0 = 0$. Then for $0 < \epsilon \leq t \leq \tau$, from (3.4) in Lemma 3.2, there holds

$$\sup_{B_{\eta/2}} u(x, t) \leq C(N, q) \eta^{-q'} t + \sup_{B_\eta} u(x, \epsilon).$$

Then from (4.18) with $\rho = \eta/2$, we deduce that for any $(x, t) \in Q_{\mathbb{R}^N, \epsilon, \tau}$,

$$u(x, t) \leq C(N, q) \eta^{-q'} t + \sup_{B_{\eta/2}} u(\cdot, \epsilon) + C(1 + (t - \epsilon)^{-\frac{1}{q-1}}) |x|^{q'},$$

with $C = C(N, q, \nu, \eta, \tau)$. Next we take $\epsilon = t/2$. Then for any $t \in (0, \tau]$, from (3.8) in Theorem 3.3,

$$u(x, t) \leq C(N, q, \eta)t + Ct^{-1(q-1)}|x|^{q'} + Ct^{-\frac{N}{2}}(t + \int_{B_\eta} du_0).$$

with $C = C(N, q, \nu, \eta, \tau)$ and we obtain (4.12). And (4.13), (4.14) follow from (3.9) and (3.4). \blacksquare

Next we show our main Theorem 1.2. We use a *local* Bernstein technique, as in [26]. The idea is to compute the equation satisfied by the function $v = u^{(q-1)/q}$, introduced in [9], and the equation satisfied by $w = |\nabla v|^2$, to obtain estimates of w in a cylinder $Q_{B_M, T}$, $M > 0$. The difficulty is that this equation involves an elliptic operator $w \mapsto w_t - \Delta w + b \cdot \nabla w$, where b depends on v , and may be unbounded. However it can be controlled by the estimates of v obtained at Theorem 4.4. Then as $M \rightarrow \infty$, we can prove nonuniversal L^∞ estimates of w . Finally we obtain universal estimates of w by application of the maximum principle in $Q_{\mathbb{R}^N, T}$, valid because w is bounded. First we give a slight improvement of a comparison principle shown in [26, Proposition 2.2].

Lemma 4.5 *Let Ω be any domain of \mathbb{R}^N , and $\tau, \kappa \in (0, \infty)$, $A, B \in \mathbb{R}$. Let $U \in C([0, \tau]; L^2_{loc}(\overline{\Omega}))$ such that $U_t, \nabla u, D^2u \in L^2_{loc}(\overline{\Omega} \times (0, \tau))$, $\text{ess sup}_{Q_{\Omega, \tau}} U < \infty$, $U \leq B$ on the parabolic boundary of $Q_{\Omega, \tau}$, and*

$$U_t - \Delta U \leq \kappa(1 + |x|)|\nabla U| + f \quad \text{in } Q_{\Omega, \tau}$$

where $f = f(x, t)$ such that $f(\cdot, t) \in L^2_{loc}(\overline{\Omega})$ for a.e. $t \in (0, \tau)$ and $f \leq 0$ on $\{(x, t) \in Q_{\Omega, \tau} : U(x, t) \geq A\}$. Then $\text{ess sup}_{Q_{\Omega, \tau}} U \leq \max(A, B)$.

Proof. We set $\varphi(x, t) = \Lambda t + \ln(1 + |x|^2)$, $\Lambda > 0$. Then $\nabla \varphi = 2x/(1 + |x|^2)$, $0 \leq \Delta \varphi \leq 2N/(1 + |x|^2) \leq 2N$. Let $\epsilon > 0$ and $Y = U - \max(A, B) - \epsilon \varphi$. Taking $\Lambda = 2\sqrt{2}\kappa + 2N$, we obtain

$$Y_t - \Delta Y - f - \kappa(1 + |x|)|\nabla Y| \leq \epsilon(K(1 + |x|)|\nabla \varphi| - \varphi_t + \Delta \varphi) \leq \epsilon(2\sqrt{2}\kappa + 2N - \Lambda) = 0.$$

Since $\text{ess sup}_{Q_{\Omega, \tau}} U < \infty$, for R large enough, and any $t \in (0, \tau)$, we have $Y(\cdot, t) \leq 0$ a.e. in $\Omega \cap \{|x| > R\}$. And $Y^+ \in C([0, \tau]; L^2(\Omega)) \cap W^{1,2}((0, \tau); L^2(\Omega))$, $Y^+(0) = 0$ and $Y^+(\cdot, t) \in W^{1,2}(\Omega \cap B_R)$ for a.e. $t \in (0, \tau)$, and $fY^+(\cdot, t) \leq 0$. Then

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} Y^{+2}(\cdot, t) \right) \leq - \int_{\Omega} |\nabla Y^+(\cdot, t)|^2 + \kappa(1 + R) \int_{\Omega} |\nabla Y(\cdot, t)| Y^+(\cdot, t) \leq \frac{\kappa^2(1 + R)^2}{4} \int_{\Omega} Y^{+2}(\cdot, t),$$

hence by integration $Y \leq 0$ a.e. in $Q_{\Omega, \tau}$. We conclude as $\epsilon \rightarrow 0$. \blacksquare

Proof of Theorem 1.2. We can assume $x_0 = 0$. By setting $u(x, t) = \nu^{q'/2} U(x/\sqrt{\nu}, t)$, for proving (4.4) we can suppose that u is a classical solution of (1.1) with $\nu = 1$. We set

$$\delta + u = v^{\frac{q}{q-1}}, \quad \delta \in (0, 1).$$

(i) **Local problem relative to $|\nabla v|^2$.** Here u is any classical solution u of equation (1.1) in a cylinder $Q_{B_M, T}$ with $M > 0$. Then v satisfies the equation

$$v_t - \Delta v = \frac{1}{q-1} \frac{|\nabla v|^2}{v} - cv|\nabla v|^q, \quad c = (q')^{q-1}. \quad (4.19)$$

Setting $w = |\nabla v|^2$, we define

$$\mathcal{L}w = w_t - \Delta w + b \cdot \nabla w, \quad b = (qcvw^{\frac{q-2}{2}} - \frac{2}{q-1} \frac{1}{v}) \nabla v.$$

Differentiating (4.19) and using the identity $\Delta w = 2\nabla(\Delta v) \cdot \nabla v + 2|D^2v|^2$, we obtain the equation

$$\mathcal{L}w + 2cw^{\frac{q+2}{2}} + 2|D^2v|^2 + \frac{2}{q-1} \frac{w^2}{v^2} = 0. \quad (4.20)$$

As in [26], for $s \in (0, 1)$, we consider a test function $\zeta \in C^2(\overline{B}_{3M/4})$ with values in $[0, 1]$, $\zeta = 0$ for $|x| \geq 3M/4$ and $|\nabla \zeta| \leq C(N, s)\zeta^s/M$ and $|\Delta \zeta| + |\nabla \zeta|^2/\zeta \leq C(N, s)\zeta^s/M^2$ in $B_{3M/4}$. We set $z = w\zeta$. We have

$$\mathcal{L}z = \zeta \mathcal{L}w + w \mathcal{L}\zeta - 2\nabla w \cdot \nabla \zeta \leq \zeta \mathcal{L}w + w \mathcal{L}\zeta + |D^2v|^2 \zeta + 4w \frac{|\nabla \zeta|^2}{\zeta}.$$

It follows that in $Q_{B_M, T}$,

$$\mathcal{L}z + 2cw^{\frac{q+2}{2}} \zeta + \frac{2}{q-1} \frac{w^2}{v^2} \zeta \leq \frac{C\zeta^s w}{M^2} + \frac{C\zeta^s w^{\frac{3}{2}}}{M} \left| cqvw^{\frac{q-2}{2}} - \frac{2}{q-1} \frac{1}{v} \right| \leq C\zeta^s \left(\frac{w}{M^2} + \frac{vw^{\frac{q+1}{2}}}{M} + \frac{w^{\frac{3}{2}}}{Mv} \right),$$

with constants $C = C(N, q, s)$. Since $\zeta \leq 1$, from the Young inequality, taking $s \geq \max(q+1, 3)/(q+2)$, for any $\varepsilon > 0$,

$$\frac{C}{M} \zeta^s vw^{\frac{q+1}{2}} = \frac{C}{M} \zeta^{\frac{q+1}{q+2}} \zeta^{s-\frac{q+1}{q+2}} vw^{\frac{q+1}{2}} \leq \varepsilon \zeta w^{\frac{q+2}{2}} + C(N, q, \varepsilon) \frac{v^{q+2}}{M^{q+2}},$$

and

$$\frac{C}{M^2} \zeta^s w \leq \varepsilon \zeta w^{\frac{q+2}{2}} + C(N, q, \varepsilon) \frac{1}{M^{\frac{2(q+2)}{q}}},$$

$$\frac{C}{M} \zeta^s \frac{w^{\frac{3}{2}}}{v} \leq \frac{1}{\delta M} \zeta^s w^{\frac{3}{2}} = \frac{1}{\delta M} \zeta^{s-\frac{3}{q+2}} \zeta^{\frac{3}{q+2}} w^{\frac{3}{2}} \leq \varepsilon \zeta w^{\frac{q+2}{2}} + C(N, q, \varepsilon) \frac{1}{(\delta M)^{\frac{q+2}{q-1}}}.$$

Then with a new $C = C(N, q, \delta)$

$$\mathcal{L}z + cz^{\frac{q+2}{2}} \leq C \left(\frac{v^{q+2}}{M^{q+2}} + \frac{1}{M^{\frac{2(q+2)}{q}}} + \frac{1}{M^{\frac{q+2}{q-1}}} \right). \quad (4.21)$$

(ii) Nonuniversal estimates of w . Here we assume that u is a classical solution of (1.1) in whole $Q_{\mathbb{R}^N, T}$, such that $u \in C(\mathbb{R}^N \times [0, T])$. From Theorem 4.4, for any $\tau \in (0, T)$, there holds in $Q_{\mathbb{R}^N, \tau}$

$$v(x, t) = (\delta + u(x, t))^{\frac{q-1}{q}} \leq C(t^{-\frac{1}{q}} |x| + (t + \sup_{B_{2\eta}} u_0)^{\frac{q-1}{q}}), \quad C = C(N, q, \eta, \tau). \quad (4.22)$$

hence for $M \geq M(q, \sup_{B_{2\eta}} u_0, \tau) \geq 1$, we deduce

$$v(x, t) \leq 2Ct^{-\frac{1}{q}} M, \quad \text{in } Q_{B_M, \tau}.$$

Then with a new constant $C = C(N, q, \eta, \tau, \delta)$, there holds in $Q_{B_{3M/4}, \tau}$

$$\mathcal{L}z + cz^{\frac{q+2}{2}} \leq Ct^{-\frac{q+2}{q}}. \quad (4.23)$$

Next we consider $\Psi(t) = Kt^{-2/q}$. It satisfies

$$\Psi_t + c\Psi^{\frac{q+2}{2}} = (cK^{\frac{q+2}{2}} - 2q^{-1}K)t^{-\frac{q+2}{q}} \geq Ct^{-\frac{q+2}{q}}$$

if $K \geq \bar{K} = \bar{K}(N, q, \eta, \tau, \delta)$. Fixing $\epsilon \in (0, T)$ such that $\tau + \epsilon < T$, there exists $\tau_\epsilon \in (0, \epsilon)$ such that $\Psi(\theta) \geq \sup_{B_M} z(\cdot, \epsilon)$ for any $\theta \in (0, \tau_\epsilon)$. We have

$$\begin{aligned} z_t(\cdot, t + \epsilon) - \Delta z(\cdot, t + \epsilon) + b(\cdot, t + \epsilon) \cdot \nabla(z, t + \epsilon) + cz^{\frac{q+2}{2}}(t + \epsilon) \\ \leq C(t + \epsilon)^{-\frac{q+2}{q}} \leq C(t + \theta)^{-\frac{q+2}{q}} \leq \Psi_t(t + \theta) + c\Psi^{\frac{q+2}{2}}(t + \theta). \end{aligned}$$

Therefore, setting $\tilde{z}(\cdot, t) = z(\cdot, t + \epsilon) - \Psi(t + \theta)$, on the set $\mathcal{V} = \{(x, t) \in Q_{B_{3M/4}, \tau + \epsilon} : \tilde{z}(x, t) \geq 0\}$,

$$\tilde{z}(\cdot, t) - \Delta \tilde{z}(\cdot, t) + b(\cdot, t + \epsilon) \cdot \nabla \tilde{z}(\cdot, t) \leq 0;$$

and $\tilde{z}(\cdot, t) \leq 0$ for sufficiently small $t > 0$, and $\tilde{z} \leq 0$ on $\partial B_{3M/4} \times [0, \tau]$. Then from Lemma 4.5, we get $z(\cdot, t + \epsilon) \leq \Psi(t + \theta)$ in $Q_{B_{3M/4}, \tau}$, since $|b| \leq (qcvw^{\frac{q-1}{2}} + \frac{2}{q-1} \frac{1}{\delta} w^{1/2})$, hence bounded on $Q_{B_{3M/4}, \tau + \epsilon}$. Going to the limit as $\theta, \epsilon \rightarrow 0$, we deduce that $z(\cdot, t) \leq \bar{K}t^{-\frac{2}{q}}$ in $Q_{B_{3M/4}, \tau}$, thus $w(\cdot, t) \leq \bar{K}t^{-\frac{2}{q}}$ in $Q_{B_{M/2}, \tau}$. Next we go to the limit as $M \rightarrow \infty$ and deduce that $w(\cdot, t) \leq \bar{K}t^{-\frac{2}{q}}$ in $Q_{\mathbb{R}^N, \tau}$, namely

$$(q')^q |\nabla v(\cdot, t)|^q = \frac{|\nabla u|^q}{\delta + u}(\cdot, t) \leq Ct^{-1}, \quad C = C(N, q, \eta, \delta, \tau).$$

In turn for any ϵ as above, *there holds* $w \in L^\infty(Q_{\mathbb{R}^N, \epsilon, T})$, that means $|\nabla v| \in L^\infty(Q_{\mathbb{R}^N, \epsilon, \tau})$.

(iii) Universal estimate (4.4) for $u \in C(\mathbb{R}^N \times [0, T])$: we prove the universal estimate (4.4). Taking again $\Psi(t) = Kt^{-2/q}$, with now $K = K(N, q) = q^{-2}(q-1)^{2/q'}$, we have

$$\Psi_t + 2c\Psi^{\frac{q+2}{2}} \geq (2cK^{\frac{q+2}{2}} - 2q^{-1}K)t^{-\frac{q+2}{q}} \geq 0.$$

And $\mathcal{L}w + 2cw^{\frac{q+2}{2}} \leq 0$ from (4.20). Moreover there exists $\tau_\epsilon \in (0, \tau)$ such that $\Psi(\theta) \geq \sup_{\mathbb{R}^N} w(\cdot, \epsilon)$ for any $\theta \in (0, \tau_\epsilon)$. Setting $y(\cdot, t) = w(\cdot, t + \epsilon) - \Psi(\cdot, t + \theta)$, hence on the set $\mathcal{U} = \{(x, t) \in Q_{\mathbb{R}^N, \tau} : y(x, t) \geq 0\}$, there holds in the same way

$$y(\cdot, t) - \Delta y(\cdot, t) + b(\cdot, t + \epsilon) \cdot \nabla y(\cdot, t) \leq 0.$$

Here we only have from (4.22)

$$|b| \leq (qcvw^{\frac{q-1}{2}} + \frac{2}{q-1} \frac{1}{\delta} w^{1/2}) \leq \kappa_\epsilon(1 + |x|)$$

on $Q_{\mathbb{R}^N, \epsilon, \tau}$, for some $\kappa_\epsilon = \kappa_\epsilon(N, q, \eta, \sup_{B_{2\eta}} u_0, \tau, \epsilon)$. It is sufficient to apply Lemma 4.5. We deduce that $w(\cdot, t + \epsilon) \leq \Psi(t + \theta)$ on $(0, \tau)$. As $\theta, \epsilon \rightarrow 0$ we obtain that $w(\cdot, t) \leq \Psi(t) = q^{-2}(q-1)^{2/q'} t^{-2/q}$, which shows now that in $(0, T)$

$$|\nabla v(\cdot, t)|^q = (q')^{-q} \frac{|\nabla u|^q}{\delta + u}(\cdot, t) \leq q^{-q}(q-1)^{(q-1)} t^{-1}.$$

As $\delta \rightarrow 0$, we obtain (4.4).

(iv) General universal estimate. Here we relax the assumption $u \in C(\mathbb{R}^N \times [0, T])$: For any $\epsilon \in (0, T)$ such that $\tau + \epsilon < T$, we have $u \in C(\mathbb{R}^N \times [\epsilon, \tau + \epsilon])$, then from above,

$$|\nabla v(\cdot, t + \epsilon)|^q \leq \frac{1}{q-1} \frac{1}{t},$$

and we obtain (4.4) as $\epsilon \rightarrow 0$, on $(0, \tau)$ for any $\tau < T$, hence on $(0, T)$. ■

Proof of Theorem 1.3. It is a direct consequence of Theorems 1.2 and 4.1. ■

5 Existence and nonuniqueness results

Next we mention some known uniqueness and comparison results, for the Cauchy problem, see [11, Theorems 2.1, 4.1, 4.2 and Remark 2.1], [13, Theorem 2.3, 4.2, 4.25, Proposition 4.26], and for the Dirichlet problem, see [1, Theorems 3.1, 4.2], [6], [13, Proposition 5.17], [24].

Theorem 5.1 *Let $\Omega = \mathbb{R}^N$ (resp. Ω bounded). (i) Let $1 < q < q_*$, and $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$ (resp. $u_0 \in \mathcal{M}_b(\Omega)$). Then there exists a unique weak solution u of (1.1) with trace u_0 (resp. a weak solution of $(D_{\Omega, T})$, such that $\lim_{t \rightarrow 0} u(\cdot, t) = u_0$ weakly in $\mathcal{M}_b(\Omega)$). If $v_0 \in \mathcal{M}_b(\Omega)$ and $u_0 \leq v_0$, and v is the solution associated to v_0 , then $u \leq v$.*

(ii) Let $u_0 \in L^R(\Omega)$, $1 \leq R \leq \infty$. If $1 < q < (N+2R)/(N+R)$, or if $q = 2$, $R < \infty$, there exists a unique weak solution u of (1.1) (resp. $(D_{\Omega, T})$) such that $u \in C([0, T]; L^R(\Omega))$ and $u(0) = u_0$. If $v_0 \in L^R(\mathbb{R}^N)$ and $u_0 \leq v_0$, then $u \leq v$. If u_0 is nonnegative, then for any $1 < q \leq 2$, there still exists at least a weak nonnegative solution u satisfying the same conditions.

Next we prove Theorem 1.4. We begin by the subcritical case:

Remark 5.2 *From [3, Lemma 3.3], for any reals $s < \tau$, the Dirichlet problem*

$$\begin{cases} u_t - \Delta u = g, & \text{in } \mathcal{D}'(Q_{\Omega, \tau}), \\ u = 0, & \text{on } \partial\Omega \times (0, \tau), \\ u(\cdot, s) = u_s, & \text{in } \Omega, \end{cases}$$

in a bounded domain Ω , with data $g \in L^1(Q_{\Omega, \tau})$ and $u_s \in L^1(\Omega)$, has a unique solution $u \in C([0, \tau], L^1(\Omega)) \cap L^1((0, \tau); W_0^{1,1}(\Omega))$. And $u \in L^k((s, \tau); W_0^{1,k}(\Omega))$, for any $k \in [1, q_)$, and*

$$\|u\|_{L^k((s, \tau); W_0^{1,k}(\Omega))} \leq C(k, \Omega) (\|u(\cdot, s)\|_{L^1(\Omega)} + \|g\|_{L^1(Q_{\Omega, s, \tau})}). \quad (5.1)$$

This implies local estimates in any domain Ω of \mathbb{R}^N : for any $g \in L^1_{loc}((0, T), L^1(\Omega))$, and $u \in C((0, T), L^1(\Omega)) \cap L^1_{loc}((0, T); W^{1,1}_{loc}(\Omega))$, such that

$$u_t - \Delta u = g, \quad \text{in } \mathcal{D}'(Q_{\Omega, T}),$$

there holds $u \in L^1_{loc}((0, T); W^{1,k}_{loc}(\Omega))$. And for any domain $\omega \subset\subset \Omega$, and any $0 < s < \tau < T$

$$\|u\|_{L^k((s, \tau); W^{1,k}(\omega))} \leq C(k, \omega)(\|u(s, \cdot)\|_{L^1(\omega)} + \|g + |\nabla u| + |u|\|_{L^1(Q_{\omega, s, \tau})}). \quad (5.2)$$

Proposition 5.3 *Let $1 < q < q_*$, and $\Omega = \mathbb{R}^N$ (resp. Ω bounded). Then for any $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$ (resp. $\mathcal{M}^+(\Omega)$), there exists a weak solution u of equation (1.1) (resp. of $(D_{\Omega, T})$) with trace u_0 .*

Proof. Assume $\Omega = \mathbb{R}^N$ (resp. Ω bounded). Let $u_{0,n} = u_0 \llcorner B_n$ (resp. $u_{0,n} = u_0 \llcorner \overline{\Omega}'_{1/n}$, where $\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n\}$, for n large enough). From Theorem 5.1, there exists a unique weak solution u_n of (1.1) (resp. of $(D_{\Omega, T})$) with trace $u_{0,n}$, and (u_n) is nondecreasing; and $u_n \in C^{2,1}(Q_{\mathbb{R}^N, T})$ since $q \leq 2$. From (3.1), (3.5), for any $\xi \in C^{1+}_c(\Omega)$,

$$\int_{\Omega} u_n(\cdot, t) \xi^{q'} + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla u_n|^q \xi^{q'} \leq Ct \int_{\Omega} |\nabla \xi|^{q'} + \int_{\Omega} \xi^{q'} du_0. \quad (5.3)$$

Hence (u_n) is bounded in $L^\infty_{loc}([0, T]; L^1_{loc}(\Omega))$, and $(|\nabla u_n|^q)$ is bounded in $L^1_{loc}([0, T]; L^1_{loc}(\Omega))$. In turn (u_n) is bounded in $L^\infty_{loc}((0, T); L^\infty(\Omega))$, from Theorem 3.3. From Remark 2.4, (u_n) converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N, T})$ (resp. $C^{2,1}_{loc}(Q_{\Omega, T}) \cap C^{1,0}(\overline{\Omega} \times (0, T))$) to a weak solution u of (1.1) in $Q_{\mathbb{R}^N, T}$ (resp. of $(D_{\Omega, T})$). Also (u_n) is bounded in $L^k_{loc}([0, T]; W^{1,k}_{loc}(\mathbb{R}^N))$ (resp. $L^k_{loc}([0, T]; W^{1,k}_0(\Omega))$) for any $k \in [1, q^*]$ from Remark 5.2. Since $q < q_*$, $(|\nabla u_n|^q)$ is equiintegrable in $Q_{B_R, \tau}$ for any $R > 0$ (resp. in $Q_{\Omega, \tau}$) and $\tau \in (0, T)$, then $(|\nabla u|^q) \in L^1_{loc}([0, T]; L^1_{loc}(\Omega))$. From (2.6),

$$\int_{\Omega} u_n(t, \cdot) \xi + \int_0^t \int_{\Omega} |\nabla u_n|^q \xi = - \int_0^t \int_{\Omega} \nabla u_n \cdot \nabla \xi + \int_{\Omega} \xi du_0. \quad (5.4)$$

As $n \rightarrow \infty$ we obtain

$$\int_{\Omega} u(t, \cdot) \xi + \int_0^t \int_{\Omega} |\nabla u|^q \xi = - \int_0^t \int_{\Omega} \nabla u \cdot \nabla \xi + \int_{\Omega} \xi du_0.$$

Thus $\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t) \xi = \int_{\Omega} \xi du_0$, for any $\xi \in C^{1+}_c(\Omega)$, hence for any $\xi \in C^+_c(\Omega)$; hence u admits the trace u_0 . \blacksquare

Next we consider the supercritical case $q \geq q_*$. From [1], if the Dirichlet problem $(P_{\Omega, T})$ has a solution with $u_0 \in \mathcal{M}_b(\Omega)$, then u_0 does not charge the sets of $W^{\frac{2-q}{q}, q'}(\Omega)$ capacity 0 if $q < 2$. And $u_0 \in L^1(\Omega)$ if $q \geq 2$, see another proof in [12]. In the same way, if the Cauchy problem $(P_{\mathbb{R}^N, T})$ has a solution with trace $u_0 \in \mathcal{M}(\mathbb{R}^N)$, then u_0 does not charge the sets of $W^{\frac{2-q}{q}, q'}(\mathbb{R}^N)$ -capacity 0 if $q < 2$, and $u_0 \in L^1_{loc}(\mathbb{R}^N)$ if $q \geq 2$. The converse question is to know what are the "admissible" measures for which the problem has a solution. It is widely open, and we give here a few results in that direction, extending some results of [11].

Theorem 5.4 (i) Let $1 < q \leq 2$, and $\Omega = \mathbb{R}^N$ (resp. Ω bounded). For any nonnegative $u_0 \in L^1_{loc}(\mathbb{R}^N)$ (resp. $u_0 \in L^1_{loc}(\Omega)$), there exists a weak nonnegative solution of equation (1.1) in $Q_{\Omega,T}$ with trace u_0 . And then $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N))$ (resp. $u \in C([0, T]; L^1_{loc}(\Omega))$).

(ii) Let $q > 2$. The existence is valid for any nonnegative $u_0 \in L^1_{loc}(\mathbb{R}^N)$ (resp. $L^1_{loc}(\Omega)$) which is a limit of an nondecreasing sequence of nonnegative functions in $C_b(\mathbb{R}^N)$ (resp. in $C_0(\Omega)$).

Proof. (i) Let $\Omega = \mathbb{R}^N$ (resp. Ω bounded). As in Proposition 5.3, we set $u_{0,n} = \min(u_0, n)\chi_{B_n}$ (resp. $u_{0,n} = \min(u_0, n)\chi_{\overline{\Omega}'_{1/n}}$ for n large enough). Then $u_{0,n} \in L^r(\Omega)$ for any $r \geq 1$. From Theorem 5.1, the problem admits a solution u_n , and it is unique in $C([0, T]; L^r(\Omega))$ for any $r > (2 - q)/N(q - 1)$ and then (u_n) is nondecreasing. As in Proposition 5.3, (u_n) is bounded in $L^\infty_{loc}([0, T]; L^1_{loc}(\Omega))$, and $(|\nabla u_n|^q)$ is bounded in $L^1_{loc}([0, T]; L^1_{loc}(\Omega))$. Moreover, (u_n) is bounded in $L^\infty_{loc}([0, T]; L^\infty_{loc}(\Omega))$ from Lemma 3.3. From Theorem 2.3, (u_n) converges in $C^{2,1}_{loc}(Q_{\Omega,T})$ to a weak solution u of (1.1) in $Q_{\Omega,T}$, such that $u \in L^\infty_{loc}([0, T]; L^1_{loc}(\Omega))$ and $|\nabla u|^q \in L^1_{loc}([0, T]; L^1_{loc}(\Omega))$.

Then from Remark 2.5, u admits a trace $\mu_0 \in \mathcal{M}^+(\Omega)$ as $t \rightarrow 0$. Applying (5.4) to u_n , since $u_n \leq u$, we get

$$\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t)\xi = \int_{\Omega} \xi d\mu_0 \geq \lim_{t \rightarrow 0} \int_{\Omega} u_n(\cdot, t)\xi = \int_{\Omega} \xi du_0,$$

for any $\xi \in C_c^+(\Omega)$; thus $u_0 \leq \mu_0$. Moreover

$$\int_{\Omega} u_n(t, \cdot)\xi + \int_0^t \int_{\Omega} |\nabla u_n|^q \xi = \int_0^t \int_{\Omega} u_n \Delta \xi dx + \int_{\Omega} \xi du_0.$$

And (u_n) is bounded in $L^k(Q_{\omega,\tau})$ for any $k \in (1, q_*)$; then for any domain $\omega \subset\subset \Omega$, (u_n) converges strongly in $L^1(Q_{\omega,\tau})$; then from the convergence *a.e.* of the gradients, and the Fatou Lemma,

$$\int_{\mathbb{R}^N} u(t, \cdot)\xi + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \xi \leq \int_0^t \int_{\mathbb{R}^N} u \Delta \xi dx + \int_{\mathbb{R}^N} \xi du_0.$$

But from Remark 2.5,

$$\int_{\mathbb{R}^N} u(t, \cdot)\xi + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \xi = \int_0^t \int_{\mathbb{R}^N} u \Delta \xi dx + \int_{\mathbb{R}^N} \xi d\mu_0,$$

then $\mu_0 \leq u_0$, hence $\mu_0 = u_0$. Finally we prove the continuity: Let $\xi \in \mathcal{D}^+(\Omega)$ and $\omega \subset\subset \Omega$ containing the support of ξ . Then $z = u\xi$ is solution of the Dirichlet problem

$$\begin{cases} z_t - \Delta z = g, & \text{in } Q_{\omega,T}, \\ z = 0, & \text{on } \partial\omega \times (0, T), \\ \lim_{t \rightarrow 0} z(\cdot, t) = \xi u_0, & \text{weakly in } \mathcal{M}_b(\omega), \end{cases}$$

with $g = -|\nabla u|^q \xi + v(-\Delta \psi) - 2\nabla v \cdot \nabla \psi \in L^1(Q_{\omega,T})$. The solution is unique, see [6, Proposition 2.2]. Since $u_0 \in L^1_{loc}(\Omega)$, there also exists a unique solution such that $z \in C([0, T], L^1(\omega))$ from [3, Lemma 3.3], hence $u \in C([0, T], L^1_{loc}(\Omega))$.

(ii) As above, by taking for $(u_{0,n})$ a nondecreasing sequence in $C_b(\mathbb{R}^N)$ (resp. in $C_0(\Omega)$), converging to u_0 , and using Remark 2.4 for classical solutions. \blacksquare

In particular this ends the proof of Theorem 1.4.

Next we show the nonuniqueness of the weak solutions when $q > 2$: here the coefficient a defined at (1.7) is negative, and $|a| = (q - 2)/(q - 1) < 1$.

Proof of Theorem 1.5. Since $q > 2$ and $N \geq 2$, the function \tilde{U} is a solution in $\mathcal{D}'(\mathbb{R}^N)$ of the stationary equation

$$-\Delta u + |\nabla u|^q = 0$$

Indeed $\tilde{U} \in W_{loc}^{1,q}(\mathbb{R}^N) \cap W_{loc}^{2,1}(\mathbb{R}^N)$ because $N > q'$, and \tilde{U} is a classical solution in $\mathbb{R}^N \setminus \{0\}$. Then it is a weak solution of $(P_{\mathbb{R}^N, \infty})$, and $\tilde{U} \notin C^1(Q_{\mathbb{R}^N, \infty})$. Since $\tilde{U} \in C(\mathbb{R}^N)$, from Theorem 5.4, or from [5], there exists also a classical solution $U_{\tilde{C}} \in C^{2,1}(Q_{\mathbb{R}^N, \infty})$ of the problem, thus $U_{\tilde{C}} \neq U_0$.

More generally, for any $C > 0$, there exists a classical solution U_C with trace $C|x|^{|a|}$. And U_C is obtained as the limit of the nondecreasing sequence of the unique solutions $U_{n,C}$ with trace $\min(C|x|^{|a|}, n)$, then it is radial. Moreover for any $\lambda > 0$, the function $U_{n,C,\lambda}(x, t) = \lambda^{-a}U_{n,C}(\lambda x, \lambda^2 t)$ admits the trace $\min(C|x|^{|a|}, n\lambda^{-a})$. Therefore, denoting by $k_{\lambda,n}$ the integer part of $n\lambda^{-a}$, there holds $U_{k_{\lambda,n},C} \leq U_{n,C,\lambda} \leq U_{k_{\lambda,n}+1}$ from the comparison principle. And $U_{n,C,\lambda}(x, t)$ converges everywhere to $\lambda^{-a}U_C(\lambda x, \lambda^2 t)$, thus $U_C(x, t) = \lambda^{-a}U_C(\lambda x, \lambda^2 t)$, that means U_C is self-similar. Then U_C has the form (1.14), where $f \in C^2([0, \infty))$, $f(0) \geq 0$, $f'(0) = 0$, and $\lim_{\eta \rightarrow \infty} \eta^{-|a|/2} f(\eta) = C$, and for any $\eta > 0$,

$$f''(\eta) + \left(\frac{N-1}{\eta} + \frac{\eta}{2}\right)f'(\eta) - \frac{|a|}{2}f(\eta) - |f'(\eta)|^q = 0. \quad (5.5)$$

From the Cauchy-Lipschitz Theorem, we find $f(0) > 0$, since $f \not\equiv 0$, hence $f''(0) > 0$. The function f is increasing: indeed if there exists a first point $\eta_0 > 0$ such that $f'(\eta_0) = 0$, then $f''(\eta_0) > 0$, which is contradictory. \blacksquare

6 Second local regularizing effect

Here we show the second regularizing effect. We prove an estimate, playing the role of the sub-caloricity estimate (2.4). Our proof follows the general scheme of Stampacchia's method, developed by many authors, see [17] and references there in, and [19].

First we write estimate (3.1) in another form, and from Gagliardo estimate, we obtain the following:

Lemma 6.1 *Let $q > 1$. Let $\eta > 0, r \geq 1$. Let u be any nonnegative weak subsolution of equation (1.1) in $Q_{\Omega, T}$. Let $B_{2\eta} \subset \subset \Omega$, $0 < \theta < \tau < T$, and $\xi \in C^1((0, T), C_c^1(\Omega))$, with values in $[0, 1]$, such that $\xi(\cdot, t) = 0$ for $t \leq \theta$. Let $\lambda \geq \max(2, q')$.*

Then for any $\nu \in (0, 1]$,

$$\sup_{[\theta, \tau]} \int_{\Omega} u^r(\cdot, t) \xi^\lambda + \frac{\int_{\theta}^{\tau} \int_{\Omega} u^{(q+r-1)(1+\frac{\mu}{N})} \xi^{\lambda(1+\frac{\mu}{N})}}{(\sup_{t \in [\theta, \tau]} \int_{\Omega} u^r \xi^{\frac{\lambda r}{q+r-1}})^{\frac{q}{N}}} \leq C \int_{\theta}^{\tau} \int_{\Omega} (u^r |\xi_t| + u^{r-1} |\nabla \xi|^{q'} + u^{q+r-1} |\nabla \xi|^q), \quad (6.1)$$

where $\mu = rq/(q+r-1)$, $C = C(N, q, r, \lambda)$.

Proof. From Remark 2.2, $u \in L_{loc}^\infty(Q_{\Omega,T})$, and hence $u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda}{q}} \in W^{1,q}(Q_{\Omega,\theta,t})$ and

$$\begin{aligned} \int_{\theta}^t \int_{\Omega} |\nabla(u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda}{q}})|^q &= \int_{\theta}^t \int_{\Omega} \left| \frac{q+r-1}{q} u^{\frac{r-1}{q}} \xi^{\frac{\lambda}{q}} \nabla u + \frac{\lambda}{q} u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda-q}{q}} \nabla \xi \right|^q \\ &\leq C \left(\int_{\theta}^t \int_{\Omega} u^{r-1} |\nabla u|^q \xi^{\lambda} + \int_{\theta}^t \int_{\Omega} u^{q+r-1} |\nabla \xi|^q \xi^{\lambda-q} \right), \end{aligned}$$

with $C = C(q, r, \lambda)$. From (3.1), since $\nu \leq 1$, we get

$$\sup_{[\theta,\tau]} \int_{\Omega} u^r(\cdot, t) \xi^{\lambda} + \int_{\theta}^{\tau} \int_{\Omega} |\nabla(u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda}{q}})|^q \leq C \int_{\theta}^{\tau} \int_{\Omega} (u^r |\xi_t| + u^{r-1} |\nabla \xi|^{q'} + u^{q+r-1} |\nabla \xi|^q), \quad (6.2)$$

where $C = C(q, r, \lambda)$. Next we use a Galliaro type estimate, see [17, Proposition 3.1]: for any $\mu \geq 1$, and any $w \in L_{loc}^\infty((0, T), L^\mu(\Omega)) \cap L_{loc}^q((0, T), W^{1,q}(\Omega))$,

$$\int_{\theta}^{\tau} \int_{\Omega} w^{q(1+\frac{\mu}{N})} \leq C \left(\int_{\theta}^{\tau} \int_{\Omega} |\nabla w|^q \right) \left(\sup_{t \in [\theta,\tau]} \int_{\Omega} |w|^\mu \right)^{\frac{q}{N}}, \quad C = C(N, q, \mu).$$

Taking $w = u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda}{q}}$ and $\mu = qr/(q+r-1) \geq r \geq 1$, setting $s = 1 + \mu/N$, it comes

$$\int_{\theta}^{\tau} \int_{\Omega} u^{(q+r-1)s} \xi^{\lambda s} \leq C \left(\int_{\theta}^{\tau} \int_{\Omega} |\nabla w|^q \right) \left(\sup_{t \in [\theta,\tau]} \int_{\Omega} u^r \xi^{\frac{\lambda r}{q+r-1}} \right)^{\frac{q}{N}},$$

hence (6.1) follows. ■

Theorem 6.2 *Let $q > 1$. Let u be any nonnegative weak solution of equation (1.1) in $Q_{\Omega,T}$. Let $B(x_0, \rho) \subset\subset \Omega$. Let $R > q - 1$ (in particular any $R \geq 1$ if $q < 2$). Then there exists $C = C(N, q, R)$ such that, for any t, θ such that $0 < t - 2\theta < t < T$,*

$$\begin{aligned} \sup_{B(x_0, \frac{\rho}{2}) \times [t-\theta, t]} u &\leq C \theta^{-\frac{N+q}{qR+N(q-1)}} \left(\int_{t-2\theta}^t \int_{B(x_0, \rho)} u^R \right)^{\frac{q}{qR+N(q-1)}} \\ &+ C \rho^{-\frac{N+q}{(q-1)(R+N+1)}} \left(\int_{t-2\theta}^t \int_{B(x_0, \rho)} u^R \right)^{\frac{1}{R+N+1}} + C \rho^{-\frac{N+q}{R+1-q}} \left(\int_{t-2\theta}^t \int_{B(x_0, \rho)} u^R \right)^{\frac{1}{R+1-q}}. \end{aligned} \quad (6.3)$$

Proof. Since $u \in C((0, T); L_{loc}^R(Q_{\Omega,T}))$, by regularization we can assume that u is a classical solution in $Q_{\Omega,T}$. Let t, θ such that $0 < t - 2\theta < t < T$. We can assume $x_0 = 0 \in \Omega$. By translation of $t - \theta$, we are lead to prove that for any solution in $Q_{\Omega, -\tau/2, \tau/2}$ ($\tau < T$),

$$\begin{aligned} \sup_{Q_{B_{\rho/2}, 0, \theta}} u &\leq C \theta^{-\frac{N+q}{qR+N(q-1)}} \left(\int_{-\theta}^{\theta} \int_{B_{\rho}} u^R \right)^{\frac{q}{qR+N(q-1)}} \\ &+ C \rho^{-\frac{N+q}{(q-1)(R+N+1)}} \left(\int_{-\theta}^{\theta} \int_{B_{\rho}} u^R \right)^{\frac{1}{R+N+1}} + C \rho^{-\frac{N+q}{R+1-q}} \left(\int_{-\theta}^{\theta} \int_{B_{\rho}} u^R \right)^{\frac{1}{R+1-q}}. \end{aligned} \quad (6.4)$$

For given $k > 0$ we set $u_k = (u - k)^+$. Then $u_k \in C(0, T); L_{loc}^R(Q_{\Omega, T})$, and u_k is a weak subsolution of equation (1.1), from the Kato inequality. We set

$$\begin{aligned} \rho_n &= (1 + 2^{-n})\rho/2, & t_n &= -(1 + 2^{-n})\theta/2, \\ Q_n &= B_{\rho_n} \times (t_n, \theta), & Q_0 &= B_\rho \times (-\theta, \theta), & Q_\infty &= B_{\rho/2} \times (-\theta/2, \theta), \\ k_n &= (1 - 2^{-(n+1)})k, & \tilde{k} &= (k_n + k_{n+1})/2. \end{aligned}$$

and set $M_\sigma = \sup_{Q_\infty} u$, $M = \sup_{Q_0} u$. Let $\xi(x, t) = \xi_1(x)\xi_2(t)$ where $\xi_1 \in C_c^1(\Omega)$, $\xi_2 \in C^1(\mathbb{R})$, with values in $[0, 1]$, such that

$$\begin{aligned} \xi_1 &= 1 \quad \text{on } B_{\rho_{n+1}}, & \xi_1 &= 0 \quad \text{on } \mathbb{R}^N \setminus B_{\rho_n}, & |\nabla \xi_1| &\leq C(N)2^{n+1}/\rho; \\ \xi_2 &= 1 \quad \text{on } [\theta_{n+1}, \infty), & \xi_2 &= 0 \quad \text{on } (-\infty, \theta_n], & |\xi_{2,t}| &\leq C(N)2^{n+1}/\theta. \end{aligned}$$

From Lemma 6.1 we get, with $\mu = qr/(q + r - 1)$,

$$\begin{aligned} &\sup_{t \in [t_{n+1}, \theta]} \int_{B_{\rho_{n+1}}} u_{k_{n+1}}^r(\cdot, t) + \frac{\int_{t_{n+1}}^\theta \int_{B_{\rho_{n+1}}} u_{k_{n+1}}^{(q+r-1)(1+\frac{\mu}{N})}}{(\sup_{t \in [t_n, \theta]} \int_{B_{\rho_n}} u_{k_n}^r)^{\frac{q}{N}}} \leq CX_n, \text{ where} \\ X_n &= \int_{t_n}^\theta \int_{B_{\rho_n}} (u_{k_{n+1}}^r |\zeta_t| + u_{k_{n+1}}^{r-1} |\nabla \xi|^{q'} + u_{k_{n+1}}^{q+r-1} |\nabla \xi|^q). \end{aligned}$$

Let us define

$$Y_n = \int_{t_n}^\theta \int_{B_{\rho_n}} u_{k_n}^{q+r-1}, \quad Z_n = \sup_{t \in [t_n, \theta]} \int_{B_{\rho_n}} u_{k_n}^r, \quad W_n = \int_{t_n}^\theta \int_{B_{\rho_n}} \chi_{\{u \geq k_n\}}.$$

Thus, from the Hölder inequality,

$$Z_{n+1} + Z_n^{-\frac{q}{N}} W_{n+1}^{-\frac{\mu}{N}} Y_{n+1}^{1+\frac{\mu}{N}} \leq CX_n. \quad (6.5)$$

Moreover, for any $\gamma, \beta > 0$,

$$\begin{aligned} \int_{t_n}^\theta \int_{B_{\rho_n}} u_{k_n}^{\gamma+\beta} &\geq \int_{t_n}^\theta \int_{B_{\rho_n}} (k_n - k_{n+1})^{\gamma+\beta} \chi_{\{u \geq k_{n+1}\}} \\ &\geq (k2^{-(n+2)})^{\gamma+\beta} \int_{t_n}^\theta \int_{B_{\rho_n}} \chi_{\{u \geq k_{n+1}\}} \geq (k2^{-(n+2)})^{\gamma+\beta} \int_{t_{n+1}}^\theta \int_{B_{\rho_{n+1}}} \chi_{\{u \geq k_{n+1}\}}, \end{aligned}$$

and from the Hölder inequality,

$$\begin{aligned} \int_{t_n}^\theta \int_{B_{\rho_n}} u_{k_{n+1}}^\gamma &\leq \left(\int_{t_n}^\theta \int_{B_{\rho_n}} u_{k_{n+1}}^{\gamma+\beta} \right)^{\frac{\gamma}{\gamma+\beta}} \left(\int_{t_n}^\theta \int_{B_{\rho_n}} \chi_{\{u \geq k_{n+1}\}} \right)^{\frac{\beta}{\gamma+\beta}} \\ &\leq \left(\int_{t_n}^\theta \int_{B_{\rho_n}} u_{k_n}^{\gamma+\beta} \right) (k^{-1}2^{(n+2)})^\beta \left(\int_{t_n}^\theta \int_{B_{\rho_n}} u_{k_n}^{\gamma+\beta} \right)^{\frac{\beta}{\gamma+\beta}} \\ &\leq (k^{-1}2^{(n+2)})^\beta \int_{t_n}^\theta \int_{B_{\rho_n}} u_{k_n}^{\gamma+\beta}. \end{aligned}$$

Thus in particular

$$W_{n+1} \leq C \left(\frac{2^{n+1}}{k} \right)^{q+r-1} Y_n, \quad \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_{n+1}}^r \leq C \left(\frac{2^{n+1}}{k} \right)^{q-1} Y_n, \quad \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_{n+1}}^{r-1} \leq C \left(\frac{2^{n+1}}{k} \right)^q Y_n. \quad (6.6)$$

Otherwise

$$X_n \leq \int_{t_n}^{\theta} \int_{B_{\rho_n}} (2^{n+1} \theta^{-1} u_{k_{n+1}}^r + 2^{q'(n+1)} \rho^{-q'} u_{k_{n+1}}^{r-1} + 2^{q(n+1)} \rho^{-q} u_{k_{n+1}}^{q+r-1}),$$

then from (6.6),

$$X_n \leq C b_0^n f(\theta, \rho, k) Y_n, \quad \text{where } f(\theta, \rho, k) = (\theta^{-1} \frac{1}{k^{q-1}} + \frac{1}{k^q} \rho^{-q'} + \rho^{-q}). \quad (6.7)$$

for some b_0 depending on q, r . Then from (6.5), (6.6) and (6.7),

$$Z_{n+1} \leq C b_0^n f(\theta, \rho, k) Y_n, \quad Y_{n+1}^{1+\frac{\mu}{N}} \leq C Z_n^{\frac{q}{N}} \left(\frac{2^{n+1}}{k} \right)^{(q+r-1)\frac{\mu}{N}} b_0^n f(\theta, \rho, k) Y_n^{1+\frac{\mu}{N}}.$$

Since $Y_{n+1} \leq Y_n$, setting $\alpha = q/(N + \mu)$ and denoting by b_1, b some new constants depending on N, q, r ,

$$\begin{aligned} Y_{n+2} &\leq C Z_{n+1}^{\frac{q}{N+\mu}} b_1^{n+1} k^{-(q+r-1)\frac{\mu}{N+\mu}} f^{\frac{N}{N+\mu}}(\theta, \rho, k) Y_{n+1} \\ &\leq C (b_0^n f(\theta, \rho, k) Y_n)^{\frac{q}{N+\mu}} b_1^{n+1} k^{-(q+r-1)\frac{\mu}{N+\mu}} f^{\frac{N}{N+\mu}}(\theta, \rho, k) Y_n \\ &\leq C b^n f^{\frac{N+q}{N+\mu}} k^{-(q+r-1)\frac{\mu}{N+\mu}} Y_n^{1+\frac{q}{N+\mu}} := D b^n Y_n^{1+\alpha}. \end{aligned}$$

From [17, Lemma 4.1], $Y_n \rightarrow 0$ if

$$Y_0^\alpha \delta^{1/\alpha} \leq D^{-1} = C^{-1} k^{(q+r-1)\frac{\mu}{N+\mu}} f^{-\frac{N+q}{N+\mu}},$$

that means

$$k^{qr} \geq c Y_0^q \left((\theta^{-1} \frac{1}{k^{q-1}} + \frac{1}{k^q} \rho^{-q'} + \rho^{-q}) \right)^{N+q}. \quad (6.8)$$

For getting (6.8) it is sufficient that

$$k^{qr+(q-1)(N+q)} \geq \frac{c}{2} Y_0^q \theta^{-(N+q)}, \quad k^{(r+N+q)} \geq \left(\frac{c}{2} \right)^{1/q} Y_0 \rho^{-\frac{N+q}{q-1}}, \quad \text{and } k^r \geq \frac{c}{2} Y_0 \rho^{-(N+q)}.$$

Thus we deduce that

$$\begin{aligned} \sup_{Q_\infty} u &\leq C \theta^{-\frac{N+q}{qr+(N+q)(q-1)}} \left(\int_{-\theta}^{\theta} \int_{B_\rho} u^{q+r-1} \right)^{\frac{q}{qr+(N+q)(q-1)}} \\ &\quad + C \rho^{-\frac{N+q}{(q-1)(r+N+q)}} \left(\int_{-\theta}^{\theta} \int_{B_\rho} u^{q+r-1} \right)^{\frac{1}{r+N+q}} + C \rho^{-\frac{N+q}{r}} \left(\int_{-\theta}^{\theta} \int_{B_\rho} u^{q+r-1} \right)^{\frac{1}{r}}. \end{aligned} \quad (6.9)$$

If we set $q + r - 1 = R$, we obtain (6.4) for any $R \geq q$.

Next we consider the case $R < q$. From (6.9) we get

$$\begin{aligned}
\sup_{B_{\sigma\rho} \times (-\theta/2, \theta)} u &\leq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} \left(\int_0^\theta \int_{B_\rho} u^q \right)^{\frac{q}{q+(q-1)(N+q)}} \\
&\quad + C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^q \right)^{\frac{1}{1+N+q}} + C\rho^{-(N+q)} \int_{-\theta}^\theta \int_{B_\rho} u^q \\
&\leq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} \left(\sup_{B_\rho \times 0, \theta} u \right)^{\frac{q(q-R)}{q+(q-1)(N+q)}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{q}{q+(q-1)(N+q)}} \\
&\quad + C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} \left(\sup_{B_\rho \times 0, \theta} u \right)^{\frac{q(q-R)}{1+N+q}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{1}{1+N+q}} \\
&\quad + C\rho^{-(N+q)} \left(\sup_{B_\rho \times 0, \theta} u \right)^{(q-R)} \int_{-\theta}^\theta \int_{B_\rho} u^R.
\end{aligned}$$

We define

$$\tilde{\rho}_n = (1 + 2^{-(n+1)})\rho, \quad \theta_n = -(1 + 2^{-(n+1)})\theta, \quad \tilde{Q}_n = B_{\tilde{\rho}_n} \times (\theta_n, \theta), \quad M_n = \sup_{\tilde{Q}_n} u,$$

hence $M_0 = \sup_{B_{\rho/2} \times (-\theta/2, \theta)} u$. We find

$$\begin{aligned}
M_n &\leq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} M_{n+1}^{\frac{q(q-R)}{q+(q-1)(N+q)}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{q}{q+(q-1)(N+q)}} \\
&\quad + C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} M_{n+1}^{\frac{q(q-R)}{1+N+q}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{1}{1+N+q}} + C\rho^{-(N+q)} M_{n+1}^{q-R} \int_{-\theta}^\theta \int_{B_\rho} u^R.
\end{aligned}$$

We set

$$\begin{aligned}
I &= C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{q}{q+(q-1)(N+q)}}, \\
J &= C\rho^{-(N+q)} \int_{-\theta}^\theta \int_{B_\rho} u^R, \quad L = C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{1}{1+N+q}}.
\end{aligned}$$

Note that $R > q - 1$, that means $q - R < 1$. Then from Hölder inequality,

$$M_n \leq \frac{1}{2} M_{n+1} + C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}}), \quad \sigma = \frac{q + (q-1)(N+q)}{N(q-1) + qR}, \quad \delta = \frac{1 + N + q}{R + N + 1}.$$

Thus $M_0 \leq 2^{-n} M_n + 2C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}})$, and finally

$$\begin{aligned}
M_0 = \sup_{Q_0} u &\leq C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}}) = C\theta^{-\frac{N+q}{N(q-1)+qR}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{q}{N(q-1)+qR}} \\
&\quad + C\rho^{-\frac{N+q}{(q-1)(R+N+1)}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{1}{R+N+1}} + C\rho^{-\frac{N+q}{R+1-q}} \left(\int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{1}{R+1-q}},
\end{aligned}$$

which shows again (6.4). Then (6.4) holds for any $R > q - 1$, in particular for any $R \geq 1$ if $q < 2$. ■

Now we prove our second regularizing effect due to the effect of the gradient:

Proof of Theorem 1.6. We assume $x_0 = 0$. Let $\kappa > 0$ be a parameter. From (6.3), for any $\rho \in (0, \eta)$ such that $\rho^\kappa \leq t < \tau$,

$$\begin{aligned} \sup_{B_{\frac{\rho}{2}} \times [t-\rho^\kappa, t]} u &\leq C \rho^{-\frac{\kappa(N+q)}{qR+N(q-1)}} \left(\int_{t-\rho^\kappa}^t \int_{B_\rho} u^R \right)^{\frac{q}{qR+N(q-1)}} \\ &+ C \rho^{-\frac{N+q}{(q-1)(R+N+1)}} \left(\int_{t-\rho^\kappa}^t \int_{B_\rho} u^R \right)^{\frac{1}{R+N+1}} + C \rho^{-\frac{N+q}{R+1-q}} \left(\int_{t-\rho^\kappa}^t \int_{B_\rho} u^R \right)^{\frac{1}{R+1-q}}, \end{aligned}$$

where $C = C(N, q, R)$. Now from estimate (3.3) of Lemma 3.2,

$$\begin{aligned} \sup_{B_{\eta/2}} u(\cdot, t) &\leq C \rho^{-\frac{\kappa N}{qR+N(q-1)}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{LR(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ C \rho^{-\frac{N+q}{(q-1)(R+N+1)} + \frac{\kappa}{R+N+1}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{LR(B_\eta)})^{\frac{R}{R+N+1}} \\ &+ C \rho^{-\frac{-(N+q)+\kappa}{R+1-q}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{LR(B_\eta)})^{\frac{R}{R+1-q}}. \end{aligned}$$

Let $\tau < T$, and $k_0 \in \mathbb{N}$ such that $k_0 \eta^\kappa / 2 \geq \tau$. For any $t \in (0, \tau]$, there exists $k \in \mathbb{N}$ with $k \leq k_0$ such that $t \in (k\eta^\kappa/2, (k+1)\eta^\kappa/2]$. taking $\rho^\kappa = t/(k+1)$, we find for any $0 < t < \tau$, and $C = C(N, q, R)$,

$$\begin{aligned} \sup_{B_{\eta/2}} u(\cdot, t) &\leq C \left(\frac{1 + \eta^{-\kappa} \tau}{t} \right)^{\frac{N}{qR+N(q-1)}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{LR(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ C \left(\frac{1 + \eta^{-\kappa} \tau}{t} \right)^{\frac{N+q}{R+N+1} - 1} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{LR(B_\eta)})^{\frac{R}{R+N+1}} \\ &+ C \left(\frac{1 + \eta^{-\kappa} \tau}{t} \right)^{\frac{N+q}{R+1-q} - 1} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{LR(B_\eta)})^{\frac{R}{R+1-q}}. \end{aligned} \quad (6.10)$$

If we choose κ such that $\kappa \varepsilon (N+q) q' \geq 1$, we obtain, with $C = C(N, q, R, \eta, \varepsilon, \tau)$,

$$\begin{aligned} \sup_{B_{\eta/2}} u(\cdot, t) &\leq C t^{-\frac{N}{qR+N(q-1)}} (t + \|u_0\|_{LR(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ C t^{\frac{1-\varepsilon}{R+N+1}} (t + \|u_0\|_{LR(B_\eta)})^{\frac{R}{R+N+1}} + C t^{\frac{1-\varepsilon}{R+1-q}} (t + \|u_0\|_{LR(B_\eta)})^{\frac{R}{R+1-q}} \end{aligned} \quad (6.11)$$

And in fact the second term can be absorbed by the first one, with a new constant depending on τ , and we finally obtain (1.15). \blacksquare

Remark 6.3 These estimate in $t^{-N/(qR+N(q-1))}$ improves the estimate in $t^{-N/2R}$ of the first regularizing effect when $q > q_*$. And it appears to be sharp. Indeed consider for example the particular solutions given in [25] of the form $u_C(x, t) = C t^{-a/2} f(|x|/\sqrt{t})$, where $\eta \mapsto f(\eta)$ is bounded, $f'(0) = 0$ and $\lim_{\eta \rightarrow \infty} \eta^a f(\eta) = C$. Then u_C is solution of (1.1) in $Q_{\mathbb{R}^N \setminus \{0\}, \infty}$, with initial data $C|x|^{-a}$. When $a < N$, that means $q > q_*$, then $|x|^{-a} \in L_{loc}^R(\mathbb{R}^N)$ for any $R \in [1, N/a)$, and u_C is solution in $Q_{\mathbb{R}^N, \infty}$. We have $\sup_{B_1} u(\cdot, t) = C f(0) t^{-a/2}$. Taking $N/R = a(1 + \delta)$, for small $\delta > 0$ our estimate near $t = 0$ gives $\sup_{B_1} u(\cdot, t) \leq C_\delta t^{-\frac{a}{2}(1+\delta)}$.

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