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Motives and automorphic representations

L. Clozel

Introduction

These are, slightly expanded, the notes of my 3-hour lecture at IHES in July 2006.

The organizers had assigned me two tasks. First, to give an overview of the conjectural relations between the motives defined by algebraic varieties over number fields and automorphic forms – here, automorphic forms on $GL(n)$. (The relation is mediated by two other kinds of objects, namely, Galois representations and L -functions). Secondly, to furnish – in a more leisurely way, at least for the topics I chose to present, than in his paper – the prerequisites necessary to understand the automorphic part of R. Taylor’s beautiful lecture, “Galois representations”, at the Beijing ICM (2002)¹.

These are rather succinct notes, so it was certainly not possible to present in detail, with proofs, the basic material: the reader will have to consult the standard references [36, 44, 18]. My purpose was rather to introduce a listener, not necessarily familiar with the modern theory of automorphic forms, with the basic objects of the theory – at least those pertinent in relation with the theory of motives, and the standard “dictionary” relating automorphic representations and motives.

The automorphic data, and their properties, are presented in Lecture 1; the dictionary in Lecture 2.

In both I have tried, not only to summarize *results*, but to state *problems* which may catch the fancy of young mathematicians. This is already true in Lecture 1 – see (1.12), (1.14), the end of § 1.2, as well as (1.19). In Lecture 2, I have seized the opportunity given by these lectures to discuss in some detail the relation between classical and automorphic objects (§ 2.3), answering some questions often asked by arithmetic geometers reading [18]. § 2.4-2.6 discuss some interesting problems which may be considered using both sides of the dictionary: the existence of motives with small ramification, and the degeneracy of totally even geometric Galois representations (over \mathbb{Q}). Consequently these notes are rather lopsided – Lecture 2 is heavier – but I hope that the new material will be found interesting.

My course of lectures took place between those by L. Fargues (on the theory of complex multiplication) and M. Harris (on the proof of the Sato-Tate Conjecture). Fargues’ lectures deal with the motives that are “potentially Abelian” and the associated automorphic objects and L -functions. This may be seen as a special case but, as the reader will see in his text, the theory is, there, essentially complete. On the other hand, my lectures were also intended as introducing part of the material needed by Harris, and reviewed by him in his § 1, 2. In particular, Lecture 3 introduces the Galois representations associated to self-dual automorphic representations, which play a crucial role in the proof of Sato-Tate. There is some duplication with Harris’ notes.

In conclusion I want to thank the organizers, and particularly J.-B. Bost and J.-M. Fontaine, for a very exciting conference; the audience for their sustained interest; and J.-P. Serre, M. Harris, J.-M. Fontaine and N. Ratazzi for useful discussions. I also thank Bost and Fontaine for allowing me to include here a long-overdue Errata to [18].

¹The reader will consult, not the ICM text, but the full version published in Ann. Fac. Sc. Toulouse, and available on his web site.

Addendum (July 2011)

Given the long delay in publication, there has been impressive progress in the field since these notes were written. It was not possible to rewrite them in order to include the new developments. I will simply refer to the recent papers, most of them in preprint form.

- The construction of (compatible systems of) Galois representations associated to suitable cuspidal representations for totally real or CM-fields is now complete. See [49] as well as the volume announced at the end of our chapter 3, [4] .

- The existence of cuspidal representations associated to compatible systems of Galois representations has now been proved in considerable generality. (One proves it only "potentially", i.e., one obtains the sought representation of $GL(n)$ only for a suitable Galois extension.) For the most general results to date see [8]. This relies on important work done in the meantime, in particular by Harris and the authors of this paper.

- In particular the Sato-Tate conjecture is now known with no ramification condition [9].

- The Ramanujan conjecture (Theorem 3.10) is now known under the natural assumptions (F totally real or CM, π cohomological and essentially self-dual.) See [49], [14] . For a simple proof at the unramified primes see [20].

Lecture 1: Algebraic representations

1.1. In this lecture F is a number field (the *ground field*),

$$G = GL(n).$$

If F is given I simply note

$$\mathbb{A} = \mathbb{A}_F = \text{adèles of } F.$$

I use standard notations: thus $\mathbb{A}_F = \prod'_v F_v$ (restricted product) where v denotes a prime (finite or Archimedean) of F , F_v is the completion at v ; \mathcal{O}_v is the ring of integers for v finite, ϖ_v a uniformizing parameter.

I want to define in full generality the *automorphic* objects that should eventually be identified with motives.

Fix $\omega =$ continuous character of $F^\times \backslash \mathbb{A}^\times$ – *not necessarily unitary*.

We will be interested in automorphic forms for the group G , i.e. functions on $G(F) \backslash G(\mathbb{A})$ (verifying certain conditions).

We identify $Z =$ center of G with \mathbb{G}_m , $Z(\mathbb{A}) = \mathbb{A}^\times$, and set :

Definition. –

$$\mathcal{A}_G^{\text{cusp}}(\omega) = \{f : G(F) \backslash G(\mathbb{A}) \rightarrow \mathbb{C}, f(zg) \equiv \omega(z) f(g) \quad (z \in Z(\mathbb{A})), f \text{ cuspidal}\},$$

with the right representation of $G(\mathbb{A})$.

Recall that “cuspidal” functions = cusp forms are, in this context, defined by the condition

$$\int_{N(F) \backslash N(\mathbb{A})} f(n g) dn \equiv 0 \quad (\text{identically in } g),$$

$N =$ unipotent radical of a proper parabolic subgroup of G .

The last condition means simply that, in block-matrix form,

$$N = \begin{pmatrix} 1 & * & * & \dots & * \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & \ddots & * \\ & & & & 1 \end{pmatrix}$$

$\underbrace{\hspace{1.5cm}}_{n_1} \quad \underbrace{\hspace{1.5cm}}_{n_2} \quad \dots \quad \underbrace{\hspace{1.5cm}}_{n_r}$

with $n = \sum n_i$, $(n_i) \neq n$.

There is also a mild “ L^2 ” condition which is a little complicated to define, because ω is not unitary: use that

$$G(\mathbb{A}) = A_G G(\mathbb{A})^1$$

where $A_G = \mathbb{R}_+^\times \subset \mathbb{A}^\times$ (diagonal embedding),

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) : |\det g| = 1\},$$

and that $G(F) \backslash G(\mathbb{A})^1$ has finite volume for the invariant measure.

With these definitions,

$$(1.1) \quad \mathcal{A}_G^{\text{cusp}}(\omega) = \widehat{\bigoplus} \pi$$

(countable direct sum of irreducible representations of $G(\mathbb{A})$ in Hilbert spaces) and by definition, each summand

$$\pi = \text{cuspidal representation of } G(\mathbb{A}), \text{ with central character } \omega.$$

Remark. – $G(\mathbb{A}_f) = \prod'_{v \text{ finite}} G(F_v)$ (restricted product) has compact-open subgroups, of the form

$$(1.2) \quad K = \prod_{\substack{\text{finite set} \\ S \text{ of } v}} K_v \times \prod_{\text{all other } v} G(\mathcal{O}_v)$$

with $K_v \subset G(F_v)$ compact-open ($v \in S$). Classically an automorphic form f is as before, but we impose moreover:

- For some (or all) K , f is K -finite under right translation
- At the Archimedean primes, f is K_v -finite where $K_v (= O(n) \text{ or } U(n))$ is maximal compact in $G(F_v)$ ($= GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$).

If f belongs to only one π in (1.1), f is then \mathfrak{Z} -finite for the action of the center \mathfrak{Z} of the enveloping algebra. I will not dwell on such matters, cf. [11].

If we consider the full decomposition

$$G(\mathbb{A}) = \prod'_{\text{all } v} G(F_v)$$

then each π in (1.1) is decomposable:

$$(1.3) \quad \pi = \bigotimes_v \pi_v$$

with a more or less obvious notion of \bigotimes – cf. [24]. The structure (1.2) of compact-open subgroups forces, too:

$$(1.4) \quad \text{For almost all } v \text{ (finite) } \pi_v \text{ has a vector fixed by } K_v^{\max} = G(\mathcal{O}_v).$$

Moreover we have the fundamental result of *multiplicity one*:

(1.5) *Any representation π of $G(\mathbb{A})$ occurs at most once in (1.1). In fact, if $S =$ finite set of primes and if $\pi^S = \bigotimes_{v \notin S} \pi_v$ is given, there is at most one summand ρ in (1.1) such that $\rho^S \cong \pi^S$ (strong multiplicity one).*

1.2. We can now define *algebraic representations*. Assume $v = \text{Arch. place of } F$, so $F_v = \mathbb{R}$ or $F_v \cong \mathbb{C}$. Langlands has classified the representations of $GL(n, F_v)$.

Let W_{F_v} be the Weil group

$$\begin{aligned} &= \mathbb{C}^\times \quad (F_v = \mathbb{C}) \\ &= \text{non-trivial extension of } \text{Gal}(\mathbb{C}/\mathbb{R}) \text{ by } \mathbb{C}^\times \quad (F_v = \mathbb{R}). \end{aligned}$$

Then (Langlands, B. Speh) there is a 1 – 1 correspondence

$$\left\{ \begin{array}{l} \text{Irreducible (admissible)} \\ \text{representations of } GL(n, F_v) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Semi-simple representations} \\ W_{F_v} \rightarrow GL(n, \mathbb{C}) \end{array} \right\}.$$

Of course both sides are taken modulo equivalence. “Admissible” is a technical condition, see [11]. It is true for our $\pi_v(v \mid \infty)$.

If $F_v = \mathbb{R}$ we “forget” the non-connected component of W_{F_v} . So in all cases π_v gives

$$\mathbb{C}^\times \xrightarrow{r_v} GL(n, \mathbb{C}), \text{ semi-simple } (v \mid \infty).$$

(1.6) **Definition.** – π is algebraic if $\forall v \mid \infty$,

$$r_v : \mathbb{C}^\times \ni z \mapsto |z|_{\mathbb{C}}^{\frac{1-n}{2}} (\chi_1 \oplus \dots \oplus \chi_n)$$

with χ_i algebraic: $\chi_i(z) = z^{p_i}(\bar{z})^{q_i}$, $p_i, q_i \in \mathbb{Z}$.

π is algebraic regular (\equiv cohomological) if

$$\forall v, \forall i \neq j : (p_i, q_i) \neq (p_j, q_j).$$

We make a few remarks, some of them non-trivial (see [18]).

(1.7) For $n = 1$, $\pi (= \omega)$ is an algebraic Grössencharakter in the sense of Hecke: for the basic properties see Weil [57].

(1.8) One shows that in fact, π being given, there exists $w \in \mathbb{Z}$ such that

$$p_i + q_i = w \quad (\text{all } v \mid \infty, \text{ all } i).$$

We will say π is of (pure) weight w .

(1.9) The factor $|z|_{\mathbb{C}}^{\frac{1-n}{2}}$ ($|z|_{\mathbb{C}} = z\bar{z}$) in (1.6) is annoying but cannot be avoided. This is first because the Langlands parametrization is “transcendental” and behaves badly w.r. to the rationality properties (see below). If we take it away (“Tate parametrization”, cf. [51]) we have problems w.r. to functorial properties, e.g. direct sums of motives. Note that $|z|_{\mathbb{C}}^{1/2}$ is a “half Tate twist”!

(1.10) Write $\pi = \pi_\infty \otimes \pi_f$ (representation of $G(F_\infty) \times G(\mathbb{A}_f)$). We look at the K -finite vectors of π_f , so we have a smooth representation of $G(\mathbb{A}_f)$. This is a totally algebraic notion, so π_f has a field of rationality: the smallest field E such that

$$E \subset \mathbb{C}, \pi_f^\sigma \cong \pi_f \quad \forall \sigma \in \text{Aut}(\mathbb{C}/E).$$

Besides purity (1.8), the next result shows that (at least under the “regular” assumption) algebraicity is a good notion:

(1.11) **Theorem.** – π algebraic regular. Then π_f is defined over a number field $E \subset \mathbb{C}$.

For the proof see [18].

In general if π is a representation of an (abstract) group G , we can define as above a field of rationality $E \subset \mathbb{C}$. It is not necessarily a field of definition, i.e., if V/\mathbb{C} is the space of π , we cannot necessarily find a representation V_E over E such that $V_E \otimes \mathbb{C} = V$ (as G -module).

The stronger property *is true* for $GL(n, \mathbb{A}_f)$ [18].

Also note that for $n = 1$, (1.11) is a theorem of Weil [57] – which would not have surprised Hecke. We also note that for algebraic regular representations one can prove not only the rationality over E of the local components of π – this is equivalent, at unramified primes, to the rationality of the conjugacy class of the Hecke matrix $t(\pi_v)$, see 1.3 – but an integrality property, depending on the system of coefficients V associated to π – see (2.14). This integrality property corresponds to the inequality between the Hodge and Newton polygons for the corresponding motive (Lecture 2). See V. Lafforgue [35].

A very interesting open question is

(1.12) **Conjecture.** – π algebraic. Then π_f is defined over a number field.

At this point it will be useful to review these objects in the “classical” case: $n = 2$, $F = \mathbb{Q}$. At the Archimedean prime we get a 2-dimensional representation of $W_{\mathbb{R}}$, given on $\mathbb{C}^\times = W_{\mathbb{C}}$ by

$$(1.13) \quad z \mapsto (z\bar{z})^{-1/2} (z^p \bar{z}^q, z^q \bar{z}^p)$$

with $w = p + q$ (= geometric weight).

(We get two conjugate characters in (1.13) since the representation extends to $W_{\mathbb{R}}$.) There are 3 cases:

(i) $p \neq q$. We are in the regular case. We know that π_f is rational (on some number field); in fact in this case π is associated to a classical cusp form \mathbf{f} , of weight $k = |p - q| + 1$ and this is essentially a reformulation of the Eichler-Shimura theory [48].

(ii) $p = q$. Up to a power of the half-Tate twist the representation of $W_{\mathbb{C}}$ is trivial. If we extend it to an *odd* representation r of $W_{\mathbb{R}}$:

$$w \mapsto (1, \operatorname{sgn} w) \quad (w \in W_{\mathbb{R}})$$

π is associated to classical holomorphic cusp forms, of weight $k = 1$. Again (1.12) is classical.

(iii) $p = q$, r *even*. Then π is associated to Maass forms (of eigenvalue $\lambda = \frac{1}{4}$: see [18] for details). The rationality result is unknown.

In general (say for $F = \mathbb{Q}$) the conjecture is accessible in some cases if the parameter $p = (p_i)$ of (1.6) is quasi-regular, i.e. there exists at most a pair $i \neq j$ with $p_i = p_j$. (Note that $q = (q_i)$ is defined up to order by $p_i + q_i = w$.) Otherwise little is known.

Since our purpose is to formulate as many *problems* as possible for the young generation, we also state:

(1.14) **Conjecture.** – Assume π is an arbitrary cuspidal representation, and π_f is defined over a number field $E \subset \mathbb{C}$. Then π is algebraic.

This (natural) conjecture was stated 20 years ago. For $n = 1$ it is a theorem of Waldschmidt [56]. Nobody seems to know even how to attack such non-Abelian transcendence questions.

Let us return to Theorem (1.11) and to this Conjecture, in the case $n = 1$. In this case Weil proves a weaker result when the infinity type (p_v, q_v) ($v \mid \infty$) of the Größencharakter χ is only *rational*: $p_v, q_v \in \mathbb{Q}$. Then some power m of χ is algebraic, and therefore “ χ_f is defined over $\bar{\mathbb{Q}}$ ”, i.e. $\chi(\varpi_v) \in \bar{\mathbb{Q}}$ for almost all v . The converse is true (Waldschmidt [56]): χ_f defined over $\bar{\mathbb{Q}} \Rightarrow$ the infinity type of χ is rational.

It would be interesting to study this question for $GL(n)$. I *do not know* what the answer should be. In this case it is not possible to consider m -th powers – even assuming Langlands functoriality, see 2.2 (*Exercise*: Why?). So even the fact that the direct implication:

$$\left. \begin{array}{l} \text{infinity type of } \pi \text{ rational} \\ (p_i, q_i \in \mathbb{Q} \quad \forall v) \end{array} \right\} \stackrel{?}{\implies} \pi_f \text{ def}/\bar{\mathbb{Q}}$$

should be reasonable (conjecturally) is not clear. However (?) will be true when π is automorphically induced from a Grössencharakter of an extension of F (see 2.2):

$$\pi = \text{ind}_K^F \chi,$$

$[K : F] = n$, $\chi =$ Abelian character. In this case even the converse (i.e., the analogue of Conjecture 1.14, with $\pi_f = \text{def}/\mathbb{Q}$ and π rational) is true, by reduction to the Abelian case. It is not clear, however, that these rather degenerate representations give evidence as to the general case.

1.3. In preparation for the next lecture we now review the data given by an algebraic representation π . By definition we have representations of $W_{\mathbb{R}}$ or $W_{\mathbb{C}}$ at the Archimedean primes. Conjecturally (1.12) we have a field of coefficients E – a number field.

Now assume v is a finite prime such that π_v is *unramified*, i.e., has a vector fixed by $G(\mathcal{O}_v)$. The unramified representation theory [15, 10] tells us that π_v is a subquotient of

$$\rho_v = \text{ind}_{B_v}^{G_v}(\chi_1, \dots, \chi_n).$$

Here $G_v = G(F_v)$, B_v is the Borel subgroup

$$B_v = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix},$$

χ_i is an unramified character of F_v^\times , (χ_i) defines a character of the diagonal subgroup of B_v and thus of B_v ; ρ_v is the induced representation [15]. We set

$$t(\pi_v) = \text{diag}(\chi_1(\varpi_v), \dots, \chi_n(\varpi_v))$$

where ϖ_v is a uniformizing parameter. It can be seen as a diagonal matrix (mod \mathfrak{S}_n) or as a conjugacy class in $GL(n, \mathbb{C})$.

(1.15) **Definition.** – $t(\pi_v)$ is the Hecke matrix of π at the unramified (for π) prime v .

At the ramified primes, there are more delicate data associated to π . Recall the Weil-Deligne group discussed by Kim and Sujatha. The local Langlands conjecture, now a theorem of Harris and Taylor [29], asserts that

(1.16) **Theorem.** – There is a natural bijection between

$$\left\{ \begin{array}{l} \text{Reps of } WD_{F_v}, \text{ Frob-semi-} \\ \text{simple, of dimension } n \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Admissible irreducible} \\ \text{representations of } GL(n, F_v) \end{array} \right\}.$$

Of course the bijection is between *isomorphism classes* on both sides. So given π we get a collection (r_v) of representations of WD_{F_v} , at the ramified primes.

1.4. One of the crucial properties of automorphic representations, and in particular cuspidal representations of $GL(n)$, is that they define L -functions with the usual properties (meromorphic continuation to the complex s -plane, functional equation).

Thus let π be a cuspidal representation of $GL(n, \mathbb{A})$. The next fundamental theorem is due to Godement and Jacquet, extending the method of Tate for $GL(1)$. See [30].

Assume $\pi = \otimes_v \pi_v$. For each (Archimedean or finite) prime they define a local L -function

$$L(s, \pi_v)$$

– a Γ -factor in the Archimedean case, a local Euler factor of the usual type in the p -adic case. There is then a local ε -factor

$$\varepsilon(s, \pi_v)$$

(whose choice depends in fact on an additive non-trivial character ψ_v of F_v); $\varepsilon(s, \pi_v)$ is associated to a local functional equation.

(1.17) **Theorem.** –

(1) *The Euler product $L(s, \pi) = \prod_v L(s, \pi_v)$ is absolutely convergent (thus holomorphic) for $\operatorname{Re}(s)$ sufficiently large.*

(2) *If π is not isomorphic to a power $|\cdot|^{s_0}$ ($s_0 \in \mathbb{C}$) of the idèle norm – in particular if $n > 1$ – $L(s, \pi)$ extends to a holomorphic function in the whole s -plane.*

(3) *It satisfies the functional equation*

$$L(s, \pi) = \varepsilon(s, \pi) L(1 - s, \tilde{\pi})$$

where $\tilde{\pi}$ is the dual representation and

$$\varepsilon(s, \pi) = \prod_{\text{all } v} \varepsilon(s, \pi_v).$$

The last product is trivially convergent for $\varepsilon(s, \pi_v) \equiv 1$ for almost all v (the local factors ε depend on local additive characters ψ_v ; if $\psi = \otimes_v \psi_v$ is a character of \mathbb{A}/F the product $\varepsilon(s, \pi)$ is independent of ψ).

The deep arguments relative to the Artin conjecture given by Taylor at the beginning of [52] rely on Weil's philosophy of the converse theorem. Assume now for definiteness π (cuspidal) unitary. If χ is a unitary character of $\mathbb{A}^\times/F^\times$, (1.17) implies in particular a functional equation

$$(1.18) \quad L(s, \pi \otimes \chi) = \varepsilon(s, \pi \otimes \chi) L(1 - s, \tilde{\pi} \otimes \chi^{-1})$$

where ε , a product, is explicitly determined. Conversely, Weil and Jacquet-Langlands showed for $GL(2)$ that if $\pi = \otimes_v \pi_v$, a unitary representation of $G(\mathbb{A})$ – not supposed to be automorphic – and if $L(s, \pi)$ and $L(s, \tilde{\pi})$ exist as holomorphic functions (with a few properties) and satisfy (1.18) for all χ , then π is indeed cuspidal – occurs in the space of cusp forms.

For $n \geq 4$ this is not true; conditions relative to higher twists have to be imposed (see [44] for a useful discussion). Piatetski-Shapiro has made the following conjecture. If π is given, say that π' is a modification of π if $\pi' = \otimes_v \pi'_v$ differs from π only at a finite number of primes. Also, we will accept the notion of automorphic representation – these are essentially representations of $G(\mathbb{A})$ occurring in spaces of automorphic forms on $G(F)\backslash G(\mathbb{A})$ that are not (necessarily) cuspidal. (For precise definitions see [37]).

(1.19) **Conjecture.** – *Assume $\pi = \otimes_v \pi_v$ is given, and the central character ω_π of \mathbb{A}^\times is trivial on F^\times . Assume the twisted L -functions $L(s, \pi \otimes \chi)$, for Abelian characters χ , are absolutely convergent in some half-plane $\operatorname{Re}(s) > \sigma_0$, with the expected properties of holomorphic continuation and verifying (1.18). Then a modification π' of π is automorphic.*

For a more precise statement of the Conjecture – which implies ours – see [21], p. 166. (The authors impose that π' should be isomorphic to π at all finite, unramified primes.)

To the writer's knowledge no approach is known to this very deep conjecture – even its plausibility is not clear. For a spectacular application, again, see [52]. In the full version of the paper, Taylor shows that (essentially) Conjecture 1.19 will imply, in many cases, Artin's Conjecture on the holomorphy of the L -functions associated to irreducible Artin representations.

[Actually Taylor's statement of a variant of Conjecture 1.19 – Conjecture 3.2 in his paper – is incorrect. Taylor asserts that under the hypothesis of (1.19) the completed L -function of π can be written

$$(1.20) \quad L(s, \pi) = \prod_{i=1}^r L(s, \pi_i)$$

where $n = \sum_{i=1}^r n_i$ and π_i is a cuspidal representation of $GL(n_i, \mathbb{A})$. This is false in general. For instance, if $n = 2$, consider the representation

$$\rho = \text{ind}_{B_2(\mathbb{A})}^{GL(2, \mathbb{A})} (| \cdot |^{1/2} \otimes | \cdot |^{-1/2}) = \bigotimes_v \rho_v$$

where B_2 is the Borel subgroup in $GL(2)$, $| \cdot |$ is the idèle norm, and induction is unitary. At any prime, ρ_v has length 2, with the trivial representation $\mathbf{1}_v$ as a quotient and the Steinberg representation St_v as a submodule. Any constituent

$$\pi(S) = \bigotimes_{v \in S} \text{St}_v \otimes \bigotimes_{v \notin S} \mathbf{1}_v \quad (S \text{ finite})$$

of ρ is automorphic [37]. The L -function is

$$L(\pi(S), s) = \pi^{-s} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s-1}{2}\right) \prod_{v \in S} \frac{1}{1-p^{-1/2-s}} \cdot \prod_{v \notin S} \frac{1}{1-p^{-1/2-s}} \frac{1}{1-p^{1/2-s}}.$$

(We have assumed $\pi_\infty \cong \mathbf{1}_{\mathbb{R}}$.) This L -function and its twists satisfy the conditions in (1.19) (Jacquet [30]), but it is not of the form (1.20).

In order to correct Taylor's argument, first note that the argument about automorphic induction in § 5.3.1 of [52] (complete version) remains correct if we remove a finite set S of primes, containing the Archimedean primes, and use (1.19) or Piatetski-Shapiro's variant. One then gets an identity²

$$L_K^S(\text{ind } \pi, s) = L^S(\Pi, s) = \prod_{i=1}^r L_K^S(\pi_i, s)$$

(Taylor's notations in § 5.3.1). In the argument of his § 5.3.2, concerning a pole of

$$L(R \otimes R^*, s)$$

at $s = 1$, replace L by L^S , which does not change the order of the pole for an Artin L -function.

Similarly, in the identity

$$(1.21) \quad L(R, s) = \prod_i L(\pi_i, s)^{m_i}$$

of Taylor, § 5.3.2, replace the L -functions by the partial L -functions. (I have omitted Taylor's ι , assuming that the Artin representation is complex-valued. . .). By construction the cuspidal representations π_i in (1.21) are unitary – because π_i is automorphically induced from an Artin character: the central character of π_i is then an Artin character. This implies again that the order of the pole, at $s = 1$, of $L(s, \pi_i \times \tilde{\pi}_j)$ and $L^S(s, \pi_i \times \tilde{\pi}_j)$ coincide.

Now Taylor's argument in the first paragraph of 5.3.1 shows that $L^S(R, s) = L^S(\pi, s)$ for a cuspidal representation π . By a result of Henniart – see Taylor, § 5.3.1 – this implies $L(R, s) = L(\pi, s)$, whence the Artin conjecture in the case considered there. The correction was indicated by Taylor.]

²The following correction is written for a reader who is studying Taylor's paper, and I refer to it for the pertinent definitions.

Lecture 2: The conjectural dictionary

2.1. Fix a number field F and let M be a (pure) motive over F with coefficients in E , a number field. In agreement with Lecture 1 we assume $E \subset \mathbb{C}$; let $\bar{\mathbb{Q}}$ be the algebraic closure of \mathbb{Q} in \mathbb{C} . Assume M absolutely irreducible. There are a number of (local) data associated to M .

First consider an Archimedean embedding $\iota : F \hookrightarrow \mathbb{C}$. If M is a piece of $H^i(X)$ for a smooth, projective variety X/F (cut out by correspondences with coefficients in E), we can consider $M_{\iota,B} \subset H_B^i(X_{F,\iota} \times \mathbb{C}, E) =$ Betti cohomology, with coefficients in E , of the variety over \mathbb{C} obtained from X by extension of scalars. Write X_ι for $X_{F,\iota} \times \mathbb{C}$. Then $M_{\iota,B} \otimes \mathbb{C} \subset H_B^i(X_\iota, \mathbb{C})$. This subspace of $H_B^i(X_\iota, \mathbb{C})$ inherits a Hodge decomposition. If $n = \dim(M)$, we then have, for all embeddings ι , a Hodge structure (over \mathbb{C}) of dimension n . Thus defines an algebraic representation of $W_{\mathbb{C}} = \mathbb{C}^\times : z \in \mathbb{C}^\times$ acts on the summand M_{pq} by $z \mapsto z^{-p}(\bar{z})^{-q}$.

Assume ι is a complex (non real) embedding, defining a prime v of F . Then ι is defined by v and the choice of an isomorphism $F_v \xrightarrow{\sim} \mathbb{C}$. If we replace ι by its complex conjugate $\bar{\iota}$, the isomorphism is changed by conjugation, and the new representation of $W_{\mathbb{C}}$ also. Thus M defines, for each complex prime, a well-defined representation of $W_{F_v} = F_v^\times$. (The choice of isomorphism $F_v \cong \mathbb{C}$ was left implicit in Lecture 1).

If ι is a real embedding, the complex conjugation acts on X_ι and preserves $M_{\iota,B} \otimes \mathbb{C}$ (because correspondences are defined *over* F). The prescriptions of Serre [46], § 3.2 show that the representation of $W_{\mathbb{C}}$ extends to a canonically defined representation of $W_{\mathbb{R}} = W_{F_v}$ (where v is the prime defined by ι).

Now let ℓ be a finite prime, and λ a prime of E dividing ℓ . If v is a finite prime of F , we get a representation of $\text{Gal}(\bar{F}_v/F_v)$ on $M_\lambda \subset H_{\text{ét}}^i(X \times_F \bar{F}_v, E_\lambda)$ (étale cohomology) in a similar manner. If $r_{\lambda,v}$ is this n -dimensional representation, and v does not divide ℓ , $r_{\lambda,v}$ defines in fact a representation of the local Weil-Deligne group. We will denote, by abuse of notation, by $r_{\lambda,v}$ the *Frobenius semi-simplification* of this representation of WD_{F_v} .

Globally, we get a system (r_λ) of λ -adic representations of $\text{Gal}(\bar{F}/F)$, and this system is expected to be *strictly compatible*. For fixed v , and different primes λ, μ of E (prime to v), this implies that the representations $r_{\lambda,v}$ and $r_{\mu,v}$ are essentially isomorphic, for instance in the following sense: there exists a number field $E' : E \subset E' \subset \mathbb{C}$ such that $r_{\lambda,v}$ and $r_{\mu,v}$ are defined and isomorphic over E' . In particular, we have a well-defined representation r_v over \mathbb{C} .

Finally, if v is a place of good reduction for M (i.e., for instance, for X), r_v is just the collection of eigenvalues of the (geometric) Frobenius endomorphism Frob_v , a diagonal matrix in $GL(n, \bar{\mathbb{Q}})$ modulo conjugation. We will denote it by $f_{M,v}$. (We have discussed the data associated to an *effective* motive $M \subset H^\bullet(X)$. A general motive is of the form $L^{-n} \otimes M$ where M is of this type and L , the Lefschetz or Tate motive, is $H^2(\mathbb{P}^1)$. The previous constructions then extend in obvious ways to arbitrary motives).

We now return to Langlands' "local conjecture", now a theorem (Theorem. 1.16). The same problem as in (1.9) arises as to the normalization of the Langlands correspondence – for a precise discussion see Tate [51]: we will need to twist the local representation of WD_{F_v} associated to a cuspidal representation of $GL(n, \mathbb{A})$.

For this, normalize, for a p -adic field L , the bijection given by local class field theory:

$$(L^\times)_c \xrightarrow{\sim} \text{Gal}(L^{ab}/L)$$

(where $(L^\times)_c$ is the profinite completion of L^\times) so that a uniformizing parameter corresponds to the geometric Frobenius. This fixes uniquely a map

$$W_L \rightarrow L^\times,$$

so the absolute value $|\cdot|_L$ defines a representation of W_L . Finally, if r is the (Frob-semisimple) representation of W_L associated to a representation π of $GL(n, L)$, define by $r^T = r^T(\pi)$ the representation:

$$r^T(\pi) = |\cdot|_L^{\frac{1-n}{2}} r(\pi).$$

2.2. We can now state the general conjecture associating motives and cuspidal representations. We will consider “motives with coefficients in $\bar{\mathbb{Q}}$ ”, i.e., motives over some number field F , with coefficients in a (finite but unspecified) $E \subset \bar{\mathbb{Q}}$ and irreducible over $\bar{\mathbb{Q}}$. Note that the local representations of WD_{F_v} defined in 2.1 required a *finite* field of coefficients (finite over \mathbb{Q} !) for their definition; however they are defined over $\bar{\mathbb{Q}}$ and then independent of the choice of E (in particular, at the unramified primes, the Frobenius matrix $f_{M,v}$ is algebraic over \mathbb{Q}).

(2.1) **Conjecture.** – *There is a 1 – 1 correspondence between irreducible motives over F , with coefficients in $\bar{\mathbb{Q}}$, and cuspidal, algebraic representations of $GL(n, \mathbb{A})$. Moreover, if M and π are associated:*

(2.1.1) *The representations of W_{F_v} defined by the Hodge structure of M , and by $|\cdot|_v^{\frac{1-n}{2}} r_v$, coincide at the Archimedean primes.*

(2.1.2) *At a finite prime, the representations $|\cdot|_v^{\frac{1-n}{2}} r_v(\pi) = r_v^T(\pi)$ and $r_v(M) = r_{\lambda,v}$ (for some E, λ) are isomorphic (over $\bar{\mathbb{Q}}$).*

(2.1.3) *In particular, the unramified primes for π and M coincide and, at such a prime:*

$$t(\pi_v) = f_{M,v}$$

(identity of semi-simple $\bar{\mathbb{Q}}$ -matrices in $GL(n, \bar{\mathbb{Q}})$, up to conjugation).

We now discuss the obvious consequences of the Conjecture, in order to insist on its power (and, probably, its inaccessibility in this generality, despite the extraordinary recent advances). The first one is obvious. Since local L -functions can be associated to representations of the Weil-Deligne groups (Langlands, cf. Tate [51]) as well as ε -factors, we can define the L -function of M (recall that $\bar{\mathbb{Q}} \subset \mathbb{C}$):

$$(2.2) \quad \begin{cases} L(M, s) &= \prod_v L_v(M, s), \\ L_v(M, s) &= L(r_v(M), s). \end{cases}$$

(2.3) **Corollary.** (of Conjecture 2.1) – *The Hasse-Weil L -function $L(M, s)$ extends to a meromorphic function in the complex plane. For M irreducible, it is holomorphic except if M is a Tate twist of the trivial motive $\mathbf{1}_F$. (For $M = \mathbf{1}_F$ it is the zeta function $\zeta_F(s)$, completed with its Archimedean factors, and with a simple pole at $s = 1$).*

It verifies the functional equation:

$$L(M, s) = \varepsilon(M, s) L(\tilde{M}, 1 - s)$$

where \tilde{M} is the dual motive.

Indeed this is well-known for cuspidal representations.

For much more detailed information on $L(M, s)$ and the arithmetic properties of its special values, see Fontaine and Perrin-Riou [27].

In the other direction, we must now introduce the *principle of functoriality* discovered by Langlands, but only in the simplest case.

Assume π is a cuspidal representation of $GL(n, \mathbb{A})$ and let $(t(\pi_v))_{v \notin S}$ be its collection of Hecke matrices – at almost all finite primes. Thus $t(\pi_v) \in GL(n, \mathbb{C})$. Let R be an arbitrary (algebraic, finite-dimensional) representation of $GL(n)$. Set

$$T_v = R(t(\pi_v)) \quad (v \notin S).$$

We think of T_v as a conjugacy class in $GL(N, \mathbb{C})$ where $N = \dim(R)$. Then we get a candidate for a family of Hecke matrices, for a representation Π of $GL(N, \mathbb{A})$.

(2.4) **Conjecture.** (Langlands) – *There exists an automorphic representation Π of $GL(n, \mathbb{A})$ whose Hecke matrices are almost everywhere equal to T_v .*

We have written “automorphic representation” and not “cuspidal representation” and this should be natural in view of what we know so far. If we form a higher tensor (i.e., R) of a motive M , $R(M)$ will generally not be irreducible even if M is. Thus Conjecture 2.1 forces us to have a wider collection of automorphic representations. Note also that Langlands’ principle is not restricted to algebraic representations³.

If we were able to construct the (Tannakian) category of (Grothendieck) motives, tensor operations would be available, tautologically. Thus we get the following approximation to the Functoriality Principle, which we state rather loosely:

(2.5) *Assume Conjecture 2.1. If π is cuspidal algebraic, $\Pi = R(\pi)$ exists.*

The conclusion of this discussion is this: motives and automorphic representations have complementary (known) properties. When we are able (in some cases) to use the dictionary, we obtain very deep new properties of either side.

To prepare for the next lecture, we discuss another case of Langlands functoriality, namely, base change. Assume K/F is a finite (Galois) extension of number fields.

Let first π_F be a cuspidal representation of $GL(n, \mathbb{A}_F)$. If v is a prime of F , $w|v$ a prime of K , $r_v(\pi)$ defines by restriction a representation of WD_{K_w} ; so a representation Π_w of $GL(n, K_w)$. Langlands functoriality predicts the existence of a representation Π of $GL(n, \mathbb{A}_K)$ (*automorphic* representation) such that the local representations of the WD -groups are so associated. It is denoted by

$$\Pi = \text{Res}_F^K \pi$$

(*automorphic restriction*) – assuming it exists. It is known to exist for K/F solvable [7].

Similarly, if $d = [K : F]$, we can start with $\pi =$ cuspidal representation of $GL(n, \mathbb{A}_K)$ and define a representation of $GL(nd, \mathbb{A}_F)$ by *induction* on the Weil side. For the correct definition of induction in this case (taking into account the decomposition of primes) see [7], p. 215. Note that the local representations there are supposed unramified, but this extends in an obvious fashion to (Frob-semi-simple representations of the WD groups at) all primes. We write

$$\Pi = \text{Ind}_K^F \pi.$$

Again Π is known to exist for solvable extensions. (The corresponding operations for motives are more or less transparent).

Finally, we note that the dictionary is essentially known in the case of potentially Abelian motives: this is discussed in the lectures of Fargues.

2.3. We now discuss some examples.

Of course the first interesting one is an elliptic curve \mathcal{E}/\mathbb{Q} , which defines (unconditionally) a motive M over \mathbb{Q} , with \mathbb{Q} -coefficients: this is “ $M = H^1(\mathcal{E})$ ”.

Conjecture 2.1 is known in this case, due to the work of Wiles, Taylor [58, 53] and to [13] in the most general case. The associated algebraic representation π (of Hodge type (z^{-1}, \bar{z}^{-1}) and geometric weight 1) is associated to classical modular forms of weight 2 and with rational coefficients.

³If we start with a cuspidal π , Π will belong (up to a Tate twist) in a rather simple set of representations, called “induced from cuspidal” in [7].

We use this example to explain the role of coefficients, and the necessity of considering them. Assume \mathcal{E} is a CM-curve over a number field F : this means that $\text{End}(\mathcal{E} \times \mathbb{C}) \otimes \mathbb{Q} = E$ is different from \mathbb{Q} , and it is then a quadratic imaginary field. Let $F = H$ be the Hilbert class field of a given quadratic imaginary E . Then there exist elliptic curves over F , with complex multiplication by E , the complex multiplications being defined over F . (The best reference for this is Gross [28].)

If we think of $M = H^1(\mathcal{E})$ as a motive over F , and form $M_E = M \otimes_{\mathbb{Q}} E$ – extension of the *coefficients* – we obtain a motive with coefficients in E , which is reducible: $M_E = M' \oplus M''$, each summand having rank 1, the corresponding projectors being given by the action of E and the decomposition $E \otimes E \cong E \oplus E$.

Thus M_E is the direct sum of two irreducible motives, associated to Größencharakterer of \mathbb{A}_F^\times . This, and deep generalizations, is the object of Fargues' lectures. I emphasize that M is irreducible as a motive with \mathbb{Q} -coefficients (*even over F*) but that $M \otimes E$ is reducible (*over F*), and thus associated to cuspidal representations (= characters) of $GL(1)$ not $GL(2)$. If E has class number 1, there are such examples where \mathcal{E} is defined over \mathbb{Q} , the complex multiplications (by E) being defined over E – not \mathbb{Q} , why?

Now I want to say something about Abelian varieties. So fix $g \geq 2$ and consider an Abelian variety A over \mathbb{Q} , of dimension g . Then $M = H^1(A)$ is a motive of rank $2g$. For “generic” A it will be irreducible, even after extension of scalars (ground field) or coefficients. Here “generic” means “with no complex multiplications”, i.e. $\text{End}(A \times \mathbb{C}) = \mathbb{Z}$. One then expects that the image of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $GL(H^1(A \times \overline{\mathbb{Q}}, \mathbb{Q}_\ell))$ should be of finite index in $\text{GSp}(g, \mathbb{Q}_\ell)$; if g is odd this is a theorem of Serre [47, 17]. This should imply the irreducibility of $H^1(A) \otimes E$ (any E) over any ground field F (finite over \mathbb{Q} ...).

Now return to elliptic curves, and the theorem of Wiles and Taylor. In this case the construction of the algebraic representation π is obtained by showing that M occurs in $H^1(X)$, where X is a *modular curve* over \mathbb{Q} . In the next lecture we will show that, under many restrictions, we can essentially associate to an algebraic representation π a motive M occurring in $H^\bullet(X)$ where X is a Shimura variety, associated to a unitary group. (For Shimura varieties, see the lectures by Genestier and Ngô).

So, at least in principle, the question arises: given an abelian variety A of higher dimension, can we find, in the cohomology of a Shimura variety X , a motive M isomorphic to $H^1(A)$ – perhaps after extending F , this would be good enough for some purposes?

Unfortunately, the answer is a very clear NO. For instance, in the next lecture, we will be considering Shimura varieties associated to unitary groups G/\mathbb{Q} such that $G(\mathbb{R}) \cong U(1, n-1)$. The parts of the cohomology of X cut out by Hecke correspondences – our best approximation to the motive M – are *regular* in the sense of Definition 1.6 – i.e., if the Hodge structure on $M \otimes \mathbb{C}$ is given by the Hodge numbers (p_i, q_i) , each Hodge number (p, q) occurs with *multiplicity one*. But the Hodge structure of $H^1(A)$ is of type (z^{-1}, \bar{z}^{-1}) , each character of \mathbb{C}^\times occurring with multiplicity g .

The conclusion is that the theory of Shimura varieties will at best yield a rather small subset in the collection of all motives, those with regular Hodge structure. There is another strong restriction, of “self duality”.

This will be central in Lecture 3, so let us say a few words about it from the motivic point of view. Assume X is a smooth projective variety over F , and consider $M = H^i(X)$.

Lefschetz theory gives $L^{d-i} : H^i \rightarrow H^{2d-i}$, which we can think of as a tensor product $(d-i)$ times with the Lefschetz motive L , of weight 2. By Poincaré duality we get $H^i \otimes H^{2d-i} \rightarrow H^{2d} = \bigotimes^d L$ (X connected). This is non-degenerate, so finally H^i is isomorphic to its dual, up to a suitable “Lefschetz” or “Tate” twist. The corresponding pairing is a polarization on H^i – see Deligne [22].

So a motive equal to the full $H^i(X)$ – for some X – is “*essentially* (= up to a twist) *self-dual*”. This should be true for the “generic” motive, even if we extend coefficients. In some “degenerate” cases, however, even if we consider $M = H^i(X)$, $M \otimes E$ may break up into 2 or more pieces, each being paired with its dual by the polarization. This is definitely known to happen, and in the absence of self-duality (up to a twist), we

do not know so far how to exhibit corresponding motives, using Shimura varieties⁴. For a beautiful example, where the existence of an algebraic representation of $GL(3)$ associated to a non-self-dual motive M with coefficients in $E = \mathbb{Q}(\sqrt{-1})$ is proved *experimentally*, see [55]. (Here a natural 6-dimensional motive M in the cohomology $H^2(S)$ of an algebraic surface splits into two 3-dimensional motives when coefficients are taken in E .)

2.4. In this paragraph we discuss another consequence of the main Conjecture 2.1, concerning the existence of motives with small ramification (ideally: motives unramified at all primes). The ground field is \mathbb{Q} .

The first problem of this kind is:

(2.6) *Fix g . Does there exist an Abelian variety A , of dimension g , over \mathbb{Q} , and having good reduction everywhere?*

The condition imposed, then, is that for any p the local representation of $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $H^1(A \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ should be unramified (if $\ell \neq p$). (This is equivalent to the existence of an Abelian scheme over \mathbb{Z} [43] giving A by extension of scalars.)

As is well-known the answer is NO. For $g = 1$ this is a theorem of Tate ([50], Exercise 8.15). For arbitrary g , it is a famous theorem of Fontaine [25] and Abrashkin [5]:

(2.7) **Theorem.** – *There exists no Abelian variety over \mathbb{Q} with good reduction at all primes.*

Fontaine’s proof relies on the study of the group of torsion points $A[p^n]$, a finite, flat commutative group scheme over \mathbb{Z} . For odd $p \leq 17$, he shows that such a group is a direct sum $(\mathbb{Z}/p^n\mathbb{Z})^a \oplus (\mu_{p^n})^b$; for an Abelian variety this implies by self-duality that

$$A[p^n] \cong (\mathbb{Z}/p^n\mathbb{Z})^g \oplus (\mu_{p^n})^g \quad (p \leq 17 \text{ odd}).$$

But this implies the existence of p^{ng} points of torsion in $A(\mathbb{Q})[p^n]$ (all n) contradicting, for instance, the Mordell-Weil theorem, or, more simply, the fact that the kernel of $A(\mathbb{Q}_p) \rightarrow \mathcal{A}(\mathbb{F}_p)$ has no p -torsion, \mathcal{A} being the Néron model of A .

Now *assume* Conjecture 2.1. Then, associated to $H^1(A)$, there should exist a cuspidal representation π of $GL(2g, \mathbb{A})$ with the following properties:

- $\pi \cong \tilde{\pi}$
- For any finite prime, π_p is unramified, with Hecke matrix

$$t(\pi_p) = \text{diag}(\alpha_p^1, \dots, \alpha_p^{2g}), \quad |\alpha_p^i| = 1.$$

- π_∞ is associated, by the Langlands correspondence (1.2) to the representation of $W_{\mathbb{R}}$ $r_\infty = \bigoplus^g r_0$, where

$$r_0 = \text{ind}_{W_{\mathbb{C}} = \mathbb{C}^\times}^{W_{\mathbb{R}}} (z/\sqrt{z\bar{z}}).$$

⁴If M is an irreducible motive (irreducible over $\bar{\mathbb{Q}}$) with \mathbb{Q} -coefficients, M is self-dual modulo a twist. This was explained to us by Deligne: “Ce serait une conséquence des conjectures standard, qui donnent une polarisation de la catégorie des motifs: pour M de poids n , on obtient une forme bilinéaire $(-1)^n$ -symétrique “définie positive”. Côté Hodge: $\langle x, h(i)x \rangle > 0$ pour $x \in H_{\mathbb{Q}} \otimes \mathbb{R}$, h étant défini par $h(z)x = z^{-p}\bar{z}^{-q}$ pour x dans $H^{p,q}$. Pour la partie primitive (au sens de Lefschetz difficile) du H^i , à un signe près, le cup-produit et L fournissent une forme “définie positive”. Comme son nom le suggère, si $\psi : M \otimes M \rightarrow \mathbb{Q}(-n)$ est défini positif, la restriction de ψ à $N \subset M$ l’est aussi. Tout motif à coefficients dans \mathbb{Q} (absolument irréductible ou non) est donc isomorphe à son dual à un twist de Tate près. Pour un motif à coefficients dans un corps CM E , on a une E -dualité (valeurs $E(n)$) entre M et le E -motif $M \otimes_{E, \sigma} E$ (σ la conjugaison complexe) On renvoie à [45] pour le formalisme.”

For $g = 1$ this just means that π is associated to classical modular forms of weight 2 (by the condition on π_∞ , see 1.2) on $\Gamma(1) = SL(2, \mathbb{Z})$ (by the “unramified” condition). Since there are no such forms (in weight $k < 12$) we see that A cannot exist, in conformity with Tate’s result. For general g , consider the L -function $L(s, \pi)$. We can compute the Archimedean factor $L_\infty(s, \pi)$, and this yields [51]:

$$(2.8) \quad L_\infty(s, \pi) = \left\{ (2\pi)^{-\frac{1}{2}-s} \Gamma\left(\frac{1}{2} + s\right) \right\}^g .$$

(We have omitted a factor 2^g from [51], here irrelevant.) With

$$(2.9) \quad L_p(s, \pi) = \frac{1}{(1 - \alpha_p^1 p^{-s}) \dots (1 - \alpha_p^{2g} p^{-s})} ,$$

the function $L(s, \pi) = L_\infty(s, \pi) \prod_p L_p(s, \pi)$ should be (*modulo* Conjecture 2.1, or the Hasse-Weil Conjecture) holomorphic in the whole s -plane, with functional equation

$$(2.10) \quad \begin{cases} L(\pi, s) = \varepsilon(\pi, s) L(\pi, 1 - s) \\ \text{with } \varepsilon(\pi, s) = (-1)^g . \end{cases}$$

Now the behaviour of such an L -function can be studied using analytic number theory – specifically, Weil’s explicit formulas. And Mestre was able to prove [40]⁵:

(2.11) **Theorem.** – *There exists no L -function, given by the Euler factors (2.8) and (2.9) and verifying the functional equation (2.10).*

Thus the existence of π (so of the corresponding L -functions), if it were given by the dictionary, would imply a different proof of (2.7).

2.5. We continue this discussion in order to state a very interesting problem along the same lines. The proof of Fermat’s last theorem – through the arguments of Ribet and Wiles – relies on the non-existence of modular cusp forms of weight 2 and level 1 (or even 2). Analytically this can be verified by Riemann-Roch, or by the arguments of Theorem 2.11 (again, 2 very different proofs). This same fact – no form of weight 2 and level N for small N – is also crucial in the proof by Khare and Wintenberger of Serre’s conjecture for modular forms mod ℓ (for odd levels: see [32]).

The Hodge structure associated to a form of weight 2 – rather, to the associated cuspidal representation, so normalized as to be *unitary* – is expressed by r_0 , whose restriction to $W_{\mathbb{C}} = \mathbb{C}^\times$ is

$$z \mapsto \begin{pmatrix} z/\sqrt{z\bar{z}}, \\ \bar{z}/\sqrt{z\bar{z}} \end{pmatrix}$$

so $\{(p, q)\} = \{(1, 0), (0, 1)\}$, cf. (1.13). This is a regular Hodge structure so π_∞ is regular algebraic, or cohomological (1.6). In § 1 we had not specified what “cohomological” meant. Here it means that

$$(2.12) \quad H^\bullet(GL(2, \mathbb{R}), \mathcal{H}_{\pi_\infty}) \neq 0$$

for a suitable cohomology theory, cf. [12]. (In this case the relevant degree is 1.) It is this fact, combined with the relation between the cohomology (2.12) and the Betti cohomology of the associated arithmetic quotients of the symmetric spaces, which allows one to prove Theorem 1.11. See [12, 18].

As we pointed out in § 2.3, for an Abelian variety A of dimension > 1 the corresponding representation $r_\infty = \bigoplus^g r_0$ is no longer regular. On the other hand, in the proof of (2.11), the precise form of the Archimedean factors is capital.

⁵After Serre, cf. [40].

Now consider cuspidal, algebraic representations of $GL(n, \mathbb{A}_{\mathbb{Q}})$, still unramified at all finite primes (if they exist!). We want to impose the condition similar to (2.12):

$$(2.13) \quad H^{\bullet}(GL(n, \mathbb{R}), \mathcal{H}_{\pi_{\infty}}) \neq 0.$$

Here we must insist on one point. Condition (2.12) characterizes cuspidal representations of $GL(2)$ associated to forms of weight 2, or to the cohomology of modular curves with a *trivial* local system. As is well-known (see Saito’s lectures) forms of weight $k > 2$ will correspond to the cohomology of modular curves, with coefficients in a local system defined by the representation $\text{Sym}^{k-2}(\mathbb{C}^2)$ of $GL(2, \mathbb{R})$.

Similarly, if we want to obtain all *algebraic regular* cuspidal representations of $GL(n, \mathbb{A})$ (Def. 1.6) we must consider more generally the condition

$$(2.14) \quad H^{\bullet}(GL(n, \mathbb{R}), \mathcal{H}_{\pi_{\infty}} \otimes V) \neq 0$$

where V is an arbitrary (complex) representation of finite dimension of $GL(n)$. See again [12, 18]. The more restricted (2.13) corresponds, then, to “forms of weight 2” for $GL(2)$.

Now for $GL(2)$ there exist (classical) cusp forms of weight k , everywhere unramified (\equiv of level 1) for k sufficiently large, in fact $k \geq 12$. Similarly for $GL(n)$ we can expect the existence of cuspidal representations π (even self-dual), everywhere unramified, and verifying (2.14) for a “sufficiently twisted” coefficient system V . It seems much harder to obtain cohomology with trivial coefficients (2.13).

The problem has been raised by C. Khare:

(2.15) **Problem.** – For n sufficiently large, do there exist cuspidal, self-dual, (algebraic) representations of $GL(n, \mathbb{A})$, unramified at all finite primes and verifying (2.13)?

The motivation is that this would give the “reduction to level 1” argument used by Khare and Wintenberger, but for modular Galois representations of degree n^6 .

In order to solve this question, it is natural to try to apply Mestre’s method of proof. This has been done by Fermigier and Miller. However, the Archimedean factor of the L -functions becomes, for large n , more and more different from that in the “Abelian motive” case and the method fails for large n . It works for sufficiently small n . In fact, Fermigier [23] and S.J. Miller [41] show that there exist no cuspidal representations having cohomology with trivial coefficients for $n \leq 23$.

In the proofs of Fermigier and Miller self-duality is not used. I have imposed it in (2.15) because only in this case can we hope (in the present state of our knowledge) to obtain the representation of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ associated to π . Under suitable ramification assumptions on π this is actually known – see § 3. It is expected that these conditions can be dispensed with, cf. § 3.4.

Assuming, then, the existence of the (compatible system of) Galois representations associated to π , say (r_{ℓ}) , each r_{ℓ} will be unramified at all primes $p \neq \ell$ with specified (small) Hodge-Tate weights. One may hope to prove the *non-existence* of such a representation by an extension of Fontaine’s method; whether this is possible is unclear.

On the other hand, self-dual representations are amenable to an analogue of the Riemann-Roch theorem, namely, the twisted trace formula. (See the program formulated at the very end of Arthur’s notes [6], § 30.) This may lead to an answer, positive or negative⁷.

For Khare’s own discussion of these questions see [31].

2.6. I conclude this tour of the problems motivated by the dictionary by discussing a very striking consequence for Galois representations. For this I must introduce the Fontaine-Mazur Conjecture [26].

⁶Note that “algebraic” is between brackets because it is implied by (2.13).

⁷For the reader familiar with Arthur’s work – up to the last line of [6]! – note that the difficulty will be to show the existence of a π that is *cuspidal*, rather than simply belonging to the discrete spectrum.

Let L denote an ℓ -adic field – a finite extension of \mathbb{Q}_ℓ – and consider an n -dimensional, continuous irreducible representation

$$r : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL(n, L).$$

We make the following assumptions on r :

(2.16 i) For almost all $p \neq \ell$, r is unramified at p .

Let

$$G_\ell = \text{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \subset \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

the injection being given by the choice of a prime of $\bar{\mathbb{Q}}$ over ℓ .

(2.16 ii) $r|_{G_\ell}$ is geometric, i.e., potentially semi-stable.

For this notion we refer to [26]. It is expected that irreducible representations occurring in the cohomology of arbitrary algebraic varieties over \mathbb{Q} are potentially semi-stable. Conversely:

(2.17) **Conjecture.** (Fontaine-Mazur) – *If r is geometric (i.e. verifies 2.16 i-ii) r occurs in $H_{\text{ét}}^\bullet(X \times \bar{\mathbb{Q}}, L)$ for some algebraic variety X/\mathbb{Q} (perhaps after a Tate twist)⁸.*

This is true if $n = 1$ (easy if \mathbb{Q} is the ground field) and is now proved in a large number of cases if $n = 2$ (and \mathbb{Q} is the ground field). If this is true r should be realized on $M \otimes_E L$, $M \subset H^\bullet(X, E)$ being a motive over E and $L = E_\lambda$ a completion of E . According to Conjecture 2.1, there should be an algebraic, cuspidal representation π of $GL(n, \mathbb{A})$ associated to r .

In particular there is an n -dimensional representation of $W_{\mathbb{R}}$ (1.2) coming from π . As stated in 2.1 this is determined by Serre [46], but we now review the construction.

Recall that our coefficient fields E are contained in \mathbb{C} . Then $M \otimes_E \mathbb{C} \subset H_B^w(X, \mathbb{C})$, and this has a Hodge decomposition:

$$H := M \otimes \mathbb{C} = \bigoplus_{p,q} H^{p,q}, \quad p + q = w.$$

Moreover $c =$ complex conjugation (on $X \times_{\mathbb{Q}} \mathbb{C}$, not on the coefficients!) acts on H .

Suppose first $p \neq q$. Then c exchanges $M^{p,q}$ and $M^{q,p}$; the corresponding representation of $W_{\mathbb{R}}$ is

$$\text{ind}_{W_{\mathbb{C}} = \mathbb{C}^\times}^{W_{\mathbb{R}}} (z^{-p}(\bar{z})^{-q}) \quad (\text{of degree } 2)$$

with the suitable multiplicity. The eigenvalues of c are $(+1, -1)$ with the same multiplicity.

If $p = q$ (so $w = 2p$ is even) σ preserves $H^{p,p}$. Let

$$H^{p,\varepsilon} = \{x \in H^{p,p} : \sigma(x) = (-1)^p \varepsilon x\} \quad (\varepsilon = \pm 1).$$

Since $W_{\mathbb{R}}$ is generated by \mathbb{C}^\times and ι such that $\iota z \iota^{-1} = \bar{z}$, $\iota^2 = -1$, we define a character of $W_{\mathbb{R}}$ by

$$\begin{aligned} z &\mapsto (z \bar{z})^{-p} \quad (z \in \mathbb{C}^\times) \\ \iota &\mapsto \varepsilon \end{aligned}$$

with multiplicity $\dim H^{p,\varepsilon}$.

Suppose now that r is *totally even*, i.e., c act trivially. Then $w = 2p$ is even and the representation of $W_{\mathbb{R}}$ associated to π_∞ is n times the character

$$\begin{aligned} z &\mapsto (z \bar{z})^{-p} \\ \iota &\mapsto (-1)^p \end{aligned}$$

⁸A Tate twist may be needed to bring r to a positive weight! (Recall the difference between motives and effective motives.)

of $W_{\mathbb{R}}$, associated to the character $x \mapsto x^p$ of \mathbb{R}^{\times} .

According to Conjecture 2.1, we see then that the representation $\Pi_{\mathbb{R}}$ should be, up to a twist by an integral power of the determinant and a sign, associated to the trivial representation of $W_{\mathbb{R}}$; in other terms, up to twist:

$$(2.18) \quad \Pi_{\mathbb{R}} = \text{ind}_{B_n(\mathbb{R})}^{GL_n(\mathbb{R})}(\mathbf{1})$$

where B_n is the Borel subgroup. It has generally been assumed that representations

$$\Pi = \Pi_{\mathbb{R}} \otimes \bigotimes_p \Pi_p$$

such that $\Pi_{\mathbb{R}}$ is of the form (2.18) are associated to (irreducible) Artin representations of degree n of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, i.e., representations with finite image⁹. (See the comments at the end of [10], as well as [18].) This would lead to:

(2.19) **Conjecture.** – *Assume r is an irreducible, geometric, ℓ -adic representation of $G_{\mathbb{Q}}$ and r is totally even. Then r is the tensor product of an Artin representation by a power of the cyclotomic character.*

It would be interesting to see what should be the case if r is “purely of type (p, p) ”; this can presumably be formulated directly in terms of the geometric representation $r|_{G_{\ell}}$.

2.7. The approach that led to the proof of the Tanyama conjecture for elliptic curves uses Galois representations (and their reduction mod ℓ , which will not be considered in these lectures). It relies on the availability of an *ample supply* of Galois representations associated to classical modular forms.

In order to attack these questions in higher rank, we need again – for arbitrary n – a large collection of Galois representations. In the present approach (Harris, Taylor and others) these representations are associated to automorphic forms on higher unitary groups. Because these are closely related to $GL(n)$, they also give a partial proof of Conjecture 2.1, in the direction

$$\pi \underset{?}{\rightsquigarrow} \text{motive } M/F.$$

Here π must be considerably restricted, and “motive” essentially interpreted as “Galois representation”.

⁹But compare Langlands’s remarks in [38], p. 9.

Lecture 3: Galois representations associated to self-dual automorphic representations

3.1. We now return in earnest to the main Conjecture of § 2.1, in the easier direction: associating a motive (in fact a Galois representation, if possible a compatible system) to a cuspidal representation. As recalled in Lecture 2, this is possible for $GL(2)$ (over \mathbb{Q}) because a cusp form (of weight 2) essentially “is” a piece of H^1 (modular curve) cut out by Hecke operators (we will not recall the classical theory of Hecke correspondences, see [48] and T. Saito’s lectures).

In higher rank the only natural algebraic varieties associated to automorphic forms are the Shimura varieties

$$(3.1) \quad S = \Gamma \backslash X$$

where we choose a reductive group G/\mathbb{Q} , $X = G(\mathbb{R})/K_\infty Z(\mathbb{R})$ (here $K_\infty \subset G(\mathbb{R})$ is a maximal compact subgroup, Z is the center), and $\Gamma \subset G(\mathbb{Q})$ is a congruence subgroup. This should describe the \mathbb{C} -points of an algebraic variety defined over a number field (call it R), so in particular X – the *symmetric space* – must have a complex structure invariant by $G(\mathbb{R})$.

This is of course true for $G = GL(2)$, but for $n \geq 3$, $X =$ space of positive definite symmetric matrices is *not* a complex variety. The usual “classical” description of modular (or automorphic) forms does not apply.

On the other hand, assume that G is a \mathbb{Q} -group such that $G(\mathbb{R}) = U(1, n-1)$. Then X is the unit ball in \mathbb{C}^n , and the quotients $S = \Gamma \backslash X$ are complex algebraic varieties. They are defined over a suitable number field R . So there is some hope of associating “motives” in $H^\bullet(S)$ with automorphic forms.

Assume now we start with a cuspidal representation π of $GL(n, \mathbb{A})$ where $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$. In § 2.2 we have described the theory of base change. Choose a quadratic imaginary field E . Then from π we get $\Pi = \text{Res}_{\mathbb{Q}}^E \pi$. (In the cases we will describe Π will in fact be cuspidal).

Now we consider a *standard* unitary group in n variables G – we will no longer use G for $GL(n)$. This is the unitary group of a Hermitian form on E^n , relative to Galois conjugation w.r.t. \mathbb{Q} . If we restrict scalars to E , $G_E = G \times_{\mathbb{Q}} E$ is isomorphic to $GL(n)/E$. Not surprisingly, Langlands functoriality still predicts a relation between automorphic forms on $GL(n, E) \backslash GL(n, \mathbb{A}_E)$ and on $G(\mathbb{Q}) \backslash G(\mathbb{A})$. We describe this by a diagram:

$$(3.2) \quad \begin{array}{ccc} GL(n)/E & = & GL(n)/E \\ | & & | \\ G/\mathbb{Q} & & GL(n)/\mathbb{Q} \end{array}$$

On automorphic forms, the vertical maps should be associated to base change functoriality. Given π , then, we try to get an automorphic representation of $G(\mathbb{A})$ by:

$$(3.3) \quad \begin{array}{ccc} \Pi & = & \Pi \\ ? \downarrow & & \uparrow \\ \tau & & \pi \end{array}$$

The right-hand side is unconditional [7] and its image Π is uniquely defined (essentially by strong multiplicity one, cf. Lecture 1). The problem is descent in the LHS.

Now part of [7] is the characterization of those Π which are images of π by base change. Let σ be the generator of $\text{Gal}(E/\mathbb{Q})$. Then σ acts on $G(E)$, $G(\mathbb{A}_E)$ and gives an operator:

$$R(\sigma) : L^2(G(E) \backslash G(\mathbb{A}_E), \omega) \circlearrowleft$$

with the notations of 1.1. The cuspidal representations Π in the image are exactly those stable by $R(\sigma)$, which we write

$$\Pi \cong \Pi^\sigma .$$

(By strong multiplicity one, this is equivalent to

$$\Pi_w \cong \Pi_w^\sigma \quad (\text{almost all } w)$$

where w ranges over the primes of E .)

Similarly we expect descent in the LHS if

$$(3.4) \quad \Pi \cong \Pi^\tau$$

where τ is now the Galois action defined by G – a different \mathbb{Q} -structure on $G(E)$. However τ acts (up to an inner automorphism) as $\sigma \circ \theta$ where θ is the automorphism $g \mapsto {}^t g^{-1}$ of $GL(n)$. Thus we are forced to assume

$$(3.5) \quad \pi \cong \pi^\theta ,$$

i.e., π is self-dual \equiv isomorphic to its contragredient. This is the condition on π mentioned in § 2.4.

From now on, then, we will consider only self-dual algebraic cuspidal representations of $GL(n)/\mathbb{Q}$. Note also that if we start with representations of $GL(n, \mathbb{A}_E)$, the pertinent condition is (3.4). Such a representation will be called *conjugate self-dual*. There is an obvious generalization: \mathbb{Q} is replaced by a totally real number field F , E by a CM-extension (quadratic!) of F . Similar definitions (and results) obtain. To keep things simple we will stick with \mathbb{Q} – and E .

Note that we have introduced a strong restriction on the ground fields F for which we can approach Conjecture 2.1: F must be totally real or CM. Again, without such assumptions, nothing is known in reasonable generality.

3.2. We now introduce Shimura varieties, and a bird’s eye view of the relation between automorphic forms and Galois representations occurring in them. (This will be described in much fuller detail in the lectures by Genestier and Ngô: these few pages may serve as motivation.)

We denote by G the unitary *similitude* group of a Hermitian form on E^n , of type $(1, n-1)$ at the Arch. prime¹⁰. (Then $G \times E \cong GL(n) \times \mathbb{G}_m/E$.) The varieties S will be as follows: fix $K \subset G(\mathbb{A}_f)$ – a compact – open subgroup, let $K_\infty = (\text{maximal compact subgroup in } G(\mathbb{R})) \times Z(\mathbb{R})$, and take

$$(3.6) \quad S = S_K = G(\mathbb{Q}) \backslash G(\mathbb{R}) / K_\infty K .$$

This is not quite of the form (3.1) – it is generally not connected – but it is a finite union of quotients (3.1). The advantage of working with (3.6) is that the field of definition of S_K is now uniformly (for all K) equal to E . Thus $\text{Gal}(\bar{E}/E)$ acts on $H_{\text{ét}}^\bullet(S_K, \mathbb{Q}_\ell)$ or $H_{\text{ét}}^\bullet(S_K, L)$ where L is any ℓ -adic (finite) field of coefficients¹¹. (We have simply written $H_{\text{ét}}^\bullet(S_K, L)$ for the cohomology of S_K seen as a variety over \bar{E} .)

Now let τ be an automorphic representation of $G(\mathbb{A})$ – this can be defined and generalizes cuspidal representations, cf. § 1.4 and [37]. In fact we will be only interested in representations τ that are summands of

$$(3.7) \quad L_{\text{dis}}^2(G(\mathbb{Q}) A_G \backslash G(\mathbb{A}))$$

¹⁰The similitude group is natural from the point of view of moduli problems (of Abelian varieties) and behaves better than the unitary group in other respects.

¹¹Note the change of notation: E does not denote a field of *coefficients*, but the field of *definition*.

where $A_G = \mathbb{R}_+^\times$, diagonally embedded in $G(\mathbb{R})$; L_{dis}^2 denotes the “discrete” part of L^2 . (Again, this can be neglected at first reading.) We assume that τ_∞ is *cohomological*, i.e.

$$H^\bullet(G(\mathbb{R}), \mathcal{H}(\tau_\infty) \otimes V) \neq 0$$

where $\mathcal{H}(\tau_\infty)$ is the Hilbert space realizing τ_∞ , V is an irreducible representation of $G(\mathbb{R})$ and H^\bullet is a suitable cohomology theory – cf. [12], and again neglect this if not familiar. The point is that “cohomological representations” correspond to “algebraic regular (cuspidal) representations” for $GL(n)$. From τ we deduce by base change a representation of $G(\mathbb{A}_E)$, call it τ_E , with Hecke matrices $t(\tau_E, w) := t(\tau, w)$ at almost all primes w of E .

Langlands’ fundamental idea describing the relation between the cohomology of S_K and automorphic forms was the following. Assume w is a prime of good reduction for S_K . Let

$$\text{trace}(\text{Frob}_w^\alpha | H^\bullet(S_K))$$

denote the usual alternating sum, Frob_w being the geometric Frobenius, and $\alpha \geq 1$. For the clarity of the following discussion, assume moreover that the signature of G is (p, q) , $p+q = n$ (we will *later* assume $p = 1$). Then

$$(??3.8) \quad \text{trace}(\text{Frob}_w^\alpha | H^\bullet(S_K)) = \sum_{\tau} \text{trace}(\Lambda^p t(\tau, w)^\alpha \otimes \omega(\varpi_w^\alpha)).$$

We must explain the RHS. Here τ runs over cohomological representations of $G(\mathbb{A})$ occurring in (3.7). For a diagonal matrix t , $\Lambda^p t$ is its image by the representation of $GL(n, \mathbb{C})$ given by the p -th exterior power. Finally, ω is an Abelian character of $E^\times \backslash \mathbb{A}_E^\times$ – including a Tate twist – which we will not describe. Of course, in the sum Σ , representations τ occur with their multiplicities in (3.7).

As, I hope, will be explained in Ngo’s lectures, (??3.8) is only a first approximation to the real picture. (In some sense, it gives, however, a correct expression of “the main part” of the cohomology.) If τ or τ_E is given, we see that in order to get the “right” Galois representation, we will have to assume p (or q) = 1. Also, we must be able to isolate in (??3.8) the part of the full trace of Frobenius corresponding to our given τ .

We will return to the first problem in the next §. For the second, let us just say that Langlands’ theory – now considerably expanded – allows one to compute not only $\text{trace}(\text{Frob}_w^\alpha)$ but $\text{trace}(\text{Frob}_w^\alpha \times T)$ where T is a Hecke operator acting as a correspondence on S_K . Since, for fixed K , there are only a finite number of terms in (??3.8) we may expect to isolate the part relative to τ . This is not quite true because we do not have (strong) multiplicity 1 for G . If $M_\lambda(\tau_E)$ is the λ -adic representation conjecturally associated to τ_E , we are then able to obtain only a *multiple* of $M_\lambda(\tau_E)$ in the LHS of (??3.8). An ingenious argument of Taylor (cf. [29]) allows one to extract the n -dimensional Galois representations. As a *motive* however we would only, in any sense, obtain the (possible) multiple.

3.3. We now review the obstructions to obtaining the Galois representations associated to π (or π_E , a conjugate self-dual representation) and show how they are bypassed.

First, if we follow the tentative approach in 3.2, we will have to define the unitary (similitude) group G , i.e., a Hermitian form. If we want to make no particular assumption on π , we will want the group to be unramified at all primes w of E (except those ramified for E/\mathbb{Q}). (Without explaining details, we just state that otherwise certain representations of $GL(n, \mathbb{A}_\mathbb{Q})$ – self-dual, etc. – will not transfer to G .) Furthermore we want $G(\mathbb{R})$ to be of type $(1, n-1)$. But the Hasse principle gives a relation between the local Hasse invariants which describe the \mathbb{Q}_v -structure of $G \times_{\mathbb{Q}} \mathbb{Q}_v$ ($v = p, \infty$). This introduces a first obstruction. This may be lifted by replacing \mathbb{Q} by a (totally real) extension F , but we obtain a (smaller) Galois representation associated to the number field EF (at best: the computation of reflex field is more complicated).

The next obstruction arises in realizing descent (from $GL(n)/E$ to G) and proving the identity (?3.8). It turns out that descent cannot be as simple as in [7], and that (?3.8) is generally false. The solution to both problems is given by *endoscopy*. It so happens that the deviation between having a simple identity of Selberg trace formulas relative to G and $G \times E$, and between the two sides of (?3.8), is given by a sum of traces relative to smaller groups H , the *endoscopic groups* of our problem. I cannot describe this theory here, see [33] and references therein.

What are we to do? Kottwitz [34] discovered that there were particular unitary groups G for which (?3.8) was true with no correction terms. They are defined as follows: let D be a division algebra of degree n^2 (rank n) over E , admitting an involution $*$ (anti-automorphism of order 2) of the second kind ($z^* = z^\sigma$ for $z \in \text{Cent}(D) = E$, σ being the generator of $\text{Gal}(E/\mathbb{Q})$). Then the set $\{d \in D : dd^* = 1\}$ defines a unitary group over \mathbb{Q}, G ; we have $G \times_{\mathbb{Q}} E = D^\times$. (As usual, we then work in fact with the group of unitary similitudes). Moreover the author [19] showed that in this case there is no obstruction to descent from $D^\times = G \times E$ to G .

We can then implement the scheme described in 3.2. If we start with the representation π or π_E , we then have to transfer it to $G(\mathbb{A}_E) = D(\mathbb{A}_E^\times)$. This is yet another case of Langlands functoriality, but it is known in many cases due to [7], and further work of Vignéras and, recently, of Badulescu. However, the transfer to $G(\mathbb{A}_E)$ imposes *conditions* on π or π_E : we have to assume that, at a finite prime, π_p (or $\pi_{E,w}$) belongs to the discrete series of $GL(n, \mathbb{Q}_p)$ (or $GL(n, E_w)$). (For the notion of discrete series, see [15]. This is a ramification condition at p : the associated representation of the Weil-Deligne group should be indecomposable.) Finally, this will force the non-triviality of the Hasse invariant of G at those primes, and the vanishing of the global invariant must be compatible to the type $(1, n-1)$ at ∞ .

The (provisional) end result is:

(3.9) **Theorem.** – $\pi =$ *cuspidal, self-dual, algebraic, regular representation of $GL(n, \mathbb{A}_{\mathbb{Q}})$.*

Assume π_p belongs to the discrete series (some p).

Then, for any ℓ , there exists an extension $E_\lambda/\mathbb{Q}_\ell$ (finite) and a Galois representation

$$r_\lambda : \text{Gal}(\bar{\mathbb{Q}}/E) \rightarrow GL(n, E_\lambda)$$

such that, for $p \neq \ell$, π_p is associated (by (1.16)) to the $WD_{\mathbb{Q}_p}$ -representation deduced from $r_\lambda|_{\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$.

There is a similar result for \mathbb{Q} replaced by E , “self-dual” replaced by “conjugate self-dual”. The theorem is essentially proved in [19] – for almost all unramified primes – then, in this form, in [29] but, at the ramified primes, Harris and Taylor control only the restriction to $W_{\mathbb{Q}_p}$ of the (semi-simplified) representation of $WD_{\mathbb{Q}_p}$. The full result follows from [54].

3.4. We end with some comments, and remarks on the stronger results which may be (soon?) expected.

First, we would like to have a *compatible system* of Galois representations; note that this is not asserted in (3.9). The weaker form in (3.9) arises because, due to all the obstructions present, one has in general to patch together r_λ from families of representations associated to extensions F of \mathbb{Q} (and FE of E). In some cases this is unnecessary. However, the remark at the end of 3.2 concerning multiplicities still arises. In many cases, one would unconditionally get a compatible system associated to dM , where d is a multiplicity and M the expected motive. Moreover, dM is essentially realized as a motive – a part of $H^\bullet(S_K)$ cut out by a (Hecke) projector, independent of ℓ .

Second, if one starts with π , one would want to get a representation of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Since this representation should exist, the Galois obstruction to extending r_λ to the full Galois group must vanish, but this writer does not see why – some people at the conference may...

There is now active progress in understanding the Galois representations associated to the Shimura varieties of unitary groups: Laumon and Ngo have proved the “fundamental lemma” needed to understand

the endoscopic terms, both for descent and for the correction terms to (?3.8) [39]. The work of Arthur gives then some hope to complete the analysis of descent, and of the Galois representations in $H^\bullet(S_K)$. In general, S_K is not compact (it is compact in the case of the unitary groups associated to division algebras) but, at least for the left-hand side of (?3.8), the full analysis in the non-compact case has been achieved by Sophie Morel [42]. Thus it can safely be assumed that the local condition in (3.9) will be dispensed with. This is the object of an ongoing program by Harris, Labesse, Ngô, the author and others. For the particular representations in (3.9), (but *without* the discrete series assumption at some prime) this will give a proof of the Ramanujan conjecture. Under the assumptions of Theorem 3.9, it is a theorem:

(3.10) **Theorem.** – *Assume that π verifies the assumptions of Theorem 3.9. Then, for any p , π_p is tempered. In particular, at an unramified prime,*

$$t(\pi_p) = \text{diag}(t_1, \dots, t_n) \quad \text{with } |t_i| = 1.$$

For the meaning of “tempered” at ramified places, see [16]. For the theorem – in much greater generality – see [29].

References

- [1] Automorphic forms, representations, and L -functions, Borel, Casselman eds., Proc. Symp. Pure Math. **33**, AMS, 1979 (vol I-II).
- [2] Motives, Jannsen, Kleiman, Serre eds., Proc. Symp. Pure Math. **55**, AMS, 1994 (vol I-II).
- [3] Automorphic forms, Shimura varieties, and L -functions, Clozel, Milne eds., Perspectives in Math., Acad. Press, 1988 (vol I-II).
- [4] Stabilization of the trace formula, Shimura varieties, and arithmetic applications, vol. I: On the Stabilization of the Trace Formula, Clozel, Harris, Labesse, Ngô eds., International Press, 2011.
- [5] ABRASHKIN, Group schemes of period p over the ring of Witt vectors, Dokl. Akad. Nauk. SSSR **283** (1985), 1289-1294.
- [6] ARTHUR, An introduction to the trace formula, in Harmonic Analysis, the trace formula, and Shimura varieties, Clay Math. Proc. **4**, A.M.S. (2005), 1-263.
- [7] ARTHUR, CLOZEL, Simple algebras, base change, and the advanced theory of the trace formula, Princeton U. Press, 1989.
- [8] BARNET-LAMB, GEE, GERAGHTY, TAYLOR, Potential automorphy and change of weight, preprint.
- [9] BARNET-LAMB, GERAGHTY, HARRIS, TAYLOR, A family of Calabi-Yau varieties and potential automorphy II, P.R.I.M.S **47** (2011), 29-98.
- [10] BOREL, in [1], II, 27-62.
- [11] BOREL, JACQUET, Automorphic forms and automorphic representations, in [1], I, 189-202.
- [12] BOREL, WALLACH, Continuous cohomology, discrete subgroups, and representations of reductive groups, Princeton U. Press, 1980.
- [13] BREUIL, CONRAD, DIAMOND, TAYLOR, On the modularity of elliptic curves over \mathbb{Q} : wild 3-adic exercises, J.A.M.S. **14** (2001), 843-939.
- [14] CARAIANI, Local-global compatibility and the action of monodromy on nearby cycles, preprint.
- [15] CARTIER, in [1], I, 111-155.
- [16] CASSELMAN, Introduction to the theory of admissible representations of p -adic reductive groups, unpublished manuscript.
- [17] CHI, ℓ -adic and λ -adic representations associated to abelian varieties defined over number fields, Am. J. Math. **114** (1992), 315-353.
- [18] CLOZEL, in [3], I, 77-159.
- [19] CLOZEL, Représentations galoisiennes associées aux représentations automorphes autoduales de $GL(n)$, Publ. IHES **73** (1991), 97-145.
- [20] CLOZEL, Purity reigns supreme (provisional title), preprint.
- [21] COGDELL, PIATETSKI-SHAPIRO, Converse theorems for GL_n , Publ. IHES **79** (1994), 157-214.
- [22] DELIGNE, La conjecture de Weil pour les surfaces K3, Inv. Math. **15** (1972), 206-226.

- [23] FERMIGIER, Annulation de la cohomologie cuspidale de sous-groupes de congruence de $GL(n, \mathbb{Z})$, Math. Ann. **306** (1996), 247-256.
- [24] FLATH, in [1], I, 179-183.
- [25] FONTAINE, Il n'y a pas de variété abélienne sur \mathbb{Z} , Inv. math. **81** (1985), 515-538.
- [26] FONTAINE, MAZUR, Geometric Galois representations, in Elliptic curves, Modular forms, and Fermat's Last Theorem, Coates and Yau eds., International Press, 1995, 41-78.
- [27] FONTAINE, PERRIN-RIOU, in [2], I, 599-706.
- [28] GROSS, Arithmetic on elliptic curves with complex multiplication, Springer LN 776, 1980.
- [29] HARRIS, TAYLOR, The geometry and cohomology of some simple Shimura varieties, Princeton U. Press, 2001.
- [30] JACQUET, Principal L -functions of the linear group, in [1], II, 63-86.
- [31] KHARE, Modularity of Galois representations and motives with good Galois properties, preprint.
- [32] KHARE, WITENBERGER, Serre's modularity conjecture, Inv. Math. **178** (2009), I: 485-504; II: 505-586.
- [33] KOTTWITZ, in [3], I, 161-209.
- [34] KOTTWITZ, On the λ -adic representations associated to some simple Shimura varieties, Inv. Math. **108** (1992), 653-665.
- [35] LAFFORGUE, Estimées pour les valuations p -adiques des valeurs propres des opérateurs de Hecke, preprint.
- [36] LANGLANDS, in [1], II, 205-246.
- [37] LANGLANDS, in [1], I, 203-207.
- [38] LANGLANDS, Base change for $GL(2)$, Annals of Math. Series, Princeton U. Press, 1980.
- [39] LAUMON, NGO, Le lemme fondamental pour les groupes unitaires, Annals of Math., to appear.
- [40] MESTRE, Formules explicites et minorations de conducteurs de variétés algébriques, Compositio Math. **58** (1986), 209-232.
- [41] MILLER, Spectral and cohomological applications of the Rankin-Selberg method, Int. Math. Res. Notices **1** (1996), 115-154.
- [42] MOREL, Complexes d'intersection des compactifications de Baily-Borel, thesis, Univ. Paris-XI, 2005.
- [43] MUMFORD, Abelian Varieties, Oxford University Press, Oxford 1970.
- [44] RAMAKRISHNAN, in [2], II, 411-446.
- [45] SAAVEDRA RIVANO, Catégories tannakiennes, Lecture Notes in Math. **265**, Springer-Verlag, 1972.
- [46] SERRE, Facteurs locaux..., in Œuvres, t. II, Springer, 581-592.
- [47] SERRE, Résumés des cours, Collège de France, 1984-1985.
- [48] SHIMURA, Introduction to the arithmetic theory of automorphic functions, Princeton UP/Iwanami Shoten, 1971.

- [49] SHIN, Galois representations arising from some compact Shimura varieties, to appear in *Ann. Math.*
- [50] SILVERMAN, *The arithmetic of elliptic curves*, Springer-Verlag, 1986.
- [51] TATE, in [1], II, 3-26.
- [52] TAYLOR, Galois representations, *Proc. ICM (Beijing 2002) vol. I*, 449-474 (= *Ann. Fac. Sc. Toulouse* **13** (2004), 73-119, complete version; complete version available at his web site).
- [53] TAYLOR, WILES, Ring-theoretic properties of certain Hecke algebras, *Ann. Math.* **141** (1995), 553-572.
- [54] TAYLOR, YOSHIDA, Compatibility of local and global Langlands correspondences, preprint.
- [55] VAN GEMEEN, TOP, A non-selfdual automorphic representation of $GL(3)$ and a Galois representation, *Inv. Math.* **117** (1991), 391-401.
- [56] WALDSCHMIDT, Transcendence et exponentielles en plusieurs variables, *Inv. Math.* **63** (1981), 97-127.
- [57] WEIL, On a certain type of characters of the idèle-class group of an algebraic number field, *Proc. Int. Symp. on algebraic number theory, Tokyo-Nikko, 1955*, 1-7 (= *Œuvres Scientifiques*, vol. II, Springer 1979, 252-261).
- [58] WILES, Modular elliptic curves and Fermat's last theorem, *Ann. Math.* **141** (1995), 443-551.

The basic references for the material covered in these notes are [36], [44] and [18].

ERRATA¹²

Motifs et formes automorphes : Applications du principe de functorialité

L. Clozel

In : **Automorphic Forms, Shimura Varieties, and L-functions**,

Clozel, Milne eds, Academic Press, Boston 1988

Cet article contient plusieurs erreurs plus ou moins sérieuses ; je remercie A. Genestier, B. Gross, J. Mahnkopf et R. Taylor de me les avoir indiquées et J. Rohlfs et J. Schwermer pour des indications précieuses. On n'a pas corrigé les erreurs typographiques dues entre autres à la traduction (primitive à l'époque) d'une version de TEX à une autre.

78³ : Seules les traces

91₁₃ : (k_1, \dots, k_r)

112¹ : $\pi \in \mathcal{A}lg^0(n)$

112₁₀ : $\pi_{\mathbb{C}}(z^{a_1} \bar{z}^{b_1}, \dots, z^{a_n} \bar{z}^{b_n})$

113⁵ : $\delta = (\frac{n-1}{2}, \frac{n-3}{2}, \dots)$

114₂ : $\dots, a_{2m-1} < a_{2m}$

117₉ : $s^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2m-1 & 2m \\ 1 & 2m & 2 & 2m-1 & 3 & \dots & m & m+1 \end{pmatrix}$

120₇ : $\sigma_i \cong \sigma_i \otimes \epsilon$

123₄ : $X^1 = A \setminus X$

124₆ : $\overline{d\Omega_{(2)}^{i-1}}$

124₂ : $\overline{H_{(2)}^i}$

126 : Remarque et Prop. 3.18 : Le lemme 3.17 n'implique pas que $H_c^i/E_c^i \hookrightarrow \overline{H_{(2)}^i}$; comme me l'a fait remarquer Genestier il n'y a pas de flèche naturelle $H_!^i \rightarrow \overline{H_{(2)}^i}$. La Prop. 3.18 n'a pas de sens en général.

Soit en revanche $H_d^i \subset H^i$ le sous-espace des classes représentées par des formes de carré intégrable.

Soit par ailleurs \mathcal{H}^i l'espace des formes harmoniques de carré intégrable. D'après Kodaira l'application naturelle $\mathcal{H}^i \rightarrow H_d^i$ est surjective ; par ailleurs $H_!^i$ s'injecte naturellement dans H_d^i . Pour ces faits voir Borel [I, 2].) On a donc

$$H_!^i \hookrightarrow \mathcal{H}^i/N^i$$

où $N^i = \ker(\mathcal{H}^i \rightarrow H_d^i)$ est l'espace des formes harmoniques représentant 0 en cohomologie. C'est cet énoncé (correct) que démontre Harder [30a, Satz 3.7.13.4].

Enfin d'après Borel et Garland [II] (voir aussi [11])

$$H_d^i \cong H^i(\tilde{\mathfrak{g}}, K_\infty; (L_{\text{dis}}^2)_{\infty}^K) \cong \overline{H_{(2)}^i}$$

avec les notations de la p. 127 (voir aussi Schwermer [VII]).

¹²Les références bibliographiques en chiffres arabes sont celles de l'article corrigé; les références en chiffres romains sont à la bibliographie de l'Errata.

Pour un niveau K fixé tous ces espaces sont de dimension finie ; puisque L_{dis}^2 est somme directe de représentations unitaires, H_d^i ou $\overline{H}_{(2)}^i$ sont sommes directes de représentations irréductibles de l'algèbre de Hecke associée à K (cf. les arguments après la Prop. 3.16). Avec les notations de la p. 127, on en déduit alors que $\sigma\Pi_f$ (vue si l'on préfère comme représentation de l'algèbre de Hecke de niveau K) apparaît dans $\overline{H}_{(2)}^i$. Le reste de l'argument (p. 127-128) est correct, donc aussi le Théorème 3.13.

$$129_{14} : H_{\text{cusp}}^i(\tilde{S}, \mathcal{V}_E) = H_c^i(\tilde{S}, \mathcal{V}_E) \cap H_{\text{cusp}}^i(\tilde{S}, \mathcal{V})$$

139₅ : On peut renforcer la conjecture en tenant compte des représentations semi-simplifiées de W'_{F_v} .

141⁵ : combinaison linéaire

142 : les articles [6a,6b] sont erronés comme l'a remarqué R. Taylor. Voir Henniart [VI]. L'algébricité des valeurs propres pour les formes de Maass telles que $\lambda = \frac{1}{4}$ n'est pas connue à ce jour. Pour une approche nouvelle du problème voir Carayol [III].

150 : Théorème 5.1 : Les conditions de "parité" (i), (ii) sont incorrectes (B. Gross). L'énoncé correct est dans [IV]. Le §4 de [IV] contient lui-même des erreurs, corrigées dans [V].

151, 5.2.2 : La remarque est erronée. Cf. correction p. 142.

Bibliographie

- [I] A. BOREL, *Stable real cohomology of arithmetic groups*, Ann. Sc. É. N. S. (4^e s.) **7** (1974), 235-272.
- [II] A. BOREL, H. GARLAND, *Laplacian and the discrete spectrum of an arithmetic group*, Amer. J. Math **105** (1983), 309-335.
- [III] H. CARAYOL, *Limites dégénérées de séries discrètes, formes automorphes et variétés de Griffiths-Schmid : le cas du groupe $U(2,1)$* , Compositio Math **111** (1998), 51-88.
- [IV] L. CLOZEL, *Représentations galoisiennes associées aux représentations automorphes autoduales de $GL(n)$* , Publ. IHES **73** (1991), 97-145.
- [V] L. CLOZEL, J.-P. LABESSE *Changement de base pour les représentations cohomologiques de certains groupes unitaires*, Appendice à J.-P. Labesse, *Cohomologie, stabilisation et changement de base*, Astérisque **257** (1999).
- [VI] G. HENNIART, Erratum à l'Exposé n^o 711, Séminaire Bourbaki, vol. 1990/91, Astérisque **201-202-203** (1991), 485-486.
- [VII] J. SCHWERMER, *Cohomology of arithmetic groups, automorphic forms and L-functions*, in Labesse, Schwermer eds., *Cohomology of Arithmetic Groups and Automorphic Forms*, Springer, Berlin (1990), 1-30.

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