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Giacomo Canevari, Antonio Segatti, Marco Veneroni

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# MORSE'S INDEX FORMULA IN VMO FOR COMPACT MANIFOLDS WITH BOUNDARY

GIACOMO CANEVARI, ANTONIO SEGATTI, AND MARCO VENERONI

ABSTRACT. In this paper, we study Vanishing Mean Oscillation vector fields on a compact manifold with boundary. Inspired by the work of Brezis and Nirenberg, we construct a topological invariant — the index — for such fields, and establish the analogue of Morse's formula. As a consequence, we characterize the set of boundary data which can be extended to nowhere vanishing VMO vector fields. Finally, we show briefly how these ideas can be applied to (unoriented) line fields with VMO regularity, thus providing a reasonable framework for modelling a surface coated with a thin film of nematic liquid crystals.

## 1. INTRODUCTION

The starting point of the investigations developed in this paper is the analysis of a variational model for nematic shells. Nematic shells are the datum of a two-dimensional surface (for simplicity, at a first step, without boundary)  $N \subset \mathbb{R}^3$  coated with a thin film of nematic liquid crystal ([16, 18, 21, 22, 23, 27, 28, 30]). This line of research has attracted a lot of attention from the physics community due to its vast technological applications (see [23]). From the mathematical point of view, nematic shells offer an interesting and nontrivial interplay between calculus of variations, partial differential equations, geometry and topology. The basic mathematical description of nematic shells consists in an energy defined on tangent vector fields with unit length, named directors. This energy, in the simplest situation, takes the form

$$(1.1) \quad \mathcal{E}(n) := \int_N |\nabla n|^2 dS,$$

where  $\nabla$  stands for the covariant derivative of the surface  $N$ . If one is interested in the minimization of this energy, the first step is to understand whether there are competitors for the minimization process. For this type of energy, the natural functional space where to look for minimizers is the space of tangent vector fields with  $H^1$  regularity. This means, recalling that we are looking for vector fields with unit norm, the space defined in this way

$$(1.2) \quad H_{\text{tan}}^1(N, \mathbb{S}^2) := \{n \in H^1(N, \mathbb{R}^3) : n(x) \in T_x(N) \text{ and } |n| = 1 \text{ a.e.}\}.$$

Now, the problem turns into the understanding of the topological conditions on  $N$ , if any, that make  $H_{\text{tan}}^1(N, \mathbb{S}^2)$  empty or not. Note that this problem, in the case  $N = \mathbb{S}^2$ , is indeed a Sobolev version of the celebrated hairy ball problem concerning the existence of a tangent vector field with unit norm on the two-dimensional sphere. The answer, when dealing with continuous fields, is negative. This is a consequence of a more general result, the Poincaré-Hopf Theorem, that relates the existence of a smooth tangent vector field with unit norm to the topology of  $N$ . More precisely, a smooth vector field with unit norm exists if and only if  $\chi(N) = 0$ , where  $\chi$  is the Euler characteristic of  $N$ . In case  $N$  is a compact surface in  $\mathbb{R}^3$ , the Euler characteristic can be written as a function of the topological genus  $k$ :

$$\chi(N) = 2(1 - k).$$

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In [27] it has been proved, using calculus of variations tools, that the very same result holds for vector fields with  $H^1$  regularity. Therefore, up to diffeomorphisms, the only compact surface in  $\mathbb{R}^3$  which admits a unit norm vector field in  $H^1$  is the torus, corresponding to  $k = 1$ . On the other hand, it is easy to comb the sphere with a field  $v \in W_{\text{tan}}^{1,p}(\mathbb{S}^2, \mathbb{S}^2)$  for all  $1 \leq p < 2$ . It is interesting to note that this result could be seen as a "non flat" version of a well know result of Bethuel that gives conditions for the non emptiness of the space

$$H_g^1(\Omega; \mathbb{S}^1) := \{v \in H^1(\Omega; \mathbb{R}^2) : |v(x)| = 1 \text{ a.e. in } \Omega \text{ and } v \equiv g \text{ on } \partial\Omega\},$$

where  $\Omega$  is a simply connected bounded domain in  $\mathbb{R}^2$  and  $g$  is a prescribed smooth boundary datum with  $|g| = 1$ . The non-emptiness of  $H_g^1(\Omega; \mathbb{S}^1)$  is related to a topological condition on the Dirichlet datum  $g$  (see [2] and [3]) while in the result in [27] the topological constraint is on the genus of the surface.

Instead of using the standard Sobolev theory, we reformulate this problem in the space of Vanishing Mean Oscillation (VMO) functions, introduced by Sarason in [26], which constitute a special subclass of Bounded Mean Oscillations functions, defined by John and Nirenberg in [15]. We recall the definitions and some properties of these objects in Section 2, but we immediately note that VMO contains the critical spaces with respect to Sobolev embeddings, that is,

$$(1.3) \quad W^{s,p}(\mathbb{R}^n) \subset \text{VMO}(\mathbb{R}^n) \quad \text{when } sp = n, \quad 1 < s < n.$$

In a sense, VMO functions are a good surrogate for the continuous functions, because some classical topological constructions can be extended, in a natural way, to the VMO setting. In particular, we recall here the VMO degree theory, which has been developed after Brezis and Nirenberg's seminal papers [4] and [5].

Besides relaxing the regularity on the vector field, we will consider  $n$ -dimensional compact and connected submanifolds of  $\mathbb{R}^{n+1}$  and, instead of fixing the length of the vector field to be 1, we will look for vector fields which are bounded and uniformly positive.

Thus, the problem of combing a two-dimensional surface with  $H^1$  vector fields can be generalized in the following way.

**Question 1.** *Let  $N$  be a compact, connected submanifold of  $\mathbb{R}^{n+1}$ , without boundary, of dimension  $n$ . Does a vector field  $v \in \text{VMO}(N, \mathbb{R}^{n+1})$ , satisfying*

$$(1.4) \quad v(x) \in T_x N \quad \text{and} \quad c_1 \leq |v(x)| \leq c_2$$

*for a.e.  $x \in N$  and some constants  $c_1, c_2 > 0$ , exists?*

The first outcome of this paper is to provide a complete answer to Question 1. By means of the Brezis and Nirenberg's degree theory, we can show that the existence of nonvanishing vector fields in VMO is subject to the same topological obstruction as in the continuous case, that is, we prove the following

**Proposition 1.1.** *Let  $N$  be a compact, connected submanifold of  $\mathbb{R}^{n+1}$ , without boundary. There exists a function  $v \in \text{VMO}(N, \mathbb{R}^{n+1})$  satisfying (1.4) if and only if  $\chi(N) = 0$ .*

After addressing manifolds without boundary, we consider the case where  $N$  is a manifold with boundary, and we prescribe Dirichlet boundary conditions to the vector field  $v$  on  $N$ . The main issue of this paper is to understand which are the topological conditions on the manifold  $N$  and on the Dirichlet boundary datum that guarantee the existence of a nonvanishing and bounded tangent vector field on  $N$  extending the boundary condition. Applications of these results can be found in variational problems for vector fields that satisfy a prescribed boundary condition of Dirichlet type, e.g., in the framework of liquid crystal shells.

More precisely, we address the following problem:

**Question 2.** *Let  $N \subset \mathbb{R}^d$  be a compact, connected and orientable  $n$ -submanifold with boundary. Let  $g: \partial N \rightarrow \mathbb{R}^d$  be a boundary datum in VMO, satisfying*

$$(1.5) \quad g(x) \in T_x N \quad \text{and} \quad c_1 \leq |g(x)| \leq c_2$$

for  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial N$  and some constants  $c_1, c_2 > 0$ . Does a field  $v \in \text{VMO}(N, \mathbb{R}^d)$ , which fulfills (1.4) and has trace  $g$  (in some sense, to be specified), exist?

When working in the continuous setting, a similar issue can be investigated with the help of a topological tool: the index of a vector field. In particular, even in this weak framework, we expect conditions that relate the index of the boundary conditions with the index of the tangent vector field and the Euler characteristic of  $N$ . In order to understand the difficulties and to ease the presentation, we recall here some definitions related to the degree theory and an important property.

First, we recall Brouwer's definition of degree. Let  $N$  be as in Question 2 and let  $M$  be a connected, orientable manifold without boundary, of the same dimension as  $N$ . Let  $\varphi: N \rightarrow M$  be a smooth map, and let  $p \in M \setminus \varphi(\partial N)$  be a regular value for  $\varphi$  (that is, the Jacobian matrix  $D\varphi(x)$  is non-singular for all  $x \in \varphi^{-1}(p)$ ). We define the degree of  $\varphi$  with respect to  $p$  as

$$\deg(\varphi, N, p) := \sum_{x \in \varphi^{-1}(p)} \text{sign}(\det D\varphi(x)).$$

This sum is finite, because  $\varphi^{-1}(p)$  is a discrete set (as  $\varphi$  is locally invertible around each point of  $\varphi^{-1}(p)$ ) and  $N$  is compact.

It can be proved that, if  $p_1$  and  $p_2$  are two regular values in the same component of  $M \setminus \varphi(\partial N)$ , then  $\deg(\varphi, N, p_1) = \deg(\varphi, N, p_2)$ . Since the regular values of  $\varphi$  are dense in  $M$  (by Sard lemma), the definition of  $\deg(\varphi, N, p)$  can be extended to every  $p \in M \setminus \varphi(\partial N)$ . Moreover, by approximation it is possible to define the degree when  $\varphi$  is just continuous. In case  $N$  is a manifold without boundary,  $\deg(\varphi, N, p)$  does not depend on the choice of  $p \in M$ , so we will denote it by  $\deg(\varphi, N, M)$ . Let us mention also that, if  $N$  and  $M$  are compact and without boundary, the following formula holds:

$$(1.6) \quad \deg(\varphi, N, M) = \frac{1}{\tau(M)} \int_N \varphi^*(d\tau) = \frac{1}{\tau(M)} \int_N \det D\varphi(x) d\sigma(x),$$

where  $\sigma, \tau$  are the Riemannian metrics on  $N$  and  $M$ , respectively.

Ideally, given a continuous vector field  $v$ , one would like to define its index by

$$\text{ind}(v, N) = \deg(v, N, 0).$$

However, this is not possible, because in order to define the degree it is essential that the domain and the target manifold have the same dimension. This is not the case here, since the domain manifold  $N \subset \mathbb{R}^d$  has dimension strictly less than the target manifold  $\mathbb{R}^d$ . To overcome this issue, there are at least two different strategies. The one we consider in this paper, which is also the most widely studied in the literature (see, e.g., [20, 10, 17, 19, 29]), is to use coordinate charts to represent  $v$ , locally around its zeros, as a map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ . This requires an additional assumption, namely that the zero set of  $v$  is discrete. Thus, within this approach, an approximation technique is needed in order to extend the definition of index to any continuous field. This construction, based on the Transversality Theorem, is explained in detail in Section 3. Another possibility is to consider an open neighbourhood  $U \subset \mathbb{R}^d$  of  $N$ , and extend  $v$  to a map  $w: U \rightarrow \mathbb{R}^d$ , in a suitable way. Then, it would make sense to write

$$\text{ind}(v, N) := \deg(w, U, 0),$$

and this would give an equivalent definition of the index. This approach is inspired by a classical proof of the Poincaré-Hopf theorem, which can be found in [19, Theorem 1, p. 38]). Some details of this construction are given in Remark 3.4.

Once the index has been properly defined, it can be used to establish a precise relation between the behaviour of a vector field  $v$  and the topological properties of  $N$ . Denote by  $\partial_- N$  the subset of the boundary where  $v$  points inward (that is, letting  $\nu(x)$  be the outward unit normal to  $\partial N$  in  $T_x N$ , we have  $x \in \partial_- N$  if and only if  $v(x) \cdot \nu(x) < 0$ ). Call  $P_{\partial N} v$  the vector field on  $\partial N$  defined by

$$P_{\partial N} v(x) := \text{proj}_{T_x \partial N} v(x) \quad \text{for all } x \in \partial N.$$

Morse proved the following equality (see [20]), which was later rediscovered and generalized by Pugh (see [24]) and Gottlieb (see [8, 9]).

**Proposition 1.2** (Morse's index formula). *If  $v$  is a continuous vector field over  $N$  satisfying  $0 \notin v(\partial N)$ , with finitely many zeros, and if  $P_{\partial N}v$  has finitely many zeros, then*

$$(1.7) \quad \text{ind}(v, N) + \text{ind}(P_{\partial N}v, \partial_- N) = \chi(N),$$

where  $\chi(N)$  is the Euler characteristic of  $N$ .

In figure 1 we plot some examples on  $N = \overline{B_r(0)}$ . In this case  $\chi(N) = 1$ .

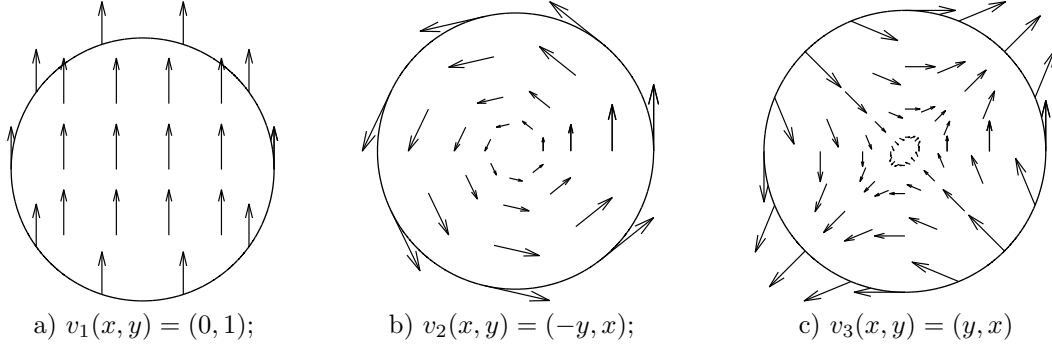


FIGURE 1. a)  $\text{ind}(v_1, N) = 0$ ,  $\text{ind}(P_{\partial N}v, \partial_- N) = 1$ ; b)  $\text{ind}(v_2, N) = 1$ ,  $\text{ind}(P_{\partial N}v, \partial_- N) = 0$ ; c)  $\text{ind}(v_3, N) = -1$ ,  $\text{ind}(P_{\partial N}v, \partial_- N) = 2$ .

Identity (1.7) can be seen as a generalization of the Poincaré-Hopf index formula. As an immediate corollary, we obtain a necessary condition for the existence of nowhere vanishing vector fields which extends in  $N$  a given a boundary datum.

**Corollary 1.3.** *Let  $g: \partial N \rightarrow \mathbb{R}^d$  be a continuous function, satisfying (1.5), and assume that  $P_{\partial N}g$  has finitely many zeros. If there exists a continuous vector field  $v$ , satisfying (1.4), such that  $v|_{\partial N} = g$  then*

$$\text{ind}(P_{\partial N}g, \partial_- N) = \chi(N).$$

This Corollary gives an answer to Question 2 in case we consider smooth vector fields.

Our aim in this paper is to extend Proposition 1.2 to the VMO setting. For this purpose, we extend the definition of index to arbitrary VMO fields, with a trace at the boundary. We introduce another quantity, which we call “inward boundary index” and denote by  $\text{ind}_-(v, \partial N)$ , playing the role of  $\text{ind}(P_{\partial N}v, \partial_- N)$ . (The reader is referred to Section 4 for the definitions).

Then, our main result is

**Theorem 1.4.** *Let  $N$  be a compact, connected and orientable submanifold of  $\mathbb{R}^d$ , with boundary. Let  $g \in \text{VMO}(\partial N, \mathbb{R}^d)$  be a boundary datum which fulfills*

$$g(x) \in T_x N \quad \text{and} \quad c_1 \leq |g(x)| \leq c_2$$

for some constants  $c_1, c_2 > 0$  and  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial N$ . If  $v \in \text{VMO}(N, \mathbb{R}^d)$  is a map with trace  $g$  at the boundary, satisfying

$$v(x) \in T_x N$$

for a.e.  $x \in N$ , then

$$\text{ind}(v, N) + \text{ind}_-(v, \partial N) = \chi(N).$$

Note that this Theorem is the analogous of Proposition 1.2 for VMO vector fields. Finally, regarding Question 2, we have the following answer.

**Proposition 1.5.** *Let  $g \in \text{VMO}(\partial N, \mathbb{R}^d)$  satisfy the assumption (1.5). A field  $v \in \text{VMO}(N, \mathbb{R}^d)$  that satisfies (1.4) and has trace  $g$  exists if and only if*

$$(1.8) \quad \text{ind}_-(g, \partial N) = \chi(N).$$

We conclude this introduction with an outline of the paper. In Section 2 we provide some preliminary material on the VMO space. Then, in Section 3 we introduce the notion of index for a continuous vector field, starting with the basic case of a field with a finite number of zeros and then moving to an arbitrary number of zeros by Thom's Transversality Theorem. In Section 4, by means of an approximation argument, this extension allows us to give a notion of index for a VMO vector field and to prove Theorem 1.4. Finally, in Section 5, we apply these results to the existence of line fields with VMO regularity. Interestingly, such an existence result shares the same topological obstruction as the existence result for vector fields. As a side result of the existence of VMO  $Q$ -tensor fields, we obtain topological conditions for the existence of line fields with VMO regularity, thus extending to this weaker setting a classical result due to Poincaré and Kneser.

**Notation.** In the following sections either  $N = \mathbb{R}^n$ , or  $N$  is a compact, connected and oriented manifold with boundary, of dimension  $n$ , embedded as a submanifold of  $\mathbb{R}^d$  for some  $d \in \mathbb{N}$ .

- The injectivity radius of  $N$  (see, e.g., do Carmo [7]) is called  $r_0$ .
- We denote geodesic balls in  $N$  by  $B_r^N(x)$  or simply  $B_r(x)$ , when it is clear from the context that we work in  $N$ . In case  $N = \mathbb{R}^n$ , we write  $B_r^n(x)$  or  $B^n(x, r)$ .
- For  $\varepsilon > 0$ , we set

$$N_\varepsilon := \{x \in N : \text{dist}(x, \partial N) \geq \varepsilon\}.$$

- For each  $x \in \partial N$ , we denote by  $\nu(x)$  the outward unit normal to  $\partial N$  in  $T_x N$ .
- Given a non-empty, convex and closed set  $K \subset \mathbb{R}^d$ , we denote the nearest-point projection on  $K$  by  $\text{proj}_K$ .
- Given a manifold  $X \subset \mathbb{R}^d$  and a continuous map  $v: X \rightarrow \mathbb{R}^d$ , we denote the tangential component of  $v$  by

$$P_X v(x) := \text{proj}_{T_x X} v(x) \quad \text{for } x \in X.$$

## 2. PRELIMINARY MATERIAL: VMO FUNCTIONS

For the reader's convenience, we recall here the basic definitions about VMO functions, following the presentation of [5] (to which the reader is referred, for more details). All the functions we consider here take values in  $\mathbb{R}^d$ , so functional spaces such as, e.g.,  $L^1(N, \mathbb{R}^d)$  or  $\text{VMO}(N, \mathbb{R}^d)$  will be simply written as  $L^1(N)$  or  $\text{VMO}(N)$ .

Recall that  $N$  is endowed with a Riemannian measure  $\sigma$ . For  $u \in L^1(N)$  (with respect to  $\sigma$ ), define

$$(2.1) \quad \|u\|_{\text{BMO}} := \sup_{\varepsilon \leq r_0, x \in N_{2\varepsilon}} \int_{B_\varepsilon(x)} |u(y) - \bar{u}_\varepsilon(x)| \, d\sigma(y),$$

where

$$(2.2) \quad \bar{u}_\varepsilon(x) := \int_{B_\varepsilon(x)} u(y) \, d\sigma(y), \quad \text{for } x \in N_{2\varepsilon}.$$

The set of functions with  $\|u\|_{\text{BMO}} < +\infty$  will be denoted  $\text{BMO}(N)$ , and (2.1) defines a norm on  $\text{BMO}(N)$  modulo constants. Using cubes instead of balls leads to an equivalent norm. Moreover, if  $\varphi: X_1 \rightarrow X_2$  is a  $\mathcal{C}^1$  diffeomorphism between two unbounded manifolds, then  $u \in \text{BMO}(X_2)$  implies  $u \circ \varphi \in \text{BMO}(X_1)$  and

$$\|u \circ \varphi\|_{\text{BMO}(X_1)} \leq C \|u\|_{\text{BMO}(X_2)}.$$

Bounded functions (in particular, continuous functions) belong to  $\text{BMO}$ . Following Sarason, we define  $\text{VMO}(N)$  as the closure of  $\mathcal{C}(N)$  with respect to the  $\text{BMO}$  norm. Functions in  $\text{VMO}(N)$  can be characterized by means of this lemma (see [4, Lemma 3]):

**Lemma 2.1.** *A function  $u \in \text{BMO}(N)$  is in  $\text{VMO}(N)$  if and only if*

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in N_{2\varepsilon}} \int_{B_\varepsilon(x)} |u(y) - \bar{u}_\varepsilon(x)| \, d\sigma(y) \rightarrow 0.$$

Sobolev spaces provide an interesting class of functions in VMO, since, for critical exponents, the embeddings which fail to be in  $L^\infty$  hold true in VMO:

$$W^{s,p}(N) \subset \text{VMO}(N) \quad \text{whenever } 0 < s < n, \, sp = n.$$

In general, VMO functions do not have a trace on the boundary. However, it is possible to introduce a subclass of VMO for which traces are well defined. We sketch here the construction.

First, we need to embed  $N$  as a domain of a bigger manifold  $X$ , smooth and without boundary. Here, we take  $X$  as the double of  $N$ , that is, the manifold we obtain by gluing two copies of  $N$  along their boundaries. Modifying, if necessary, the value of  $d$  we can assume that  $X \subset \mathbb{R}^d$ . Also, let  $U$  be a tubular neighbourhood of  $\partial N$  in  $X$ , and assume that the nearest-point projection  $\pi: U \rightarrow \partial N$  is well defined. Now, we fix  $g \in \text{VMO}(\partial N)$  and we extend it to a function  $G$ , by the formula

$$(2.3) \quad G(x) = \begin{cases} g(\pi(x))\chi(x) & \text{if } x \in X \cap U \\ 0 & \text{if } x \in X \setminus U \end{cases}$$

where  $\chi$  is a cut-off function, which is equal to 1 near  $\partial N$  and vanishes outside  $U$ . It can be checked that  $G \in \text{VMO}(X)$ .

We say that a function  $u \in \text{VMO}(N)$  has trace  $g$  on  $\partial N$ , and we write  $u \in \text{VMO}_g(N)$ , if and only if the function defined by

$$\begin{cases} u & \text{in } N \\ G & \text{in } X \setminus N \end{cases}$$

is in  $\text{VMO}(X)$ . This definition is independent on the choice of  $\chi$  and of  $X$  (see [5, Property 6]). The notion of  $\text{VMO}_g$  is stable under diffeomorphism: suppose  $\varphi: X_1 \rightarrow X_2$  is a  $\mathcal{C}^1$  diffeomorphism between bounded manifolds, mapping diffeomorphically  $\partial X_1$  onto  $\partial X_2$ . If  $g \in \text{VMO}(\partial X_2)$  and  $u \in \text{VMO}_g(X_2)$ , then

$$u \circ \varphi \in \text{VMO}_{g \circ \varphi}(X_1).$$

As an example of VMO functions with trace, let us mention that every map in  $W^{1,n}(X)$  has a trace in the sense of VMO, which coincides with the Sobolev trace.

**2.1. Combing an unbounded manifold in VMO.** In this section, we prove Proposition 1.1. Of course, it could be obtained as a corollary of our main result, Theorem 1.4. Anyway, it can be proved independently, and we present here an elementary argument inspired by [11, Theorem 2.28]. We assume that  $N$  is a compact, connected  $n$ -manifold *without boundary*, embedded as an hypersurface of  $\mathbb{R}^{n+1}$ .

*Proof of Proposition 1.1.* It is well-known that, if  $\chi(N) = 0$ , then a nowhere vanishing, smooth (hence VMO) vector field on  $N$  exists. The idea of the proof is the following: One picks an arbitrary continuous field, approximates it with a field  $v$  having a finite number of zeros, then uses the Poincaré-Hopf formula and the hypothesis  $\chi(N) = 0$  to show that  $\text{ind}(v, N) = 0$ , so  $v$  can be modified into a nowhere vanishing field. This argument is given in detail in the proof of Corollary 1.5, in case  $N$  is a manifold with boundary, and it is even simpler when  $\partial N = \emptyset$ .

Let us prove the other side of the proposition: we suppose that a tangent vector field  $v \in \text{VMO}(N)$  such that  $\text{ess inf}_N |v| > 0$  exists, and we claim that  $\chi(N) = 0$ . Every compact hypersurface of  $\mathbb{R}^{n+1}$  is orientable, so there is a smooth unit vector field  $\gamma: N \rightarrow \mathbb{R}^{n+1}$  such that  $\gamma(x) \perp T_x N$  for all  $x \in N$ . The choice of such a map induces an orientation on  $N$ , and  $\gamma$  is called the Gauss map of the oriented manifold  $N$ . We can also assume that  $n$  is even, since  $\chi(N) = 0$  whenever  $N$  is a compact, unbounded manifold of odd dimension (see, e.g., [11, Corollary 3.37]).

Consider the function  $H: N \times [0, \pi] \rightarrow \mathbb{R}^{n+1}$  given by

$$H(x, t) := (\cos t)\gamma(x) + (\sin t)v(x).$$

It is readily checked that  $|H(x, t)|^2 = 1$  for all  $(x, t) \in N \times [0, \pi]$ . We claim that

$$(2.4) \quad H \in \mathcal{C}([0, \pi], \text{VMO}(N, \mathbb{S}^n)).$$

Indeed,  $H(\cdot, t)$  is the linear combination of functions in  $\text{VMO}(N)$  and hence belongs to  $\text{VMO}(N)$ , for all  $t$ . On the other hand, for all  $t_1, t_2 \in [0, \pi]$

$$\|H(\cdot, t_1) - H(\cdot, t_2)\|_{\text{BMO}} \leq |\cos t_1 - \cos t_2| \|\gamma\|_{\text{BMO}} + |\sin t_1 - \sin t_2| \|v\|_{\text{BMO}},$$

whence the claimed continuity (2.4) follows.

Since the degree is a continuous function  $\text{VMO}(N, \mathbb{S}^n) \rightarrow \mathbb{Z}$  (see [4, Theorem 1]), we infer that

$$\deg(H(\cdot, 0), N, \mathbb{S}^n) = \deg(H(\cdot, \pi), N, \mathbb{S}^n).$$

On the other hand,  $H(\cdot, 0) = \gamma$  and  $H(\cdot, \pi) = -\gamma$ . By standard properties of the degree (in particular, [11, Properties (d, f) p. 134]), and since we have assumed that  $n$  is even, we have

$$\deg(-\gamma, N, \mathbb{S}^n) = (-1)^{n+1} \deg(\gamma, N, \mathbb{S}^n) = -\deg(\gamma, N, \mathbb{S}^n),$$

hence

$$\deg(\gamma, N, \mathbb{S}^n) = -\deg(\gamma, N, \mathbb{S}^n).$$

By the degree formula (1.6) and Gauss-Bonnet Theorem (see, e.g., [10, page 196]), for an even-dimensional hypersurface  $N$

$$\deg(\gamma, N, \mathbb{S}^n) = \deg(\gamma, N, \mathbb{S}^n) \int_{\mathbb{S}^n} d\sigma_n = \frac{1}{\omega_n} \int_N \gamma^*(d\sigma_n) = \frac{1}{\omega_n} \int_N \kappa d\sigma = \frac{1}{2} \chi(N),$$

where  $d\sigma_n$  is the volume form of  $\mathbb{S}^n$ ,  $\omega_n := \int_{\mathbb{S}^n} d\sigma_n$  is the volume of  $\mathbb{S}^n$ , and  $\kappa$  is the Gaussian curvature of  $N$ . Since  $\deg(\gamma, N, \mathbb{S}^n) = 0$  by the above construction, this shows that  $\chi(N) = 0$  and thus completes the proof.  $\square$

*Remark 2.2.* When  $\chi(N) \neq 0$ , Proposition 1.1 shows that there is no unit vector field in the critical Sobolev space  $W^{s,p}(N)$ , for  $0 < s < n$  and  $sp = n$ . In contrast, when  $sp < n$  it is not difficult to construct unit vector fields in  $W^{s,p}(N)$ . For instance, on  $N = \mathbb{S}^{2k}$  one may consider a field with two ‘‘hedgehog’’ singularities, of the form  $x \mapsto x/|x|$ , located at the opposite poles of the sphere.

### 3. THE INDEX OF A CONTINUOUS FIELD

We aim to extend Morse formula to the VMO setting. As a preliminary step, we need to define the index for any continuous vector field, dropping out the assumption of finitely many zeros. This goal can be achieved quite straightforwardly, by applying a fundamental tool of differential geometry: the transversality theorem. Such a construction is usually given for granted but, for the reader’s convenience, in this section we present it in detail. As a consequence of the transversality theorem, we are able to extend some properties of the classical index of a vector field, namely excision, invariance under homotopy, and stability, to continuous vector fields with any number of zeros. In Propositions 3.5 and 3.6 and in Corollary 3.7 we give the corresponding statements.

Let us start by recalling the definition of transversality. Throughout this section, we denote by  $X \subset \mathbb{R}^d$  a compact, connected and oriented manifold without boundary (in what follows we will take as  $X$  either the double of  $N$  or  $\partial N$ ). Also, let  $E$  be a smooth manifold (without boundary),  $\varphi: X \rightarrow E$  a map of class  $\mathcal{C}^1$ , and  $Y \subset E$  a submanifold.

**Definition 3.1.** The map  $\varphi$  is said to be transverse to  $Y$  if and only if, for all  $x \in \varphi^{-1}(Y)$ , we have

$$d\varphi_x(T_x X) \oplus T_{\varphi(x)} Y = T_{\varphi(x)} E.$$



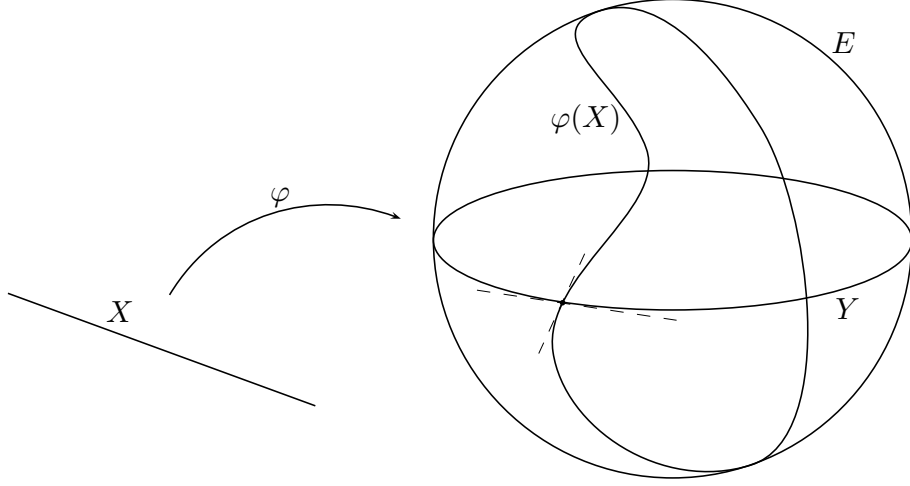


FIGURE 2. An example of transversality: The curve  $\varphi$  on the sphere  $E$  is transverse to the equator  $Y$ . Their tangent lines generate the tangent plane to  $E$  in the points of intersection.

In other words, we ask the image of  $\varphi$  to “cross transversally” the submanifold  $Y$ , at each point of intersection. In our case of interest,  $E = TX$  is the tangent bundle of  $X$ , equipped with the natural projection  $\pi: E \rightarrow X$  given by  $(x, w) \mapsto x$ . We take  $\varphi$  to be a section of  $\pi$  — that is, a map  $\varphi: X \rightarrow E$  such that  $\pi \circ \varphi = \text{Id}_X$ . Notice that given a vector field  $v: X \rightarrow \mathbb{R}^d$ , there exists a unique section of  $\pi$  induced by  $v$ , that is

$$\varphi: x \in X \mapsto (x, v(x)) \in E.$$

Vice-versa, each section of  $\pi$  induces a unique vector field  $X \rightarrow \mathbb{R}^d$ , because each tangent plane  $T_x X \in E$  can be regarded as a hyperplane of  $\mathbb{R}^d$ . Finally, we take  $Y$  as the image of the zero section, that is,

$$Y := \{(x, 0) : x \in X\} \subset E.$$

Clearly,  $Y$  is a submanifold of  $E$ , diffeomorphic to  $X$ , and  $\varphi(x) \in Y$  if and only if  $v(x) = 0$ .

Transverse sections can be characterized in terms of the corresponding vector fields. To do so, fix a point  $x \in X$  and a chart  $f: V \rightarrow \mathbb{R}^n$  in a neighbourhood of  $x$ . Then, define the (smooth) map  $f_*v: f(V) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$f_*v(y) := df_{f^{-1}(y)}(v(f^{-1}(y))) \quad \text{for all } y \in f(V) \subset \mathbb{R}^n.$$

Since  $T_{\varphi(x)}Y = T_x M \oplus \{0\}$ , then  $\varphi$  is transverse to  $Y$  in  $x$  if and only if  $d\varphi_x$  is invertible, i.e., if and only if  $dv_x$  is invertible. As

$$d(f_*v)_{f(x)} = df_x \circ dv_x \circ (df_x)^{-1},$$

we obtain the following

**Proposition 3.1.** *The map  $\varphi$  is transverse to  $Y$  if and only if for all  $x \in v^{-1}(0)$  the differential  $d(f_*v)_{f(x)}$  is invertible.*

If  $f, g$  are two local charts around  $x$ , then  $d(f_*v)_{f(x)}$  is invertible if and only if  $d(g_*v)_{g(x)}$  is, so this characterization is independent of the choice of the chart. Vector fields in these conditions will simply be called transverse fields. Remark that, for a transverse field  $v$ , the set  $v^{-1}(0)$  is discrete (by the local inversion theorem), hence is finite because  $X$  is compact. Moreover, given two coordinate charts  $f$  and  $g$  which agree with the fixed orientation of  $X$ , the Jacobians

$\det d(f_*v)_{f(x)}$  and  $\det d(g_*v)_{g(x)}$  have the same sign. Thus, if  $U \subset X$  is an open set and  $v$  a vector field on  $X$  satisfying

$$0 \notin v(\partial U),$$

the index of  $v$  on  $U$  is well-defined by the formula

$$(3.1) \quad \text{ind}(v, U) := \sum_{x \in v^{-1}(0) \cap U} \text{sign} \det d(f_*v)_{f(x)}.$$

This formula can be expressed in an equivalent way. Pick a geodesic ball  $B_r(x) \subset\subset U$  around each zero  $x$ , so small that no other zero is contained in  $B_r(x)$ . Then,  $\frac{f_*v}{|f_*v|}$  is well-defined as a map  $\partial B_r(x) \simeq \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$ , and

$$(3.2) \quad \text{ind}(v, U) = \sum_{x \in v^{-1}(0) \cap U} \text{deg} \left( \frac{f_*v}{|f_*v|}, \partial B_r(x), \mathbb{S}^{n-1} \right).$$

The equivalence of (3.1) and (3.2) follows, e.g., from [5, Equation (4.1), p. 25].

Since we want to extend the definition of index to any continuous field, it is natural to ask whether a continuous field can be approximated by transverse fields. The transversality theorem gives a positive answer. This result, due to Thom (see [31, 32]), states that transverse mappings are a dense subset of continuous mappings. The statement that we present here is [6, Theorem 14.6]. This formulation is convenient for our purposes, because it guarantees that if  $\varphi$  is a section of  $\pi$ , then the approximating transverse maps can be chosen to be sections as well.

**Theorem 3.2** (Transversality theorem). *Let  $\pi: E \rightarrow X$  be a smooth vector bundle,  $Y$  a submanifold of  $E$ , and  $\varphi: X \rightarrow E$  a smooth section of  $\pi$ . Then, given any continuous function  $\varepsilon: X \rightarrow (0, +\infty)$ , there exists a section  $\psi$  of  $\pi$  which is transverse to  $Y$  and satisfies*

$$\|\varphi(x) - \psi(x)\|_{T_x E} \leq \varepsilon(x) \quad \text{for all } x \in X.$$

Moreover, if  $A \subset X$  is a closed set such that  $\varphi|_A$  is of class  $\mathcal{C}^1$  and transverse to  $Y$ , then one can choose  $\psi$  so that  $\psi|_A = \varphi|_A$ .

The smoothness assumption on  $\varphi$  is not really a restriction, because every continuous section can be approximated with smooth sections (e.g., working in coordinate charts which trivialize  $\pi$ ). Hence, from this theorem we immediately obtain the result we need about vector fields.

**Corollary 3.3.** *Let  $U$  be an open subset of  $X$ , and let  $v$  be a continuous vector field defined on  $\bar{U}$ . If  $v$  satisfies  $0 \notin v(\partial U)$ , then there exists a transverse field  $u$  on  $\bar{U}$ , such that*

$$(3.3) \quad u \text{ has finitely many zeros,}$$

$$(3.4) \quad \sup_{x \in \bar{U}} |v(x) - u(x)| < \inf_{x \in \partial U} |v(x)|.$$

Now we can define the index of an arbitrary field.

**Definition 3.2.** Let  $v$  be a continuous vector field on  $U$ , such that  $0 \notin v(\partial U)$ . If  $v$  is transverse, we define  $\text{ind}(v, U)$  by formula (3.1). Otherwise, we define

$$\text{ind}(v, U) := \text{ind}(u, U),$$

where  $u$  is any transverse field satisfying (3.4).

The well-posedness of this definition follows directly from the stability of the index with respect to uniform convergence of continuous vector fields. We comment on this after Corollary 3.7.

The definition of index closely resembles Brouwer's construction of the degree. This similarity is not coincidental. Indeed, as we mentioned in the Introduction, an equivalent way of making sense of the index for an arbitrary continuous field is to define it as the degree of an appropriate map.

*Remark 3.4.* More precisely, consider a tubular neighbourhood  $M \subset \mathbb{R}^d$  of the manifold  $X$ , i.e., an open neighbourhood of  $X$  in  $\mathbb{R}^d$  such that any point  $y \in M$  can be uniquely decomposed as  $y = x + \nu$ , where  $x \in X$  and  $\nu$  is orthogonal to  $T_x X$ . Let  $\tau: M \rightarrow X$  be the map given by  $y \mapsto x$ , which is smooth if  $M$  is small enough. Consider the normal extension of  $v$ , that is, the continuous function  $w: M \rightarrow \mathbb{R}^d$  given by

$$w(y) := v(\tau(y)) + y - \tau(y) \quad \text{for all } y \in M.$$

Then, we can set

$$(3.5) \quad \text{ind}(v, U) := \text{deg}(w, \tau^{-1}(U), 0).$$

It is not hard to see that this quantity coincides with the index in the sense of Definition 3.2. Actually, by means of Brezis and Nirenberg degree theory, the right-hand side in this formula makes sense when  $v$  is just VMO (and satisfies a suitable nonvanishing condition near the boundary). Thus, one could consider taking (3.5) as a general definition of index. However, for a VMO field  $v$  this approach does not allow to define the quantity  $\text{ind}_-(P_{\partial N} v, \partial_- N[v])$ , which occurs in Morse's formula, because  $\partial_- N[v]$  may not be open. Henceforth, one would still have to consider continuous fields at first, then take care of the VMO case by an approximation procedure.

Due to this strong link between the index and the degree, it is not surprising that some important properties of the degree have a counterpart for the index. The first property we consider here is excision.

**Proposition 3.5** (Excision). *Let  $U_1 \subset U$ ,  $U_2 \subset U$  be two disjoint open sets in  $X$ , and let  $v$  be a continuous vector field on  $X$ . If  $0 \notin v(\overline{U} \setminus (U_1 \cup U_2))$ , then*

$$\text{ind}(v, U) = \text{ind}(v, U_1) + \text{ind}(v, U_2).$$

*Proof.* Using Theorem 3.2, we construct a transverse field  $u$  which satisfies

$$\sup_{x \in N} |v(x) - u(x)| < \inf_{x \in \overline{U} \setminus (U_1 \cup U_2)} |v(x)|.$$

In particular,  $u$  vanishes nowhere on  $\overline{U} \setminus (U_1 \cup U_2)$ . By Formula (3.1), which defines the index for a transverse field, we deduce

$$\text{ind}(u, U) = \text{ind}(u, U_1) + \text{ind}(u, U_2),$$

hence the lemma is proved.  $\square$

The second property is the invariance of the index under a continuous homotopy. We state a first version of this principle, in which we allow both the vector field and the underlying domain to vary continuously.

**Proposition 3.6** (General homotopy principle). *Let  $\{M_t\}_{0 \leq t \leq 1}$  be a family of compact, oriented  $n$ -manifolds in  $\mathbb{R}^d$ , without boundary, such that the set*

$$M := \coprod_{0 \leq t \leq 1} M_t \times \{t\}$$

*is a  $(n+1)$ -submanifold of  $\mathbb{R}^d \times [0, 1]$ . Let  $V$  be an open, connected subset of  $M$ , and set  $V_t := V \cap (\mathbb{R}^d \times \{t\})$ . Let  $v: \overline{V} \rightarrow \mathbb{R}^d$  be a continuous map such that, for each  $0 \leq t \leq 1$ ,*

- (i)  $v(\cdot, t)$  is a tangent field to  $M_t$ , and
- (ii)  $0 \notin v(\partial V_t)$ .

*Then, for any  $0 \leq t_1, t_2 \leq 1$  such that  $V_{t_1} \neq \emptyset$ ,  $V_{t_2} \neq \emptyset$ , we have*

$$\text{ind}(v(\cdot, t_1), V_{t_1}) = \text{ind}(v(\cdot, t_2), V_{t_2}).$$

*Proof.* Without loss of generality, we assume  $t_1 = 0$ ,  $t_2 = 1$ . Then, the assumption  $V_0 \neq \emptyset$ ,  $V_1 \neq \emptyset$  and the connectedness of  $V$  ensure that  $V_t \neq \emptyset$  for all  $0 < t < 1$ . Using (ii) and the transversality theorem, we can take two smooth, transverse fields  $u_0, u_1$ , satisfying

$$\sup_{x \in V_i} |u_i(x) - v(x, i)| < \inf_{x \in \partial V_i} |v(x, i)|$$

for  $i \in \{0, 1\}$ . Moreover, we introduce the sets

$$E := \coprod_{0 \leq t \leq 1} TM_t \times \{t\},$$

$$Y := \{(x, 0, t) : 0 \leq t \leq 1, x \in M_t\} \subset E$$

and the map  $\pi: E \rightarrow M$ , by setting

$$\pi(x, w, t) := (x, t) \quad \text{for all } 0 \leq t \leq 1, x \in M_t, w \in T_x M_t.$$

Then,  $E$  is a vector bundle over  $M$ , with fiber  $\mathbb{R}^n$  (remark:  $E \neq TM!$ ), and  $Y$  is a submanifold of  $E$ . Moreover, thanks to our assumption (i), the function  $\varphi: V \rightarrow E$  given by

$$\varphi(x, t) := (x, v(x, t), t)$$

is a continuous section of  $\pi$ , and  $\varphi(x, t) \in Y$  if and only if  $v(x, t) = 0$ . By smoothing  $v$ , then applying the transversality theorem as we did in the proof of Corollary 3.3, we approximate  $v$  by a section  $\psi: V \rightarrow E$  which is transverse to  $Y$ . Denoting by  $u(\cdot, t)$  the vector field on  $V_t$  induced by  $\psi(\cdot, t)$ , we can assume that

$$\sup_{x \in \bar{V}_t} |u(x, t) - v(x, t)| < \inf_{x \in \partial V_t} |v(x, t)| \quad \text{for all } 0 \leq t \leq 1$$

(which is possible, thanks to (ii)) and that  $u(\cdot, i) = u_i$  for  $i \in \{0, 1\}$  (because  $u_0, u_1$  are transverse fields already). In particular,  $\text{ind}(u(\cdot, t), V_t) = \text{ind}(v(\cdot, t), V_t)$  for all  $t$ . Then one can argue, e.g. as in [25], to check that  $\text{ind}(u_0, V_0) = \text{ind}(u_1, V_1)$ . Here is a sketch of the argument. A standard result about transversal maps entails that the set  $\psi^{-1}(Y)$  is a smooth submanifold of  $M$ , of dimension

$$\dim M - \dim E + \dim Y = (n + 1) - (2n + 1) + (n + 1) = 1,$$

hence a disjoint, finite union of smooth curves.

A closed curve in  $\psi^{-1}(Y)$  cannot touch  $V_0$  nor  $V_1$ . Indeed, assume by contradiction that there is a curve in  $\psi^{-1}(Y)$  touching, say,  $V_0$ . Consider a parametrization  $\gamma: S^1 \rightarrow V$  by a multiple of arc length. Let  $\theta \in S^1$  be such that  $p := \gamma(\theta) \in V_0$  and denote by  $\sigma: M \rightarrow [0, 1]$  the projection  $(x, t) \mapsto t$ . One has  $\gamma'(\theta) \in T_p M \simeq T_{\sigma(p)} M_0 \oplus \mathbb{R}$ . In fact,  $\gamma'(\theta) \in T_{\sigma(p)} M_0$  because

$$d_p \sigma(\gamma'(\theta)) = \left. \frac{d}{dt} \right|_{t=\theta} \sigma(\gamma(t)) = 0,$$

as  $\sigma \circ \gamma$  attains its minimum at  $\theta$ . On the other hand, since  $u(\gamma(t)) \equiv 0$ , we have

$$d_p u(\gamma'(\theta)) = \left. \frac{d}{dt} \right|_{t=\theta} u(\gamma(t)) = 0,$$

which contradicts the transversality of  $u_0$  because  $\gamma'(\theta) \neq 0$ ,  $\gamma'(\theta) \in T_{\sigma(p)} M_0$ .

Thus,  $\psi^{-1}(Y)$  is the union of smooth curves in  $V \setminus (V_0 \cup V_1)$  and arcs whose endpoints are in  $V_0 \cup V_1$ . These endpoints are exactly the zeros of  $u_0, u_1$ . By considering moving tangent frames along the arcs, one sees that if an arc has both endpoints on  $V_0$ , then their contributions to the index of  $u_0$  are opposite and cancel each other. An analogous property holds if the arc has both the endpoints on  $V_1$ . On the other hand, the two endpoints of an arc connecting  $V_0$  to  $V_1$  have the same local index. Thus, summing up over all the arcs, we conclude that  $\text{ind}(u_0, U_0) = \text{ind}(u_1, U_1)$ .  $\square$

In case the domain is fixed, from this general principle we can derive the stability of the index with respect to small perturbations of the fields.

**Corollary 3.7** (Stability). *Let  $v_0, v_1$  be two continuous vector fields on  $\bar{U}$ , satisfying  $0 \notin v_0(\partial U)$ ,  $0 \notin v_1(\partial U)$ . If*

$$(3.6) \quad |v_0(x) - v_1(x)| < |v_0(x)| \quad \text{for all } x \in \partial U,$$

then  $\text{ind}(v_0, U) = \text{ind}(v_1, U)$ .

*Proof.* Set  $M := X \times [0, 1]$ ,  $V := U \times [0, 1]$  and let  $v: V \rightarrow \mathbb{R}^d$  be given by

$$v(x, t) := (1 - t)v_0(x) + tv_1(x) \quad \text{for all } (x, t) \in V.$$

Then  $v$  is a continuous function, which satisfies the hypothesis (i) of Proposition 3.6 because  $v(\cdot, t)$  is just a linear combination of  $v_0$  and  $v_1$ . In addition, using (3.6) we see that

$$|(1 - t)v_0(x) + tv_1(x)| \geq |v_0(x)| - t|v_1(x) - v_0(x)| > 0$$

for all  $x \in \partial U$  and all  $0 \leq t \leq 1$ . Hence the condition (ii) is met, so that we can invoke Proposition 3.6 and conclude the proof.  $\square$

Corollary 3.7 implies that all the continuous vector fields have the same index on  $X$ . This agrees with the Poincaré-Hopf formula, which yields  $\text{ind}(v, X) = \chi(X)$ .

Now, come back to our manifold  $N$  with boundary, and take a continuous vector field  $v: N \rightarrow \mathbb{R}^d$  such that  $0 \notin v(\partial N)$ . The well-posedness of  $\text{ind}(v, N)$  in Definition 3.2 simply follows by taking  $X$  as the topological double of  $N$  and  $U := N \setminus \partial N$ .

We introduce the set

$$(3.7) \quad \partial_- N[v] := \{x \in \partial N : v(x) \cdot \nu(x) < 0\},$$

called the inward boundary, which is open in  $\partial N$ . (We simply write  $\partial_- N$ , when  $v$  is clear from the context). The tangential component  $P_{\partial N}v$  defines a vector field over  $\partial_- N$  and, despite  $0 \notin v(\partial N)$ , it is possible that  $P_{\partial N}v$  vanishes at some point. However,  $P_{\partial N}v$  does not vanish on  $\partial(\partial_- N)$ . Indeed,

$$\partial(\partial_- N) = \{x \in \partial N : v(x) \cdot \nu(x) = 0\},$$

hence if  $x \in \partial(\partial_- N)$  we have  $P_{\partial N}v(x) = v(x) \neq 0$ . Thus, the following definition is well-posed.

**Definition 3.3.** Let  $v$  be a continuous vector field on  $N$ , such that  $0 \notin v(\partial N)$ . We define the inward boundary index of  $v$  by

$$\text{ind}_-(v, \partial N) := \text{ind}(P_{\partial N}v, \partial_- N).$$

Notice that the inward boundary index depends only on  $v|_{\partial N}$ . Hence, it make sense to compute it for a continuous map  $g$  defined only on  $\partial N$ , provided that  $g$  is tangent to  $N$  and vanishes nowhere.

Also the inward boundary index is stable, with respect to small perturbations of the field.

**Lemma 3.8.** *Let  $v: \partial N \rightarrow \mathbb{R}^d$  be a continuous function, nowhere vanishing, such that*

$$(3.8) \quad v(x) \in T_x N \quad \text{for all } x \in \partial N.$$

*There exists  $\varepsilon_1 = \varepsilon_1(v) > 0$  such that for all  $\varepsilon \in (0, \varepsilon_1)$ , if  $w: \partial N \rightarrow \mathbb{R}^d$  is another continuous function satisfying (3.8) and*

$$(3.9) \quad \|v - w\|_{\mathcal{C}(\partial N)} \leq \varepsilon,$$

then  $\text{ind}_-(v, \partial N) = \text{ind}_-(w, \partial N)$ . For example, an admissible choice of  $\varepsilon_1$  is

$$\varepsilon_1 := \frac{\sqrt{5} - 1}{4} \min_{\partial N} |v|.$$

*Proof.* Denote by  $\nu(x)$  the outward unit normal to  $\partial N$  in  $T_x N$ . Since  $v$  is continuous and  $\partial N$  is compact, there exists a constant  $c > 0$  such that

$$|v(x)| \geq c \quad \text{for all } x \in \partial N.$$

Then, combining this lower bound with (3.9), we deduce

$$(3.10) \quad \left| \frac{v(x) \cdot \nu(x)}{|v(x)|} - \frac{w(x) \cdot \nu(x)}{|w(x)|} \right| \leq \frac{2\varepsilon}{c - \varepsilon} \quad \text{for all } x \in \partial N.$$

Indeed, for a fixed  $x \in \partial N$  we suppose, e.g., that  $|w(x)| \leq |v(x)|$ . Then

$$\begin{aligned} \left| \frac{v(x) \cdot \nu(x)}{|v(x)|} - \frac{w(x) \cdot \nu(x)}{|w(x)|} \right| &\leq \left| \frac{v(x) \cdot \nu(x)}{|v(x)|} - \frac{v(x) \cdot \nu(x)}{|w(x)|} \right| + \left| \frac{v(x) \cdot \nu(x)}{|w(x)|} - \frac{w(x) \cdot \nu(x)}{|w(x)|} \right| \\ &\leq |v(x)| \left( \frac{1}{|w(x)|} - \frac{1}{|v(x)|} \right) + \frac{|v(x) - w(x)|}{|w(x)|} \\ &= \frac{|v(x)| - |w(x)|}{|w(x)|} + \frac{|v(x) - w(x)|}{|w(x)|} \\ &\leq 2 \frac{|v(x) - w(x)|}{|w(x)|}, \end{aligned}$$

whence the desired inequality (3.10). Thus, setting

$$U_+ := \left\{ x \in \partial N : \frac{w(x) \cdot \nu(x)}{|w(x)|} < \frac{2\varepsilon}{c - \varepsilon} \right\}$$

and

$$U_- := \left\{ x \in \partial N : \frac{w(x) \cdot \nu(x)}{|w(x)|} < -\frac{2\varepsilon}{c - \varepsilon} \right\},$$

from (3.10) it follows that

$$U_- \subset \partial_- N[v] \subset U_+ \quad \text{and} \quad \partial(\partial_- N[v]) \subset \overline{U_+} \setminus U_-.$$

Moreover, for all  $x \in \overline{U_+} \setminus U_-$  the conditions (3.9) and (3.10) imply

$$(3.11) \quad |P_{\partial N} w(x)| \geq |w(x)| \sqrt{1 - \frac{4\varepsilon^2}{(c - \varepsilon)^2}} \geq \sqrt{(c - \varepsilon)^2 - 4\varepsilon^2}.$$

Let  $\varepsilon_1$  be the solution to

$$\varepsilon_1 = \sqrt{(c - \varepsilon_1)^2 - 4\varepsilon_1^2},$$

i.e.,  $\varepsilon_1 = c(\sqrt{5} - 1)/4$ . As the map  $t \mapsto \sqrt{(c - t)^2 - 4t^2}$  is monotone decreasing on  $[0, c/3]$ , for all  $\varepsilon \in (0, \varepsilon_1)$  we have

$$|P_{\partial N} v - P_{\partial N} w| \leq |v - w| \leq \varepsilon < \sqrt{(c - \varepsilon)^2 - 4\varepsilon^2} \stackrel{(3.11)}{\leq} |P_{\partial N} w|.$$

The condition (3.6) is thus satisfied, so that we can apply Corollary 3.7 to  $P_{\partial N} v$ ,  $P_{\partial N} w$ , to infer

$$\text{ind}_-(v, \partial N) = \text{ind}(P_{\partial N} v, \partial_- N[v]) = \text{ind}(P_{\partial N} w, \partial_- N[v]).$$

On the other hand, by (3.11) there is no zero of  $P_{\partial N} w$  in the region  $\overline{U_+} \setminus U_-$ , which contains the symmetric difference between  $\partial_- N[v]$  and  $\partial_- N[w]$ . Hence, Proposition 3.5 gives

$$\text{ind}(P_{\partial N} w, \partial_- N[v]) = \text{ind}(P_{\partial N} w, \partial_- N[w]) = \text{ind}_-(w, \partial N).$$

This concludes the proof.  $\square$

We can now prove that Morse's index formula holds true for arbitrary continuous fields.

**Proposition 3.9.** *Let  $v$  be a continuous vector field on  $N$ , such that  $0 \notin v(\partial N)$ . Then,*

$$\text{ind}(v, N) + \text{ind}_-(v, \partial N) = \chi(N).$$

*Proof.* We show that it is possible to approximate both  $v$  and  $P_{\partial N}v$  using the same transverse field  $u$ . Then, the proposition will follow by applying the classical Morse's formula to  $u$ .

Owing to the continuity of  $v$ , we find a number  $c > 0$  and a neighbourhood  $U$  of  $\partial N$  in  $N$  such that

$$(3.12) \quad |v(x)| \geq c \quad \text{for all } x \in U.$$

Let  $\varepsilon > 0$  be a small parameter, to be chosen later. We fix a smooth vector field  $\tilde{v}$  on  $N$  such that

$$(3.13) \quad \|v - \tilde{v}\|_{\mathcal{C}(N)} \leq \varepsilon.$$

Then, by Theorem 3.2, we approximate  $P_{\partial N}\tilde{v}$  with a transverse vector field  $\xi$  on  $\partial N$ , such that  $\xi$  has finitely many zeros on  $\partial N$  and

$$(3.14) \quad \|P_{\partial N}\tilde{v} - \xi\|_{\mathcal{C}(\partial N)} \leq \varepsilon.$$

We claim that there exists a continuous vector field  $w$  on  $N$ , which is smooth on  $U$ , satisfies

$$w = \begin{cases} \xi + \tilde{v} - P_{\partial N}\tilde{v} & \text{on } \partial N \\ \tilde{v} & \text{on } N \setminus U \end{cases}$$

and

$$(3.15) \quad \|v - w\|_{\mathcal{C}(N)} \leq C\varepsilon,$$

for some constant  $C$  depending only on  $N$ . (Remark that the prescribed boundary value for  $w$  is compatible with the condition (3.15), as it follows from (3.13) and (3.14)). We are giving the details of this construction in a moment, but first, we show how to conclude the proof.

By construction,  $w|_{\partial N}$  is a smooth function satisfying (3.8). For  $\varepsilon$  small enough, (3.15) and Lemma 3.8 entail that

$$(3.16) \quad \text{ind}_-(v, \partial N) = \text{ind}_-(w, \partial N).$$

Take  $\varepsilon < c/C$ . Then, (3.12) and (3.15) together imply that  $w$  does not vanish on  $U$ . In particular,  $w$  is vacuously transverse on  $U$ . Using Theorem 3.2, we modify  $w$  out of  $U$  to get a transverse vector field  $u$ , such that  $u|_U = w|_U$ . As  $u$  can be taken arbitrarily close to  $w$  in the  $\mathcal{C}$ -norm, we can assume that (3.4) is satisfied. Hence,

$$(3.17) \quad \text{ind}(v, N) = \text{ind}(u, N).$$

Since  $u$  is a transverse field, with finitely many zeros, Morse's identity applies to  $u$ . Then, using (3.16) and (3.17), the proposition follows.

Now, let us explain how to construct the map  $w$ . Taking a smaller  $U$  if necessary, we can assume that  $U$  is a collar of  $\partial N$ . This means,  $U$  is of the form

$$U = \{x \in N : \text{dist}(x, \partial N) \leq \delta\}$$

for some  $\delta > 0$ , each point  $x \in U$  has a unique nearest projection  $\sigma(x) \in \partial N$ , and the mapping  $\varphi$  given by

$$\varphi(x) := (\sigma(x), |x - \sigma(x)|) \quad \text{for } x \in U$$

is a diffeomorphism  $U \rightarrow \partial N \times [0, \delta]$ . For each  $x \in U$ , the differential  $d\varphi_x$  is an isomorphism

$$T_x N \simeq T_{\sigma(x)} \partial N \oplus \mathbb{R},$$

so  $T_x N$  can be decomposed into a tangential and a normal subspace, with respect to  $\partial N$ . To keep the notation simple, we assume here that  $U = \partial N \times [0, \delta]$ , and  $\varphi = \text{Id}_U$ .

To define  $w$ , we interpolate linearly between  $\xi$  and the tangential component of  $\tilde{v}$ , but we leave the normal component of  $\tilde{v}$  unchanged. More precisely, given  $x = (y, t) \in \partial N \times [0, \delta]$  we

define

$$\begin{aligned} w(x) &:= \left(1 - \frac{t}{\delta}\right) \xi(y) + \frac{t}{\delta} P_{\partial N} \tilde{v}(x) + \tilde{v}(x) - P_{\partial N} \tilde{v}(x) \\ &= \left(1 - \frac{t}{\delta}\right) \left(\xi(y) - P_{\partial N} \tilde{v}(x)\right) + \tilde{v}(x) \end{aligned}$$

whereas we set

$$w(x) := \tilde{v}(x) \quad \text{for } x \in N \setminus U.$$

Then  $w$  is of class  $\mathcal{C}^1$  on  $U$ , continuous on  $N$ , satisfies  $w = \xi + \tilde{v} - P_{\partial N} \tilde{v}$  on  $\partial N$ . Moreover, for  $x = (y, t) \in U$  we have

$$\begin{aligned} |\tilde{v}(x) - w(x)| &\leq \left(1 - \frac{t}{\delta}\right) |\xi(y) - P_{\partial N} \tilde{v}(x)| \\ &\leq \left(1 - \frac{t}{\delta}\right) (|\xi(y) - P_{\partial N} \tilde{v}(y, 0)| + |P_{\partial N} \tilde{v}(y, 0) - P_{\partial N} \tilde{v}(y, t)|) \\ &\stackrel{(3.14)}{\leq} \left(1 - \frac{t}{\delta}\right) \varepsilon + t \left(1 - \frac{t}{\delta}\right) \text{Lip}_U(P_{\partial N} \tilde{v}) \\ &\stackrel{(3.14)}{\leq} \varepsilon + \delta C. \end{aligned}$$

By choosing  $\delta$  small, and combining this inequality with (3.13), we deduce (3.15).  $\square$

#### 4. THE INDEX IN THE VMO SETTING

We have now all the necessary tools to define and study the index of a VMO field, which is the aim of this section. From now on,  $X$  will be taken to be the topological double of  $N$ , as in Section 2. Moreover, throughout this section we consider a function  $g \in \text{VMO}(\partial N)$  such that

$$(4.1) \quad g(x) \in T_x N \quad \text{and} \quad c_1 \leq |g(x)| \leq c_2 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial N$$

for some constants  $c_1, c_2 > 0$ . Let  $v$  be a VMO vector field with trace  $g$ , that is,

$$(4.2) \quad v \in \text{VMO}_g(N), \quad v(x) \in T_x N \quad \text{for a.e. } x \in N.$$

By definition of  $\text{VMO}_g(N)$ , the function  $u$  given by

$$u := \begin{cases} v & \text{on } N \\ G & \text{on } X \setminus N, \end{cases}$$

where  $G$  is the extension of  $g$  defined in (2.3), is in  $\text{VMO}(X)$ . Denote the local averages of  $u$  and  $g$  by

$$\bar{u}_\varepsilon(x) := \int_{B_\varepsilon^X(x)} u(y) \, d\sigma(y), \quad \text{for } x \in X.$$

and

$$\bar{g}_\varepsilon(x) := \int_{B_\varepsilon^{\partial N}(x)} g(y) \, d\mathcal{H}^{n-1}(y), \quad \text{for } x \in \partial N.$$

Consider the functions

$$(4.3) \quad u_\varepsilon := P_X \bar{u}_\varepsilon \quad \text{and} \quad g_\varepsilon := P_X \bar{g}_\varepsilon,$$

defined on  $X$  and  $\partial N$ , respectively, which are continuous and tangent to  $X$ . As we will prove in the following Lemma 4.1, Lemma 4.2, and Lemma 4.5, the quantities  $\text{ind}(u_\varepsilon, N)$  and  $\text{ind}_-(g_\varepsilon, \partial N)$  are well-defined and constant with respect to  $\varepsilon$ , for  $\varepsilon$  small enough.

**Definition 4.1.** Given  $g \in \text{VMO}(\partial N)$  and  $v$  which satisfy (4.1)–(4.2), we define the index and the inward boundary index of  $v$  by

$$\text{ind}(v, N) := \text{ind}(u_\varepsilon, N) \quad \text{and} \quad \text{ind}_-(v, \partial N) := \text{ind}_-(g_\varepsilon, \partial N),$$

where  $\varepsilon$  is fixed arbitrarily in  $(0, \varepsilon_0)$  and  $\varepsilon_0$  is given by Lemma 4.5.



Once we have checked that the index, in the sense of Definition 4.1, is well-defined, Theorem 1.4 will follow straightforwardly from Proposition 3.9. However, before directing our attention to the main theorem, there are some facts which need to be checked.

The next two lemmas compare the behaviour of  $g_\varepsilon$  and  $u_\varepsilon|_{\partial N}$ .

**Lemma 4.1.** *For every  $\delta > 0$ , there exists  $\varepsilon_0 \in (0, r_0)$  so that, for all  $\varepsilon \in (0, \varepsilon_0)$  and all  $x \in \partial N$ , we have*

$$c_1 - \delta \leq |g_\varepsilon(x)| \leq c_2 + \delta.$$

**Lemma 4.2.** *It holds that*

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \partial N} |u_\varepsilon(x) - g_\varepsilon(x)| = 0.$$

Combining Lemmas 4.1 and 4.2, we deduce that there exist constants  $\varepsilon_0, c > 0$  such that

$$|u_\varepsilon(x)| \geq c, \quad |g_\varepsilon(x)| \geq c \quad \text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and all } x \in \partial N.$$

In particular,

$$0 \notin u_\varepsilon(\partial N) \quad \text{and} \quad 0 \notin g_\varepsilon(\partial N)$$

so  $\text{ind}(u_\varepsilon, N)$  and  $\text{ind}_-(g_\varepsilon, \partial N)$  are well-defined, according to Definition 3.2 and Definition 3.3, for all  $\varepsilon \in (0, \varepsilon_0)$ .

Before proving Lemmas 4.1 and 4.2, we need a useful property.

**Lemma 4.3.** *It holds that*

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in X} |u_\varepsilon(x) - \bar{u}_\varepsilon(x)| = \lim_{\varepsilon \rightarrow 0} \sup_{x \in \partial N} |g_\varepsilon(x) - \bar{g}_\varepsilon(x)| = 0.$$

*Proof.* We present the proof for  $u_\varepsilon$  only, as the same argument applies to  $g_\varepsilon$  as well. Consider a finite atlas  $\mathcal{A} = \{U_\alpha\}_{\alpha \in A}$  for  $X$  and, for each  $\alpha \in A$ , let  $\nu_1^\alpha, \dots, \nu_{d-n}^\alpha$  be a smooth moving frame for the normal bundle of  $X$ , defined on  $U_\alpha$  (i.e.,  $(\nu_i^\alpha(y))_{1 \leq i \leq d-n}$  is an orthonormal base for  $T_y X^\perp$ , for all  $y \in U_\alpha$ ). Set

$$(4.4) \quad C_N := \max_{\substack{\alpha \in A \\ 1 \leq i \leq d-n}} \|D\nu_i^\alpha\|_{L^\infty(U_\alpha)} < +\infty.$$

For all  $\alpha \in A$  and  $x \in U_\alpha$ , we write

$$(4.5) \quad u_\varepsilon(x) - \bar{u}_\varepsilon(x) = \sum_{i=1}^{d-n} (\bar{u}_\varepsilon(x) \cdot \nu_i^\alpha(x)) \nu_i^\alpha(x)$$

and, since  $u(y) \cdot \nu_i^\alpha(y) = 0$  for a.e.  $y \in U_\alpha$ , we have

$$\bar{u}_\varepsilon(x) \cdot \nu_i^\alpha(x) = \int_{B_\varepsilon(x)} u(y) \cdot (\nu_i^\alpha(x) - \nu_i^\alpha(y)) \, d\sigma(y).$$

Taking into account (4.4), we infer

$$|\bar{u}_\varepsilon(x) \cdot \nu_i^\alpha(x)| \leq C_N \int_{B_\varepsilon(x)} |u(y)| |x - y| \, d\sigma(y).$$

To bound the right-side of this inequality, we exploit the injection  $\text{BMO}(X) \hookrightarrow L^p(X)$ , which holds true for all  $1 \leq p < +\infty$ , and the Hölder inequality. For a fixed  $p$ , we obtain

$$(4.6) \quad |\bar{u}_\varepsilon(x) \cdot \nu_i^\alpha(x)| \leq C_N \sigma(B_\varepsilon(x))^{-1} \|x - y\|_{L^p(B_\varepsilon(x))} \|u\|_{L^{p'}(X)} \leq C_{N,n,p} \varepsilon^{1+n/p-n} \|u\|_{L^{p'}(X)},$$

for some constant  $C_{N,n,p}$  depending only on  $C_N, n$  and  $p$ . Whenever  $p' < +\infty$ , the  $L^{p'}$  norm of  $u$  can be bounded using only the BMO norm of  $u$  and  $\int_X u$  (with the help of [4, Lemmas A.1 and B.3]). Thus, choosing  $p = p(n) > 1$  so small that  $1 + n/p - n > 0$ , from (4.5) and (4.6) we conclude the proof.  $\square$

*Proof of Lemma 4.1.* Setting

$$S_x := \{v \in T_x N : c_1 \leq |v| \leq c_2\}$$

we have, for all  $x \in \partial N$ ,

$$(4.7) \quad \text{dist}(\bar{g}_\varepsilon(x), S_x) \leq \int_{B_\varepsilon(x)} |\bar{g}_\varepsilon(x) - g(y)| \, d\sigma(y) + \int_{B_\varepsilon(x)} \text{dist}(g(y), S_x) \, d\sigma(y).$$

The first term in the right-hand side tends to zero as  $\varepsilon \rightarrow 0$ , uniformly in  $x$ , due to Lemma 2.1. On the other hand, it holds

$$(4.8) \quad \sup_{\substack{x, y \in \partial N \\ \text{dist}(x, y) \leq \varepsilon}} \sup_{v \in S_y} \text{dist}(v, S_x) \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

since  $N$  is compact and smooth up to the boundary. Formula (4.8) can be easily proved, e.g., by contradiction: Assume that (4.8) does not hold. Then, we find a number  $\eta > 0$ , a sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$  of positive numbers s.t.  $\varepsilon_k \searrow 0$ , two sequences  $(x_k)_{k \in \mathbb{N}}$ ,  $(y_k)_{k \in \mathbb{N}}$  in  $N$  and one  $(v_k)_{k \in \mathbb{N}}$  in  $\mathbb{R}^d$ , which satisfy

$$v_k \in S_{y_k}, \quad \text{dist}(x_k, y_k) \leq \varepsilon_k, \quad \text{dist}(v_k, S_{x_k}) \geq \eta.$$

By compactness of  $N$ , up to subsequences we can assume that

$$x_k \rightarrow x \in N, \quad y_k \rightarrow y \in N, \quad v_k \rightarrow v \in \mathbb{R}^d,$$

where  $c_1 \leq |v| \leq c_2$ . Let  $\nu_i, \nu_2, \dots, \nu_{d-n}$  be a moving frame for the normal bundle of  $N$ , defined on a neighbourhood of  $y$ . Passing to the limit in the condition

$$v_k \cdot \nu_i(y_k) = 0 \quad \text{for all } i$$

we find that  $v \in T_y N$ , hence  $v \in S_y$ . But  $y = x$ , because  $\text{dist}(x_k, y_k) \leq \varepsilon_k \rightarrow 0$ . Thus, we have found  $v \in S_x$  so that  $\text{dist}(v, S_{x_k}) \geq \eta/2 > 0$ . On the other hand, if  $\varphi: U \subset N \rightarrow \mathbb{R}^n$  is a coordinate chart near  $x$  then

$$w_k := d\varphi_{\varphi(x_k)}^{-1}(d\varphi_x v), \quad \tilde{w}_k := \min\{\max\{|w_k|, c_1\}, c_2\} \frac{w_k}{|w_k|}$$

are well-defined for  $k \gg 1$  and  $\tilde{w}_k \in S_{x_k}$ ,  $\tilde{w}_k \rightarrow v$ . This leads to a contradiction.

Thus, we can take advantage of (4.1) and (4.8) to estimate the second term in the right-hand side of (4.7). We deduce that

$$\sup_{x \in \partial N} \text{dist}(\bar{g}_\varepsilon(x), S_x) \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

and, invoking Lemma 4.3, we conclude the proof.  $\square$

*Proof of Lemma 4.2.* In view of Lemma 4.3, proving that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \partial N} |\bar{u}_\varepsilon(x) - \bar{g}_\varepsilon(x)| = 0$$

is enough to conclude. In addition, it holds

$$(4.9) \quad |\bar{u}_\varepsilon(x) - \bar{g}_\varepsilon(x)| \leq |\bar{u}_\varepsilon(x) - \bar{G}_\varepsilon(x)| + |\bar{G}_\varepsilon(x) - \bar{g}_\varepsilon(x)|,$$

so we can study each term in the right-hand side and prove that they converge to zero as  $\varepsilon \rightarrow 0$ .

Let us focus on the first term. We remark that  $\bar{u}_\varepsilon - \bar{G}_\varepsilon = \overline{(u - G)}_\varepsilon$  and that

$$u - G = \begin{cases} v - G & \text{on } N \\ 0 & \text{on } X \setminus N. \end{cases}$$

Thus, for all  $x \in \partial N$  we have (recall that  $(u - G)(y) = 0$  for almost any  $y \in X \setminus N$ .)

$$\begin{aligned} \frac{\sigma(B_\varepsilon^X(x) \setminus N)}{\sigma(B_\varepsilon^X(x))} \left| \overline{(u - G)}_\varepsilon(x) \right| &\leq \frac{1}{\sigma(B_\varepsilon^X(x))} \int_{B_\varepsilon^X(x) \setminus N} \left| (u - G)(y) - \overline{(u - G)}_\varepsilon(x) \right| \, d\sigma(y) \\ &\leq \int_{B_\varepsilon^X(x)} \left| (u - G)(y) - \overline{(u - G)}_\varepsilon(x) \right| \, d\sigma(y), \end{aligned}$$

where  $\sigma$  is the Riemannian measure on  $X$ . Now, assume for a while that there exist two numbers  $\alpha, \varepsilon_0 > 0$  such that

$$(4.10) \quad \frac{\sigma(B_\varepsilon^X(x) \setminus N)}{\sigma(B_\varepsilon^X(x))} \geq \alpha$$

for all  $x \in \partial N$  and all  $\varepsilon \in (0, \varepsilon_0)$ . Therefore, when  $\varepsilon < \varepsilon_0$  we deduce

$$\sup_{x \in \partial N} \left| \overline{(u-G)_\varepsilon}(x) \right| \leq \alpha^{-1} \sup_{x \in \partial N} \int_{B_\varepsilon^X(x)} \left| (u-G)(y) - \overline{(u-G)_\varepsilon}(x) \right| d\sigma(y)$$

and, since  $u-G \in \text{VMO}(X)$ , the right-hand side tends to 0 as  $\varepsilon \rightarrow 0$ , by Lemma 2.1. To conclude, we have to prove the validity of (4.10). To this end we assume without loss of generality that  $N$  is a smooth, bounded domain in  $X = \mathbb{R}^n$ . For a fixed  $x_0 \in \partial N$ , we can locally write  $\partial N$  as the graph of a smooth function  $\varphi: B_{r_0}^n(0) \subseteq \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ . Then, letting  $L_{x_0}(x) := \varphi(x_0) + d\varphi(x_0)(x - x_0)$  be the linear approximation of  $\varphi$ , considering the region between the graphs of  $\varphi$  and  $L_{x_0}$  we deduce

$$\left| |N \cap B_\varepsilon^n(x_0)| - \frac{1}{2} |B_\varepsilon^n(x_0)| \right| \leq \int_{B_\varepsilon^{n-1}(x_0)} |\varphi(x) - L_{x_0}(x)| dx.$$

By the Taylor-Lagrange formula, we have  $|\varphi(x) - L_{x_0}(x)| \leq M|x - x_0|^2$ , for a suitable constant  $M$  controlling the hessian of  $\varphi$ . Thus

$$\left| |N \cap B_\varepsilon^n(x_0)| - \frac{1}{2} |B_\varepsilon^n(x_0)| \right| \leq M\omega_n \varepsilon^{n+2},$$

where  $\omega_n := \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n|B_1^n(0)|$ , and

$$(4.11) \quad \left| \frac{|N \cap B_\varepsilon^n(x_0)|}{|B_\varepsilon^n(x_0)|} - \frac{1}{2} \right| \leq nM\varepsilon^2.$$

The constant  $M$  depends on  $\varphi$ , which is defined just locally, in a neighbourhood of  $x_0$ . Nevertheless, owing to the compactness of  $\partial N$ , one needs to consider a *finite* number of functions  $\varphi$  only, and hence it is possible to choose a constant  $M$  which satisfies (4.11) for all  $x_0 \in N$ . Therefore, (4.10) follows.

Now, we have to deal with the second term in (4.9). We can assume, without loss of generality, that  $X = \mathbb{R}^n$  and

$$N = \mathbb{R}_+^n := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}.$$

We can always reduce to this case by composing with local coordinates, with the help of a partition of the unity argument. For the sake of simplicity, denote the variable in  $\mathbb{R}^n$  by  $x = (t, y)$ , where  $t \in \mathbb{R}$  and  $y \in \mathbb{R}^{n-1}$ .

Call  $\alpha_n$  the volume of the unit ball of  $\mathbb{R}^n$ . Using Fubini's theorem and the definition (2.3) of  $G$ , for  $x_0 = (0, y_0)$  and  $\varepsilon$  small enough (so that  $\chi(t, y) \equiv 1$ , for  $|t| \leq \varepsilon$ ) we compute

$$\begin{aligned} \overline{G}_\varepsilon(x_0) &= \frac{1}{\alpha_n \varepsilon^n} \int_{-\varepsilon}^{\varepsilon} \left( \int_{B^{n-1}(y_0, \sqrt{\varepsilon^2 - t^2})} G(t, y) dy \right) dt \\ &= \frac{\alpha_{n-1}}{\alpha_n \varepsilon^n} \int_{-\varepsilon}^{\varepsilon} (\varepsilon^2 - t^2)^{\frac{n-1}{2}} \left( \int_{B^{n-1}(y_0, \sqrt{\varepsilon^2 - t^2})} g(y) dy \right) dt \\ &= \frac{\alpha_{n-1}}{\alpha_n \varepsilon^n} \int_{-\varepsilon}^{\varepsilon} (\varepsilon^2 - t^2)^{\frac{n-1}{2}} \overline{g}_{\sqrt{\varepsilon^2 - t^2}}(y_0) dt \\ &= \frac{\alpha_{n-1}}{\alpha_n \varepsilon^n} \int_{-1}^1 (\varepsilon^2 - (\varepsilon s)^2)^{\frac{n-1}{2}} \overline{g}_{\sqrt{\varepsilon^2 - (\varepsilon s)^2}}(y_0) \varepsilon ds \\ &= \frac{\alpha_{n-1}}{\alpha_n} \int_{-1}^1 (1 - s^2)^{\frac{n-1}{2}} \overline{g}_{\varepsilon \sqrt{1 - s^2}}(y_0) ds. \end{aligned}$$

On the other hand, Fubini's theorem also implies that

$$\alpha_n = \alpha_{n-1} \int_{-1}^1 (1-t^2)^{\frac{n-1}{2}} dt,$$

thus

$$(4.12) \quad |\overline{G}_\varepsilon(x_0) - \overline{g}_\varepsilon(x_0)| \leq \frac{\alpha_{n-1}}{\alpha_n} \int_{-1}^1 (1-t^2)^{\frac{n-1}{2}} |\overline{g}_{\varepsilon\sqrt{1-t^2}}(y_0) - \overline{g}_\varepsilon(y_0)| dt.$$

For all  $-1 < t < 1$ , since  $B^{n-1}(y_0, \varepsilon\sqrt{1-t^2}) \subset B^{n-1}(y_0, \sqrt{1-t^2})$  we infer that

$$\begin{aligned} |\overline{g}_{\varepsilon\sqrt{1-t^2}}(y_0) - \overline{g}_\varepsilon(y_0)| &\leq \int_{B^{n-1}(y_0, \varepsilon\sqrt{1-t^2})} |g(y) - \overline{g}_\varepsilon(y_0)| dy \\ &\leq (1-t^2)^{\frac{1-n}{2}} \int_{B^{n-1}(y_0, \varepsilon)} |g(y) - \overline{g}_\varepsilon(y_0)| dy \end{aligned}$$

and, injecting this information into (4.12), we deduce

$$|\overline{G}_\varepsilon(x_0) - \overline{g}_\varepsilon(x_0)| \leq \frac{2\alpha_{n-1}}{\alpha_n} \int_{B^{n-1}(y_0, \varepsilon)} |g(y) - \overline{g}_\varepsilon(y_0)| dy.$$

Hence, applying once again Lemma 2.1, we conclude that the second term in the right-hand side of (4.9) converges to zero as  $\varepsilon \rightarrow 0$ , uniformly in  $x \in \partial N$ .  $\square$

*Remark 4.4.* Setting

$$\text{ind}_-(g, \partial N) := \text{ind}_-(u_\varepsilon, \partial N)$$

gives another possibility to define the inward boundary index of  $g$ , just as natural as our Definition 4.1. However, thanks to Lemma 4.2 and to the stability of the inward boundary index (Lemma 3.8), we deduce that the two definitions agree.

**Lemma 4.5.** *There exists  $\varepsilon_0 \in (0, r_0)$  so that the functions*

$$\varepsilon \mapsto \text{ind}(u_\varepsilon, N), \quad \varepsilon \mapsto \text{ind}_-(g_\varepsilon, \partial N)$$

*are constant on  $(0, \varepsilon_0)$ .*

*Proof.* We have already remarked that  $\text{ind}(u_\varepsilon, N)$  and  $\text{ind}_-(g_\varepsilon, \partial N)$  are well-defined for  $\varepsilon$  small, as a consequence of Lemmas 4.1 and 4.2. Consider the functions  $H: N \times (0, \varepsilon_0) \rightarrow \mathbb{R}^d$  and  $G: \partial N \times (0, \varepsilon_0) \rightarrow \mathbb{R}^d$  given by

$$H(x, \varepsilon) := u_\varepsilon(x) = \text{proj}_{T_x N} \overline{u}_\varepsilon(x)$$

and

$$G(x, \varepsilon) := g_\varepsilon(x) = \text{proj}_{T_x N} \overline{g}_\varepsilon(x).$$

These maps are well-defined and continuous. Indeed, it follows from the dominated convergence theorem that  $(x, \varepsilon) \mapsto \overline{u}_\varepsilon(x)$  and  $(x, \varepsilon) \mapsto \overline{g}_\varepsilon(x)$  are continuous, whereas the family of projections  $\text{proj}_{T_x N}$  depends continuously on  $x$ . Applying Corollary 3.7 and Lemma 3.8 to  $H$  and  $G$  respectively, we conclude that  $\text{ind}(u_\varepsilon, N)$  and  $\text{ind}_-(g_\varepsilon, \partial N)$  are constant with respect to  $\varepsilon$ .  $\square$

After these preliminary lemmas, the proof of our main result, Theorem 1.4, is straightforward.

*Proof of Theorem 1.4.* Let  $v$  be given and let  $u_\varepsilon$  and  $g_\varepsilon$  be the continuous approximants of  $v$  and of its trace, as defined in (4.3). Let  $\varepsilon_0 > 0$  be the constant given by Lemma 4.5. Up to choosing a smaller value of  $\varepsilon_0$ , owing to Lemma 4.1 and Lemma 4.2, for all  $\varepsilon \in (0, \varepsilon_0)$  and all  $x \in \partial N$  we have

$$\begin{aligned} |u_\varepsilon(x)| &\geq \frac{c_1}{2}, \quad |g_\varepsilon(x)| \geq \frac{c_1}{2}, \\ |u_\varepsilon(x) - g_\varepsilon(x)| &< \frac{\sqrt{5}-1}{8} c_1. \end{aligned}$$

Therefore, by Lemma 3.8, for all  $\varepsilon \in (0, \varepsilon_0)$

$$(4.13) \quad \text{ind}_-(u_\varepsilon, \partial N) = \text{ind}_-(g_\varepsilon, \partial N).$$

To conclude, by Definition 4.1 and Proposition 3.9 we obtain

$$\begin{aligned} \text{ind}(v, N) + \text{ind}_-(v, \partial N) &= \text{ind}(u_\varepsilon, N) + \text{ind}_-(g_\varepsilon, \partial N) \\ &\stackrel{(4.13)}{=} \text{ind}(u_\varepsilon, N) + \text{ind}_-(u_\varepsilon, \partial N) = \chi(N), \end{aligned}$$

which proves Theorem 1.4. □

We can finally give the proof of Proposition 1.5.

*Proof of Proposition 1.5.* Since a field  $v \in \text{VMO}_g(N)$  satisfying (1.4) has  $\text{ind}(v, N) = 0$ , Theorem 1.4 directly implies (1.8). In order to prove the converse implication, let a field  $g \in \text{VMO}(\partial N)$  be given, such that (1.5) and (1.8) hold. By Lemma 4.5 there exists  $\varepsilon > 0$  such that the continuous field  $g_\varepsilon$ , defined by

$$\bar{g}_\varepsilon(x) := \int_{B_{\partial N}^{\varepsilon}(x)} g(y) d\mathcal{H}^{n-1}(y), \quad g_\varepsilon(x) := P_X \bar{g}_\varepsilon(x), \quad \text{for } x \in \partial N,$$

satisfies

$$(4.14) \quad \text{ind}_-(g_\varepsilon, \partial N) = \text{ind}_-(g, \partial N) \stackrel{(1.8)}{=} \chi(N).$$

Moreover, by Lemma 4.1, we can choose  $\varepsilon$  such that

$$\frac{c_1}{2} \leq |g_\varepsilon| \leq 2c_2.$$

Let  $G_\varepsilon \in \text{VMO}_{g_\varepsilon}(X)$  be its standard extension (defined in (2.3)). As  $0 \notin G_\varepsilon(\partial N)$ , by the Transversality Theorem 3.2 there exists a smooth tangent vector field  $F$  on  $X$  such that  $F$  has finitely many zeros in  $N$ ,  $F|_{\partial N} = g_\varepsilon$  and, by stability (Corollary 3.7) and by Theorem 1.4, that

$$\text{ind}(F, N) = \text{ind}(G_\varepsilon, N) = \chi(N) - \text{ind}_-(G_\varepsilon, \partial N) = \chi(N) - \text{ind}_-(g_\varepsilon, \partial N) \stackrel{(4.14)}{=} 0.$$

Assume for the moment that  $|F(x)| > 0$  for all  $x \in X$ , then we just need to modify the trace of  $F$  on  $\partial N$  and to rescale it. Let  $U$  be the tubular neighbourhood used in the definition of the extension  $G$  of  $g$  and let  $d_\sigma(x, \partial N)$  be the geodesic distance from  $\partial N$ . Define

$$\tilde{F}(x) := \begin{cases} F(x) & \text{if } x \in X \setminus U, \\ d_\sigma(x, \partial N)F(x) + G(x) & \text{if } x \in U. \end{cases}$$

The new field clearly satisfies  $\tilde{F}(x) \in \text{VMO}_g$ , as  $d(x, \partial N)F(x)$  is continuous with zero trace on  $\partial N$  and  $G \in \text{VMO}_g$  by definition. Moreover, as the mapping

$$\gamma : t \mapsto (1-t)\tilde{F} + tF$$

is continuous from  $[0, 1]$  to  $\text{BMO}(X)$  (indeed  $\|\gamma(t) - \gamma(s)\|_{\text{BMO}} \leq |t-s| \|F - \tilde{F}\|_{\text{BMO}}$ ) by the general homotopy principle (Proposition 3.6)

$$\text{ind}(\tilde{F}, X) = \text{ind}(F, X) = 0.$$

Finally, let

$$A_1 := \{x \in X : |F(x)| < c_1\}, \quad A_2 := \{x \in X : |F(x)| > c_2\}.$$

Again, we assume for the moment that  $|\tilde{F}(x)| > 0$  for all  $x \in X$ . The field defined by

$$v(x) := \begin{cases} \frac{\tilde{F}(x) c_i}{|\tilde{F}(x)|} & \text{if } x \in A_i, \quad i = 1, 2, \\ \tilde{F}(x) & \text{otherwise.} \end{cases}$$

belongs to  $\text{VMO}_g(N)$  and satisfies (1.4).

To conclude, we note that there is a standard technique to modify a continuous field  $u$  such that

$$\text{ind}(u, N) = 0, \quad \text{and} \quad \#\{x \in N : u(x) = 0\} < +\infty$$

into a continuous field  $\tilde{u}$  such that  $|\tilde{u}| > 0$  and  $\tilde{u} = u$  on  $\partial N$ . We sketch here the idea. First, up to a continuous transformation, we can assume that all the zeros are contained in one coordinate neighbourhood  $U$ , with chart  $\phi: U \subset N \rightarrow D \subset \mathbb{R}^n$ , so we can reduce to study the vector field in coordinates: let  $D = B_1(0)$ ,  $D_{1/2} = B_{1/2}(0)$ , assume that  $u: D \rightarrow \mathbb{R}^n$  and  $|u| > 0$  in  $D \setminus D_{1/2}$ . Then,

$$0 = \text{ind}(u, D) = \text{deg} \left( \frac{u}{|u|}, \partial D, \mathbb{S}^{n-1} \right).$$

By a classical result (see, e.g., [13]), there exists a harmonic field  $\psi: D \rightarrow \mathbb{S}^{n-1}$  such that  $\psi|_{\partial D} = \frac{u}{|u|}$ . Define

$$\tilde{\psi}(x) := \begin{cases} \psi(x) & \text{if } x \in D_{1/2} \\ \psi(x)(2d(x, \partial D) + (1 - 2d(x, \partial D)|u(x)|)) & \text{if } x \in D \setminus D_{1/2}, \end{cases}$$

so that  $\tilde{\psi}(x)$  is continuous on  $D$ , nowhere zero, and it agrees with  $u$  on  $\partial D$ . To conclude, the field

$$\tilde{u}(x) := \begin{cases} u & \text{if } x \in N \setminus U \\ \phi^* \tilde{\psi} & \text{if } x \in U \end{cases}$$

is continuous and nowhere zero on  $N$ . Here  $\phi^* \tilde{\psi}(x) := d\phi_{\phi(x)}^{-1} \tilde{\psi}(\phi(x))$  denotes the usual pullback of  $\tilde{\psi}$  via  $\phi$ .  $\square$

## 5. AN APPLICATION: Q-TENSOR FIELDS AND LINE FIELDS.

In the mathematical modelling of Liquid Crystals two different theories are eminent. In the Frank-Oseen theory the molecules are represented by the unit vector field  $n$  which appears in the energy (1.1). The main drawback of this approach is to neglect the natural head-to-tail symmetry of the crystals. The theory of Landau-de Gennes takes this symmetry into account by introducing a tensor-valued field, called *Q-tensor*, to which is associated a scalar parameter  $s$  that represents the local average ordered/disordered state of the molecules. In the particular, but physically relevant, case when the order parameter is a positive constant, there is a bijection between Q-tensors and line fields. The differences between the vector-based and the line field-based theory have been studied in [1], in two- and three-dimensional Euclidean domains. In this Section we have to aims: firstly we apply the results obtained in Section 4 to line fields on a compact surface, obtaining the VMO-analogue of Poincaré-Kneiser Theorem (see Theorem 5.2 below); secondly we show how the question of orienting a line field, studied in [1], has generally a negative answer on a compact surface. As it happens for liquid crystals in Euclidean domains, the elastic part of the Landau-de Gennes energy for nematic shells is, at least in some simplified situations, proportional to a Dirichlet type energy. See on this regard [16] and [22]. Therefore, owing to the embedding of Sobolev spaces in VMO spaces, (1.3), Proposition 5.3 establishes a relation between the existence of finite energy Q-tensors with strictly positive order parameter and the topology of the underlying surface, thus extending our application scope from the Frank-Oseen theory to the (constrained) Landau-de Gennes one, for uniaxial nematic shells.

**5.1. Q-tensors and line fields.** Nematic shells are the datum of a compact, connected and without boundary surface  $N \subset \mathbb{R}^3$  coated with a thin film of rod-shaped, head-to-tail symmetric particles of nematic liquid crystal. At a given point  $x \in N$ , the local configuration is represented by a probability measure  $\mu_x$  on the unit circle  $S_x$  in  $T_x N$ . More precisely, for each Borel set  $A \subset S_x$ ,  $\mu_x(A)$  is the probability of finding a particle at  $x$ , with direction contained in  $A$ . To account for the symmetry of the particles, we require

$$(5.1) \quad \mu_x(A) = \mu_x(-A)$$

for each Borel set  $A \subset S_x$ . Due to this constraint, the first-order momentum of  $\mu_x$  vanishes. Hence, we are naturally led to consider the second-order momentum

$$(5.2) \quad Q = \sqrt{2} \int_{S_x} \left( p^{\otimes 2} - \frac{1}{2} \mathbb{P}_x \right) d\mu_x(p),$$

where  $(p^{\otimes 2})_{ij} := p^i p^j$  and  $\mathbb{P}_x$  denotes the orthogonal projection on  $T_x N$ . Note that  $Q$  has been suitably renormalized, so that  $Q = 0$  when  $\mu_x$  is the uniform measure, and  $|Q| = 1$  when  $\mu_x$  is a Dirac measure concentrated on one direction (see (5.6) and (5.7)). This formula defines a real  $3 \times 3$  symmetric and traceless matrix called *Q-tensor*. As we are interested in fields on surfaces, we replaced the usual three-dimensional renormalization term  $-\frac{1}{3} Id$  by  $-\frac{1}{2} \mathbb{P}_x$  (see, e.g., [16]). Once we have fixed an orientation on  $N$ , we let  $\gamma$  denote the Gauss map. By definition (5.2),  $Q\gamma(x) = 0$ , which translates the intuitive fact that the probability of finding a particle in the normal direction of the surface is zero. We call this type of anchoring a *degenerate (tangent) anchoring* (see [22]).

For any  $x \in N$  we define the class of “admissible tensors” at  $x$  as

$$(5.3) \quad \mathbf{Q}_x := \{Q \in \mathcal{S}_0 : Q\gamma(x) = 0\},$$

where  $\mathcal{S}_0$  is the space of  $3 \times 3$  real, symmetric, and traceless matrices, endowed with the scalar product  $Q \cdot P = \sum_{ij} Q_{ij} P_{ij}$ . It is clear from the definition that  $\mathbf{Q}_x$  is a linear subspace of  $\mathcal{S}_0$  of dimension 2 (this can be easily checked, e.g., by proving that the map  $\mathcal{S}_0 \rightarrow \mathbb{R}^3$  given by  $Q \mapsto Q\gamma(x)$  is surjective). Moreover,  $\mathbf{Q}_x$  varies smoothly with  $x$ .

**Lemma 5.1.** *The set*

$$\mathbf{Q} := \coprod_{x \in N} \mathbf{Q}_x,$$

*equipped with the natural projection  $(x, Q) \mapsto x$ , is a smooth vector bundle on  $N$ .*

*Proof.* Consider a smooth orthonormal frame  $(n, m, \gamma)$  defined on a coordinate neighbourhood of  $N$ , where  $(n, m)$  is a basis for the tangent bundle of  $N$ . With straightforward computations, one can see that the matrices

$$\begin{aligned} X_{ij} &:= n_i n_j - m_i m_j, & Y_{ij} &:= n_i m_j + m_i n_j, \\ E_{ij} &:= \gamma_i \gamma_j - \frac{1}{3} \delta_{ij}, & F_{ij} &:= n_i \gamma_j + \gamma_i n_j, & G_{ij} &:= m_i \gamma_j + \gamma_i m_j \end{aligned}$$

define an orthogonal frame for  $\mathcal{S}_0$ . Moreover,  $(X(x), Y(x))$  is a basis for  $\mathbf{Q}_x$ , at each point  $x$  (see [14] for a use of this basis with a particular choice for  $(n, m, \gamma)$ ). The lemma follows easily.  $\square$

We can now analyze the special structure of the matrices in  $\mathbf{Q}_x$ . Fix  $Q \in \mathbf{Q}_x$ , from (5.3) it follows that  $\gamma(x)$  is an eigenvector of  $Q$ , corresponding to the zero eigenvalue. Since  $Q$  is symmetric and traceless, there exists an orthonormal basis  $(n, m)$  of  $T_x N$ , whose elements are eigenvectors of  $Q$ , and the corresponding eigenvalues are opposite. Thus, denoting by  $n$  the eigenvector corresponding to the positive eigenvalue,  $Q$  can be written in the form

$$(5.4) \quad Q = \frac{s}{2} (n^{\otimes 2} - m^{\otimes 2})$$

for some  $s \geq 0$  (If  $s = 0$ , then  $Q = 0$  and any choice of  $n$  is allowed). Using the identity  $n^{\otimes 2} + m^{\otimes 2} = \mathbb{P}_x$ , we conclude that for each  $Q \in \mathbf{Q}_x$  there exist a number  $s \geq 0$  and a unit vector  $n \in T_x N$  such that

$$(5.5) \quad Q = s \left( n^{\otimes 2} - \frac{1}{2} \mathbb{P}_x \right).$$

The number  $s$ , called the order parameter, is uniquely determined, and from (5.4) we obtain

$$(5.6) \quad |Q|^2 = Q \cdot Q = \frac{s^2}{4} (n^{\otimes 2} - m^{\otimes 2}) \cdot (n^{\otimes 2} - m^{\otimes 2}) = \frac{s^2}{4} (n^{\otimes 2} \cdot n^{\otimes 2} + m^{\otimes 2} \cdot m^{\otimes 2}) = \frac{s^2}{2}.$$

When  $Q \neq 0$ ,  $n$  is also uniquely determined, up to a sign. Thus, each  $Q \in \mathbf{Q}_x \setminus \{0\}$  identifies a positive number and a (un oriented) direction in  $T_x N$ , that is, a *line field*.

A *line field* on  $N$  (also called *1-distribution*) is an assignment of a (non zero) tangent direction — but not an orientation — to each point of the submanifold  $N$ . More precisely, following [29, Chapter 6] a line field  $L$  is a function that assigns to each point  $x$  of a manifold  $N$  a one-dimensional subspace  $L(x) \subset T_x N$ . Then  $L$  is spanned by a vector field *locally*; that is, we can choose a vector field  $v$  such that  $0 \neq v(x) \in L(x)$  for all  $x$  in some neighbourhood of  $x$ . We say that  $L$  is a smooth (continuous) 1-distribution if the vector field  $v$  can be chosen to be smooth (continuous) in a neighbourhood of each point.

Conversely, to a given line  $\ell \subset T_x N$  generated by a unit vector  $\xi \in T_x N$  it is possible to associate the measure  $\mu_x := \frac{1}{2}\delta_\xi + \frac{1}{2}\delta_{-\xi}$  and thus by (5.2) the direction  $\xi$  corresponds to

$$(5.7) \quad Q = \sqrt{2} \left( \xi^{\otimes 2} - \frac{1}{2}\mathbb{P} \right),$$

which is a unit  $Q$ -tensor. The reason for associating to the direction  $\xi$  the measure  $\mu_x = \frac{1}{2}\delta_\xi + \frac{1}{2}\delta_{-\xi}$ , instead of simply  $\delta_\xi$ , is to be found in the head-to-tail symmetry of the molecules expressed by (5.1). Thus, line fields on  $N$  can be identified with sections of the bundle  $\mathbf{Q}$ , having modulus one.

In the following, we relax the condition  $|Q| = 1$ , by requiring  $|Q|$  to be bounded and uniformly positive.

**5.2. Existence of VMO line fields.** In what follows, we assume that  $N \subset \mathbb{R}^3$  is a smooth, compact, connected surface, without boundary. Based on Proposition 1.1 and on the results of Section 4, in Proposition 5.3 we prove that the existence of a VMO line field is subject to the same topological obstruction that holds for continuous vector fields. If we restrict to the continuous setting, the following result is classical (see, e.g., [12, Theorem 2.4.6, p. 24])

**Theorem 5.2** (Poincaré-Kneiser). *Let  $N$  be a compact, connected submanifold of  $\mathbb{R}^{n+1}$ . Then a continuous line field exists if and only if  $\chi(N) = 0$ .*

**Definition 5.1.** A VMO line field on  $N$  is a map  $Q \in \text{VMO}(N, \mathcal{S}_0)$ , such that

$$(5.8) \quad Q(x) \in \mathbf{Q}_x \quad \text{and} \quad c_1 \leq |Q(x)| \leq c_2$$

for some constants  $c_1, c_2 > 0$  and  $\mathcal{H}^2$ -a.e.  $x \in N$ .

The condition  $Q \in \text{VMO}(N, \mathcal{S}_0)$  makes perfectly sense, because  $\mathcal{S}_0 \simeq \mathbb{R}^5$  is a finite-dimensional linear space.

**Proposition 5.3.** *If a VMO line field on  $N$  exists, then  $\chi(N) = 0$ , that is,  $N$  has genus 1.*

*Proof.* The proof is based on the arguments of Section 4, with straightforward adaptations. We approximate  $Q$  with a family of continuous functions, by setting

$$\bar{Q}_\varepsilon(x) := \int_{B_\varepsilon^n(x)} Q(y) \, d\sigma(y)$$

for each  $x \in N$  and  $\varepsilon \in (0, r_0)$ . Then, we define

$$Q_\varepsilon(x) := \text{proj}_{\mathbf{Q}_x} \bar{Q}_\varepsilon(x) \quad \text{for } x \in N.$$

The functions  $Q_\varepsilon$  are continuous, since the  $Q_x$ 's vary smoothly (see Lemma 5.1). Owing to (5.8), and arguing as in Lemma 4.1, it can be proved that

$$\frac{c_1}{2} \leq |Q_\varepsilon(x)| \leq 2c_2$$

for all  $x \in N$  and  $\varepsilon$  small enough. In view of formula (5.5), each  $Q_\varepsilon$  induces a continuous line field on  $N$ . In fact, the continuity of  $Q_\varepsilon$  gives the continuity of  $|Q_\varepsilon|$ . Consequently, we have that  $s$  is a continuous function, thanks to (5.6). On the other hand, the representation formula (5.5) gives that

$$n^{\otimes 2}(x) = \frac{Q(x)}{s(x)} + \frac{1}{2}\mathbb{P}_x,$$



which implies the continuity of  $n^{\otimes 2}$  thanks to the assumed strict positivity of  $s$  and thanks to the continuity of the projection operator. The tensor  $n^{\otimes 2}$  is the line field we were looking for. Thus, by Theorem 5.2, it must be  $\chi(N) = 0$ .  $\square$

**5.3. Orientability of line fields.** A typical problem in the study of line fields is to understand in which circumstances a Q-tensor can be described in terms of a vector, that is when, given a tensor field  $Q$  with a specified regularity, one can find a unit vector field  $n$  with the same regularity, such that (in three dimensions)

$$(5.9) \quad Q = s \left( n^{\otimes 2} - \frac{1}{3} \mathbb{I} \right)$$

for some positive constant  $s$ . In other words, we are trying to prescribe an orientation for the Q-tensor without creating artificial discontinuities in the vector  $n$ . If for a given tensor  $Q$  we can find a vector  $n$  for which the representation (5.9) holds, we say that  $Q$  is *orientable*, otherwise *non-orientable*. The problem of the orientability of a Q-tensor has been addressed and solved by Ball and Zarnescu in [1], in the case of two- and three-dimensional Euclidean domains. They showed that the conditions for orienting a given tensor field are of topological as well as of analytical nature. Precisely, they require a Sobolev-type regularity, i.e.  $Q \in W^{1,p}(\Omega)$  with  $p \geq 2$ , together with the condition that the domain  $\Omega$  be simply connected.

Regarding Q-tensor fields on manifolds (which we assume here to be compact, connected, without boundary), we observe that there exists no two-dimensional surface  $N$  and exponent  $p \geq 2$  such that

$$Q \in W^{1,p}(N) \quad \Rightarrow \quad Q \text{ is orientable.}$$

Indeed, by Proposition 1.1 the only surface which allows for the existence of a unit vector field with regularity at least  $W^{1,2}$  is the torus, which is not simply connected, and on which simple examples of smooth nonorientable line fields can easily be constructed (see Fig. 3).

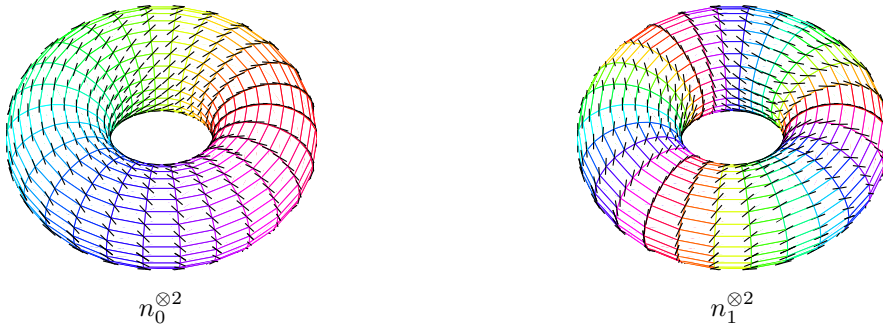


FIGURE 3. The case of an axisymmetric torus, with radii  $R = 2, r = 1$ , parametrized by  $X(\theta, \phi) = ((2 + \cos \theta) \cos \phi, (2 + \cos \theta) \sin \phi, \sin \theta)$ , on  $[0, 2\pi[ \times [0, 2\pi[$ . Let  $e_\theta := \partial_\theta X / |\partial_\theta X|$ ,  $e_\phi := \partial_\phi X / |\partial_\phi X|$ . We give a schematic representation of the two line fields defined via

$$n_i(\theta, \phi) := \cos \left( \frac{2i+1}{2} \phi \right) e_\theta + \sin \left( \frac{2i+1}{2} \phi \right) e_\phi, \quad i = 0, 1.$$

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SORBONNE UNIVERSITÉS, UPMC — UNIVERSITÉ PARIS 06, UMR 7598, LABORATOIRE JACQUES-LOUIS LIONS,  
F-75005, PARIS, FRANCE

*E-mail address*, G. Canevari: [canevari@ann.jussieu.fr](mailto:canevari@ann.jussieu.fr)

DIPARTIMENTO DI MATEMATICA “F. CASORATI”, UNIVERSITÀ DI PAVIA, PAVIA, ITALY

*E-mail address*, A. Segatti: [antonio.segatti@unipv.it](mailto:antonio.segatti@unipv.it)

*E-mail address*, M. Veneroni: [marco.veneroni@unipv.it](mailto:marco.veneroni@unipv.it)