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► **To cite this version:**

Dalia Ibrahim, Frédéric Abergel. Non-linear filtering and optimal investment under partial information for stochastic volatility models. 2014. hal-01018869v2

HAL Id: hal-01018869

<https://hal.science/hal-01018869v2>

Preprint submitted on 19 Nov 2014 (v2), last revised 22 Dec 2015 (v6)

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Non linear filtering and optimal investment under partial information for stochastic volatility models

Dalia Ibrahim; Frédéric Abergel*

July 6, 2014

Abstract

This paper studies the question of filtering and maximizing terminal wealth from expected utility in a stochastic volatility models. The special feature is that the only information available to the investor is the one generated by the asset prices and, in particular, the return processes cannot be observed directly and assumed to be modelled by a stochastic differential equation. Using stochastic non-linear filtering and change of measure techniques, the partial observation context can be transformed into a full information context such that coefficients depend only on past history of observed prices (filters processes). The main difficulty is that these filters are valued in infinite-dimensional space: it satisfy a stochastic partial differential equations named "Kushner-Stratonovich equations". We also show that we need to introduce an a priori models for the trend and the stochastic volatility in order to evaluate the filters processes. The dynamic programming or maximum principle are still applicable and the associated Bellman equation or Hamiltonian system are now in infinite dimension.

Keywords 0.1. *Partial information, stochastic volatility, utility maximization, non-linear filtering, Kushner-Stratonovich, infinite dimensional systems, dynamic programming.*

1 Introduction

The aim of a model in finance is to be able to predict, in the probabilistic sense, the evolution of the movements of a financial stock. The interest is to build a investment quantitative strategies or to cover a financial product against the fluctuations in the market. Louis Bachelier introduced the use in finance of the stochastic process now called "Brownian motion", in order to answer the questions about the price of derivatives. He proposed that the asset is modeled by a Brownian motion with a trend. After that, several attempts of modeling have subsequently been introduced.

Many practitioners in today's financial industry believe that most stock prices and indices are best modeled by continuous time stochastic processes, and in particular by

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diffusion processes. In the early 1970's Black, Scholes and Merton were the first to acknowledge this fact, by modeling the asset by a geometric Brownian motion where the stock's mean rate of return and volatility are constants. But the problem of this model is that the parameters are constant. So, a natural generalization of this model was to model the volatility by a stochastic process. The most popular stochastic volatility model is the Heston model (1993) where the volatility is modeled by a square root process introduced in finance by Cox Ingersoll and Ross (1985) in a very general context.

The basic problem of mathematical finance is the problem of an economic agent who invests in a financial market so as to maximize the expected utility of his terminal wealth. In the framework of continuous time model, the utility maximization problem has been studied for the first time by Merton (1971) in a Black-Scholes environment (full information) via the Hamilton-Jacobi-Bellman equation and dynamic programming. As in financial market models, we do not have in general a complete knowledge of all the parameters, which may be driven by unobserved random factors. So, we are in the situation of the utility maximization problem with partial observation, which has been studied extensively in the literature by Detemple (1986), Feldman(1986), Lakner (1995, 1998) and Karatzas and Xue (1991), etc.

In order to solve this problem with partial observation, the common way is to use the stochastic non-linear filtering and change of measure techniques, so as the partial observation context can be transformed into a full information context. Then it is possible to solve this problem either with the martingale approach or via dynamic programming approach. Models with incomplete information have been investigated by Dothan and Feldman using dynamic programming methods in a linear Gaussian filtering, Lakner [2] has solved the partial optimization problem via martingale approach and worked out the special case of the linear Gaussian filtering. Pham and Quenez [4] treated the case of partial information stochastic volatility model where they have combined stochastic filtering techniques and a martingale duality approach to characterize the value function and the optimal portfolio of the utility maximization problem. They have considered two cases studied by Lakner and Karatzas-Zhao to resolve the filtering problem. Firstly, they have assumed that the risk of the model are Gaussian processes modeled by a system of linear stochastic differential equations. Secondly, they have assumed that the risks are unobservable independent random variables with known probability distribution.

In our paper, we will be also interested by optimal investment problem under partial information for stochastic volatility models, but here we assume that the unobservable drift is modeled by a stochastic differential equation. In section 2, we describe the model and formulate the optimization problem. In section 3, using non-linear filtering techniques and change of measure techniques, the partial observation context can be transformed into a full information context such that coefficients depend only on past history of observed prices (filters processes). In section 4, we show that the filters estimations depend on a priori models for the trend and the stochastic volatility and these filters are valued in infinite-dimensional space: it satisfy a stochastic partial differential equations named "Kushner-Stratonovich equations". We give an example in a Bayesian setting for the filters and two others examples where the filters satisfy a Kushner-Stratonovich equations. Finally, in section 5, we obtain an explicit formulas for the optimal portfolio and optimal wealth of the utility maximization problem in the Bayesian case. For the

other cases, we use dynamic programming approach or maximum principle where the associated Bellman equation is in infinite dimension. The special cases of power and logarithmic utility functions are studied.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ satisfying the usual conditions, where $T > 0$ is a fixed time horizon. The financial market consists of one risky asset and a bank account (bond). The price of the bond is assumed for simplicity to be 1 over the entire continuous time-horizon $[0, T]$ and the risky asset has dynamics:

$$\frac{dS_t}{S_t} = \mu_t dt + g(V_t) dW_t^1, \quad (1)$$

$$dV_t = f(\beta_t, V_t) dt + h(V_t) dW_t, \quad (2)$$

$$d\mu_t = \zeta(\mu_t) dt + \vartheta(\mu_t) dW_t^3. \quad (3)$$

The processes W^1 and W are two Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and they are correlated with correlation coefficient ρ with $-1 \leq \rho \leq 1$, so we can write $dW_t = \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2$, where W^2 is independent of W^1 . W^3 is a standard Brownian motion independent of W^1 and W^2 . $\beta = \{\beta_t, 0 \leq t \leq T\}$ is a \mathbb{R} -adapted process or a constant. The drift $\mu = \{\mu_t, 0 \leq t \leq T\}$ is not observable and follows a stochastic differential equation.

We assume that g , f and h are such that unique strong solutions to the stochastic differential equations (1) and (2) exist. A Lipschitz condition is sufficient, but we do not impose this on the parameters at this stage, as we do not wish to exclude some well-known stochastic volatility models from the outset.

In the sequel, we denote by $\mathbb{F}^S = \{F_t^S, 0 \leq t \leq T\}$ (resp. $\mathbb{F}^V = \{F_t^V, 0 \leq t \leq T\}$) the filtration generated by the price process S (resp. by the stochastic volatility V). Also we denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the \mathbb{P} -augmentation of the market filtration generated by the price process S .

Remark 2.1. Notice that the \mathbb{P} -augmentation filtration \mathbb{G} is equal to the enlarged progressive filtration $\mathbb{F}^S \vee \mathbb{F}^V$. The first inclusion is obvious and the other inclusion $\mathbb{F}^S \vee \mathbb{F}^V \subset \mathbb{G}$ is deduced from the fact that V_t can be estimated from the quadratic variation of $\log(S_t)$.

2.1 The optimization problem

Let π_t be the fraction of the wealth that the trader decides to invest in the risky asset at time t , and $1 - \pi_t$ is the fraction of wealth invested in the bond. We assume that the trading strategy is self-financing, then the wealth process corresponding to a portfolio π is defined by $R_0^\pi = x$ and satisfies the following *S.D.E*:

$$dR_t^\pi = R_t^\pi (\pi_t \mu_t dt + \pi_t g(V_t) dW_t^1).$$

A function $U : \mathbb{R} \rightarrow \mathbb{R}$ is called a utility function if it is strictly increasing, strictly concave of class C^2 . We assume that the investor wants to maximize the expected utility of his terminal wealth. The optimization problem thus reads as

$$J(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(R_T^\pi)], \quad x > 0, \quad (4)$$

where \mathcal{A} denotes the set of the admissible controls $(\pi_t, 0 \leq t \leq T)$ which are \mathbb{F}^S -adapted, and satisfies the integrability condition:

$$\int_t^T g^2(V_s) \pi_s^2 ds < \infty \quad \mathbb{P} - a.s. \quad (5)$$

We are in a context when an investor wants to maximize the expected utility from terminal wealth, where the only information available to the investor is the one generated by the asset prices, therefore leading to a utility maximization problem in partially observed incomplete model. In order to solve it, we aim to reduce it to a maximization problem with full information. For that, it becomes important to exploit all the information coming from the market itself in order to continuously update the knowledge of the not fully known quantities and this is where stochastic filtering becomes useful.

3 Reduction to a full observation context

Let us consider the following processes:

$$\tilde{\mu}_t := \frac{\mu_t}{g(V_t)}, \quad (6)$$

$$\tilde{\beta}_t := \left(\sqrt{1 - \rho^2 h(V_t)} \right)^{-1} (f(\beta_t, V_t) - \rho h(V_t) \tilde{\mu}_t), \quad (7)$$

we assume that they verify the integrability condition:

$$\int_0^T |\tilde{\mu}_t|^2 + |\tilde{\beta}_t|^2 dt < \infty \quad a.s.$$

Here $\tilde{\mu}_t$ and $\tilde{\beta}_t$ are the unobservable processes that account for the market price of risk. The first is related to the asset's Brownian component. The second to the stochastic volatility's Brownian motion.

Also we introduce the following process:

$$L_t = 1 - \int_0^t L_s \left[\tilde{\mu}_s dW_s^1 + \tilde{\beta}_s dW_s^2 \right]. \quad (8)$$

We shall make the usual standing assumption on filtering theory.

Assumption 3.1. *The process L is a martingale, that is, $\mathbb{E}[L_T] = 1$.*

Under this assumption, we can now define a new probability measure $\tilde{\mathbb{P}}$ equivalent to \mathbb{P} on (Ω, \mathbb{F}) characterized by:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = L_t, \quad 0 \leq t \leq T. \quad (9)$$

Then Girsanov's transformation ensures that

$$\begin{aligned} \tilde{W}_t^1 &= W_t^1 + \int_0^t \tilde{\mu}_s ds \text{ is a } (\tilde{\mathbb{P}}, \mathbb{F})\text{-Brownian motion,} \\ \tilde{W}_t^2 &= W_t^2 + \int_0^t \tilde{\beta}_s ds \text{ is a } (\tilde{\mathbb{P}}, \mathbb{F})\text{-Brownian motion.} \end{aligned}$$

Also, we have that $(\tilde{\mu}_t, \tilde{\beta}_t)$ is independent of the Brownian motion $(\tilde{W}_t^1, \tilde{W}_t^2)$.

Therefore, the dynamics of (S, V) under $\tilde{\mathbb{P}}$ become:

$$\begin{aligned} \frac{dS_t}{S_t} &= g(V_t) d\tilde{W}_t^1, \\ dV_t &= \rho h(V_t) d\tilde{W}_t^1 + \sqrt{1 - \rho^2} h(V_t) d\tilde{W}_t^2. \end{aligned}$$

We now state a lemma which will highly relevant in the following. The proof of this lemma is similar to lemma 3.1 in Pham and Quenez [4].

Lemma 3.2. *The filtration \mathbb{G} is the augmented filtration of $(\tilde{W}^1, \tilde{W}^2)$.*

We now make the following assumption on the risk processes $(\tilde{\mu}, \tilde{\beta})$.

$$\forall t \in [0, T], \quad \mathbb{E}|\tilde{\mu}_t| + \mathbb{E}|\tilde{\beta}_t| < \infty \quad (10)$$

Under this assumption, we can introduce the filter estimates of $(\tilde{\mu}, \tilde{\beta})$:

$$\bar{\mu}_t := \mathbb{E}[\tilde{\mu}_t | \mathcal{G}_t], \quad (11)$$

$$\bar{\beta}_t := \mathbb{E}[\tilde{\beta}_t | \mathcal{G}_t]. \quad (12)$$

Now, we aim to construct the restriction of \mathbb{P} equivalent to $\tilde{\mathbb{P}}$ on (Ω, \mathbb{G}) . First, let us consider the conditional version of Baye's formula: for any $\tilde{\mathbb{P}}$ integrable random variable X ($X \in L^1(\tilde{\mathbb{P}})$), we have:

$$\tilde{\mathbb{E}}[X | \mathcal{G}_t] = \frac{\mathbb{E}[X L_t | \mathcal{G}_t]}{\mathbb{E}[L_t | \mathcal{G}_t]}. \quad (13)$$

Then by taking $X = \frac{1}{L_t}$ in the above equation, we get:

$$\tilde{L}_t := \tilde{\mathbb{E}}\left[\frac{1}{L_t} | \mathcal{G}_t\right] = \frac{1}{\mathbb{E}[L_t | \mathcal{G}_t]}. \quad (14)$$

Therefore, from (9) (14), we have the following restriction of \mathbb{P} to \mathbb{G} :

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{G}_t} = 1/\tilde{L}_t.$$

Finally, from Bain and Crisan [3, P.33] (proposition 2.30) and Pardoux [5, P.85](proposition 2.2.7), we have the following result:

Proposition 3.3. *The following processes \overline{W}^1 and \overline{W}^2 are independent (\mathbb{P}, \mathbb{G}) -Brownian motions.*

$$\begin{aligned}\overline{W}_t^1 &= W_t^1 + \int_0^t (\tilde{\mu}_s - \bar{\mu}_s) ds := \tilde{W}_t^1 - \int_0^t \bar{\mu}_s ds, \\ \overline{W}_t^2 &= W_t^2 + \int_0^t (\tilde{\beta}_s - \bar{\beta}_s) ds := \tilde{W}_t^2 - \int_0^t \bar{\beta}_s ds.\end{aligned}$$

These processes are called the innovation processes in filtering theory. They include the distances between the true values of $\tilde{\mu}$ and $\tilde{\beta}$ and their estimates:

Then, by means of the innovation processes, we can describe the dynamics of (S, V, R) within a framework of full observation model:

$$(Q) = \begin{cases} \frac{dS_t}{S_t} = g(V_t)\bar{\mu}_t dt + g(V_t)d\overline{W}_t^1, \\ dV_t = \left(\rho h(V_t)\bar{\mu}_t + \sqrt{1-\rho^2} h(V_t)\bar{\beta}_t \right) dt + \rho h(V_t)d\overline{W}_t^1 + \sqrt{1-\rho^2}h(V_t)d\overline{W}_t^2, \\ dR_t^\pi = R_t^\pi \pi_t \left(g(V_t) \bar{\mu}_t dt + g(V_t)d\overline{W}_t^1 \right). \end{cases}$$

4 Filtering

We have showed that conditioning arguments can be used to replace the initial partial information problem by a full information problem one which depends only on the past history of observed prices. But the reduction procedure involves the filter estimates $\bar{\mu}_t$ and $\bar{\beta}_t$.

There exists different cases for the modeling of the unobservable risk premia. Firstly, the Bayesian case, when the risk premia $\tilde{\mu}_t$ and $\tilde{\beta}_t$ are random variables with known prior distribution. This case was firstly studied by [1] and then by Pham and Quenez in [4] for the stochastic volatility case. We show in the application section, with which dynamics for the drift and the volatility, we are in the Bayesian case. Secondly, the case where the risk premia $\tilde{\mu}_t$ and $\tilde{\beta}_t$ are supposed to be Gaussian processes modeled by a system of stochastic differential equations. In their article, they have considered the case where $\tilde{\mu}_t$ and $\tilde{\beta}_t$ are independent, then by Kalman-Bucy filter they have deduced the estimates filters $\bar{\mu}_t$ and $\bar{\beta}_t$. But as we have mentioned in the introduction, Pham and Quenez [4] didn't assume any dynamics for the drift. Because, if we assume that the drift is modeled by certain dynamics, then we can't obtain a Gaussian modeling for the risk $\tilde{\mu}_t$. we will explain this point in details in our general result below.

Now, we aim to present our principal result concerning the general case, where we assume that the processes $\tilde{\mu}_t$ and $\tilde{\beta}_t$ are solutions of the following stochastic differential equations:

$$d \begin{pmatrix} \tilde{\mu}_t \\ \tilde{\beta}_t \end{pmatrix} = \begin{pmatrix} a \\ \bar{a} \end{pmatrix} dt + \begin{pmatrix} g_1 & g_2 \\ \bar{g}_1 & \bar{g}_2 \end{pmatrix} d \begin{pmatrix} N_t^1 \\ N_t^2 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ \bar{b}_1 & \bar{b}_2 \end{pmatrix} d \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \quad (15)$$

where we denote for simplification the functions $a := a(\tilde{\mu}_t, \tilde{\beta}_t)$, $\bar{a} := \bar{a}(\tilde{\mu}_t, \tilde{\beta}_t)$, $\bar{b}_2 = \bar{b}_2(\tilde{\mu}_t, \tilde{\beta}_t)$, and the Brownian motion (N_t^1, N_t^2) is independent of (W_t^1, W_t^2) .

4.1 Estimate $\bar{\mu}_t$ and $\bar{\beta}_t$

From lemma 3.2, we have $\mathbb{G} = \mathbb{F}^{\tilde{W}^1} \vee \mathbb{F}^{\tilde{W}^2}$. Then the vector $(\tilde{W}^1, \tilde{W}^2)$ corresponds to the observation process and its dynamics is given as follows:

$$d \begin{pmatrix} \tilde{W}_t^1 \\ \tilde{W}_t^2 \end{pmatrix} = d \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} + \begin{pmatrix} \tilde{\mu}_t \\ \tilde{\beta}_t \end{pmatrix} dt \quad (16)$$

and the dynamics of the signal process $(\tilde{\mu}_t, \tilde{\beta}_t)$ is give above.

Notations Let us denote by:

$$X_t = (\tilde{\mu}_t, \tilde{\beta}_t), \quad Y_t = (\bar{W}_t^1, \bar{W}_t^2), \quad A = \begin{pmatrix} a \\ \bar{a} \end{pmatrix}, \quad G = \begin{pmatrix} g_1 & g_2 \\ \bar{g}_1 & \bar{g}_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ \bar{b}_1 & \bar{b}_2 \end{pmatrix}$$

$$N_t = \begin{pmatrix} N_t^1 \\ N_t^2 \end{pmatrix}, \quad W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

where for $x = (m, b)$, $h_1(x) = m$ and $h_2(x) = b$.

With these notations, let us re-write the dynamics of the signal process X_t and observation process Y_t as follows:

$$dX_t = A(X_t)dt + G(X_t)dN_t + B(X_t)dW_t \quad (17)$$

$$dY_t = dW_t + h(X_t)dt \quad (18)$$

Now, we will be interested by filtering problem which consist in evaluating the conditional expectation of the unobservable process having the observations. In the sequel, we denote this conditional expectation by $\alpha_t(\phi) = \mathbb{E}[\phi(X_t)|\mathcal{G}_t]$.

Then one of the approaches to obtain the evolution equation for α_t is to change the measure. Using the change of measure $\tilde{\mathbb{P}}$ given in (9), we can define a new measure $\tilde{\mathbb{P}}$, such that the observation process becomes a $\tilde{\mathbb{P}}$ Brownian motion independent of the signal variable X_t . For that we need to discuss some conditions under which the process L is a martingale.

The classical condition is Navikov's condition:

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^t h_1(X_s)^2 ds + \frac{1}{2} \int_0^t h_2(X_s)^2 ds \right) \right] < \infty.$$

Normally Navikov's condition is quite difficult to verify directly, so we will present some results given in [3], to use an alternative conditions under which the process L is a martingale.

From lemma 3.9 in [3, P.52], we can deduce that L is a martingale if the following conditions are satisfied:

$$\mathbb{E} \left[\int_0^t (\|h(X_s)\|^2) ds \right] < \infty, \quad \mathbb{E} \left[\int_0^t L_s \|h(X_s)\|^2 ds \right] < \infty \quad \forall t > 0. \quad (19)$$

Denote by Λ_t the $(\tilde{\mathbb{P}}, \mathbb{F})$ -martingale given by $\Lambda_t = \frac{1}{L_t}$. We then have:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \Lambda_t, \quad 0 \leq t \leq T.$$

Therefore the computation of $\alpha_t(\phi)$ is obtained by the so-called Kallianpur-Striebel formula, which is related to Bayes formula. For every $\phi \in B(\mathbb{R}^d)$, we have the following representation:

$$\alpha_t(\phi) := \mathbb{E} [\phi(X_t) | \mathcal{F}_t^Y] = \frac{\tilde{\mathbb{E}} [\phi(X_t) \Lambda_t | \mathcal{F}_t^Y]}{\tilde{\mathbb{E}} [\Lambda_t | \mathcal{F}_t^Y]} := \frac{\psi_t(\phi)}{\psi_t(1)}, \quad (20)$$

with $\psi_t(f) := \mathbb{E}^0[\phi(X_t) \Lambda_t | \mathcal{F}_t^Y]$ is the unnormalized conditional distribution of $\phi(X_t)$, given \mathcal{F}_t^Y , $\psi_t(1)$ can be viewed as the normalising factor and $B(\mathbb{R}^d)$ is the space of bounded measurable functions $\mathbb{R}^2 \rightarrow \mathbb{R}$.

Remark 4.1. *The Kallianpur-Striebel formula can also be viewed as the abstract version of Bayes' identity in this filtering framework. Also this formula holds true for any Borel measurable, not necessarily bounded, such that $\mathbb{E} [|f(X_t)|] < \infty$ (see. exercise 5.1 in [3]).*

In the following, we assume that for all $t \geq 0$,

$$\tilde{\mathbb{P}} \left[\int_0^t [\psi_s(\|h\|)]^2 ds < \infty \right] = 1, \quad \text{for all } t > 0. \quad (21)$$

The following proposition shows that the conditional distribution of the signal is a solution of a nonlinear stochastic and parabolic type partial differential equation often called the Kushner-Stratonovich equation. This result due to Bain and Crisan [3] and Pardoux [5].

Proposition 4.2. *Assume that the signal and observation processes satisfy (17) and (18). If conditions (19) and (21) are satisfied then the conditional distribution $\alpha_t = \mathbb{E}[\phi(X_t) | \mathcal{G}_t]$ satisfies the following Kushner-Stratonovich equation:*

$$\begin{aligned} d\alpha_t(\phi) = & \alpha_t(A\phi)dt + [\alpha_t((h^1 + \mathcal{B}^1)\phi) - \alpha_t(h^1)\alpha_t(\phi)] [d\tilde{W}_t^1 - \alpha_t(h^1)dt] \\ & + [\alpha_t((h^2 + \mathcal{B}^2)\phi) - \alpha_t(h^2)\alpha_t(\phi)] [d\tilde{W}_t^2 - \alpha_t(h^2)dt]. \end{aligned} \quad (22)$$

for any $\phi \in B(\mathbb{R}^2)$.

Where for $x = (m, b) \in \mathbb{R}^2$, we have :

$$A\phi(m, b) = a\partial_m\phi + \bar{a}\partial_b\phi + \frac{1}{2} (b_1^2 + b_2^2 + g_1^2 + g_2^2) \partial_m^2\phi + (b_1\bar{b}_1 + b_2\bar{b}_2 + g_1\bar{g}_1 + g_2\bar{g}_2) \partial_{mb}^2\phi \\ + \frac{1}{2} (\bar{b}_1^2 + \bar{b}_2^2 + \bar{g}_1^2 + \bar{g}_2^2) \partial_b^2\phi,$$

$\mathcal{B}^1, \mathcal{B}^2$ are the operators given by:

$$\mathcal{B}^1\phi(m, b) = b_1\partial_m\phi + \bar{b}_1\partial_b\phi, \\ \mathcal{B}^2\phi(m, b) = b_2\partial_m\phi + \bar{b}_2\partial_b\phi.$$

Remark 4.3. *The above Kushner-Stratonovich equation holds true for any Borel measurable ϕ_1 , not necessarily bounded. In fact, we cannot replace ϕ by ϕ_1 because it is an unbounded function. For that, we proceed by cutting of ϕ_1 at a fixed level which we let tend to infinity. For this, let us introduce the functions $(\psi^k)_{k>0}$ defined as*

$$\psi^k(x) = \psi(x/k), \quad x \text{ in } \mathbb{R}^2,$$

where

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ \exp(\frac{|x|^2-1}{|x|^2-4}) & \text{if } 1 < |x| < 2 \\ 2 & \text{if } |x| \geq 2. \end{cases}$$

Then by using the following relations given in [3, P.151]:

$$\lim_{k \rightarrow \infty} \phi_1\psi^k(x) = \phi_1(x), \quad |\phi_1(x)\psi^k(x)| \leq |\phi_1(x)|, \\ \lim_{k \rightarrow \infty} A_s(\phi_1\psi^k)(x) = A_s\phi_1(x).$$

Then by replacing in equation (22) ϕ by $\phi_1\psi^k$ and from dominated convergence theorem, we may pass to the limit as $k \rightarrow \infty$ and then we deduce that $\alpha_t(\phi_1)$ satisfies equation (22).

Let ϕ_1 and ϕ_2 be the following functions:

$$\text{for } x = (m, b), \quad \phi_1(x) = m \quad \text{and} \quad \phi_2(x) = b.$$

Now, we can deduce our main result concerning the filter estimates $\bar{\mu}_t$ and $\bar{\beta}_t$ in the general case of models for the drift and the stochastic volatility.

Corollary 4.4. *If conditions (19) and (21) are satisfied, then the processes $\bar{\mu}$ and $\bar{\beta}$ are solution of the following stochastic differential equations:*

$$d\bar{\mu}_t = \alpha_t(a)dt + [\alpha_t(h^1\phi_1 + b_1) - \alpha_t(h^1)\alpha_t(\phi_1)]d\bar{W}_t^1 + [\alpha_t(h^2\phi_1 + b_2) - \alpha_t(h^2)\alpha_t(\phi_1)]d\bar{W}_t^2, \\ d\bar{\beta}_t = \alpha_t(\bar{a})dt + [\alpha_t(h^1\phi_2 + \bar{b}_1) - \alpha_t(h^1)\alpha_t(\phi_2)]d\bar{W}_t^1 + [\alpha_t(h^2\phi_2 + \bar{b}_2) - \alpha_t(h^2)\alpha_t(\phi_2)]d\bar{W}_t^2.$$

Proof. From the definition of $\bar{\mu}_t$ and $\bar{\beta}_t$, we have that:

$$\bar{\mu}_t = \alpha_t(\phi_1) \quad \text{and} \quad \bar{\beta}_t = \alpha_t(\phi_2)$$

Therefore from proposition 4.2, we can deduce the above equations for $\bar{\mu}_t$ and $\bar{\beta}_t$. \square

Remark 4.5. *Generally, these equations are infinite-dimensional and cannot be solved explicitly. Infinite-dimensional filters have to be solved numerically, but in concrete application, the filter could thus never be implemented exactly, so in order to avoid infinite dimensionality, some approximation schemes have been proposed, yielding finite-dimensional filters for the the unobserved state. A well-known approximation method is the extended Kalman filter, which is based upon linearization of the state equation around the current estimate, see e.g Pardoux [5].*

4.2 Existence of a regular density to the filter

In this filter, we show that under some conditions, the conditional distribution of X_t given \mathcal{F}_t^S has a density with respect to a reference measure, in particular with respect to Lebesgue measure.

The existence of a regular density for the filter in the case of uncorrelated system (decorrelation between the signal and the observations processes) with bounded coefficients has been proved by means of Malliavin calculus, by pardoux [5], Bain and Crisan [3]. The case of correlated systems with bounded coefficients has been studied by Michel and Bismut and Michel. Under some growth conditions on the sensor function h appeared in the observation process, Ferreyra has studied the existence of a regular density for the case of uncorrelated system where the observation coefficients are unbounded.

In our context, we will be interested by the case of correlated system where the observation coefficients are unbounded, more precisely where the sensor function h is not bounded.

Let us now make the following assumption:

Assumption 4.6. *We assume that the law of X_t given \mathcal{F}_t^S admits a density $p_t(x)$ relative to some dominating measure $m(dx)$.*

Given the assumption above we can thus write the conditional expectation of X_t given \mathcal{F}_t^S as follows:

$$\mathbb{E}[\phi(X_t)|\mathcal{F}_t^S] = \int_{\mathbb{R}^2} \phi(x)p_t(x)m(dx),$$

and it is known that p_t satisfies the Kushner-Stratonovich equation, see e.g. [3] and [5]:

$$dp_t(x) = A^*p_t(x)dt + \vartheta(p_t(x))d\bar{W}_t^1 + \nu(p_t(x))d\bar{W}_t^2 \quad (23)$$

where

$$\begin{aligned} \vartheta(p_t(x)) &= p_t(x)(h^1(x) - \alpha_t(h^1)) + \mathcal{B}^{1,*}p_t(x), \\ \nu(p_t(x)) &= p_t(x)(h^2(x) - \alpha_t(h^2)) + \mathcal{B}^{2,*}p_t(x). \end{aligned}$$

and A^* (resp. $\mathcal{B}^{1,*}$, $\mathcal{B}^{2,*}$) design the adjoint of the operators A (resp. \mathcal{B}^1 , \mathcal{B}^2) given as follows, for $x = (m, b)$:

$$\begin{aligned} A^* \phi(m, b) &= \partial_m(a\phi) + \partial_b(\bar{a}\phi) + \frac{1}{2} \partial_m^2((b_1^2 + b_2^2 + g_1^2 + g_2^2)\phi) + \partial_{mb}^2((b_1\bar{b}_1 + b_2\bar{b}_2 + g_1\bar{g}_1 + g_2\bar{g}_2)\phi) \\ &\quad + \frac{1}{2} \partial_b^2((\bar{b}_1^2 + \bar{b}_2^2 + \bar{g}_1^2 + \bar{g}_2^2)\phi), \end{aligned}$$

$$\mathcal{B}^{1,*} \phi(m, b) = \partial_m(b_1\phi) + \partial_b(\bar{b}_1\phi), \quad \mathcal{B}^2 \phi(m, b) = \partial_m(b_2\phi) + \partial_b(\bar{b}_2\phi).$$

The conditional density process (p_t) is Markov with respect to \mathbb{F}^S with infinite dimension.

4.3 Applications

We show in this section that the filtering problem depends on the dynamics of the drift and the stochastic volatility. Let us firstly study the Bayesian case. For this case, we will be interested by the decorrelated case ($\rho = 0$). Let us assume that the drift μ_t is modeled by an Ornstein Uhlenbeck as follows:

$$d\mu_t = \eta(v - \mu_t) dt + \sigma_\mu dW_t^3. \quad (24)$$

where W^3 is a standard Brownian motion independent of W^1 and W^2 .

Also we assume that the model of diffusion is given by the following GARCH model:

$$\frac{dS_t}{S_t} = \mu_t dt + \sqrt{V_t} dW_t^1, \quad (25)$$

$$dV_t = -\lambda V_t dt + \sigma_V V_t dW_t^2. \quad (26)$$

So we are in the context of dynamics (1), (2) and (3) with $g(x) = \sqrt{x}$, $\beta_t = -\lambda$ is a constant, $f(\lambda, x) = \lambda x$, $h(x) = \sigma_V x$, $\rho = 0$, $\xi(x) = \eta(v - x)$ and $\vartheta(x) = \sigma_\mu$ is a constant.

So from (6) and (7), the risk $\tilde{\mu}_t$ related to the asset's Brownian component and the risk $\tilde{\beta}_t$ related to the stochastic volatility Brownian motion are given by:

$$\tilde{\mu}_t = \frac{\mu_t}{\sqrt{V_t}} \quad \text{and} \quad \tilde{\beta}_t = \frac{-\lambda}{\sigma_V}. \quad (27)$$

Therefore the filter estimate $\bar{\beta}_t$ is a constant and it remains to calculate $\bar{\mu}_t := \mathbb{E}[\tilde{\mu}_t | \mathcal{F}_t^S]$.

Explicit formula for $\bar{\mu}_t$:

In this case of models, $\tilde{\mu}$ has a prior known distribution which will be denoted in the sequel by $q(dx)$. So we are in the Bayesian framework studied by Karatzas and Zhao [1] and also by Pham and Quenez [4] which give us an explicit formula of the filter estimate $\bar{\mu}_t$ as a function of the Brownian motion \tilde{W}^1 and the distribution $q(dx)$.

Let us introduce the new probability $\tilde{\mathbb{P}}$ as follows:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} | \mathcal{F}_t = L_t, \quad 0 \leq t \leq T.$$

where:

$$L_t = 1 - \int_0^t L_s \left[\tilde{\mu}_s dW_s^1 - \frac{\lambda}{\sigma_V} dW_s^2 \right].$$

Notice that Assumption 3.1 is satisfied if conditions (19) are satisfied. Here for $x = (m, b)$, $h(x) = (m, \frac{-\lambda}{\sigma_V})$. From theorem 3.11 in [3, P.54], in order to verify condition (19) it remains to verify that $h(x)$ has linear growth, which is the case.

Now, let $\Lambda_t = \frac{1}{L_t}$, then we have:

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \Big|_{\mathcal{F}_t} = \Lambda_t.$$

So from Ito's formula and the definition of \tilde{W}^1 and \tilde{W}^2 , we have the following dynamics of Λ :

$$d\Lambda_t = \Lambda_t \left[\tilde{\mu}_t d\tilde{W}_t^1 - \frac{\lambda}{\sigma_V} d\tilde{W}_t^2 \right]. \quad (28)$$

Now from Kallianpur-Striebel formula (20), we have:

$$\bar{\mu}_t := \mathbb{E} [\tilde{\mu}_t | \mathcal{G}_t] = \frac{\mathbb{E}^{\tilde{\mathbb{P}}} [\tilde{\mu}_t \Lambda_t | \mathcal{G}_t]}{\mathbb{E}^{\tilde{\mathbb{P}}} [\Lambda_t | \mathcal{G}_t]}. \quad (29)$$

i) Calculate $\mathbb{E}^{\tilde{\mathbb{P}}} [\tilde{\mu}_t \Lambda_t | \mathcal{G}_t]$:

From lemma 3.2, \mathbb{G} is the augmented filtration of $(\tilde{W}^1, \tilde{W}^2)$, that is $\forall t \geq 0$, $\mathcal{G}_t = \mathcal{F}_t^{\tilde{W}^1} \vee \mathcal{F}_t^{\tilde{W}^2}$. Also from Girsanov transformation, the random variable $\tilde{\mu}$ is independent of the Brownian motion \tilde{W}^1 under $\tilde{\mathbb{P}}$. So from equation (28), we have the following:

$$\mathbb{E}^{\tilde{\mathbb{P}}} [\tilde{\mu}_t \Lambda_t | \mathcal{G}_t] = \int_{\mathbb{R}} x \exp \left(x \tilde{W}_t^1 - \frac{\lambda}{\sigma_V} \tilde{W}_t^2 - \frac{1}{2} x^2 t - \frac{1}{2} \frac{\lambda^2}{\sigma_V^2} t \right) q(dx). \quad (30)$$

ii) Calculate $\mathbb{E}^{\tilde{\mathbb{P}}} [\Lambda_t | \mathcal{G}_t]$:

Similarly as above, we have:

$$\mathbb{E}^{\tilde{\mathbb{P}}} [\Lambda_t | \mathcal{G}_t] = \int_{\mathbb{R}} \exp \left(x \tilde{W}_t^1 - \frac{\lambda}{\sigma_V} \tilde{W}_t^2 - \frac{1}{2} x^2 t - \frac{1}{2} \frac{\lambda^2}{\sigma_V^2} t \right) q(dx). \quad (31)$$

Finally from (29), we have the following explicit formula for $\bar{\mu}_t$:

$$\begin{aligned} \bar{\mu}_t &= \frac{\int_{\mathbb{R}} x \exp \left(x \tilde{W}_t^1 - \frac{\lambda}{\sigma_V} \tilde{W}_t^2 - \frac{1}{2} x^2 t - \frac{1}{2} \frac{\lambda^2}{\sigma_V^2} t \right) q(dx)}{\int_{\mathbb{R}} \exp \left(x \tilde{W}_t^1 - \frac{\lambda}{\sigma_V} \tilde{W}_t^2 - \frac{1}{2} x^2 t - \frac{1}{2} \frac{\lambda^2}{\sigma_V^2} t \right) q(dx)} \\ &= N(t, \tilde{W}_t^1). \end{aligned} \quad (32)$$

Where the function N is given by:

$$N(t, w) := \frac{\partial_w M(t, w)}{M(t, w)}, \quad (t, w) \in [0, T] \times \mathbb{R},$$

and $M(t, w) = \int_{\mathbb{R}} \exp\left(xw - \frac{1}{2}x^2t\right) q(dx)$.

For the general case where the risks $\tilde{\mu}_t$ and $\tilde{\beta}_t$ are dependent and the correlation coefficient $\rho \neq 0$, we aim to use the non-linear filtering theory described in section 4 in order to deduce a stochastic differential equations for their filters.

For that we need to apply the following two steps:

- **First step**: Describe the dynamics of $(\tilde{\mu}_t, \tilde{\beta}_t)$ as in equation (15).
- **Second step**: We deduce from section 4.1 the filters $\bar{\mu}_t$ and $\bar{\beta}_t$.

The following result shows that we need to introduce an a priori models for the trend and the stochastic volatility in order to describe the dynamics of $(\tilde{\mu}_t, \tilde{\beta}_t)$ as in (15).

Proposition 4.7. *The dynamics of $(\tilde{\mu}_t, \tilde{\beta}_t)$ depends on the models for the trend and the stochastic volatility. Moreover, if there exists a function $\Upsilon : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $V_t = \Upsilon(\tilde{\mu}_t, \tilde{\beta}_t)$, then the dynamics of $(\tilde{\mu}_t, \tilde{\beta}_t)$ satisfies (15).*

Proof. From Itô's formula we have:

$$\tilde{\mu}_t = \tilde{\mu}_0 + \int_0^t \frac{1}{g(V_s)} d\mu_s - \int_0^t \frac{\mu_s}{g^2(V_s)} d(g(V_s)) + \int_0^t \frac{\mu_s}{g^3(V_s)} d \langle g(V) \rangle_s$$

So depending on the model for the trend and the stochastic volatility, we can appear $\tilde{\mu}_t$ in the above terms. Also, in some terms V_s still appear. So if there exists a function Υ such that $V_s = \Upsilon(\tilde{\mu}_s, \tilde{\beta}_s)$, then V_s is replaced by something depending on $(\tilde{\mu}_t, \tilde{\beta}_t)$ and therefore the dynamics of $\tilde{\mu}_t$ is described only in terms of $(\tilde{\mu}_t, \tilde{\beta}_t)$.

In the same way, one can describe the dynamics of $\tilde{\beta}_t$ in terms of $(\tilde{\mu}_t, \tilde{\beta}_t)$. \square

Let us consider two models which satisfy proposition 4.7 and then deduce the corresponding filters: Garch and log-Ornstein Models.

Garch Model:

Let us consider the following Garch-model:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + V_t dW_t^1, \\ dV_t &= \lambda(\theta - V_t) dt + \sigma_V V_t \left(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right), \\ d\mu_t &= \eta(v - \mu_t) dt + \sigma_\mu dW_t^3. \end{aligned}$$

Here β is a constant. $\lambda, \theta, \eta, v, \sigma_V$ and σ_μ are constants, and W^3 is a standard Brownian motion independent of W^1 and W^2 .

Here the risks of the model are given by:

$$\tilde{\mu}_t = \frac{\mu_t}{V_t} \quad \text{and} \quad \tilde{\beta}_t := \frac{\lambda(\theta - V_t)}{V_t \sigma_V \sqrt{1 - \rho^2}} - \frac{\rho}{\sqrt{1 - \rho^2}} \tilde{\mu}_t. \quad (33)$$

In the sequel, we aim to estimate the filters $\bar{\mu}_t$ and $\bar{\beta}_t$ using the above two steps.

Remark 4.8. Notice that V_t can be estimated from the quadratic variation of the log of the price process, so we have that V is a \mathbb{G} -adapted. However:

$$\bar{\beta}_t = \frac{\lambda}{\sigma_V \sqrt{1 - \rho^2}} \left[\frac{\theta}{V_t} - 1 \right] - \frac{\rho}{\sqrt{1 - \rho^2}} \bar{\mu}_t.$$

Let us begin by finding the stochastic differential equations satisfied by $\tilde{\mu}_t$ and $\tilde{\beta}_t$.

From Itô's formula, $\tilde{\mu}_t$ satisfies the following *S.D.E*:

$$\begin{aligned} \tilde{\mu}_t &= \tilde{\mu}_0 + \int_0^t \frac{1}{V_s} d\mu_s - \int_0^t \frac{\mu_s}{V_s^2} dV_s + \int_0^t \frac{\mu_s}{V_s^3} d\langle V \rangle_s \\ &= \tilde{\mu}_0 + \int_0^t \frac{1}{V_s} \eta (v - \mu_s) ds + \int_0^t \frac{\sigma_\mu}{V_s} dW_s^3 - \int_0^t \frac{\mu_s}{V_s^2} \lambda (\theta - V_s) ds - \int_0^t \frac{\mu_s}{V_s^2} \rho \sigma_V V_s dW_s^1 \\ &\quad - \int_0^t \frac{\mu_s}{V_s^2} \sqrt{1 - \rho^2} \sigma_V V_s dW_s^2 + \int_0^t \frac{\mu_s}{V_s^3} \sigma_V^2 V_s^2 ds, \\ &= \tilde{\mu}_0 + \int_0^t \left(\frac{1}{V_s} (\eta v - \lambda \theta \tilde{\mu}_s) + (\sigma_V^2 - \eta + \lambda) \tilde{\mu}_s \right) ds + \int_0^t \frac{\sigma_\mu}{V_s} dW_s^3 - \int_0^t \sigma_V \sqrt{1 - \rho^2} \tilde{\mu}_s dW_s^2 \\ &\quad - \int_0^t \rho \sigma_V \tilde{\mu}_s dW_s^1. \end{aligned} \tag{34}$$

Also from Itô's formula, $\tilde{\beta}_t$ satisfies the following *S.D.E*:

$$\begin{aligned} \tilde{\beta}_t &= \tilde{\beta}_0 - \int_0^t \frac{\lambda \theta}{\sigma_V \sqrt{1 - \rho^2}} \frac{1}{V_s^2} dV_s - \int_0^t \frac{\rho}{\sqrt{1 - \rho^2}} d\tilde{\mu}_s + \frac{\lambda \theta}{\sigma_V \sqrt{1 - \rho^2}} \int_0^t \frac{1}{V_s^3} d\langle V \rangle_s, \\ &= \tilde{\beta}_0 - \int_0^t \frac{\lambda \theta}{\sigma_V \sqrt{1 - \rho^2}} \left(\frac{\lambda \theta}{V_s^2} - \frac{\lambda}{V_s} - \frac{\sigma_V^2}{V_s} \right) ds - \int_0^t \frac{\rho}{\sqrt{1 - \rho^2}} d\tilde{\mu}_s \\ &\quad - \int_0^t \frac{\lambda \theta \rho}{\sqrt{1 - \rho^2}} \frac{1}{V_s} dW_s^1 - \int_0^t \lambda \theta \frac{1}{V_s} dW_s^2. \end{aligned} \tag{35}$$

On the other hand, from the expression of $\tilde{\beta}_t$ given in (33), we can deduce that V_t is a function in terms of $\tilde{\mu}_t$ and $\tilde{\beta}_t$ as follows:

$$\frac{1}{V_t} = \frac{\sigma_V \sqrt{1 - \rho^2}}{\lambda \theta} \tilde{\beta}_t + \frac{1}{\theta} + \frac{\rho \sigma_V}{\lambda \theta} \tilde{\mu}_t,$$

By replacing the above equation of $\frac{1}{V_t}$ in (34) and (35), one obtains the following dynamics of $\tilde{\mu}_t$ and $\tilde{\beta}_t$:

$$\tilde{\mu}_t = \tilde{\mu}_0 + \int_0^t a(\tilde{\mu}_s, \tilde{\beta}_s) ds + \int_0^t b_1(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^1 + \int_0^t b_2(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^2 + \int_0^t g(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^3, \tag{36}$$

$$\tilde{\beta}_t = \tilde{\beta}_0 + \int_0^t \bar{a}(\tilde{\mu}_s, \tilde{\beta}_s) ds + \int_0^t \bar{b}_1(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^1 + \int_0^t \bar{b}_2(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^2 + \int_0^t \bar{g}(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^3. \tag{37}$$

Where:

$$a(m, b) = (\eta v - \lambda \theta m) \left(\frac{\sigma_V \sqrt{1 - \rho^2}}{\lambda \theta} b + \frac{1}{\theta} + \frac{\rho \sigma_V}{\lambda \theta} m \right) + (\sigma_V^2 - \eta + \lambda) m,$$

$$b_1(m, b) = -\rho \sigma_V m,$$

$$b_2(m, b) = -\sigma_V \sqrt{1 - \rho^2} m,$$

$$g(m, b) = \frac{\sigma_\mu \sigma_V \sqrt{1 - \rho^2}}{\lambda \theta} b + \frac{\sigma_\mu}{\theta} + \frac{\sigma_\mu \rho \sigma_V}{\lambda \theta} m$$

$$\bar{a}(m, b) = - \left(b + \frac{\lambda}{\sigma_V \sqrt{1 - \rho^2}} + \frac{\rho}{\sqrt{1 - \rho^2}} m \right) \left(\lambda \theta \sigma_V \sqrt{1 - \rho^2} b + \lambda^2 \theta - \lambda - \sigma_V^2 + (\lambda \theta \rho \sigma_V) m \right),$$

$$\bar{b}_1(m, b) = -\rho \sigma_V b - \frac{\lambda \rho}{\sqrt{1 - \rho^2}},$$

$$\bar{b}_2(m, b) = -\sigma_V \sqrt{1 - \rho^2} b - \lambda,$$

$$\bar{g}(m, b) = -\frac{\rho \sigma_\mu \sigma_V}{\lambda \theta} b - \frac{\rho}{\sqrt{1 - \rho^2}} \frac{\sigma_\mu}{\theta} - \frac{\sigma_\mu \sigma_V}{\lambda \theta} \frac{\rho}{\sqrt{1 - \rho^2}} m.$$

We are in the context of a signal process $(\tilde{\mu}_t, \tilde{\beta}_t)$ which satisfy (17) and an observation process which satisfy (18). So we deduce the filters $\bar{\mu}_t$ and $\bar{\beta}$ from corollary 4.4. For that, let us firstly check if conditions (19) and (21) are satisfied. From exercise 3.11 in [3, P.54], it remains to verify that h has a linear growth, that is, there exists C such that:

$$\|h(x)\|^2 \leq C(1 + \|x\|^2).$$

which is the case for our model. Thus from corollary 4.4 we can deduce the stochastic differential equations satisfied by the filters:

$$\begin{aligned} d\bar{\mu}_t &= \alpha_t(a)dt + [\alpha_t(h^1 \phi_1 + b_1) - \alpha_t(h^1) \alpha_t(\phi_1)] d\bar{W}_t^1 + [\alpha_t(h^2 \phi_1 + b_2) - \alpha_t(h^2) \alpha_t(\phi_1)] d\bar{W}_t^2, \\ d\bar{\beta}_t &= \alpha_t(\bar{a})dt + [\alpha_t(h^1 \phi_2 + \bar{b}_1) - \alpha_t(h^1) \alpha_t(\phi_2)] d\bar{W}_t^1 + [\alpha_t(h^2 \phi_2 + \bar{b}_2) - \alpha_t(h^2) \alpha_t(\phi_2)] d\bar{W}_t^2. \end{aligned}$$

where the functions $a, b_1, b_2, \bar{a}, \bar{b}_1, \bar{b}_2$ are defined above. .

Log-Ornstein model

Let us consider the following model:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + e^{V_t} dW_t^1, \\ dV_t &= \lambda(\theta - V_t) dt + \sigma_V \left(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right), \\ d\mu_t &= \eta(v - \mu_t) dt + \sigma_\mu dW_t^3. \end{aligned}$$

In this case of model, we have:

$$\tilde{\mu}_t = \frac{\mu_t}{e^{V_t}} \quad \text{and} \quad \tilde{\beta}_t := \frac{\lambda(\theta - V_t)}{\sigma_V \sqrt{1 - \rho^2}} - \frac{\rho}{\sqrt{1 - \rho^2}} \tilde{\mu}_t. \quad (38)$$

As in the case of GARCH model. Firstly, we describe the dynamics of $\tilde{\mu}_t$ and $\tilde{\beta}_t$. Then, using the filtering theory, we can deduce the stochastic differential equations satisfied by $\bar{\mu}_t$ and $\bar{\beta}_t$.

Let $M_t = e^{V_t}$, then by Itô's formula, one obtains the following dynamics of M ;

$$M_t = M_0 + \int_0^t M_s \left[\lambda(\theta - \log(M_s)) + \frac{1}{2} \sigma_V^2 \right] ds + \sigma_V M_s \left(\rho dW_s^1 + \sqrt{1 - \rho^2} dW_s^2 \right).$$

Also, from Itô's formula, $\tilde{\mu}_t := \frac{\mu_t}{M_t}$ satisfies the following *S.D.E*:

$$\begin{aligned} \tilde{\mu}_t &= \tilde{\mu}_0 + \int_0^t \frac{\eta v}{M_s} ds + \tilde{\mu}_s \left(\sigma_V^2 - \eta - \left[\lambda(\theta - \log(M_s)) + \frac{1}{2} \sigma_V^2 \right] \right) ds + \int_0^t \frac{\sigma_V}{M_s} dW_s^3 - \int_0^t \rho \sigma_V \tilde{\mu}_s dW_s^1 \\ &\quad - \int_0^t \sqrt{1 - \rho^2} \sigma_V \tilde{\mu}_s dW_s^2 \\ &= \tilde{\mu}_0 + \int_0^t \eta v e^{-V_s} ds + \tilde{\mu}_s \left(\sigma_V^2 - \eta - \left[\lambda(\theta - V_s) + \frac{1}{2} \sigma_V^2 \right] \right) ds + \int_0^t \sigma_V e^{-V_s} dW_s^3 - \int_0^t \rho \sigma_V \tilde{\mu}_s dW_s^1 \\ &\quad - \int_0^t \sqrt{1 - \rho^2} \sigma_V \tilde{\mu}_s dW_s^2. \end{aligned} \quad (39)$$

and $\tilde{\beta}_t$ satisfies the following *S.D.E*:

$$\begin{aligned} \tilde{\beta}_t &= \tilde{\beta}_0 - \int_0^t \frac{\lambda}{\sigma_V \sqrt{1 - \rho^2}} dV_s - \int_0^t \frac{\rho}{\sqrt{1 - \rho^2}} d\tilde{\mu}_s \\ &= - \int_0^t \frac{\lambda^2(\theta - V_s)}{\sigma_V \sqrt{1 - \rho^2}} ds - \int_0^t \frac{\lambda \rho}{\sqrt{1 - \rho^2}} dW_s^1 - \int_0^t \lambda dW_s^2 - \frac{\rho}{\sqrt{1 - \rho^2}} d\tilde{\mu}_s. \end{aligned} \quad (40)$$

On the other hand, from (38), we can express V_t in terms of $\tilde{\mu}_t$ and $\tilde{\beta}_t$ as follows:

$$V_t = - \frac{\sigma_V \sqrt{1 - \rho^2}}{\lambda} \tilde{\beta}_t - \frac{\sigma_V \rho}{\lambda} \tilde{\mu}_t + \theta. \quad (41)$$

By replacing the above equation of $\frac{1}{V_t}$ in (39) and (40), one obtains the following dynamics of $\tilde{\mu}_t$ and $\tilde{\beta}_t$:

$$\tilde{\mu}_t = \tilde{\mu}_0 + \int_0^t a(\tilde{\mu}_s, \tilde{\beta}_s) ds + \int_0^t b_1(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^1 + \int_0^t b_2(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^2 + \int_0^t g(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^3, \quad (42)$$

$$\tilde{\beta}_t = \tilde{\beta}_0 + \int_0^t \bar{a}(\tilde{\mu}_s, \tilde{\beta}_s) ds + \int_0^t \bar{b}_1(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^1 + \int_0^t \bar{b}_2(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^2 + \int_0^t \bar{g}(\tilde{\mu}_s, \tilde{\beta}_s) dW_s^3. \quad (43)$$

Where

$$a(m, b) = \eta v \exp \left(\frac{\sigma_V \sqrt{1 - \rho^2}}{\lambda} b + \frac{\sigma_V \rho}{\lambda} m - \theta \right) + \left(\frac{1}{2} \sigma_V^2 - \eta - \sigma_V \sqrt{1 - \rho^2} b - \sigma_V \rho m \right) m$$

$$b_1(m, b) = -\rho \sigma_V m$$

$$b_2(m, b) = -\sigma_V m \sqrt{1 - \rho^2}$$

$$g(m, b) = \sigma_V \exp \left(\frac{\sigma_V \sqrt{1 - \rho^2}}{\lambda} b + \frac{\sigma_V \rho}{\lambda} m - \theta \right).$$

and

$$\bar{a}(m, b) = -\lambda b - \frac{\lambda \rho}{\sqrt{1 - \rho^2}} m - \frac{\rho}{\sqrt{1 - \rho^2}} a(m, b)$$

$$\bar{b}_1(m, b) = -\frac{\lambda \rho}{\sqrt{1 - \rho^2}} - \frac{\rho^2 \sigma_V}{\sqrt{1 - \rho^2}} m$$

$$\bar{b}_2(m, b) = -\lambda + \rho \sigma_V m$$

$$\bar{g}(m, b) = \frac{-\rho}{\sqrt{1 - \rho^2}} \exp \left(\frac{\sigma_V \sqrt{1 - \rho^2}}{\lambda} b + \frac{\sigma_V \rho}{\lambda} m - \theta \right).$$

Similarity as in the above example for Garch model, we can deduce that the filters $\bar{\mu}_t$ and $\bar{\beta}_t$ satisfy the following stochastic differential equations:

$$\begin{aligned} d\bar{\mu}_t &= \alpha_t(a)dt + [\alpha_t(h^1 \phi_1 + b_1) - \alpha_t(h^1) \alpha_t(\phi_1)] d\bar{W}_t^1 + [\alpha_t(h^2 \phi_1 + b_2) - \alpha_t(h^2) \alpha_t(\phi_1)] d\bar{W}_t^2, \\ d\bar{\beta}_t &= \alpha_t(\bar{a})dt + [\alpha_t(h^1 \phi_2 + \bar{b}_1) - \alpha_t(h^1) \alpha_t(\phi_2)] d\bar{W}_t^1 + [\alpha_t(h^2 \phi_2 + \bar{b}_2) - \alpha_t(h^2) \alpha_t(\phi_2)] d\bar{W}_t^2. \end{aligned}$$

Where we have now a new functions $a, b_1, b_2, \bar{a}, \bar{b}_1, \bar{b}_2$ given above.

5 Martingale and PDE approaches

Let us present in general these two approaches and how we can describe the value function and the optimal wealth and strategy. Also, we show in which cases of utility functions we can derive an explicit solutions for these quantities.

From section 3, we can reduced our optimization problem from partial information context to full information context as follows:

$$\begin{cases} \frac{dS_t}{S_t} = g(V_t) \bar{\mu}_t dt + g(V_t) d\bar{W}_t^1, \\ dV_t = f(\beta_t, V_t) dt + \rho h(V_t) d\bar{W}_t^1 + \sqrt{1 - \rho^2} h(V_t) d\bar{W}_t^2, \\ dR_t^\pi = R_t^\pi \pi_t \left(g(V_t) \bar{\mu}_t dt + g(V_t) d\bar{W}_t^1 \right). \end{cases}$$

and the trader's objective is to solve the following optimization problem:

$$J(x) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}[U(R_T^\pi)], \quad x > 0, \quad (44)$$

where \mathcal{A}_t is the set of admissible controls π_t which are \mathbb{F}^S -adapted process, take their value in a compact $U \subset \mathbb{R}$, and satisfies the integrability condition:

$$\int_t^T g^2(V_s)\pi_s^2 < \infty \quad \mathbb{P} \text{ a.s.} \quad (45)$$

5.1 Martingale approach

The martingale approach was introduced by Karatzas et al [14], Cox and Huang [13], for dealing with more general price process and in particular for relaxing Markov assumption required in the Ballman approach. With this approach, the optimal portfolio is characterized through a representation theorem for martingale but explicit solutions of the value function and strategy can be produced in very few cases. Moreover, in the case of incomplete markets, we have to solve a dual optimization problem.

The martingale approach in incomplete market is based on a dual formulation of the optimization problem in terms of a suitable family of (\mathbb{P}, \mathbb{G}) -local martingales. The important result for the dual formulation is the martingale representation theorem given in [4] for (\mathbb{P}, \mathbb{G}) -local martingales with respect to the innovation processes \overline{W}^1 and \overline{W}^2 .

Lemma 5.1 (Martingale representation theorem). *Let A be any (\mathbb{P}, \mathbb{G}) -local martingale. Then, there exist a \mathbb{G} -adapted processes ϕ and ψ , \mathbb{P} a.s. square-integrable and such that*

$$A_t = \int_0^t \phi_s d\overline{W}_s^1 + \int_0^t \psi_s d\overline{W}_s^2. \quad (46)$$

Now, we aim to describe the dual formulation of the optimization problem.

5.1.1 Equivalent martingales measures and duality theory

A probability measure \mathbb{Q} equivalent to \mathbb{P} is an equivalent martingales measure (EMM) if the discounted asset process is a martingale under this measure.

Now for any \mathbb{G} -adapted process $\nu = \{\nu_t, 0 \leq t \leq T\}$, which satisfies $\int_0^T \nu_s^2 ds < \infty$, we introduce the (\mathbb{P}, \mathbb{G}) -local martingale strictly positive:

$$Z_t^\nu = \exp \left(- \int_0^t \overline{\mu}_s d\overline{W}_s^1 - \int_0^t \nu_s d\overline{W}_s^2 - \frac{1}{2} \int_0^t \overline{\mu}_s^2 ds - \frac{1}{2} \int_0^t \nu_s^2 ds \right) \quad (47)$$

When, $\mathbb{E}[Z_T^\nu] = 1$, the process Z is a martingale and then there exists a probability measure \mathbb{Q} equivalent to \mathbb{P} with:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{G}_t} = Z_t^\nu.$$

Here $\overline{\mu}$ is the risk related to the asset's Brownian motion W^1 , which is chosen such that \mathbb{Q} is a EMM, that is, the process $Z^\nu S$ is a (\mathbb{P}, \mathbb{G}) -local martingale. On the other hand, ν is the risk related to the stochastic volatility's Brownian motion and this risk will be determined as the optimal solution of the dual problem defined below.

Consequently, from Itô's formula, the process Z^ν satisfies:

$$dZ_t^\nu = -Z_t^\nu \left(\bar{\mu}_s d\bar{W}_s^1 + \nu_s d\bar{W}_s^2 \right). \quad (48)$$

As shown by Karatzas et al [7] and Kramkov and Schachemayer [6], the solution of the primal problem (44) relying upon solving the dual optimization problem:

$$\tilde{J}(z) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E} \left[\tilde{U} \left(z \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] := \inf_{\nu \in \mathcal{K}} \mathbb{E} \left[\tilde{U} (z Z_T^\nu) \right], \quad z > 0 \quad (49)$$

Where:

- \mathcal{Q} is the set of equivalent martingale measures given by:

$$\mathcal{Q} = \{ \mathbb{Q} \sim \mathbb{P} \mid S \text{ is a local } (\mathbb{Q}, \mathbb{G}) - \text{martingale} \}. \quad (50)$$

- \tilde{U} is the convex dual of U given by:

$$\tilde{U}(y) = \sup_{m > 0} [U(m) - ym], \quad m > 0. \quad (51)$$

- \mathcal{K} is the Hilbert space of \mathbb{G} -adapted process ν such that $\mathbb{E} \left[\int_0^T |\nu_t^2| dt \right] < \infty$.

In the sequel, we denote by $I :]0, \infty[\rightarrow]0, \infty[$ the inverse function of U' on $]0, \infty[$. It's a decreasing function and verifies $\lim_{x \rightarrow 0} I(x) = \infty$ and $\lim_{x \rightarrow \infty} I(x) = 0$.

We henceforth impose the following assumptions on the utility functions in order to guarantee that the dual problem admits a solution $\tilde{\nu} \in \mathcal{H}$, see Karatzas et al [7], sections 11 and 12.

Assumption 5.2. • For some $p \in (0, 1), \gamma \in (1, \infty)$, we have

$$pU'(x) \geq U'(\gamma x) \quad \forall x \in (0, \infty).$$

- $x \rightarrow xU'(x)$ is nondecreasing on $(0, \infty)$.
- For every $z \in (0, \infty)$, there exists $\nu \in \mathcal{K}$ such that $\tilde{J}(z) < \infty$.

Now from Karatzas, Lehoczky, Shreve and Xu [7] and Owen [8], we have the following result about the solution of the primal utility maximization problem (4).

Theorem 5.3. The optimal wealth for the utility maximization problem (4) is given by

$$\tilde{R}_t = \mathbb{E} \left[\frac{Z_T^{\tilde{\nu}}}{Z_t^{\tilde{\nu}}} I(z_x Z_T^{\tilde{\nu}}) \mid \mathcal{G}_t \right]$$

where $\tilde{\nu}$ is the solution of the dual problem and z_x is the Lagrange multiplier such that $\mathbb{E} [Z_T^{\tilde{\nu}} I(z_x Z_T^{\tilde{\nu}})] = x$. Also the optimal portfolio $\tilde{\pi}$ is implicitly determined by the equation

$$d\tilde{R}_t = \tilde{\pi}_t g(V_t) d\tilde{W}_t^1. \quad (52)$$

Remark 5.4. The constraint $\mathbb{E} [Z_T^{\tilde{\nu}} I(z_x Z_T^{\tilde{\nu}})] = x$ to choose z_x is satisfied if

$$z_x \in \operatorname{argmin}_{z > 0} \{ \tilde{J}(z) + xz \}. \quad (53)$$

5.1.2 Optimal equivalent martingale measure

We remark from theorem 5.3 that optimal wealth depends on the optimal choice of ν , that is, depends on the equivalent martingale measure \mathbb{Q} . So we are interested in the following by finding the optimal risk ν which is solution of (49).

Here we present two cases. Firstly, we show that in the case when the price risk $\tilde{\mu}_t \in \mathcal{F}_t^{\tilde{W}^1}$, the infimum over all equivalent martingale measures of the dual problem is reached for $\tilde{\nu} = 0$. Secondly, for the general case, the idea is to derive a Hamilton-Jacobi-Bellman equation for dual problem, which involves the volatility risk ν as control process.

Lemma 5.5. *Assume that $\bar{\mu}_t \in \mathcal{F}_t^{\tilde{W}^1}$, then the infimum of the dual problem is reached for $\tilde{\nu} = 0$, that is:*

$$\tilde{J}(z) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E} \left[\tilde{U} \left(z \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] = \mathbb{E} \left[\tilde{U} \left(z Z_T^0 \right) \right]. \quad (54)$$

Proof. From equation (47), the definition of the conditional expectation and Jensen's inequality, it follows for any $\nu \in \mathcal{H}$:

$$\begin{aligned} & \mathbb{E} \left[\tilde{U} \left(z Z_T^0 \right) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\tilde{U} \left(z \exp \left(- \int_0^T \bar{\mu}_s d\bar{W}_s^1 - \frac{1}{2} \int_0^T \bar{\mu}_s^2 ds - \int_0^T \nu_s d\bar{W}_s^2 - \frac{1}{2} \int_0^T \nu_s^2 ds \right) \right) \middle| \mathcal{F}_T^{\tilde{W}^1} \right] \right] \\ &\geq \mathbb{E} \left[\tilde{U} \left(z \exp \left(- \int_0^T \bar{\mu}_s d\bar{W}_s^1 - \frac{1}{2} \int_0^T \bar{\mu}_s^2 ds \right) \mathbb{E} \left[\exp \left(- \int_0^T \nu_s d\bar{W}_s^2 - \frac{1}{2} \int_0^T \nu_s^2 ds \right) \middle| \mathcal{F}_T^{\tilde{W}^1} \right] \right) \right]. \end{aligned}$$

On the other hand, $\mathbb{E} \left[\exp \left(- \int_0^T \nu_s d\bar{W}_s^2 - \frac{1}{2} \int_0^T \nu_s^2 ds \right) \right] = 1$ a.s. In fact, from the definition of the conditional expectation, it remains to prove that for each positive function h , for each, $t_1, \dots, t_k \in [0, T]$, we have:

$$\mathbb{E} \left[\exp \left(- \int_0^T \nu_s d\bar{W}_s^2 - \frac{1}{2} \int_0^T \nu_s^2 ds \right) h \left(\tilde{W}_{t_1}^1, \dots, \tilde{W}_{t_k}^1 \right) \right] = \mathbb{E} \left[h \left(\tilde{W}_{t_1}^1, \dots, \tilde{W}_{t_k}^1 \right) \right].$$

As ν is a \mathbb{G} -adapted, we can define a new probability measure \mathbb{P}^ν equivalent to \mathbb{P} on \mathcal{G}_T given by:

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}} = \exp \left(- \int_0^T \nu_u d\bar{W}_u^2 - \frac{1}{2} \int_0^T \nu_u^2 du \right)$$

By Girsanov theorem, N is a \mathbb{G} Brownian motion under \mathbb{P}^ν . On the other hand, from the dynamic of \tilde{W}^1 given by $d\tilde{W}_t^1 = dN_t + \bar{\mu}_t dt$ and the assumption that $\bar{\mu}_t \in \mathcal{F}_t^{\tilde{W}^1}$, we deduce that the law of \tilde{W}^1 remains the same under \mathbb{P} and \mathbb{P}^ν . Thus:

$$\begin{aligned} \mathbb{E}^\nu \left[h \left(\tilde{W}_{t_1}^1, \dots, \tilde{W}_{t_k}^1 \right) \right] &:= \mathbb{E} \left[\exp \left(- \int_0^T \nu_s d\bar{W}_s^2 - \frac{1}{2} \int_0^T \nu_s^2 ds \right) h \left(\tilde{W}_{t_1}^1, \dots, \tilde{W}_{t_k}^1 \right) \right] \\ &= \mathbb{E} \left[h \left(\tilde{W}_{t_1}^1, \dots, \tilde{W}_{t_k}^1 \right) \right]. \end{aligned}$$

Therefore $\mathbb{E} \left[\exp \left(- \int_0^T \nu_s d\bar{W}_s^2 - \frac{1}{2} \int_0^T \nu_s^2 ds \right) \right] = 1$ and then one obtains:

$$\mathbb{E} \left[\tilde{U} (z Z_T^\nu) \right] \geq \mathbb{E} \left[\tilde{U} \left(z \exp \left(- \int_0^t \bar{\mu}_s d\bar{W}_s^1 - \frac{1}{2} \int_0^t \bar{\mu}_s^2 ds \right) \right) \right] := \mathbb{E} \left[\tilde{U} (z Z_T^0) \right].$$

On the other hand, we have from the definition of the dual problem that $\tilde{J}(z) \leq \mathbb{E} \left[\tilde{U} (z Z_T^0) \right]$, so we conclude that

$$\tilde{J}(z) = \mathbb{E} \left[\tilde{U} (z Z_T^0) \right].$$

□

However, in the general case where the risk premia don't satisfy lemma 5.5, it's a difficult problem to derive an explicit characterization for the solution of the dual problem and therefore for the optimal wealth and portfolio. So, here we present a stochastic control approach to the dual problem of determining the optimal martingale measures.

Note that the underlying dynamics for the equivalent martingale measure are the stochastic differential equation (48) and the stochastic volatility equation given in system (Q). So for initial time $t \in [0, T]$ and for fixed z , the dual value function is defined by the dual stochastic control problem

$$\tilde{J}(z, t, \bar{z}, \bar{v}) := \inf_{\nu} \mathbb{E} \left[\tilde{U} (z Z_T^\nu) | Z_t^\nu = \bar{z}, V_t = \bar{v} \right]. \quad (55)$$

Remark that the dual value function in (49) is simply deduced from $\tilde{J}(z) = \tilde{J}(z, 0, \bar{z}, \bar{v})$. Notice that in the case where \tilde{U} satisfies the following iso-elasticity property:

$$\tilde{U}(kx) = f(k)\tilde{U}(x) + g(k), \quad k > 0,$$

for any functions f and g , then:

$$\begin{aligned} \tilde{J}(z, t, \bar{z}, \bar{v}) &= f(z) \inf_{\nu} \mathbb{E} \left[\tilde{U} (Z_T^\nu) | Z_t^\nu = \bar{z}, V_t = \bar{v} \right] + g(z) \\ &:= f(z) \inf_{\nu} \Psi(t, \bar{z}, \bar{v}) + g(z). \end{aligned}$$

Now, we are in the setting of a stochastic control problem with controlled process (Z^ν, V) and control process ν . Let ζ be a smooth function on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$. The Hamilton-Jacobi-Bellman equation associated to this control problem is written as:

$$\partial_t \zeta(t, \bar{z}, \bar{v}) + \inf_{\nu} \mathcal{L} \zeta(t, \bar{z}, \bar{v}) = 0.$$

where

$$\begin{aligned} \mathcal{L} \zeta &= f(\beta_t, V_t) \partial_{\bar{v}} \zeta + \frac{1}{2} h^2(\bar{v}) \partial_{\bar{v}\bar{v}}^2 \zeta + \frac{1}{2} \bar{z}^2 (\bar{\mu}_t^2 + \nu^2) \partial_{\bar{z}\bar{z}}^2 \zeta \\ &\quad - \left(\rho h(\bar{v}) \bar{\mu}_t + \sqrt{1 - \rho^2} h(\bar{v}) \nu \right) \bar{z} \partial_{\bar{z}\bar{v}}^2 \zeta. \end{aligned}$$

Under smoothness assumptions on the dual value function, we can show by verification theorem that a smooth solution of the above Hamilton-Jacobi-Bellman equation coincides with the value function Ψ .

Proposition 5.6. *Suppose that there exists a function $\zeta(t, \bar{z}, \bar{v}) \in C^{1,2}([0, T] \times \mathbb{R}^+ \times \mathbb{R}^+)$, satisfying the following HJB equation:*

$$\partial_t \zeta(t, \bar{z}, \bar{v}) + \inf_{\nu} \mathcal{L} \zeta(t, \bar{z}, \bar{v}) = 0. \quad (56)$$

with terminal condition

$$\zeta(T, \bar{z}, \bar{v}) = \tilde{U}(\bar{z}).$$

Then, for all $t \in [0, T], \bar{z} \in \mathbb{R}^+, \bar{v} \in \mathbb{R}^+$

$$\psi(t, \bar{z}, \bar{v}) \leq \zeta(t, \bar{z}, \bar{v}).$$

Assume there exists a minimizer $\nu_t^* := \nu^*(t, \bar{z}, \bar{v})$ of (56) such that:

$$\partial_t \zeta(t, \bar{z}, \bar{v}) + \mathcal{L} \zeta(t, \bar{z}, \bar{v}) = 0, \quad (57)$$

Then $\psi(t, \bar{z}, \bar{v}) = \zeta(t, \bar{z}, \bar{v})$ for all $t \in [0, T]$, and the optimal risk $\tilde{\nu}_t$ is given by

$$\tilde{\nu}_t = -\frac{\sqrt{1-\rho^2} h(V)}{Z^\nu} \frac{\partial_{\bar{z}\bar{v}}^2 \psi(t, Z^\nu, V)}{\partial_{\bar{z}\bar{z}}^2 \psi(t, Z^\nu, V)}.$$

5.1.3 Application

Let assume that the model of diffusion is given by (25) and (26) and the trader's objective is to solve the optimization problem given by (44).

Firstly, notice from (32) that $\tilde{\mu}_t$ is adapted to the filtration generated by \tilde{W}^1 , then from lemma 5.5, the dual problem is reached for $\tilde{\nu} = 0$. Therefore from theorem 5.3 the optimal wealth is given by:

$$\tilde{R}_t = \mathbb{E} \left[\frac{Z_T^0}{Z_t^0} I(z_x Z_T^0) | \mathcal{G}_t \right].$$

Now, from the definition of the innovation process \bar{W}^1 and (32), we have that:

$$Z_t^0 = \exp \left(- \int_0^t N(s, \tilde{W}_s^1) d\tilde{W}_s^1 + \frac{1}{2} \int_0^t \left(N(s, \tilde{W}_s^1) \right)^2 ds \right).$$

On the other hand, as in Lakner [2], we can deduce that the random variable $\tilde{\Lambda}_t := \mathbb{E}^{\tilde{\mathbb{P}}} [\Lambda_t | \mathcal{G}_t]$ satisfies the following *S.D.E*:

$$\begin{aligned} \tilde{\Lambda}_t &= 1 + \int_0^t \bar{\mu}_s \tilde{\Lambda}_s d\tilde{W}_s^1 - \frac{\lambda}{\sigma_V} \int_0^t \tilde{\Lambda}_s d\tilde{W}_s^2 \\ &= 1 + \int_0^t N(s, \tilde{W}_s^1) \tilde{\Lambda}_s d\tilde{W}_s^1 - \frac{\lambda}{\sigma_V} \int_0^t \tilde{\Lambda}_s d\tilde{W}_s^2 \quad (\text{from(32)}) \end{aligned}$$

So notice that $Z_t^0 = \mathbb{E} \left[\frac{1}{\tilde{\Lambda}_t} | \mathcal{F}_t^{\tilde{W}^1} \right]$. Moreover from (31) we have that:

$$\tilde{\Lambda}_t = M(t, \tilde{W}_t^1) \exp \left(-\frac{\lambda}{\sigma_V} \tilde{W}_t^2 - \frac{1}{2} \frac{\lambda^2}{\sigma_V^2} t \right)$$

Then, as \tilde{W}^2 is independent of \tilde{W}^1 , one obtains:

$$Z_t^0 = \mathbb{E} \left[\frac{1}{\tilde{\Lambda}_t} | \mathcal{F}_t^{\tilde{W}^1} \right] = \frac{1}{M(t, \tilde{W}_t^1)}.$$

Now from the restriction of \mathbb{P} to \mathbb{G} given in (14), we have the following lemma proved by Pham and Quenez [4]:

Lemma 5.7. *For all $t \in [0, T]$ and for all measurable function Υ on \mathbb{R}_+ , such that $Z_T^0 \Upsilon(Z_T^0)$ is \mathbb{P} -integrable, we have $\Upsilon(Z_T^0)$ is $\tilde{\mathbb{P}}$ -integrable and:*

$$\tilde{\mathbb{E}} [\Upsilon(Z_T^0) | \mathcal{G}_t] = \mathbb{E} \left[\frac{Z_T^0}{Z_t^0} \Upsilon(Z_T^0) | \mathcal{G}_t \right].$$

Now as in Karatzas and Zhao [1], let us assume that the function

$$\Psi(z, t, w) := \begin{cases} \int_{\mathbb{R}} I \left(\frac{z}{M(T, w+v)} \right) \phi_{T-t}(v) dv; & k > 0, t \in [0, T[, w \in \mathbb{R} \\ I \left(\frac{z}{M(T, w)} \right); & k > 0, t = T, w \in \mathbb{R} \end{cases}$$

is finite for every $(z, t, w) \in (0, \infty) \times [0, T] \times \mathbb{R}$. The function ϕ is the Gaussian density given by:

$$\phi_t(z) := \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t}.$$

We also assume that $\Psi(z, s, w)$ has finite first derivatives with respect to the arguments z, s and w and finite second derivatives with respect to the arguments z and w .

Under this assumption, the strictly decreasing function

$$z \rightarrow \int_{\mathbb{R}} I \left(\frac{z}{M(T, v)} \right) \phi_{T-t}(v) dv$$

is continuous. Finally, from lemma 5.7, the optimal wealth satisfies:

$$\tilde{R}_t = \tilde{\mathbb{E}} \left[I \left(\frac{z_x}{M(T, \tilde{W}_T^1)} \right) | \mathcal{G}_t \right] = \Psi(z_x, t, \tilde{W}_t^1), \quad (58)$$

where the Lagrange multiplier z_x is the unique solution of $\tilde{J}'(z) = -x$, where $\tilde{J}(z)$ is given as follows:

$$\tilde{J}(z) = \mathbb{E} \left[\tilde{U}(z Z_T^0) \right] = \tilde{\mathbb{E}} \left[M(T, \tilde{W}_T^1) \tilde{U} \left(\frac{z}{M(T, \tilde{W}_T^1)} \right) \right].$$

It remains to make explicit the optimal strategy $\tilde{\pi}$.

In fact, as $d\tilde{R}_t = \tilde{R}_t \tilde{\pi}_t \sqrt{V_t} d\tilde{W}_t^1$, then as in Karatzas and Zhao [1], we can deduce that the optimal strategy $\tilde{\alpha}$ is given by:

$$\tilde{\pi}_t = (V_t)^{-1/2} \frac{\partial_w \Psi}{\Psi}(z_x, t, \tilde{W}_t^1), \quad (59)$$

where

$$\partial_w \Psi(z_x, t, w) = -z_x \int_{\mathbb{R}} \frac{\partial_w M(T, w+v)}{(M(T, w+v))^2} I' \left(\frac{z_x}{M(T, w+v)} \right) \phi_{T-t}(v) dv.$$

Special cases for utility functions

Let us consider the more standard utility functions:

$$U(y) = \begin{cases} \log(y), & y \in \mathbb{R}^+ \\ \frac{y^p}{p}, & y \in \mathbb{R}^+, p \in (0, 1) \end{cases}$$

Then we can deduce the following formulae for the convex function \tilde{U} :

$$\tilde{U}(y) = \begin{cases} -(1 + \log(y)), & y \in \mathbb{R}^+ \\ -\frac{y^\delta}{\delta}, & y \in \mathbb{R}^+, \delta = \frac{p}{p-1} \end{cases}$$

and

$$I(y) = \begin{cases} \frac{1}{y}, & y \in \mathbb{R}^+ \\ y^{-q}, & y \in \mathbb{R}^+, q = \frac{1}{1-p} \end{cases}$$

Logarithm function

In this case, the Lagrange multiplier z_x is given by $z_x = 1/x$. Then from (58), one obtains:

$$\tilde{R}_t = x \int M(T, \tilde{W}_T^1 + v) \phi_{T-t}(v) dv.$$

and from (59), the optimal strategy is given as follows:

$$\begin{aligned} \tilde{\pi}_t &= (V_t)^{-1/2} \frac{\int \partial_w M(T, \tilde{W}_T^1 + v) \phi_{T-t}(v) dv}{\int M(T, \tilde{W}_T^1 + v) \phi_{T-t}(v) dv} \\ &= (V_t)^{-1/2} N(t, \cdot). \end{aligned}$$

Power function

In this case, the Lagrange multiplier z_x is given by:

$$z_x = \left(\frac{\int (M(T, v))^q \phi_T(v) dv}{x} \right)^{1-p}.$$

Then from (58), one obtains:

$$\tilde{R}_t = x \frac{\int (M(T, \tilde{W}_T^1 + v))^q \phi_{T-t}(v) dv}{\int (M(T, v))^q \phi_T(v) dv}.$$

and from (59), the optimal strategy is given as follows:

$$\tilde{\pi}_t = q (V_t)^{-1/2} \frac{\int \partial_w M(T, \tilde{W}_t^1 + v) \left(M(T, \tilde{W}_t^1 + v) \right)^{q-1} \phi_{T-t}(v) dv}{\int (M(T, \tilde{W}_t^1 + v))^q \phi_{T-t}(v) dv}.$$

Remark 5.8. Notice that in general it is not easy to deduce an explicit formulas for the optimal wealth and strategy as in the above example. Because in general cases, the solution of the dual problem is not zero and then we can't have a simplification in the above formulas which leads to make explicit the calculus. So for general case, martingale approach is not useful to resolve our optimization problem. For that, we need to use another approach, like dynamic programming approach or PDE approach.

5.2 PDE approach

In our framework, we are in the context of portfolio optimization problem with stochastic volatility. These problems have been studied by a lot of authors like Zariphilou [9], Pahl [10] and Kraft [11]...etc.

Under smooth assumptions on the value function J , this approach leads to a characterization of J as solution of a associated HJB equation which in general is nonlinear, but in the case of CARA's utility functions and via a suitable transformations, we can make this equation linear and then we can deduce an explicit solutions for the value function and the optimal strategy. Also for this approach, a verification result is necessary to guarantee that a solution of the PDE coincides with the value function. Thus, we need to make regularity assumptions on the coefficients of the asset dynamic process and the volatility dynamic process, in order to guarantee the regularity of the value function.

Now, let us assume that we have the following dynamics of (R, V) within the full observation framework:

$$dR_t^\pi = R_t^\pi \pi_t \left(g(V_t) \bar{\mu}_t + g(V_t) d\bar{W}_t^1 \right) \quad (60)$$

$$dV_t = f(V_t) dt + \rho k(V_t) d\bar{W}_t^1 + \sqrt{1 - \rho^2} k(V_t) d\bar{W}_t^2 \quad (61)$$

$$(62)$$

The coefficients g, f and k are assumed to satisfy all the required regularity assumptions in order to guarantee a unique solution to the above stochastic differential equations.

Here we take the general case, when the dynamics of the risk processes are given as in (15). So from section filtering, we can deduce the stochastic differential equation satisfied by $\bar{\mu}_t$. Also, we assume that the law of $X_t := (\tilde{\mu}_t, \tilde{\beta}_t)$ given \mathcal{F}_t^S satisfy assumption 4.6, that is we have:

$$\mathbb{E}[\phi(X_t) | \mathcal{F}_t^S] = \int_{\mathbb{R}^2} \phi(x) p_t(x) m(dx),$$

where from (63), the density $p_t(x)$ satisfies:

$$dp_t(x) = A^* p_t(x) dt + \vartheta(p_t(x)) d\bar{W}_t^1 + \nu(p_t(x)) d\bar{W}_t^2 \quad (63)$$

Then

$$\bar{\mu}_t := \mathbb{E}[\phi_1(X_t) | \mathcal{F}_t^S] = \int_{\mathbb{R}^2} \phi_1(x) p_t(x) m(dx). \quad (64)$$

Note that the process (R_t^π, V_t, p_t) is a Markov process as is immediate from the dynamics in (60), (62) and (63). Now we recall that the trader's objective is to solve the optimization problem (44). In general, we can rewrite this problem as follows:

$$J(t, x, v, m) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}[U(R_T^\pi) | R_t^\pi = x, V_t = v, p_t = p]. \quad (65)$$

Observe that now our optimization problem has not only one state variable, as in the classical Merton's optimization problem, but three variables, corresponding to the three sources of randomness in our model.

Formally, the Hamilton-Jacobi-Bellman equation associated to the above stochastic control problem is the following nonlinear partial differential equation:

$$\begin{aligned} & \frac{\partial J}{\partial t} + f(v) \frac{\partial J}{\partial v} + \frac{1}{2} k^2(v) \frac{\partial^2 J}{\partial v^2} + A^* p \frac{\partial J}{\partial p} + \frac{1}{2} [\vartheta^2(p) + \nu^2(p)] \frac{\partial^2 J}{\partial p^2} \\ & + \left[\rho k(v) \vartheta(p) + \sqrt{1 - \rho^2} k(v) \nu(p) \right] \frac{\partial^2 J}{\partial v, p} \\ & + \sup_{\pi} \left[\pi g(v) \bar{\mu}_t(p) x \frac{\partial J}{\partial x} + \frac{1}{2} \pi^2 g^2(v) x^2 \frac{\partial^2 J}{\partial x^2} + x \pi g(v) \vartheta(p) \frac{\partial^2 J}{\partial x, p} \right. \\ & \quad \left. + x \pi g(v) \rho k(v) \frac{\partial^2 J}{\partial x, v} \right]. \end{aligned}$$

together with the terminal boundary condition $J(t, x, v, p) = U(x)$.

The associated optimal strategy is given by the Markov control $\{\tilde{\pi} = \tilde{\pi}(t, V_t, p_t)\}$ with:

$$\tilde{\pi}_t(t, x, v, p) = - \frac{\bar{\mu}_t(p)}{x g(v)} \frac{\partial_x J}{\partial_x^2 J} - \frac{\rho k(v)}{x g(v)} \frac{\partial_{x,v}^2 J}{\partial_x^2 J} - \frac{\vartheta(p)}{x g(v)} \frac{\partial_{x,p}^2 J}{\partial_x^2 J}.$$

where $\bar{\mu}_t$ is defined by (64).

5.2.1 To do for the second version of this paper

We have several theoretical difficulties concerning the above HJB: In fact, as p is generally infinite-dimensional, the above HJB equation is a PDE with infinite-dimensional state variable. With this remark, we have to use the maximum principle or dynamic programming for an infinite dimensional system. The situation is very similar to that for systems governed by partial differential equations. For this systems, the treatment varies with the choice of the functional space on which the conditional probability is defined. We will discuss this points in next version. Also we aim to study the above HJB in the cases of logarithmic and power utility function.

If we consider the case of CARA's utility functions and via a suitable transformation, we can make the above PDE a semilinear. Let us firstly consider the case of logarithmic function. Due to the homogeneity of the utility function together with the fact that the wealth process R^π and the control π appear linearly in the wealth dynamic, we can suggest that the value function must be of the form:

$$J(t, x, v, p) = \log(x) + \Phi(t, y, p). \quad (66)$$

Then direct substitution of (66) in the above HJB, gives us a semilinear *P.D.E* for Φ , and the associated optimal strategy is now given by:

$$\tilde{\pi}_t = \frac{\bar{\mu}_t(p)}{g(v)}.$$

where for notation $\bar{\mu}_t = \bar{\mu}_t(p)$ is given by (64).

On the other hand, for the case of power utility function $U(x) = \frac{x^\delta}{\delta}$, we can suggest that the value function must be of the form:

$$J(t, x, v, p) = \frac{x^\delta}{\delta} \exp(\Phi(t, y, p))$$

also with this form we can obtain an semilinear *P.D.E* for Φ . Here the optimal strategy is given by:

$$\tilde{\pi}_t = \frac{\bar{\mu}_t(p)}{(1-\delta)g(v)} + \frac{\rho k(v)\partial_v \Phi}{(1-\delta)g(v)} + \frac{\vartheta(p)\partial_p \Phi}{(1-\delta)g(v)}$$

Remark 5.9. *For the logarithmic case, we notice that in the case of partial information, the optimal portfolio can be formally derived from the full information case by replacing the unobservable risk premium $\tilde{\mu}_t$ by its estimate $\bar{\mu}_t$. But on the other hand, in the power utility function, this property does not hold and the optimal strategy cannot be derived from the full information case by replacing the risk $\tilde{\mu}_t$ by its best estimate $\bar{\mu}_t$ due to the last additional term which depend on the filter.*

This property corresponds to the so called separation principle. It is proved in Kuwana that certainty equivalence holds if and only if the utilities functions are logarithmic.

References

- [1] Karatzas Ioannis and Zhao Xiaoliang. Bayesian adaptive portfolio optimization, *In Option pricing, interest rates and risk management, Handb. Math. Finance*, 632–669. Cambridge University Press, Cambridge, 2001.
- [2] Lakner Peter. Optimal trading strategy for an investor: the case of partial information. *Stochastic Process. Appl.*, 76(1):77-97, 1998.
- [3] Bain Alan and Crisan Dan. *Fundamentals of stochastic filtering*, volume 60 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2009.
- [4] Pham Huyên and Quenez Marie-Claire. *Optimal portfolio in partially observed stochastic volatility models*. *Ann. Appl. Probab.*, 11(1):210-238, 2001.
- [5] Étienne Pardoux. *Filtrage non linéaire et équations aux dérivées partielles stochastiques associées* In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 67-163. Springer, Berlin, 1991.
- [6] D.Kramkov and W.Schachermayer. *The asymptotic elasticity of utility functions and optimal investment in incomplete markets* *Ann. Appl. Probab.*, 9(3):904-950, 1999.
- [7] Ioannis Karatzas, John P.Lehoczky, Steven E .Shreve and Gan-Lin Xu . *Martingale and duality methods for utility maximization in an incomplete market* *SIAM J. Control Optim.*, 29(3):702–730, 1991.
- [8] M.P Owen. *Utility based optimal hedging in incomplete markets* *Ann. Appl. Probab.*, 12(2):691-709, 2002.
- [9] Thaleia Zariphopoulou. *A solution approach to valuation with unhedgeable risks* *Finance Stoch.*, 5(1):61-82, 2001.
- [10] Huyên Pham. *Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints* *Appl. Math. Optim.*, 46(1):55-78, 2002.
- [11] Holger Kraft. *Optimal portfolios and Heston's stochastic volatility model: an explicit solution for power utility* *Quant. Finance*, 5(3):303-313, 2005.
- [12] Ioannis Karatzas, John P.Lehoczky and Steven E. Shreve. *Optimal portfolio and consumption decisions for a "small investor" on a finite horizon* *SIAM J. Control Optim.*, 25(6):1557-1987.
- [13] John C.Cox and Chi-fu Huang. *Optimal consumption and portfolio policies when asset prices follow a diffusion process* *J. Econom. Theory*, 49(1):33-83, 1989.
- [14] Ioannis Karatzas, John P.Lehoczky and Steven E .Shreve. *Optimal portfolio and consumption decisions for a "small investor" on a finite horizon* *SIAM J. Control Optim.*, 25(6):1586-1586, 1987.