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Coupling Linear Sloshing with Six Degrees of Freedom Rigid Body Dynamics

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Abstract

Fluid motion in tanks is usually described in space industry with the so-called Lomen hypothesis which assumes the vorticity is null in the moving frame. We establish in this contribution that this hypothesis is valid only for uniform rotational motions. We give a more general formulation of this coupling problem, with a compact formulation.

We consider the mechanical modeling of a rigid body with a motion of small amplitude, containing an incompressible fluid in the linearized regime. We first establish that the fluid motion remains irrotational in a Galilean referential if it is true at the initial time. When continuity of normal velocity and pressure are prescribed on the free surface, we establish that the global coupled problem conserves an energy functional composed by three terms. We introduce the Stokes - Zhukovsky vector fields, solving Neumann problems for the Laplace operator in the fluid in order to represent the rotational rigid motion with irrotational vector fields. Then we have a good framework to consider the coupled problem between the fluid and the rigid motion. The coupling between the free surface and the *ad hoc* component of the velocity potential introduces a “Neumann to Dirichlet” operator that allows to write the coupled system in a very compact form. The final expression of a Lagrangian for the coupled system is derived and the Euler-Lagrange equations of the coupled motion are presented.

Keywords:

Stokes, Zhukovsky, Fraeijs de Veubeke, vector fields, integral boundary operator.

AMS classification: 70E99, 76B07.

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Scope of the problem

Sloshing of liquid in tanks is an important phenomenon for space and terrestrial applications. We think for example of sloshing effects in road vehicles and ships carrying liquid cargo. The question is to know the magnitude of the wave and the total effort on the structure due to the movement of the fluid. For this kind of problematics, a lot of references exist and we refer the reader *i.e.* to the book of H. Morand and R. Ohayon [31], to the review proposed by R. Ibrahim, V. Pilipchuk and T. Ikeda [24], to the book of R.A. Ibrahim [23], the book of O.M. Faltinsen and A.N. Timokha [17] or to the review article of G. Hou *et al.* [22].

Moreover, for industrial applications, we would have a movable rigid tank with liquid free surface with six possible rigid movements and without needing a complete study of the elastic body, as studied *e.g.* in H. Bauer *et al.* [6], S. Piperno *et al.* [34], C. Farhat *et al.* [14], J.F. Gerbeau and M. Vidrascu [21], K.J. Bathe and H. Zhang [4], T.E. Tezduyar *et al.* [38] and the previous references.

In our case relative to space applications, the fundamental hypothesis of this contribution is the existence of some propulsion. We do not consider in this study the very complicated and nonlinear movement due to the quasi-disparition of gravity field. We refer for such studies to the contributions of F. Dodge and L. Garza [11, 12], S. Ostrach [33], H. Snydera [36], C. Falcón *et al.* [15] and P. Behruzi, *et al.* [7] among others. On the contrary, a gravity field is supposed to be present in our contribution and moreover an extra-gravity field is added due to the propulsion system. Then it is legitimus to linearize all the geometrical deformations and the equations of dynamics. In this kind of situation, the knowledge of the action of the fluid on the structure is mandatory. The question has been intensively studied during the sixties under the impulsion of NASA (see *e.g.* H. Bauer [5], D. Lomen [27, 28], H. Abramson [1], L. Fontenot [18]) and in European countries in the seventies (see *e.g.* J.P. Leriche [25]) or in the context of Ariane 5 studies (B. Chemoul *et al.* [8]).

We observe that due to its own intrinsic movement, the structure has also some influence on the fluid displacement. This question has been rigorously studied by the Russian school in the sixties (N. Moiseev and V. Rumiantsev [30]). It is sufficient in a first approach to consider the solid as a rigid body and to neglect all the flexible deformations.

In fact, we are in front of a complete coupled problem. The fluid is linearized and has an action on the solid, considered as a rigid body. The solid is a “six degrees of freedom” system that can also be considered as linearized around a given configuration. This coupled problem does not seem to have been considered previously under this form in the literature. We observe that this quite old problem raises actually an intensive scientific activity. As examples, we mention the contributions of O. Faltinsen, O. Rognebakke, I. Lukovsky and A. Timokha [16, 20] who derived a variational method to analyze the sloshing with finite water depth. Note also that K. London [29] analyzed the case of a multi-body model with applications to the Triana spacecraft, and J. Vierendeels *at al.*

[39] proposed to use the Flow3D computer software (Fluent, Inc) to analyze numerically nonlinear effects involved in the coupling of a rigid body with sloshing fluid, L. Diebold *at al.* [10] studied the effects on sloshing pressure due to the coupling between seakeeping and tank liquid motion. In the thesis of A. Ardakani, the general rigid-body motion with interior shallow-water sloshing is studied in great detail and we refer to the communication of A. Ardakani and T. Bridges [2]. A time-independent finite difference method to solve the problem of sloshing waves and resonance modes of fluid in a tridimensional tank is also considered by C. Wu and B. Chen [40].

We begin this article with classical considerations on sloshing in a fixed solid. We focus on the free surface and to usual physical ingredients: the continuity of normal velocity and the continuity of pressure. The coupling between the free surface and the velocity potential introduces a “Neumann to Dirichlet” integral operator that allows to write the coupled system in a very compact form. Then in Section 2, we recall fundamental aspects of the dynamics of a six degrees of freedom rigid body dynamics: description of the rigid body and its infinitesimal motion, the incompressible fluid and its linearization. We discuss the so-called “Lomen hypothesis” intensively used for industrial space applications and prove that, with a good generality, the fluid motion remains irrotational in a Galilean referential. We introduce some special vectorial functions that we call the “Stokes - Zhukovsky vector fields”, independently rediscovered by multiple generations of great scientists during the two last centuries (see *e.g.* G. Stokes [37], N. Zhukovsky [41] and B. Fraeijs de Veubeke [19]). These vector fields solve Neumann problems for the Laplace operator in the fluid and allow the representation of a rigid body displacement by an irrotational field. It is a good framework to consider the coupled problem. Then the dynamics equations of the rigid body in the presence of an internal sloshing fluid are established. In Section 3, we study the coupled problem. We do not incorporate any dissipation and in consequence we establish the conservation of energy for this simple case. We propose a compact set of variables to describe the entire coupled dynamics. Then the coupled system appears in a very simple form formally analogous to a scalar harmonic oscillator! Finally, the expression of a Lagrangian for the coupled system is proposed.

When the explanations of the mathematical results are not detailed, we refer the reader to the classical books of N. Moiseev and V. Rumiantsev [30], H. Morand and R. Ohayon [31], R.A. Ibrahim [23], O.M. Faltinsen and A.N. Timokha [17] or to the preliminary edition [13] of this contribution.

1) Sloshing in a fixed solid

In this section, the studied mechanical system is the fluid. The liquid is contained inside the solid \mathcal{S} , it occupies a volume $\Omega(t)$ variable with time, with a **constant** density ρ_L . The total mass m_L of liquid is the integral of the density ρ_L on the volume $\Omega(t)$. At the boundary $\partial\Omega$ of liquid, we have a contact surface $\Sigma(t)$ between liquid and solid and $\Sigma(t) = \partial\Omega \cap \partial\mathcal{S}$, as described in Figure 1, and a free surface $\Gamma(t)$ where the liquid is in

thermodynamical equilibrium with its vapor. The liquid is submitted to a gravity field g_0 . This vector is collinear to an “absolute” vertical direction associated with a vector e_3 , third coordinate of a Galilean referential (e_1, e_2, e_3) :

$$g_0 = -g e_3.$$

Note that $g > 0$ with this choice, as illustrated in Figure 1. The velocity field of the liquid $u(t)$ is measured relatively to an absolute referential, following *e.g.* the work of L. Fontenot [18]. The liquid is assumed incompressible:

$$(1) \quad \operatorname{div} u = 0 \quad \text{in } \Omega(t).$$

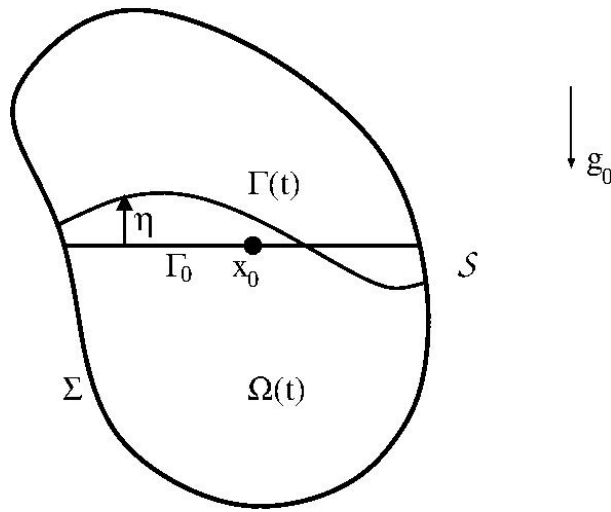


Figure 1. General view of the sloshing problem in a solid at rest. The free boundary $\Gamma(t)$ is issued from the equilibrium free boundary at rest Γ_0 with the help of the elongation η .

- **Liquid as a perfect linearized fluid**

The pressure field $p(x)$ is defined in the liquid domain $\Omega \ni x \mapsto p(x) \in \mathbb{R}$. The conservation of momentum for a perfect fluid is written with the Euler equations of hydrodynamics: $\frac{\partial u}{\partial t} + (\operatorname{curl} u) \times u + \nabla \left(\frac{p}{\rho_L} + \frac{1}{2} |u|^2 \right) = g_0$ in $\Omega(t)$. In this contribution, we make a **linearization hypothesis**. In particular, we neglect the nonlinear terms in fluid dynamics and replace the previous equation by:

$$(2) \quad \frac{\partial u}{\partial t} + \frac{1}{\rho_L} \nabla p = g_0 \quad \text{in } \Omega(t).$$

- **Velocity potential**

We suppose moreover that the fluid is irrotational:

$$(3) \quad \operatorname{curl} u = 0.$$

If the domain $\Omega(t)$ is simply connected (be **careful** with this hypothesis for **toric** geometries !), the simple hypothesis (3) implies that the velocity field can be generated by a potential φ :

$$(4) \quad u(x) = \nabla \varphi(x), \quad x \in \Omega(t)$$

• In order to have precise information concerning this velocity potential, we recall the Bernoulli theorem. We inject the velocity field $u = \nabla\varphi$ in the dynamical equations. We introduce a point P : $\nabla\left(\frac{\partial\varphi}{\partial t} + \frac{p}{\rho_L} - g_0 \bullet (x - x_P)\right) = 0$ for $x \in \Omega$. We add some time function to the scalar potential of velocity (and assume that the domain Ω is connected). Then:

$$(5) \quad \frac{\partial\varphi}{\partial t} + \frac{p}{\rho_L} - g_0 \bullet (x - x_P) = 0, \quad x \in \Omega.$$

• We take now into consideration the incompressibility hypothesis (1) together with the potential representation of the velocity field (4). We then obtain the Laplace equation:

$$(6) \quad \Delta\varphi = 0 \quad \text{in } \Omega(t).$$

A first boundary condition for this equation is a consequence of the continuity of the normal velocity $u \bullet n$ at the interface Σ between solid and liquid:

$$(7) \quad \frac{\partial\varphi}{\partial n} = 0, \quad x \in \Sigma(t).$$

• **Free surface**

Consider as a reference situation the solid at rest. Then the free surface at equilibrium has a given position Γ_0 as presented in Figure 1. We note ηn_0 the displacement of the free boundary at position $y \in \Gamma_0$, where n_0 denotes the outward normal direction to Γ_0 . In this case of a fixed solid, we have $n_0 = e_3$. The point x new position takes into account the variation of the free surface:

$$(8) \quad x = y + \eta(y) n_0, \quad y \in \Gamma_0, \quad x \in \Gamma.$$

We denote by x_0 the center of gravity of the frozen free surface Γ_0 :

$$(9) \quad \int_{\Gamma_0} (y - x_0) d\gamma = 0.$$

Note that due to incompressibility condition, we have:

$$(10) \quad \int_{\Gamma_0} \eta d\gamma \equiv 0.$$

Moreover, with x given on $\Gamma(t)$ according to (8), we have,

$$\int_{\Gamma_0} (x - x_0) d\gamma = 0.$$

We observe that, thanks to (10), $\int_{\Gamma_0} (x - x_0) d\gamma = \int_{\Gamma_0} (y - x_0 + \eta(y) n_0) d\gamma = \int_{\Gamma_0} (y - x_0) d\gamma = 0$ due to the definition (9). We introduce also the coordinates X_1, X_2, X_3 , of a point x in the referential (x_0, e_1, e_2, e_3) : $x - x_0 = X_1 e_1 + X_2 e_2 + X_3 e_3$.

• **Proposition 1. Neumann boundary condition on the free surface**

If we keep only the first order linear terms, the boundary condition for the velocity potential on the free surface can be written as a kinematic condition:

$$(11) \quad \frac{\partial\varphi}{\partial n} = \frac{\partial\eta}{\partial t}, \quad x \in \Gamma_0.$$

Then equations (7) and (11) can be written in a synthetic form:

$$(12) \quad \frac{\partial\varphi}{\partial n} = \begin{cases} 0 & \text{on } \Sigma \\ \frac{\partial\eta}{\partial t} & \text{on } \Gamma_0. \end{cases}$$

- **Proof of Proposition 1.**

We introduce the equation $F(X_1, X_2, X_3, t) = 0$ of the free surface. We take the total derivative relative to time of this constraint and replace the velocity $\frac{dX}{dt}$ by the gradient $\nabla\varphi$ of the potential. We obtain $\nabla F \cdot \nabla\varphi + \frac{\partial F}{\partial t} = 0$. The normal vector n can be written as $n = \nabla F / |\nabla F|$ and we have $\frac{\partial\varphi}{\partial n} \equiv \nabla\varphi \cdot n = \nabla\varphi \cdot \frac{\nabla F}{|\nabla F|}$. The previous equation can be written as

$$(13) \quad \frac{\partial\varphi}{\partial n} + \frac{1}{|\nabla F|} \frac{\partial F}{\partial t} = 0.$$

We parameterize the surface with an explicit function η , *id est*

$$(14) \quad F(X_1, X_2, X_3, t) \equiv X_3 - \eta(X_1, X_2, t).$$

Then linearizing the problem, we suppose that the free surface is close to its reference value Γ_0 at rest and we can neglect the gradient $\nabla\eta$ of the free surface equation (14) compared to the unity. Thus we have $|\nabla F| = 1 + O(|\eta|^2)$. Due to the particular form (14), we deduce that we have $\frac{\partial F}{\partial t} = -\frac{\partial\eta}{\partial t}$ on the free boundary and the relation (11) is a direct consequence of (13) and the fact that the norm of ∇F is of order unity.

- **Proposition 2. Pressure continuity across the free surface**

On the free surface Γ , the continuity of the stress tensor can be written for a perfect fluid as a dynamic condition:

$$(15) \quad p = 0 \quad \text{on } \Gamma.$$

It takes the following linearized form:

$$(16) \quad \frac{\partial\varphi}{\partial t} + g\eta = 0, \quad x \in \Gamma_0.$$

- **Proof of Proposition 2.**

The proof is classical and is explained in classic books as [17, 23, 31]. We give it here for completeness of the study. We choose the point P for the Bernoulli equation (5) on the frozen free surface Γ_0 equal to the center x_0 introduced in (9). Due to Bernoulli theorem (5) and continuity of the pressure on Γ , we deduce the following relation on the free surface:

$$(17) \quad \frac{\partial\varphi}{\partial t} - g_0 \cdot (x - x_0) = 0, \quad x \in \Gamma(t).$$

We have the following calculus: $-g_0 \cdot (x - x_0) = g e_3 \cdot (X_1 e_1 + X_2 e_2 + \eta e_3) = g\eta$, and the condition $p = 0$ of pressure continuity on the free surface is expressed by $\frac{\partial\varphi}{\partial t} + g\eta = 0$ which is exactly relation (16). The proof is established. \square

- **Free surface potential and Neumann to Dirichlet operator**

We introduce the “free surface potential” $\Omega \ni x \mapsto \psi(x) \in \mathbb{R}$ satisfying the following Neumann boundary-value problem for the Laplace equation:

$$(18) \quad \begin{cases} \Delta\psi = 0 & \text{in } \Omega \\ \frac{\partial\psi}{\partial n} = \begin{cases} 0 & \text{on } \Sigma \\ \eta & \text{on } \Gamma_0. \end{cases} \end{cases}$$

We consider a free surface η such that the global incompressibility condition (10) holds. We introduce the functional space

$$F^{1/2}(\Gamma_0) \equiv \left\{ \eta : \Gamma_0 \longrightarrow \mathbb{R}, \int_{\Gamma_0} \eta \, d\gamma = 0 \right\}.$$

We consider the “free surface potential” ψ associated to a given $\eta \in F^{1/2}(\Gamma_0)$ in the following way. The function $\Omega \ni x \longmapsto \psi(x) \in \mathbb{R}$ is uniquely defined by the Neumann problem (18) with the additional condition

$$(19) \quad \int_{\Gamma_0} \psi \, d\gamma = 0.$$

We consider the restriction ζ (the trace) of the function ψ on the surface Γ_0

$$(20) \quad \Gamma_0 \ni x \longmapsto \zeta(x) \equiv \psi(x) \in \mathbb{R}$$

The mapping $F^{1/2}(\Gamma_0) \ni \eta \longmapsto \zeta \in F^{1/2}(\Gamma_0)$ is the “Neumann to Dirichlet” operator. We denote it with the letter W :

$$(21) \quad \zeta \equiv W \bullet \eta.$$

A precise mathematical definition of the space $F^{1/2}(\Gamma_0)$ in the context of Sobolev spaces can be found in [26] or [32].

• **Proposition 3. Positive self adjoint operator**

The operator $W : F^{1/2}(\Gamma_0) \ni \eta \longmapsto \zeta \in F^{1/2}(\Gamma_0)$ with ζ defined by the relations (18), (19), (20) and (21) is self-adjoint. If we denote by (\bullet, \bullet) the L^2 scalar product on the linearized free surface Γ_0 , *id est*

$$(22) \quad (\eta, \zeta) \equiv \int_{\Gamma_0} \eta \zeta \, d\gamma, \quad \eta, \zeta \in F^{1/2}(\Gamma_0),$$

we have: $(\eta', W \bullet \eta) = (W \bullet \eta', \eta)$, for all $\eta, \eta' \in F^{1/2}(\Gamma_0)$. In particular, with the free surface potential ψ defined in (18), we have

$$(23) \quad (\eta, W \bullet \eta) = \int_{\Omega} (\nabla \psi \bullet \nabla \psi) \, dx \geq 0.$$

• **Proof of Proposition 3.**

We have with the previous notations:

$$\begin{aligned} (\eta', W \bullet \eta) &= \int_{\Gamma_0} \eta' \psi \, d\gamma = \int_{\partial\Omega} \frac{\partial \psi'}{\partial n} \psi \, d\gamma \quad \text{because } \frac{\partial \psi'}{\partial n} = 0 \text{ on } \Sigma \text{ and } \frac{\partial \psi'}{\partial n} = \eta' \text{ on } \Gamma_0 \\ &= \int_{\Omega} \operatorname{div}(\psi \nabla \psi') \, dx \quad \text{due to Green formula} \\ &= \int_{\Omega} (\nabla \psi' \bullet \nabla \psi) \, dx \quad \text{because } \Delta \psi' = 0 \\ &= \int_{\partial\Omega} \psi' \frac{\partial \psi}{\partial n} \, d\gamma \quad \text{because } \Delta \psi = 0 \\ &= \int_{\Gamma_0} \psi' \eta \, d\gamma = (W \bullet \eta', \eta). \end{aligned}$$

We observe that if $\eta = \eta'$, then $\psi = \psi'$ and the scalar product $(W \bullet \eta, \eta)$ is given by the relation (23) and is positive. The proof is complete. \square

- In conclusion of the section, the velocity potential φ satisfies (12), then its value on the free boundary Γ may be described in terms of the operator W . We have

$$\varphi = W \bullet \frac{\partial \eta}{\partial t} \quad \text{on } \Gamma_0.$$

Then the evolution equation (16) can be formulated only in terms of the *a priori* unknown free surface, parameterized by the function η and the operator W :

$$(24) \quad \rho_L W \bullet \frac{\partial^2 \eta}{\partial t^2} + \rho_L g \eta = 0 \quad \text{on } \Gamma_0.$$

This equation is usually presented as a family of harmonic oscillators, through a diagonalization of the operator W with usual spectral methods [17, 23, 31]. Our formulation with a Neumann to Dirichlet operator can be solved with boundary element methods. See *e.g.* the books of J.C. Nédélec [32] or O.M. Faltinsen and A.N. Timokha [17]. For an explicit implementation of boundary integral methods, we refer to the work of one of us [9]. We observe also that equation (24) will be modified by the coupling with the rigid movement.

2) Sloshing in a body with a rigid movement

In their book [30], N. Moiseev and V. Rumiantsev study the problem of a completely fluid-filled reservoir. They introduce special functions to represent the effect of the fluid on the solid motion. In the present study, the reservoir is partially filled and the fluid has a free surface. Nevertheless, the system is now the rigid body submitted to various forces. In this section, we derive the evolution equations of momentum and kinetic momentum of the solid. We adapt also the previous section in order to describe the fluid movement inside the moving rigid body.

• Rigid body

We consider a rigid moving solid \mathcal{S} , of density ρ_S and total mass m_S . We introduce the center of gravity ξ . This solid is submitted to three forces. The first one is the gravity described previously. The weight of the solid \mathcal{S} is then equal to $m_S g_0$. Secondly a force R at a fixed point A on the boundary $\partial\mathcal{S}$. We can suppose that this force is a given function of time. Last but not least, the surface forces f on the boundary $\partial\mathcal{S}$ due to the internal fluid. We introduce a local referential ε_j associated to the rigid body and issued from Galilean referential e_j .

• Infinitesimal motion of the rigid body

The linearization hypothesis acts now in a geometrical manner. The center of gravity is a function of time $\xi = \xi(t)$ and an infinitesimal rotation of angle $\theta = \theta(t)$ allows to write a simple algebraic relation between the vectors ε_j and e_j : $\varepsilon_j = e_j + \theta \times e_j$. Then $\frac{d\varepsilon_j}{dt} = \frac{d\theta}{dt} \times \varepsilon_j$ for $1 \leq j \leq 3$. The solid velocity field $u_S(x)$ satisfies:

$$u_S(x) = \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi(t)), \quad x \in \mathcal{S}.$$

The kinetic momentum σ_S and the tensor of inertia I_S are defined as usual: $\sigma_S \equiv \int_S \rho_S (x - \xi) \times u_S(x) dx$ and $I_S \bullet y \equiv \int_S \rho_S (x - \xi) \times (y \times (x - \xi)) dx$ for $y \in \mathbb{R}^3$. For a rigid body, we have the classical relation:

$$\sigma_S = I_S \bullet \frac{d\theta}{dt}.$$

- **Dynamics equations of the rigid body**

By integration of the classical Newton laws of motion, the conservation of momentum takes the form:

$$(25) \quad m_S \frac{d^2\xi}{dt^2} = m_S g_0 + R + \int_{\partial S} f d\gamma.$$

The momentum \mathcal{M}_S of the surface forces relatively to the center of gravity is given according to: $\mathcal{M}_S = \int_{\partial S} (x - \xi) \times f d\gamma$. Then the conservation of kinetic momentum takes the form:

$$(26) \quad I_S \bullet \frac{d^2\theta}{dt^2} = (x_A - \xi) \times R + \mathcal{M}_S.$$

- The force f on the boundary of the solid surface Σ admits the expression

$$f = \begin{cases} pn, & x \in \Sigma \\ 0, & x \in \partial S \setminus \Sigma. \end{cases}$$

Then: $\int_{\partial S} f d\gamma = \int_{\Sigma} pn d\gamma = \int_{\partial\Omega} pn d\gamma = \int_{\Omega} \nabla p dx$. Due to the previous expression of the momentum of pressure forces, we have after an elementary calculus:

$$(27) \quad \mathcal{M}_S = \int_{\Omega} (x - \xi) \times \nabla p dx.$$

- **About the fluid irrotationality hypothesis**

In the monograph [27], D. Lomen suppose the irrotationality for the motion of the liquid **relatively** to the motion of the rigid body. Then the velocity field of the liquid satisfies the conditions:

$$u(x) = \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi(t)) + v, \quad \text{curl } v \equiv 0.$$

By taking the curl of this relation: $\text{curl } u = 2 \frac{d\theta}{dt}$. Consider now the time derivative of the previous relation and the curl of relation (2). Then we obtain:

$$\frac{d^2\theta}{dt^2} = 0$$

and the hypothesis of irrotationality in the **relative** referential done in [27] is physically correct **only** if the rotation of solid referential is **uniform** in time.

- **Irrotationality in the Galilean referential**

We observe that under an assumption of linearized dynamics, if vorticity $\text{curl } u$ of liquid measured **in the Galilean referential** at initial time is null, then it remains identically null for all times:

$$\text{curl } u \equiv 0, \quad t \geq 0, \quad x \in \Omega(t).$$

To prove the previous relation, just take the curl of the linearized dynamics equation (2). Then $\frac{\partial}{\partial t}(\text{curl } u) = 0$ and the property is established if it is true at $t = 0$. In the following, it is assumed that the fluid is irrotational in the **Galilean** referential. In this case, one can find a velocity potential even if the angular velocity is an arbitrary function of time.

- **Stokes - Zhukovsky vector fields for fluid potential decomposition**

A natural question is the incorporation of the movement of a rigid body inside the expression of the potential φ of velocities. In other words, we have put in evidence the very particular role of the rigid movement for the determination of the velocity potential. We recall that this dynamics is a six degrees of freedom system described by the two vectors $\xi(t)$ and $\theta(t)$. The remaining difficulty concerns the solid body velocity field which is rotational. Following an old idea due independently (at our knowledge) to G. Stokes [37], N. Zhukovsky [41] and B. Fraeijs de Veubeke [19], we introduce a function $\tilde{\varphi}$ such that:

$$(28) \quad \begin{cases} \Delta \tilde{\varphi} = 0 & \text{in } \Omega \\ \frac{\partial \tilde{\varphi}}{\partial n} = \left(\frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi) \right) \cdot n & \text{on } \partial\Omega. \end{cases}$$

Due to **linearity** of the problem (28), we can decompose the vector field $\tilde{\varphi}$ under the form:

$$(29) \quad \tilde{\varphi} \equiv \alpha \cdot \frac{d\xi}{dt} + \beta \cdot \frac{d\theta}{dt}.$$

The vector fields $\Omega \ni x \mapsto \alpha(x) \in \mathbb{R}^3$ for translation and $\Omega \ni x \mapsto \beta(x) \in \mathbb{R}^3$ for rotation only depend on the three-dimensional geometry of the liquid. Observe that $\alpha(x)$ is homogeneous to a length and $\beta(x)$ to a surface. We call them the ‘‘Stokes-Zhukovsky vector fields’’ in this contribution, as in the reference [17].

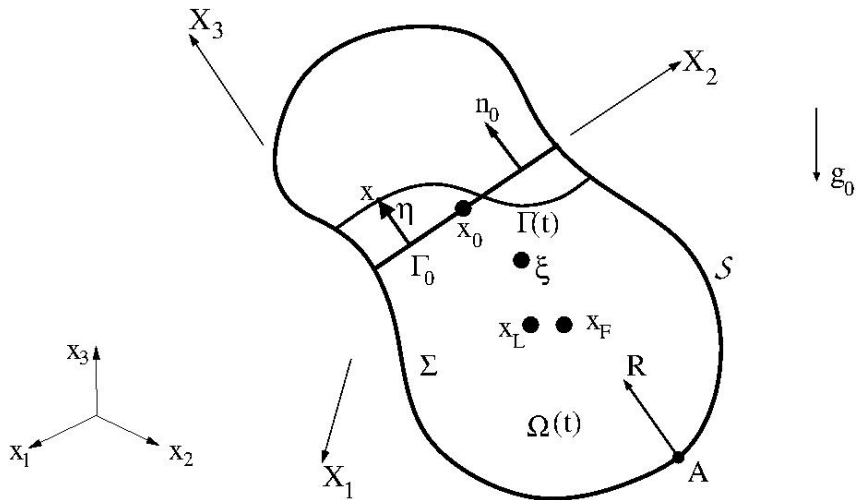


Figure 2. General view of the sloshing problem in a six degrees of freedom rigid body. The free boundary $\Gamma(t)$ is issued from the equilibrium free boundary at rest Γ_0 with the help of the elongation η . The other notations are explained in the corpus of the text.

- **Free surface**

Consider as a reference situation the fluid at rest relative to the solid. Then the free surface has a given position Γ_0 as presented in Figure 2. During sloshing, two processes have now to be taken into account. First, the rigid motion of the surface Γ_0 and secondly the free displacement ηn_0 of the free boundary measured in the relative referential, where n_0 denotes the normal direction to Γ_0 at position $y \in \Gamma_0$. The local coordinates X_j are defined by the relation $x - x_0 = \sum_{j=1}^3 X_j \varepsilon_j$, where x_0 is the barycenter of the free surface Γ_0 defined in (9). We introduce the center of gravity x_L of the liquid: $m_L x_L \equiv \int_{\Omega} \rho_L x \, dx$, which is *a priori* a function of time. We introduce also the center of gravity x_F of the “frozen fluid” Ω_0 at rest. Note that Γ_0 is a part of the boundary of Ω_0 : $m_L x_F \equiv \int_{\Omega_0} \rho_L x \, dx$ and we refer to Figure 2 for a representation of this point.

- **Stokes-Zhukovsky vector fields for translation**

We set $\alpha \equiv \sum_{j=1}^3 \alpha_j \varepsilon_j$. Then the scalar function $\alpha_j(x)$ satisfies clearly the following Neumann problem for the Laplace equation:

$$(30) \quad \begin{cases} \Delta \alpha_j = 0 & \text{in } \Omega_0 \\ \frac{\partial \alpha_j}{\partial n} = n_j & \text{on } \partial \Omega_0. \end{cases}$$

The problem (30) has a unique solution up to a scalar constant if the domain Ω is connected. It has an analytical solution. We consider the center of gravity x_0 of the linearized free surface Γ_0 according to (9). Then: $\alpha_j(x) = (x - x_0) \cdot \varepsilon_j$ for $j = 1, 2, 3$ if the condition $\int_{\Gamma_0} \alpha_j \, d\gamma = 0$. holds. Then in consequence,

$$\nabla \alpha_j = \varepsilon_j, \quad j = 1, 2, 3$$

- We introduce $\tilde{\alpha}$ by rotating the Stokes-Zhukovsky translation vector field α :

$$\tilde{\alpha} \equiv (x - x_0) \times \varepsilon_3 = \alpha \times \varepsilon_3$$

Then: $\int_{\Gamma_0} \tilde{\alpha} \, d\gamma = 0$ and $\tilde{\alpha} = X_2 \varepsilon_1 - X_1 \varepsilon_2$. Moreover $\int_{\Gamma_0} \eta (X_2 \varepsilon_1 - X_1 \varepsilon_2) \, d\gamma = \int_{\Gamma_0} \eta \tilde{\alpha} \, d\gamma$ and

$$(31) \quad \tilde{\alpha} \cdot \theta = (-X_1 \theta_2 + X_2 \theta_1).$$

- **Stokes-Zhukovsky vector fields for rotation**

Analogously to the definition (30) of Stokes-Zhukovsky vector fields for translation, we set $\beta \equiv \sum_{j=1}^3 \beta_j \varepsilon_j$. The scalar function $\beta_j(x)$ satisfies the equations:

$$(32) \quad \begin{cases} \Delta \beta_j = 0 & \text{in } \Omega_0 \\ \frac{\partial \beta_j}{\partial n} = ((x - \xi) \times n)_j & \text{on } \partial \Omega_0. \end{cases}$$

It is elementary (see *e.g.* P.A. Raviart and J.M. Thomas [35]) to verify that the Neumann problem (32) is well set up to an additive constant. But, oppositely to the Stokes-Zhukovsky vector field for translation, we have **no analytical expression** for the Stokes-Zhukovsky functions β_j for rotation. Nevertheless, the following relations show that beautiful algebra can be developed for the Stokes-Zhukovsky vector fields. They are proven in detail in [13]:

$$(33) \quad \rho_L \int_{\Omega} \nabla \alpha_j \, dx = m_L \varepsilon_j, \quad \rho_L \int_{\Omega} \nabla \beta_j \, dx = m_L \varepsilon_j \times (x_F - \xi), \quad j = 1, 2, 3.$$

- **Liquid inertial tensor**

With B. Fraeijns de Veubeke [19], we introduce the so-called “liquid inertial tensor” I_ℓ defined according to

$$(34) \quad (I_\ell)_{jk} \equiv \rho_L \int_{\Omega} \nabla \beta_j \bullet \nabla \beta_k \, dx.$$

We have the complementary results, proven also in [13]:

$$(35) \quad \begin{cases} \rho_L \int_{\Omega} (x - \xi) \times \nabla \alpha_j \, dx = m_L (x_F - \xi) \times \varepsilon_j, & j = 1, 2, 3, \\ \rho_L \int_{\Omega} (x - \xi) \times \nabla \beta_j \, dx = I_\ell \bullet \varepsilon_j, & j = 1, 2, 3. \end{cases}$$

Moreover, the liquid inertial tensor I_ℓ defined in (34) is positive definite. We have $(\theta, I_\ell \bullet \theta) = \rho_L \int_{\Omega} |\nabla(\beta \bullet \theta)|^2 \, dx \geq 0$ for $\theta \in \mathbb{R}^3$. Moreover, if $(\theta, I_\ell \bullet \theta) = 0$, then $\theta = 0$ in \mathbb{R}^3 .

- **Decomposition of the fluid velocity potential**

The fluid velocity potential φ satisfies a continuity condition across the solid interface $\Sigma(t)$ due to the non-penetration of the fluid inside the solid:

$$(36) \quad \frac{\partial \varphi}{\partial n} = \left(\frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi) \right) \bullet n, \quad x \in \Sigma.$$

Therefore we subtract to φ the potential $\tilde{\varphi}$ introduced in (29) and the difference satisfies a homogeneous boundary condition on the solid interface. In an analogous way, Proposition 1 can be derived in the relative referential. Then the fluid velocity potential φ can be decomposed according to

$$(37) \quad \varphi \equiv \alpha \bullet \frac{d\xi}{dt} + \beta \bullet \frac{d\theta}{dt} + \frac{\partial \psi}{\partial t}.$$

The free surface potential ψ introduced in (37) still satisfies the relations (18):

$$\begin{cases} \Delta \psi = 0 & \text{in } \Omega \\ \frac{\partial \psi}{\partial n} = \begin{cases} 0 & \text{on } \Sigma \\ \eta & \text{on } \Gamma_0. \end{cases} \end{cases}$$

Due to the definition (21) of the Neumann to Dirichlet operator, this last expression admits the compact form

$$(38) \quad \psi = W \bullet \eta \quad \text{on } \Gamma_0.$$

- We introduce the position ℓ_0 of the center of gravity of the fluid relatively to the solid center of gravity:

$$(39) \quad \ell_0 \equiv x_F - \xi.$$

We observe that this vector is linked to the solid and we have in particular $d\ell_0 = d\theta \times \ell_0$. We have the following relations, with $\eta \in F^{1/2}(\Gamma_0)$ and α, β defined in (30)(32) and ψ by the relation (18). The proofs are detailed in our report [13].

$$(40) \quad \begin{cases} \rho_L \int_{\Omega} \nabla \left(\frac{\partial^2 \psi}{\partial t^2} \right) \, dx = \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} (X_1 \varepsilon_1 + X_2 \varepsilon_2) \, d\gamma \\ \rho_L \int_{\Omega} (x - \xi) \times \nabla \left(\frac{\partial^2 \psi}{\partial t^2} \right) \, dx = \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} \beta \, d\gamma. \end{cases}$$

- **Proposition 4. Pressure continuity across the free surface**

The continuity of pressure on the free boundary Γ takes the following linearized form:

$$(41) \quad \frac{\partial \varphi}{\partial t} + g(\tilde{\alpha} \bullet \theta + \eta) = 0, \quad x \in \Gamma(t).$$

In an equivalent way with the potential ψ introduced in (37):

$$(42) \quad \frac{\partial^2 \psi}{\partial t^2} + \alpha \bullet \frac{d\xi^2}{dt^2} + \beta \bullet \frac{d^2 \theta}{dt^2} + g(\tilde{\alpha} \bullet \theta + \eta) = 0 \quad \text{on } \Gamma_0.$$

Compared to the relation (16), the new term $g \tilde{\alpha} \bullet \theta$ expresses the action of the external gravity field driving the fluid due to the rotational movement of the solid. In the stationary case, the local displacement η of the free boundary compensates exactly the rotational displacement of the solid.

- **Proof of Proposition 4.**

The proof is a small variation of the one proposed for Proposition 2. On the free surface Γ , the continuity of the stress tensor can be written $p = 0$ as previously. We choose the point P for the Bernoulli equation (5) on the frozen free surface Γ_0 equal to the center x_0 introduced in (9). Then $x - x_0 = X_1 \varepsilon_1 + X_2 \varepsilon_2 + \eta \varepsilon_3 + O(|\eta|^2)$. Due to Bernoulli theorem (5) and continuity (15) of the pressure on Γ , we deduce the following relation (17) on the free surface. In order to show the angular displacement of the solid, we have the following calculus:

$$\begin{aligned} -g_0 \bullet (x - x_0) &= g \varepsilon_3 \bullet \left(X_1 (\varepsilon_1 + \theta \times \varepsilon_1) + X_2 (\varepsilon_2 + \theta \times \varepsilon_2) + \eta \varepsilon_3 \right), \\ &= g(-X_1 \theta_2 + X_2 \theta_1 + \eta) + O(|\eta|^2), \end{aligned}$$

and the condition (17) of pressure continuity on the free surface is expressed by

$$\frac{\partial \varphi}{\partial t} + g(-X_1 \theta_2 + X_2 \theta_1 + \eta) + O(|\eta|^2) = 0, \quad x \in \Gamma(t),$$

which is exactly relation (41) due to the relation (31): $\tilde{\alpha} \bullet \theta = (-X_1 \theta_2 + X_2 \theta_1)$. If we determine the velocity potential φ according to the left hand side of the relation (37), the continuity of the pressure field (15) across the free surface takes exactly the form (42) and the proof is completed. \square

- **Proposition 5. Conservation of the solid momentum**

The conservation of the solid momentum conducts to the coupled relation:

$$(43) \quad (m_S + m_L) \frac{d^2 \xi}{dt^2} - m_L \ell_0 \times \frac{d^2 \theta}{dt^2} + \rho_L \int_{\Gamma_0} \alpha \frac{\partial^2 \eta}{\partial t^2} d\gamma = (m_S + m_L) g_0 + R.$$

- **Proof of Proposition 5.**

Due to the linearized Euler equations (2), the pressure field action can be stated as

$$\nabla p = \rho_L \left[g_0 - \nabla \left(\frac{\partial \varphi}{\partial t} \right) \right], \quad x \in \Omega,$$

the conservation of momentum of the solid can be written as:

$$(44) \quad m_S \frac{d^2\xi}{dt^2} = (m_S + m_L) g_0 + R - \int_{\Omega} \rho_L \nabla \left(\frac{\partial\varphi}{\partial t} \right) dx.$$

With the help of the Stokes-Zhukovsky vector fields, we can express the last term in the right hand side of (44) with the free surface potential ψ :

$$\rho_L \int_{\Omega} \nabla \left(\frac{\partial\varphi}{\partial t} \right) dx = \rho_L \int_{\Omega} \left[\sum_j \left(\nabla\alpha_j \frac{d^2\xi_j}{dt^2} + \nabla\beta_j \frac{d^2\theta_j}{dt^2} \right) + \nabla \left(\frac{\partial^2\psi}{\partial t^2} \right) \right] dx \quad c.f. (37)$$

$$= m_L \frac{d^2\xi}{dt^2} + m_L \frac{d^2\theta}{dt^2} \times (x_F - \xi) + \rho_L \int_{\Omega} \frac{\partial^2\eta}{\partial t^2} (X_1 \epsilon_1 + X_2 \epsilon_2) d\gamma$$

due to (33), (35) and (40). Then the conservation of impulsion of the solid (44) takes the form:

$$\left\{ \begin{array}{l} (m_S + m_L) \frac{d^2\xi}{dt^2} + m_L \frac{d^2\theta}{dt^2} \times (x_F - \xi) + \rho_L \int_{\Gamma_0} \frac{\partial^2\eta}{\partial t^2} (X_1 \epsilon_1 + X_2 \epsilon_2) d\gamma = \\ = (m_S + m_L) g_0 + R. \end{array} \right.$$

Then due to (39), the equations (25) for the conservation of impulsion of the solid takes the following expression (43) for the coupled problem. The proof is completed. \square

- **Proposition 6. Conservation of the solid angular momentum**

The conservation of the solid angular momentum conducts to the coupled relation

$$(45) \quad \left\{ \begin{array}{l} m_L \ell_0 \times \frac{d^2\xi}{dt^2} + (I_S + I_\ell) \cdot \frac{d^2\theta}{dt^2} + \rho_L \int_{\Gamma_0} \beta \frac{\partial^2\eta}{\partial t^2} d\gamma + \rho_L g \int_{\Gamma_0} \tilde{\alpha} \eta d\gamma = \\ = m_L \ell_0 \times g_0 + (x_A - \xi) \times R. \end{array} \right.$$

The term $m_L \ell_0 \times \frac{d^2\xi}{dt^2}$ of dynamic evolution of the kinetic momentum in the relation (45) is due to the non-coincidence of the center of gravity ξ of the solid and the frozen center of gravity x_F of the frozen liquid (see Figure 2). With the help of (40), the kinetic momentum of the fluid relative to the center of gravity of the solid is represented by the term $\rho_L \int_{\Gamma_0} \beta \cdot \frac{\partial^2\eta}{\partial t^2} d\gamma$. Last but not least, the restoring torque of the gravity field due to the weight of the liquid displaced by the movement of the free boundary is described by $\rho_L g \int_{\Gamma_0} \tilde{\alpha} \eta d\gamma$. We can interpret the elongation of the free surface by a continuous distribution of small harmonic oscillators that have an impact on the global conservation of the solid angular momentum.

- **Proof of Proposition 6.**

We explain in the following how the center of gravity x_L of the liquid depends on the position η relative to the free boundary. The conservation of kinetic momentum (26) takes now the form:

$$I_S \cdot \frac{d^2\theta}{dt^2} = (x_A - \xi) \times R + m_L (x_L - \xi) \times g_0 - \int_{\Omega} \rho_L (x - \xi) \times \nabla \left(\frac{\partial\varphi}{\partial t} \right) dx.$$

We consider also the center of gravity x_F of the “frozen fluid” and we denote by X_3^0 the vertical coordinate of the frozen free surface Γ_0 . Relatively to the rigid referential, we have the following calculus:

$$\begin{aligned}
 m_L x_L &= \int_{\Gamma_0} dX_1 dX_2 \left(\int_{X_{3\min}}^{X_3^0} \rho_L x dX_3 + \int_{X_3^0}^{X_3^0+\eta} \rho_L x dX_3 \right) \\
 &= m_L x_F + \int_{\Gamma_0} dX_1 dX_2 \int_0^\eta \rho_L \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} dX_3 = m_L x_F + \int_{\Gamma_0} \rho_L \begin{pmatrix} X_1 \eta \\ X_2 \eta \\ \frac{1}{2}\eta^2 \end{pmatrix} d\gamma.
 \end{aligned}$$

At first order: $m_L x_L \times g_0 = m_L x_F \times g_0 + \rho_L g \int_{\Gamma_0} \eta (-X_2 \varepsilon_1 + X_1 \varepsilon_2) d\gamma$.

- In consequence the conservation of kinetic momentum can be written under the form:

$$(46) \quad \begin{cases} \mathbf{I}_S \bullet \frac{d^2\theta}{dt^2} &= (x_A - \xi) \times R + m_L (x_F - \xi) \times g_0 \\ &+ \rho_L g \int_{\Gamma_0} \eta (-X_2 \varepsilon_1 + X_1 \varepsilon_2) d\gamma - \int_{\Omega} \rho_L (x - \xi) \times \nabla \left(\frac{\partial\varphi}{\partial t} \right) dx. \end{cases}$$

In an analogous way, the last term of the right hand side of (46) can be developed with the help of the decomposition (37):

$$\begin{aligned}
 \rho_L \int_{\Omega} (x - \xi) \times \nabla \left(\frac{\partial\varphi}{\partial t} \right) dx &= \\
 &= \rho_L \int_{\Omega} (x - \xi) \times \left[\sum_j \left(\nabla \alpha_j \frac{d^2\xi_j}{dt^2} + \nabla \beta_j \frac{d^2\theta_j}{dt^2} \right) + \nabla \left(\frac{\partial^2\psi}{\partial t^2} \right) \right] dx \\
 &= m_L (x_F - \xi) \times \frac{d^2\xi}{dt^2} + \mathbf{I}_\ell \bullet \frac{d^2\theta}{dt^2} + \rho_L \int_{\Omega} \frac{\partial^2\eta}{\partial t^2} \beta d\gamma
 \end{aligned}$$

due to (33), (35) and (40). In consequence, the motion (46) of the solid around its center of gravity can be written as:

$$\begin{cases} (\mathbf{I}_S + \mathbf{I}_\ell) \bullet \frac{d^2\theta}{dt^2} + m_L (x_F - \xi) \times \frac{d^2\xi}{dt^2} + \rho_L \int_{\Gamma_0} \frac{\partial^2\eta}{\partial t^2} \beta d\gamma + \\ + \rho_L g \int_{\Gamma_0} \eta (X_2 \varepsilon_1 - X_1 \varepsilon_2) d\gamma &= m_L (x_F - \xi) \times g_0 + (x_A - \xi) \times R. \end{cases}$$

and due to (39), this is exactly the relation (45) and the proof is completed. \square

• Towards a synthetic formulation

With the help of the relation (38) on the boundary Γ_0 , the continuity (42) of the pressure field across the free surface is simply written as:

$$(47) \quad \rho_L W \bullet \frac{\partial^2\eta}{\partial t^2} + \rho_L \alpha \bullet \frac{d^2\xi}{dt^2} + \rho_L \beta \bullet \frac{d^2\theta}{dt^2} + \rho_L g (\tilde{\alpha} \bullet \theta + \eta) = 0 \quad \text{on } \Gamma_0.$$

The term $\rho_L g \tilde{\alpha} \bullet \theta$ is due to the weight of the fluid working in the rigid movement associated to the solid rotation. The coupled problem (43) (45) (47) is now formulated in an attractive mathematical point of view. The unknown is composed of the triple $(\xi(t), \theta(t), \eta(t))$, with $\xi(t) \in \mathbb{R}^3$, $\theta(t) \in \mathbb{R}^3$, $\eta(t) \in F^{1/2}(\Gamma_0)$ and the three equations (43) (45) (47) are considered in \mathbb{R}^3 , \mathbb{R}^3 and on Γ_0 respectively. The mathematical difficulty is due to the term $W \bullet \frac{\partial^2\eta}{\partial t^2}$ because W is an integral operator.

3) Coupled system structure

• We are now in position to aggregate the previous results. The solid movement is a six degrees of freedom motion described by the velocity $\frac{d\xi}{dt}$ of its center of gravity and its instantaneous rotation $\frac{d\theta}{dt}$. The motion of the solid around its center of gravity has been obtained in relation (46). The two equations (44) and (46) admit as a source term the gradient of the velocity potential. The partial differential equation that governs this potential is simply the incompressibility of the liquid, expressed by the Laplace equation. The boundary conditions are the non-penetration (36) of the fluid inside the solid, the normal movement (18) of the fluid relatively to the free surface and the continuity (15) of the pressure field across the free surface expressed by (42).

• Energy conservation

We can now consider the three terms of the total energy: the uncoupled kinetic energy

$$T \equiv \frac{1}{2} \frac{d\xi}{dt} \bullet m_S \frac{d\xi}{dt} + \frac{1}{2} \frac{d\theta}{dt} I_S \bullet \frac{d\theta}{dt} + \frac{1}{2} \int_{\Omega} \rho_L |\nabla\varphi|^2 dx ,$$

the energy of interaction with gravity

$$U \equiv \frac{1}{2} \rho_L g \int_{\Gamma_0} |\eta|^2 d\gamma + \rho_L g \int_{\Gamma_0} \eta (X_2 \theta_1 - X_1 \theta_2) d\gamma$$

and the gravity potential $V \equiv -m_S g_0 \bullet \xi - m_L g_0 \bullet x_F$. With kinetic energy T , energy of interaction with gravity U and gravity potential V defined previously respectively, we have the following detailed expressions:

$$(48) \quad \left\{ \begin{array}{l} T = \frac{1}{2} (m_S + m_L) \left| \frac{d\xi}{dt} \right|^2 + \frac{1}{2} \left(\frac{d\theta}{dt}, (I_S + I_\ell) \frac{d\theta}{dt} \right) + \frac{\rho_L}{2} \int_{\Gamma_0} \left(\frac{\partial\eta}{\partial t}, W \bullet \frac{\partial\eta}{\partial t} \right) d\gamma \\ \quad + m_L \left(\ell_0, \frac{d\xi}{dt}, \frac{d\theta}{dt} \right) + \rho_L \int_{\Gamma_0} \left(\alpha \frac{d\xi}{dt} + \beta \frac{d\theta}{dt} \right) \frac{\partial\eta}{\partial t} d\gamma \end{array} \right.$$

$$(49) \quad U = \frac{1}{2} \rho_L g \int_{\Gamma_0} \eta^2 d\gamma + \rho_L g \int_{\Gamma_0} (\tilde{\alpha} \bullet \theta) \eta d\gamma$$

$$(50) \quad V = -(m_S + m_L) g_0 \bullet \xi - m_L g_0 \bullet \ell_0 .$$

We recognize the kinetic energy of the solid with the translation and rotation decoupled terms $\frac{1}{2} (m_S + m_L) \left| \frac{d\xi}{dt} \right|^2$, $\frac{1}{2} \left(\frac{d\theta}{dt}, (I_S + I_\ell) \frac{d\theta}{dt} \right)$, the coupling between translation and rotation $m_L \left(\ell_0, \frac{d\xi}{dt}, \frac{d\theta}{dt} \right)$, the kinetic energy of the free surface $\frac{\rho_L}{2} \int_{\Gamma_0} \left(\frac{\partial\eta}{\partial t}, W \bullet \frac{\partial\eta}{\partial t} \right) d\gamma$ and the coupling $\rho_L \int_{\Gamma_0} \left(\alpha \frac{d\xi}{dt} + \beta \frac{d\theta}{dt} \right) \frac{\partial\eta}{\partial t} d\gamma$ between the solid movement and the free boundary.

• Proposition 7. Energy conservation

Due to the lack of knowledge concerning the external force R , the conservation of energy takes the following form, with the previous notations:

$$(51) \quad \frac{d}{dt} (T + U + V) = R \bullet u_A .$$

The proof is detailed in [13].

- **Operator matrices**

We consider now the global vector $q(t)$ according to:

$$(52) \quad q \equiv (\eta, \xi, \theta)^t.$$

Remark that when $t \geq 0$, $q(t)$ belongs to the functional space $F^{1/2}(\Gamma_0) \times \mathbb{R}^3 \times \mathbb{R}^3$, an infinite dimensional vector space denoted by $Q_0(\Omega, \mathcal{S})$ in the following:

$$(53) \quad Q_0(\Omega, \mathcal{S}) \equiv F^{1/2}(\Gamma_0) \times \mathbb{R}^3 \times \mathbb{R}^3.$$

With this notation, the **interaction** between the liquid Ω and the solid \mathcal{S} , through the free boundary Γ_0 , is defined through global operator matrices M and K . The mass matrix M is defined according to:

$$(54) \quad M = \begin{pmatrix} \rho_L W \bullet & \rho_L \alpha \bullet & \rho_L \beta \bullet \\ \rho_L \int_{\Gamma_0} d\gamma \alpha \bullet & m_S + m_L & -m_L \ell_0 \times \bullet \\ \rho_L \int_{\Gamma_0} d\gamma \beta \bullet & m_L \ell_0 \times \bullet & I_S + I_\ell \end{pmatrix}.$$

Remark that this matrix is composed by operators. In particular the operator W at the position (1, 1) is defined in (21). Moreover, if $q \in Q_0(\Omega, \mathcal{S})$, $M \bullet q \in Q_0(\Omega, \mathcal{S})$ and M is an operator $Q_0(\Omega, \mathcal{S}) \rightarrow Q_0(\Omega, \mathcal{S})$. In an analogous way, we define the global rigidity matrix K :

$$(55) \quad K = \begin{pmatrix} \rho_L g & 0 & \rho_L g \tilde{\alpha} \bullet \\ 0 & 0 & 0 \\ \rho_L g \int_{\Gamma_0} d\gamma \tilde{\alpha} \bullet & 0 & 0 \end{pmatrix}$$

and we obtain as previously an operator $Q_0(\Omega, \mathcal{S}) \ni q \mapsto K \bullet q \in Q_0(\Omega, \mathcal{S})$. We introduce also a global right hand side vector $F(t)$:

$$(56) \quad F(t) = \begin{pmatrix} 0 \\ (m_S + m_L) g_0 + R \\ m_L \ell_0 \times g_0 + (x_A - \xi) \times R \end{pmatrix}$$

and the relation $F(t) \in Q_0(\Omega, \mathcal{S})$ is natural. We remark with these relatively complicated definitions (53), (54), (55), (56) that the global dynamical system composed by the relations (43), (45), (47) admits finally a very simple form:

$$(57) \quad M \bullet \frac{d^2 q}{dt^2} + K \bullet q = F(t).$$

This equation is the extension of the previous free fluid oscillators equation (24) to the coupling with the solid motion.

- **Properties of the mass matrix**

The matrix M defined in (54) is symmetric and “positive definite”. We have the following expression for the quadratic form:

$$(q, M \bullet q) = m_S |\xi|^2 + (\theta, I_S \bullet \theta) + \rho_L \int_{\Omega} |\nabla \alpha \bullet \xi + \nabla \beta \bullet \theta + \nabla \psi|^2 d\gamma.$$

In other words, we have the expression $T = \frac{1}{2} \left(\frac{dq}{dt}, M \bullet \frac{dq}{dt} \right)$ for the kinetic energy developed in (48). The proof of this proposition is detailed in [13].

- We consider now the same questions for the rigidity operator K . We recall that the tangential coordinates X_j on the linearized free surface Γ_0 such that $x = \sum_{j=1}^3 X_j \varepsilon_j$ satisfy $\int_{\Gamma_0} X_j d\gamma = 0$ for $j = 1, 2$. We introduce a length a characteristic of this surface Γ_0 . Precisely, we suppose that

$$(58) \quad \int_{\Gamma_0} |X_j|^2 d\gamma \leq a^4, \quad j = 1, 2.$$

We introduce also the L^2 norm $\|\eta\| \equiv \sqrt{\int_{\Gamma_0} |\eta|^2 d\gamma}$ of the free surface, in coherence with the scalar product proposed in the relation (22).

- **Proposition 8. Properties of the rigidity matrix**

The matrix K is symmetric: $(q, K \bullet q') = (K \bullet q, q')$ for arbitrary global vectors q and q' in the space $Q_0(\Omega, \mathcal{S})$. The matrix K is positive: $(q, K \bullet q) \geq 0$ if the rotation θ of the solid is sufficiently small relatively to the mean quadratic value of the free surface, *id est*

$$(59) \quad |\theta_1| + |\theta_2| \leq \frac{1}{2a^2} \|\eta\|.$$

This relation is quite precise concerning the validity of linearity hypotheses.

- **Proof of Proposition 8.**

The symmetry of the matrix K is elementary to establish. We refer to [13]. We have also:

$$(q, K \bullet q) = \rho_L g \left(2 \int_{\Gamma_0} \theta \bullet \tilde{\alpha} \eta d\gamma + \|\eta\|^2 \right) = \rho_L g \left[2 \int_{\Gamma_0} \eta (X_2 \theta_1 - X_1 \theta_2) d\gamma + \|\eta\|^2 \right].$$

Then

$$\begin{aligned} \left| \int_{\Gamma_0} \eta X_2 \theta_1 d\gamma \right| &\leq |\theta_1| \int_{\Gamma_0} |\eta| |X_2| d\gamma && \text{because } \theta_1 \text{ is a constant on } \Gamma_0 \\ &\leq |\theta_1| \|\eta\| \|X_2\| && \text{using the Cauchy-Schwarz inequality} \\ &\leq |\theta_1| \|\eta\| a^2 && \text{by hypothesis (58)} \end{aligned}$$

and the analogous inequality $\left| \int_{\Gamma_0} \eta X_1 \theta_2 d\gamma \right| \leq |\theta_2| \|\eta\| a^2$ for the other component. We deduce from the previous assessment the minoration:

$$\begin{aligned} (q, K \bullet q) &\geq \rho_L g \left[\|\eta\|^2 - 2a^2 \|\eta\| (|\theta_1| + |\theta_2|) \right] \\ &\geq \rho_L g \|\eta\| \left[\|\eta\| - 2a^2 (|\theta_1| + |\theta_2|) \right] \end{aligned}$$

and this expression is positive when $|\theta_1| + |\theta_2| \leq \frac{1}{2a^2} \|\eta\|$ which is exactly the hypothesis (59). The proposition is established. \square

- **Lagrangian function for the coupled system**

With the reduction of the coupled sloshing problem to the unknown $q \equiv (\eta, \xi, \theta) \in Q_0(\Omega, \mathcal{S})$ we first specify the energies according to this global field. The conservation of energy (51) has been established again from the compact form (57) of the evolution equation. If the external force $R(t)$ is equal to zero, it is natural to introduce the Lagrangian \mathcal{L} according to the usual definition:

$$(60) \quad \mathcal{L} = T - (U + V).$$

Then this Lagrangian is a functional of the state q defined in (52) and of its first time derivative. We have the final Proposition:

- **Proposition 9. Euler-lagrange equations**

With the above notations when the right hand side $F(t)$ is reduced to the gravity term, the equations of motion (57) take the form

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{dq}{dt} \right)} \right) = \frac{\partial \mathcal{L}}{\partial q}.$$

The proof of this proposition is elementary. We omit it and refer to [13].

- With this general framework, the Lagrangian formulation is simple to use. It is sufficient for the applications to evaluate carefully the Lagrangian \mathcal{L} given by the relations (48), (49), (50) and (60).

Conclusion

In this contribution, we started from our industrial practice of sloshing for rigid bodies submitted to an acceleration. We first set the importance of the irrotational hypothesis of the flow in the external Galilean reference frame. Then we derived carefully the mechanics of the solid motion (conservation of momentum and conservation of kinetic momentum) and of the fluid motion (Laplace equation for the velocity potential), with a particular emphasis for the coupling with the continuity of the normal velocity field and the continuity of pressure across the fluid surface. A first difficulty is the representation of the solid rotational velocity vector field with potential functions. This can be achieved with the Stokes-Zhukovsky vector fields that are particular harmonic functions associated to the geometry of the fluid. Efficient numerical methods like integral methods (see *e.g.* [32]) could be used to go one step further. A much well known mathematical difficulty is the reduction of the fluid problem to a Neumann to Dirichlet operator for the Laplace equation. The use of integral methods is also natural for this kind of coupling (see *e.g.* [30] and [31]). In particular, the integral algorithm used *e.g.* in our contribution [3] is appropriate for such numerical computation. The degrees of freedom of both Stokes-Zhukovsky and modal functions are located on the interface between solid and liquid and the total computational cost of such approach is reasonable. Last but not least, we have derived a general expression for the Lagrangian of this coupled system. The next step is

to look to simplified systems and confront our rigorous mathematical analysis with the state of the art in the engineering community. In particular, we are interested in developing appropriate methodologies to define equivalent simplified mechanical systems as the ones presented in [1]. We plan also to apply our formulation with a Neumann to Dirichlet operator with boundary element methods.

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