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# Coupling Linear Sloshing with Six Degrees of Freedom Rigid Body Dynamics

François Dubois <sup>ab\*</sup> and Dimitri Stoliaroff <sup>c</sup>

<sup>a</sup> *Conservatoire National des Arts et Métiers, Paris, France.*

<sup>b</sup> *Department of Mathematics, University Paris Sud, Orsay, France.*

<sup>c</sup> *Airbus Defence and Space, Les Mureaux, France.*

*\* corresponding author*

*francois.dubois@cnam.fr, dimitri.stoliaroff@astrium.eads.net*

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## Abstract

Fluid motion in tanks is usually described in space industry with the so-called Lomen hypothesis which assumes the vorticity is null in the moving frame. We establish in this contribution that this hypothesis is valid only for uniform rotational motions. We give a more general formulation of this coupling problem, with a compact formulation.

We consider the mechanical modelling of a rigid body with a motion of small amplitude, containing an incompressible fluid in the linearized regime. We first establish that the fluid motion remains irrotational in a Galilean referential if it is true at the initial time. When continuity of normal velocity and pressure are prescribed on the free surface, we establish that the global coupled problem conserve an energy functional composed by three terms. We introduce the Stokes - Zhukovsky vector fields, solving Neumann problems for the Laplace operator in the fluid in order to represent the rotational rigid motion with irrotational vector fields. Then we have a good framework to consider the coupled problem between the fluid and the rigid motion. The coupling between the free surface and the *ad hoc* component of the velocity potential introduces a “Neumann to Dirichlet” operator that allows to write the coupled system in a very compact form. The final expresion of a Lagrangian for the coupled system is derived and the Euler-Lagrange equations of the coupled motion are presented.

## Keywords:

Stokes, Zhukovsky, Fraeijs de Veubeke, vector fields, integral boundary operator.

**AMS classification:** 70E99, 76B07.

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<sup>1</sup> Contribution submitted to publication.

## 1) Scope of the problem

Sloshing of liquid in tanks is an important phenomenon for terrestrial applications. We think for example of sloshing effects in road vehicles and ships carrying liquid cargo. The question is to know the magnitude of the wave and the total effort on the structure due to the movement of the fluid. For this kind of problematics, a lot of references exist and we refer the reader *i.e.* to the book of H. Morand and R. Ohayon [26], to the review proposed by R. Ibrahim, V. Pilipchuk and T. Ikeda [20], to the book of O.M. Faltinsen and A.N. Timokha [14] or to the review article of G. Hou *et al.* [19].

Moreover, for industrial applications, we would have a movable rigid tank with liquid free surface with six possible rigid movements and without needing a complete study of the elastic body, as studied *e.g.* in Bauer *et al.* [5], S. Piperno *et al.* [29], C. Farhat *et al.* [11], J.F. Gerbeau and M. Vidrascu [18], K.J. Bathe and H. Zhang [3], T.E. Tezduyar *et al.* [33] and the previous references.

In our case relative to space applications, the fundamental hypothesis of this contribution is the existence of some propulsion. We do not consider in this study the very complicated and nonlinear movement due to the quasi-disparition of gravity field. We refer for such studies to the contributions of F. Dodge and L. Garza [9, 10], S. Ostrach [28], H. Snydera [31], C. Falcón *et al.* [12] and P. Behruzi, *et al.* [6] among others. On the contrary, a gravity field is supposed to be present in our contribution and moreover an extra-gravity field is added due to the propulsion system. Then it is legitimus to linearize all the geometrical deformations and the equations of dynamics. In this kind of situation, the knowledge of the action of the fluid on the structure is mandatory. The question has been intensively studied during the sixties under the impulsion of NASA (see *e.g.* H. Bauer [4], D. Lomen [22, 23], H. Abramson [1], L. Fontenot [15]) and in European countries in the seventies (see *e.g.* J.P. Leriche [21]) or more recently (B. Chemoul *et al.* [7]).

We observe that due to its own intrinsic movement, the structure has also some influence on the fluid displacement. This question has been rigorously studied by the russian school in the sixties (N. Moiseev and V. Rumiantsev [25]). It is sufficient in a first approach to consider the solid as a rigid body and to neglect all the flexible deformations.

In fact, we are in front of a complete coupled problem. The fluid is linearized and has an action on the solid, considered as a rigid body. The solid is a “six degrees of freedom” system that can also be considered as linearized around a given configuration. This coupled problem does not seem to have been considered previously under this form in the litterature. We are happy to see that this quite old problem raises actually an intensive scientific activity. As examples, we mention the contributions of O. Faltinsen, O. Rognebakke, I. Lukovsky and A. Timokha [13, 17] who derived a variational method to analyse the sloshing with finite water depth. Note also that K. London [24] analysed the case of a multy-body model with applications to the Triana spacecraft, and J. Vierendeels *et al.* [34] proposed to use the Flow3D computer software (Fluent, Inc) to analyse numerically nonlinear effects involved in the coupling of a rigid body with sloshing fluid,



## 2) Mechanical Modelling

- **Rigid body**

We consider a solid  $\mathcal{S}$ , of density  $\rho_S$ . Then the solid mass is given by the relation:

$$(1) \quad m_S = \int_{\mathcal{S}} \rho_S dx.$$

We introduce the center of gravity  $\xi$  according to:

$$(2) \quad \int_{\mathcal{S}} \rho_S (x - \xi) dx = 0.$$

This solid is submitted to three forces. The first one is due to gravity field  $g_0$ . This vector is colinear to an “absolute” vertical direction associated with a vector  $e_3$ , third coordinate of a Galilean referential:

$$(3) \quad g_0 = -g e_3.$$

Note that  $g > 0$  with this choice, as illustrated on Figure 1. The weight of the solid  $\mathcal{S}$  is then equal to  $m_S g_0$ . Secondly a force  $R$  at a fixed point  $A$  on the boundary  $\partial\mathcal{S}$ . We can suppose that this force is a given function of time. Last but not least, the surfacic forces  $f$  on the boundary  $\partial\mathcal{S}$  due to the internal fluid.

- **Infinitesimal motion of the rigid body**

The center of gravity is a function of time:  $\xi = \xi(t)$ . We introduce a local referential  $\varepsilon_j$  issued from Galilean referential  $e_j$  thanks to an infinitesimal rotation of angle  $\theta = \theta(t)$ :

$$(4) \quad \varepsilon_j = e_j + \theta \times e_j.$$

Then

$$\frac{d\varepsilon_j}{dt} = \frac{d\theta}{dt} \times \varepsilon_j, \quad 1 \leq j \leq 3.$$

The solid velocity field  $u_S(x)$  satisfies:

$$(5) \quad u_S(x) = \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi(t)), \quad x \in \mathcal{S}.$$

The kinetic momentum  $\sigma_S$  is defined according to:

$$(6) \quad \sigma_S \equiv \int_{\mathcal{S}} \rho_S (x - \xi) \times u_S(x) dx.$$

The tensor of inertia  $I_S$  is defined by:

$$(7) \quad I_S \bullet y \equiv \int_{\mathcal{S}} \rho_S (x - \xi) \times (y \times (x - \xi)) dx, \quad y \in \mathbb{R}^3.$$

For a rigid body, we have the classical relation:

$$\sigma_S = I_S \bullet \frac{d\theta}{dt}.$$

- **Dynamics equations of the rigid body**

By integration of the classical Newton laws of motion, the conservation of momentum takes the form:

$$(8) \quad m_S \frac{d^2\xi}{dt^2} = m_S g_0 + R + \int_{\partial\mathcal{S}} f d\gamma.$$

The momentum  $\mathcal{M}_S$  of the surfacic forces relatively to the center of gravity is given according to:

$$(9) \quad \mathcal{M}_S = \int_{\partial S} (x - \xi) \times f \, d\gamma.$$

Then the conservation of kinetic momentum takes the form:

$$(10) \quad \mathbf{I}_S \bullet \frac{d^2\theta}{dt^2} = (x_A - \xi) \times R + \mathcal{M}_S.$$

- **Incompressible liquid**

The liquid is contained inside the solid  $\mathcal{S}$ , it occupies a volume  $\Omega(t)$  variable with time, with a **constant** density  $\rho_L$ . The total mass of liquid can be easily expressed:

$$(11) \quad m_L = \int_{\Omega} \rho_L \, dx.$$

At the boundary  $\partial\Omega$  of liquid, we have a contact surface  $\Sigma(t)$  between liquid and solid (see Figure 1):

$$\Sigma(t) = \partial\Omega \cap \partial S$$

and a free surface  $\Gamma(t)$  where the liquid is in thermodynamical equilibrium with its vapor:

$$\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}, \quad \Gamma \cap \Sigma = \emptyset.$$

The velocity field of the liquid  $u(t)$  is measured relatively to an **absolute** referential, following *e.g.* the work of L. Fontenot [15] but oppositely to the hypothesis done by D. Lomen [22]. The liquid is assumed incompressible. We write it usually:

$$(12) \quad \operatorname{div} u = 0 \quad \text{in} \quad \Omega(t).$$

- **Liquid as a perfect linearized fluid**

The pressure field  $p(x)$  is defined in the liquid domain  $\Omega \ni x \mapsto p(x) \in \mathbb{R}$ . The conservation of momentum for a perfect fluid is written with the Euler equations of hydrodynamics:

$$(13) \quad \frac{\partial u}{\partial t} + (\operatorname{curl} u) \times u + \nabla \left( \frac{p}{\rho_L} + \frac{1}{2} |u|^2 \right) = g_0 \quad \text{in} \quad \Omega(t).$$

We make a **linearization hypothesis** and replace the previous equation by:

$$(14) \quad \frac{\partial u}{\partial t} + \frac{1}{\rho_L} \nabla p = g_0 \quad \text{in} \quad \Omega(t).$$

- **About Lomen hypothesis**

In the monograph [22], D. Lomen suppose the irrotationality for the motion of the liquid **relatively** to the motion of the rigid body. Then the velocity field of the liquid satisfies the conditions:

$$u(x) = \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi(t)) + v, \quad \operatorname{curl} v \equiv 0.$$

By taking the curl of this relation:

$$\operatorname{curl} u = 2 \frac{d\theta}{dt}.$$

Consider now the time derivative of the previous relation and the curl of relation (14). Then we obtain:

$$\frac{d^2\theta}{dt^2} = 0$$

and the hypothesis done by Lomen in [22] is physically correct **only** if the rotation of solid referential is **uniform** with time.

• **Proposition 1. Irrotationality in the Galilean referential** <sup>2</sup>

Under an assumption of linearized dynamics, if vorticity  $\text{curl } u$  of liquid measured **in the Galilean referential** at initial time is null, then it remains identically null for all time:

$$\text{curl } u \equiv 0, \quad t \geq 0, \quad x \in \Omega(t).$$

• **Velocity potential**

If the domain  $\Omega(t)$  is simply connected (be **careful** with this hypothesis for **toric** geometries !), the velocity field can be generated by a potential  $\varphi$ :

$$(15) \quad u(x) = \nabla\varphi, \quad x \in \Omega(t)$$

because  $\text{curl } u = 0$ .

• In order to have precise informations concerning this velocity potential, we recall the Bernoulli theorem. We inject the velocity field  $u = \nabla\varphi$  in the dynamical equations. We introduce a point  $P$  associated to the solid motion:

$$\nabla \left( \frac{\partial\varphi}{\partial t} + \frac{p}{\rho_L} - g_0 \bullet (x - x_P) \right) = 0, \quad x \in \Omega.$$

We add some time function to the scalar potential of velocity (and assume that the domain  $\Omega$  is connected). Then:

$$(16) \quad \frac{\partial\varphi}{\partial t} + \frac{p}{\rho_L} - g_0 \bullet (x - x_P) = 0, \quad x \in \Omega.$$

• We take now into consideration the incompressibility hypothesis (12) together with the potential representation of the velocity field (15). We then obtain the Laplace equation:

$$(17) \quad \Delta\varphi = 0 \quad \text{in } \Omega(t).$$

A first boundary condition for this equation is a consequence of the continuity of the normal velocity  $u \bullet n$  at the interface  $\Sigma$  between solid and liquid:

$$(18) \quad \frac{\partial\varphi}{\partial n} = \left( \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi) \right) \bullet n, \quad x \in \Sigma.$$

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<sup>2</sup>The proof of Proposition 1 and of all technical results of this contribution are detailed in the Annex.

- **Free surface**

Consider as a reference situation the solid at rest. Then the free surface has a given position  $\Gamma_0$  as presented on Figure 1. During sloshing, two processes have to be taken into account. First, the rigid motion of the surface  $\Gamma_0$  and secondly the free displacement  $\eta n_0$  of the free boundary measured in the relative referential, where  $n_0$  denotes the normal direction to  $\Gamma_0$  at position  $y \in \Gamma_0$ . The point  $x$  new position takes into account the variation of the free surface:

$$(19) \quad x = y + \eta(y) n_0, \quad y \in \Gamma_0, \quad x \in \Gamma.$$

Note that due to incompressibility condition, we have:

$$(20) \quad \int_{\Gamma_0} \eta \, d\gamma \equiv 0.$$

- **Proposition 2. Neumann boundary condition on the free surface**

If we keep only the first order terms, the boundary condition for the velocity potential on the free surface can be written as:

$$(21) \quad \frac{\partial \varphi}{\partial n} = \frac{d\xi}{dt} \cdot n + \left( \frac{d\theta}{dt} \times (x - \xi) \right) \cdot n + \frac{\partial \eta}{\partial t}, \quad x \in \Gamma_0.$$

- We remark that the equations (18) and (21) can be written in a synthetic form:

$$(22) \quad \frac{\partial \varphi}{\partial n} = \frac{d\xi}{dt} \cdot n + \frac{d\theta}{dt} \times (x - \xi) \cdot n + \begin{cases} 0 & \text{on } \Sigma \\ \frac{\partial \eta}{\partial t} & \text{on } \Gamma_0. \end{cases}$$

- **Proposition 3. Pressure continuity across the free surface**

We denote by  $x_0$  the center of gravity of the frozen free surface  $\Gamma_0$ :

$$(23) \quad \int_{\Gamma_0} (x - x_0) \, d\gamma = 0.$$

The continuity of pressure on the free boundary  $\Gamma$  takes the following linearized form:

$$(24) \quad \frac{\partial \varphi}{\partial t} + g(-X_1 \theta_2 + X_2 \theta_1 + \eta) = 0, \quad x \in \Gamma(t)$$

with local coordinates  $X_j$  defined by the relation

$$(25) \quad x = \sum_{j=1}^3 X_j \varepsilon_j.$$

- **Pressure field action**

The force  $f$  on the boundary of the solid surface  $\Sigma$  admits the expression

$$(26) \quad f = \begin{cases} p n, & x \in \Sigma \\ 0, & x \in \partial \mathcal{S} \setminus \Sigma. \end{cases}$$

Then:

$$\int_{\partial \mathcal{S}} f \, d\gamma = \int_{\Sigma} p n \, d\gamma = \int_{\partial \Omega} p n \, d\gamma = \int_{\Omega} \nabla p \, dx.$$

Due to Bernoulli equation:

$$(27) \quad \nabla p = \rho_L \left[ g_0 - \nabla \left( \frac{\partial \varphi}{\partial t} \right) \right], \quad x \in \Omega,$$

the conservation of momentum can be written as:

$$(28) \quad m_S \frac{d^2 \xi}{dt^2} = (m_S + m_L) g_0 + R - \int_{\Omega} \rho_L \nabla \left( \frac{\partial \varphi}{\partial t} \right) dx.$$

Due to the expression (9) of the momentum of pressure forces, we have also:

$$\mathcal{M}_S^i = \sum_{j,k} \varepsilon_{ijk} \int_{\partial \Omega} (x - \xi)_j p n_k d\gamma.$$

After an elementary calculus:

$$(29) \quad \mathcal{M}_S = \int_{\Omega} (x - \xi) \times \nabla p dx.$$

- **Liquid center of gravity**

We introduce the center of gravity  $x_L$  of the liquid:

$$(30) \quad m_L x_L \equiv \int_{\Omega} \rho_L dx.$$

**Be careful!** The variable  $x_L$  is *a priori* a function of time. We explicit in the following how this point depends on the position  $\eta$  relative to the free boundary. The conservation of kinetic momentum (10) takes now the form:

$$(31) \quad \mathbf{I}_S \bullet \frac{d^2 \theta}{dt^2} = (x_A - \xi) \times R + m_L (x_L - \xi) \times g_0 - \int_{\Omega} \rho_L (x - \xi) \times \nabla \left( \frac{\partial \varphi}{\partial t} \right) dx.$$

We introduce the center of gravity  $x_F$  of the “frozen fluid”  $\Omega_0$  at rest. Note that  $\Gamma_0$  is a part of the boundary of  $\Omega_0$ :

$$(32) \quad m_L x_F \equiv \int_{\Omega_0} \rho_L x dx$$

and we refer to Figure 1 for a representation of this point. We denote by  $X_3^0$  the vertical coordinate of the frozen free surface  $\Gamma_0$ . Relatively to the rigid referential, we have the following calculus:

$$\begin{aligned} m_L x_L &= \int_{\Gamma_0} dX_1 dX_2 \left( \int_{X_3^{\min}}^{X_3^0} \rho_L x dX_3 + \int_{X_3^0}^{X_3^0 + \eta} \rho_L x dX_3 \right) \\ &= m_L x_F + \int_{\Gamma_0} dX_1 dX_2 \int_0^{\eta} \rho_L \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} dX_3 = m_L x_F + \int_{\Gamma_0} \rho_L \begin{pmatrix} X_1 \eta \\ X_2 \eta \\ \frac{1}{2} \eta^2 \end{pmatrix} d\gamma. \end{aligned}$$

At first order:

$$(33) \quad m_L x_L \times g_0 = m_L x_F \times g_0 + \rho_L g \int_{\Gamma_0} \eta (-X_2 \varepsilon_1 + X_1 \varepsilon_2) d\gamma.$$

In consequence the conservation of kinetic momentum can be written under the form:

$$(34) \quad \begin{cases} I_S \bullet \frac{d^2\theta}{dt^2} &= (x_A - \xi) \times R + m_L (x_F - \xi) \times g_0 \\ &+ \rho_L g \int_{\Gamma_0} \eta (-X_2 \varepsilon_1 + X_1 \varepsilon_2) d\gamma - \int_{\Omega} \rho_L (x - \xi) \times \nabla \left( \frac{\partial \varphi}{\partial t} \right) dx. \end{cases}$$

- **Conservation of energy**

We are now in position to aggregate the previous results. The solid movement is a six degrees of freedom motion described by the velocity  $\frac{d\xi}{dt}$  of its center of gravity and its instantaneous rotation  $\frac{d\theta}{dt}$ . Due to (8), the evolution of the center of gravity of the solid takes now the form (28):

$$m_S \frac{d^2\xi}{dt^2} = (m_S + m_L) g_0 + R - \int_{\Omega} \rho_L \nabla \left( \frac{\partial \varphi}{\partial t} \right) dx.$$

The motion of the solid around its center of gravity has been obtained in relation (34). The two equations (28) and (34) admit as a source term the gradient of the velocity potential. The partial differential equation that governs this potential is simply the incompressibility of the liquid, expressed by the Laplace equation (17). The boundary conditions are the non-penetration of the fluid inside the solid (18), the normal movement (21) of the fluid relatively to the free surface and the continuity of the pressure field across the free surface expressed by (24).

We can now consider the three terms of the total energy: the uncoupled kinetic energy

$$(35) \quad T \equiv \frac{1}{2} \frac{d\xi}{dt} \bullet m_S \frac{d\xi}{dt} + \frac{1}{2} \frac{d\theta}{dt} I_S \bullet \frac{d\theta}{dt} + \frac{1}{2} \int_{\Omega} \rho_L |\nabla \varphi|^2 dx,$$

the energy of interaction (see a justification in Proposition 4)

$$(36) \quad U \equiv \frac{1}{2} \rho_L g \int_{\Gamma_0} |\eta|^2 d\gamma + \rho_L g \int_{\Gamma_0} \eta (X_2 \theta_1 - X_1 \theta_2) d\gamma$$

and the gravity potential

$$(37) \quad V \equiv -m_S g_0 \bullet \xi - m_L g_0 \bullet x_F.$$

Due to the lack of knowledge concerning the external force  $R$ , the conservation of energy takes the following form

- **Proposition 4. Energy conservation**

With the previous notations,

$$(38) \quad \frac{d}{dt} (T + U + V) = R \bullet u_A.$$

- **Coupled system**

At this step of the study, we have obtained a system of coupled equations. The unknowns are the center of gravity  $\xi(t) \in \mathbb{R}^3$  and the infinitesimal rotation  $\theta(t) \in \mathbb{R}^3$  of the solid, the displacement  $\eta(t, x) \in \mathbb{R}$  for  $x \in \Gamma(t)$  of the free boundary and the potential

$\varphi(t, x) \in \mathbb{R}$  for  $x \in \Omega(t)$  of the fluid velocity field. The equations express the conservation of momentum (28), the conservation of kinetic momentum (34), the incompressibility of the fluid (17), the non-penetration (18) (21) of the fluid across the solid boundary  $\Sigma$  and the free boundary  $\Gamma(t)$  and the continuity of the pressure field (24). It is not clear at this step that this system of equations can be mathematically correctly stated. In particular, the link between the fields  $\varphi$ ,  $\eta$ ,  $\xi$ , and  $\theta$  has to be explicated. It is the object of the next section.

### 3) Stokes - Zhukovsky vector fields

- **Velocity potential decomposition for a rigid body**

A natural question is the incorporation of the movement of a rigid body inside the expression of the potential  $\varphi$  of velocities. In other words, we have put in evidence the very particular role of the rigid movement for the determination of the velocity potential. We recall that this dynamics is a six degrees of freedom system described by the two vectors  $\xi(t)$  and  $\theta(t)$ . The remaining difficulty concerns the solid body velocity field which is rotational. Following an old idea due independently (at our knowledge) to G. Stokes [32], N. Zhukovsky [36] and B. Fraeijs de Veubeke [16], we introduce a function  $\tilde{\varphi}$  such that:

$$(39) \quad \begin{cases} \Delta \tilde{\varphi} = 0 & \text{in } \Omega \\ \frac{\partial \tilde{\varphi}}{\partial n} = \left( \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (s - \xi) \right) \cdot n & \text{on } \partial\Omega. \end{cases}$$

Due to **linearity** of the problem (39), we can decompose the vector field  $\tilde{\varphi}$  under the form:

$$\tilde{\varphi} \equiv \alpha \cdot \frac{d\xi}{dt} + \beta \cdot \frac{d\theta}{dt}.$$

The vector fields  $\Omega \ni x \mapsto \alpha(x) \in \mathbb{R}^3$  for translation and  $\Omega \ni x \mapsto \beta(x) \in \mathbb{R}^3$  for rotation only depend on the three-dimensional geometry of the liquid. Observe that  $\alpha(x)$  is homogeneous to a length and  $\beta(x)$  to a surface. We call them the ‘‘Stokes-Zhukovsky vector fields’’ in this contribution. They are studied in detail in this section.

- **Stokes-Zhukovsky vector fields for translation**

We set  $\alpha \equiv \sum_{j=1}^3 \alpha_j \varepsilon_j$ . Then the scalar function  $\alpha_j(x)$  satisfies clearly the following Neumann problem for the Laplace equation:

$$(40) \quad \begin{cases} \Delta \alpha_j = 0 & \text{in } \Omega \\ \frac{\partial \alpha_j}{\partial n} = n_j & \text{on } \partial\Omega. \end{cases}$$

The problem (40) has a unique solution up to a scalar constant if the domain  $\Omega$  is connected. It has an analytical solution. We consider the center of gravity  $x_0$  of the linearized free surface  $\Gamma_0$  according to (23). Then:

$$(41) \quad \alpha_j(x) = (x - x_0) \cdot \varepsilon_j, \quad j = 1, 2, 3.$$

In consequence,

$$(42) \quad \nabla \alpha_j = \varepsilon_j, \quad j = 1, 2, 3$$

and

$$(43) \quad \int_{\Gamma_0} \alpha_j d\gamma = 0.$$

- **Stokes-Zhukovsky vector fields for rotation**

Analogously to the definition (40) of Stokes-Zhukovsky vector fields for translation, we set  $\beta \equiv \sum_{j=1}^3 \beta_j \varepsilon_j$ . The scalar function  $\beta_j(x)$  satisfies the equations:

$$(44) \quad \begin{cases} \Delta \beta_j = 0 & \text{in } \Omega \\ \frac{\partial \beta_j}{\partial n} = ((x - \xi) \times n)_j & \text{on } \partial\Omega. \end{cases}$$

It is elementary (see *e.g.* P.A. Raviart and J.M. Thomas [30]) to verify that the Neumann problem (44) is well set up to an additive constant. But, oppositely to the Stokes-Zhukovsky vector field for translation, we have **no analytical expression** for the Stokes-Zhukovsky functions  $\beta_j$  for rotation.

- **Liquid inertial tensor**

We introduce the so-called “liquid inertial tensor” (see *e.g.* [16])  $I_\ell$  defined according to

$$(45) \quad (I_\ell)_{jk} \equiv \rho_L \int_{\Omega} \nabla \beta_j \bullet \nabla \beta_k dx.$$

- **Proposition 5. The liquid inertial tensor is positive definite**

With the previous notations, we have the following results:

$$(46) \quad \rho_L \int_{\Omega} \nabla \alpha_j dx = m_L \varepsilon_j, \quad j = 1, 2, 3,$$

$$(47) \quad \rho_L \int_{\Omega} \nabla \beta_j dx = m_L \varepsilon_j \times (x_F - \xi), \quad j = 1, 2, 3,$$

$$(48) \quad \rho_L \int_{\Omega} (x - \xi) \times \nabla \alpha_j dx = m_L (x_F - \xi) \times \varepsilon_j, \quad j = 1, 2, 3,$$

$$(49) \quad \rho_L \int_{\Omega} (x - \xi) \times \nabla \beta_j dx = I_\ell \bullet \varepsilon_j, \quad j = 1, 2, 3.$$

Moreover, with the liquid inertial tensor  $I_\ell$  defined in (45), we have

$$(50) \quad (\theta, I_\ell \bullet \theta) = \rho_L \int_{\Omega} |\nabla(\beta \bullet \theta)|^2 dx \geq 0, \quad \theta \in \mathbb{R}^3,$$

and if  $(\theta, I_\ell \bullet \theta) = 0$ , then  $\theta = 0$  in  $\mathbb{R}^3$ : the liquid inertial matrix  $I_\ell$  is positive definite.

- **Free surface potential**

We introduce the “free surface potential”  $\Omega \ni x \mapsto \psi(x) \in \mathbb{R}$  satisfying the following Neumann boundary-value problem for the Laplace equation:

$$(51) \quad \begin{cases} \Delta \psi = 0 & \text{in } \Omega \\ \frac{\partial \psi}{\partial n} = \begin{cases} 0 & \text{on } \Sigma \\ \eta & \text{on } \Gamma_0. \end{cases} \end{cases}$$

Then due to (18), (21), (22) and (39), we have a new expression for the potential of liquid velocity:

$$(52) \quad \varphi \equiv \alpha \bullet \frac{d\xi}{dt} + \beta \bullet \frac{d\theta}{dt} + \frac{\partial\psi}{\partial t}.$$

• **Proposition 6. First moments of the free surface potential**

We have the following relations:

$$(53) \quad \rho_L \int_{\Omega} \nabla \left( \frac{\partial^2 \psi}{\partial t^2} \right) dx = \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} (X_1 \varepsilon_1 + X_2 \varepsilon_2) d\gamma$$

$$(54) \quad \rho_L \int_{\Omega} (x - \xi) \times \nabla \left( \frac{\partial^2 \psi}{\partial t^2} \right) dx = \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} \beta d\gamma.$$

#### 4) Coupled problem

• **A new expression of the coupled problem**

With the help of the Stokes-Zhukovsky vector fields, we can express the last term in the right hand side of (28) with the free surface potential  $\psi$ :

$$\begin{aligned} \rho_L \int_{\Omega} \nabla \left( \frac{\partial \varphi}{\partial t} \right) dx &= \rho_L \int_{\Omega} \left[ \sum_j \left( \nabla \alpha_j \frac{d^2 \xi_j}{dt^2} + \nabla \beta_j \frac{d^2 \theta_j}{dt^2} \right) + \nabla \left( \frac{\partial \psi}{\partial t} \right) \right] dx \quad c.f. (52) \\ &= m_L \frac{d^2 \xi_j}{dt^2} + m_L \frac{d^2 \theta_j}{dt^2} \times (x_F - \xi) + \rho_L \int_{\Omega} \frac{\partial^2 \eta}{\partial t^2} (X_1 \varepsilon_1 + X_2 \varepsilon_2) d\gamma \end{aligned}$$

due to (46), (48) and (53). Then the conservation of impulsion of the solid (28) takes the form:

$$(55) \quad \begin{cases} (m_S + m_L) \frac{d^2 \xi}{dt^2} + m_L \frac{d^2 \theta}{dt^2} \times (x_F - \xi) + \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} (X_1 \varepsilon_1 + X_2 \varepsilon_2) d\gamma = \\ = (m_S + m_L) g_0 + R. \end{cases}$$

In an analogous way, the last term of the right hand side of (34) can be developed:

$$\begin{aligned} \rho_L \int_{\Omega} (x - \xi) \times \nabla \left( \frac{\partial \varphi}{\partial t} \right) dx &= \\ &= \rho_L \int_{\Omega} (x - \xi) \times \left[ \sum_j \left( \nabla \alpha_j \frac{d^2 \xi_j}{dt^2} + \nabla \beta_j \frac{d^2 \theta_j}{dt^2} \right) + \nabla \left( \frac{\partial^2 \psi}{\partial t^2} \right) \right] dx \\ &= m_L (x_F - \xi) \times \frac{d^2 \xi_j}{dt^2} + I_\ell \bullet \frac{d^2 \theta_j}{dt^2} + \rho_L \int_{\Omega} \frac{\partial^2 \eta}{\partial t^2} \bullet \beta d\gamma \end{aligned}$$

due to (47), (49) and (54). In consequence, the motion (34) of the solid around its center of gravity can be written as:

$$(56) \quad \begin{cases} (I_S + I_\ell) \bullet \frac{d^2 \theta}{dt^2} + m_L (x_F - \xi) \times \frac{d^2 \xi}{dt^2} + \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} \beta d\gamma + \\ + \rho_L g \int_{\Gamma_0} \eta (X_2 \varepsilon_1 - X_1 \varepsilon_2) d\gamma = m_L (x_F - \xi) \times g_0 + (x_A - \xi) \times R. \end{cases}$$

If we explicit the velocity potentiel  $\varphi$  in the left hand side of the relation (24), we have thanks to (52), the following form for the continuity of the pressure field (24) across the free surface:

$$(57) \quad \frac{\partial^2 \psi}{\partial t^2} + \alpha \bullet \frac{d\xi^2}{dt^2} + \beta \bullet \frac{d^2 \theta}{dt^2} + g(-X_1 \theta_2 + X_2 \theta_1 + \eta) = 0 \quad \text{on } \Gamma_0.$$

- **Neumann to Dirichlet operator**

We consider a free surface  $\eta$  such that the global incompressibility condition (20) holds. We introduce the functional space

$$(58) \quad F^{1/2}(\Gamma_0) \equiv \left\{ \eta : \Gamma_0 \longrightarrow \mathbb{R}, \int_{\Gamma_0} \eta \, d\gamma = 0 \right\}.$$

We consider the “free surface potential”  $\psi$  associated to a given  $\eta \in F^{1/2}(\Gamma_0)$  according to the following way. The function  $\Omega \ni x \longmapsto \psi(x) \in \mathbb{R}$  is uniquely defined by the Neumann problem (51) with the additional condition

$$(59) \quad \int_{\Gamma_0} \psi \, d\gamma = 0.$$

We consider the restriction  $\zeta$  (the trace) of the function  $\psi$  on the surface  $\Gamma_0$

$$(60) \quad \Gamma_0 \ni x \longmapsto \zeta(x) \equiv \psi(x) \in \mathbb{R}$$

The mapping  $F^{1/2}(\Gamma_0) \ni \eta \longmapsto \zeta \in F^{1/2}(\Gamma_0)$  is the “Neumann to Dirichlet” operator. We denote it with the letter  $W$ :

$$(61) \quad \zeta \equiv W \bullet \eta.$$

- **Proposition 7. Self adjoint operator**

The operator  $W : F^{1/2}(\Gamma_0) \ni \eta \longmapsto \zeta \in F^{1/2}(\Gamma_0)$  with  $\zeta$  defined by the relations (51), (59), (60) and (61) is self-adjoint. If we denote by  $(\bullet, \bullet)$  the  $L^2$  scalar product on the linearized free surface  $\Gamma_0$ , *id est*

$$(62) \quad (\eta, \zeta) \equiv \int_{\Gamma_0} \eta \zeta \, d\gamma, \quad \eta, \zeta \in F^{1/2}(\Gamma_0),$$

we have:

$$(63) \quad (\eta', W \bullet \eta) = (W \bullet \eta', \eta), \quad \forall \eta, \eta' \in F^{1/2}(\Gamma_0).$$

- **Proposition 8. A technical property**

We introduce the position  $\ell_0$  of the center of gravity of the fluid relatively to the solid center of gravity:

$$(64) \quad \ell_0 \equiv x_F - \xi.$$

We have the following relations, with  $\eta \in F^{1/2}(\Gamma_0)$ :

$$(65) \quad \int_{\Gamma_0} \eta \alpha_3 \, d\gamma = 0$$

$$(66) \quad \int_{\Gamma_0} \eta (\alpha \bullet \xi) \, d\gamma = \xi \bullet \int_{\Gamma_0} \eta (X_1 \varepsilon_1 + X_2 \varepsilon_2) \, d\gamma$$

$$(67) \quad \rho_L \int_{\Omega} \nabla \alpha_j \bullet \nabla \alpha_k \, dx = m_L \varepsilon_j \bullet \varepsilon_k, \quad 1 \leq j, k \leq 3$$

$$(68) \quad \rho_L \int_{\Omega} \nabla \alpha_j \bullet \nabla \beta_k \, dx = m_L (\varepsilon_j, \varepsilon_k, \ell_0), \quad 1 \leq j, k \leq 3$$

$$(69) \quad \int_{\Omega} \nabla \alpha_j \bullet \nabla \psi \, dx = \int_{\Gamma_0} \alpha_j \eta \, d\gamma, \quad 1 \leq j \leq 3$$

$$(70) \quad \int_{\Omega} \nabla \beta_k \bullet \nabla \psi \, dx = \int_{\Gamma_0} \beta_k \eta \, d\gamma, \quad 1 \leq k \leq 3$$

$$(71) \quad \int_{\Omega} |\nabla \psi|^2 \, dx = \int_{\Gamma_0} (\eta, W \bullet \eta) \, d\gamma,$$

with  $\eta \in F^{1/2}(\Gamma_0)$ ,  $\alpha$ ,  $\beta$  defined in (40)(44) and  $\psi$  by the relation (51).

• **Towards a synthetic formulation**

We introduce  $\tilde{\alpha}$  by rotating the Stokes-Zhukovsky translation vector field  $\alpha$ :

$$(72) \quad \tilde{\alpha} \equiv (x - x_0) \times \varepsilon_3 = \alpha \times \varepsilon_3$$

where  $x_0$  is the barycentre of the free surface  $\Gamma_0$ . Then:

$$(73) \quad \int_{\Gamma_0} \tilde{\alpha} \, d\gamma = 0$$

and

$$(74) \quad \tilde{\alpha} = X_2 \varepsilon_1 - X_1 \varepsilon_2.$$

Moreover  $\int_{\Gamma_0} \eta (X_2 \varepsilon_1 - X_1 \varepsilon_2) \, d\gamma = \int_{\Gamma_0} \eta \tilde{\alpha} \, d\gamma$  and

$$(75) \quad \tilde{\alpha} \bullet \theta = (-X_1 \theta_2 + X_2 \theta_1).$$

Then the equations (55), (56) of the coupled problem take the form

$$(76) \quad (m_S + m_L) \frac{d^2 \xi}{dt^2} - m_L \ell_0 \times \frac{d^2 \theta}{dt^2} + \rho_L \int_{\Gamma_0} \alpha \bullet \frac{\partial^2 \eta}{\partial t^2} \, d\gamma = (m_S + m_L) g_0 + R$$

for the conservation of impulsion of the solid and

$$(77) \quad \left\{ \begin{array}{l} m_L \ell_0 \times \frac{d^2 \xi}{dt^2} + (I_S + I_\ell) \bullet \frac{d^2 \theta}{dt^2} + \rho_L \int_{\Gamma_0} \beta \bullet \frac{\partial^2 \eta}{\partial t^2} \, d\gamma + \rho_L g \int_{\Gamma_0} \tilde{\alpha} \eta \, d\gamma = \\ = m_L \ell_0 \times g_0 + (x_A - \xi) \times R. \end{array} \right.$$

for the motion of solid around its center of gravity. The continuity (57) of the pressure field across the free surface is simply written as:

$$(78) \quad \rho_L \alpha \bullet \frac{d\xi^2}{dt^2} + \rho_L \beta \bullet \frac{d^2 \theta}{dt^2} + \rho_L W \bullet \frac{\partial^2 \eta}{\partial t^2} + \rho_L g (\tilde{\alpha} \bullet \theta + \eta) = 0 \quad \text{on } \Gamma_0.$$

The coupled problem (76) (77) (78) is now formulated in an attractive mathematical point of view. The unknown is composed of the triple  $(\xi(t), \theta(t), \eta(t))$ , with  $\xi(t) \in \mathbb{R}^3$ ,  $\theta(t) \in \mathbb{R}^3$ ,  $\eta(t) \in F^{1/2}(\Gamma_0)$  and the three equations (76) (77) (78) are considered in  $\mathbb{R}^3$ ,  $\mathbb{R}^3$  and on  $\Gamma_0$  respectively. The mathematical difficulty is due to the term  $W \bullet \frac{\partial^2 \eta}{\partial t^2}$  because  $W$  is an integral operator.

• **Operator matrices**

We consider now the global vector  $q(t)$  according to:

$$(79) \quad q \equiv (\xi, \theta, \eta)^t.$$

Remark that when  $t \geq 0$ ,  $q(t)$  belongs to the functional space  $\mathbb{R}^3 \times \mathbb{R}^3 \times F^{1/2}(\Gamma_0)$ , an infinite dimensional vector space denoted by  $Q_0(\Omega, \mathcal{S})$  in the following:

$$(80) \quad Q_0(\Omega, \mathcal{S}) \equiv \mathbb{R}^3 \times \mathbb{R}^3 \times F^{1/2}(\Gamma_0).$$

With this notation, the **interaction** between the liquid  $\Omega$  and the solid  $\mathcal{S}$ , through the free boundary  $\Gamma_0$ , is defined through global operator matrices  $M$  and  $K$ . The mass matrix  $M$  is defined according to:

$$(81) \quad M = \begin{pmatrix} m_S + m_L & -m_L \ell_0 \times \bullet & \rho_L \int_{\Gamma_0} d\gamma \alpha \bullet \\ m_L \ell_0 \times \bullet & I_S + I_\ell & \rho_L \int_{\Gamma_0} d\gamma \beta \bullet \\ \rho_L \alpha & \rho_L \beta & \rho_L W \bullet \end{pmatrix}.$$

Remark that this matrix is composed by operators. In particular the operator  $W$  at the position (3, 3) is defined in (61). Moreover, if  $q \in Q_0(\Omega, \mathcal{S})$ ,  $M \bullet q \in Q_0(\Omega, \mathcal{S})$  and  $M$  is an operator  $Q_0(\Omega, \mathcal{S}) \rightarrow Q_0(\Omega, \mathcal{S})$ . In an analogous way, we define the global rigidity matrix  $K$ :

$$(82) \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho_L g \int_{\Gamma_0} d\gamma \tilde{\alpha} \bullet \\ 0 & \rho_L g \tilde{\alpha} & \rho_L g \end{pmatrix}$$

and we obtain as previously an operator  $Q_0(\Omega, \mathcal{S}) \ni q \mapsto K \bullet q \in Q_0(\Omega, \mathcal{S})$ . We introduce also a global right hand side vector  $F(t)$ :

$$(83) \quad F(t) = \begin{pmatrix} (m_S + m_L) g_0 + R \\ m_L \ell_0 \times g_0 + (x_A - \xi) \times R \\ 0 \end{pmatrix}$$

and the relation  $F(t) \in Q_0(\Omega, \mathcal{S})$  is natural. We remark that with these relatively complicated definitions (80), (81), (82), (83) that the global dynamical system composed by the relations (55), (56), (57) admits finally a very simple form:

$$(84) \quad M \bullet \frac{d^2 q}{dt^2} + K \bullet q = F(t).$$

- **Proposition 9. Properties of the mass matrix**

The matrix  $M$  defined in (81) is symmetric and “positive definite”. We have the following expression for the quadratic form:

$$(85) \quad (q, M \bullet q) = m_S |\xi|^2 + (\theta, I_S \bullet \theta) + \rho_L \int_{\Omega} |\nabla \alpha \bullet \xi + \nabla \beta \bullet \theta + \nabla \psi|^2 d\gamma.$$

In other words, we have the expression

$$(86) \quad T = \frac{1}{2} \left( \frac{dq}{dt}, M \bullet \frac{dq}{dt} \right)$$

for the kinetic energy introduced in (35).

- We consider now the same questions for the rigidity operator  $K$ . We recall that the tangential coordinates  $X_j$  on the linearized free surface  $\Gamma_0$  introduced in (25) satisfy  $\int_{\Gamma_0} X_j d\gamma = 0$  for  $j = 1, 2$ . We introduce a length  $a$  characteristic of this surface  $\Gamma_0$ . Precisely, we suppose that

$$(87) \quad \int_{\Gamma_0} |X_j|^2 d\gamma \leq a^4, \quad j = 1, 2.$$

We introduce also the  $L^2$  norm  $\|\eta\|$  of the free surface by the relation

$$(88) \quad \|\eta\| \equiv \sqrt{\int_{\Gamma_0} |\eta|^2 d\gamma},$$

in coherence with the scalar product proposed in the relation (62).

- **Proposition 10. Properties of the rigidity mass matrix**

The matrix  $K$  is symmetric:

$$(89) \quad (q, K \bullet q') = (K \bullet q, q')$$

for arbitrary global vectors  $q$  and  $q'$  in the space  $Q_0(\Omega, \mathcal{S})$ .

The matrix  $K$  is positive:  $(q, K \bullet q) \geq 0$  if the rotation  $\theta$  of the solid is sufficiently small relatively to the mean quadratic value of the free surface, *id est*

$$(90) \quad |\theta_1| + |\theta_2| \leq \frac{1}{2a^2} \|\eta\|.$$

## 5) Lagrangian for the coupled dynamics

With the reduction of the coupled sloshing problem to the unknown  $q \equiv (\xi, \theta, \eta) \in Q_0(\Omega, \mathcal{S})$  we first explicit the energies according to this global field.

- **Proposition 11. Detailed expressions of various energies**

With kinetic energy  $T$ , potential energy  $U$  and gravitational external energy  $V$  defined in (35), (36) and (37) respectively, we have the following detailed expressions:

$$(91) \quad \left\{ \begin{array}{l} T = \frac{1}{2} (m_S + m_L) \left| \frac{d\xi}{dt} \right|^2 + \frac{1}{2} \left( \frac{d\theta}{dt}, (I_S + I_\ell) \frac{d\theta}{dt} \right) + \frac{\rho_L}{2} \int_{\Gamma_0} \left( \frac{\partial \eta}{\partial t}, W \bullet \frac{\partial \eta}{\partial t} \right) d\gamma \\ \quad + m_L \left( \ell_0, \frac{d\xi}{dt}, \frac{d\theta}{dt} \right) + \rho_L \int_{\Gamma_0} \left( \alpha \frac{d\xi}{dt} + \beta \frac{d\theta}{dt} \right) \frac{\partial \eta}{\partial t} d\gamma \end{array} \right.$$

$$(92) \quad U = \frac{1}{2} \rho_L g \int_{\Gamma_0} \eta^2 d\gamma + \rho_L g \int_{\Gamma_0} (\tilde{\alpha} \cdot \theta) \eta d\gamma$$

$$(93) \quad V = -(m_S + m_L) g_0 \cdot \xi - m_L (g_0 \cdot \ell_0).$$

We recognize the kinetic energy of the solid with the translation and rotation decoupled terms  $\frac{1}{2}(m_S + m_L) \left| \frac{d\xi}{dt} \right|^2$ ,  $\frac{1}{2} \left( \frac{d\theta}{dt}, (I_S + I_\ell) \frac{d\theta}{dt} \right)$ , the coupling between translation and rotation  $m_L \left( \ell_0, \frac{d\xi}{dt}, \frac{d\theta}{dt} \right)$ , the kinetic energy of the free surface  $\frac{\rho_L}{2} \int_{\Gamma_0} \left( \frac{\partial \eta}{\partial t}, W \cdot \frac{\partial \eta}{\partial t} \right) d\gamma$  and the coupling  $\rho_L \int_{\Gamma_0} \left( \alpha \frac{d\xi}{dt} + \beta \frac{d\theta}{dt} \right) \frac{\partial \eta}{\partial t} d\gamma$  between the solid movement and the free boundary.

- **Lagrangian function for the coupled system**

The conservation of energy (38) has been established again from the compact form (84) of the evolution equation. If the generalized external force  $F(t)$  is equal to zero, it is natural to introduce the Lagrangian  $\mathcal{L}$  according to the usual definition:

$$(94) \quad \mathcal{L} = T - (U + V).$$

Then this Lagrangian is a functional of the state  $q$  defined in (79) and of its first time derivative. We have our final Proposition:

- **Proposition 12. Euler-lagrange equations**

With the above notations when the right hand side  $F(t)$  is reduced to zero, the equations of motion (84) take the form

$$(95) \quad \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{dq}{dt} \right)} \right) = \frac{\partial \mathcal{L}}{\partial q}.$$

- With this general framework, the Lagrangian formulation is simpler to use. It is sufficient for the applications to evaluate carefully the Lagrangian  $\mathcal{L}$  given by the relations (91), (92), (93) and (94).

## 6) Conclusion

In this contribution, we started from our industrial practice of sloshing for rigid bodies submitted to an acceleration. We first set the importance of the irrotational hypothesis of the flow in the external Galilean reference frame. Then we derived carefully the mechanics of the solid motion (conservation of momentum and conservation of kinetic momentum) and of the fluid motion (Laplace equation for the velocity potential), with a particular emphasis for the coupling with the continuity of the normal velocity field and the continuity of pressure across the fluid surface. A first difficulty is the representation of the solid rotational velocity vector field with potential functions. This can be achieved with the Stokes-Zhukovsky vector fields that are particular harmonic functions associated to the geometry of the fluid. Efficient numerical methods like integral methods

(see *e.g.* [27]) could be used to go one step further. A much well known mathematical difficulty is the reduction of the fluid problem to a Neumann to Dirichlet operator for the Laplace equation. The use of integral methods is also natural for this kind of coupling (see *e.g.* [25] and [26]). Last but not least, we have derived a general expression for the Lagrangian of this coupled system. The next step is to look to simplified systems and confront our rigorous mathematical analysis with the state of the art in the engineering community. In particular, we are interested in developing appropriate methodologies to define equivalent simplified mechanical systems as the ones presented in [1].

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## Annex. Proof of the technical results

- **Proof of Proposition 1.**

Take the curl of the linearized dynamics equation (14). Then  $\frac{\partial}{\partial t}(\text{curl } u) = 0$  and the property is established if it is true at  $t = 0$ .  $\square$

- **Proof of Proposition 2.**

The coordinates  $X_j$  of a point  $x \in \Gamma$  relatively to the solid referential have been introduced in (25). Then the equation of the reference free surface  $\Gamma_0$  in the solid referential is  $X_3 = \text{constant}$  and for the free surface  $\Gamma$  it takes the form  $X_3 = \eta(X_1, X_2)$ . The function  $\eta$  and all its derivatives are supposed to be first order infinitesimals. We determine now the normal direction  $n$  relative to the free boundary  $\Gamma$ . We have relatively to the basis  $\varepsilon_j$ :  $\frac{\partial x}{\partial X_1} = (1, 0, \frac{\partial \eta}{\partial X_1})^t$  and  $\frac{\partial x}{\partial X_2} = (0, 1, \frac{\partial \eta}{\partial X_2})^t$ . Then  $\frac{\partial x}{\partial X_1} \times \frac{\partial x}{\partial X_2} = (-\frac{\partial \eta}{\partial X_1}, -\frac{\partial \eta}{\partial X_2}, 1)^t$  for  $x \in \Gamma(t)$ , and we have at second order accuracy  $\|\frac{\partial x}{\partial X_1} \times \frac{\partial x}{\partial X_2}\| = 1 + O(|\eta|^2)$ . We deduce the following expression for the normal vector:

$$n \equiv \frac{\frac{\partial x}{\partial X_1} \times \frac{\partial x}{\partial X_2}}{\|\frac{\partial x}{\partial X_1} \times \frac{\partial x}{\partial X_2}\|} = -\frac{\partial \eta}{\partial X_1} \varepsilon_1 - \frac{\partial \eta}{\partial X_2} \varepsilon_2 + \varepsilon_3 + O(|\eta|^2)$$

and finally:

$$n = n_0 - \frac{\partial \eta}{\partial X_1} \varepsilon_1 - \frac{\partial \eta}{\partial X_2} \varepsilon_2.$$

We introduce now the normal velocity  $u_\Gamma \cdot n$  on the free surface  $\Gamma$ . Observe that the free boundary is a contact discontinuity between liquid and gas. The continuity condition expresses that the normal velocity of the free surface is equal to the fluid normal velocity.

But the normal velocity of the free surface has two terms: on one hand, the velocity of the rigid body at the point  $y \in \Gamma_0$  and, on the other hand, the time derivative of relative altitude  $\eta(t)$ . Then:

$$u_\Gamma \bullet n = \left( \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi) \right) \bullet n + \frac{\partial \eta}{\partial t} + O(|\eta|^2)$$

and

$$u \bullet n = \frac{\partial \varphi}{\partial n}, \quad x \in \Gamma.$$

Due to the continuity condition of normal velocity:

$$u_\Gamma \bullet n = u \bullet n,$$

we obtain from the two previous equations a Neumann boundary condition for the potential  $\varphi$  on the free boundary:

$$\frac{\partial \varphi}{\partial n} = \frac{d\xi}{dt} \bullet n + \left( \frac{d\theta}{dt} \times (x - \xi) \right) \bullet n + \frac{\partial \eta}{\partial t} + O(|\eta|^2)$$

and the relation (21) is established.  $\square$

• **Proof of Proposition 3.**

On the free surface  $\Gamma$ , the continuity of the stress tensor can be written for a perfect fluid:

$$(96) \quad p = 0 \quad \text{on } \Gamma.$$

We choose the point  $P$  for the Bernoulli equation (16) on the frozen free surface  $\Gamma_0$  equal to the center  $x_0$  introduced in (23). Then:

$$x - x_0 = X_1 \varepsilon_1 + X_2 \varepsilon_2 + \eta \varepsilon_3 + O(|\eta|^2).$$

Due to Bernoulli theorem (16) and continuity (96) of the pressure on  $\Gamma$ , we deduce the following relation on the free surface:

$$(97) \quad \frac{\partial \varphi}{\partial t} - g_0 \bullet (x - x_0) = 0, \quad x \in \Gamma(t).$$

In order to show the angular displacement of the solid, we have the following calculus:

$$\begin{aligned} -g_0 \bullet (x - x_0) &= g \varepsilon_3 \bullet \left( X_1 (\varepsilon_1 + \theta \times \varepsilon_1) + X_2 (\varepsilon_2 + \theta \times \varepsilon_2) + \eta \varepsilon_3 \right), \\ &= g (-X_1 \theta_2 + X_2 \theta_1 + \eta) + O(|\eta|^2), \end{aligned}$$

and the condition (97) of pressure continuity on the free surface is expressed by

$$\frac{\partial \varphi}{\partial t} + g (-X_1 \theta_2 + X_2 \theta_1 + \eta) + O(|\eta|^2) = 0, \quad x \in \Gamma(t),$$

which is exactly relation (24). The proof is established.  $\square$

• **Proof of Proposition 4.**

We use the evolution equations to evaluate the time derivative of the total energy:

$$\begin{aligned}
 \frac{d}{dt}(T + V + U) &= m_S \frac{d\xi}{dt} \cdot \frac{d^2\xi}{dt^2} + \frac{d\theta}{dt} \cdot \frac{d}{dt}(\mathbb{I}_S \cdot \frac{d\theta}{dt}) + \int_{\Omega} \rho_L \nabla\varphi \cdot \nabla\left(\frac{\partial\varphi}{\partial t}\right) dx \\
 &\quad - m_S g_0 \cdot \frac{d\xi}{dt} - m_L g_0 \cdot u_F + \rho_L g \int_{\Gamma_0} \eta \frac{\partial\eta}{\partial t} d\gamma \\
 &\quad + \rho_L g \int_{\Gamma_0} \frac{\partial\eta}{\partial t} (X_2 \theta_1 - X_1 \theta_2) d\gamma + \rho_L g \int_{\Gamma_0} \eta \left( X_2 \frac{d\theta_1}{dt} - X_1 \frac{d\theta_2}{dt} \right) d\gamma \\
 \frac{d}{dt}(T + V + U) &= \frac{d\xi}{dt} \cdot \left[ (m_S + m_L) g_0 + R - \int_{\Omega} \rho_L \nabla\left(\frac{\partial\varphi}{\partial t}\right) dx \right] \\
 &\quad + \frac{d\theta}{dt} \cdot \left[ m_L (x_F - \xi) \times g_0 + (x_A - \xi) \times R \right. \\
 &\quad \left. + \rho_L g \int_{\Gamma_0} \eta (-X_2 \varepsilon_1 + X_1 \varepsilon_2) d\gamma - \int_{\Omega} \rho_L (x - \xi) \times \nabla\left(\frac{\partial\varphi}{\partial t}\right) dx \right] \\
 &\quad + \int_{\Omega} \rho_L \nabla\varphi \cdot \nabla\left(\frac{\partial\varphi}{\partial t}\right) dx - m_S g_0 \cdot \frac{d\xi}{dt} - m_L g_0 \cdot u_F + \rho_L g \int_{\Gamma_0} \eta \frac{\partial\eta}{\partial t} d\gamma \\
 &\quad + \rho_L g \int_{\Gamma_0} \frac{\partial\eta}{\partial t} (X_2 \theta_1 - X_1 \theta_2) d\gamma + \rho_L g \int_{\Gamma_0} \eta \left( X_2 \frac{d\theta_1}{dt} - X_1 \frac{d\theta_2}{dt} \right) d\gamma \\
 \frac{d}{dt}(T + V + U) &= m_L g_0 \cdot \left[ \xi + \frac{d\theta}{dt} \times (x_F - \xi) - u_F \right] \\
 &\quad + R \cdot \left[ \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x_A - \xi) \right] - \rho_L \int_{\Omega} \left[ \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi) \right] \cdot \nabla\left(\frac{\partial\varphi}{\partial t}\right) dx \\
 &\quad + \rho_L \int_{\Omega} \nabla\varphi \cdot \nabla\left(\frac{\partial\varphi}{\partial t}\right) dx + \rho_L g \int_{\Gamma_0} \frac{\partial\eta}{\partial t} [\eta + X_2 \theta_1 - X_1 \theta_2] d\gamma \\
 \frac{d}{dt}(T + V + U) &= R \cdot u_A + \int_{\Omega(t)} \operatorname{div} \left( \frac{d\theta}{dt} \times x \right) \frac{\partial\varphi}{\partial t} dx \\
 &\quad - \rho_L \int_{\partial\Omega} \left( \frac{d\xi}{dt} + \frac{d\theta}{dt} \times (x - \xi) \right) \cdot n \frac{\partial\varphi}{\partial t} d\gamma + \rho_L \int_{\partial\Omega} \frac{\partial\varphi}{\partial n} \frac{\partial\varphi}{\partial t} d\gamma - \rho_L \int_{\Gamma_0} \frac{\partial\eta}{\partial t} \frac{\partial\varphi}{\partial t} d\gamma
 \end{aligned}$$

and the result is established due to the identity  $\operatorname{div} \left( \frac{d\theta}{dt} \times x \right) \equiv 0$  and to the Neuman boundary condition (22) for the potential  $\varphi$ .  $\square$

• **Proof of Proposition 5.**

The identity (46) is a direct consequence of relation (42):

$$\rho_L \int_{\Omega} \nabla\alpha_j = \rho_L \int_{\Omega} \varepsilon_j dx = m_L \varepsilon_j.$$

Using the Laplace equation  $\Delta\beta_j = 0$  and the totally antisymmetric tensor  $\varepsilon_{ijk}$  we have

$$\begin{aligned}
 \rho_L \int_{\Omega} \nabla\beta_j \cdot \varepsilon_i dx &= \rho_L \int_{\Omega} \nabla\beta_j \cdot \nabla x_i dx = \rho_L \int_{\partial\Omega} \frac{\partial\beta_j}{\partial n} x_i d\gamma = \rho_L \int_{\partial\Omega} \varepsilon_{j\ell q} (x - \xi)_{\ell} n_q x_i d\gamma \\
 &= \rho_L \int_{\Omega} \frac{\partial}{\partial x_q} (\varepsilon_{j\ell q} (x - \xi)_{\ell} x_i) dx = \rho_L \int_{\Omega} \varepsilon_{j\ell q} (x - \xi)_{\ell} \delta_{iq} dx = m_L \varepsilon_{ij\ell} (x_F - \xi)_{\ell}
 \end{aligned}$$

$$= m_L \varepsilon_{iq\ell} \delta_{qj} (x_F - \xi)_\ell = i^\circ \text{ component of } m_L \varepsilon_j \times (x_F - \xi)$$

and the relation (47) is established.

We have also, due to (42):

$$\rho_L \int_{\Omega} (x - \xi) \times \nabla \alpha_j dx = \rho_L \int_{\Omega} (x - \xi) \times \varepsilon_j dx = m_L (x_F - \xi) \times \varepsilon_j$$

which expresses the relation (48).

The proof of the fourth technical proposition (49) can be conducted as follows:

$$\begin{aligned} i^\circ \text{ component of } \rho_L \int_{\Omega} (x - \xi) \times \nabla \beta_j dx &= \rho_L \int_{\Omega} \varepsilon_{ik\ell} (x - \xi)_k \frac{\partial \beta_j}{\partial x_\ell} dx \\ &= \rho_L \int_{\partial\Omega} \varepsilon_{ik\ell} (x - \xi)_k n_\ell \beta_j d\gamma = \rho_L \int_{\partial\Omega} ((x - \xi) \times n)_i \beta_j d\gamma = \rho_L \int_{\partial\Omega} \frac{\partial \beta_i}{\partial n} \beta_j d\gamma \\ &= \rho_L \int_{\partial\Omega} (\nabla \beta_i \cdot n) \beta_j d\gamma = \rho_L \int_{\Omega} \operatorname{div}(\beta_j \nabla \beta_i) dx = \rho_L \int_{\Omega} \nabla \beta_j \cdot \nabla \beta_i dx \\ &\hspace{15em} \text{because } \Delta \beta_i = 0 \\ &= i^\circ \text{ component of } I_\ell \cdot \varepsilon_j. \end{aligned}$$

Due to the definition (44) of the Stokes-Zhukovsky vector fields for rotation, the field  $u \equiv \beta \cdot \theta$  is solution of the problem

$$(98) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \theta \times (x - \xi) \cdot n & \text{on } \partial\Omega. \end{cases}$$

Then taking into account the definition (45) of the liquid inertial tensor  $I_\ell$  we have

$$\begin{aligned} (\theta, I_\ell \cdot \theta) &= \rho_L \int_{\Omega} \theta \cdot [(x - \xi) \times \nabla u] dx = \rho_L \int_{\Omega} (\theta \times (x - \xi)) \cdot \nabla u dx \\ &= \rho_L \int_{\partial\Omega} (\theta \times (x - \xi) \cdot n) u d\gamma && \text{because } \operatorname{div}(\theta \times (x - \xi)) \equiv 0 \\ &= \rho_L \int_{\partial\Omega} u \frac{\partial u}{\partial n} d\gamma && \text{due to (98)} \\ &= \rho_L \int_{\partial\Omega} \frac{\partial}{\partial n} \left( \frac{u^2}{2} \right) d\gamma = \rho_L \int_{\Omega} \Delta \left( \frac{u^2}{2} \right) dx = \rho_L \int_{\Omega} |\nabla u|^2 dx \end{aligned}$$

because  $\Delta u = 0$  and the property (50) is established.

If the scalar product  $(\theta, I_\ell \cdot \theta)$  is null, then  $\nabla u = 0$  and the field  $u$  is a constant function. In particular the normal derivative  $\frac{\partial u}{\partial n}$  is null on the boundary  $\partial\Omega$  and due to (98),  $\theta \times (x - \xi) \cdot n_x = 0$  for all boundary points  $x \in \partial\Omega$ . Then the (constant) vector  $\theta \in \mathbb{R}^3$  is necessarily equal to zero. The proposition 5 is established.  $\square$

### • Proof of Proposition 6.

The proof of relation (53) can be conducted as follows:

$$\begin{aligned} \rho_L \int_{\Omega} \nabla \left( \frac{\partial^2 \psi}{\partial t^2} \right) \cdot \varepsilon_i dx &= \rho_L \int_{\Omega} \nabla \left( \frac{\partial^2 \psi}{\partial t^2} \right) \cdot \nabla \alpha_i dx = \rho_L \int_{\partial\Omega} \frac{\partial}{\partial n} \left( \frac{\partial^2 \psi}{\partial t^2} \right) \alpha_i d\gamma \\ &= \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} \alpha_i d\gamma. \end{aligned}$$

Then:

$$\rho_L \int_{\Omega} \nabla \left( \frac{\partial^2 \psi}{\partial t^2} \right) dx = \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} (x - x_0) d\gamma = \rho_L \int_{\Gamma_0} \frac{\partial^2 \eta}{\partial t^2} (X_1 \varepsilon_1 + X_2 \varepsilon_2) d\gamma$$

because  $x \bullet \varepsilon_3 = \text{const}$  on  $\Gamma_0$  and  $\int_{\Gamma_0} \eta d\gamma \equiv 0$ .

For the relation (54) we have

$$\begin{aligned} i^\circ \text{ component of } \rho_L \int_{\Omega} (x - \xi) \times \nabla \left( \frac{\partial^2 \psi}{\partial t^2} \right) dx &= \rho_L \int_{\Omega} \varepsilon_{ikl} (x - \xi)_k \partial_l \left( \frac{\partial^2 \psi}{\partial t^2} \right) dx \\ &= \rho_L \int_{\partial\Omega} \varepsilon_{ikl} (x - \xi)_k n_l \frac{\partial^2 \psi}{\partial t^2} d\gamma = \rho_L \int_{\partial\Omega} \frac{\partial \beta_i}{\partial n} \frac{\partial^2 \psi}{\partial t^2} d\gamma = \rho_L \int_{\partial\Omega} \nabla \beta_i \bullet n \frac{\partial^2 \psi}{\partial t^2} d\gamma \\ &= \rho_L \int_{\Omega} \text{div} \left( \frac{\partial^2 \psi}{\partial t^2} \nabla \beta_i \right) dx = \rho_L \int_{\Omega} \nabla \left( \frac{\partial^2 \psi}{\partial t^2} \right) \bullet \nabla \beta_i dx = \rho_L \int_{\partial\Omega} \beta_i \frac{\partial}{\partial n} \left( \frac{\partial^2 \psi}{\partial t^2} \right) d\gamma \\ &= \rho_L \int_{\Gamma_0} \beta_i \frac{\partial^2 \eta}{\partial t^2} d\gamma. \quad \square \end{aligned}$$

• **Proof of Proposition 7.**

We have with the previous notations:

$$\begin{aligned} (\eta', W \bullet \eta) &= \int_{\Gamma_0} \eta' \psi d\gamma = \int_{\partial\Omega} \frac{\partial \psi'}{\partial n} \psi d\gamma = \int_{\Omega} \text{div}(\psi \nabla \psi') dx && \text{due to Green formula} \\ &= \int_{\Omega} (\nabla \psi' \bullet \nabla \psi) dx && \text{because } \Delta \psi' = 0 \\ &= \int_{\partial\Omega} \psi' \frac{\partial \psi}{\partial n} d\gamma && \text{because } \Delta \psi = 0 \\ &= \int_{\Gamma_0} \psi' \eta d\gamma && \text{because } \frac{\partial \psi}{\partial n} = 0 \text{ on } \Sigma \text{ and } \frac{\partial \psi}{\partial n} = \eta \text{ on } \Gamma_0 \\ &= (W \bullet \eta', \eta). \quad \square \end{aligned}$$

• **Proof of Proposition 8.**

We have seen with (41) that  $\alpha_3 \equiv (x - x_0) \bullet \varepsilon_3$ . Then  $\alpha_3$  is a constant on the linearized free surface  $\Gamma_0$  and the property (65) is a direct consequence of the global incompressibility condition (20).

The relation (66) is proven as follows. The vector  $\xi \in \mathbb{R}^3$  is a constant on the surface  $\Gamma_0$ . Moreover  $\alpha_j \equiv (x - x_0) \bullet \varepsilon_j$  for  $j = 1, 2$ . Then

$$\int_{\Gamma_0} \eta (\alpha_j - X_j) d\gamma = 0$$

for each index  $j$  and by summation the result (66) is a consequence of the property (65).

The relation (68) is a direct consequence of the explicit expression (41). We have also

$$\begin{aligned} \int_{\Omega} \nabla \alpha_j \bullet \nabla \beta_k dx &= - \int_{\Omega} \alpha_j \bullet \Delta \beta_k dx + \int_{\partial\Omega} \alpha_j \frac{\partial \beta_k}{\partial n} d\gamma \\ &= \int_{\partial\Omega} \alpha_j \frac{\partial \beta_k}{\partial n} d\gamma && \text{because } \Delta \beta_k = 0 \end{aligned}$$

$$\begin{aligned}
 &= \int_{\partial\Omega} (x - x_0)_j ((x - \xi) \times n)_k \, d\gamma = \varepsilon_{k\ell m} \int_{\partial\Omega} (x - x_0)_j (x - \xi)_\ell n_m \, d\gamma \\
 &= \varepsilon_{k\ell m} \int_{\Omega} \frac{\partial}{\partial x_m} [(x - x_0)_j (x - \xi)_\ell] \, dx && \text{due to Green formula} \\
 &= \varepsilon_{k\ell m} \delta_{jm} \int_{\Omega} (x - \xi)_\ell \, dx = \varepsilon_{k\ell j} |\Omega| f_{0\ell} && \text{with } \ell_0 \equiv x_F - \xi \\
 &= |\Omega| (\varepsilon_j, \varepsilon_k, \ell_0)
 \end{aligned}$$

and the relation (68) is established. We have analogously:

$$\int_{\Omega} \nabla \alpha_j \cdot \nabla \psi \, dx = \int_{\partial\Omega} \alpha_j \frac{\partial \psi}{\partial n} \, d\gamma = \int_{\Gamma_0} \alpha_j \eta \, d\gamma$$

and the relation (69) is established. We have also:

$$\int_{\Omega} \nabla \beta_k \cdot \nabla \psi \, dx = \int_{\partial\Omega} \beta_k \frac{\partial \psi}{\partial n} \, d\gamma = \int_{\Gamma_0} \beta_k \eta \, d\gamma$$

and the relation (70) is clear. Finally, we have

$$\int_{\Omega} |\nabla \psi|^2 \, dx = \int_{\partial\Omega} \psi \frac{\partial \psi}{\partial n} \, d\gamma = \int_{\Gamma_0} \psi \eta \, d\gamma = \int_{\Gamma_0} (\eta, W \cdot \eta) \, d\gamma$$

and (71) is proven.  $\square$

### • Proof of Proposition 9.

We first establish the symmetry of the matrix  $M$ . We have:

$$\begin{aligned}
 (q', M \cdot q) &= \xi' \cdot (m_S + m_L) \xi - \xi' \cdot (m_L \ell_0 \times \theta) \\
 &\quad + \rho_L \xi' \cdot \int_{\Gamma_0} \alpha \eta \, d\gamma + \theta' \cdot (m_L \ell_0 \times \xi) + (\theta', (I_S + I_\ell) \cdot \theta) \\
 &\quad + \rho_L \int_{\Gamma_0} \theta' \cdot \beta \eta \, d\gamma + \rho_L \int_{\Gamma_0} \eta' (\alpha \cdot \xi + \beta \cdot \theta + \psi) \, d\gamma \\
 &= \xi' \cdot (m_S + m_L) \xi + m_L ((\ell_0, \xi', \theta) + (\ell_0, \xi, \theta')) \\
 &\quad + \rho_L \int_{\Gamma_0} (\eta \alpha \cdot \xi' + \eta' \alpha \cdot \xi) \, d\gamma + (\theta', (I_S + I_\ell) \cdot \theta) \\
 &\quad + \rho_L \int_{\Gamma_0} (\eta \beta \cdot \theta' + \eta' \beta \cdot \theta) \, d\gamma + \rho_L (\eta', W \cdot \eta) = (M \cdot q', q)
 \end{aligned}$$

because the previous expression is clearly symmetric.

From the expression (52) of the velocity potential we can omit the time derivatives and replace  $\varphi$  by the simple expression  $\tilde{\varphi} \equiv \alpha \cdot \xi + \beta \cdot \theta + W \cdot \eta$ . Then we have

$$\begin{aligned}
 \rho_L \int_{\Omega} |\nabla \tilde{\varphi}|^2 \, dx &= \rho_L \int_{\Omega} \left( \sum_j (\xi_j \nabla \alpha_j + \theta_j \nabla \beta_j) + \nabla \psi \right) \cdot \left( \sum_k (\xi_k \nabla \alpha_k + \theta_k \nabla \beta_k) + \nabla \psi \right) \, dx \\
 &= m_L \sum_{jk} \left[ (\varepsilon_j \cdot \varepsilon_k) \xi_j \xi_k + 2 \xi_j \theta_k (\varepsilon_j, \varepsilon_k, \ell_0) \right] + I_\ell \theta_j \theta_k \\
 &\quad + \rho_L \int_{\Gamma_0} \eta \left[ \psi + 2 \sum_j (\alpha_j \xi_j + \beta_j \theta_j) \right] \, d\gamma
 \end{aligned}$$

$$\begin{aligned}
 &= m_L |\xi|^2 + 2m_L(\xi, \theta, \ell_0) + (\theta, \mathbf{I}_\ell \bullet \theta) + \rho_L \int_{\Gamma_0} \eta (\psi + 2\alpha \bullet \xi + 2\beta \bullet \theta) d\gamma \\
 &= (q, M \bullet q) - m_S |\xi|^2 - (\theta, \mathbf{I}_S \bullet \theta)
 \end{aligned}$$

and the first relation (85) is established. When we replace the variable  $q \equiv (\xi, \theta, \eta)^t$  by its time derivative, the relation (86) relative to the kinetic energy is straightforward. It is clear that the operator matrix  $M$  is positive because the quadratic form  $(q, M \bullet q)$  is the sum of three positive terms. We establish now that  $M$  is a “definite operator”, that is if  $(q, M \bullet q)$  is null, then the vector  $q$  itself is reduced to zero. If we have  $(q, M \bullet q) = 0$ , each term of the sum (85) is null. Then, due to Proposition 5,  $\xi = \theta = 0$  and  $\nabla\psi = 0$  with  $\psi$  the free surface potential associated with the free surface  $\eta$ . Then  $\psi = 0$  and  $\eta = 0$  and  $M$  is positive definite.  $\square$

• **Proof of Proposition 10.**

The symmetry of the matrix  $K$  is elementary to establish. We have:

$$\begin{aligned}
 (q', K \bullet q) &= \rho_L g \left[ \theta' \bullet \int_{\Gamma_0} \eta \tilde{\alpha} d\gamma + \int_{\Gamma_0} \eta' \tilde{\alpha} \bullet \theta d\gamma + \int_{\Gamma_0} \eta' \eta d\gamma \right] \\
 &= \rho_L g \int_{\Gamma_0} (\theta' \bullet \tilde{\alpha} \eta + \eta \tilde{\alpha} \bullet \theta) d\gamma + \rho_L g (\eta', \eta)
 \end{aligned}$$

and this expression is symmetric when we exchange  $q$  and  $q'$ . When  $q' = q$  we have:

$$\begin{aligned}
 (q, K \bullet q) &= \rho_L g \left( 2 \int_{\Gamma_0} \theta \bullet \tilde{\alpha} \eta d\gamma + \|\eta\|^2 \right) \\
 &= \rho_L g \left[ 2 \int_{\Gamma_0} \eta (X_2 \theta_1 - X_1 \theta_2) d\gamma + \|\eta\|^2 \right].
 \end{aligned}$$

We have also:

$$\begin{aligned}
 \left| \int_{\Gamma_0} \eta X_2 \theta_1 d\gamma \right| &\leq |\theta_1| \int_{\Gamma_0} |\eta| |X_2| d\gamma && \text{because } \theta_1 \text{ is a constant on } \Gamma_0 \\
 &\leq |\theta_1| \|\eta\| \|X_2\| && \text{using the Cauchy-Schwarz inequality} \\
 &\leq |\theta_1| \|\eta\| a^2 && \text{by hypothesis (87)}
 \end{aligned}$$

and the analogous inequality  $\left| \int_{\Gamma_0} \eta X_1 \theta_2 d\gamma \right| \leq |\theta_2| \|\eta\| a^2$  for the other component. We deduce from the previous assessment the minoration:

$$\begin{aligned}
 (q, K \bullet q) &\geq \rho_L g \left[ \|\eta\|^2 - 2a^2 \|\eta\| (|\theta_1| + |\theta_2|) \right] \\
 &\geq \rho_L g \|\eta\| \left[ \|\eta\| - 2a^2 (|\theta_1| + |\theta_2|) \right]
 \end{aligned}$$

and this expression is positive when  $|\theta_1| + |\theta_2| \leq \frac{1}{2a^2} \|\eta\|$  which is exactly the hypothesis (90). The proposition is established.  $\square$

• **Proof of Proposition 11.**

We have:

$$T = \frac{1}{2} \left( \frac{dq}{dt}, M \bullet \frac{dq}{dt} \right) = \frac{1}{2} \left\{ \frac{d\xi}{dt} \bullet \left[ (m_S + m_L) \frac{d\xi}{dt} - m_L \ell_0 \times \frac{d\theta}{dt} + \rho_L \int_{\Gamma_0} \alpha \frac{\partial \eta}{\partial t} d\gamma \right] \right\}$$

$$\begin{aligned}
 & + \frac{d\theta}{dt} \cdot \left[ m_L \ell_0 \times \frac{d\xi}{dt} + (I_S + I_\ell) \frac{d\theta}{dt} + \rho_L \int_{\Gamma_0} \beta \frac{\partial \eta}{\partial t} d\gamma \right] \\
 & + \rho_L \int_{\Gamma_0} \frac{\partial \eta}{\partial t} \left[ \alpha \frac{d\xi}{dt} + \beta \frac{d\theta}{dt} + W \cdot \frac{\partial \eta}{\partial t} \right] d\gamma \}
 \end{aligned}$$

and the relation (91) is established. We have also:

$$\begin{aligned}
 U &= \frac{1}{2} (q, K \cdot q) = \frac{1}{2} \theta \cdot (\rho_L g) \int_{\Gamma_0} \tilde{\alpha} \eta d\gamma + \frac{1}{2} \rho_L \int_{\Gamma_0} \eta (\tilde{\alpha} \theta + \eta) g d\gamma \\
 &= \frac{1}{2} \rho_L g \int_{\Gamma_0} \eta^2 d\gamma + \rho_L g \int_{\Gamma_0} (\tilde{\alpha} \theta) \eta d\gamma
 \end{aligned}$$

and (92) follows. For the last term, we remark that the vector  $\ell_0$  is linked to the solid and

$$(99) \quad \frac{d\ell_0}{dt} = \frac{d\theta}{dt} \times \ell_0.$$

Then the right hand side of the relation (84) can be expressed in the following way:

$$\begin{aligned}
 F(t) \cdot \frac{dq}{dt} - \left[ R \cdot \frac{d\xi}{dt} + \left( x_A - \xi, R, \frac{d\theta}{dt} \right) \right] &= (m_S + m_L) g_0 \cdot \frac{d\xi}{dt} + m_L \left( \ell_0 \times g_0, \frac{d\theta}{dt} \right) \\
 &= \frac{d}{dt} [(m_S + m_L) g_0 \xi] + m_L g_0 \cdot \frac{d\ell_0}{dt} = \frac{d}{dt} [(m_S + m_L) g_0 \xi + m_L g_0 \ell_0]
 \end{aligned}$$

and the expression of the interaction potentiel with the external field is given according to (93). The proposition is established.  $\square$

### • Proof of Proposition 12.

It is sufficient to explicit the different terms of the equation (95) of the dynamics. We detail the three components of the expression  $\frac{\partial \mathcal{L}}{\partial \left( \frac{dq}{dt} \right)}$  relative to the vector  $\frac{dq}{dt} = \left( \frac{d\xi}{dt}, \frac{d\theta}{dt}, \frac{d\eta}{dt} \right)^t$ .

We have:

$$\begin{aligned}
 \frac{\partial \mathcal{L}}{\partial \left( \frac{d\xi}{dt} \right)} &= (m_S + m_L) \frac{d\xi}{dt} - m_L \left( \ell_0 \times \frac{d\theta}{dt} \right) + \rho_L \int_{\Gamma_0} \alpha \cdot \frac{\partial \eta}{\partial t} d\gamma \\
 \frac{\partial \mathcal{L}}{\partial \left( \frac{d\theta}{dt} \right)} &= (I_S + I_\ell) \cdot \frac{d\theta}{dt} + m_L \left( \ell_0 \times \frac{d\xi}{dt} \right) + \rho_L \int_{\Gamma_0} \beta \cdot \frac{\partial \eta}{\partial t} d\gamma \\
 \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \eta}{\partial t} \right)} &= \rho_L \int_{\Gamma_0} W \cdot \frac{\partial \eta}{\partial t} d\gamma + \rho_L \int_{\Gamma_0} \left( \alpha \cdot \frac{\partial \xi}{\partial t} + \beta \cdot \frac{\partial \theta}{\partial t} \right) d\gamma.
 \end{aligned}$$

Then the first term of the the equations of motion (84) is clear. On the other hand, we detail, in an analogous manner, the three components of the expression  $\frac{\partial \mathcal{L}}{\partial q} = -\frac{\partial U}{\partial q} -$

$\frac{\partial V}{\partial q}$  relative to the vector  $q = (\xi, \theta, \eta)^t$ . We have:

$$\begin{aligned}
 \frac{\partial \mathcal{L}}{\partial \xi} &= (m_S + m_L) g_0 \\
 \frac{\partial \mathcal{L}}{\partial \theta} &= -\rho_L g \int_{\Gamma_0} \tilde{\alpha} \eta d\gamma + m_L \frac{\partial}{\partial \theta} (g_0 \times \ell_0) = -\rho_L g \int_{\Gamma_0} \tilde{\alpha} \eta d\gamma - m_L g_0 \times \ell_0
 \end{aligned}$$

because, as  $\ell_0$  is linked to the solid, we have  $\delta \ell_0 = \delta \theta \times \ell_0$ . Last but not least,

$$\frac{\partial \mathcal{L}}{\partial \eta} \bullet \delta \eta = \rho_L g \int_{\Gamma_0} \eta \delta \eta d\gamma \quad \text{and the proposition is established.} \quad \square$$

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