Asymptotic-preserving methods for an anisotropic model of electrical potential in a tokamak

Philippe Angot, Thomas Auphan, Olivier Guès

Asymptotic-preserving methods for an anisotropic model of electrical potential in a tokamak

Philippe Angot, Thomas Auphan and Olivier Guès

Abstract A 2D nonlinear model for the electrical potential in the edge plasma in a tokamak generates a stiff problem due to the low resistivity in the direction parallel to the magnetic field lines. An asymptotic-preserving method based on a micro-macro decomposition is studied in order to have a well-posed problem, even when the parallel resistivity goes to 0. Numerical tests with a finite difference scheme show a bounded condition number for the linearised discrete problem solved at each time step, which confirms the theoretical analysis on the continuous problem.

Key words: Evolution problem, nonlinear anisotropic model, asymptotic-preserving method, numerical tests
MSC2010: 00B25, 41A60, 65M30

1 Introduction

The fusion reaction can be performed using a tokamak, a machine whose shape is toroidal. The plasma is confined and warmed in the core of the tokamak to produce the fusion reaction. This technique is expected to maintain the fusion reaction during a long time (more than five minutes, for the ITER project).

One of the main challenges for this objective is to control the wall-plasma interactions. Indeed, the magnetic confinement is not perfect and the plasma is in contact with the wall. In a tokamak such as TORE SUPRA, an obstacle called the limiter, is settled at the bottom of the machine. Due to the strong magnetic confinement, the plasma transport essentially occurs along the magnetic field lines. Thus, the parallel resistivity η is very small (typically, $\eta = 10^{-6}$), generating a strong anisotropy in the model. The area where the magnetic lines are interrupted by the limiter is called

Philippe Angot, Thomas Auphan · Olivier Guès
Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France
e-mail: [thomas.auphan, philippe.angot, olivier.gues]@univ-amu.fr

the scrape-off layer. The numerical simulation of the edge plasma transport allows us to better understand the interactions with the wall.

2 Anisotropic model of the electrical potential

In this paper, we focus on a 2D model of the electrical potential of the edge plasma ϕ_η in a tokamak with a limiter configuration. A schematic representation of the domain is given in Fig. 1. The x axis corresponds to the curvilinear coordinates along a magnetic field line and the y axis is the radial direction. In the following equations, the curvature terms have been neglected. As the magnetic field lines above the limiter set are closed, periodic boundary conditions are imposed at $x = \pm 0.5$. The dimensionless problem for the electrical potential reads:

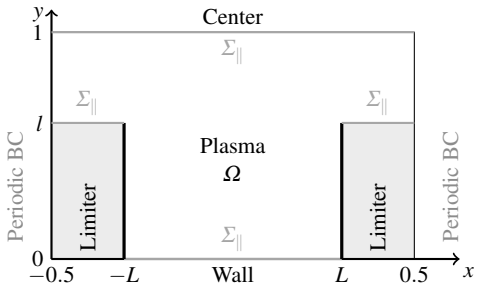

Fig. 1 Schematic representation of the 2D domain.

$$
\begin{cases}
-\partial_t \partial_y^2 \phi_\eta - \dfrac{1}{\eta}\partial_x^2 \phi_\eta + \nu \partial_y^4 \phi_\eta = S & \text{in }]0,T[\times\Omega \\
\partial_y \phi_{\eta|t=0} = \partial_y \phi_{ini} & \text{in } \Omega \\
\partial_y \phi_{\eta|\Sigma_\parallel} = 0 \quad \text{and} \quad \partial_y^3 \phi_{\eta|\Sigma_\parallel} = 0 & \text{on }]0,T[\times\Sigma_\parallel \\
\partial_x \phi_{\eta|x=-L} = \eta\left(1 - \mathrm{e}^{\Lambda - \phi_{\eta|x=-L}}\right) & \text{on }]0,T[\times]0,l[\times\{-L\} \\
\partial_x \phi_{\eta|x=L} = -\eta\left(1 - \mathrm{e}^{\Lambda - \phi_{\eta|x=L}}\right) & \text{on }]0,T[\times]0,l[\times\{L\},
\end{cases}
\tag{1}
$$

where ν corresponds to the ionic viscosity in the perpendicular direction and Λ stands for the reference potential inside the limiter. The initial condition is $\partial_y \phi_{\eta|t=0} = \partial_y \phi_{ini}$. Negulescu *et al.* [4] proved that, for a fixed value of $\eta > 0$, the problem (1) admits a unique weak solution, under suitable hypotheses on the data ϕ_{ini} and S.

The boundary conditions at the limiter interface $x = \pm L$, are nonlinear. Setting directly $\eta = 0$ in the system (1) (after multiplying the first equation by η) leads to an

under-determined problem since there are only homogeneous Neumann boundary conditions at the limiter surface $x = \pm L$. Thus, when η is small the numerical resolution of the problem (1) becomes stiff. This issue can be avoided by reformulating the problem (1) thanks to asymptotic-preserving methods.

3 The micro macro asymptotic-preserving method

We study the Asymptotic-Preserving (AP) method introduced by Degond *et al.* [3] for a linear anisotropic elliptic problem. It consists in a decomposition of the solution ϕ_η as $\phi_\eta = p_\eta + \eta q_\eta$ where $\partial_x p_\eta = 0$ and $q_{\eta|x=-L} = 0$. Then, it yields the problem below where the unknowns are (ϕ_η, q_η):

$$\begin{cases} -\partial_t \partial_y^2 \phi_\eta - \partial_x^2 q_\eta + \nu \partial_y^4 \phi_\eta = S & \text{in }]0,T[\times\Omega \\ \partial_x^2 \phi_\eta = \eta \partial_x^2 q_\eta & \text{in }]0,T[\times\Omega \\ \partial_x \phi_{\eta|x=-L} = \eta \partial_x q_{\eta|x=-L} & \text{on }]0,T[\times]0,l[\times\{-L\} \\ \partial_x \phi_{\eta|x=L} = \eta \partial_x q_{\eta|x=L} & \text{on }]0,T[\times]0,l[\times\{L\} \\ \partial_x \phi_{\eta|x=-0.5} = \eta \partial_x q_{\eta|x=-0.5} & \text{on }]0,T[\times]l,1[\times\{-0.5\} \\ \partial_x \phi_{\eta|x=0.5} = \eta \partial_x q_{\eta|x=0.5} & \text{on }]0,T[\times]l,1[\times\{0.5\} \\ \partial_y \phi_{\eta|t=0} = \partial_y \phi_{ini} & \text{in } \Omega \\ \partial_y \phi_{\eta|\Sigma_\parallel} = 0 \quad \text{and} \quad \partial_y^3 \phi_{|\Sigma_\parallel} = 0 & \text{on }]0,T[\times\Sigma_\parallel \\ \partial_x q_{\eta|x=-L} = \left(1 - \mathrm{e}^{\Lambda - \phi_{\eta|x=-L}}\right) & \text{on }]0,T[\times]0,l[\times\{-L\} \\ \partial_x q_{\eta|x=L} = -\left(1 - \mathrm{e}^{\Lambda - \phi_{\eta|x=L}}\right) & \text{on }]0,T[\times]0,l[\times\{L\}, \end{cases} \tag{2}$$

One important advantage of this AP method is that it can be easily implemented even if the mesh is not aligned with the directions (Ox) and (Oy). The main drawback is the need to compute two unknowns (ϕ_η and q_η) on the 2D domain though only ϕ_η is interesting for the physics.

Let us give the theoretical result which ensures that the modified problem is well-posed for $\eta = 0$, and that ϕ_η converges towards ϕ_0. First, we provide the definitions of the spaces used for the variational formulation of the problem (2).

Definition 1. Let us define the following Hilbert spaces:

- $V = \left\{ f \in H^1(\Omega), \partial_y^2 f \in L^2(\Omega), f \text{ periodic on } \{-0.5, 0.5\}\times]l,1[, \partial_y f = 0 \text{ on } \Sigma_\parallel \right\}$ with the scalar product:

$$\langle f, u \rangle_V = \int_\Omega \partial_x f \, \partial_x u \, dy dx + \int_\Omega \partial_y^2 f \, \partial_y^2 u \, dy dx + 2 \int_0^l f_{|x=L} \, u_{|x=L} \, dy.$$

- $Q = \left\{ f \in L^2(\Omega), \partial_x f \in L^2(\Omega), f_{|x=-L} = 0 \text{ on }]0,1[\right\}$, with the scalar product:

$$\langle f,u\rangle_Q = \int_\Omega \partial_x f\, \partial_x u\, dydx.$$

Definition 2. The space $\mathscr{A}$ is the set of functions ϕ such that:

- $\phi \in L^2(0,T;V)$.
- $\partial_y\phi \in L^\infty(0,T;L^2(\Omega))$.
- $\partial_y\phi \in L^2\left(0,T;\{f\in H^1(\Omega), \partial_y^2 f\in L^2(\Omega), f_{|\Sigma_\parallel}=0\}\right)$.
- $\partial_y^2\phi \in L^\infty(0,T;L^2(\Omega))$.
- $\partial_t\phi \in L^2(0,T;V)$.
- $\partial_y\partial_t\phi \in L^\infty(0,T;L^2(\Omega))$.

The weak solution ϕ_η of (2) is then searched in the space $\mathscr{A}$.

Assumption 3.1 *Assume that S and ϕ_{ini} verify:*

1. $S, \partial_y S, \partial_y^2 S, \partial_t S, \partial_t^2 S \in L^2(]0,T[\times\Omega)$, $\|S\|_{L^\infty(]0,T[\times\Omega)} \le C_s$ *and* $\|S_{|t=T}\|_{L^\infty(\Omega)} \le C_s$ *with C_s sufficiently small.*
2. $\phi_{ini} \in H^4(\Omega)$.
3. ϕ_{ini} *does not depend on x.*
4. $\displaystyle\int_\Omega S_{|t=0}\, dydx = \nu \int_\Omega \partial_y^4\phi_{ini}\, dydx + 2\int_0^l \left(1-\mathrm{e}^{\Lambda-\phi_{ini|x=L}}\right) dy.$

The two last hypotheses are compatibility conditions for the initial and boundary conditions with the source term.

We can now write the theorem which asserts the convergence of ϕ_η to ϕ_0 when η goes to 0:

Theorem 1. *With the assumption 3.1, the weak formulation of (2):*
find $(\phi_\eta, q_\eta) \in \mathscr{A}\times L^2(0,T;Q)$ verifying

$$\begin{cases} \forall \xi \in H^1(]0,T[), \forall u \in V\cap H^2(\Omega), \forall w \in Q, \\ \displaystyle\int_\Omega \partial_y\phi_{\eta|t=T}\, \partial_y u\, dydx\xi(T) - \int_0^T\int_\Omega \partial_y\phi_{\eta|t=T}\, \partial_y u\, dydx\, \xi'\, dt \\ \quad + \displaystyle\int_0^T\int_\Omega \partial_x q_\eta\, \partial_x u\, dydx\, \xi\, dt + \nu\int_0^T\int_\Omega \partial_y^2\phi_\eta\, \partial_y^2 u\, dydx\, \xi\, dt \\ \quad + \displaystyle\int_0^T\int_0^l \left(1-\mathrm{e}^{\Lambda-\phi_{\eta|x=-L}}\right) u_{|x=-L}\, dy\, \xi\, dt + \int_0^T\int_0^l \left(1-\mathrm{e}^{\Lambda-\phi_{\eta|x=L}}\right) u_{|x=L}\, dy\, \xi\, dt \\ \quad = \displaystyle\int_\Omega \partial_y\phi_{ini}\, \partial_y u\, dydx\, \xi(0) + \int_0^T\int_\Omega S u\, dydx\, \xi\, dt \\ \eta\displaystyle\int_0^T\int_\Omega \partial_x q_\eta\, \partial_x w\, dydx\, \xi\, dt = \int_0^T\int_\Omega \partial_x\phi_\eta\, \partial_x w\, dydx\, \xi\, dt, \end{cases} \tag{3}$$

admits a unique solution. Besides, (ϕ_η, q_η) converges weakly in $L^2(]0,T[\times\Omega)^2$, towards $(\phi_0,q_0)\in\mathscr{A}\times L^2(0,T;Q)$ the solution of (3) when η equals 0.

Finally, the following error estimate holds:

$$\|\phi_\eta - \phi_0\|_{L^1(0,T;L^2(\Omega))} \leq c(T,\Omega,\phi_0,S,\Lambda)\sqrt{\eta},$$

where $c(T,\Omega,\phi_0,S,\Lambda) > 0$ *does not depend on* η.

Theorem 1 provides an error estimate for the norm in $L^1(0,T;L^2(\Omega))$, but not for the $L^2(]0,T[\times\Omega)$ norm. This point can be subject to further improvements.

This result is shown in [1, 2]. The proof of the existence and uniqueness of ϕ_0 follows the same steps of [4], based on a fixed point method. The existence and uniqueness of q_0 and the convergence of (ϕ_η, q_η) when η goes to 0 are shown by extending to a nonlinear case the proof provided in [3] for a linear elliptic problem.

4 Numerical experiments

In this section, some numerical tests are presented for the system (2). The space discretisation is done by the centred finite difference scheme. The time resolution uses Euler semi-implicit method.

At first glance, a directional splitting method seems to be interesting. But, the discrete problems obtained in the directions x and y are not invertible. The problem is thus discretised implicitly, except for the nonlinear term. At each time step, a linear system has to be solved to compute the approximations of ϕ_η and q_η.

Let us consider a rectangular mesh of the space domain Ω with a constant mesh step δx (for the direction (Ox)) and δy (for the direction (Oy)). The time step writes δt. The scalar quantities $\phi^n_{i,j}, q^n_{i,j}$ stands respectively for the approximations of $\phi_\eta(n\delta t, -0.5 + i\delta x, j\delta y)$ and $q_\eta(n\delta t, -0.5 + i\delta x, j\delta y)$. The boundary condition at $x = -L$ is discretised as:

$$\frac{q^{n+1}_{I_1+1,j} - q^{n+1}_{I_1-1,j}}{2\delta x} - \phi^{n+1}_{I_1,j} = \left(1 - \mathrm{e}^{\Lambda - \phi^n_{I_1,j}} - \phi^n_{I_1,j}\right),$$

where I_1 is the index such that $-0.5 + I_1\delta x = -L$.

For the boundary condition at $x = L$, the same technique is used. This time linearisation enables us to have an invertible matrix which is the same at each time step.

The mesh convergence test is performed using a configuration where the limiter goes up to the top of the computational domain, *i.e.* $l = 1$. This does not change the results proven for $l < 1$. For $L = 0.4$, the chosen manufacturated solution is

$$\phi_\eta(t,x,y) = \eta\left(\frac{t}{\pi}\right)^2 \cos(\pi y)\cos(1.25\pi x) - \ln\left(1 - \frac{1.25t^2}{\pi\cos(\pi y)}\right) + \Lambda. \qquad (4)$$

Let us note that the source term S associated to the manufactured solution (4) depends on η but is not singular when η goes to 0. This differs from the hypotheses made for Theorem 1.

The plot of the approximated solution is shown in Fig. 2. Studying the L^2 error in Fig. 3, we observe that the numerical scheme is of second-order accuracy in space.

In Fig. 4, we observe that the condition number obtained with the AP method is high but it is bounded independently from η. This is not the case for the matrix obtained for the resolution of (1) without the asymptotic-preserving method. In order to avoid the issues due to the bad conditioning, we choose a LU method to solve the linear problem at each time step, which is faster than a GMRES solver with PETSc library. Finding an efficient preconditioner in order to use iterative methods is a future enhancement of this work.

For the convergence when η tends to 0, the same domain is considered ($l = 1, L = 0.4$) but another source term is chosen :

$$S(t,x,y) = 40t\cos(2\pi y)\sin\left(\frac{\pi}{2L}x\right) \quad , \quad \phi_{ini}(x,y) = \Lambda = 0 \tag{5}$$

This configuration (5) with $l = 1$ leads to $\phi_0(t,x,y) = 0$, which enables us to compute numerically $\|\phi_\eta - \phi_0\|_{L^1(0,T;L^2(\Omega))}$ and $\|\phi_\eta - \phi_0\|_{L^2(0,T;L^2(\Omega))}$. For these two norms, we observe a convergence in $\mathcal{O}(\eta)$, see Fig. 5. This suggests that the estimate of Theorem 1 might be improved.

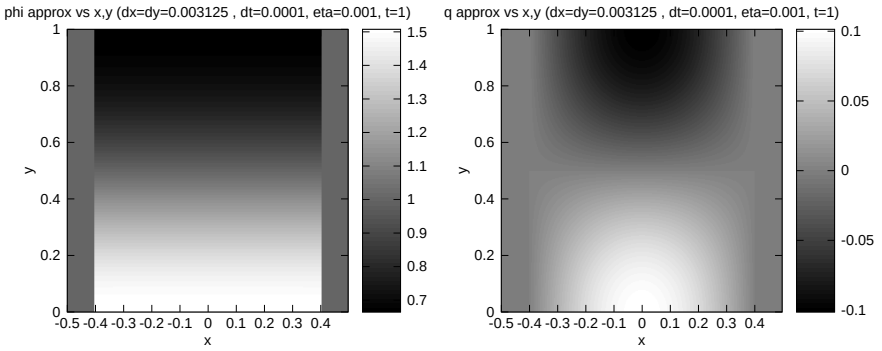

Fig. 2 Approximate fields of ϕ_η and q_η for $\delta x = \delta y = 0.003125$, $\delta t = 0.0001$ and $\eta = 0.001$. The reference solution is given by (4). Recall that the limiter area corresponds to $x \leq -0.4$ and $x \geq 0.4$: the values of ϕ_η do not have any physical sense in this zone.

5 Conclusion

The high anisotropy of the 2D model for the edge plasma electrical potential in a tokamak leads to an ill-conditioned matrix for the numerical approximation using classical methods. The micro-macro decomposition induced by Degond *et al.* [3] for a linear anisotropic elliptic problem is studied and analysed for the nonlinear evolution problem of the electrical potential. This method yields a weak formulation which is not degenerated when the parallel resistivity η tends to 0. Moreover, we

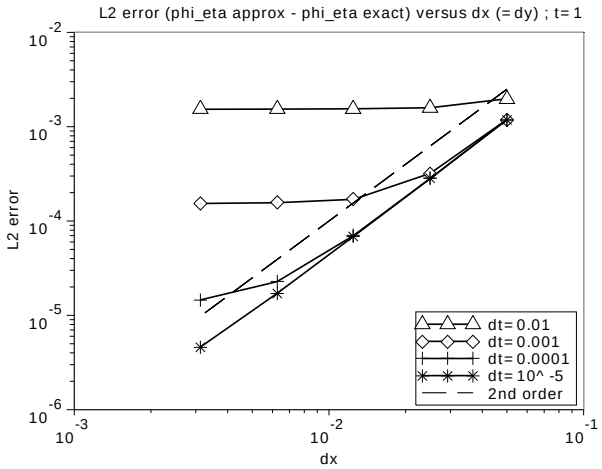

Fig. 3 $\|\phi_\eta^{approx} - \phi_\eta\|_{L^2(\Omega)}$ at $t = 1$ as a function of the space step $\delta x = \delta y$ for different values of the time step and $\eta = 0.001$. The reference solution is given by (4).

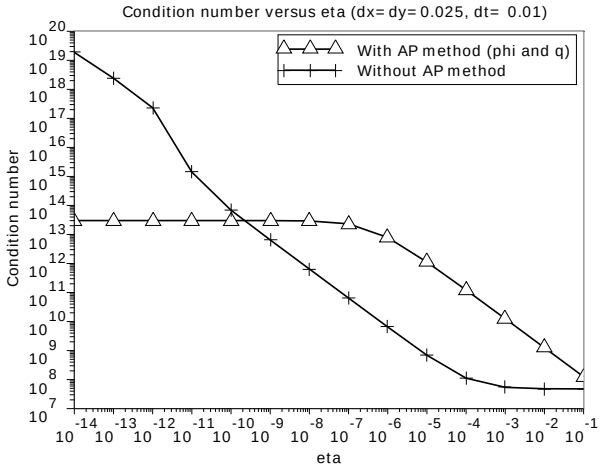

Fig. 4 Condition number in the Euclidean norm as a function of the parallel resistivity η for the linear system approaching (the same at each time step) the solution (4) with $\delta x = \delta y = 0.025$ and $\delta t = 0.001$.

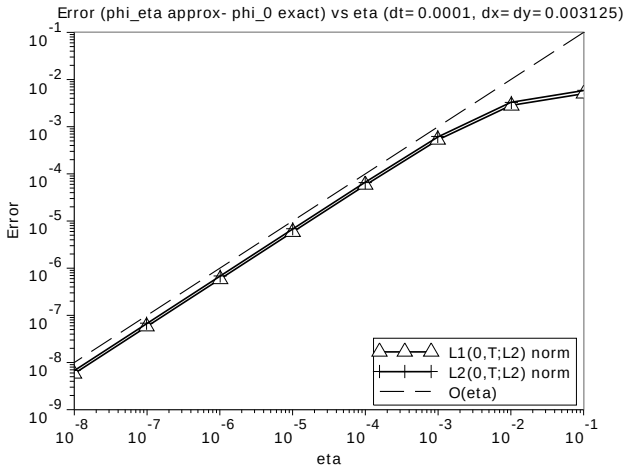

Fig. 5 $\|\phi_\eta - \phi_0\|_{L^1(0,T;L^2(\Omega))}$ (Δ) and $\|\phi_\eta - \phi_0\|_{L^2(]0,T[\times\Omega)}$ ($+$) as a function of η. The configuration is given by Eqs. (5) with $T = 1$, $\delta x = \delta y = 0.003125$ and $\delta t = 0.0001$.

have the estimate

$$\|\phi_\eta - \phi_0\|_{L^1(0,T,L^2(\Omega))} = \mathscr{O}(\sqrt{\eta}),$$

which can probably be improved, as suggested by the numerical results.

Acknowledgements: This work has been funded by the ANR ESPOIR (Edge Simulation of the Physics Of ITER Relevant turbulent transport)and the *Fédération nationale de Recherche Fusion par Confinement Magnétique* (FR-FCM). We thank Eric Serre, Frédéric Schwander, Guillaume Chiavassa, Philippe Ghendrih and Patrick Tamain for fruitful discussions.

References

1. Angot, P., Auphan, T., Guès, O.: Analysis of asymptotic preserving methods for nonlinear anisotropic models of electrical potential in plasma. in preparation (2014)
2. Auphan, T.: Analyse de modèles pour ITER ; Traitement des conditions aux limites de systèmes modélisant le plasma de bord dans un tokamak. Ph.d. thesis in mathematics, Aix Marseille Université (2014)
3. Degond, P., Lozinski, A., Narski, J., Negulescu, C.: An asymptotic-preserving method for highly anisotropic elliptic equations based on a micromacro decomposition. Journal of Computational Physics **231**(7), 2724 – 2740 (2012)
4. Negulescu, C., Nouri, A., Ghendrih, P., Sarazin, Y.: Existence and uniqueness of the electric potential profile in the edge of tokamak plasmas when constrained by the plasma-wall boundary physics. Kinetic and Related Models **1**(4) (2008)