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# APPROXIMATION OF ALMOST TIME AND BAND LIMITED FUNCTIONS I: HERMITE EXPANSIONS

PHILIPPE JAMING, ABDERRAZEK KAROU, RON KERMAN, SUSANNA SPEKTOR

ABSTRACT. The aim of this paper is to investigate the quality of approximation of almost time and band limited functions by its expansion in the Hermite and scaled Hermite basis. As a corollary, this allows us to obtain the rate of convergence of the Hermite expansion of function in the  $L^2$ -Sobolev space with fixed compact support.

## 1. INTRODUCTION

The aim of this paper is to investigate the quality of approximation of almost time and band limited functions by its expansion in the Hermite basis. As a corollary, this allows us to obtain the rate of convergence of the Hermite expansion of function in the  $L^2$ -Sobolev space with fixed compact support.

Time-limited functions and band-limited functions play a fundamental role in signal and image processing. The time-limiting assumption is natural as a signal can only be measured over a finite duration. The band-limiting assumption is natural as well due to channel capacity limitations. It is also essential to apply sampling theory. Unfortunately, the simplest form of the uncertainty principle tells us that a signal can not be simultaneously time and band limited. A natural assumption is thus that a signal is almost time and band limited in the following sense:

**Definition.** Let  $T, \Omega > 0$  and  $\varepsilon_T, \varepsilon_\Omega > 0$ . A function  $f \in L^2(\mathbb{R})$  is said to be

- $\varepsilon_T$  almost-time limited to  $[-T, T]$  if

$$\int_{|t|>T} |f(t)|^2 dt \leq \varepsilon_T^2 \|f\|_{L^2(\mathbb{R})}^2;$$

- $\varepsilon_\Omega$  almost-band limited to  $[-\Omega, \Omega]$  if

$$\int_{|\omega|>\Omega} |\widehat{f}(\omega)|^2 d\omega \leq \varepsilon_\Omega^2 \|f\|_{L^2(\mathbb{R})}^2.$$

Here and throughout this paper the Fourier transform is normalized so that, for  $f \in L^1(\mathbb{R})$ ,

$$\widehat{f}(\omega) := \mathcal{F}[f](\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\omega} dt.$$

Of course, given  $f \in L^2(\mathbb{R})$ , for every  $\varepsilon_T, \varepsilon_\Omega > 0$  there exist  $T, \Omega > 0$  such that  $f$  is  $\varepsilon_T$  almost-time limited to  $[-T, T]$  and  $\varepsilon_\Omega$  almost-time limited to  $[-\Omega, \Omega]$ . The point here is that we consider  $T, \Omega, \varepsilon_T, \varepsilon_\Omega$  as fixed parameters. A typical example we have in mind is that  $f \in H^s(\mathbb{R})$  and is time-limited to  $[-T, T]$ . Such an hypothesis is common in tomography, see e.g. [Na], where it is required in the proof of the convergence of the filtered back-projection

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algorithm for approximate inversion of the Radon transform. But, if  $f \in H^s(\mathbb{R})$  with  $s > 0$ , that is if

$$\|f\|_{H^s(\mathbb{R})}^2 := \int_{\mathbb{R}} (1 + |\omega|)^{2s} |\widehat{f}(\omega)|^2 d\omega < +\infty,$$

then

$$\begin{aligned} \int_{|\omega| > \Omega} |\widehat{f}(\omega)|^2 d\omega &\leq \int_{|\omega| > \Omega} \frac{(1 + |\omega|)^{2s}}{(1 + |\Omega|)^{2s}} |\widehat{f}(\omega)|^2 d\omega \\ &\leq \frac{\|f\|_{H^s(\mathbb{R})}^2}{(1 + |\Omega|)^{2s}}. \end{aligned}$$

Thus  $f$  is  $\frac{1}{(1 + |\Omega|)^s} \frac{\|f\|_{H^s}}{\|f\|_{L^2(\mathbb{R})}}$ -almost band limited to  $[-\Omega, \Omega]$ .

An alternative to the back projection algorithms in tomography are the Algebraic Reconstruction Techniques (that is variants of Kaczmarz algorithm, *see* [Na]). For those algorithms to work well it is crucial to have a good representing system (basis, frame...) of the functions that one wants to reconstruct. Thanks to the seminal work of Landau, Pollak and Slepian, the optimal orthogonal system for representing almost time and band limited functions is known. The system in questions consists of the so called prolate spheroidal wave functions  $\psi_k^T$  and has many valuable properties (see [SP, LP1, LP2, Sl1]). Among the most striking properties they have is that, if a function is almost time limited to  $[-T, T]$  and almost band limited to  $[-\Omega, \Omega]$  then it is well approximated by its projection on the first  $8\Omega T$  terms of the basis:

$$(1.1) \quad f \simeq \sum_{0 \leq k < 8\Omega T} \langle f, \psi_k^T \rangle \psi_k^T.$$

This is a remarkable fact as this is exactly the heuristics given by Shannon's sampling formula (note that to make this heuristics clearer, the functions are usually almost time-limited to  $[-T/2, T/2]$  and this result is then known as the  $4\Omega T$ -theorem).

However, there is a major difficulty with prolate spheroidal wave functions that has attracted a lot of interest recently, namely the difficulty to compute them as there is no inductive nor closed form formula (see e.g. [BK1, BK2, Bo, LKL, XRY]). One approach is to explicitly compute the coefficients of the prolate spheroidal wave functions in terms of a basis of orthogonal polynomials like the Legendre polynomials or in the Hermite basis. The question that then arises is that of directly approximating almost time and band limited functions by the (truncation of) their expansion in the Hermite basis. This is the question we address here. We postpone the same question concerning Legendre polynomials for which we use different methods.

An other motivation for this work comes from the work of the first author [JP] on uncertainty principles for orthonormal bases. There it is shown that an orthonormal basis  $(e_k)$  of  $L^2(\mathbb{R})$  can not have uniform time-frequency localization. Several ways of measuring localization were considered, and for most of them, the Hermite functions provided the optimal behavior. However, in one case, the proof relied on (1.1): this shows that the set of functions that are  $\varepsilon_T$ -time limited to  $[-T, T]$  and  $\varepsilon_\Omega$ -band limited to  $[-\Omega, \Omega]$  is almost of dimension  $8\Omega T$ . In particular, this set can not contain more than a fixed number of elements of an orthonormal sequence. As this proof shows, the optimal basis here consists of prolate spheroidal wave functions. As the Hermite basis is optimal for many uncertainty principles, it is thus natural to ask how far it is from optimal in this case.

Let us now be more precise and describe the main results of the paper.

Recall that the hermite basis  $(h_k)_{k \geq 0}$  is an orthonormal basis of  $L^2(\mathbb{R})$  given by  $h_k = \alpha_k e^{x^2/2} \partial^k e^{-x^2}$  where  $\alpha_k$  is a normalization constant. Recall also that the  $h_k$ 's are eigenfunction of the Fourier transform. Moreover, as is well known the  $h_k$ 's satisfy a second order differential equation. This allows us to use the standard WKB method to approximate the Hermite functions as follows: let  $\lambda = \sqrt{2n+1}$ ,  $p(x) = \sqrt{\lambda^2 - x^2}$  and  $\varphi(x) = \int_0^x p(t) dt$ , then, for  $|x| < \lambda$ ,

$$(1.2) \quad h_k(x) = h_n(0) \sqrt{\frac{\lambda}{p(x)}} \cos \varphi(x) + h'_n(0) \frac{\sin \varphi(x)}{\sqrt{\lambda p(x)}} + \text{error}.$$

This formula is not new (e.g. [Do, KT, LC]). However, we will need a precise estimate of the error term, both in the  $L^\infty$  sense for which we improve the one given in [BKH] and the Lipschitz bound.

A first consequence of this formula is that the  $L^2$ -mass of  $h_n$  is essentially concentrated in an annulus of radius  $\sqrt{2n+1}$  and width  $\leq 1$  of the time-frequency plane. The heuristic is then as follows. Assume that  $f$  is almost time and band limited in  $[-T, T] \times [-\Omega, \Omega]$ . Thus  $f$  is only ‘‘correlated’’ to the first  $\sim N := \max(T^2, \Omega^2)$  Hermite functions:  $|\langle f, h_k \rangle|$  is small if  $k > N$ . One may thus expect that  $f = \sum_{0 \leq k \leq N} \langle f, h_k \rangle h_k + \text{error}$ . This seems unfortunately not to be the case. We establish that for  $n \geq N$ , the  $\text{error} = f - \sum_{0 \leq k \leq N} \langle f, h_k \rangle h_k$  has an  $L^2$ -norm bounded by  $\lesssim T^3 / \sqrt{2n+1}$ :

**Theorem 1.1.** *Let  $\Omega_0, T_0 \geq 2$  and  $\varepsilon_T, \varepsilon_\Omega > 0$ . Assume that*

$$\int_{|t| > T_0} |f(t)|^2 dt \leq \varepsilon_T^2 \|f\|_{L^2(\mathbb{R})}^2 \quad \text{and} \quad \int_{|\omega| > \Omega_0} |\widehat{f}(\omega)|^2 d\omega \leq \varepsilon_\Omega^2 \|f\|_{L^2(\mathbb{R})}^2.$$

*For  $n$  an integer, let  $K_n f$  be the orthogonal projection of  $f$  on the span of  $h_0, \dots, h_n$ .*

*Assume that  $n \geq \max(2T^2, 2\Omega^2)$ . Then, for  $T \geq T_0$ ,*

$$(1.3) \quad \|f - K_n f\|_{L^2([-T, T])} \leq \left( \varepsilon_\Omega + \frac{34T^3}{\sqrt{2n+1}} + 2\varepsilon_T \right) \|f\|_{L^2(\mathbb{R})}$$

In particular, one would need  $\sim T^6 / \varepsilon^2$  terms to reach an error  $\lesssim \varepsilon$ . The above heuristics suggest that the right power of  $T$  may (1.3) should be closer to 1. In order to prove this result, we approximate the projection on the  $n$  first Hermite functions by the time-frequency localization operator: as is well known, this projection is given by an integral kernel:

$$\sum_{k=0}^n \langle h_k, f \rangle h_k(x) = \int_{\mathbb{R}} k_n(x, y) f(y) dy.$$

We then use (1.2) and the Christoffel-Darboux formula to approximate the kernel  $k_n$  on  $[-T, T] \times [-T, T]$  when  $n \geq 2T^2$ :

$$(1.4) \quad k_n(x, y) = \frac{1}{\pi} \frac{\sin \sqrt{2n+1}(x-y)}{x-y} + \text{error}.$$

Again, this approximation is not new [Sa, Us] but we improve the error estimates.

In order to improve on our result and obtain a reasonable approximation we *rescale* the problem. Indeed, if we consider  $f_T(x) = T^{1/2} f(Tx)$  then  $f_T$  is almost time and band limited to  $[-1, 1] \times [\Omega T]$ . Now the factor  $T^3$  in the previous result becomes harmless as we have rescaled  $T$  to 1. We will indeed show that  $f_T$  is well approximated by its projection on the first  $(\Omega T)^2$  scaled Hermite functions  $h_k^T(x) = T^{-1/2} h_k(x/T)$ :

**Theorem 1.2.** *Let  $\Omega, T \geq 2$  and  $\varepsilon_T, \varepsilon_\Omega > 0$ . Assume that*

$$\int_{|t|>T} |f(t)|^2 dt \leq \varepsilon_T^2 \|f\|_{L^2(\mathbb{R})}^2 \quad \text{and} \quad \int_{|\omega|>\Omega} |\widehat{f}(\omega)|^2 d\omega \leq \varepsilon_\Omega^2 \|f\|_{L^2(\mathbb{R})}^2.$$

*Then, for  $n \geq 2T^2\Omega^2$ .*

$$\|f - K_n^T f\|_{L^2([-T, T])} \leq \left( \varepsilon_T + \varepsilon_\Omega + \frac{34}{\sqrt{2n+1}} \right) \|f\|_{L^2(\mathbb{R})}.$$

As an immediate corollary, we obtain the following:

**Corollary 1.3.** *Let  $s > 0$  and  $T \geq 2$  and let  $f \in H^s(\mathbb{R})$  with support in  $[-T, T]$ . Then for every  $n \geq T^2\Omega^2$*

$$\|f - K_n^T f\|_{L^2([-T, T])} \leq \left( \frac{1}{(1+|\Omega|)^s} \frac{\|f\|_{H^s}}{\|f\|_{L^2(\mathbb{R})}} + \frac{34}{\sqrt{2n+1}} \right) \|f\|_{L^2(\mathbb{R})}.$$

The remaining of this paper is organized as follows. The next section is devoted to the approximation of Hermite functions by the WKB method. We then devote Section 3.1 and 3.2 to establish properties of the kernel of the projection on the Hermite functions. We then prove Theorem 1.1 in Section 3.3. Finally, Theorem 1.2 is proved in Section 3.4

## 2. APPROXIMATING HERMITE FUNCTIONS WITH THE WKB METHOD

**2.1. The WKB method.** Let  $H_n$  be the  $n$ -th Hermite polynomial, that is

$$H_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Define the Hermite functions as

$$h_n(x) = \alpha_n H_n(x) e^{-x^2/2} \quad \text{where} \quad \alpha_n = \frac{1}{\pi^{1/4} \sqrt{2^n n!}}.$$

As is well known:

- (i)  $(h_n)_{n \geq 0}$  is an orthonormal basis of  $L^2(\mathbb{R})$ .
- (ii)  $h_n$  is even if  $n$  is even and odd if  $n$  is odd.
- (iii)  $h_{2p}(0) = \frac{(-1)^p}{\pi^{1/4}} \sqrt{\frac{(2p-1)!!}{(2p)!!}} = \frac{(-1)^p}{\sqrt{\pi} p^{1/4}} \left(1 - \frac{\eta_{2p}}{8p}\right)$  with  $0 < \eta_{2p} < 1$ , while  $h'_{2p}(0) = 0$ .
- (iv)  $h_{2p+1}(0) = 0$  while  $h'_{2p+1}(0) = \frac{(-1)^p \sqrt{4p+2}}{\pi^{1/4}} \sqrt{\frac{(2p-1)!!}{(2p)!!}} = \frac{(-1)^p \sqrt{4p+3}}{\sqrt{\pi} p^{1/4}} \left(1 - \frac{\eta_{2p+1}}{4p}\right)$  with  $|\eta_{2p+1}| < 1$ .
- (v)  $h_n$  satisfies the differential equation

$$(2.5) \quad h_n''(x) + (2n+1-x^2)h_n(x) = 0.$$

We will now follow the WKB method to obtain an approximation of  $h_n$ .

In order to simplify notation, we will fix  $n$  and drop all subscripts during the computation. Let  $h = h_n$ ,  $\lambda = \sqrt{2n+1}$ , and define  $p(x)$  and  $\varphi(x)$  for  $|x| < \lambda$  as

$$p(x) = \sqrt{\lambda^2 - x^2} \quad \text{and} \quad \varphi(x) = \int_0^x p(t) dt.$$

Note that (2.5) reads  $h''(x) + p(x)h(x) = 0$ . Let us define

$$\psi_\pm(x) = \frac{1}{\sqrt{p(x)}} \exp \pm i\varphi(x) \quad |x| < \lambda.$$

**Remark.** These functions are introduced according to the standard WKB method. The factor  $\exp \pm i\varphi(x)$  would be the solution of (2.5) if  $p$  were a constant. The factor  $p^{-1/2}$  is here to make the wronskian of  $\psi_+, \psi_-$  constant. Indeed, as  $\varphi' = p$ , a simple computation shows that

$$(2.6) \quad \psi'_{\pm}(x) = \left( -\frac{1}{2} \frac{p'(x)}{p(x)} \pm ip(x) \right) \psi_{\pm}(x).$$

It follows that

$$(2.7) \quad \begin{aligned} \psi_+(x)\psi'_-(x) - \psi_-(x)\psi'_+(x) &= \psi_+(x)\psi_-(x) \left| -\frac{1}{2} \frac{p'(x)}{p(x)} + ip(x) \quad -\frac{1}{2} \frac{p'(x)}{p(x)} - ip(x) \right| \\ &= \frac{1}{p(x)} \times (-2ip(x)) = -2i. \end{aligned}$$

Using (2.6) it is not hard to see that  $\psi_{\pm}$  both satisfy the differential equation

$$(2.8) \quad y'' + (p^2 - q)y = 0 \quad \text{where } q = \frac{1}{2} \left( \frac{p'}{p} \right)' - \frac{1}{4} \left( \frac{p'}{p} \right)^2.$$

A simple computation shows that  $q(x) = -\frac{2\lambda^2 + 3x^2}{4p(x)^4}$ . We will frequently use that  $|q(x)| \leq \frac{5\lambda^2}{4p(x)^4}$ . Note also that, if  $0 < \eta < 1$  and  $|x| \leq \lambda(1 - \eta)$  then  $p(x) \geq \lambda\sqrt{2\eta - \eta^2} \geq \lambda\sqrt{\eta}$  while  $q(x) \leq \frac{5}{4\lambda^2\eta^2}$ .

Now multiplying (2.8) by  $h$ , (2.5) by  $\psi_{\pm}$  and subtracting both results, we obtain

$$h''\psi_{\pm} - \psi''_{\pm}h + qh\psi_{\pm} = 0.$$

On the other hand,  $h''\psi_{\pm} - \psi''_{\pm}h = (h'\psi_{\pm} - \psi'_{\pm}h)'$ . Therefore,

$$(2.9) \quad (h'\psi_{\pm} - \psi'_{\pm}h)' = -qh\psi_{\pm}.$$

Let us now define

$$Q_{\pm}(x) = \int_0^x q(t)h(t)\psi_{\pm}(t) dt.$$

Integrating (2.9) between 0 and  $x$ , we obtain the system

$$\begin{cases} h'(x)\psi_+(x) - h(x)\psi'_+(x) &= h'(0)\psi_+(0) - h(0)\psi'_+(0) - Q_+(x) \\ h'(x)\psi_-(x) - h(x)\psi'_-(x) &= h'(0)\psi_-(0) - h(0)\psi'_-(0) - Q_-(x) \end{cases}.$$

According to (2.7) the determinant of this system is  $-2i$ , we can thus solve it for  $h(x)$ . This leads to

$$\begin{aligned} h(x) &= h(0) \frac{\psi'_+(0)\psi_-(x) - \psi'_-(0)\psi_+(x)}{2i} \\ &\quad + h'(0) \frac{\psi_-(0)\psi_+(x) - \psi_+(0)\psi_-(x)}{2i} \\ &\quad + \frac{Q_+(x)\psi_-(x) - Q_-(x)\psi_+(x)}{2i}. \end{aligned}$$

It remains to identify those 3 terms. First, note that  $\psi_+(0) = \psi_-(0) = 1/\sqrt{p(0)} = 1/\sqrt{\lambda}$  while  $\psi'_+(0) = \overline{\psi'_-(0)} = \left(-\frac{1}{2} \frac{p'(0)}{p(0)} + ip(0)\right) \psi_+(0) = i\sqrt{\lambda}$ . From this, we get

$$\begin{aligned} \frac{\psi'_+(0)\psi_-(x) - \psi'_-(0)\psi_+(x)}{2i} &= \sqrt{\frac{\lambda}{p(x)}} \frac{e^{i\varphi(x)} + e^{-i\varphi(x)}}{2} \\ &= \sqrt{\frac{\lambda}{p(x)}} \cos \varphi(x). \end{aligned}$$

Further,

$$\begin{aligned} \frac{\psi_-(0)\psi_+(x) - \psi_+(0)\psi_-(x)}{2i} &= \frac{1}{\sqrt{\lambda p(x)}} \frac{e^{i\varphi(x)} - e^{-i\varphi(x)}}{2i} \\ &= \frac{1}{\sqrt{\lambda p(x)}} \sin \varphi(x). \end{aligned}$$

Finally,

$$\begin{aligned} \frac{Q_+(x)\psi_-(x) - Q_-(x)\psi_+(x)}{2i} &= \frac{1}{\sqrt{p(x)}} \int_0^x \frac{q(t)}{\sqrt{p(t)}} h(t) \frac{e^{i\varphi(t)}e^{-i\varphi(x)} - e^{-i\varphi(t)}e^{i\varphi(x)}}{2i} dt \\ &= \frac{1}{\sqrt{p(x)}} \int_0^x \frac{q(t)}{\sqrt{p(t)}} h(t) \sin(\varphi(t) - \varphi(x)) dt. \end{aligned}$$

We are now in position to prove the following theorem:

**Theorem 2.1.** *Let  $n \geq 0$ . Assume that  $|x| \leq \sqrt{2n+1}$ , then*

$$(2.10) \quad \begin{aligned} h_n(x) &= h_n(0) \left( \frac{2n+1}{2n+1-x^2} \right)^{1/4} \cos \varphi_n(x) + h'_n(0) \frac{\sin \varphi_n(x)}{((2n+1)(2n+1-x^2))^{1/4}} \\ &+ E_n(x) \end{aligned}$$

where

$$(2.11) \quad \varphi_n(x) = \int_0^x \sqrt{2n+1-t^2} dt \quad \text{and} \quad |E_n(x)| \leq \frac{5}{4} \left( \frac{\sqrt{2n+1}}{2n+1-x^2} \right)^{5/2}.$$

Moreover, for  $(2n+1)^{-a} < \eta < 1$ ,  $a < 3/20$  and  $x, y \leq (1-\eta)\sqrt{2n+1}$

$$(2.12) \quad |E_n(x) - E_n(y)| \leq \frac{5}{(2n+1)^{3/4-5a}} |x-y|.$$

Further, if  $|x|, |y| \leq T \leq \frac{\sqrt{2n+1}}{2}$ ,

$$\varphi_n(x) = \sqrt{2n+1}x - e_n(x),$$

where

$$(2.13) \quad |e_n(x)| \leq \frac{T^3}{3\sqrt{2n+1}} \quad \text{and} \quad |e_n(x) - e_n(y)| \leq \frac{T^2}{\sqrt{2n+1}} |x-y|,$$

while

$$(2.14) \quad |E_n(x)| \leq \frac{2}{(2n+1)^{3/2}} \quad \text{and} \quad |E_n(x) - E_n(y)| \leq \frac{8}{(2n+1)^{5/4}} |x-y|.$$

**Remark.** One may explicitly compute  $\varphi$ :

$$\varphi_n(x) = \frac{2n+1}{2} \arcsin \frac{x}{\sqrt{2n+1}} + \frac{x}{2} \sqrt{2n+1-x^2} = \sqrt{2n+1}x - e(x),$$

where

$$e_n(x) = \frac{1}{2} \left[ (2n+1) \left( \frac{x}{\sqrt{2n+1}} - \arcsin \frac{x}{\sqrt{2n+1}} \right) + x \left( \sqrt{2n+1} - \sqrt{2n+1-x^2} \right) \right].$$

Also,  $\varphi_n$  has a geometric interpretation: it is the area of the intersection of a disc of radius  $\sqrt{2n+1}$  centered at 0 with the strip  $[0, x] \times \mathbb{R}^+$ . In particular, when  $x \rightarrow \sqrt{2n+1}$ ,  $\varphi_n(x) \sim \frac{\pi}{4}(2n+1)$ .

*Proof.* We will fix  $n$  and use the same notation as previously, e.g.  $\lambda = \sqrt{2n+1}$ ,  $p(x) = \sqrt{\lambda^2 - x^2}, \dots$

Let us first establish the bounds on  $e$ . Note that

$$\begin{aligned} e(x) &= \int_0^x \lambda - \sqrt{\lambda^2 - t^2} dt = \lambda \int_0^x 1 - \sqrt{1 - (t/\lambda)^2} dt \\ &= \lambda^2 \int_0^{x/\lambda} 1 - \sqrt{1 - s^2} ds = \lambda^2 \int_0^{x/\lambda} \frac{s^2}{1 + \sqrt{1 - s^2}} ds. \end{aligned}$$

But,  $\left| \int_a^b \frac{s^2}{1 + \sqrt{1 - s^2}} ds \right| \leq \left| \int_a^b s^2 ds \right| = \frac{|b^3 - a^3|}{3}$  the estimate of  $e(x)$  and  $e(x) - e(y)$  follow immediately.

Consider

$$E(x) = \frac{1}{\sqrt{p(x)}} \int_0^x \frac{q(t)}{\sqrt{p(t)}} h(t) \sin(\varphi(x) - \varphi(t)) dt.$$

Using Cauchy-Schwarz, we obtain

$$|E(x)| \leq \frac{1}{\sqrt{p(x)}} \left( \int_0^x \frac{q(t)^2}{p(t)} dt \right)^{1/2} \left( \int_0^x h(t)^2 dt \right)^{1/2} \leq \frac{1}{\sqrt{p(x)}} \left( \int_0^x \frac{25\lambda^4}{16p(t)^9} dt \right)^{1/2}$$

since  $\|h_n\|_2 = 1$ . As  $|x| < \lambda$ , and  $p$  decreases, the estimate  $|E(x)| \leq \frac{5\lambda^{5/2}}{4p(x)^5}$  follows.

Note that, if  $|x| \leq \lambda/2$ , then a slightly better estimate holds:

$$|E(x)| \leq \frac{10}{4\lambda\sqrt{3}} \left( \int_0^{\lambda/2} \frac{\lambda^4}{(\lambda^2 - t^2)^{9/2}} dt \right)^{1/2} = \frac{10}{4\sqrt{3}\lambda^3} \left( \int_0^{1/2} \frac{1}{(1 - s^2)^{9/2}} ds \right)^{1/2}.$$

A numerical computation shows that  $|E(x)| \leq \frac{2}{\lambda^3}$ .

**Remark.** Note that the bound on  $E$  allows to obtain a bound on  $h_n$ . For instance, if  $n \geq 2$  is even

$$\begin{aligned} |h_{2n}(x)| &\leq \sqrt{\frac{\lambda}{p(x)}} |h_{2n}(0)| + \frac{5}{4} \left( \frac{\lambda^{1/2}}{p(x)} \right)^5 \\ &\leq \left( \frac{(2n+1)^{1/4}}{\sqrt{\pi n^{1/4}}} + \frac{5}{4} \frac{\lambda^{5/2}}{p(x)^{9/2}} \right) \frac{1}{\sqrt{p(x)}} \\ &\leq \left( \frac{(2n+1)^{1/4}}{\sqrt{\pi n^{1/4}}} + \frac{5}{4} \frac{1}{\lambda^2 \eta^{9/4}} \right) \frac{1}{\sqrt{p(x)}} \leq \frac{1}{\sqrt{p(x)}}, \end{aligned}$$



provided  $|x| \leq (1 - \eta)\lambda$  with  $\eta \geq \frac{2}{\lambda^{8/9}}$ .

The same estimate is valid in the case when  $n$  is odd.

In order to prove the Lipschitz bound on  $E$ , let us introduce some further notation:

$$\chi(x, t) = \frac{q(t)}{\sqrt{p(t)}} h(t) \sin(\varphi(x) - \varphi(t))$$

and

$$\Phi(x, y) = \int_0^x \chi(y, t) dt.$$

Thus, we have proved that for  $|x| < \lambda$ ,

$$|\Phi(x, x)| = |E(x)| \leq \begin{cases} \frac{5}{4} \left( \frac{\lambda^{1/2}}{p(x)} \right)^5 & \text{if } |x| < \lambda \\ \frac{2}{\lambda^3} & \text{if } |x| < \lambda/2 \end{cases}.$$

Now, if  $x \leq y < \lambda$ ,

$$\begin{aligned} E(y) - E(x) &= \left( \frac{1}{\sqrt{p(y)}} - \frac{1}{\sqrt{p(x)}} \right) \Phi(y, y) \\ &\quad + \frac{1}{\sqrt{p(x)}} [\Phi(y, y) - \Phi(x, y)] \\ &\quad - \frac{1}{\sqrt{p(x)}} [\Phi(x, y) - \Phi(x, x)] \\ &= E_1 + E_2 + E_3. \end{aligned}$$

Note, that

$$\begin{aligned} \left| \frac{1}{\sqrt{p(y)}} - \frac{1}{\sqrt{p(x)}} \right| &\leq \frac{1}{2} |x - y| \sup_{t \in [x, y]} \left| \frac{p'(t)}{p(t)^{3/2}} \right| \\ &= \frac{1}{2} |x - y| \sup_{t \in [x, y]} \left| \frac{t}{p(t)^{5/2}} \right| \leq \frac{\lambda}{2p(y)^{5/2}} |x - y|. \end{aligned}$$

Thus, we obtain that  $|E_1| \leq \frac{5\lambda^{7/2}}{8p(y)^{15/2}} |x - y|$ .

In the case when  $|x|, |y| < \lambda/2$ , the same reasoning leads to the estimate  $|E_1| \leq \frac{|x - y|}{\lambda^{9/2}}$ .

Next, if  $|x|, |y| \leq (1 - \eta)\lambda$  one can estimate  $E_2$  as follows:

$$\begin{aligned} |\Phi(y, y) - \Phi(x, y)| &\leq \int_x^y |\chi(y, t)| dt \leq |x - y| \sup_{t \in [x, y]} \frac{q(t)}{\sqrt{p(t)}} \sup_{|t| \leq |y| \lambda} |h(t)| \\ &\leq \frac{5\lambda^2}{4p(y)^5} |x - y|. \end{aligned}$$

Therefore,  $|E_2| \leq \frac{5\lambda^2}{4p(y)^{11/2}} |x - y|$ .

In general, we will bootstrap the approximation of  $h$ . Let us first assume that  $n$  is even, so that

$$h(t) = h_n(0) \sqrt{\frac{\lambda}{p(x)}} \cos \varphi(x) + E(x)$$

Then

$$\begin{aligned}\chi(x, t) &= \frac{q(t)}{\sqrt{p(t)}} h(t) \sin(\varphi(x) - \varphi(t)) \\ &= h(0) \sqrt{\lambda} \frac{q(t)}{p(t)} \cos \varphi(x) \sin(\varphi(x) - \varphi(t)) + \frac{q(t)}{\sqrt{p(t)}} E(t) \sin(\varphi(x) - \varphi(t)) \\ &= \chi_1(x, t) + \chi_2(x, t).\end{aligned}$$

Therefore, we may write  $|E_2| \leq E_2^1 + E_2^2$  where  $E_2^j = \frac{1}{\sqrt{p(x)}} \int_x^y |\chi_j(y, t)| dt$ .

For  $E_2^2$  we use the estimate  $E(t) \leq \frac{5}{4} \left( \frac{\lambda^{1/2}}{p(t)} \right)^5$  that we established above. It follows that

$$E_2^2 \leq \frac{1}{\sqrt{p(x)}} \int_x^y \frac{5\lambda^2}{4p^{9/2}(t)} \frac{5}{4} \left( \frac{\lambda^{1/2}}{p(t)} \right)^5 dt \leq \frac{25\lambda^{9/2}}{16p(y)^{10}} |x - y|.$$

If  $|x|, |y| \leq \lambda/2$ , we may use  $|E(t)| \leq 2\lambda^{-3}$ ,  $q(t) \leq 5\lambda^{-2}$ ,  $p(x) \geq \sqrt{3}\lambda/2$  to obtain  $E_2^2 \leq \frac{12}{\lambda^6} |x - y|$ .

On the other hand,

$$E_2^1 \leq \frac{|h(0)|\sqrt{\lambda}}{\sqrt{p(x)}} \int_x^y \left| \frac{q(t)}{p(t)} \right| dt \leq \frac{5 \times 2^{1/4} \lambda^{5/2}}{4\sqrt{\pi} n^{1/4} p(y)^{11/2}} |x - y|.$$

If  $n$  is odd,  $h(0) = 0$  while  $|h'(0)| \leq \frac{2^{1/4}\lambda}{\sqrt{\pi} n^{1/4}}$  and we have to replace  $\chi_1$  by

$$\chi_1(x, t) = h'(0) \frac{q(t)}{\sqrt{\lambda} p(t)} \sin \varphi(x) \sin(\varphi(x) - \varphi(t)),$$

from which we deduce that

$$E_2^1 \leq \frac{|h'(0)|}{\sqrt{\lambda} \sqrt{p(x)}} \int_x^y \left| \frac{q(t)}{p(t)} \right| dt \leq \frac{5 \times 2^{1/4} \lambda^{5/2}}{4\sqrt{\pi} n^{1/4} p(y)^{11/2}} |x - y| \leq \frac{\lambda^{5/2}}{n^{1/4} p(y)^{11/2}} |x - y|.$$

If  $|x|, |y| \leq \lambda/2$ , there is again a slight improvement:

$$E_2^1 \leq \frac{10^{7/4}}{\sqrt{\pi} n^{1/4} 3^{3/4} \lambda^{13/2}} |x - y| \leq \frac{11}{\lambda^7} |x - y|,$$

since  $n \geq 3^{-1/4} \lambda^{1/2}$  if  $n \geq 1$ .

Finally,

$$\begin{aligned}\Phi(x, y) - \Phi(x, x) &= \int_0^x \frac{q(t)}{\sqrt{p(t)}} h(t) \left[ \sin(\varphi(y) - \varphi(t)) - \sin(\varphi(x) - \varphi(t)) \right] dt \\ &= 2 \int_0^x \frac{q(t)}{\sqrt{p(t)}} h(t) \cos \frac{\varphi(x) + \varphi(y) - 2\varphi(t)}{2} dt \sin \frac{\varphi(y) - \varphi(x)}{2}.\end{aligned}$$

The integral is estimated in the same way as we estimated  $\Phi(x, x)$ , while for  $\varphi$  we use the mean value theorem and the fact that  $\varphi' = p$ . We, thus, get

$$|E_3| \leq \frac{5\lambda^{5/2} p(x)}{4p(y)^5} |x - y| \leq \frac{5\lambda^{7/2}}{4p(y)^5} |x - y|.$$

If  $|x|, |y| \leq \lambda/2$ , there is again a slight improvement:  $|E_3| \leq \frac{2}{\lambda^3} p(x) |x - y| \leq \frac{2}{\lambda^{5/2}} |x - y|$ .

Summarizing,

$$\begin{aligned} |E(x) - E(y)| &\leq \left[ \frac{p(y)^{5/2}}{2\lambda} + \frac{5}{4} + \frac{4p(y)^{9/2}}{5n^{1/4}\lambda^2} + \frac{p(y)^5}{\lambda} \right] \frac{5\lambda^{9/2}}{4p(y)^{10}} |x - y| \\ &\leq \left[ \frac{p(y)^{3/2}}{2} + \frac{5}{4} + p(y)^2 + p(y)^4 \right] \frac{5\lambda^{9/2}}{4p(y)^{10}} |x - y|, \end{aligned}$$

since  $p(y) \leq \lambda$  and  $n^{-1/4} \geq \frac{5}{4}p(y)^{1/2}$ . Now, assume that  $|x|, |y| \leq (1 - \eta)\lambda$ , with  $\lambda^{-2a} < \eta < 1$ ,  $a > 0$ . In particular,  $\lambda \geq p(y) \geq \lambda\sqrt{\eta} \geq \lambda^{1-a}$ . Thus,

$$|E(x) - E(y)| \leq 5 \frac{\lambda^{4+9/2}}{\lambda^{10(1-a)}} |x - y| = \frac{5}{\lambda^{3/2-10a}} |x - y|.$$

If  $|x|, |y| \leq \lambda/2$ , then

$$|E(x) - E(y)| \leq \left[ \frac{1}{\lambda^{9/2}} + \frac{12}{\lambda^6} + \frac{11}{\lambda^7} + \frac{2}{\lambda^{5/2}} \right] \leq \frac{8}{\lambda^{5/2}} |x - y|,$$

since  $\lambda \geq \sqrt{3}$ . □

**2.2. Two technical lemmas.** We will now prove two technical lemmas. The first one concerns the function  $\varphi_n$ :

**Lemma 2.2.** *If  $|x|, |y| \leq T \leq \frac{1}{2}\sqrt{2n+1}$ , then*

$$(2.15) \quad |\varphi_{n+1}(x) - \varphi_n(x)| \leq \frac{3T}{\sqrt{2n+1}},$$

$$(2.16) \quad |\varphi_{n+1}(x) - \varphi_{n+1}(y) - \varphi_n(x) + \varphi_n(y)| \leq \frac{3}{\sqrt{2n+1}} |x - y|,$$

$$(2.17) \quad |\varphi_{n+1}(x) - \varphi_n(x) + \varphi_{n+1}(y) - \varphi_n(y)| \leq \frac{5T}{\sqrt{2n+1}},$$

$$(2.18) \quad \varphi_{n+1}(x) + \varphi_n(x) - \varphi_{n+1}(y) - \varphi_n(y) = (\sqrt{2n+1} + \sqrt{2n+3})(x - y) + \varepsilon_n(x, y),$$

with  $|\varepsilon_n(x, y)| \leq \frac{T^2}{\sqrt{2n+1}} |x - y|$  and

$$(2.19) \quad |\varphi_n(x) - \varphi_n(y)| \leq \frac{5}{4}\sqrt{2n+1} |x - y|.$$

*Proof.* Note that (2.15) is a direct consequence of (2.16) with  $y = 0$ .

Recall that  $\varphi_n(x) = \int_0^x \sqrt{2n+1-t^2} dt$ . We have

$$\begin{aligned} |\varphi_{n+1}(x) - \varphi_n(x) - \varphi_{n+1}(y) + \varphi_n(y)| &= \left| \int_y^x \sqrt{2n+3-t^2} - \sqrt{2n+1-t^2} dt \right| \\ &= \left| \int_y^x \frac{2}{\sqrt{2n+3-t^2} + \sqrt{2n+1-t^2}} dt \right| \\ &\leq \left| \int_y^x \frac{2}{\sqrt{2n+1-t^2}} dt \right| \\ &= 2 \left| \arcsin \frac{x}{\sqrt{2n+1}} - \arcsin \frac{y}{\sqrt{2n+1}} \right|. \end{aligned}$$

But, arcsin is  $\frac{1}{\sqrt{\eta}}$ -Lipschitz on  $[-(1-\eta), (1-\eta)]$ , thus,

$$|\varphi_{n+1}(x) + \varphi_n(x) - \varphi_{n+1}(y) - \varphi_n(y)| \leq 2\sqrt{2} \frac{|x-y|}{\sqrt{2n+1}}.$$

Next,

$$\begin{aligned} |\varphi_{n+1}(x) - \varphi_n(x) + \varphi_n(y) - \varphi_{n+1}(y)| &\leq \left| \int_0^x \sqrt{2n+3-t^2} - \sqrt{2n+1-t^2} dt \right| \\ &\quad + \left| \int_0^y \sqrt{2n+3-t^2} - \sqrt{2n+1-t^2} dt \right| \\ &\leq 2 \int_0^T \frac{2}{\sqrt{2n+1-t^2}} dt \leq \frac{8}{\sqrt{3}} \frac{T}{\sqrt{2n+1}}. \end{aligned}$$

Set  $N = \sqrt{2n+1} + \sqrt{2n+3}$ . Then,

$$\begin{aligned} \varphi_{n+1}(x) + \varphi_n(x) - \varphi_{n+1}(y) - \varphi_n(y) &= \int_y^x \sqrt{2n+3-t^2} + \sqrt{2n+1-t^2} dt \\ &= N(x-y) + \int_y^x \sqrt{2n+3-t^2} + \sqrt{2n+1-t^2} - N dt. \end{aligned}$$

Therefore,

$$\varepsilon(x, y) = \int_y^x \sqrt{2n+3-t^2} - \sqrt{2n+3} dt + \int_y^x \sqrt{2n+1-t^2} - \sqrt{2n+1} dt.$$

Let us estimate the second integral, the first being estimated in the same way:

$$\begin{aligned} \left| \int_y^x \sqrt{2n+1-t^2} - \sqrt{2n+1} dt \right| &= \left| \int_y^x \frac{t^2}{\sqrt{2n+1-t^2} + \sqrt{2n+1}} dt \right| \\ &\leq \frac{|x^3 - y^3|}{3(1 + \sqrt{3}/2)\sqrt{2n+1}} \leq \frac{T^2}{2\sqrt{2n+1}} |x-y|, \end{aligned}$$

since  $\sqrt{2n+1-t^2} \geq \sqrt{3}/2$ , when  $|t| \leq T \leq \sqrt{2n+1}/2$ .

Finally, (2.18) implies (2.19):

$$\begin{aligned} |\varphi_n(x) - \varphi_n(y)| &\leq \sqrt{2n+1}|x-y| + |e_n(x) - e_n(y)| \leq \left( \sqrt{2n+1} + \frac{T^2}{2n+1} \right) |x-y| \\ &\leq \frac{5}{4} \sqrt{2n+1} |x-y|, \end{aligned}$$

since  $T \leq \sqrt{2n+1}/2$ . □

**Remark.** Geometrically,  $|\varphi_{n+1}(x) - \varphi_n(x) - \varphi_{n+1}(y) + \varphi_n(y)|$  is the area of the intersection of the annulus of inner radius  $\sqrt{2n+1}$  an outer radius  $\sqrt{2n+3}$  with a vertical strip with first coordinate in  $[x, y]$ . The annulus has width  $o(n^{-1/2})$  so that its intersection with the strip has area  $o(n^{-1/2}|x-y|)$  as long as this strip is not “tangent” to the annulus. The lemma is a quantitative statement of this simple geometric fact.

The next result is a simplification of Theorem 2.1:

**Corollary 2.3.** *Let  $T \geq 2$  and let  $n \geq 2T^2$ . Then, for  $|x| \leq T$ , we obtain that*  
*- if  $n$  is even,  $n = 2p$*

$$(2.20) \quad h_{2p}(x) = \frac{(-1)^p}{\sqrt{\pi p^{1/4}}} \cos \varphi_{2p}(x) + \tilde{E}_{2p}(x);$$

– if  $n$  is odd,  $n = 2p + 1$

$$(2.21) \quad h_{2p+1}(x) = \frac{(-1)^p}{\sqrt{\pi p^{1/4}}} \sin \varphi_{2p+1}(x) + \tilde{E}_{2p+1}(x),$$

where, for  $|x|, |y| \leq T$ ,

$$(2.22) \quad |\tilde{E}_n(x)| \leq \frac{2T^2}{(2n+1)^{5/4}} \quad \text{and} \quad |\tilde{E}_n(x) - \tilde{E}_n(y)| \leq 3 \frac{T^2}{(2n+1)^{3/4}} |x-y|$$

*Proof.* First, we consider the case when  $n$  is even,  $n = 2p$ . Then,  $h_{2p}(0) = \frac{(-1)^p}{\sqrt{\pi p^{1/4}}} \left(1 - \frac{\eta_{2p}}{8p}\right)$  and  $h'_{2p+1}(0) = 0$ . Therefore, (2.10) reads

$$\begin{aligned} h_{2p}(x) &= \frac{(-1)^p}{\sqrt{\pi p^{1/4}}} \left(1 - \frac{\eta_{2p}}{8p}\right) \left(\frac{4p+1}{4p+1-x^2}\right)^{1/4} \cos \varphi_{2p}(x) + \tilde{E}_{2p}(x) \\ &= \frac{(-1)^p}{\sqrt{\pi p^{1/4}}} \cos \varphi_{2p}(x) + \tilde{E}_{2p}(x), \end{aligned}$$

where

$$\begin{aligned} \tilde{E}_{2p}(x) &= \frac{(-1)^p}{\sqrt{\pi p^{1/4}}} \left(1 - \frac{\eta_{2p}}{8p}\right) \left[ \left(\frac{4p+1}{4p+1-x^2}\right)^{1/4} - 1 \right] \cos \varphi_{2p}(x) + \frac{(-1)^p}{\sqrt{\pi}} \frac{\eta_{2p}}{8p^{5/4}} \cos \varphi_{2p}(x) \\ &\quad + E_{2p}(x). \end{aligned}$$

But,  $(1+a)^{1/4} - 1 \leq \frac{a}{4}$ , which for  $0 \leq a := \frac{x^2}{4p+1-x^2} \leq \frac{T^2}{4p+1-T^2} \leq \frac{4}{3} \frac{T^2}{4p+1}$  gives

$$(2.23) \quad \left| \left(\frac{4p+1}{4p+1-x^2}\right)^{1/4} - 1 \right| \leq \frac{1}{3} \frac{T^2}{4p+1}.$$

It follows that

$$\begin{aligned} |\tilde{E}_{2p}(x)| &\leq \frac{1}{\sqrt{\pi p^{1/4}}} \left| \left(1 + \frac{x^2}{4p+1-x^2}\right)^{1/4} - 1 \right| + \frac{1}{8\sqrt{\pi} p^{5/4}} + |E_{2p}(x)| \\ &\leq \frac{1}{\sqrt{\pi p^{1/4}}} \frac{1}{3} \frac{T^2}{4p+1} + \frac{1}{8\sqrt{\pi} p^{5/4}} + \frac{2}{(4p+1)^{3/2}} \leq \frac{2T^2}{(4p+1)^{5/4}}. \end{aligned}$$

Further,

$$\begin{aligned} |\tilde{E}_{2p}(x) - \tilde{E}_{2p}(y)| &\leq \frac{1}{\sqrt{\pi p^{1/4}}} \left| \left(1 - \frac{x^2}{4p+1}\right)^{-1/4} - \left(1 - \frac{y^2}{4p+1}\right)^{-1/4} \right| \\ &\quad + \frac{1}{\sqrt{\pi p^{1/4}}} \left[ \left| \left(\frac{4p+1}{4p+1-y^2}\right)^{1/4} - 1 \right| + \frac{1}{8p} \right] |\cos \varphi_{2p}(x) - \cos \varphi_{2p}(y)| \\ &\quad + |E_{2p}(x) - E_{2p}(y)| = E_{2p}^1(x, y) + E_{2p}^2(x, y) + E_{2p}^3(x, y). \end{aligned}$$

We have already established that  $E_{2p}^3(x, y) \leq \frac{8}{(4p+1)^{5/4}} |x-y|$ . Further, if  $0 \leq X, Y \leq \frac{T^2}{4p+1} \leq \frac{1}{4}$ , then

$$|(1-X)^{-1/4} - (1-Y)^{-1/4}| \leq \frac{5}{4} |X-Y| \sup_{0 \leq t \leq 1/4} (1-t)^{-5/4} = \frac{5\sqrt{2}}{3^{1/4}} |X-Y|.$$

Therefore

$$E_{2p}^1(x, y) \leq \frac{1}{\sqrt{\pi}p^{1/4}} \frac{5\sqrt{2}}{3^{1/4}} \frac{|x^2 - y^2|}{4p+1} \leq 4 \frac{T}{4p+1} |x - y|.$$

Finally, for  $E_{2p}^2$  we use the fact that  $\cos$  is 1-Lipschitz, (2.19) and (2.23), to obtain

$$E_{2p}^2(x, y) \leq \frac{1}{\sqrt{\pi}p^{1/4}} \left[ \frac{1}{3} \frac{T^2}{4p+1} + \frac{1}{8p} \right] \frac{5}{4} \sqrt{4p+1} |x - y| \leq 2 \frac{T^2}{(4p+1)^{3/4}} |x - y|.$$

Thus,

$$|\tilde{E}_{2p}(x) - \tilde{E}_{2p}(y)| \leq \left( 4 \frac{T}{4p+1} + 2 \frac{T^2}{(4p+1)^{3/4}} + \frac{8}{(4p+1)^{5/4}} \right) |x - y| \leq 3 \frac{T^2}{(4p+1)^{3/4}} |x - y|.$$

Let us now consider the case when  $n$  is odd,  $n = 2p + 1$ . Then,  $h_{2p+1}(0) = 0$  and  $h'_{2p+1}(0) = \frac{(-1)^p \sqrt{4p+3}}{\sqrt{\pi}p^{1/4}} \left( 1 - \frac{\eta_{2p+1}}{4} \right)$ . Therefore (2.10) reads

$$\begin{aligned} h_{2p+1}(x) &= \frac{(-1)^p \sqrt{4p+3}}{\sqrt{\pi}p^{1/4}} \left( 1 - \frac{\eta_{2p+1}}{4} \right) \frac{\sin \varphi_{2p+1}(x)}{((4p+3)(4p+3-x^2))^{1/4}} + E_{2p+1}(x) \\ &= \frac{(-1)^p}{\sqrt{\pi}p^{1/4}} \left( 1 - \frac{\eta_{2p+1}}{4} \right) \left( \frac{4p+3}{(4p+3-x^2)} \right)^{1/4} \sin \varphi_{2p+1}(x) + E_{2p+1}(x) \\ &= \frac{(-1)^p}{\sqrt{\pi}p^{1/4}} \sin \varphi_{2p+1}(x) + \tilde{E}_{2p+1}(x), \end{aligned}$$

where

$$\begin{aligned} \tilde{E}_{2p+1}(x) &= \frac{(-1)^p}{\sqrt{\pi}p^{1/4}} \left( 1 - \frac{\eta_{2p+1}}{4} \right) \left[ \left( \frac{4p+3}{(4p+3-x^2)} \right)^{1/4} - 1 \right] \sin \varphi_{2p+1}(x) \\ &\quad + \frac{(-1)^p \eta_{2p+1}}{4\sqrt{\pi}p^{1/4}} \sin \varphi_{2p+1}(x) + E_{2p+1}(x). \end{aligned}$$

The remaining of the proof is the same as for  $\tilde{E}_{2p}$ .  $\square$

**Remark.** The assumption  $T \geq 2$  is here to make it easier to group terms in the estimates of the errors. For  $T \geq 1$  the constants are slightly worse. The reader may check that

$$(2.24) \quad |\tilde{E}_n(x)| \leq \frac{3T^2}{(2n+1)^{5/4}} \quad \text{and} \quad |\tilde{E}_n(x) - \tilde{E}_n(y)| \leq 8 \frac{T^2}{(2n+1)^{3/4}} |x - y|.$$

### 3. $L^2$ -APPROXIMATION OF FUNCTIONS BY HERMITE FUNCTIONS

**3.1. The kernel of the projection onto the Hermite functions.** As  $(h_n)_{n \geq 0}$  forms an orthonormal basis of  $L^2(\mathbb{R})$ , every  $f \in L^2(\mathbb{R})$  can be written as

$$f(x) = \lim_{n \rightarrow +\infty} \sum_{k=0}^n \langle f, h_k \rangle h_k(x),$$

where the limit is in the  $L^2(\mathbb{R})$  sense. Further,

$$\sum_{k=0}^n \langle f, h_k \rangle h_k(x) = \sum_{k=0}^n \int_{\mathbb{R}} f(y) h_k(y) dy h_k(x) = \int_{\mathbb{R}} f(y) \sum_{k=0}^n h_k(x) h_k(y) dy = \int_{\mathbb{R}} k_n(x, y) f(y) dy,$$

with the kernel  $k_n(x, y) = \sum_{k=0}^n h_k(x)h_k(y)$ . According to the Christoffel-Darboux Formula,

$$k_n(x, y) = \sqrt{\frac{n+1}{2}} \frac{h_{n+1}(x)h_n(y) - h_{n+1}(y)h_n(x)}{y-x}.$$

We will now use Corollary 2.3 to approximate this kernel:

**Theorem 3.1.** *Let  $T \geq 2$ ,  $n \geq 2T^2$  and  $N = \frac{\sqrt{2n+1} + \sqrt{2n+3}}{2}$ . Then, for  $|x|, |y| \leq T$ ,*

$$k_n(x, y) = \frac{1}{\pi} \frac{\sin N(x-y)}{x-y} + R_n(x, y),$$

with  $|R_n(x, y)| \leq \frac{17T^2}{\sqrt{2n+1}}$ .

**Remark.** The same estimate holds for  $T = 1$  provided  $n \geq 6$ .

*Proof.* For sake of simplicity, we will only prove the theorem in the case when  $n$  is even and write  $n = 2p$ .

Let  $\lambda = \sqrt{2n+1}$ ,  $\mu = \sqrt{2n+3}$ ,  $\alpha = \frac{1}{\sqrt{\pi p^{1/4}}}$ ,  $\beta = \frac{1}{\sqrt{\pi p^{1/4}}}$ ,  $E = (-1)^p \tilde{E}_{2p}$  and  $F = (-1)^p \tilde{E}_{2p+1}$ . Then, according to Lemma 2.2,

$$\begin{cases} h_{2p}(x) &= (-1)^p \left( \frac{1}{\sqrt{\pi p^{1/4}}} \cos \varphi_{2p}(x) + E(x) \right) \\ h_{2p+1}(x) &= (-1)^p \left( \frac{1}{\sqrt{\pi p^{1/4}}} \sin \varphi_{2p+1}(x) + F(x) \right) \end{cases}.$$

Therefore,

$$\begin{aligned} h_{2p+1}(x)h_{2p}(y) - h_{2p+1}(y)h_{2p}(x) &= \frac{1}{\pi p^{1/2}} (\sin \varphi_{2p+1}(x) \cos \varphi_{2p}(y) - \sin \varphi_{2p+1}(y) \cos \varphi_{2p}(x)) \\ &\quad + \frac{1}{\sqrt{\pi p^{1/4}}} (F(x) \cos \varphi_{2p}(y) - F(y) \cos \varphi_{2p}(x)) \\ &\quad + \frac{1}{\sqrt{\pi p^{1/4}}} (\sin \varphi_{2p+1}(x) E(y) - \sin \varphi_{2p+1}(y) E(x)) \\ &\quad + F(x) E(y) - F(y) E(x) \\ &= H_1(x, y) + H_2(x, y) + H_3(x, y) + H_4(x, y). \end{aligned}$$

— The first term in the equation above is the principal one. Let us start by computing

$$\begin{aligned} A := \sin \varphi_{2p+1}(x) \cos \varphi_{2p}(y) - \sin \varphi_{2p+1}(y) \cos \varphi_{2p}(x) &= \frac{1}{2} [\sin(\varphi_{2p+1}(x) + \varphi_{2p}(y)) - \sin(\varphi_{2p+1}(x) - \varphi_{2p}(y)) \\ &\quad - \sin(\varphi_{2p+1}(y) + \varphi_{2p}(x)) + \sin(\varphi_{2p+1}(y) - \varphi_{2p}(x))] \\ &= \sin \frac{\varphi_{2p+1}(x) - \varphi_{2p+1}(y) - \varphi_{2p}(x) + \varphi_{2p}(y)}{2} \\ &\quad \times \cos \frac{\varphi_{2p+1}(x) + \varphi_{2p+1}(y) + \varphi_{2p}(x) + \varphi_{2p}(y)}{2} \\ &\quad + \sin \frac{\varphi_{2p+1}(y) + \varphi_{2p}(y) - \varphi_{2p}(x) - \varphi_{2p+1}(x)}{2} \\ &\quad \times \cos \frac{\varphi_{2p+1}(x) - \varphi_{2p}(x) - \varphi_{2p}(y) + \varphi_{2p+1}(y)}{2} \\ &= S_1 C_1 + S_2 C_2. \end{aligned}$$

Now, according to (2.16),

$$|S_1 C_1| \leq |S_1| \leq \frac{|\varphi_{2p+1}(x) - \varphi_{2p+1}(y) - \varphi_{2p}(x) + \varphi_{2p}(y)|}{2} \leq \frac{3}{2\sqrt{2n+1}}|x-y|,$$

while  $S_2 C_2 = S_2(1 + C_2 - 1)$ . But, with (2.17),

$$|C_2 - 1| \leq \frac{|\varphi_{2p+1}(x) - \varphi_{2p}(x) - \varphi_{2p}(y) + \varphi_{2p+1}(y)|^2}{2} \leq \frac{25T^2}{2(2n+1)}.$$

Thus, with (2.18),

$$|S_2(C_2 - 1)| \leq \left(N + \frac{T^2}{\sqrt{2n+1}}\right) |x-y| \frac{25T^2}{2(2n+1)} \leq \frac{16T^2}{\sqrt{2n+1}}|x-y|.$$

Finally, using again Lemma 2.2,

$$\begin{aligned} \sin(N(y-x) + \varepsilon_n(y,x)) &= \sin N(y-x) + \sin N(y-x)(\cos \varepsilon_n(y,x) - 1) + \cos N(y-x) \sin \varepsilon_n(x,y) \\ &= \sin N(y-x) + E_2(x,y), \end{aligned}$$

where

$$|E_2(x,y)| \leq |\varepsilon_n(x,y)| + \frac{|\varepsilon_n(x,y)|^2}{2} \leq \frac{2T^2}{\sqrt{2n+1}}|x-y|.$$

Grouping those estimates leads to

$$A = \sin N(y-x) + E_3(x,y) \quad \text{with} \quad |E_3(x,y)| \leq \frac{39T^2}{2\sqrt{2n+1}}|x-y|$$

Notice, that

$$\frac{1}{\pi p^{1/2}} = \frac{1}{\pi} \sqrt{\frac{2}{n+1}} \sqrt{1 + \frac{1}{n}} = \sqrt{\frac{2}{n+1}} \frac{1}{\pi} \left(1 + \frac{\xi_n}{n}\right)$$

with  $|\xi_n| \leq 1/2$ .

We, thus, conclude that  $H_1(x,y) = \sqrt{\frac{2}{n+1}} \left(\frac{1}{\pi} \sin N(y-x) + E_4(x,y)\right)$ , with

$$|E_4(x,y)| \leq \frac{1}{\pi} \left(1 + \frac{\xi_n}{n}\right) |E_3(x,y)| + \frac{\xi_n}{\pi n} N|x-y| \leq \frac{5T^2}{\sqrt{2n+1}}|x-y|.$$

— Consider

$$F(x) \cos \varphi_{2p}(y) - F(y) \cos \varphi_{2p}(x) = F(x)(\cos \varphi_{2p}(y) - \cos \varphi_{2p}(x)) + (F(x) - F(y)) \cos \varphi_{2p}(x).$$

Then, according to (2.22),  $|(F(x) - F(y)) \cos \varphi_{2p}(x)| \leq |F(x) - F(y)| \leq \frac{3T^2}{(2n+1)^{3/4}}|x-y|$ , while

$$\begin{aligned} |F(x)(\cos \varphi_{2p}(y) - \cos \varphi_{2p}(x))| &\leq \frac{2T^2}{(2n+1)^{5/4}} |\varphi_{2p}(y) - \varphi_{2p}(x)| \\ &\leq \frac{2T^2}{(2n+1)^{5/4}} \frac{5}{4} \sqrt{2n+1} |x-y| = \frac{5T^2}{2(2n+1)^{3/4}} |x-y|, \end{aligned}$$

with (2.19). Therefore,

$$|H_2(x,y)| \leq \frac{1}{\sqrt{\pi} p^{1/4}} \frac{(5/2 + 3)T^2}{(2n+1)^{3/4}} |x-y| \leq \sqrt{\frac{2}{n+1}} \frac{3T^2}{2(2n+1)^{1/2}} |x-y|.$$

Similarly, the estimate  $|H_3(x,y)| \leq \sqrt{\frac{2}{n+1}} \frac{3T^2}{2(2n+1)^{1/2}} |x-y|$  holds.



Note that for  $T = 1$ , we have to use (2.24) instead of (2.22) which gives  $|H_2(x, y)|, |H_3(x, y)| \leq \sqrt{\frac{2}{n+1} \frac{5T^2}{(2n+1)^{1/2}} |x-y|}$

— Finally, according to (2.22),

$$\begin{aligned} |F(x)E(y) - F(y)E(x)| &\leq |F(x)||E(y) - E(x)| + |E(x)||F(x) - F(y)| \\ &\leq \frac{12T^4}{(2n+1)^2} |x-y| \leq \sqrt{\frac{2}{n+1} \frac{2T^2}{\sqrt{2n+1}}} |x-y|. \end{aligned}$$

Note that for  $T = 1$ , we have to use (2.24) instead of (2.22) which gives  $|H_4(x, y)| \leq \sqrt{\frac{2}{n+1} \frac{24}{(2n+1)^{3/2}} |x-y|}$

Grouping terms together, we obtain,

$$h_{2p+1}(x)h_{2p}(y) - h_{2p+1}(y)h_{2p}(x) = \sqrt{\frac{2}{n+1}} \left( \frac{1}{\pi} \sin N(y-x) + E_5(x, y) \right),$$

with  $|E_5(x, y)| \leq \frac{9T^2}{\sqrt{2n+1}} |x-y|$ . □

**3.2. A tail estimate.** Let us now establish a tail estimate for  $k_n$ .

**Proposition 3.2.** *Let  $T \geq 2$  and  $n \geq 2T^2$ . Then, for  $|x| \leq T$ ,*

$$\int_{|y| \geq 2T} k_n(x, y)^2 dy \leq \frac{2}{\pi^2 T} + \frac{12T^2}{\sqrt{2n+1}} \ln(2n+1).$$

*Proof.* First, using the reproducing kernel property of  $k_n$ ,

$$\int_{\mathbb{R}} k_n(x, y)k_n(z, y) dy = k_n(x, z).$$

But, since  $k_n(x, y) = \sum_{k=0}^n h_k(x)h_k(y)$  and  $h_k = H_k e^{-x^2/2}$ , with  $H_k$  a polynomial of degree  $k$ , there exists a constant  $C_n$ , such that

$$|k_n(x, y)| \leq C_n(1+|x|)^n(1+|y|)^n e^{-(x^2+y^2)/2}.$$

Applying Lebesgue's Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}} k_n(x, y)k_n(z, y) dy \rightarrow \int_{\mathbb{R}} k_n(x, y)^2 dy,$$

when  $z \rightarrow x$ . On the other hand, Theorem 3.1 shows that

$$k_n(x, z) \rightarrow \frac{N}{\pi} + R_n(x, x)$$

uniformly in  $x \in [-T, T]$ . Therefore,

$$(3.25) \quad \int_{\mathbb{R}} k_n(x, y)^2 dy = \frac{N}{\pi} + R_n(x, x), \quad |R_n(x)| \leq \frac{9T^2}{\sqrt{2n+1}}.$$

Now, for  $|x| \leq T$ , Theorem 3.1 shows that

$$\begin{aligned} \int_{[-2T, 2T]} k_n(x, y)^2 dy &= \int_{[-2T, 2T]} \left( \frac{1}{\pi} \frac{\sin N(y-x)}{y-x} + R_n(x, y) \right)^2 dy \\ &= \frac{1}{\pi^2} \int_{[-2T, 2T]} \frac{\sin^2 N(y-x)}{(y-x)^2} dy + R_n(x). \end{aligned}$$

The estimation of the first term is classical: for  $|x| \leq T$ ,

$$\begin{aligned} \frac{1}{\pi^2} \int_{-2T}^{2T} \frac{\sin^2 N(y-x)}{(y-x)^2} dy &= \frac{N}{\pi^2} \int_{-N(2T+x)}^{N(2T-x)} \frac{\sin^2 z}{z^2} dz \\ &= \frac{N}{\pi^2} \int_{\mathbb{R}} \frac{\sin^2 z}{z^2} dz - \frac{N}{\pi^2} \left( \int_{-\infty}^{-N(2T+x)} + \int_{N(2T-x)}^{+\infty} \right) \frac{\sin^2 z}{z^2} dz \\ &= \frac{N}{\pi} - R_N^1(x), \end{aligned}$$

where  $0 \leq R_N^1(x) \leq \frac{2N}{\pi^2} \int_{NT}^{+\infty} \frac{dz}{z^2} = \frac{2}{\pi^2 T}$ .

Next, we write  $R_n(x) = R_n^2(x) + R_n^3(x)$ , where  $R_n^2(x) = \int_{[-2T, 2T]} R_n(x, y)^2 dy \geq 0$  and

$$\begin{aligned} |R_n^3| &= \int_{[-T, T]} \frac{2}{\pi} \left| \frac{\sin N(y-x)}{y-x} R_n(x, y) \right| dy \\ &\leq \frac{18T^2}{\pi\sqrt{2n+1}} \int_{-N(T-x)}^{N(T-x)} \left| \frac{\sin z}{z} \right| dz \\ &\leq \frac{36T^2}{\pi\sqrt{2n+1}} \int_0^{2NT} \min(1, z^{-1}) dz \leq \frac{12T^2}{\sqrt{2n+1}} \ln(2n+1). \end{aligned}$$

It follows that for  $|x| \leq T$

$$\int_{|y| \geq 2T} k_n(x, y)^2 dy \leq R_n^1 + |R_n^3| \leq \frac{2}{\pi^2 T} + \frac{12T^2}{\sqrt{2n+1}} \ln(2n+1)$$

as announced.  $\square$

### 3.3. Approximating almost time and band limited functions by Hermite functions.

We can now prove Theorem 1.1:

**Theorem 3.3.** *Let  $\Omega_0, T_0 \geq 2$  and  $\varepsilon_T, \varepsilon_\Omega > 0$ . Assume that*

$$\int_{|t| > T_0} |f(t)|^2 dt \leq \varepsilon_T^2 \|f\|_{L^2(\mathbb{R})}^2 \quad \text{and} \quad \int_{|\omega| > \Omega_0} |\widehat{f}(\omega)|^2 d\omega \leq \varepsilon_\Omega^2 \|f\|_{L^2(\mathbb{R})}^2.$$

For  $n$  an integer, let  $K_n f$  be the orthogonal projection of  $f$  on the span of  $h_0, \dots, h_n$ .

Assume that  $n \geq \max(2T^2, 2\Omega^2)$ . Then, for  $T \geq T_0$ ,

$$(3.26) \quad \|f - K_n f\|_{L^2([-T, T])} \leq \left( \varepsilon_\Omega + \frac{34T^3}{\sqrt{2n+1}} + 2\varepsilon_T \right) \|f\|_{L^2(\mathbb{R})}$$

and, for  $T \geq 2T_0$ ,

$$(3.27) \quad \|f - K_n f\|_{L^2(\mathbb{R} \setminus [-T, T])} \left( 2\varepsilon_T + \frac{1}{2T^{1/2}} + \frac{12T^{5/2}}{\sqrt{2n+1}} \ln(2n+1) \right)^{1/2} \|f\|_{L^2(\mathbb{R})}.$$

**Remark.** As the proof of (3.26) only depends on Theorem 3.1, this estimate holds for  $T = 1$ , provided we assume that  $n \geq 6$  (see the remark following Theorem 3.1).

*Proof.* We will introduce several projections. For  $T, \Omega > 0$ , let

$$P_T f = \mathbf{1}_{[-T, T]} f \quad \text{and} \quad Q_\Omega f = \mathcal{F}^{-1}[\mathbf{1}_{[-\Omega, \Omega]} \widehat{f}].$$

A simple computation shows that

$$Q_\Omega f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin \Omega(x-y)}{x-y} f(y) dy.$$

The hypothesis on  $f$  is that  $\|f - P_T f\|_{L^2(\mathbb{R})} \leq \varepsilon_T \|f\|_{L^2(\mathbb{R})}$  for  $T \geq T_0$  and  $\|f - Q_\Omega f\|_{L^2(\mathbb{R})} \leq \varepsilon_\Omega \|f\|_{L^2(\mathbb{R})}$  for  $\Omega \geq \Omega_0$ .

Finally, recall that the projection on the  $n$  first Hermite functions, is given by

$$K_n f(x) = \sum_{k=0}^n \langle f, h_k \rangle h_k(x) = \int_{\mathbb{R}} k_n(x, y) f(y) dy.$$

It is enough to prove (3.26) for  $T = T_0$ . Let us define a last integral operator by

$$\mathcal{R}_n f(x) = \int_{[-T, T]} R_n(x, y) f(y) dy,$$

where  $R_n(x, y)$  are defined in Theorem 3.1. Notice that  $k_n(x, y) = k_n(y, x)$  so that  $R_n(x, y) = R_n(y, x)$ . We may then reformulate Theorem 3.1 as following:

$$P_T K_n P_T f = P_T Q_N P_T f + P_T \mathcal{R}_n P_T f,$$

where  $N = \frac{\sqrt{2n+1} + \sqrt{2n+3}}{2}$ . Note that  $N \geq \Omega_0$ .

Using Schur's test,

$$\begin{aligned} \|P_T T_n P_T\| &\leq \sup_{|x| \leq T} \int_{|y| \leq T} |R_n(x, y)| dy \\ &\leq 2T \sup_{|x| \leq T} \sup_{|y| \leq T} |R_n(x, y)| \leq \frac{34T^3}{\sqrt{2n+1}}. \end{aligned}$$

Now, using the fact that projections are contractive and  $N \geq \Omega_0$ , we have

$$\begin{aligned} \|f - K_n f\|_{L^2([-T, T])} &= \|P_T f - P_T K_n f\|_{L^2(\mathbb{R})} \\ &\leq \|P_T f - P_T K_n P_T f\|_{L^2(\mathbb{R})} + \|P_T K_n (f - P_T f)\|_{L^2(\mathbb{R})} \\ &\leq \|P_T f - P_T Q_N P_T f + P_T \mathcal{R}_n P_T f\|_{L^2(\mathbb{R})} + \|f - P_T f\|_{L^2(\mathbb{R})} \\ &\leq \|P_T f - P_T Q_N P_T f\|_{L^2(\mathbb{R})} + \|P_T \mathcal{R}_n P_T f\|_{L^2(\mathbb{R})} + \|f - P_T f\|_{L^2(\mathbb{R})}. \end{aligned}$$

Now, write  $P_T Q_N P_T f = P_T Q_N f + P_T Q_N (f - P_T f)$ , then

$$\begin{aligned} \|P_T f - P_T Q_N P_T f\|_{L^2(\mathbb{R})} &\leq \|P_T f - P_T Q_N f\|_{L^2(\mathbb{R})} + \|P_T Q_N (f - P_T f)\|_{L^2(\mathbb{R})} \\ &\leq \|f - Q_N f\|_{L^2(\mathbb{R})} + \|f - P_T f\|_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f - K_n f\|_{L^2([-T, T])} &\leq \|f - Q_N f\|_{L^2(\mathbb{R})} + \frac{34T^3}{\sqrt{2n+1}} \|f\|_{L^2(\mathbb{R})} + 2\|f - P_T f\|_{L^2(\mathbb{R})} \\ &\leq \left( \varepsilon_\Omega + \frac{34T^3}{\sqrt{2n+1}} + 2\varepsilon_T \right) \|f\|_{L^2(\mathbb{R})}, \end{aligned}$$

since  $N \geq \Omega_0$ .

Let us now prove (3.27). It is enough to prove it for  $T = 2T_0$ . Note that

$$\begin{aligned} \|f - K_n f\|_{L^2(\mathbb{R} \setminus [-2T_0, 2T_0])} &\leq \|f\|_{L^2(\mathbb{R} \setminus [-2T_0, 2T_0])} + \|K_n P_T f\|_{L^2(\mathbb{R} \setminus [-2T_0, 2T_0])} + \|K_n (f - P_T f)\|_{L^2(\mathbb{R})} \\ &\leq 2\varepsilon_T \|f\|_{L^2(\mathbb{R})}^2 + \|K_n P_T f\|_{L^2(\mathbb{R} \setminus [-2T_0, 2T_0])}. \end{aligned}$$

We, therefore, need to estimate

$$\|K_n P_T f\|_{L^2(\mathbb{R} \setminus [-2T_0, 2T_0])} = \left( \int_{|x| \geq 2T_0} \left| \int_{|y| \leq T_0} k_n(x, y) f(y) dy \right|^2 dx \right)^{1/2}.$$

Using Minkowski's inequality, this quantity is bounded by

$$\begin{aligned} \int_{|y| \leq T_0} \left( \int_{|x| \geq 2T_0} |k_n(x, y) f(y)|^2 dx \right)^{1/2} dy &= \int_{|y| \leq T_0} \left( \int_{|x| \geq 2T_0} |k_n(x, y)|^2 dx \right)^{1/2} |f(y)| dy \\ &\leq \left( \sup_{|y| \leq T_0} \int_{|x| \geq 2T_0} |k_n(x, y)|^2 dx \right)^{1/2} \left( \int_{|y| \leq T_0} |f(y)| dy \right)^{1/2} \\ &\leq 2 \left( \frac{2}{\pi^2 T_0} + \frac{6T_0^2}{\sqrt{2n+1}} \ln(2n+1) \right)^{1/2} \|f\|_{L^1([-T_0, T_0])} \\ &\leq 2\sqrt{2} \left( \frac{1}{\pi^2 T_0^{1/2}} + \frac{6T_0^{5/2}}{\sqrt{2n+1}} \ln(2n+1) \right)^{1/2} \|f\|_{L^2(\mathbb{R})}, \end{aligned}$$

which is, with Proposition 3.2, complete the proof.  $\square$

**3.4. Approximating almost time and band limited functions by scaled Hermite functions.** In order to improve on the dependence on  $T, \Omega$  in the previous theorem, it is better to use the *scaled* Hermite basis defined by  $h_k^c(x) = c^{1/4} h_k(\sqrt{c}x)$ . Then  $h_k^c$  is also an orthonormal basis of  $L^2(\mathbb{R})$  and this basis is better adapted to approximate functions that are almost time limited to  $[-T, T]$  and  $\varepsilon_\Omega$ -bandlimited to  $[-\Omega, \Omega]$ . For  $f \in L^2(\mathbb{R})$ , define

$$(3.28) \quad K_n^c f = \sum_{k=0}^n \langle P_T f, h_k^c \rangle h_k^c.$$

**Theorem 3.4.** *Let  $\Omega, T \geq 2$  and  $\varepsilon_T, \varepsilon_\Omega > 0$ . Assume that*

$$\int_{|t| > T} |f(t)|^2 dt \leq \varepsilon_T^2 \|f\|_{L^2(\mathbb{R})}^2 \quad \text{and} \quad \int_{|\omega| > \Omega} |\widehat{f}(\omega)|^2 d\omega \leq \varepsilon_\Omega^2 \|f\|_{L^2(\mathbb{R})}^2.$$

*Then, for  $n \geq 2T^2\Omega^2$ .*

$$\|f - K_n^c f\|_{L^2([-T, T])} \leq \left( \varepsilon_T + \varepsilon_{\sqrt{c}\Omega} + \frac{34(\sqrt{c}T)^3}{\sqrt{2n+1}} \right) \|f\|_{L^2(\mathbb{R})}.$$

*Proof.* For  $f \in L^2(\mathbb{R})$ ,

$$\begin{aligned} K_n^c f(x) &= \sum_{k=0}^n \langle f, h_k^c \rangle h_k^c(x) = \int_{-T}^T f(y) \sqrt{c} \sum_{k=0}^n h_k(\sqrt{c}x) h_k(\sqrt{c}y) dy \\ &= \int_{-\sqrt{c}T}^{\sqrt{c}T} f(\sqrt{c}y) \sum_{k=0}^n h_k(\sqrt{c}x) h_k(y) dy. \end{aligned}$$

Then

$$\begin{aligned}
\|f - K_n^c f_c\|_{L^2([-T, T])} &= \left( \int_{-T}^T \left| f(x) - \int_{-\sqrt{c}T}^{\sqrt{c}T} f(y/\sqrt{c}) \sum_{k=0}^n h_k(\sqrt{c}x) h_k(y) dy \right|^2 dx \right)^{1/2} \\
&= \left( \int_{-\sqrt{c}T}^{\sqrt{c}T} \left| c^{-1/4} f(x/\sqrt{c}) - \int_{-\sqrt{c}T}^{\sqrt{c}T} c^{-1/4} f(y/\sqrt{c}) \sum_{k=0}^n h_k(x) h_k(y) dy \right|^2 dx \right)^{1/2} \\
&= \|f_c - K_n f_c\|_{L^2([- \sqrt{c}T, \sqrt{c}T])}
\end{aligned}$$

where  $f_c(x) = c^{-1/4} f(x/\sqrt{c}) \chi_{[-\sqrt{c}T, \sqrt{c}T]}(x) = \delta_c f(x) \chi_{[-\sqrt{c}T, \sqrt{c}T]}(x)$  with  $\delta_c f(x) = c^{-1/4} f(x/\sqrt{c})$ .

Now note that  $f_c$  is  $\varepsilon_T$  time-limited to  $[-\sqrt{c}T, \sqrt{c}T]$  and that

$$\begin{aligned}
\left( \int_{|\omega| > \Omega} |\widehat{f}_c(\omega)|^2 d\omega \right)^{1/2} &\leq \left( \int_{|\omega| > \Omega} |\widehat{\delta_c f}(\omega)|^2 d\omega \right)^{1/2} \\
&\quad + \left( \int_{|\omega| > \Omega} |\widehat{\delta_c f - f_c}(\omega)|^2 d\omega \right)^{1/2} \\
&\leq \left( \int_{|\omega| > \Omega} |c^{1/4} \widehat{f}(\sqrt{c}\omega)|^2 d\omega \right)^{1/2} \\
&\quad + \left( \int_{\mathbb{R}} |\widehat{\delta_c f - f_c}(\omega)|^2 d\omega \right)^{1/2} \\
&= \left( \int_{|\omega| > \sqrt{c}\Omega} |\widehat{f}(\omega)|^2 d\omega \right)^{1/2} \\
&\quad + \left( \int_{\mathbb{R}} |\delta_c f - f_c(x)|^2 dx \right)^{1/2} \\
&= \varepsilon_{\sqrt{c}\Omega} \|f\|_{L^2(\mathbb{R})} + \left( \int_{|x| \geq \sqrt{c}T} |c^{-1/4} f(x/\sqrt{c})|^2 dx \right)^{1/2} \\
&\leq (\varepsilon_{\sqrt{c}\Omega} + \varepsilon_T) \|f\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Thus  $f_c$  is  $\varepsilon_{\sqrt{c}\Omega} + \varepsilon_T$ -almost band limited in  $[-\Omega, \Omega]$ . It remains to apply Theorem 3.3 to  $f_c$  (with  $T_0 = \sqrt{c}T$  and  $\Omega$  replaced by  $\Omega/\sqrt{c}$ ).  $\square$

#### 4. NUMERICAL RESULTS

In this paragraph, we give several examples that illustrate the different results of this work.

**Example 1.** In this example, we check numerically that the actual error of the uniform approximation of the kernel  $k_n(x, y) = \sum_{k=0}^n h_k(x) h_k(y)$  is far much smaller than the theoretical error given by Theorem 3.1. For this purpose, we have considered the value of  $T = 1$  and various values of the integer  $10 \leq n \leq 100$ . For each value of  $n$ , we have used a uniform discretization  $\Lambda$  the square  $[-1, 1]^2$  with equidistant 6400 nodes. Then, we have computed approximation error over these grid points,  $\tilde{E}_n = \sup_{x, y \in \Lambda} \left| k_n(x, y) - \frac{\sin N(x-y)}{\pi(x-y)} \right|$  in order to

estimate the uniform error  $E_n = \sup_{x,y \in [-1,1]} \left| k_n(x,y) - \frac{\sin N(x-y)}{\pi(x-y)} \right|$ . The obtained results are given in the following table:

$n$	10	25	50	75	100
$\bar{E}_n$	0.067	0.039	0.025	0.023	0.022

**Example 2.** In this example, we illustrate the quality of approximation by scaled Hermite functions of a time-limited and an almost band-limited function. For this purpose, we consider the function  $f(x) = \mathbf{1}_{[-1/2,1/2]}(x)$ . From the Fourier transform of  $f$ , one can easily check that  $f \in H^s(\mathbb{R})$  for any  $s < 1/2$ . Note that  $f$  is 0-concentrated in  $[-1/2, 1/2]$  and since  $f \in H^s(\mathbb{R})$ , then  $\epsilon_\Omega$ -band concentrated in  $[-\Omega, +\Omega]$ , for any  $\epsilon_\Omega < M_s \Omega^{-s}$  with  $M_s$  a positive constant. We have considered the value  $c = 100$  and we have used (3.28) to compute the scaled Hermite approximations  $K_n^c(f)$  of  $f$  with  $n = 40$  and  $n = 80$ . The graphs of  $f$  and its scaled Hermite approximation are given by Figure 1. In Figure 2, we have given the approximation errors  $f(x) - K_n^c f(x)$ .

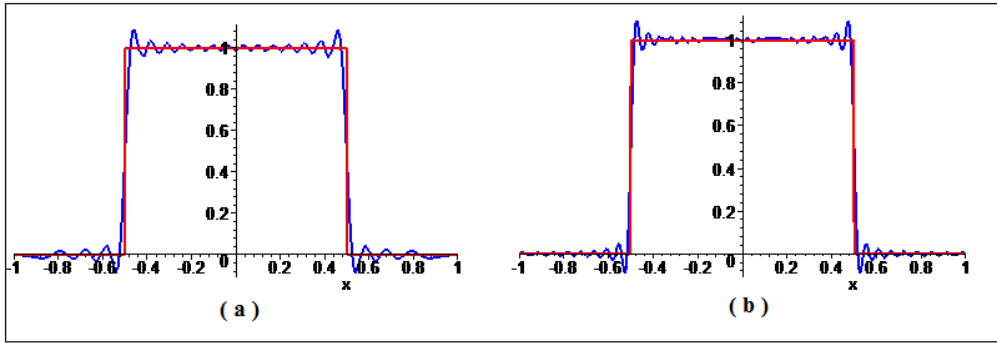


FIGURE 1. The Graph of  $f(x) = \mathbf{1}_{[-1/2,1/2]}(x)$  (red) and of  $K_n^c f(x)$  (blue) with (a)  $n = 40, c = 100$  and (b)  $n = 80, c = 100$ . Note the Gibbs phenomena that appears.

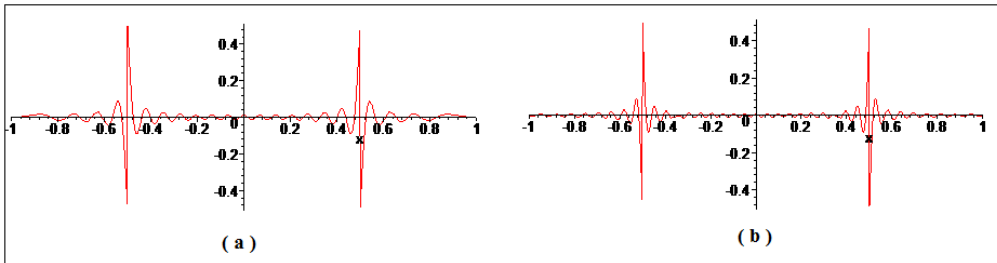


FIGURE 2. Graph of the approximation error  $f(x) - K_n^c f(x), c = 100$ , (a)  $n = 40$  (b)  $n = 80$ .

Also, to illustrate the fact that the scaled Hermite approximation outperforms the usual Hermite approximation, we have repeated the previous numerical tests without the scaling

factor (*i.e.* with  $c = 1$ ). The result is shown in Figure ?? Figure 3 shows the graphs of  $f$  and  $K_n f$ .

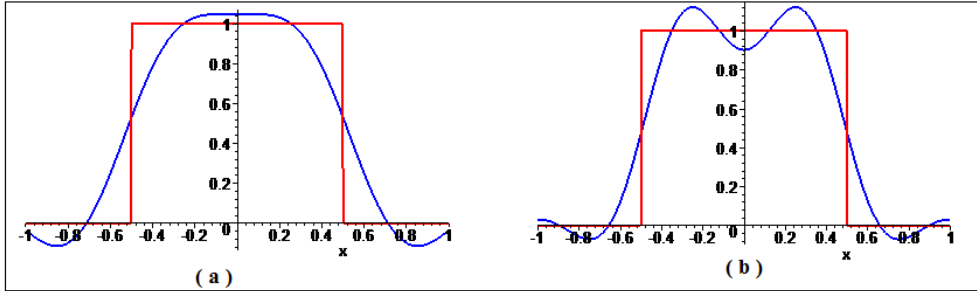


FIGURE 3. The Graph of  $f(x) = \mathbf{1}_{[-1/2, 1/2]}(x)$  (red) and of  $K_n^c f(x)$  (blue) with (a)  $n = 40$ ,  $c = 1$  and (b)  $n = 80$ ,  $c = 1$ .

**Example 3.** In this last example, we illustrate the quality of approximation of almost band-limited and time-limited function by the scaled Hermite functions for the function  $g$  given by  $g(x) = (1 - |x|)\chi_{[-1, 1]}(x)$ . As is easily seen by expressing the Fourier transform of  $g$ ,  $g \in H^s(\mathbf{R})$  for any  $s < 3/2$ . Moreover since  $g$  is supported on  $[-1, 1]$ ,  $g$  is 0-concentrated in  $[-1, 1]$ . Moreover, as in the previous example  $g$  is  $\epsilon_\Omega$ -band concentrated in  $[-\Omega, +\Omega]$ , for any  $\epsilon_\Omega < M_s \Omega^{-s}$ . We have considered the four couples  $(c, n) = (10, 20), (10, 50), (50, 20), (50, 50)$  and computed  $K_n^c g$ . The numerical results are given by Figures 4 and 5. These numerical results suggest again that the scaled Hermite functions are well suited for the approximation of almost band-limited and almost time-limited functions. In this sense, they have similar approximation properties as the PSWFs. The actual approximation error is much smaller than the theoretical error given by Theorem 3.4 This actual approximation error depends on the truncation order  $n$  as well as on the parameter  $c$ .

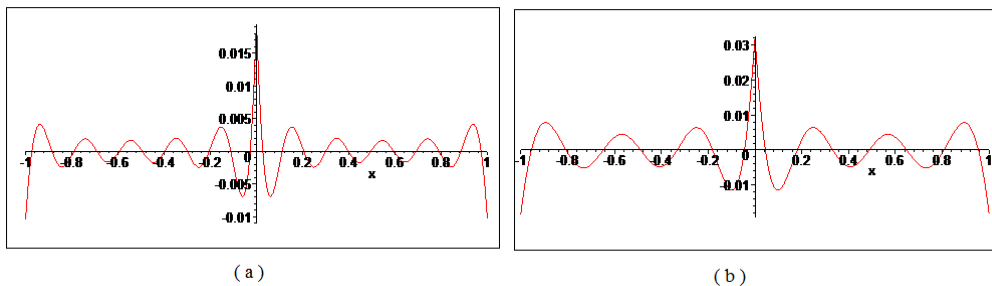


FIGURE 4. Graph of the approximation error  $g(x) - K_n^c g(x)$  with (a)  $c = 10$ ,  $n = 20$  and (b)  $c = 10$ ,  $n = 50$ .

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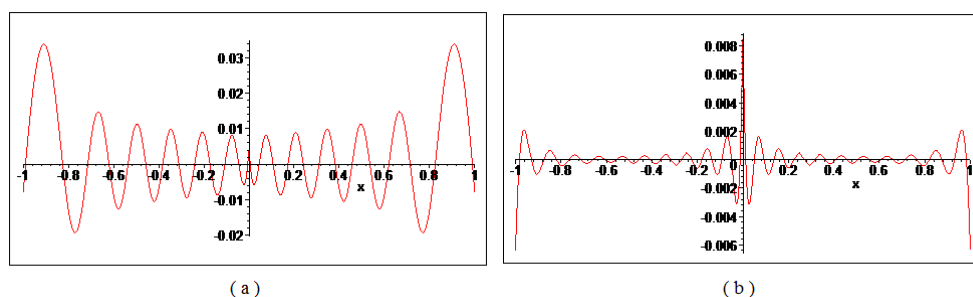


FIGURE 5. Graph of the approximation error  $g(x) - K_n^c g(x)$  with (a)  $c = 50$ ,  $n = 20$  and (b)  $c = 50$ ,  $n = 50$ .

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