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Abdallah Al-Amrani

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HISTORY / PREFACE

As far as the author of the present monograph knows, weighted projective spaces (and implicitly weighted lens spaces) were , first of all , studied in the Ph.D. thesis [TR] (not published) ([1]) :

The Cohomology Ring of Pseudo-Projective Spaces, by Henry J. TRAMER. Johns Hopkins University (Baltimore , Maryland) (1965).

The supervisor was Jun-Ishi IGUSA . He mentioned a part of Tramer's work at the I.C.M. in Sweden, 1962 [IG].

Almost ten years after, Tetsuro KAWASAKI published his computation of "Cohomology of Twisted Projective Spaces and Lens Complexes" [KW] , independently of Tramer's Ph.D.

It is remarkable that both of Tramer and Kawasaki begin their computations by the case of weighted lens spaces (w.l.s.), from which they deduce the case of weighted projective spaces (w.p.s.). And so did Masato KUWATA for intersection homology [KU]. Indeed, w.p.s.'s are covered , except for a finite number of "origin" points, by w.l.s.'s as open subsets.

These spaces , well-known now as weighted (projective or lens) spaces , are also called "anisotropic" [DE, JO1]. In his very early work [MO], Shigefumi MORI used the qualifier "weak projective space" for the "good" space where to embed generalized (i.e. weighted) complete intersections . It was shown in [AA1983] , that this is nothing else but the regular locus of the ambient w.p.s. Ourselves we prefer the adjective "twisted" because of its geometric meaning ("tordu" in French) [KW].

Immediately after the publication of Kawasaki's article, Jean-Pierre JOUANOLOU asked his student A.A. to achieve a systematic cohomological study of these weighted spaces (projective as well as lens) ,

[1] In 2013, we asked the Mathematics Library (Université de Strasbourg) to do everything possible to get a copy from Johns Hopkins University. It took some longtime, and cost 65$! Many thanks to our librarian Christine DISDIER.

in the algebraic sitting (i.e. as algebraic varieties, or , more generally, as schemes). Algebraic K-theory was included in the question ([2]) .

At the same times (around 1974), Christophe DELORME [DE] made a large study of the "Espaces projectifs anisotropes" as schemes (over any ground ring), and Shigefumi Mori (loc.cit.) found where to embed his generalized complete intersections. Delorme was the first to establish a global reduction of the weights of a w.p.s. We checked that Delorme's weights reduction is the best possible [AA1983].

Weighted projective and lens spaces belong to the realm of the so popular toric varieties. However their essence is not toric : they do not need any fancy fan to exist , nor to be wonderfully studied. They are simple and nice examples of non-smooth algebraic varieties, living in the pleasant sitting of G.I.T. , à la MUMFORD [MU]. Structure of their cohomological theories has to be computed in terms of their intrinsic geometry as quotients of canonical spaces by canonical algebraic groups (G_m, μ_q).

A good illustration of this is intersection homology as determined by Kuwata (loc.cit.) : no cones of any fan are needed . Certainly, for an arbitrary toric variety , these are necessary (by definition !). See FULTON & al.[FS]. An other nice example is the early study done by Igor DOLGACHEV [DO] where BOTT 's theorem (on cohomology of differential sheaves) is extended to w.p.s.'s .

Let us make precise that we do not pretend that toric geometry is of no any help to the study of w.p.s.'s. For example, in his ".... huge Grothendieck group", Joseph GUBELADZE [GU] showed some evidence that the algebraic K-group of a w.p.s. (associated to vector bundles) may not be finitely generated ! Since the seventies (1970's), when the question was asked to the author by Jouanolou, that is still an open problem (as far as we know, of course !).

Now, about *algebraic twisted lens spaces*, to the best of our knowledge, no work is known , nothing is mentioned anywhere ([3]), except for our own non-published work on the subject . In fact, under Jouanolou's supervision, we have carried out cohomological calculations for both twisted spaces, projective and lens, at the same time and in the two cases, topological and algebraic (see our Habilitation thesis, Rabat / Strasbourg , 1985) . We dealt with : étale cohomology, complex K-theory, CHOW group, coherent K-group, twisted CHERN classes [AA] (unfinished !) References may be found inside the quoted literature.

[2]Since we are dealing with some history, let us recall that JOUANOLOU was a cohomological student of GROTHENDIECK.

[3]We posted in vain a question in this regard on the MathOverflow wiki site.

Notice that some KO-theory of particular w.l.s.'s has been considered by Y. NISHIMURA and Z. YOSIMURA [NY].

Recently, in conjunction with the Manchester Toric Topology school (Nigel RAY & cie.), some colleagues made relive our topological work on w.p.s.'s. They answered some of our questionings or rediscovered certain results [BFR1, BFR2, BFR3]. This motivated and encouraged us to reconsider our non-published study of the algebraic twisted lens spaces and to rewrite it in English.

A forthcoming work will deal with twisted lens BUNDLES (inchallah !) .

The present monograph on algebraic twisted lens spaces contains :

I. Construction, properties.
II. Etale cohomology.
III. Intersection theory.
   A. CHOW group;
   B. Coherent K-theory;
   C. ℓ-adic Homology.

A.A.
Trinidad de CUBA,
April 2014.

I. Weighted lens spaces as schemes

We are going to construct *weighted lens spaces* as geometric quotients in the category of schemes ([MU, GR1]).

A. Construction.

Let k be a fixed field (any characteristic, algebraically closed or not). Fixed are also integers

$$q; q_0, q_1, \ldots, q_n \in \mathbb{Z} \setminus \{0\}.$$

1. Denote by

$$U_q = U_q(k) = \{\lambda \in k^* \mid \lambda^q = 1\}$$

the *subgroup of q-th roots of unity* in the multiplicative group $k^* = k \setminus \{0\}$. This is a cyclic group the order of which is a divisor of $|q|$.

In case k is algebraically closed and $|q|$ is prime to the characteristic exponent of k, U_q has order equal to $|q|$ ($char.exp.(k) := 1$ if $char(k) = 0; := char(k)$ if not). If $char(k) \neq 0$ and $|q|$ is a power of $char(k)$, then $U_q = \{1\}$. Put:

$$d = d(k, q) = |U_q| \qquad (\text{order of } U_q).$$

The constant algebraic group (over k), defined by the (abstract) group U_q, is *the group of d-th roots of unity* (over k):

$$\mu_d = \mu_{d,k} = Spec(k[T]/(T^d - 1))$$

(since $T^d - 1 = \prod_{\lambda \in U_q}(T - \lambda)$).

Given a k-scheme S, an action (by automorphisms) of U_q on S, is equivalent to an action of μ_d on S.

2. Consider the scheme $X = A_k^{n+1} \setminus \{0\}$ (A_k^m stands for affine space over k). Let us make U_q act by automorphisms on X. Fix $\lambda \in U_q$. This gives a k-algebra automorphism

$$\begin{array}{rccc} u = u_\lambda : & R = k[T_0, \ldots, T_n] & \longrightarrow & R \\ & T_s & \longmapsto & \lambda^{q_s} T_s \quad (0 \leq s \leq n), \end{array}$$

which induces automorphisms ($0 \leq i, j \leq n$):

$$\begin{array}{rcccc} u_i : & R_i = k[T_0, \ldots, T_n, T_i^{-1}] & \longrightarrow & R_i \\ u_{ij} : & R_{ij} = k[T_0, \ldots, T_n, T_i^{-1}, T_j^{-1}] & \longrightarrow & R_{ij}. \end{array}$$

Now these fill in obvious *commutative* diagrams:

$$\begin{array}{ccccccc} R_i & \to & R_{ij} & \tilde{\to} & R_{ji} & \leftarrow & R_j \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle u_{ij}} & & \downarrow{\scriptstyle u_{ji}} & & \downarrow{\scriptstyle u_j} \\ R_i & \to & R_{ij} & \tilde{\to} & R_{ji} & \leftarrow & R_j \end{array}$$

(canonical mappings).

The corresponding affine schemes (morphisms included) are denoted:

$$\begin{array}{ccccccc} X_i & \leftarrow & X_{ij} & \tilde{\leftarrow} & X_{ji} & \rightarrow & X_j \\ {\scriptstyle g_i}\uparrow & & \uparrow{\scriptstyle g_{ij}} & & \uparrow{\scriptstyle g_{ji}} & & \uparrow{\scriptstyle g_j} \\ X_i & \leftarrow & X_{ij} & \tilde{\leftarrow} & X_{ji} & \rightarrow & X_j \end{array}$$

Since $(X_i)_{0\leq i\leq n}$ is an open affine covering of $X = A_k^{n+1} \setminus \{0\}$, with $X_i \cap X_j \simeq X_{ij}$, the preceding commutative diagrams define an automorphism g_λ of X (which restricts to g_i on $X_i (0 \leq i \leq n)$).

Let $\lambda' \in U_q$. Then:

$$(g_{\lambda\lambda'})|X_i = (g_\lambda)|X_i \circ (g_{\lambda'})|X_i = (g_\lambda \circ g_{\lambda'})|X_i$$

(to be checked at level of R_i, k-algebra of X_i!)

Whence: $g_{\lambda\lambda'} = g_\lambda \circ g_{\lambda'}$. So we have a morphism of groups

$$g = g(q_0, \ldots, q_n) : U_q \longrightarrow Aut_k(X),$$

that is *an action* (by automorphisms) of U_q on *the scheme* X.

3. Construction of a geometric quotient of X by U_q.

First let us recall the general case of a *finite* abstract group acting on a scheme by automorphisms [GR1].

a) General case.

Consider a k-scheme Y and a finite abstract group G, acting on Y by automorphisms. This means we have a group morphism $G \to Aut_k(Y)$. It is equivalent to an action of the constant group scheme G_k on Y [GR1].

Assume Y affine:

$$Y = Spec(A) \qquad (A \text{ } k\text{-algebra}).$$

Since $Aut_k(Y) = Aut_k(A)$, the group G operates on A. Its invariants form a sub-k-algebra $A^G \subset A$.

The result we shall apply is [Exp. V, loc. cit.]:

i) The morphism $Y \to Z = Spec(A^G)$ is a geometric quotient of Y by G (i.e., by G_k).

ii) If S is a k-scheme where G operates, then a geometric quotient of S by G exists when, and only when, S is covered by affine open subsets which are G-invariant.

b) Geometric quotient of X by U_q.

Recall that $X = A_k^{n+1} \setminus \{0\}, U_q = \{\lambda \in k^* \mid \lambda^q = 1\}$, k being the fixed (under)ground field, and that μ_d is the constant scheme group over k defined by U_q (abstract group). So, a geometric quotient of X by U_q, or by μ_d, that is the same thing (by definition!).

We are going to use a) ii) above. Because of the construction of the operation $g : U_q \to Aut_k(X)$, a) ii) shows that such a geometric quotient exists. Let us give an explicit construction.

We have group homomorphisms ($0 \leq i, j \leq n$):

$$\begin{array}{c} U_q \to Aut_k(X_i) = Aut_k(R_i) : \lambda \mapsto g(\lambda)|X_i, \\ U_q \to Aut_k(X_{ij}) = Aut_k(R_{ij}) : \lambda \mapsto g(\lambda)|X_{ij}, \end{array}$$

which define operations of U_q on the algebras R_i, R_{ij}. Put:

$$\begin{array}{rl} R_i^{(q)} & = R_i^{(q)}(q_0, \ldots, q_n) = R_i^{U_q} \quad \text{(invariants under } U_q(^4)) \\ Z_i & = Z_i(q; q_0, \ldots, q_n) = Spec(R_i^{(q)}), \end{array}$$

and

$$\begin{array}{rl} R_{ij}^{(q)} & = R_{ij}^{(q)}(q_0, \ldots, q_n) = R_{ij}^{U_q} \\ Z_{ij} & = Z_{ij}(q; q_0, \ldots, q_n) = Spec(R_{ij}^{(q)}). \end{array}$$

The inclusions $R_i^{(q)} \subset R_i, R_{ij}^{(q)} \subset R_{ij}$, give morphisms $\rho_i : X_i \to Z_i$, $\rho_{ij} : X_{ij} \to Z_{ij}$ such that (Z_i, ρ_i) and (Z_{ij}, ρ_{ij}) be geometric quotients of X_i and X_{ij} by U_q (respectively) (after a)i).).

Now, open immersions $\varphi_{ij} : X_{ij} \to X_i$ and canonical isomorphisms $X_{ij} \xrightarrow{\sim} X_{ji}$ are U_q-equivariant (see §.1). This induces commutative diagrams ($0 \leq i, j \leq n$)

$$\begin{array}{ccccccc} X_i & \xleftarrow{\varphi_{ij}} & X_{ij} & \xrightarrow{\sim} & X_{ji} & \xrightarrow{\varphi_{ji}} & X_j \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z_i & \xleftarrow{\bar{\varphi}_{ij}} & Z_{ij} & \xrightarrow{\sim} & Z_{ji} & \xrightarrow{\bar{\varphi}_{ji}} & Z_j \end{array}$$

where $\bar{\varphi_{ij}}$ and $\bar{\varphi_{ji}}$ are open immersions (see definition of geometric quotients as in GIT[MU]). Let Z be the k-scheme obtained by gluing the Z_i's along the isomorphisms $Z_{ij} \xrightarrow{\sim} Z_{ji}$ above. The result is a k-scheme morphism

$$\rho : X \longrightarrow Z$$

which restricts, for each i, to the geometric quotient $\rho_i : X_i \to Z_i$. Hence (Z, ρ) *is a geometric quotient of* X *by* U_q (This is a local property on Z by GIT! [MU] (0.§1.)).

[4]An explicit computation of $R_i^{(q)} := R_q^{U_q}$ is given in next n°. It will justify the notation.

Definition.
The k-scheme Z we just constructed is denoted

$$\tilde{L}_k^n = L_k^n(q; q_0, \ldots, q_n),$$

and called *lens space* over the field k, *of type* $(q; q_0, \ldots, q_n)$. We shall see that $L_k^n(q; q_0, \ldots, q_n)$ is closely linked to the weighted projective space $P_k^{n+1}(q_0, \ldots, q_n, q)$. That is why such a lens space is also named *weighted lens space* (or *twisted* [5] lens space).

c) *Invariant subalgebra* $R_i^{(q)} = R_i^{U_q}$.
Fix a generator ξ of U_q. For $0 \leq i \leq n$, an element of the form

$$a_{\underline{\alpha}} = T_0^{\alpha_0} \cdots T_n^{\alpha_n} \in R_i = k[T_0, \ldots, T_n, T_i^{-1}] \quad (\alpha_i \in \mathbb{Z}, \alpha_s \in \mathbb{N}(s \neq i))$$

is invariant under the action of U_q, if and only if

$$\xi^\beta a_{\underline{\alpha}} = a_{\underline{\alpha}}, \quad \text{where} \quad \beta = \sum_0^n \alpha_s q_s,$$

that is $\xi^\beta = 1$. In other words β is multiple of $|U_q|$. So the invariant subalgebra $R_i^{(q)} = R_i^{U_q}$ is generated by the elements $a_{\underline{\alpha}}$ above with $\sum_0^n \alpha_s q_s$ multiple of $|U_q|$.

4. A graduation construction (à la Grothendieck).
We assume the integers $q_i (0 \leq i \leq n)$ *positive*. The k-algebra $R = k[T_0, \ldots, T_n]$ is $\mathbb{N}$-graded by :

$$deg(T_i) = q_i \ (0 \leq i \leq n), \ deg(\lambda) = 0 \ (\lambda \in k).$$

Equipped with this graduation the k-algebra R is denoted $R = R(q_0, \ldots, q_n)$.

A $\mathbb{Z}$-graduation is induced on each $R_i = R_{T_i}$ such that the invariant subalgebra $R_i^{(q)} \subset R_i$ (under U_q) is generated by homogeueous elements of degree multiple of $|U_q|$ (after 3.c).) This suggests to generalize Grothendieck's construction for Proj as follows ([GR2], (II, §.2)).

Consider a commutative ring A, with 1, $\mathbb{N}$-graded, and an integer $r \geq 0$. Then one can construct a scheme $S(A, r)$ such that

$$\begin{array}{rcl} S(A, 0) & = & Proj(A), \\ S(R(q_0, \ldots, q_n), |U_q|) & = & L_k^n(q; q_0, \ldots, q_n). \end{array}$$

So it is natural to define $Len(A, r) := S(A, r)$ where $r > 0$ (*Len* stands for lens (space) as *Proj* does for projective (space)).

Construction of the scheme $S(A, r)$. Of course we follow the construction of Proj in loc.cit. Let $f \in A$ be a homogeneous element in

[5] We prefer the qualifyer *twisted* to weighted because of its geometric meaning [KW].

A of degree $\alpha > 0$. Then A_f is $\mathbb{N}$-graded, and $(A_f)^{(r)}$ stands for its subring generated by homogeneous elements of degree multiple of r (hence $(A_f)^{(0)} = A_{(f)}$ in standard notation for homogeueous elements of degree 0.) Choose another homogeneous element $g \in A$ of degree $\beta > 0$. Let us check that we have a canonical isomorphism:

$$(A_{fg})^{(r)} \xrightarrow{\sim} ((A_f)^{(r)})g^\alpha/f^\beta$$

(in A_f $deg(g^\alpha/f^\beta) = 0$).

In case $r = 0$, this is the first part of Lemma 2.2.2 [GR2] (II§.2). If $r > 0$, since

$$((A_{f^\beta})g^\alpha/f^\beta)^{(r)} = (A_{f^\beta})^{(r)}g^\alpha/f^\beta$$

and $A_{f^\beta} = A_f$, the proof is the same as in the lemma loc.cit.

We deduce a canonical isomorphism

$$(A_f)^{(r)}g^\alpha/f^\beta \simeq (A_g)^{(r)}f^\beta/g^\alpha.$$

Put

$$S_f = Spec((A_f)^{(r)}) \text{ and } \ S_{(f,g)} = Spec((A_f)^{(r)}g^\alpha/f^\beta).$$

So we get

$$\begin{array}{lll} S_{(f,g)} & \to & S_f, \quad \text{canonical open immersion,} \\ S_{(f,g)} & \xrightarrow{\sim} & S_{(g,f)}, \quad \text{canonical isomorphism.} \end{array}$$

By definition the *scheme* $S(A, r)$ is obtained by gluing the affine schemes S_f ($f \in A$ homogeneous of degree > 0) along the isomorphisms $S_{(f,g)} \simeq S_{(g,f)}$ ($f, g \in A$ homogeneous of degree > 0).

In particular, 3.c) shows that

$$L^n_k(q; q_0, \ldots, q_n) = S(R(q_0, \ldots, q_n), |U_q|).$$

We end this paragraph by a remark on positivity hypothesis on the *weights* $q_0, \ldots, q_n$ (made at the beginning of this §.4).

Remark. Recall the group morphism (§.2)

$$g = g(q_0, \ldots, q_n) : U_q \to Aut(X) \ (q \geq 1, q_i \in \mathbb{Z}(0 \leq i \leq n))$$

which defines the operation of U_q on X the geometric quotient of which is $L^n_k(q; q_0, \ldots, q_n)$ (§.3).

For any $m \in \mathbb{Z}$ such that $q_i + mq \neq 0$ (all i) one has

$$g(q_0 + mq, \ldots, q_n + mq) = g(q_0, \ldots, q_n)$$

and, therefore,

$$L^n_k(q; q_0 + mq, \ldots, q_n + mq) = L^n_k(q; q_0, \ldots, q_n).$$

Choosing m such that

$$q'_i := q_i + mq > 0 \text{ (all } i)$$

we obtain

$$L_k^n(q; q_0, \ldots, q_n) = Len(A, r)$$

with $A = R(q'_0, \ldots, q'_n), r = |U_q|$.

B. Some properties of the scheme $\tilde{L}_k^n$

We keep the same data and notation as before:

$$\tilde{L}_k^n = L_k^n(q; q_0, \ldots, q_n) = (A_k^{n+1} \setminus \{0\})/U_q.$$

1. *The weighted lens space $\tilde{L}_k^n$ is integral and of finite type over k.*

It is integral since it is a geometric quotient of $X = A_k^{n+1} \setminus \{0\}$ (by U_q) and X is integral. It is of finite type over k since it is covered by open immersions

$$Z_i = Z_i(q; q_0, \ldots, q_n) \subset \tilde{L}_k^n (0 \leq i \leq n)$$

(given by its construction in A.3.b)), where the Z_i are of finite type over k:

$$Z_i = X_i/U_q = Spec(R_i^{U_q}).$$

2. *Link to weighted projective spaces. If the group U_q has order equal to q there exists an open immersion*

$$\tilde{L}_k^n = L_k^n(q; q_0, \ldots, q_n) \subset P_k^{n+1}(q_0, \ldots, q_n, q) = \tilde{P}_k^{n+1}$$

where $\tilde{P}_k^{n+1}$ is the well-known weighted projective space over k of type $(q_0, \ldots, q_n, q)$ (constructed as geometric quotient and cohomologically studied in [AA]).

Before going into details, let us roughly say that any *weighted projective space* is covered by affine charts which are, except for a point (the origin, fixed under the U_{q_i} in consideration), *weighted lens spaces.*

We use [AA], with its notation. Put $q_{n+1} = q$. Then (since $|U_q| = q$)

$$Y_{n+1} = Y_{n+1}(q_0, \ldots, q_{n+1}) \cong A_k^{n+1}/U_q(q_0, \ldots, q_n)$$

is the $(n+1)$-th open affine chart of $\tilde{P}_k^{n+1}$ ([AA] (II.3.c)). Hence it is enough to check that we have an open immersion

$$\tilde{L}_k^n = (A_k^{n+1} \setminus \{0\})/U_q \hookrightarrow Y_{n+1}.$$

The operation of U_q on $X = A_k^{n+1} \setminus \{0\}$ defining $\tilde{L}_k^n$ is the restriction of that of U_q on A_k^{n+1} defining Y_{n+1} above. So the open immersion $A_k^{n+1} \setminus \{0\} \subset A_k^{n+1}$ gives the one we look for.

Let us be more specific.

The origin $0 \in A_k^{n+1}$ is fixed under the action of U_q. We have

$$\tilde{L}_k^n = (A_k^{n+1} \setminus \{0\})/U_q = (A_k^{n+1}/U_q) \setminus \{0\} \overset{open}{\hookrightarrow} A^{n+1}/U_q$$

(no hypothesis is needed on $|U_q|$ for that).

If $|U_q| = q$, then we have

$$\tilde{L}_k^n \xrightarrow{\sim} Y_{n+1} \setminus \{(0,\dots,0,1)\} \xrightarrow{open} \tilde{P}_k^{n+1}$$

(where $(0,\dots,0,1)$ corresponds to $0 \in A_k^{n+1}/U_q$). We always have a morphism

$$A_k^{n+1}/U_q \to Y_{n+1} = \tilde{P}_k^{n+1} \setminus \tilde{P}_k^n(q_0,\dots,q_n),$$

which is an isomorphism if $|U_q| = q$.

Conclusion. a natural morphism $L_k^n(q;q_0,\dots,q_n) \to P_k^{n+1}(q;q_0,\dots,q_n)$ always exists. It is an open immersion if $|U_q| = q$.

3. Fix an integer $r \geq 1$ such that $|U_{rq}| = r|U_q|$.

We have a commutative diagram

$$\begin{array}{ccc} L_k^n(rq;rq_0,\dots,rq_n) & \xrightarrow{\sim} & L_k^n(q;q_0,\dots,q_n) \\ \downarrow & & \downarrow \\ P_k^{n+1}(rq_0,\dots,rq_n,rq) & \xrightarrow{\sim} & P_k^{n+1}(q_0,\dots,q_n,q) \end{array}$$

Vertical morphisms are those of 2. above.

The isomorphism down is natural (compatible with the canonical morphism (quotient by $G_m : A_k^{n+2} \setminus \{0\} \to \tilde{P}_k^{n+1}$)) (given by

$$R_{i,0}(q_0,\dots,q_n,q) = R_{i,0}(rq_0,\dots,rq_n,rq)$$

: local invariants)[6].

The top isomorphism is also compatible with the canonical projection $A_k^{n+1} \setminus \{0\} \to \tilde{L}_k^n$. It is a consequence of the equalities

$$R_i^{(q)}(q_0,\dots,q_n) = R_i^{(rq)}(rq_0,\dots,rq_n) \ (0 \leq i \leq n)$$

(because $|U_{rq}| = r|U_q|$ by hypothesis; see A.3.c).).

4. *Another link to weighted projective spaces.*

There is a natural morphism

$$\theta : \tilde{L}_k^n = L_k^n(q;q_0,\dots,q_n) \to \tilde{P}_k^n = P_k^n(q_0,\dots,q_n).$$

Indeed, we have k-algebras inclusions

$$R_i^{(0)} \subset R_i^{(q)}$$

(invariants for G_m and U_q, respectively.) where $R_i := k[T_0,\dots,T_n]_{T_i}$ is graded by: $deg(T_s) = q_s$. This induces a morphism of affine schemes

$$\theta_i : Z_i \to Y_i$$

such that

$$\theta_i | Z_i \cap Z_j = \theta_j | Z_j \cap Z_i$$

[6] $R_{i,0} := R_i^0$ stands for homogeneous elements of degree 0, after localization by T_i.

$(Z_i \cap Z_j = Z_{ij} = Spec\, R_{ij}^{(q)}, Y_i \cap Y_j = Y_{ij} = Spec\, R_{ij}^{(0)})$. So the θ_i's glue together and give an *affine morphism*

$$\theta : Z = \tilde{L}_k^n \to Y = \tilde{P}_k^n.$$

Next we make precise the geomeric nature of θ modulo a hypothesis on the type $(q; q_0, \ldots, q_n)$ of the weighted lens space $\tilde{L}_k^n$.

Proposition 1. *Assume $|U_q| = q, q_i \geq 1, (0 \leq i \leq n)$ and q divisible by each q_i. Then there exists an invertible $\mathcal{O}_{\tilde{P}_k^n}$- Module $\mathcal{L}$ with the following commutative diagram:*

$$\begin{array}{ccc} \tilde{L}_k^n & \xrightarrow{\quad\sim\quad} & V(\mathcal{L})^* \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle \text{can. proj.}} & \\ & \tilde{P}_k^n & \end{array}$$

where $\tilde{L}_k^n = L_k^n(q; q_0, \ldots, q_n), \tilde{P}_k^n = P_k^n(q_0, \ldots, q_n)$ and $V(\mathcal{L})^*$ is the complement of the zero-section of the vector bundle $V(\mathcal{L})$ associated to $\mathcal{L}$.

Proof. First of all we fix (recall) some notations.

$$Y_i^n = Y_i(q_0, \ldots, q_n)\ (0 \leq i \leq n)$$

is the complement of the closed immersion $P_k^{n-1}(q_0, \ldots, \hat{q}_i \ldots, q_n) \subset \tilde{P}_k^n$.

$$Y_i^{n+1} = Y_i(q_0, \ldots, q_{n+1})$$

with $q_{n+1} = q$ $(0 \leq i \leq n+1)$. So $\tilde{P}_k^n = \cup_{0 \leq i \leq n} Y_i^n, \tilde{P}_k^{n+1} = \cup_{0 \leq i \leq n+1} Y_i^{n+1}$; put $V = \cup_{0 \leq i \leq n} Y_i^{n+1}$ (open in $\tilde{P}_k^{n+1}$). The inclusion of graded algebras $(deg(T_j) = q_j (0 \leq j \leq n+1))$

$$R(q_0, \ldots, q_n) = k[T_0, \ldots, T_n] \subset R(q_0, \ldots, q_{n+1}) = k[T_0, \ldots, T_{n+1}]$$

defines an affine morphism $\omega : V \to \tilde{P}^n$ which fills in the commutative diagram

$$\begin{array}{ccccc} \tilde{L}^n & \xrightarrow{\quad\sim\quad} & Y_{n+1}^{n+1} \setminus \{(0, \ldots, 0, 1)\} & \hookrightarrow & V \\ & {\scriptstyle\theta}\searrow & & \swarrow{\scriptstyle\omega} & \\ & & \tilde{P}^n & & \end{array}$$

The isomorphism $\tilde{\to}$ comes from §.2(since $|U_q| = q$). The diagram commutativity is easily (locally) checked, looking at k-algebras (of invariants $\cdots$) which define θ, ω.

For instance let us explicit

$$V^* := Y_{n+1}^{n+1} \cap V = Y_{n+1}^{n+1} \setminus \{(0, \ldots, 0, 1)\} = \cup Y_{i,n+1}^{n+1} \ (0 \leq i \leq n);$$

we have:

$$Y_{i,j}^{n+1} = Y_i^{n+1} \cap Y_j^{n+1} = Spec((R_{i,j}^{n+1})^{(0)})$$

where $R_{i,j}^{n+1} = k[T_0, \ldots, T_{n+1}, T_i^{-1}, T_j^{-1}]$ $(deg(T_s) = q_s, q_{n+1} = q)$.

The following commutative diagrams are the local (affine) form of the ones considered.

$$\begin{array}{ccccc} (R_i^{n+1})^{(0)} & \hookrightarrow & (R_{i,n+1}^{n+1})^{(0)} & \xrightarrow{\sim} & (R_i^n)^{(q)} \\ & \nwarrow & & \nearrow & \\ & & (R_i^n)^{(0)} & & \end{array} \qquad (0 \leq i \leq n)$$

The isomorphism is given by

$$T_0^{\alpha_0} \cdots T_{n+1}^{\alpha_{n+1}} \mapsto T_0^{\alpha_0} \cdots T_n^{\alpha_n} \quad (\text{where } \sum_0^{n+1} \alpha_s q_s = 0)$$

Now, since ω is affine, one has

$$V = Spec(\omega_* \mathcal{O}_V), \text{ with } \mathcal{O}_V = \mathcal{O}_{\tilde{P}^{n+1}} | V.$$

The weights q_i $(0 \leq i \leq n)$ are positive; so ([A1](II,2.(d)))

$$\tilde{P}^n = Proj\ R(q_0, \ldots, q_n)$$

and ([GR2] (II.2.5.7))

$$\mathcal{O}_{\tilde{P}^n}(-q) | Y_i^n = \mathcal{O}_{\tilde{P}^n} | Y_i^n \ (0 \leq i \leq n)$$

because q is divisible by each q_i. Hence the $\mathcal{O}_{\tilde{P}^n}$-Module $\mathcal{O}_{\tilde{P}^n}(-q)$ is invertible. So we need only to prove the following.

Lemma 2. *Put $P = \tilde{P}_k^n$. If the weights q_i devide q, then there exists an isomorphism of $\mathcal{O}_P$-Algebras (quasi-coherent)*

$$\omega_* \mathcal{O}_V = S_{\mathcal{O}_P}(\mathcal{O}_P(-q)) \quad (\text{symmetric Algebra of } \mathcal{O}_P(-q)),$$

inducing an isomorphism of P-schemes

$$V^* = V(\mathcal{O}_P(-q))^* \quad (\text{complement of zero-section}).$$

Proof. Let us define an $\mathcal{O}_P$-Algebras morphism

$$\tilde{u} : S_{\mathcal{O}_P}(\mathcal{O}_P(-q)) \to \omega_* \mathcal{O}_V.$$

Here $P = \tilde{P}^n_k = Proj(R(q_0, \ldots, q_n))$. We have graded k-algebras

$$R^n = R(q_0, \ldots, q_n), R^{n+1} = R(q_0, \ldots, q_{n+1}) \ (q_{n+1} = q),$$

and the R^n-graded module $R^n[-q]$ is free, generated by 1 (of degree q). Whence a graded R^n-modules morphism:

$$\begin{array}{rccc} u: & R^n[-q] & \to & R^{n+1} \quad (= k[T_0, \ldots, T_{n+1}]) \\ & 1 & \mapsto & T_{n+1} \end{array}$$

defining an $\mathcal{O}_P$-Modules morphism:

$$\mathcal{O}_P(-q) \to \omega_* \mathcal{O}_V.$$

This induces $\tilde{u}$ above.

i) We show that $\tilde{u}$ is an isomorphism. It is enough to see that, for each $i (0 \leq i \leq n), \tilde{u}_i = \tilde{u}|Y^n_i$ is an isomorphism. Put $Y = Y^n_i$ (i being fixed). There exist natural isomorphisms

$$S_{\mathcal{O}_P}(\mathcal{O}_P(-q))|Y = S_{\mathcal{O}_P|Y}(\mathcal{O}_P|Y) = (\mathcal{O}_P|Y)[T] \ (T\text{=indeterminate})$$

$$(\omega_* \mathcal{O}_V)|Y = (\omega|Y^{n+1}_i)_*(\mathcal{O}_V|Y^{n+1}_i).$$

Hence $\tilde{u}_i$ is associated to the $(R^n_i)^{(0)}$-algebras morphism

$$\begin{array}{rccc} u_i: & (R^n_i)^{(0)}[T] & \to & (R^{n+1}_i)^{(0)} \quad (\subset k[T_0, \ldots, T_{n+1}, T_i^{-1}]) \\ & T & \mapsto & T_i^{\alpha_i} T_{n+1}, \qquad \text{with } \alpha_i = -q/q_i \end{array}$$

(q_i divides q by hypothesis).

Recall that the canonical isomorphism

$$\mathcal{O}_P(-q)|Y = \mathcal{O}_P|Y$$

corresponds to the isomorphism [GR2](II.2.5.7)

$$(R^n_i)^{(0)} \xrightarrow{\sim} (R^n[-q]_{T_i})^{(0)}$$

defined by multiplication by the invertible element $T_i^{\alpha_i}$. But u_i is bijective: it is surjective because $(R^{n+1}_i)^{(0)}$ is generated by elements of the form

$$a_{\underline{\alpha}} = T_0^{\alpha_0} \ldots T_{n+1}^{\alpha_{n+1}}$$

with $\alpha_s \in \mathbb{N} \ (s \neq i), \ \alpha_i \in \mathbb{Z}$, such that $\sum_0^{n+1} \alpha_s q_s = 0$ (recall $q_{n+1} = q$); and then

$$u_i((T_0^{\beta_0} \cdots T_n^{\beta_n}) T^{\beta}) = a_{\underline{\alpha}}$$

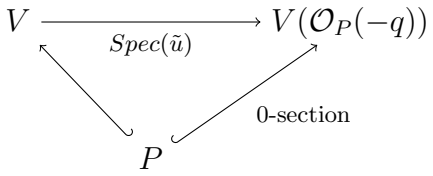
where

$$\begin{array}{rcl} \beta_s & = & \alpha_s \quad (0 \leq s \leq n, s \neq i) \\ \beta_i & = & \alpha_i + \frac{q}{q_i} \alpha_{n+1} \\ \beta & = & \alpha_{n+1} \end{array}$$

ii) To check the second assertion in the lemma, consider the isomorphism

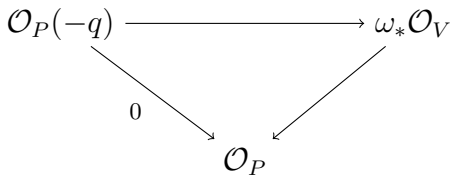
$$Spec(\tilde{u}) : V \xrightarrow{\sim} V(\mathcal{O}_P(-q)).$$

It sends V^* on $V(\mathcal{O}_P(-q)^*)$: indeed we have a closed immersion $P \subset V$ (induced by the obvious one $\tilde{P}^n \subset \tilde{P}^{n+1}$) such that $V \setminus P = V^*$. So we need only to be sure that

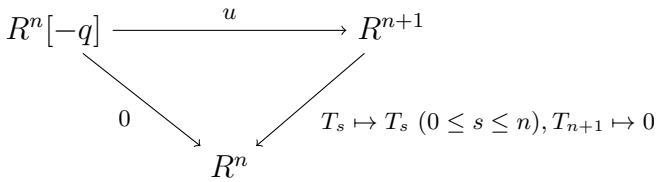


commutes.

But this is induced by the $\mathcal{O}_P$-Modules morphisms



which are associated to the R^n-modules graded morphisms (the diagram of which is commutative)



□

The proposition is proven. □

Remarks. Keep hypotheses of the preceding proposition.

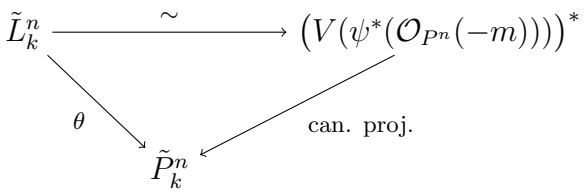
α) If we go back to the definition of the morphism ([A1] (II.3))

$$\psi : \tilde{P}_k^n = P_k^n(q_0, \ldots, q_n) \to P_k^n = P_k^n(1, \ldots, 1)$$

we see that

$$\psi^*(\mathcal{O}_{P^n}(-m)) = \mathcal{O}_{\tilde{P}^n}(-q), \text{ where } m = q/lcm(q_0, \ldots, q_n).$$

β) From the proof of the proposition we deduce (commutative diagram):



Proposition 3. *We assume $|U_q| = q, q_i \geq 1 (0 \leq i \leq n)$. The quotient-scheme (see §.2)*

$$A^{n+1}/U_q = (A^{n+1}/U_q)(q_0, \ldots, q_n)$$

is then the affine projecting cone of a projective variety (over k) the blunt cone of which is $\tilde{L}_k^n = L_k^n(q; q_0, \ldots, q_n)$.

Proof. One has (A.3.a),c))

$$A^{n+1}/U_q = Spec(B), \quad \text{with} \quad B = R^{U_q}, R = k[T_0, \ldots, T_n].$$

The subalgebra of invariants B is generated by the monomials

$$a_{\underline{\alpha}} = T_0^{\alpha_0} \cdots T_n^{\alpha_n} \quad \text{such that} \quad \sum \alpha_s q_s = 0 \pmod{q}.$$

For such elements put

$$deg(a_{\underline{\alpha}}) := (\sum \alpha_s q_s)/q.$$

This defines an $\mathbb{N}$-graduation on B (with $B_0 = k$), which is equivalent to an operation of the line $D_k = Spec(k[T])$ on A^{n+1}/U_q as we know. So the quotient-scheme A^{n+1}/U_q is an affine cone with vertex the point 0. On the other hand, we have a closed immersion

$$Proj(B) \hookrightarrow P_k^m \qquad (B \text{ being } \mathbb{N}\text{-graded})$$

for some integer $m \geq 1$. Indeed, as a subalgebra of invariants by U_q in R, B is of finite type ([GR1], Exp. V), and, for each integer $r \geq 1$, the set

$$\{T_0^{\alpha_0} \cdots T_n^{\alpha_n} \mid \sum \alpha_s q_s = rq\}$$

is finite. Then from [GR2], II. the required closed immersion follows. To conclude, see §.2. □

II. Étale cohomology

Before computing étale cohomology of the schemes $\tilde{L}^n_k$, we first compute integral cohomology of the (topological) spaces $\tilde{L}^n_{\mathbb{C}}$.

This is done in a way which extends to the case of étale cohomology.

The results are heavily based on the cohomological study done for weighted projective spaces in [AA].

1. Integral cohomology of $\tilde{L}^n_{\mathbb{C}}$.

The space $\tilde{L}^n_{\mathbb{C}} = L^n_{\mathbb{C}}(q; q_0, \ldots, q_n)$, where $q, q_0, \ldots, q_n$ are positive integrs, is definied as the topological quotient

$$\tilde{L}^n_{\mathbb{C}} = (\mathbb{C}^{n+1})^*/\mu_q$$

of the action

$$\lambda.(z_0, \ldots, z_n) = (\lambda^{q_0} z_0, \ldots, \lambda^{q_n} z_n),$$
$$\lambda \in \mu_q = \{z \in \mathbb{C} \mid z^q = 1\} \qquad (q\text{-th roots of unity}).$$

Remark. This definition of $\tilde{L}^n_{\mathbb{C}}$ works for $q_0, \ldots, q_n \in \mathbb{Z}^*$. If we replace the q_i's by $q_i + mq$, the preceding action of μ_q on $(\mathbb{C}^{n+1})^*$ does not change. That explains why the integers $q_0, \ldots, q_n$ are taken positive (choose $m \gg 0$).

To describe the cohomology ring $H^*(\tilde{L}^n_{\mathbb{C}}, \mathbb{Z})$, we need to define some integers attached to $q_0, \ldots, q_n, q$. Fix $h \in \{0, 1, \ldots, n\}$ and, for $I = \{i_0, i_1, \ldots, i_h\} \subset \{0, 1, \ldots, n\}$, put

$$\ell_I = q_{i_0} \cdots q_{i_h}/gcd\{q_{i_0}, \ldots, q_{i_h}\}.$$

Now, define

$$\ell_h = \ell_h(q_0, \ldots, q_n) = lcm\{\ell_I \mid I \subset \{0, \ldots, n\}, |I| = h + 1\}.$$

Denote $q_{n+1} = q$. Then $\ell_h(q_0, \ldots, q_n)$ divides $\ell_h(q_0, \ldots, q_{n+1})$. So we have other integers:

$$m_h = m_h(q_0, \ldots, q_{n+1}) = \ell_h(q_0, \ldots, q_{n+1})/\ell_h(q_0, \ldots, q_n)$$

(where $0 \leq h \leq n$).

To understand how these integers appear, and to see some of their properties, look at I.§5[AA].

a) *Additive structure.*

Proposition 4. *For $\tilde{L}^n_{\mathbb{C}} = \tilde{L}^n_{\mathbb{C}}(q; q_0, \ldots, q_n)$ we have*

$$H^i(\tilde{L}^n_{\mathbb{C}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } 2n+1; \\ \mathbb{Z}/m_h\mathbb{Z} & \text{if } i = 2h, 1 \leq h \leq n; \\ 0 & \text{otherwise}; \end{cases}$$

where the integers $m_h = m_h(q_0, \ldots, q_n, q)$ are defined above.

Proof. We have: $\tilde{L}^n = (\mathbb{C}^{n+1})^*/\mu_q = (\frac{\mathbb{C}^{n+1}}{\mu_q}(q_0,\dots,q_n)) \setminus \{0\}$ (obvious notation).

This gives a long cohomological exact sequence (with coefficients in $\mathbb{Z}$):

$$\cdots \to H_0^i(\mathbb{C}^{n+1}/\mu_q) \to H^i(\mathbb{C}^{n+1}/\mu_q) \to H^i(\tilde{L}^n) \to H_0^{i+1}(\mathbb{C}^{n+1}/\mu_q) \to \cdots$$

But $\mathbb{C}^{n+1}/\mu_q$ is contractile; therefore we obtain isomorphisms

$$H^i(\tilde{L}^n) \xrightarrow{\sim} H_0^{i+1}(\mathbb{C}^{n+1}/\mu_q) \qquad (i \geq 1).$$

Then the proposition follows from [AA] (I. §2 (d), §5. Corollary) ($H^0(\tilde{L}^n) = \mathbb{Z}$ since $\tilde{L}^n$ is connected.). □

b) *Multiplicative structure.*

Let us recall and fix the notation:

$$\tilde{P}^n = P^n_{\mathbb{C}}(q_0,\dots,q_n), \quad \tilde{P}^{n+1} = P^{n+1}_{\mathbb{C}}(q_0,\dots,q_n,q),$$

$$\tilde{L}^n = L^n_{\mathbb{C}}(q;q_0,\dots,q_n).$$

We have canonical maps:

$$\tilde{L}^n \xrightarrow{\theta} \tilde{P}^n \xrightarrow{\iota} \tilde{P}^{n+1}$$

(a projection and an inclusion).

Let ξ_i be the generator of $H^{2i}(\tilde{P}^n,\mathbb{Z})$ and ζ_j the generator of $H^{2j}(\tilde{P}^{n+1},\mathbb{Z})$ ($1 \leq i \leq n, 1 \leq j \leq n+1$), defined by I. §5. Theorem, in [AA]. Then one has:

$$\iota^*(\zeta_j) = m_j\xi_j \quad (1 \leq j \leq n)$$

where $m_j = m_j(q_0,\dots,q_n,q)$ (integer defined above). This is explicit in the proof of the theorem we just refered to (and in its Corollary.).

The multiplicative structure of the ring $H^*(\tilde{L}^n,\mathbb{Z})$ is described in a corollary of the following theorem.

Theorem 5. *The graded rings homomorphism*

$$\theta^* : H^{2*}(\tilde{P}^n_{\mathbb{C}},\mathbb{Z}) \to H^{2*}(\tilde{L}^n_{\mathbb{C}},\mathbb{Z}) \qquad (H^{2*} = \bigoplus_{i\geq 0} H^{2i})$$

is onto, and its kernel is generated (as an ideal) by the elements $m_i\xi_i$ ($1 \leq i \leq n$).

Before the proof, let us make precise the multiplicative structure we are interested in.

Corollary 6. *In the polynomial ring $\mathbb{Z}[T_1, \ldots, T_{n+1}]$ consider the ideal $\mathcal{B}$ generated by the elements:*

$$m_i T_i (1 \leq i \leq n), T_i T_j - e_{ij} T_{i+j} (1 \leq i, j \leq n, i + j \leq n)$$

and $T_j T_j (1 \leq i, j \leq n, i + j \geq n + 1)$.

The integers e_{ij} are defined by

$$e_{ij} = e_{ij}(q_0, \ldots, q_n) := \ell_i \ell_j / \ell_{i+j},$$

where $\ell_h = \ell_h(q_0, \ldots, q_n)$ as introduced at the beginning of this paragraph. Then we have a ring isomorphism

$$H^*(\tilde{L}^n_{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}[T_1, \ldots, T_{n+1}]/\mathcal{B}.$$

This follows from the additive structure of $H^*(\tilde{L}^n, \mathbb{Z})$ and from the multiplicative structure of $H^*(\tilde{P}^n, \mathbb{Z})$ (computed in I.§6. [AA]).

Proof of Theorem. It is enough to shwo that the following sequences of cohomology groups (with coefficients in $\mathbb{Z}$) are exact.

$$\begin{array}{ccccccccc} 0 & \to & H^{2i}(\tilde{P}^{n+1}) & \xrightarrow{\iota^*} & H^{2i}(\tilde{P}^n) & \xrightarrow{\theta_i^*} & H^{2i}(\tilde{L}^n) & \to & 0 \quad (1 \leq i \leq n) \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z} & \xrightarrow{\cdot m_i} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/m_i\mathbb{Z} & & \end{array}$$

(θ_i^* is induced by θ). Thus we have only surjectivity of the θ_i^*'s to prove. First, this is done in particular case:

Lemma 7. *If q is multiple of each $q_j (1 \leq j \leq n)$, then θ_i^* is surjective.*

This comes from the following. Since the $lcm\{q_0, \ldots, q_n\}$ divides q, there exists a vector bundle (of rank 1 over $\mathbb{C}$) E, over $\tilde{P}^n = P_{\mathbb{C}}(q_0, \ldots, q_n)$, such that:

$$\begin{array}{ccc} \tilde{L}^n = L^n_{\mathbb{C}}(q; q_0, \ldots, q_n) & \xrightarrow{\quad\sim\quad} & E^* \\ & {\scriptstyle\theta} \searrow \quad \swarrow & \\ & \tilde{P}^n & \end{array}$$

(E^* complement of 0-section).

This has been done in the algebraic case (in details) in I. B.4. Here (analytical case), E can be briefly defined by: $E = \psi^*(L^{q/m})$, where $\psi : \tilde{P}^n \to P^n, \psi(y_0, \ldots, y_n) = (y_0^{m/q_0}, \ldots, y_n^{m/q_n})$, with $m = lcm\{q_0, \ldots, q_n\}$, $L =$ canonical line bundle over P^n. See also [AA](I, §1).

Then the surjectivity of θ_i^* follows from the GYSIN exact sequence

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{2i-2}(\tilde{P}^n) & \longrightarrow & H^{2i}(\tilde{P}^n) & \longrightarrow & H^{2i}(E^*) & \longrightarrow & H^{2i-1}(\tilde{P}^n) & \longrightarrow \cdots \\ & & & \searrow^{\theta_i^*} & & \downarrow\wr & & \| & \\ & & & & & H^{2i}(\tilde{L}^n) & & 0 & \end{array}$$

Lemma 8. *Let q' be an integer > 0 multiple of q. Put*

$$'\tilde{L} = L^n_{\mathbb{C}}(q'; q_0, \ldots, q_n).$$

The natural map $\gamma : \tilde{L}^n \to {}'\tilde{L}$ induces epimorphisms (coefficients in $\mathbb{Z}$)

$$\gamma_j^* : H^{2j}({}'\tilde{L}^n) \to H^{2j}(\tilde{L}^n) \quad (1 \leq j \leq n).$$

First we check the implication:

$$\text{Lemma 7 and Lemma 8} \implies \theta_i^* \text{ surjective}$$

(for any $q > 0$, any weights $q_0, \ldots, q_n$).

Take $q' = qq_0 \cdots q_n$. Then, with obvious notation, the following triangle commutes

$$\begin{array}{ccc} \tilde{L}^n & \xrightarrow{\gamma} & {}'\tilde{L}^n \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta'} & \\ & \tilde{P}^n & \end{array}$$

and, in cohomology, gives $\theta_i^* = \gamma_i^* \circ (\theta')_i^* \quad (1 \leq i \leq n)$. Hence we are done.

Proof of Lemma 8. The map $\gamma : \tilde{L}^n \to {}'\tilde{L}^n$ is induced, since $\mu_q \subset \mu_{q'}$, by the canonical one $\mathbb{C}^{n+1}/\mu_q \to \mathbb{C}^{n+1}/\mu_{q'}$ we denote also by γ. Recall that, by definition:

$$\tilde{L}^n = L^n_{\mathbb{C}}(q; q_0, \ldots, q_n) = (\mathbb{C}^{n+1})^*/\mu_q = (\frac{\mathbb{C}^{n+1}}{\mu_q}(q_0, \ldots, q_n)) \setminus \{0\}.$$

Since $\gamma : \mathbb{C}^{n+1}/\mu_q \to \mathbb{C}^{n+1}/\mu_{q'}$ is proper, one has homomorphisms (with coefficients in $\mathbb{Z}$ as always):

$$(*) \qquad \gamma^* : H_c^{2j+1}(\mathbb{C}^{n+1}/\mu_{q'}) \to H_c^{2j+1}(\mathbb{C}^{n+1}/\mu_q) \quad (1 \leq j \leq n)$$

Now consider the commutative diagrams

$$\begin{array}{ccccc} H^{2j}({}'\tilde{L}^m) & \xrightarrow{\sim} & H_0^{2j+1}(\mathbb{C}^{n+1}/\mu_{q'}) & \xrightarrow{\sim} & H_c^{2j+1}(\mathbb{C}^{n+1}/\mu_{q'}) \\ \downarrow{\scriptstyle\gamma^*} & & \downarrow{\scriptstyle\gamma^*} & & \downarrow{\scriptstyle\gamma^*} \\ H^{2j}(\tilde{L}^n) & \xrightarrow{\sim} & H_0^{2j+1}(\mathbb{C}^{n+1}/\mu_q) & \xrightarrow{\sim} & H_c^{2j+1}(\mathbb{C}^{n+1}/\mu_q) \end{array}$$

(For the isomorphisms, see proof of Proposition above and I.§2.(d) in [AA].)

Therefore, it is enough to see that the homomorphisms $(*)$ are onto. Take $\tilde{P}^{n+1} = P^{n+1}_{\mathbb{C}}(q_0, \ldots, q_n, q)$, and ${}'\tilde{P}^{n+1} = P^{n+1}_{\mathbb{C}}(q_0, \ldots, q_n, q')$. The natural closed inclusion $\tilde{P}^n \subset \tilde{P}^{n+1}(x_{n+1} = 0)$ has $\mathbb{C}^{n+1}/\mu_q$ as open complement (see [AA] I.§1.(a)). So we have commutative squares:

$$\begin{array}{ccccc} \mathbb{C}^{n+1}/\mu_q & \overset{open}{\hookrightarrow} & \tilde{P}^{n+1} & \overset{closed}{\hookleftarrow} & \tilde{P}^n \\ \downarrow & & \downarrow & & \downarrow = \\ \mathbb{C}^{n+1}/\mu_{q'} & \hookrightarrow & {}'\tilde{P}^{n+1} & \hookleftarrow & \tilde{P}^n \end{array}$$

the vertical arrow in the middle being $(x_0, \ldots, x_{n+1}) \mapsto (x_0, \ldots, x_n, x_{n+1}^d)$ with $d = q'/q$. This gives a morphism of exact sequences

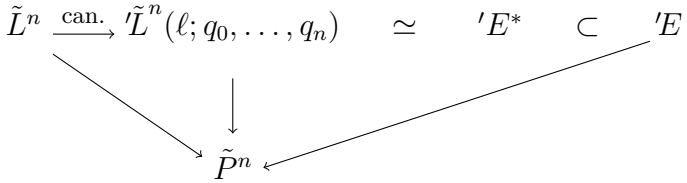
$$\begin{array}{ccccccccc} H^{2j}(\tilde{P}^{n+1}) & \to & H^{2j}(\tilde{P}^n) & \to & H_c^{2j+1}(\mathbb{C}^{n+1}/\mu_q) & \to & 0 \\ \uparrow & & \uparrow = & & \uparrow \gamma^* & & \\ H^{2j}({}'\tilde{P}^{n+1}) & \to & H^{2j}(\tilde{P}^n) & \to & H_c^{2j+1}(\mathbb{C}^{n+1}/\mu_{q'}) & \to & 0 \end{array}$$

The desired surjectivity follows. □

□

c) Remark.

The preceding proof of the Theorem in b) shows the existence of:



commutative diagram with $\ell = lcm\{q, q_0, \ldots, q_n\}$, ${}'E =$ line bundle over $\tilde{P}^n$ (arbitrary $q, q_0, \ldots, q_n$).

2. Étale cohomology of $\tilde{L}^n_k$.

In this section a ground field k is fixed. We assume k is algebraically closed, and the (positive) integers $q, q_0, \ldots, q_n$ prime to its characteristic exponent.

Consider an integer $a \geq 1$, and a prime number ℓ, both prime to the characteristic exponent of k and fixed in the sequel.

Schemes, morphisms of schemes will always mean k-schemes, morphisms of k-shcemes.

For a scheme S, we put

$$H^i(S, \mathbb{Z}/a\mathbb{Z}) = H^i(S, (\mathbb{Z}/a\mathbb{Z})_S),$$

$$H^i(S, \mathbb{Z}_\ell) = \varprojlim_{s \geq 1} H^i(S, \mathbb{Z}/\ell^s\mathbb{Z}),$$

where $H^i(S, (\mathbb{Z}/a\mathbb{Z})_S)$ is the i-th étale cohomology group of S with values in the constant sheaf defined by $\mathbb{Z}/a\mathbb{Z}$, and $\mathbb{Z}_\ell$ is the ring of ℓ-adic integers.

Here $\tilde{L}^n$ stands for the k-scheme constructed as a geometric quotient in I.A.3.b).:

$$\tilde{L}^n = L^n_k(q; q_0, \ldots, q_n).$$

a). *The additive structure* of the étale cohomology $\tilde{L}^n$ is given by:

Theorem 9. *For $0 \leq j \leq n, m_j = m_j(q_0, \ldots, q_n, q)$ is the integer defined at the beginning of §.1. Let $\alpha_j \geq 0$ be maximal such that ℓ^{α_j} divide m_j (i.e., $\alpha_j = v_\ell(m_j)$). Then we have (with $(s,t) := gcd\{s,t\}$):*

$$(u) \quad H^i(\tilde{L}^n, \mathbb{Z}/a\mathbb{Z}) = \begin{cases} \mathbb{Z}/a\mathbb{Z} & \text{if } i = 0 \text{ or } 2n+1, \\ \mathbb{Z}/(a, m_j)\mathbb{Z} & \text{if } i = 2j-1 \text{ or } 2j \ (1 \leq j \leq n), \\ 0 & \text{if } i \geq 2n+2. \end{cases}$$

$$(v) \quad H^i(\tilde{L}^n, \mathbb{Z}_\ell) = \begin{cases} \mathbb{Z}_\ell & \text{if } i = 0 \text{ or } 2n+1, \\ \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} & \text{if } i = 2j \ (1 \leq j \leq n), \\ 0 & \text{if } i \text{ is odd} \neq 2n+1 \text{ or } i > 2n+1. \end{cases}$$

Proof of (u). The last Proposition in I.B.4. says the quotient-scheme A^{n+1}/U_q ([7]) is the affine projecting cone of a projective variety the blunt cone of whih is $\tilde{L}^n$. Thus the cohomology of a cone ([AA] II.§5, [DL]) implies:

Proposition 10. *Let F be a torsion abelian group, prime to the characteristic exponent of the ground field k. Then one has*

$$\alpha) \quad H^i(A^{n+1}/U_q, F) = \begin{cases} F & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

$$\beta) \quad H^i_0(A^{n+1}/U_q, F) \xrightarrow{\sim} H^i_c(A^{n+1}/U_q, F) \quad (i \geq 0).$$

Now, from the computations of the étale cohomology groups of the twisted projective space $\tilde{P}^n_k$ ([AA]II.6(c)), we obtain:

[7] Recall that $A^{n+1}/U_q = (A^{n+1}/U_q)(q_0, \ldots, q_n)$ (I.A.3).

Proposition 11. *The quotient-scheme A^{n+1}/U_q has its cohomology with compact supports as follows (same assumptions as above).*

$$H_c^i(A^{n+1}/U_q, \mathbb{Z}/a\mathbb{Z}) = \begin{cases} \mathbb{Z}/a\mathbb{Z} & \text{if } i = 2(n+1), \\ \mathbb{Z}/(a, m_j)\mathbb{Z} & \text{if } i = 2j \text{ or } 2j+1 \ (0 \le j \le n), \\ 0 & \text{if not.} \end{cases}$$

With these two propositions one sees that the proof of part (u) in the Theorem is quite similar to that one in the analytical case (§.1.a).). □

Proof of (v). Consider the closed immersion

$$\tilde{P}^n \subset \tilde{P}^{n+1} = P_k^{n+1}(q_0, \ldots, q_n, q)$$

the complement of which is A^{n+1}/U_q (see II.3.(c) in [AA]). What follows is given by Proposition 11. (and its proof!). There is a morphism of exact sequences (induced by the projection $\mathbb{Z}/\ell^{\alpha+1}\mathbb{Z} \to \mathbb{Z}/\ell^{\alpha}\mathbb{Z} \quad (\alpha \ge 1)$):

$$\begin{array}{ccccccccc} 0 & \to & H_c^{2j}(A^{n+1}/U_q, G) & \to & H^{2j}(\tilde{P}^{n+1}, G) & \to & H^{2j}(\tilde{P}^n, G) & \to & H_c^{2j+1}(A^{n+1}/U_q, G) \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H_c^{2j}(A^{n+1}/U_q, F) & \to & H^{2j}(\tilde{P}^{n+1}, F) & \to & H^{2j}(\tilde{P}^n, F) & \to & H_c^{2j+1}(A^{n+1}/U_q, F) \to 0 \end{array}$$

where G and F stand for $\mathbb{Z}/\ell^{\alpha+1}\mathbb{Z}$ and $\mathbb{Z}/\ell^{\alpha}\mathbb{Z}$. Fix j $(0 \le j \le n)$ and $\alpha \ge \alpha_j$ $(\alpha_j = v_\ell(m_j))$. Then the preceding diagram identifies to:

$$\begin{array}{ccccccccc} & & 1 & \mapsto & \ell^{\alpha+1-\alpha_j} & & & & \\ 0 & \to & \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} & \to & \mathbb{Z}/\ell^{\alpha+1}\mathbb{Z} & \xrightarrow{\cdot m_j} & \mathbb{Z}/\ell^{\alpha+1}\mathbb{Z} & \to \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} \to 0 \\ & & \downarrow & & \downarrow proj. & & \downarrow proj. & \downarrow = & \\ 0 & \to & \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} & \to & \mathbb{Z}/\ell^{\alpha}\mathbb{Z} & \xrightarrow{\cdot m_j} & \mathbb{Z}/\ell^{\alpha}\mathbb{Z} & \to \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} \to 0 \\ & & 1 & \mapsto & \ell^{\alpha-\alpha_j} & & & & \end{array}$$

(since $(\ell^\alpha, m_j) = \ell^{\alpha_j} = (\ell^{\alpha+1}, m_j)$). Hence, by diagram commutativity, the left vertical morphism is multiplication by ℓ. On the other hand we have $(i \ge 1)$

$$H^i(\tilde{L}^n, \mathbb{Z}/a\mathbb{Z}) \xrightarrow{\sim} H_0^{i+1}(A^{n+1}/U_q, \mathbb{Z}/a\mathbb{Z}) \xrightarrow{\sim} H_c^{i+1}(A^{n+1}/U_q, \mathbb{Z}/a\mathbb{Z})$$

(Proposition 10. allows to see this similarly to the analytical case (§.1.a)). So, for $1 \le j \le n$, it comes:

$$H^{2j}(\tilde{L}^n, \mathbb{Z}_\ell) = \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z},$$

and $H^{2j-1}(\tilde{L}^n, \mathbb{Z}_\ell)$ is the limit of the projective system $(M_\alpha, u_\alpha)_{\alpha \ge \alpha_j}$:

$$\begin{array}{ccc} M_{\alpha+1} & \xrightarrow{u_\alpha} & M_\alpha \\ \| & & \| \\ \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} & \xrightarrow{\cdot \ell} & \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} \end{array}$$

Whence $H^{2j-1}(\tilde{L}^n, \mathbb{Z}_\ell) = 0$ $(1 \leq j \leq n)$. What remains follows from the assertion (u). □

b). *The multiplicative structure* of $H^*(\tilde{L}^n_k, \mathbb{Z}_\ell)$ is computed in a similar way to the analytical case §1.b).

Let η_i be the generator of $H^{2i}(\tilde{P}^n, \mathbb{Z}/a\mathbb{Z})$ $(0 \leq i \leq n)$ such that

$$\begin{array}{ccc} H^{2i}(\tilde{P}^n, \mathbb{Z}/a\mathbb{Z}) & \xrightarrow{\varphi^*} & H^{2i}(P^n, \mathbb{Z}/a\mathbb{Z}) \\ \| & & \| \\ \mathbb{Z}/a\mathbb{Z} & \xrightarrow{\cdot \ell_i} & \mathbb{Z}/a\mathbb{Z} \end{array}$$

($l_i = l_i(q_0, \ldots, q_n)$, beginning of §.1).

The morphism $\varphi : P^n = P^n_k(1, \ldots, 1) \to \tilde{P}^n = P^n_k(q_0, \ldots, q_n)$ is well defined and well studied in [AA] II.§3.,4.,6.

We have also a natural morphism, induced by the group inclusion $\mu_q \subset \mathbb{G}_m$,

$$\theta : \tilde{L}^n \to \tilde{P}^n.$$

Now we can state the analogous of Theorem in §.1.b)

Theorem 12. *The ring homomorphism*

$$\theta^* : H^{2*}(\tilde{P}^n, \mathbb{Z}/a\mathbb{Z}) \to H^{2*}(\tilde{L}^n, \mathbb{Z}/a\mathbb{Z})$$

is surjective. As an ideal, $Ker(\theta^*)$ is generated by the elements $m_i\eta_i$ $(1 \leq i \leq n)$.

Proof. By [JO2]1.5 we have a GYSIN exact sequence in the étale case at our disposal (associatd to the complement of the zero-section of a vector bundle). Therefore the proof is the same as in the analytical case (§1.b)) because of the first Proposition in I.B.4. □

Corollary 13. *1) Same statement as in Theorem above with $\mathbb{Z}_\ell$ instead of $\mathbb{Z}/a\mathbb{Z}$, and ζ_i ($\mathbb{Z}_\ell$-generator of $H^{2i}(\tilde{P}^n, \mathbb{Z}_\ell)$) instead of η_i $(1 \leq i \leq n)$ (see first preceding Theorem.).*

2) Let $\mathcal{B}$ be the ideal generated by the same elements as in Corollary §1.b) but in $\mathbb{Z}_\ell[T_1, \ldots, T_{n+1}]$. Then we have a $\mathbb{Z}_\ell$-algebra isomorphism

$$H^*(\tilde{L}^n, \mathbb{Z}_\ell) = \mathbb{Z}_\ell[T_1, \ldots, T_{n+1}]/\mathcal{B}.$$

Proof. The assertion 2) follows from the additive structure of $H^*(\tilde{L}^n, \mathbb{Z}_\ell)$ (first Theorem above) and the multiplicative structure of $H^*(\tilde{P}^n, \mathbb{Z}_\ell)$ ([AA]II.§6). □

III. Intersection theory

We compute intersection theory (à la FULTON [FU]) for the twisted lens spaces $\tilde{L}_k^n$. Because of their intimate links to twisted projective spaces $\tilde{P}_k^m$ (I.B., II.1.c)), this is done using results we obtained for $\tilde{P}_k^m$ in [A1]. Computations are origanized as follows:

A. CHOW group;
B. Coherent K-theory;
C. ℓ-adic Homology.

We always assume the ground field k algebraically closed, and the positive integers $q, q_0, \ldots, q_n$ prime to the characteristic exponent of k.

A. CHOW group of $\tilde{L}_k^n$.

Let us recall that, for an integral scheme X, of finite type (here over k), the Chow group is

$$A_*(X) = \bigoplus A_i(X) \qquad (0 \leq i \leq \dim X),$$

where $A_i(X)$ is the free abelian group generatd by the integral closed subschemes of X of dimension i, modulo rational equivalence.

A_* is a covariant functor with respect to proper morphisms. If $\alpha : U \subset X$ is an open immersion with complement $\beta : Y \subset X$, then we have an exact sequence of groups

$$A_i(Y) \xrightarrow{\beta_*} A_i(X) \xrightarrow{\alpha^*} A_i(U) \longrightarrow 0 \qquad (0 \leq i \leq \dim X).$$

For this, and for other properties used in the sequel, the reader may refer to [FU].

We consider the weighted lens space

$$\tilde{L}_k^n = L_k^n(q; q_0, \ldots, q_n)$$

as an open subscheme of the weighted projective space

$$\tilde{P}_k^{n+1} = P_k^{n+1}(q_0, \ldots, q_{n+1}), \quad \text{where } q_{n+1} = q.$$

(cf. I.B.2.) More precisely consider the standard open affine subscheme

$$U = D_+(X_{n+1}) = Spec(k[X_0, \ldots, X_{n+1}]_{(X_{n+1})})$$

where $deg(X_i) = q_i$ $(0 \leq i \leq n+1)$ (notations of [GR2] (II.)) The sub-k-algebra

$$(k[X_0, \ldots, X_n])^{(q)} \subset k[X_0, \ldots, X_n]$$

is generated by the monomials

$$X_0^{\alpha_0} \cdots X_n^{\alpha_n}, \quad \text{with} \quad \sum_0^n \alpha_i q_i \equiv 0 \pmod{q}.$$

Thus we have

$$U = Spec(k[X_0, \ldots, X_n]^{(q)})$$

and, by definition of $\tilde{L}^n_k$ (I.A.3.4),

$$\tilde{L}^n_k = U \setminus \{point\} = U \setminus V(X_0^q, \ldots, X_n^q)$$

($V(X_0^q, \ldots, X_n^q)$ is the origin defined by the equations "$X_i^q = 0$ "). So consider the canonical immersions (over the field k)

$$\sigma : \tilde{P}^n = P^n(q_0, \ldots, q_n) \subset \tilde{P}^{n+1} \text{ (closed)},$$

$$\tau : \tilde{L}^n = L^n(q; q_0, \ldots, q_n) \subset \tilde{P}^{n+1} \text{ (open)}.$$

Proposition 14. *a) For each integer i, we have an exact sequence*

$$0 \longrightarrow A_i(\tilde{P}^n) \xrightarrow{\sigma_*} A_i(\tilde{P}^{n+1}) \xrightarrow{\tau^*} A_i(\tilde{L}^n) \longrightarrow 0.$$

b) If $i \in \{0, \ldots, n\}$ and if

$$c_i \in A_i(\tilde{P}^n) \simeq \mathbb{Z}, c_i' \in A_i(\tilde{P}^{n+1}) \simeq \mathbb{Z},$$

are the canonical generators defined in [A1] *2.1.(ii), then*

$$\sigma_*(c_i) = m_i c_i',$$

where $m_i = l_i(q_0, \ldots, q_{n+1})/l_i(q_0, \ldots, q_n)$ (II. beginning of §.1).

The Chow group $A_*\tilde{L}^n_k$ follows:

Corollary 15. *We have*

$$A_i(\tilde{L}^n_k) = \mathbb{Z}/m_i\mathbb{Z} \quad (0 \leq i \leq n),$$

$$A_{n+1}(\tilde{L}^n_k) = \mathbb{Z}$$

(the other groups $A_j(\tilde{L}^n_k)$ are zero.).

The epimorphism τ^* in the proposition above gives generators of $A_*(\tilde{L}^n_k)$ from those (canonical) of $A_*(\tilde{P}^{n+1}_k)$.

Proof of Propositon. a) comes from the closed immersion $\sigma : \tilde{P}^n \subset \tilde{P}^{n+1}$ the complement of which is U, so that $\tilde{L}^n = U \setminus \{closed\ point\}$. One uses then the exact sequence associated to a closed subscheme and the theorem [A1]2.1. ($A_0(U) = 0$ since $(\sigma_*)_0$ is an isomorphism; see b) below.).

b) Notation and results are those of §.2.1 in [A1].

Let $I = \{s_0, \ldots, s_i\}, 0 \leq s_0 < \ldots < s_i \leq n$,

$$m_I = l_i(q_0, \ldots, q_n)/l_i(q_{s_0}, \ldots, q_{s_i}),$$

$$m_I' = l_i(q_0, \ldots, q_{n+1})/l_i(q_{s_0}, \ldots, q_{s_i}).$$

Consider the canonical closed immersions

$$\tilde{P}_I \subset \tilde{P}^n \subset \tilde{P}^{n+1} \qquad (\tilde{P}_I = P^i_k(q_{s_0}, \ldots, q_{s_i})).$$

By definition of the generators c_i and c'_i, one has

$$[\tilde{P}_I] = m_I c_i \quad \text{in} \quad A_i(\tilde{P}^n),$$
$$[\tilde{P}_I] = m'_I c'_i \quad \text{in} \quad A_i(\tilde{P}^{n+1}).$$

Whence

$$\sigma_*(c_i) = (m'_I/m_I)c'_i = m_i c'_i, \quad \text{as required.}$$

□

Remark. [A1]2.1. Each generator c_i of $A_i(\tilde{P}^n) \simeq \mathbb{Z}$ is such that $c_i = \sum t_I[P_I]$ for any $t_I \in \mathbb{Z}$ with $\sum_I t_I m_I = 1$.

B. Coherent K-group of $\tilde{L}^n$.

1. Let us recall some useful definitions and properties ([GR3, MA]). For a noetherian scheme X, the Grothendieck group $K_.(X)$ is defined by coherent sheaves over X. It is covariant functorial with respect to proper morphisms. Support dimension of a sheaf provides the group $K_.(X)$ with an increasing filtration $(F_iK_.(X))_i$, sometimes denoted $(F_i(X))_i$ for simplicity, which is functorial (for proper morphisms). The associated graded group is

$$Gr\ K_.(X) = \bigoplus_i F_iK_.(X)/F_{i-1}K_.(X).$$

There is a graded epimorphism

$$A_*(X) \to Gr\ K_.(X) \to 0$$

where to $[Y]$ corresponds $[\mathcal{O}_Y]$ ($Y \subset X$ integral closed subscheme).

Another link between the two groups $K_.(X)$ and $A_*(X)$ is of type Riemann-Roch theorem when, for example, X is algebraic ([FU] Chapter 18). That is a homomorphism

$$\tau_X : K_.(X) \to A_*(X) \bigotimes_{\mathbb{Z}} \mathbb{Q},$$

covariant in X (for proper morphisms), and such that

$$\tau_X[\mathcal{O}_Y] = [Y] + \text{terms of degree} < \dim Y$$

($Y \subset X$ integral closed subscheme). It induces an isomorphism

$$K_.(X) \bigotimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} A_*(X) \bigotimes_{\mathbb{Z}} \mathbb{Q}.$$

Last, as for Chow group, if $i : U \subset X$ is an open immersion with complement $j : Y \subset X$, then we have at our disposal an exact sequence

$$K.(Y) \xrightarrow{j_*} K.(X) \xrightarrow{i^*} K.(U) \longrightarrow 0$$

("Homotopy exact sequence" [GR3].)

2. Now we begin the study of the group $K.(\tilde{L}^n)$. Notation is the same as in §.A. The following exact sequence is fundamental for the calculation of $K.(\tilde{L}^n)$.

Proposition 16. *We have an exact sequence (§.A)*

$$0 \longrightarrow K.(\tilde{P}^n) \xrightarrow{\sigma_*} K.(\tilde{P}^{n+1}) \xrightarrow{\tau^*} K.(\tilde{L}^n) \longrightarrow 0$$

Proof. The sequence

$$K.(\tilde{P}^n) \xrightarrow{\sigma_*} K.(\tilde{P}^{n+1}) \xrightarrow{\tau^*} K.(\tilde{L}^n) \longrightarrow 0$$

is exact in the same way as for Chow groups in §.A. The injectivity of σ_* is given by the following lemma. The homomorphism σ_* has a factorization

$$\begin{array}{ccccc} K.(\tilde{P}^n) & \longrightarrow & F_nK.(\tilde{P}^{n+1}) & \hookrightarrow & K.(\tilde{P}^{n+1}) \\ \downarrow \wr & & \downarrow \wr & & \\ \mathbb{Z}^{n+1} & \xrightarrow{u} & \mathbb{Z}^{n+1} & & \end{array} \qquad \text{([A1]3.1.Theorem.)}$$

□

Lemma 17. *The $\mathbb{Z}$-linear map u has a triangular matrix (with respect to the canonical basis) the diagonal of which is equal to $(m_0, m_1, \ldots, m_n)$, where $m_i = m_i(q_0, \ldots, q_{n+1})$ (II.§.1).*

Proof. There are commutative squares

$$\begin{array}{ccc} A_h(\tilde{P}^n) & \longrightarrow & A_h(\tilde{P}^{n+1}) \\ \text{can.} \downarrow \wr & & \text{can.} \downarrow \wr \\ Gr_hK.(\tilde{P}^n) & \longrightarrow & Gr_hK.(\tilde{P}^{n+1}) \end{array} \qquad (0 \leq h \leq n),$$

where $Gr_hK.(X) = F_hK.(X)/F_{h-1}(X)$ (see 1. above). The vertical isomorphisms are given by [A1]3.1Corollary. Now Proposition in §.A. above and the $\mathbb{Z}$-bases of $K.(\tilde{P}^n) \simeq \mathbb{Z}^{n+1}$ and $K.(\tilde{P}^{n+1}) \simeq \mathbb{Z}^{n+2}$, as constructed in [A1]3.1., allow to conclude. To be more explicit, recall that $K.(\tilde{P}^n)$ has a basis (b_i) of the form (loc. cit.)

$$b_i = \sum_I t_I[\mathcal{O}_{\tilde{P}_I}] \quad \text{for some} \quad t_I \in \mathbb{Z} \quad \text{with} \quad \sum_I t_I m_I = 1$$

(see b) in Proof of Proposition §.A. for notation.). Remark at the end of §.A. ends the proof. □

Corollary 18. *We have* $K.(\tilde{L}^n) = \mathbb{Z} \bigoplus F_nK.(\tilde{L}^n)$*, where the subgroup* $F_nK.(\tilde{L}^n)$ *is finite of order* $\prod_1^n m_i$*.*

This is a consequence of the preceding Proposition and Lemma by Bourbaki [BO] (Alg. VII, §.4., n°7, Corollaire 3.) (m_0 is equal to 1).

This corollary shows that, for studying the group structure of $K.(\tilde{L}^n)$, we have only to do with its *finite summand* $F_nK.(\tilde{L}^n)$, which we shall denote in all what follows by

$$F_n(\tilde{L}^n) = F_nK.(\tilde{L}^n), \quad \text{where} \quad \tilde{L}^n = L_k(q; q_0, \ldots, q_n) \quad \text{(I.A.b.)}.$$

First we are going to decompose $F_n(\tilde{L}^n)$ into a direct sum of p-groups of the "same nature" in a similar way we did in the analytical case ([A2] §.3) (see next Theorem).

3. Factorization of $F_n(\tilde{L}^n)$.

Consider the primary factorization of the integer q

$$q = \prod p_i^{\alpha_i}, \quad 1 \leq i \leq s.$$

For each $i = 1, \ldots, s$, put

$$\alpha_{ij} = v_{p_i}(q_j), \quad 1 \leq j \leq n \quad (p\text{-adic valuation}).$$

The main result of this section is the following theorem.

Theorem 19. *There exists an isomorphism of groups:*

$$F_n(L(q; q_0, \ldots, q_n)) \simeq \bigoplus_{1 \leq i \leq s} F_n(L(p_i^{\alpha_i}; p_i^{\alpha_{i0}}, \ldots, p_i^{\alpha_{in}})).$$

Notice that the finite groups $F_n(L(p_i^{\alpha_i}; p_i^{\alpha_{i0}}, \ldots, p_i^{\alpha_{in}}))$ are p_i-groups after the Corollary in 2., and since the integer

$$m_i(p_i^{\alpha_{i0}}, p_i^{\alpha_{i1}}, \ldots, p_i^{\alpha_{in}}, p_i^{\alpha_i})$$

is a power of p_i (see beginning of II.§.1.). On the other hand, by definition of $\tilde{L}_k^n = L_k(q; q_0, \ldots, q_n)$ as a geometric quotient by the roots of unity group μ_q, we have (for any $s \geq 1, s_i \geq 0$)

$$L(p^s; p^{s_0}, p^{s_1}, \ldots p^{s_n}) = L(p^s; p^{r_0}, p^{r_1}, \ldots p^{r_n})$$

where $0 \leq r_i < s$ $(0 \leq i \leq n)$ are obtained by eucleadian division of the s_i's by s.

Proof of the Theorem. In its principle, that is quite similar to the case of complex topological K-theory of the analytical twisted lens space $\tilde{L}^n_{\mathbb{C}} = L(q; q_0, \ldots, q_n)$ (cf. [A2] §.3).

To prove the theorem we proceed by lemmas.

Mainly we use the fact that the order of the involved groups is known (2.,Corollary) and, on the other hand, that we dispose of the exact sequence (2. Proposition):

$$0 \longrightarrow F_n(\tilde{P}^n) = K.(\tilde{P}^n) \xrightarrow{\sigma_*} F_n(\tilde{P}^{n+1}) \xrightarrow{\tau^*} F_n(\tilde{L}^n) \longrightarrow 0$$

where $F_i(X) = F_i(K.(X))$ (see 1. above).

Lemma 20. *Let p be a prime number and β an integer ≥ 0. Put $\beta_i = v_p(q_i)$ $(0 \leq i \leq n)$. Then we have an isomorphism of groups:*

$$F_n(L(p^\beta, q_0, \ldots, q_n)) \simeq F_n(L(p^\beta; p^{\beta_0}, \ldots, p^{\beta_n})).$$

Proof. By Corollary in 2. and definition of the integers $m_i(q_0, q_1, \ldots, q_n, q)$ (II.§.1), it is clear that the two groups are p-groups of the same order.

Because of the preceding exact sequence, the proof reduces to weighted projective spaces as follows:

$$\begin{aligned}
\tilde{P}^n(p) &:= P_k(p^{\beta_0}, \ldots, p^{\beta_n}),\\
\tilde{P}^{n+1}(p) &:= P_k(p^{\beta_0}, \ldots, p^{\beta_n}, p^\beta),\\
\tilde{P}^n &:= P_k(q_0, \ldots, q_n),\\
\tilde{P}^{n+1}_0 &:= P_k(q_0, \ldots, q_n, p^\beta),
\end{aligned}$$

with the canonical closed immersions

$$\sigma(p) : \tilde{P}^n(p) \subset \tilde{P}^{n+1}(p), \quad \sigma_0 : \tilde{P}^n \subset \tilde{P}^{n+1}_0.$$

Therefore, it is enough to construct an isomorphism between the two quotient-groups:

$$\begin{aligned}
G(p) &:= F_n(\tilde{P}^{n+1}(p))/\sigma(p)_* F_n(\tilde{P}^n(p)),\\
G_0 &:= F_n((\tilde{P}^{n+1}_0))/(\sigma_0)_* F_n(\tilde{P}^n).
\end{aligned}$$

We need to define morphisms

$$f : \tilde{P}^{n+1}(p) \to \tilde{P}^{n+1}_0, \quad h : \tilde{P}^n(p) \to \tilde{P}^n.$$

This is a particular case of the following more general statement.

Lemma 21. *If $P_k(s_0, \ldots, s_n)$ and $P_k(t_0, \ldots, t_n)$ are two twisted projective spaces such that the weights satisfy*

$$s_i = \text{multiple of } t_i \ (0 \leq i \leq n),$$

then there exists a natural finite (hence proper) morphism of schemes:

$$P_k(t_0,\dots,t_n) \to P_k(s_0,\dots,s_n)$$

(Consider the corresponding graded polynomial algebras

$$S(s_0,\dots,s_n) = k[X_0,\dots,X_n] \quad (deg(X_i) = s_i),$$
$$S(t_0,\dots,t_n) = k[Y_0,\dots,Y_n] \quad (deg(Y_i) = t_i).$$

The graded homomorphism defined by $X_i \mapsto Y_i^{r_i}$ $(r_i = s_i/t_i)$ gives the result.).

Hence we get a commutative diagram

$$\begin{array}{ccc} \tilde{P}^{n+1}(p) & \xrightarrow{f} & \tilde{P}_0^{n+1} \\ {\scriptstyle\sigma(p)}\uparrow & & \uparrow{\scriptstyle\sigma_0} \\ \tilde{P}^{n}(p) & \xrightarrow{h} & \tilde{P}^{n} \end{array}$$

so that $F_n(f) : F_n(\tilde{P}^{n+1}(p)) \to F_n(\tilde{P}_0^{n+1})$ induces a morphism

$$\bar{f} : G(p) \to G_0.$$

From 2. Corollary one sees easily that the p-groups (finite) $G(p)$ and G_0 have same order (the corresponding integers m_i's have their ℓ-adic valuations equal (trivial if $\ell \neq p$ in the two cases), after their very definition (II.§.1)). Therefore Lemma 20 will be established if we see that $\bar{f}$ is injective. This follows from the next lemma.

Lemma 22. *We have a monomorphism*

$$K.(f) : K.(\tilde{P}^{n+1}(p)) \to K.(\tilde{P}_0^{n+1}).$$

Proof of Lemma 22. The two involved groups are free of same rank $n+2$ ([A1] 3.1.,Théorème). So we have only to show that $K.(f) \bigotimes_{\mathbb{Z}} \mathbb{Q}$ is bijective. Consider the commutative triangle

$$\begin{array}{ccc} \tilde{P}^{n+1}(p) & \xrightarrow{\quad f \quad} & \tilde{P}_0^{n+1} \\ & {\scriptstyle\varphi(p)}\nwarrow \quad \nearrow{\scriptstyle\varphi_0} & \\ & P^{n+1} & \end{array}$$

with

$$\varphi(p) := \varphi(p^{\beta_0},\dots,p^{\beta_n},p^{\beta}),$$
$$\varphi_0 := \varphi(q_0,\dots,q_n,p^{\beta}).$$

Here the morphism of schemes (over the ground field k)

$$\varphi = \varphi(s_0, \ldots, s_n) : P_k^m \to \tilde{P}_k^m = P_k(s_0, \ldots, s_m)$$

is given by the graded algebra homomorphism

$$\begin{aligned} k[X_0, \ldots, X_n] &\to k[Y_0, \ldots, Y_n] \\ X_i &\mapsto Y_i^{s_i} \qquad (deg(X_i) = s_i,\ deg(Y_i) = 1). \end{aligned}$$

(φ is finite, hence proper.)

All what we have to do is to be sure that $K.(\varphi) \bigotimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism (for all integers $s_i \geq 1$). But, after [A1] 3.5.c) (commutative diagram $(*)$) this is the case. Applying the result to $\varphi(p)$ and φ_0, we are done. □

□

Lemma 23. *There is a group isomorphism*

$$F_n(\tilde{L}^n) \simeq \bigoplus_{1 \leq i \leq s} F_n(\tilde{L}_i^n),$$

where :

$$\tilde{L}^n = L_k(q; q_0, \ldots, q_n), \quad \tilde{L}_i^n = L_k(p_i^{\alpha_i}; q_0, \ldots, q_n).$$

This Lemma 23 and Lemma 20 will prove the Theorem.

Proof of Lemma 23. We have canonical inclusions

$$\sigma : \tilde{P}^n \subset \tilde{P}^{n+1}, \quad \sigma_i : \tilde{P}^n \subset \tilde{P}_i^{n+1}$$

with:

$$\begin{aligned} \tilde{P}^n &= P_k(q_0, \ldots, q_n), \\ \tilde{P}^{n+1} &= P_k(q_0, \ldots, q_{n+1}), \\ \tilde{P}_i^{n+1} &= P_k(q_0, \ldots, q_n, p_i^{\alpha_i}) \quad (1 \leq i \leq s). \end{aligned}$$

The exact sequence given by §.2. (Proposition) shows that it is enough to display an isomorphism

$$\bigoplus_{1 \leq i \leq s} F_n(\tilde{P}^{n+1})/(\sigma_i)_* F_n(\tilde{P}^n) \xrightarrow{\sim} F_n(\tilde{P}^{n+1})/\sigma_* F_n(\tilde{P}^n).$$

Consider the (finite, hence proper) morphisms:

$$g_i : \tilde{P}_i^{n+1} \to \tilde{P}^{n+1},$$

defined in homogeneous coordinates by

$$\text{“}g_i(x_0, \ldots, x_{n+1}) = (x_0, \ldots, x_n, x_{n+1}^r)\text{”}, \quad \text{with } r = q_{n+1}/p_i^{\alpha_i}.$$

Recall that $q_{n+1} = q = \prod p_j^{\alpha_j}$ $(1 \leq j \leq s)$. Similarly to Lemma 22, one shows that $K.(g_i)$ is injective. Since the diagram

$$\begin{array}{ccc} \tilde{P}_i^{n+1} & \xrightarrow{\quad g_i \quad} & \tilde{P}^{n+1} \\ & {\scriptstyle\sigma_i}\nwarrow \quad \nearrow{\scriptstyle\sigma} & \\ & \tilde{P}^n & \end{array}$$

commutes, $K.(g_i)$ induces a monomorphism:

$$G_i := F_n(\tilde{P}^{n+1})/(\sigma_i)_* F_n(\tilde{P}^n) \to H := F_n(\tilde{P}^{n+1})/\sigma_* F_n(\tilde{P}^n).$$

But G_i is a p_i-group and $\bigoplus_{1\leq i\leq s} G_i$ and H are finite with same order (Corollary in 2. and [A1] §.1 (1.3.2). More details are given in [A2] §.3.8). The primes p_i's are distinct so that we obtain an isomorphism

$$\bigoplus_{1\leq i\leq s} G_i \to H, \quad \text{as desired.}$$

The Theorem is proven. □

□

4. The groups $F_n(\tilde{L}^n(p))$.

a). To study the group $K.(\tilde{L}^n)$ it is enough to do so for the groups

$$F_n(\tilde{L}^n(p)), \quad \text{where} \quad \tilde{L}^n(p) = L_k^n(p^\beta; p^{\beta_0}, \ldots, p^{\beta_n}) \quad (p \text{ prime})$$

(after the factorization Theorem §.3).

One may assume

$$0 \leq \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \leq \beta$$

(see comment after the so-called Theorem). In particular the weights satisfy in such a case a *divisibility property*

$$(dw): \quad q_{n+1}|q_n|\cdots|q_1|q_0 \qquad \text{(modulo a permutation)}.$$

This is a hypothesis in the next theorem a corollary of which gives a computation of $F_n(\tilde{L}^n(p))$ by *generators and relations*.

We have an exact sequence (see §.3 before)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_n(\tilde{P}^n(p)) & \xrightarrow{\sigma(p)_*} & F_n(\tilde{P}^{n+1}(p)) & \xrightarrow{\tau(p)^*} & F_n(\tilde{L}^n(p)) & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow\wr & & & & \\ & & \mathbb{Z}^{n+1} & \longrightarrow & \mathbb{Z}^{n+1} & & & & \end{array} \qquad ([\text{A1}]3.1)$$

which shows that the question reduces to the computation of the image of the morphism $\sigma(p)_*$.

More generally let us consider the canonical closed immersion (defined by "$X_0 = 0$"):

$$\sigma : \tilde{P}^n = P(q_1, \ldots, q_{n+1}) \subset \tilde{P}^{n+1}(q_0, \ldots, q_{n+1})$$

where the weights satisfy (dw) above. In this case, the free groups $F_n(\tilde{P}^n)$ and $F_n(\tilde{P}^{n+1})$ are endowed with *canonical bases* ([A1] Remarque 3.2). This allows us to achieve *explicit* calculations.

Comment. Recently in [HH] Tara Holm, Nigel Ray, ..., qualified weighted projective spaces with property (dw) as *divisive*! (this is unfortunate!; the correct adjective should be divisible ?) They identified their equivariant K-theory and cobordism to functional rings attached to the corresponding fan. Our feeling is that weighted projective and lens spaces do not need any fan to exist (to shine wonderfully!). Their toricity([8]) is really secondary. Their (co)homological theories must be looked for in terms of their intrinsic topology/geometry as in [TR, KW, MO, KU, A2, DO].

b) As in [A2]3.5., and under the assumption (dw), we consider, keeping the same notation (for our convenience): $(1 \leq h \leq n+1)$

$$\Delta'_h = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_h) \in \mathbb{N}^h \mid 0 \leq \alpha_i < q_0/q_i, \sum_1^h \alpha_i q_i \equiv 0(q_0)\}$$

with $\Delta'_0 = \{0\}$.

In the ring $\mathbb{Z}[X]$, we have the following polynomials ([9]) (degree of which is $\leq i$) R_i, T_{ih} $(0 \leq i \leq n, 0 \leq h \leq i)$:

$$R_i := \sum_{\underline{\alpha} \in \Delta'_{i+1}} (1-X)^{d_{i+1}}, \quad \text{where } d_{i+1} = (\alpha_1 q_1 + \cdots + \alpha_{i+1} q_{i+1})/q_0;$$

$$T_{ih} := X^{i-h} \sum_{\underline{\alpha} \in \Delta'_h} (1-X)^{d_h}.$$

Here is the main result of this §.4.

Theorem 24. *Under the divisibility hypothesis (dw),*

i) there exist unique integers

$$r_{ij} = r_{ij}(q_0, \ldots, q_{n+1}),\ 0 \leq i \leq n,\ 0 \leq j \leq i,$$

defined by the identities in $\mathbb{Z}[X]$*:*

$$R_i = r_{ii} T_{ii} + r_{i,i-1} T_{i,i-1} + \cdots + r_{is} T_{is};$$

[8] w.r.t. *toric* geometry.

[9] see also beginning of 5. further.

ii) the $\mathbb{Z}$-linear map induced by the closed immersion σ

$$F_n(\tilde{P}^n) \xrightarrow{\sigma_*} F_n(\tilde{P}^{n+1})$$

has the matrix

$$(r_{ij}) = \begin{pmatrix} r_{00} & r_{10} & \cdots & r_{n0} \\ & r_{11} & \cdots & r_{n1} \\ & & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix},$$

in the canonical bases (mentioned above).

Hence we have the corollary:

Corollary 25. *With the notations and assumptions of the beginning (see a).), we have the following exact sequence*

$$0 \longrightarrow \mathbb{Z}^{n+1} \xrightarrow{(r_{ij})} \mathbb{Z}^{n+1} \longrightarrow F_n(\tilde{L}^n(p)) \longrightarrow 0$$

where

$$r_{ij} = r_{ij}(p^\beta, p^{\beta_n}, p^{\beta_{n-1}}, \ldots, p^{\beta_0}) \qquad \text{if } 0 \leq j \leq i \leq n,$$
$$r_{ij} = 0 \qquad \text{if } 0 \leq i < j \leq n.$$

Remark. Later we shall give a method of computing the integers r_{ij}, and illustrate it by many examples (see §.5).

Now we prove the theorem.

c). *Proof of the Theorem.* Since the weights q_i satisfy the condition (dw) [A1] 3.2. shows that the free groups $F_n(\tilde{P}^n) = K_\cdot(\tilde{P}^n)$ and $F_n(\tilde{P}^{n+1})$ have $\mathbb{Z}$-basis (b_i) and (b'_i) $(0 \leq i \leq n)$, defined by

$$b_i = [\mathcal{O}_{\tilde{P}_I}], \quad \text{with } I = \{1, \ldots, i+1\},$$
$$b'_i = [\mathcal{O}_{\tilde{P}_J}], \quad \text{with } J = \{0, 1, \ldots, i\} \ (\tilde{P}_J = P(q_0, \ldots, q_i)).$$

Therefore, for each $i = 0, \ldots, n$, there exist integers r_{ij} $(0 \leq j \leq i)$, unique such that

$$(1) \qquad \sigma_*(b_i) = r_{ii} b'_i + r_{i,i-1} b'_{i-1} + \cdots + r_{i0} b'_0$$

(Recall $\sigma : P(q_1, \ldots, q_{n+1}) \subset P(q_0, \ldots, q_{n+1})$ is the canonical closed immersion.). This is equivalent to

$$\psi'_* \sigma_*(b_i) = r_{ii} \psi'_*(b'_i) + \cdots + r_{i0} \psi'_*(b'_0)$$

since ψ'_* is injective ([A1]3.5.c), where

$$\psi' = \psi(q_0, \ldots, q_{n+1}) : \tilde{P}^{n+1} \to P^{n+1}$$

is defined coordinately by "$X_i \mapsto X_i^{m/q_i}$" $(m = lcm(q_0, \ldots, q_{n+1}))$.

Take $J = \{j_0, \dots, j_i\}, 0 \le j_0 < \cdots < j_i \le n+1$. Let us compute

$$\psi'_*[\mathcal{O}_{\tilde{P}_J}] \in K.(P^{n+1}) = \mathbb{Z}[X]/(X^{n+2}),$$

where the class $\bar{X}$ corresponds to $1 - [\mathcal{O}_{P^{n+1}}(-1)]$. Applying [A1]3.5.3 to ψ' we obtain

$$\psi'_*[\mathcal{O}_{\tilde{P}_J}] = (\sum_{\underline{\alpha} \in \Delta_J} [\mathcal{O}_{P^{n+1}}(-(\alpha_0 q_{j_0} + \cdots + \alpha_i q_{j_i})/q_0)])[\mathcal{O}_{P_J}],$$

with

$$\Delta_J = \{\underline{\alpha} = (\alpha_0, \dots, \alpha_i) \mid 0 \le \alpha_h < q_0/q_j, \alpha_0 q_{j_0} + \cdots + \alpha_i q_{j_i} \equiv 0(q_0)\}.$$

So

$$\psi'_*[\mathcal{O}_{\tilde{P}_J}] = (\sum_{\underline{\alpha} \in \Delta_J} (1 - \bar{X})^{(\alpha_0 q_{j_0} + \cdots + \alpha_i q_{j_i})/q_0}) \bar{X}^{n+1-i},$$

in $\mathbb{Z}[X]/(X^{n+2})$.

Now, in $\mathbb{Z}[X]$, put $(0 \le i \le n)$

$$R'_i = X^{n+1-i} \sum_{\underline{\alpha} \in \Delta'_{i+1}} (1 - X)^{d_{i+1}},$$

$$T_i = X^{n+1-i} \sum_{\underline{\alpha} \in \Delta'_i} (1 - X)^{d_i},$$

where $d_h = (\alpha_1 q_1 + \cdots + \alpha_h q_h)/q_0$, and Δ'_h is defined in b) above. Whence we obtain, in $\mathbb{Z}[X]/(X^{n+2})$,

$$\psi'_*[\mathcal{O}_{\tilde{P}_J}] = \begin{cases} \bar{R}'_i & \text{for } J = \{1, \dots, i+1\}, \\ \bar{T}_i & \text{for } J = \{0, \dots, i\}. \end{cases}$$

This shows that the equality (1) is the same as the following congruence

$$R'_i \equiv r_{ii} T_i + r_{i,i-1} T_{i-1} + \cdots + r_{i0} T_0 \quad (X^{n+2}).$$

Since $R'_i = X^{n+1-i} R_i$ and $T_h = X^{n+1-i} T_{ih} \quad (0 \le h \le i)$, this is equivalent to

$$R_i \equiv r_{ii} T_{ii} + r_{i,i-1} T_{i,i-1} + \cdots + r_{i0} T_{i0} \quad (X^{i+1}).$$

The Theorem is proven since the polyomials R_i and T_{ih} have degree $\le i$. □

d) *Determinant of the matrix* (r_{ij})

(cf Theorem and Corollary above).

We have $\det(r_{ij}) = \prod_0^n r_{ii}$. There are two ways of computing the integers r_{ii}. We keep the preceding notations and hypotheses.

i) The commutative diagram $(0 \le i \le n)$

$$\begin{array}{ccc} A_i(\tilde{P}^n) & \xrightarrow{\sigma_*} & A_i(\tilde{P}^{n+1}) \\ {\scriptstyle \text{can.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ Gr_iK_\cdot(\tilde{P}^n) & \longrightarrow & Gr_iK_\cdot(\tilde{P}^{n+1}) \end{array}$$

and Proposition in §.A, Theorem above (b).), [A1] 2.5-3.2 show that

$$r_{ii} = r_{ii}(q_0, \ldots, q_{n+1}) = l_i(q_0, \ldots, q_{n+1})/l_i(q_1, \ldots, q_{n+1}).$$

The divisibility hypothesis (dw) implies ([A1] 1.3.2)

$$r_{ii} = q_0 q_1 \cdots q_{i-1}/q_1 q_2 \cdots q_i = q_0/q_i.$$

ii) Now we use the polynomial identity in the preceding Theorem (b).) to compute the integers r_{ii}. By definition of the polynomials R_i, T_{ih} we have

$$r_{ii} = R_i(0)/T_{ii}(0),$$

and

$$R_i(0) = \operatorname{card} \Delta'_{i+1}, \quad T_{ii}(0) = \operatorname{card} \Delta'_i.$$

But $\operatorname{card} \Delta'_h$ is calculated in [A1] 3.5.e):

$$\operatorname{card} \Delta'_h = q_0^{h-1}/q_1 \cdots q_{h-1}.$$

One concludes $r_{ii} = q_0/q_i$ and

$$\det(r_{ij}) = q_0^n/q_1 \cdots q_n.$$

5. Computations.

Notations, assumptions are the same as in the preceding §.4. First, we establish that the integers defined in the Theorem §.4.b) are the solutions of $\mathbb{Z}$-linear systems of equations we shall explicit. Then many computation examples will be given.

We have (Theorem §.4.b))

$$r_{ij} = r_{ij}(q_0, \ldots, q_{n+1}) \quad (0 \leq j \leq i \leq n), \quad \text{with } q_{h+1} | q_h \ (0 \leq h \leq n).$$

In a similar way as in [A1](3.5.d)) (with same notation), the polynomials R_i and T_{ih} $(0 \leq h \leq i \leq n)$ (see 4.b)) satisfy the following (in the ring $\mathbb{Z}[X]$):

$$R_i = \sum_{j \geq 0} \mu_{i+1,j}(1 - X)^j,$$

$$T_{ih} = X^{i-h} \sum_{j \geq 0} \mu_{hj}(1 - X)^j,$$

where

$$\mu_{hj} = \text{ coefficient of } T^{jq_0} \text{ in } \prod_{s=1}^{h} \frac{1-T^{q_0}}{1-T^{q_s}}.$$

Let us *fix* $i \in \{0, \dots, n\}$ and denote

$$u_s = \quad \text{coefficient of } X^s \text{ in } R_i \ \ (0 \le s \le i),$$
$$u_{h,s} = \quad \text{coefficiient of } X^s \text{ in } T_{ih} \ \ (i-h \le s \le i)$$

Then the formula $(1-X)^j = \sum_{s\ge 0}(-1)^s \binom{j}{s} X^s$ implies

$$u_s = (-1)^s \sum_{j\ge 0} \mu_{i+1,j} \binom{j}{s} \ (0 \le s \le i)$$

$$u_{hs} = (-1)^{s-i+h} \sum_{j\ge 0} \mu_{hj} \binom{j}{s-i+h} \ (i-h \le s \le i) \ ([10])$$

If we put $r_{ih} = r_h$ (for simplification), Theorem §.4.b) shows that $(r_0, \dots, r_i)$ is the solution of the $\mathbb{Z}$-linear system

$$\begin{pmatrix} u_{0i} & u_{1i} & \cdots & u_{ii} \\ & u_{1,i-1} & \cdots & u_{i,i-1} \\ & & \ddots & \vdots \\ & & & u_{i0} \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_i \end{pmatrix} = \begin{pmatrix} u_i \\ u_{i-1} \\ \vdots \\ u_0 \end{pmatrix}$$

(Note that $\deg(T_{ih}) < i$ for $h \neq 0$ so that $u_{1i} = \cdots = u_{ii} = 0$.)

Similarly to [A1] 3.5.e), we have

$$u_{0i} = 1,$$
$$u_{h,i-h} = \mu_{h,0} + \cdots + \mu_{h,h-1} = \operatorname{card}(\Delta'_h) \ (1 \le h \le n),$$

and hence

$$u_{h,i-h} = q_0^{h-1}/q_1 \cdots q_{h-1}.$$

Therefore the matrix of the $\mathbb{Z}$-linear system above has determinant equal to

$$q_0^t / q_1^{i-1} q_2^{i-2} \cdots q_{i-1}^1 \quad \text{with } t = i(i-1)/2.$$

Examples

We compute the polynomials R_i and T_{ih}, the degree of which is $\le i$, then we deduce the integers r_{ih} from the linear system above.

i) $n = 2, q_0 = 2, q_1 = q_2 = q_3 = 1$.

Let us compute μ_{hj} $(0 \le h \le 3)$;

In the *general case*, we have

$$\mu_{00} = 1, \mu_{0j} = 0 \quad (j \ge 1),$$

[10] Note that u_{hs} and β_j^l in [A1] 3.5.5 are such that $u_{hs} = \beta_{n+s-i}^h$.

$$\mu_{h0} = 1, \mu_{hj} = 0 \quad (j \geq h), \quad \text{for } h \geq 1.$$

Now

$$\frac{(1-T^2)^2}{(1-T)^2} = 1 + 2T + T^2,$$
$$\frac{(1-T^2)^3}{(1-T)^3} = 1 + 3T + 3T^2 + T^3.$$

Since μ_{hj} is the coefficient of T^{jq_0} in $\prod_{s=1}^{h} \frac{(1-T^{q_0})}{(1-T^{q_s})}$, we get

$$\mu_{21} = 1, \mu_{31} = 3, \mu_{32} = 0.$$

Whence

$$\begin{aligned} R_0 &= 1 \\ R_1 &= 1 + (1 - X) = 2 - X \\ R_2 &= 1 + 3(1 - X) = 4 - 3X \\ T_{00} &= 1, \qquad T_{10} = X, \quad T_{11} = X \\ T_{20} &= X^2, \quad T_{21} = X \\ T_{22} &= 1 + (1 - X) = 2 - X. \end{aligned}$$

The integers r_{ih} $(0 \leq h \leq i \leq 2)$ are then given by:

$$r_{00} = 1,$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_{10} \\ r_{11} \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \text{i.e.,} r_{10} = -1, r_{11} = 2,$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} r_{20} \\ r_{21} \\ r_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix},$$

i.e., $r_{20} = 0, r_{21} = -1, r_{22} = 2$.

ii) $n = 2, q_0 = 4, q_1 = 2, q_2 = q_3 = 1$.
We have:

$$\frac{(1-T^4)^2}{(1-T^2)(1-T)} = (1+T)(1+2T^2+T^4) = 1 + T^4 + \cdots$$
$$\begin{aligned} \frac{(1-T^4)^3}{(1-T^2)(1-T)^2} &= (1+2T+T^2)(1+3T^2+3T^4+T^6) \\ &= 1 + 6T^4 + T^8 + \cdots \end{aligned}$$

So

$$\mu_{21} = 1, \mu_{31} = 6, \mu_{32} = 1.$$

This gives

$$R_2 = 1 + 6(1 - X) + (1 - X)^2 = 8 - 8X + X^2.$$

The other polynomials R_i and T_{ih} are the same as in the preceding example and, in a similar way, one obtains

$$\begin{aligned} &r_{00} = 1, \\ &r_{10} = -1, \quad r_{11} = 2, \\ &r_{20} = 1, \quad\ \ r_{21} = -4, \quad r_{22} = 4. \end{aligned}$$

iii) $n = 3, q_0 = 6, q_1 = q_2 = 3, q_3 = q_4 = 1$.
Here we consider

$$\begin{aligned} \frac{(1-T^6)^2}{(1-T^3)^2} &= 1 + 2T^3 + T^6, \\ \frac{(1-T^6)^3}{(1-T^3)^2(1-T)} &= (1 + 2T^3 + T^6)(1 + T + T^2 + T^3 + T^4 + T^5) \\ &= 1 + 3T^6 + \cdots, \\ \frac{(1-T^6)^4}{(1-T^3)^2(1-T)^2} &= (1 + 2T^3 + T^6)(1 + T + T^2 + T^3 + T^4 + T^5)^2 \\ &= 1 + 14T^6 + 9T^{12} + \cdots. \end{aligned}$$

It follows

$$\begin{aligned} &\mu_{21} = 1, \\ &\mu_{31} = 3, \quad\ \ \mu_{32} = 0, \\ &\mu_{41} = 14, \quad \mu_{42} = 9, \quad \mu_{43} = 0 \end{aligned}$$

In particular the polynomials R_i $(i = 0, 1, 2)$, T_{ih} $(0 \le h \le i \le 2)$ are the same as in the example i). The others are:

$$\begin{aligned} R_3 &= 1 + 14(1 - X) + 9(1 - X)^2 = 24 - 32X + 9X^2, \\ T_{30} &= X^3, \qquad T_{31} = X^2 \quad \text{(always true, see i))}. \\ T_{32} &= X(1 + (1 - X)) = 2X - X^2, \\ T_{33} &= 1 + 3(1 - X) = 4 - 3X. \end{aligned}$$

So the integers r_{ih} $(0 \le h \le i \le 2)$ are equal to those in i), and the r_{3h} $(0 \le h \le 3)$ are given by the linear system:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} r_{30} \\ r_{31} \\ r_{32} \\ r_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ -32 \\ 24 \end{pmatrix}$$

Therefore: $r_{30} = 0, r_{31} = 2, r_{32} = -7, r_{33} = 6$.

Remark. After Theorem §.4.b) and its Corollary, the group structure of $K.(\tilde{L}^n)$, in case of examples i) and ii), is completedly computed. In order to make it precise in case of example iii), we need to deal with the two following situations (see Factorization Theorem §.3).

iv) $n = 3, q_0 = q_1 = q_2 = 3, q_3 = q_4 = 1$.
Consider

$$\frac{(1-T^3)^2}{(1-T^3)^2} = 1$$

$$\frac{(1-T^3)^3}{(1-T^3)^2(1-T)} = (1+T+T^2)^2 = 1+2T^3+\cdots$$

Whence

$$\begin{aligned}
&\mu_{21} = 0,\\
&\mu_{31} = \mu_{32} = 0,\\
&\mu_{41} = 2, \quad \mu_{42} = \mu_{43} = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
R_0 = R_1 = R_2 &= 1,\\
R_3 &= 1 + 2(1-X) = 3 - 2X;\\
T_{22} &= 1,\\
T_{32} &= X, \quad T_{33} = 1.
\end{aligned}$$

The other polynomials are always equal to:

$$\begin{array}{lll}
T_{00} = 1, & T_{10} = X, & T_{11} = 1,\\
T_{20} = X^2, & T_{21} = X, & \\
T_{30} = X^3, & T_{31} = X^2. &
\end{array}$$

Here the matrix $(u_{h,i-j})_{h,j}$ is the identity for all $i = 0, 1, 2, 3$. Hence

$$\begin{array}{llll}
r_{00} = 1 & r_{10} = 0 & r_{20} = 0 & r_{30} = 0\\
 & r_{11} = 1 & r_{21} = 0 & r_{31} = 0\\
 & & r_{22} = 1 & r_{32} = -2\\
 & & & r_{33} = 3
\end{array}$$

v) $n = 3, q_0 = 2, q_1 = q_2 = q_3 = q_4 = 1$.
Example i) shows that

$$\mu_{21} = 1,\ \mu_{31} = 3,\ \mu_{32} = 0.$$

To compute the integers μ_{4h} let us consider

$$\frac{(1-T^2)^4}{(1-T)^4} = (1+T)^4 = 1 + 4T + 6T^2 + 4T^3 + T^4.$$

This gives

$$\mu_{41} = 6, \quad \mu_{42} = 1, \quad \mu_{43} = 0.$$

It follows

$$R_3 = 1 + 6(1 - X) + (1 - X)^2 = 8 - 8X + X^2.$$

The integers μ_{jh} $(j = 2, 3)$ above show that the polynomials T_{3h} $(0 \leq h \leq 3)$ are the same as in example iii), and that the polynomials R_i and T_{ih} $(0 \leq h \leq i \leq 2)$ are as in example i). Hence the integers r_{ih} $(0 \leq h \leq i \leq 2)$ have same value as in i), and the r_{3h}'s are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} r_{30} \\ r_{31} \\ r_{32} \\ r_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -8 \\ 8 \end{pmatrix}$$

that is $r_{30} = r_{31} = 0, r_{32} = -1, r_{33} = 2$.

Let us make precise the group structure (given by generators and relations) of $K.(\tilde{L}^n)$ corresponding to the computations we have just acheived.

6. Groups $K.(\tilde{L}^n)$.

From the Factorization Theorem (§.3) and Theorem §.4.b) with its Corollary, we deduce the following:

i) $K.(L^2(2; 1, 1, 1)) = \mathbb{Z} \oplus \mathbb{Z}^3/(r_{ij})\mathbb{Z}^3$,
where

$$(r_{ij}) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

ii) $K.(L^2(4; 2, 1, 1)) = \mathbb{Z} \oplus \mathbb{Z}^3/(r_{ij})\mathbb{Z}^3$,
where

$$(r_{ij}) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{pmatrix}.$$

iii) $K.(L^3(6; 3, 3, 1, 1)) = \mathbb{Z} \oplus F_3(L^3(2; 1, 1, 1, 1)) \oplus F_3(L^3(3; 3, 3, 1, 1))$
where

$$F_3(L^3(2; 1, 1, 1, 1)) = \mathbb{Z}^4/(r_{ij})\mathbb{Z}^4,$$

with (5.v))

$$(r_{ij}) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix};$$

and

$$F_3(L^3(3;3,3,1,1)) = \mathbb{Z}^4/(r_{ij})\mathbb{Z}^4$$

with (5. iv))

$$(r_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Conclusion.

§.5 shows that computations are handly feasible thanks to the algorithm given before Examples (without any computer help!). About the use of computers in education, sciences and culture, nowdays, reading V.I.Arnold ([11]) would be of great benefit.

7.*Comparison with Chow group.*

No particular hypothesis is made here on the integers $q, q_0, \ldots, q_n$.

Proposition 26. *The canonical surjection (§.1)*

$$A_*(\tilde{L}^n) \longrightarrow \mathrm{Gr}\, K_.(\tilde{L}^n) \longrightarrow 0$$

is an isomorphism ($\tilde{L}^n = L_k(q; q_0, \ldots, q_n)$).

Proof. The Propositions in A. and B.2 show that we have a morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_*(\tilde{P}^n) & \xrightarrow{\sigma_*} & A_*(\tilde{P}^{n+1}) & \xrightarrow{\tau^*} & A_*(\tilde{L}^n) & \longrightarrow & 0 \\ & & \scriptstyle{can.}\downarrow & & \downarrow\scriptstyle{can.} & & \downarrow\scriptstyle{can.} & & \\ 0 & \longrightarrow & \mathrm{Gr}\, K_.(\tilde{P}^n) & \xrightarrow{Gr(\sigma_*)} & \mathrm{Gr}\, K_.(\tilde{P}^{n+1}) & \xrightarrow{Gr(\tau^*)} & \mathrm{Gr}\, K_.(\tilde{L}^n) & \longrightarrow & 0 \end{array}$$

By Corollary [A1] 3.1 we know that the two first vertical maps (from the left) are isomorphisms. Hence the third one is too. □

C. ℓ-adic Homology of $\tilde{L}^n$.

Let ℓ be a prime number different from the characteristic of the ground field k. For definition and properties of ℓ-adic homology $H_*(X, \mathbb{Z})$ (X being a k-scheme of finite type), we refer to [LA].

1. Without any hypothesis on the integers $q, q_0, \ldots, q_n$, let

$$\tilde{L}^n = L_k(q; q_0, \ldots, q_n).$$

Proposition 27. *We have*

$$H_i(\tilde{L}^n, \mathbb{Z}_\ell) = \begin{cases} \mathbb{Z}_\ell & \text{if } i = 1 \text{ or } 2(n+1), \\ \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} & \text{if } i = 2j,\ 0 \le j \le n, \\ 0 & \text{if not.} \end{cases}$$

[11] Yesterday and Long Ago (Springer / Phasis 2007)

where

$$\alpha_j = v_\ell(m_j),\ m_j = m_j(q_0, \dots, q_n, q)$$

(see beginning of II.§.1).

Proof. By definition of $\tilde{L}^n$, one has

$$\tilde{L}^n = U \setminus \{point\}, U = \text{open complement of } \tilde{P}^n \subset_\sigma \tilde{P}^{n+1}$$

($\tilde{P}^n = P^n_k(q_0, \dots, q_n), \tilde{P}^{n+1} = P^{n+1}_i(q_0, \dots, q_n, q)$). The homology exact sequence related to the closed immersion σ (with coefficients in $\mathbb{Z}_\ell$)

$$\cdots \longrightarrow H_{i+1}(U) \longrightarrow H_i(\tilde{P}^n) \longrightarrow H_i(\tilde{P}^{n+1}) \longrightarrow H_i(U) \longrightarrow \cdots$$

and [A1]1.(1.1), where $H_*(\tilde{P})$ is computed, imply (exact sequences)

$\alpha)$ $\quad \mathbb{Z}_\ell = H_{2(n+1)}(\tilde{P}^{n+1}) \xrightarrow{\sim} H_{2(n+1)}(U),$

$\beta)$
$$\begin{array}{ccccccccc} 0 & \to & H_{2i}(\tilde{P}^n) & \longrightarrow & H_{2i}(\tilde{P}^{n+1}) & \longrightarrow & H_{2i}(U) & \to & 0 \\ & & \| & & \| & & & & \\ & & \mathbb{Z}_\ell & \xrightarrow{\cdot m_i} & \mathbb{Z}_\ell & & (0 \le i \le n) & & \end{array}$$

(see proof of 2.3.(ii) in [A1])

$\gamma)$ $\quad 0 \to H_i(U) \longrightarrow H_{i-1}(\tilde{P}^n) \xrightarrow{inj.} H_{i-1}(\tilde{P}^{n+1}) \quad (i \text{ odd}).$

So the ℓ-adic homology of U is

$$H_i(U, \mathbb{Z}_\ell) = \begin{cases} \mathbb{Z}_\ell & \text{if } i = 2(n+1), \\ \mathbb{Z}/\ell^{\alpha_j}\mathbb{Z} & \text{if } i = 2j,\ 0 \le j \le n, \\ 0 & \text{otherways}, \end{cases}$$

with $\alpha_j = v_\ell(m_j)$. Now the exact sequence related to a closed point in U, and the fact that $H_j(U, \mathbb{Z}_\ell) = 0 (j = 0, 1)$, give the isomorphisms

$\delta)$

$$H_i(U, \mathbb{Z}_\ell) \xrightarrow{\sim} H_i(\tilde{L}^n, \mathbb{Z}_\ell) \qquad (i \neq 1),$$
$$H_1(\tilde{L}^n, \mathbb{Z}_\ell) \xrightarrow{\sim} H_0(\{pt\}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell.$$

This proves the Proposition. □

2. *Comparison with Chow group.*

Proposition 28. *The homomorphism fundamental class induces an isomorphism*

$$A_*(\tilde{L}^n) \bigotimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \bigoplus_i H_{2i}(\tilde{L}^n, \mathbb{Z}_\ell(-i)).$$

Proof. The fundamental class defines a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 \to & A_i(\tilde{P}^n)\bigotimes_{\mathbb{Z}}\mathbb{Z}_\ell & \to & A_i(\tilde{P}^{n+1})\bigotimes_{\mathbb{Z}}\mathbb{Z}_\ell & \to & A_i(\tilde{L}^n)\bigotimes_{\mathbb{Z}}\mathbb{Z}_\ell & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H_{2i}(\tilde{P}^n,\mathbb{Z}_\ell(-i)) & \to & H_{2i}(\tilde{P}^{n+1},\mathbb{Z}_\ell(-i)) & \to & H_{2i}(\tilde{L}^n,\mathbb{Z}_\ell(-i)) & \to 0 \end{array}$$

where the first sequence comes from Proposition III.A. (since $\mathbb{Z}_\ell$ is $\mathbb{Z}$-flat), and the second is given by β) and δ) in the proof of the preceding Proposition. So the result follows from its analogue for the weighted projective spaces $\tilde{P}^n$ and $\tilde{P}^{n+1}$ ([A1] 2.9.).

□

3. Transcendental case.

Here the ground field k is the field $\mathbb{C}$ of complex numbers.

i) We denote by $\tilde{L}^n(\mathbb{C})$ the analytical space, associated to the scheme $\tilde{L}^n = L_{\mathbb{C}}(q; q_0, \ldots, q_n)$. That is nothing else but the quotient-space of $(\mathbb{C}^{n+1})^* = \mathbb{C}^{n+1}\setminus\{0\}$ by the following action of the group μ_q (q-th roots of unity):

$$\begin{aligned} \mu_q \times (\mathbb{C}^{n+1})^* &\to (\mathbb{C}^{n+1})^* \\ (\lambda, x_0, \ldots, x_n) &\mapsto (\lambda^{q_0}x_0, \ldots, \lambda^{q_n}x_n) \end{aligned}$$

ii) The same argument as in the proof in §.1 above, shows that the homology with compact supports of $\tilde{L}^n(\mathbb{C})$ is as follows (one uses transcendental case of $\tilde{P}$, [A1] 2.10).

$$H_i^c(\tilde{L}^n(\mathbb{C}),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 1 \text{ or } 2(n+1), \\ \mathbb{Z}/m_j\mathbb{Z} & \text{if } i = 2j-1,\ 1 \le j \le n, \\ 0 & \text{otherways,} \end{cases}$$

with $m_j = m_j(q_0, \ldots, q_n, q) = l_j(q_0, \ldots, q_n, q)/l_j(q_0, \ldots, q_n)$ (II.1.).

Remark.

By restriction to the sphere $S^{2n+1} \subset (\mathbb{C}^{n+1})^*$, the μ_q-operation considered above allows to define a *compact* twisted lens space [TR, KW]

$$S^{2n+1}/\mu_q = (S^{2n+1}/\mu_q)(q_0, \ldots, q_n),$$

having the same homotopy type as $\tilde{L}^n(\mathbb{C})$.

Thanks to the universal coefficients formula, one deduces its homology from its cohomology (loc. cit.)

$$H_i(S^{2n+1}/\mu_q,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } 2(n+1), \\ \mathbb{Z}/m_j\mathbb{Z} & \text{if } i = 2j-1,\ 1 \le j \le n, \\ 0 & \text{if not.} \end{cases}$$

On the other hand, $\tilde{L}^n(\mathbb{C})$ is a trivial fibre bundle of the form

$$\tilde{L}^n(\mathbb{C}) \simeq S^{2n+1}/\mu_q \times \mathbb{R}_+^*.$$

The THOM-GYSIN isomorphism

$$H_{i+1}^c(\tilde{L}^n(\mathbb{C}), \mathbb{Z}) \simeq H_i(S^{2n+1}/\mu_q, \mathbb{Z})$$

confirms then the computation done in ii).

iii) *The homomorphism fundamental class*

$$A_*(\tilde{L}^n) \to \bigoplus_i H_{2i}^c(\tilde{L}^n(\mathbb{C}), \mathbb{Z})$$

is bijective.

Proof. This is similar to that of Proposition §.2, because of [A1] 2.10.iii) (case of $\tilde{P}^n$). □

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