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# Sheets, slice induction and $G_{2(2)}$ case

MICHAËL BULOIS \*    PASCAL HIVERT†

## Abstract

In this paper, we study sheets of symmetric Lie algebras through their Slodowy slices. In particular, we introduce a notion of *slice induction* of nilpotent orbits which coincides with the parabolic induction in the Lie algebra case. We also study in more details the sheets of the non-trivial symmetric Lie algebra of type  $G_2$ . We characterize their singular loci and provide a nice desingularisation lying in  $so_7$ .

## 1 Introduction

Let  $\mathfrak{g}$  be a reductive Lie algebra defined over an algebraically closed field of characteristic zero. Assume that  $\mathfrak{g}$  is  $\mathbb{Z}/2\mathbb{Z}$ -graded

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

with even part  $\mathfrak{k}$  and odd part  $\mathfrak{p}$ . We may refer to such a symmetric Lie algebra by the *symmetric pair*  $(\mathfrak{g}, \mathfrak{k})$ .

The algebraic adjoint group  $G$  of  $\mathfrak{g}$  acts on  $\mathfrak{g}$  and the closed connected subgroup  $K \subset G$  with Lie algebra  $\mathfrak{k}$  acts on  $\mathfrak{p}$ . A sheet of  $\mathfrak{p}$  (resp.  $\mathfrak{g}$ ) is an irreducible component of a locally closed set of the form

$$\mathfrak{p}^{(m)} := \{x \in \mathfrak{p} \mid \dim K.x = m\}, \quad (\text{resp. } \mathfrak{g}^{(m)} := \{x \in \mathfrak{g} \mid \dim G.x = m\}).$$

Sheets of  $\mathfrak{g}$  have been extensively studied in several papers in the past decades.

On one hand, the key papers of Borho and Kraft [BK, Bo] describe sheets as disjoint union of so-called *Jordan classes* (also known as *decomposition classes*). The Jordan classes form a finite partition of  $\mathfrak{g}$ , each class being irreducible, locally closed and of constant orbit dimension. In particular, there is a dense

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class in each given sheet. An important notion used in [BK, Bo] is the parabolic induction of orbits introduced in [LS] which gives rise to a notion of induction of Jordan classes. It is shown in [Bo] that a sheet  $S$  with dense Jordan class  $J$  is precisely the union of the Jordan classes induced from  $J$ . Two consequences of this are the following. Firstly, each sheet contains a unique nilpotent orbit. Secondly, there is a parameterization of sheets coming with induction [BK, §5].

On the other hand, making use of the previous parameterization, it is shown in [Kat] that sheets are parameterized by their Slodowy slices. More explicitly, let  $e$  be a representative of the nilpotent orbit of a sheet  $S$  and embed  $e$  in an  $\mathfrak{sl}_2$ -triple  $\mathcal{S} = (e, h, f)$ . The Slodowy slice of  $S$  (with respect to  $e$ ) is

$$e + X(S, \mathcal{S}) := S \cap (e + \mathfrak{g}^f),$$

where  $\mathfrak{g}^f$  stands for the centraliser of  $f$  in  $\mathfrak{g}$ . Katsylo proves that  $S = G.(e + X)$  and that a geometric quotient of  $S$  can be expressed as a finite quotient of  $e + X$ . In [IH], Im-Hof shows that the morphism  $G \times (e + X) \rightarrow S$  is smooth. This relates smoothness of  $S$  to smoothness of  $e + X$  and eventually leads to a proof of smoothness of sheets in classical Lie algebras.

To our knowledge, the only known case of a singular sheet lies in a simple Lie algebra of type  $G_2$  [Pe]. In this case, the two non-trivial sheets (*i.e.* non-regular and with more than one orbit) are the two irreducible components of the set of subregular elements. One of these subregular sheets  $S_2^{\mathfrak{g}}$  is smooth while the other  $S_1^{\mathfrak{g}}$  is singular. More precisely, we can see that three analytical germs of  $S_1^{\mathfrak{g}}$  intersect in the neighborhood of elements of the subregular nilpotent orbit. In [Hi, §2], an explicit desingularization of  $S_1^{\mathfrak{g}}$  is constructed in terms of a classical projection  $\mathfrak{so}_7 \rightarrow \mathfrak{g}$ .

We now look at the symmetric case and sheets of  $\mathfrak{p}$ . Most of the ground results the authors are aware of in this setting are gathered in [TY, §39.5-6]. An important feature is that there exists a notion of Jordan class of  $\mathfrak{p}$  and these classes share several good properties with the classes of  $\mathfrak{g}$ . In particular, there is still a unique dense Jordan class in each sheet. In addition, each sheet of  $\mathfrak{p}$  contains at least one nilpotent orbit. However the uniqueness statement no longer holds in general.

One of the obstacle rising in the study of the sheets of  $\mathfrak{p}$  is the lack of a well behaved notion of parabolic induction as one can check in the  $(\mathfrak{sl}_2, \mathfrak{so}_2)$ -case. For instance the induction theory developed in [Oh2] does not preserve orbit dimension, hence is of few help for our purpose. The main philosophy of [Bu2] consists in noting that, at least in the case  $\mathfrak{g} = \mathfrak{gl}_n$ , the Slodowy slices of a sheet  $S$  of  $\mathfrak{p}$  still seem to encode significant geometric information of  $S$ . One of the aim of the present work is to justify this assertion in a more systematic manner. For instance, we show in Section 2 that several properties of sheets, such as dimension, smoothness and orbits involved, are fully reflected

in the corresponding Slodowy slices. We state these results for wider classes of subvarieties of  $\mathfrak{p}$  in Proposition 2.4, Theorem 2.6 and Proposition 2.8.

Then we introduce in section 3 the notion of slice induction. It turns out to be precise enough to rebuilt important parts of the theory resulting from parabolic induction in the Lie algebra case. This includes (see Theorem 3.6 and Corollary 3.8):

- Construction of one parameter deformations of orbits.
- Stratification properties for Jordan classes.
- Characterization of a sheet with dense Jordan class  $J$  as the union of induced classes from  $J$ .

In the Lie algebra case, this also provides a new insight on these results.

Our second goal is the description of sheets of  $\mathfrak{p}$  in the following case:  $\mathfrak{g}$  is a simple Lie algebra of type  $G_2$  and  $\mathfrak{k} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ . This is the only non-trivial symmetric Lie algebra of type  $G_2$ . We make use of two approaches for this study. In Section 4.1, we study the subregular sheets through their Slodowy slices. This provides the set-theoretical description of the sheets and describes the behavior of the singularities of  $S_1$  through the intersection with  $\mathfrak{p}$ . In Section 4.2, we exploit the symmetry of  $\mathfrak{g}$  making use of 4-ality as described in [LM]. This allows us to construct a nice desingularization in  $\mathfrak{so}_7$  of the singular sheet (Proposition 4.7), following the guidelines of [Hi].

Note that it is plausible that most of what is stated in section 2 and 3 remains true in the more general setting of  $\theta$ -representation. However, the authors are unaware of references for a general theory of Jordan classes in this setting.

## 2 Geometry of subvarieties and Slodowy slices

We start with some notation. In the whole paper,  $\mathbb{k}$  is an algebraically closed field of characteristic 0,  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is a reductive symmetric Lie algebra over  $\mathbb{k}$ , that is a  $\mathbb{Z}/2\mathbb{Z}$ -graded reductive Lie algebra  $\mathfrak{g}$  with even part  $\mathfrak{k}$  and odd part  $\mathfrak{p}$ . In particular,  $\mathfrak{k}$  is a Lie algebra and  $\mathfrak{p}$  is a  $\mathfrak{k}$ -module. We denote by  $\mathfrak{g}'$  the semisimple part of  $\mathfrak{g}$ . Lie algebras can be seen as particular cases of symmetric Lie algebras in the following sense: given a Lie algebra  $\hat{\mathfrak{g}}$ , there exists a symmetric Lie algebra  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  such that the  $\mathfrak{k}$ -module  $\mathfrak{p}$  is isomorphic to the  $\hat{\mathfrak{g}}$ -module  $\hat{\mathfrak{g}}^1$ . As a consequence, all the statements enounced below in the symmetric setting hold for Lie algebras replacing both  $\mathfrak{k}$  and  $\mathfrak{p}$  by  $\mathfrak{g}$ , and  $K$  by  $G$ . A large part of the Lie theory have a symmetric counterpart. We refer to [KR] for the ground results on symmetric Lie algebra.

Let  $G$  be the adjoint group of  $\mathfrak{g}$  and  $K$  be the closed connected subgroup of  $G$  with Lie algebra  $\mathfrak{k} \cap \mathfrak{g}'$ . The group  $K$  acts on  $\mathfrak{p}$ . For  $x \in \mathfrak{p}$ , it follows from

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<sup>1</sup>Namely  $\mathfrak{g} = \hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$ ,  $\mathfrak{k} = \{(x, x) | x \in \hat{\mathfrak{g}}\}$  and  $\mathfrak{p} = \{(x, -x) | x \in \hat{\mathfrak{g}}\}$ .

[KR, Proposition 5] that

$$\dim K.x = \frac{1}{2} \dim G.x, \quad \dim \mathfrak{k} - \dim \mathfrak{k}^x = \dim \mathfrak{p} - \dim \mathfrak{p}^x. \quad (2.1)$$

For  $A \subset \mathfrak{g}$ , we set  $A^\bullet := \{a \in A \mid \forall a' \in A, \dim G.a \geq \dim G.a'\}$ . Note that we can replace  $G$  by  $K$  in the previous definition when  $A \subset \mathfrak{p}$  thanks to (2.1). For  $A, B \subset \mathfrak{g}$ , the centralizer in  $B$  of  $A$  is denoted by  $\mathfrak{c}_B(A) := \{b \in B \mid \forall a \in A, [a, b] = 0\}$ .

If  $\mathfrak{m}$  is a  $\mathbb{Z}/2\mathbb{Z}$ -subspace of  $\mathfrak{g}$  we write  $\mathfrak{k}_\mathfrak{m} := \mathfrak{k} \cap \mathfrak{m}$ ,  $\mathfrak{p}_\mathfrak{m} := \mathfrak{p} \cap \mathfrak{m}$  and we have

$$\mathfrak{m} = \mathfrak{k}_\mathfrak{m} \oplus \mathfrak{p}_\mathfrak{m}.$$

We say that a Levi subalgebra  $\mathfrak{l} \subset \mathfrak{g}$  *arises from*  $\mathfrak{p}$  if there exists a semisimple element  $v \in \mathfrak{p}$  such that  $\mathfrak{l} = \mathfrak{g}^v$  (it corresponds to the notion of subsymmetric pair in [PY]). In this case,  $\mathfrak{l}$  and  $\mathfrak{l}'$  are reductive and semisimple  $\mathbb{Z}/2\mathbb{Z}$ -graded Lie subalgebras of  $\mathfrak{g}$ . In addition, we can decompose  $\mathfrak{g}$  in  $\mathfrak{l}$ -modules in the following way

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}^\perp$$

where  $\mathfrak{l}^\perp$  is the orthogonal of  $\mathfrak{l}$  in  $\mathfrak{g}'$  with respect to the Killing form. More concretely, if  $\mathfrak{l} = \mathfrak{g}^v$  we can write  $\mathfrak{l}^\perp = [\mathfrak{g}, v]$ .

It is well known that  $\mathfrak{c}_\mathfrak{g}(\mathfrak{l})$  is the center of  $\mathfrak{l}$ . In particular,  $\mathfrak{p}_\mathfrak{l} = \mathfrak{c}_\mathfrak{p}(\mathfrak{l}) \oplus \mathfrak{p}_{\mathfrak{l}'}$ . Moreover, we have [TY, 38.8.4]

$$\mathfrak{c}_\mathfrak{p}(\mathfrak{l}) = \mathfrak{c}_\mathfrak{g}(\mathfrak{l}) \cap \mathfrak{p} = \mathfrak{c}_\mathfrak{p}(\mathfrak{p}_\mathfrak{l}), \quad (2.2)$$

$$\mathfrak{c}_\mathfrak{p}(\mathfrak{l})^\bullet = \mathfrak{c}_\mathfrak{g}(\mathfrak{l})^\bullet \cap \mathfrak{p} = \{u \in \mathfrak{c}_\mathfrak{p}(\mathfrak{l}) \mid \mathfrak{g}^u = \mathfrak{l}\}. \quad (2.3)$$

We denote by  $K_\mathfrak{l}$  (resp.  $K_{\mathfrak{l}'}$ ) the closed connected subgroup of  $K$  with Lie algebra  $\mathfrak{k}_\mathfrak{l} \cap \mathfrak{g}'$  (resp.  $\mathfrak{k}_{\mathfrak{l}'}$ ). Then,  $K_\mathfrak{l} = (K^v)^\circ$  and the  $K_\mathfrak{l}$ -orbits of  $\mathfrak{p}_\mathfrak{l}$  are precisely the orbits associated to the reductive symmetric Lie algebra  $\mathfrak{l} = \mathfrak{k}_\mathfrak{l} \oplus \mathfrak{p}_\mathfrak{l}$ . The same holds for  $K_{\mathfrak{l}'}$ -orbits ( $=K_\mathfrak{l}$ -orbits) of  $\mathfrak{p}_{\mathfrak{l}'}$ . Define

$$U_\mathfrak{l} := \{y \in \mathfrak{p}_\mathfrak{l} \mid \mathfrak{g}^y \subset \mathfrak{l}\}. \quad (2.4)$$

The next lemma shows that  $K$ -orbits and  $K_\mathfrak{l}$ -orbits of elements of  $U_\mathfrak{l}$  are closely related.

**Lemma 2.1.** *Let  $\mathfrak{l}$  be a Levi subalgebra arising from  $\mathfrak{p}$ . Then, the following conditions are equivalent for any  $y \in \mathfrak{p}_\mathfrak{l}$ .*

- (i)  $y \in U_\mathfrak{l}$ ,
- (ii)  $\mathfrak{g}^s \subset \mathfrak{l}$ , where  $s$  is the semisimple component of  $y$ ,
- (iii)  $\text{codim}_\mathfrak{p} K.y = \text{codim}_\mathfrak{p} K_\mathfrak{l}.y$ ,
- (iv)  $[\mathfrak{k}, y] = \mathfrak{p}_{\mathfrak{l}^\perp} \oplus [\mathfrak{k}_\mathfrak{l}, y]$ .

Moreover,  $U_{\mathfrak{l}}$  is an open subset of  $\mathfrak{p}_{\mathfrak{l}}$ .

*Proof.* We have  $y \in \mathfrak{l}$ . Hence  $[\mathfrak{g}, y] = [\mathfrak{l}^{\perp}, y] \oplus [\mathfrak{l}, y] \subset \mathfrak{l}^{\perp} \oplus [\mathfrak{l}, y]$ , with equality if and only if  $\mathfrak{g}^y \cap \mathfrak{l}^{\perp} = \{0\}$  if and only if  $\mathfrak{g}^y \subset \mathfrak{l}$ . As a consequence, (i) is equivalent to (iv'):  $[\mathfrak{g}, y] = \mathfrak{l}^{\perp} \oplus [\mathfrak{l}, y]$ . By the way, we also note that  $\mathfrak{g}^y \cap \mathfrak{l}^{\perp} = \{0\}$  is an open condition on  $y$ .

On the other hand, we see that (iv) is just (iv') intersected with  $\mathfrak{p}$ . Let  $v \in \mathfrak{p}$  such that  $\mathfrak{l} = \mathfrak{g}^v$ . With the help of (2.1), we have

$$\begin{aligned} \dim(\mathfrak{p}_{\mathfrak{l}^{\perp}} \oplus [\mathfrak{k}_{\mathfrak{l}}, y]) - \dim[\mathfrak{k}, y] &= \dim[\mathfrak{k}, v] + \dim[\mathfrak{k}_{\mathfrak{l}}, y] - \dim[\mathfrak{k}, y] \\ &= \frac{1}{2}(\dim[\mathfrak{g}, v] + \dim[\mathfrak{l}, y] - \dim[\mathfrak{g}, y]) \\ &= \frac{1}{2}(\dim(\mathfrak{l}^{\perp} \oplus [\mathfrak{l}, y]) - \dim[\mathfrak{g}, y]) \end{aligned}$$

Since we have the inclusion  $[\mathfrak{k}, y] \subset \mathfrak{p}_{\mathfrak{l}^{\perp}} \oplus [\mathfrak{k}_{\mathfrak{l}}, y]$ , we get the equivalence between (iv') and (iv). Through tangent spaces, we also see that (iii) is equivalent to (iv).

There remains to show that (i) is equivalent to (ii). Denote the nilpotent part of  $y$  by  $n$ . We have  $\mathfrak{g}^y = \mathfrak{g}^s \cap \mathfrak{g}^n$  so (ii) implies (i). Let us now assume that  $\mathfrak{g}^s$  is not included in  $\mathfrak{l}$ . Since  $s, n \in \mathfrak{l}$ , the endomorphism  $\text{ad } n$  stabilizes the non-trivial subspace  $\mathfrak{g}^s \cap \mathfrak{l}^{\perp}$ . Since  $\text{ad } n$  is nilpotent, there exists a non-zero element in  $\mathfrak{g}^n \cap \mathfrak{g}^s \cap \mathfrak{l}^{\perp}$ . Hence,  $\mathfrak{g}^y \not\subset \mathfrak{l}$ . By contraposition, (i) implies (ii).  $\square$

**Definition 2.2.** Given a Levi subalgebra  $\mathfrak{l}$  of  $\mathfrak{g}$  arising from  $\mathfrak{p}$ , a nilpotent element  $e \in \mathfrak{p}_{\mathfrak{l}}$  embedded in a normal  $\mathfrak{sl}_2$ -triple  $\mathcal{S} := (e, h, f) \subset \mathfrak{l}'$  (here *normal* means  $e, f \in \mathfrak{p}$ ,  $h \in \mathfrak{k}$  and we allow  $(0, 0, 0)$  as  $\mathfrak{sl}_2$ -triple) and a subset  $J \subset \mathfrak{p}$ , we define  $X_{\mathfrak{l}}(J, \mathcal{S}) \subset \mathfrak{p}_{\mathfrak{l}}^f$  via

$$e + X_{\mathfrak{l}}(J, \mathcal{S}) := (e + \mathfrak{p}_{\mathfrak{l}}^f) \cap U_{\mathfrak{l}} \cap J,$$

where  $U_{\mathfrak{l}}$  is as in (2.4). We say that  $e + X_{\mathfrak{l}}(J, \mathcal{S})$  is the *generalized Slodowy slice* of  $J$  with respect to  $(\mathfrak{l}, \mathcal{S})$ .

In the case  $\mathfrak{g} = \mathfrak{l}$ , we have  $U_{\mathfrak{l}} = \mathfrak{g}$  and

$$e + X(J, \mathcal{S}) := e + X_{\mathfrak{g}}(J, \mathcal{S}) = (e + \mathfrak{p}^f) \cap J$$

is the natural analogue in the symmetric setting of the ordinary Slodowy slice in Lie algebras.

In what follows, a cone of  $\mathfrak{p}$  means a subset of  $\mathfrak{p}$  stable under multiplication by  $\mathbb{k}^*$ . The next lemma is well known. For more simplicity, we only state it in the  $\mathfrak{l} = \mathfrak{g}$  case.

**Lemma 2.3.** *Let  $J$  be a  $K$ -stable cone of  $\mathfrak{p}$  and  $\mathcal{S} = (e, h, f)$  be a normal  $\mathfrak{sl}_2$ -triple such that  $X(J, \mathcal{S}) \neq \emptyset$ .*

*Then  $e$  belongs to each irreducible component of  $e + \overline{X(J, \mathcal{S})}$ . In particular,  $e \in \overline{J}$ .*

A standard proof of this lemma is based on the construction of a one-parameter subgroup of  $K \times \mathbb{k}^* \text{Id}$  which contracts  $e + X(J, \mathcal{S})$  to  $e$ . Define the characteristic grading  $\mathfrak{g} := \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i, h)$  by  $\mathfrak{g}(i, h) := \{x \in \mathfrak{g} \mid [h, x] = ix\}$ . For  $t \in \mathbb{k}^*$ , let  $F_t \in \text{GL}(\mathfrak{g})$  be such that

$$(F_t)|_{\mathfrak{g}(i, h)} = t^{-i+2} \text{Id}.$$

We have  $F_t.e = e$  and  $F_t.(e + \mathfrak{p}^f) = e + \mathfrak{p}^f$  since  $\mathfrak{p}^f$  is compatible with the characteristic grading. On the other hand, it is easy to show [TY, 38.6.2] that  $t^{-2}F_t \in K$ , hence  $F_t \in K \times \mathbb{k}^* \text{Id}$ . As a consequence,  $F_t$  normalizes the cone  $J$  and hence  $(e + X(J, \mathcal{S}))$ . Since  $e + \mathfrak{p}^f \subset e + \bigoplus_{i \leq 0} \mathfrak{g}(i, h)$ , we have

$$\lim_{t \rightarrow 0} F_t.(e + x) = e \tag{2.5}$$

for any  $x \in \mathfrak{p}^f$ . Since  $(F_t)_{t \in \mathbb{k}^*}$  is a one parameter subgroup of  $\text{GL}(\mathfrak{g})$ , each irreducible component of  $e + X(J, \mathcal{S})$  is stable under the  $F_t$ -action and Lemma 2.3 follows.

Next, we wish to enlight the strong connection linking  $J$  and  $X(J, \mathcal{S})$ . This was lacking in [Bu2] in the general case and the following Proposition renders some definitions and techniques of this paper obsolete. For example, it can easily be used together with [TY, 38.6.9(i)] to show that condition ( $\clubsuit$ ) of [Bu2, §9] is automatically satisfied.

**Proposition 2.4.** *Let  $\mathfrak{l}$  be a Levi subalgebra of  $\mathfrak{g}$  arising from  $\mathfrak{p}$  and  $\mathcal{S} = (e, h, f)$  be a normal  $\mathfrak{sl}_2$ -triple of  $\mathfrak{V}$ . Let  $J$  be an irreducible locally closed  $K$ -stable subset of  $\mathfrak{p}$  such that  $X_{\mathfrak{l}}(J, \mathcal{S}) \neq \emptyset$  and  $Y$  be a locally closed subset in  $X_{\mathfrak{l}}(\mathfrak{p}, \mathcal{S})$ . Set  $c(J) := \text{codim}_{\mathfrak{p}} J$ ,  $c(Y) := \text{codim}_{X_{\mathfrak{l}}(\mathfrak{p}, \mathcal{S})} Y$  and  $r := \text{codim}_{\mathfrak{p}} X_{\mathfrak{l}}(\mathfrak{p}, \mathcal{S})$ .*

- (i) *The orbit morphism  $\psi : K \times (e + X_{\mathfrak{l}}(\mathfrak{p}, \mathcal{S})) \rightarrow \mathfrak{p}$  is smooth of relative dimension  $\dim K - r$ .*
- (ii)  *$\text{codim}_{\mathfrak{p}} K.(e + Y) \leq c(Y)$ .*
- (iii)  *$X_{\mathfrak{l}}(J, \mathcal{S})$  is a pure locally closed variety of codimension  $c(J)$  in  $X_{\mathfrak{l}}(\mathfrak{p}, \mathcal{S})$ .*
- (iv) *Let  $X_0$  be an irreducible component of  $X_{\mathfrak{l}}(J, \mathcal{S})$ . Then  $K.(e + X_0)$  is dense in  $J$ .*
- (v)  *$\psi$  restricts to a smooth dominant morphism*

$$\psi_J : K \times (e + X_{\mathfrak{l}}(J, \mathcal{S})) \rightarrow J.$$

*Proof.* First of all,  $\psi$  is  $K$ -equivariant with respect to the  $K$ -action on the domain of  $\psi$  given by  $k'.(k, y) = (k'k, y)$ . Hence it is sufficient to check that  $\psi$  is smooth at points of the form  $(1_K, y)$ . Moreover, since the domain and codomain of  $\psi$  are smooth (recall from Lemma 2.1 that  $e + X_{\mathfrak{l}}(\mathfrak{p}, \mathcal{S})$  is open

in the affine space  $e + \mathfrak{p}_l^f$ , it is sufficient to check that the induced map on tangent spaces is surjective [AK, VII Remark 1.2]. At the point  $(1_K, y)$ , it is given by  $(d_\psi)_{(1,y)} : \begin{cases} \mathfrak{k} \times \mathfrak{p}_l^f & \rightarrow \mathfrak{p} \\ (k, x) & \mapsto [k, y] + x \end{cases}$ . Hence, thanks to Lemma 2.1, we have  $(d_\psi)_{(1,y)}(\mathfrak{k} \times \mathfrak{p}_l^f) = [\mathfrak{k}, y] \oplus \mathfrak{p}_l^f = \mathfrak{p}_{l^\perp} \oplus [\mathfrak{k}_l, y] \oplus \mathfrak{p}_l^f$ . This implies that  $d_\psi$  is smooth at  $(1, y)$  if and only if  $d_{\psi'}$  is, where  $\psi' : K_l \times (e + \mathfrak{p}_l^f) \rightarrow \mathfrak{p}_l$  is an orbit morphism. In other words, we can restrict ourselves to the case  $\mathfrak{g} = \mathfrak{l}$ .

In this case, we follow [Sl, 7.4 Corollary 1]. We easily see from graded  $\mathfrak{sl}_2$ -theory that  $\mathfrak{p} = [\mathfrak{k}, e] \oplus \mathfrak{p}^f$  so  $\psi$  is smooth at  $(1, e)$ . Consider the  $(F_t)_{t \in \mathbb{k}^*}$ -action on  $K \times (e + \mathfrak{p}^f)$ , given by  $F_t \cdot (k, x) = (F_t k F_{t^{-1}}, F_t(x))$ . Then  $\psi$  is equivariant with respect to this action. Hence the open set of smooth points of  $\psi$  is stable under  $F_t$ . Therefore, it follows from (2.5) that  $\psi$  is smooth on  $1_K \times (e + \mathfrak{p}^f)$  and hence on the whole domain  $K \times (e + \mathfrak{p}^f)$ .

(ii) We have  $K \cdot (e + Y) = \psi(K \times (e + Y))$ . Assertion (i) implies that  $\psi$  has constant fiber dimension. Hence, the dimension of any fiber of  $\psi|_{K \times (e + Y)}$  is less or equal than  $\dim K - r$  and we have  $\dim K \cdot (e + Y) \geq \dim K + \dim Y - (\dim K - r) = \dim Y + r = \dim \mathfrak{p} - c(Y)$ .

(iii-iv) [Ha, I Proposition 7.1] (which also holds for locally closed subsets of an affine space) states that  $\text{codim}_{\mathfrak{p}} X_0 \leq c(J) + r$  for any irreducible component  $X_0$  of  $X_l(J, \mathcal{S})$ . On the other hand, since  $J$  is  $K$ -stable, we have  $K \cdot (e + X_0) \subset J$  and we deduce  $\text{codim}_{X_l(\mathfrak{p}, \mathcal{S})} X_0 \geq c(J)$  from (ii). Since  $J$  is irreducible, this proves (iii) and (iv).

(v) The dominance statement lies in (iv). In order to obtain smoothness, we apply the argument of base extension by  $J \hookrightarrow \mathfrak{p}$  as in [IH, 2.8 and above]. More details can also be found in [Bu2, Proposition 3.9].  $\square$

**Proposition 2.5.** *Let  $\mathfrak{l}, J, \mathcal{S}$  be as in Proposition 2.4, omitting the assumption  $X_l(J, \mathcal{S}) \neq \emptyset$ . Then*

- (i)  $X_l(J, \mathcal{S})$  is a dense open subset of  $X_l(\overline{J}, \mathcal{S})$ ,
- (ii)  $K \cdot (e + X_l(J, \mathcal{S}))$  is an open subset of  $J$ .

*Proof.* (i) If  $X_l(\overline{J}, \mathcal{S}) = \emptyset$ , there is nothing to prove. From now on, we assume that it is non-empty. Since  $J$  is open in  $\overline{J}$ , the subset  $X_l(J, \mathcal{S})$  is open in  $X_l(\overline{J}, \mathcal{S})$ . Let  $X_0$  be an irreducible component of  $X_l(\overline{J}, \mathcal{S})$ . Then  $K \cdot (e + X_0)$  is a dense constructible subset of  $\overline{J}$  by Proposition 2.4(iv). Hence it meets  $J$  and, since  $J$  is  $K$ -stable, we have  $J \cap X_0 \neq \emptyset$ . In other words, the open subset  $X_l(J, \mathcal{S}) \subset X_l(\overline{J}, \mathcal{S})$  meets each irreducible components of  $X_l(\overline{J}, \mathcal{S})$ .

(ii) Since smooth morphisms are open [AK, V Theorem 5.1 and VII Theorem 1.8], the result is a consequence of (i) and Proposition 2.4(v).  $\square$

The equivalent statements (i), (ii) and (iii) of the following Theorem are inspired by similar properties in the Lie algebra case, which can be deduced



from parabolic induction theory when  $J$  is a Jordan class (e.g. see [Bo, §2]). For more simplicity, we restrict once more to the special case  $\mathfrak{l} = \mathfrak{g}$ .

**Theorem 2.6.** *Let  $e \in \mathfrak{p}$  be a nilpotent element embedded in a normal  $\mathfrak{sl}_2$ -triple  $\mathcal{S} := (e, h, f)$  and let  $J$  be a locally closed  $K$ -stable cone. The following conditions are equivalent:*

- (i)  $e \in \overline{J}$ .
- (ii) There exists a non-empty open set  $U \subset J$  such that  $e \in \overline{K \cdot (\mathbb{k}^* z)}$  for any  $z \in U$ .
- (iii) There exists  $z \in J$  such that  $e \in \overline{K \cdot (\mathbb{k}^* z)}$ .
- (iv)  $X(J, \mathcal{S}) \neq \emptyset$ .
- (v)  $X(\overline{J}, \mathcal{S}) \neq \emptyset$ .

*Proof.* The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (v) are obvious and (v) $\Rightarrow$ (iv) is a consequence of Proposition 2.5(i) applied to an irreducible component of  $\overline{J}$  meeting  $e + \mathfrak{p}^f$ .

Let us now prove (iv) $\Rightarrow$ (ii). Take  $U = K \cdot (e + X(J, \mathcal{S}))$ . It is open in  $J$  by Proposition 2.5(ii) and, for any  $z \in U$ , we have  $X(K \cdot (\mathbb{k}^* z), \mathcal{S}) \neq \emptyset$ . Then, our implication is a consequence of Lemma 2.3 applied to the  $K$ -stable cone  $K \cdot (\mathbb{k}^* z)$ .  $\square$

From a computational point of view, the previous theorem may be of key importance. Indeed, the existence of a degeneration from  $J$  to  $e$  reduces the existence of an element in the intersection of  $J$  with the affine space  $e + \mathfrak{p}^f$ , which might be much easier to check.

On the other hand, we have the following proposition derived from [TY, Theorem 38.6.9 (i)].

**Proposition 2.7.** *If  $J$  is a  $K$ -stable cone of  $\mathfrak{p}^{(r)} = \{x \in \mathfrak{p} \mid \dim K \cdot x = r\}$ , then there exists a nilpotent element  $e \in \mathfrak{p}$  such that*

- (i)  $e \in \overline{J}$ ,
- (ii)  $\dim K \cdot e = r$ .

**Proposition 2.8.** *Let  $J \subset \mathfrak{p}^{(r)}$  be a locally closed  $K$ -stable cone and  $(e_i)_{i \in I}$  be a set of representatives of the nilpotent  $K$ -orbits satisfying (i) and (ii) of Proposition 2.7 embeded in normal  $\mathfrak{sl}_2$ -triples  $(\mathcal{S}_i)_{i \in I}$ . Let  $U_i := K \cdot (e_i + X(J, \mathcal{S}_i))$ . Then,  $(U_i)_{i \in I}$  is an open cover of  $J$ .*

*Proof.* The open statement lies in Proposition 2.5(ii). For the covering statement, pick  $z \in J$ , then it follows from Proposition 2.7 that there exists  $i \in I$  such that  $e_i \in \overline{K \cdot (\mathbb{k}^* z)}$ . Applying Theorem 2.6 to  $K \cdot (\mathbb{k}^* z)$ , we get  $X(K \cdot (\mathbb{k}^* z), \mathcal{S}_i) \neq$

$\emptyset$ , so  $U_i \cap K.(\mathbb{k}^*z) \neq \emptyset$ . Since  $U_i$  is stable under the action of  $K \times \mathbb{k}^*\text{Id}$ , we get  $z \in U_i$ .  $\square$

**Remarks 2.9.** - A major consequence of this Proposition is that the whole geometry of  $J$  is closely related to the geometry of its different Slodowy slices. Indeed, locally, we can assume that  $J = K.(e_i + X(J, \mathcal{S}_i))$  for some  $i \in I$  and it follows from Proposition 2.4(v) that this variety is smoothly equivalent to  $X(J, \mathcal{S}_i)$ .

- This also provides a more solid ground to the philosophy of [Bu2, §9], where it is proven in some particular cases that the Slodowy slices contains enough information to describe the whole variety.
- The main drawback of this approach is that it does not yield  $|I| = 1$  when  $J$  is irreducible in the Lie algebra case, contrary to other parameterization such as [BK, §5]. However, this drawback is somehow necessary since the property  $|I| = 1$  may fail in the symmetric case (*e.g.* see [TY, 39.6.3] for the description of the regular sheet when  $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_2, \mathfrak{so}_2)$ ).

We have several examples in mind of such locally closed  $K$ -stable cone included in some  $\mathfrak{p}^{(r)}$ . We have already seen that  $K.(\mathbb{k}^*z)$  plays a role in the previous proofs. We can also consider a Jordan class  $J_1$ , or its regular closure  $\overline{J_1}^\bullet$ . Sheets are particular examples of the last type.

### 3 $K$ -Jordan classes and induction

**Definition 3.1.** The  $K$ -Jordan class (or Jordan class) of an element  $x \in \mathfrak{p}$  with Jordan decomposition  $x = s + n$  ( $s$  semisimple,  $n$  nilpotent,  $[s, n] = 0$ ) is

$$J(x) := K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{g}^s)^\bullet + n).$$

We refer to [TY, §39.5] for most of the known properties of these classes (also known as decomposition classes) in the symmetric Lie algebra case. Let us mention that Jordan classes are finitely many, locally closed and of constant orbit dimension. In particular, in each sheet  $S$  there exists a dense open Jordan class  $J_0$  and we have

$$S = \overline{J_0}^\bullet.$$

In the Lie algebra case, such regular closures  $\overline{J_0}^\bullet$  have been studied in [BK], where a very useful parameterization is given. In [Bo], this understanding has been deepened thanks to parabolic induction theory. The aim of this section is to extend a significant number of the known properties of  $G$ -Jordan classes to the symmetric Lie algebra case. For this, we use the tools introduced in section 2 in order to define *slice induction*. This new notion turns out to coincide with the parabolic induction in the Lie algebra case.

Recall that, when  $s$  is a semisimple element of  $\mathfrak{p}$ , the Levi subalgebra  $\mathfrak{l} := \mathfrak{g}^s$  arising from  $\mathfrak{p}$  is a reductive symmetric Lie algebra whose natural orbits in  $\mathfrak{p}_{\mathfrak{l}} = \mathfrak{p}^s$  are  $K_{\mathfrak{l}} = (K^s)^{\circ}$  ones.

**Definition 3.2.** If  $x = s + n$  is the Jordan decomposition of an element  $x \in \mathfrak{p}$ , the *datum* of  $x$  is the pair  $(\mathfrak{l}, K_{\mathfrak{l}}.n)$  where  $\mathfrak{l} := \mathfrak{g}^s$ .

**Remarks 3.3.** - These data are exactly the pairs  $(\mathfrak{l}, \mathcal{O})$  where  $\mathfrak{l}$  is a Levi subalgebra of  $\mathfrak{g}$  arising from  $\mathfrak{p}$  and  $\mathcal{O}$  is a nilpotent  $K_{\mathfrak{l}}$ -orbit in  $\mathfrak{p}_{\mathfrak{l}}$ .

- Two elements lie in the same Jordan class if and only if their respective data are  $K$ -conjugate. Hence Jordan classes are equivalence classes.
- If  $J$  is a  $K$ -Jordan class, we say that  $(\mathfrak{l}, \mathcal{O})$  is a datum of  $J$  if it is the datum of  $x$  for some  $x \in J$ . Given a datum  $(\mathfrak{l}, \mathcal{O})$ , the Jordan class with this datum will be denoted by

$$J(\mathfrak{l}, \mathcal{O}) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l})^{\bullet} + \mathcal{O}).$$

- The  $K$ -orbits of Levi factors arising from  $\mathfrak{p}$  are in one to one correspondance with their  $G$ -orbits which are, in turn, easily characterized through the Satake diagram of  $(\mathfrak{g}, \mathfrak{k})$ , cf. [Bu2, §7]. Hence, in order to have a functional classification of  $K$ -Jordan class, it would be enough to understand in which cases  $(\mathfrak{l}, \mathcal{O}_1)$  is  $N_K(\mathfrak{l})/K_{\mathfrak{l}}$ -conjugate to  $(\mathfrak{l}, \mathcal{O}_2)$ . This plays an important role in the classification of sheets when  $\mathcal{O}_i$ ,  $i = 1, 2$  are rigid orbits as shown in [Bo, 3.9, 3.10, 4.5, 4.6]. However, for our purposes, we will not need such a classification.

**Definition 3.4.** 1. Given a datum  $(\mathfrak{l}_1, \mathcal{O}_1)$ , and a  $K$ -orbit  $\mathcal{O}_2$  of  $\mathfrak{g}$ , we say that  $(\mathfrak{l}_1, \mathcal{O}_1)$  *slice induces*  $(\mathfrak{g}, \mathcal{O}_2)$  if

- a)  $X(J_1, \mathcal{S}_2) \neq \emptyset$  where  $J_1 := J(\mathfrak{l}_1, \mathcal{O}_1)$  and  $\mathcal{S}_2 = (n_2, h_2, m_2)$  is a normal  $\mathfrak{sl}_2$ -triple with  $n_2 \in \mathcal{O}_2$ .
- b)  $\dim K.x = \dim \mathcal{O}_2$  where  $x \in J(\mathfrak{l}_1, \mathcal{O}_1)$ .

When, there is no context of parabolic induction, the term *induction* will always refer to slice induction.

2. If **a)** is satisfied and the assumption **b)** is dropped, the induction is said to be *weak*.
3. Given two data  $(\mathfrak{l}_i, \mathcal{O}_i)$ ,  $i = 1, 2$ , we say that  $(\mathfrak{l}_1, \mathcal{O}_1)$  (weakly) induces  $(\mathfrak{l}_2, \mathcal{O}_2)$  if  $\mathfrak{l}_1 \subset \mathfrak{l}_2$  and, when considering  $\mathfrak{l}_1$  as a Levi subalgebra of  $\mathfrak{l}_2$ ,  $(\mathfrak{l}_1, \mathcal{O}_1)$  (weakly) induces  $(\mathfrak{l}_2, \mathcal{O}_2)$ .
4. Given two  $K$ -Jordan classes  $J_i$  ( $i = 1, 2$ ), we say that  $J_1$  (weakly) induces  $J_2$  if there exists a datum  $(\mathfrak{l}_i, \mathcal{O}_i)$  of each  $J_i$  such that  $(\mathfrak{l}_1, \mathcal{O}_1)$  (weakly) induces  $(\mathfrak{l}_2, \mathcal{O}_2)$

When  $J_1$  and  $J_2$  are Jordan classes, we write

$$\begin{cases} J_1 \uparrow\uparrow J_2 & \text{if } J_1 \text{ slice induces } J_2, \\ J_1 \uparrow J_2 & \text{if } J_1 \text{ weakly slice induces } J_2. \end{cases}$$

**Remarks 3.5.** - In 1, the definition does not depend on the choice of  $n_2$  or  $\mathcal{S}_2$  since all of these are  $K$ -conjugate. Also, all elements of  $J(\mathfrak{l}_1, \mathcal{O}_1)$  share the same orbit dimension.

- If  $(\mathfrak{l}_1, \mathcal{O}_1)$  (weakly) induces  $(\mathfrak{g}, \mathcal{O}_2)$  if and only if  $(\mathfrak{l}_1 \cap \mathfrak{g}', \mathcal{O}_1)$  (weakly) induces  $(\mathfrak{g}', \mathcal{O}_2)$ . Indeed,  $n_2 + \mathfrak{p}^{m_2} = \mathfrak{c}_{\mathfrak{p}}(\mathfrak{g}) + (n_2 + \mathfrak{p}_{\mathfrak{g}'}^{m_2})$  and  $J(\mathfrak{l}_1, \mathcal{O}_1) = \mathfrak{c}_{\mathfrak{p}}(\mathfrak{g}) + J'$  where  $J'$  is the Jordan class in  $\mathfrak{g}'$  with datum  $(\mathfrak{l}_1 \cap \mathfrak{g}', \mathcal{O}_1)$ .
- One can rephrase condition **a)** in the context of definition 3.4 (3) as follows: There exists an element of  $n_2 + \mathfrak{p}_{\mathfrak{l}_2}^{m_2}$  which is  $K_{\mathfrak{l}_2}$ -conjugate to  $s_1 + n_1$  with  $n_1 \in \mathcal{O}_1$  and  $s_1 \in C := \{x \in \mathfrak{c}_{\mathfrak{p}_{\mathfrak{l}_2}}(\mathfrak{l}_1) = \mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1) \mid \dim K_{\mathfrak{l}_2}.x \text{ maximal}\}$ . Note that  $C$  is a priori different from  $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1)^\bullet = \{x \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1) \mid \dim K.x \text{ maximal}\}$ . In fact,  $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1)$  can be embedded in a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and  $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1)^\bullet$  (resp.  $C$ ) is the open subset of  $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1)$  defined by non-vanishing of roots associated to  $(\mathfrak{g}, \mathfrak{h})$  (resp. to  $(\mathfrak{l}_2, \mathfrak{h})$ ). In particular, we see that  $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1)^\bullet$  is an open subset of  $C$ .

In view of Theorem 2.6, one sees that a Jordan class  $J_1$  weakly induces a nilpotent orbit  $\mathcal{O}$  (i.e. the Jordan class  $J(\mathfrak{g}, \mathcal{O})$  assuming that  $\mathfrak{g}$  is semisimple) if and only if  $\mathcal{O} \subset \overline{\mathcal{J}}$ . The purpose of the following theorem is to extend this property to induction of arbitrary Jordan classes. Recall that  $X_{\mathfrak{l}}(J, \mathcal{S})$  is introduced in Definition 2.2.

**Theorem 3.6.** *Let  $J_1, J_2$  be  $K$ -Jordan classes. The following conditions are equivalent.*

- (i)  $J_1 \uparrow J_2$ ,
- (ii)  $X_{\mathfrak{l}_2}(J_1, \mathcal{S}_2) \neq \emptyset$  where  $(\mathfrak{l}_2, \mathcal{O}_2)$  is some (any) datum of  $J_2$  and  $\mathcal{S}_2$  is a normal  $\mathfrak{sl}_2$ -triple in  $\mathfrak{l}_2 = \mathfrak{k}_{\mathfrak{l}_2} \oplus \mathfrak{p}_{\mathfrak{l}_2}$  whose nilpositive element belongs to  $\mathcal{O}_2$ .
- (iii)  $J_2 \cap \overline{J_1} \neq \emptyset$ ,
- (iv)  $J_2 \subset \overline{J_1}$ ,

*Proof.* (iv) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii): Choose  $n_2 \in \mathcal{O}_2$  and embed it in a normal  $\mathfrak{sl}_2$ -triple  $\mathcal{S}_2$  of  $\mathfrak{l}_2$ . Since  $J_2 \cap \overline{J_1}$  is  $K$ -stable we can choose an element  $x$  in this intersection such that the Jordan decomposition of  $x$  is  $z_2 + n_2$  with  $z_2 \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_2)^\bullet$  and  $n_2 \in \mathcal{O}_2$ . Therefore  $\mathfrak{g}^{z_2} = \mathfrak{l}_2$  (2.3),  $x \in U_{\mathfrak{l}_2}$  (Lemma 2.1) and  $X_{\mathfrak{l}_2}(\overline{J_1}, \mathcal{S}_2) \neq \emptyset$ . Then, we apply Proposition 2.5(i) to get  $X_{\mathfrak{l}_2}(J_1, \mathcal{S}_2) \neq \emptyset$ .

(ii) $\Rightarrow$ (i): Choose an element  $x \in X_{\mathfrak{l}_2}(J_1, \mathcal{S}_2)$  and consider its Jordan decomposition  $s_1 + n_1$ . We denote the datum of  $x$  by  $(\mathfrak{l}_1, \mathcal{O}_1)$ . Lemma 2.1 tells

us that  $\mathfrak{l}_1 = \mathfrak{g}^{s_1} \subset \mathfrak{l}_2$  so, using notation of Remark 3.5,  $s_1 \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1)^{\bullet} \subset C$ . In particular, as an element of  $\mathfrak{l}_2$ ,  $x \in n_2 + \mathfrak{p}_{\mathfrak{l}_2}^{m_2}$  has datum  $(\mathfrak{l}_1, \mathcal{O}_1)$ . Hence  $(\mathfrak{l}_1, \mathcal{O}_1)$  weakly induces  $(\mathfrak{l}_2, \mathcal{O}_2)$ .

(i) $\Rightarrow$ (iv): Choose a datum  $(\mathfrak{l}_1, K.n_1)$  of  $J_1$  weakly inducing a datum  $(\mathfrak{l}_2, K.n_2)$  of  $J_2$ . Since  $\mathfrak{l}_1 \subset \mathfrak{l}_2$ , we have  $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1) \supset \mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_2)$  and

$$\overline{J_1} = \overline{K \cdot (\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1)^{\bullet} + n_1)} \supset \overline{K_{\mathfrak{l}_2} \cdot (\mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_1) + n_1)} = \mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_2) + \overline{J_1'}$$

where  $J_1'$  is the Jordan class in  $\mathfrak{l}_2'$  with datum  $(\mathfrak{l}_1 \cap \mathfrak{l}_2', \mathcal{O}_1)$ . On the other hand, we have seen in Remark 3.5 that  $J_1'$  weakly induces  $K.n_2$  in  $\mathfrak{l}_2'$  and it follows from Proposition 2.6 that  $n_2 \in \overline{J_1'}$ . Hence  $\overline{J_1} \supset \mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_2) + n_2$  and, by  $K$ -stability of  $\overline{J_1}$ , we get  $J_2 \subset \overline{J_1}$ .  $\square$

A key point in the proof of (iii) $\Rightarrow$ (iv) is that whenever we have  $x \in J_2 \cap \overline{J_1}$ , we can manage to realize the degeneration through a one parameter family  $(y_t)_{t \in \mathbb{k}^*} \in J_1$ ,  $\lim_{t \rightarrow 0} y_t = x$  with the  $y_t$  lying in a generalized Slodowy slice centered on  $x$ . Hence the whole degeneration takes place in the Levi  $\mathfrak{g}^s$  with  $s$  the semisimple part of  $x$  and thus yield a degeneration toward any element with Jordan decomposition similar to  $x$ .

One can now rephrase the condition **b)** defining induction as follows

**Lemma 3.7.** *Let  $J_1, J_2$  be Jordan classes such that  $J_1 \uparrow J_2$ . Then  $J_1 \uparrow\uparrow J_2$  if and only if the dimension of orbits in  $J_1$  and  $J_2$  coincide.*

*Proof.* Choose a datum  $(\mathfrak{l}_2, \mathcal{O}_2)$  of  $J_2$ . We know from Theorem 3.6(ii) that there exists  $y_1 \in J_1 \cap U_{\mathfrak{l}_2}$ . Letting  $(\mathfrak{l}_1, \mathcal{O}_1)$  be the datum of  $y_1$ , we have  $\mathfrak{l}_1 \subset \mathfrak{l}_2$ . On the other hand, any element  $y_2 \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{l}_2)^{\bullet} + n_2$ ,  $n_2 \in \mathcal{O}_2$  satisfies  $y_2 \in J_2 \cap U_{\mathfrak{l}_2}$ . Then, it follows from Lemma 2.1(iii) that  $y_1$  and  $y_2$  share the same  $K$ -orbit dimension in  $\mathfrak{p}$  if and only if they share the same  $K_{\mathfrak{l}_2}$ -orbit dimension in  $\mathfrak{l}_2$ . The result follows.  $\square$

As a consequence of Theorem 3.6 and Lemma 3.7, the following holds.

**Corollary 3.8.** (i) *If  $J_0$  is a Jordan class, then*

$$\overline{J_0} = \bigcup_{J \uparrow J_0} J, \quad \overline{J_0}^{\bullet} = \bigcup_{J \uparrow\uparrow J_0} J.$$

(ii) *If  $J_1$  and  $J_2$  are Jordan classes,  $\overline{J_1} \cap \overline{J_2}$  is a union of Jordan classes.*

(iii) *Sheets are union of Jordan classes.*

(iv) *Induction (resp. weak induction) is transitive. That is,*

$$(J_1 \uparrow\uparrow J_2) \wedge (J_2 \uparrow\uparrow J_3) \Rightarrow J_1 \uparrow\uparrow J_3,$$

$$\text{resp. } (J_1 \uparrow J_2) \wedge (J_2 \uparrow J_3) \Rightarrow J_1 \uparrow J_3.$$

(v) In the Lie algebra case, parabolic induction of Jordan classes coincides with slice induction.

*Proof.* Since Jordan classes form a partition of  $\mathfrak{p}$ , the first part of (i) is an immediate consequence of the equivalence (i)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv) in Theorem 3.6. The second part of (i) follows from the first one and Lemma 3.7. Then, we easily deduce statement (iv) from (i).

In (ii), we see that  $\overline{J_1} \cap \overline{J_2}$  is just the union of Jordan classes induced both by  $J_1$  and  $J_2$ . In (iii), we use the fact that any sheet is the regular closure of a Jordan class.

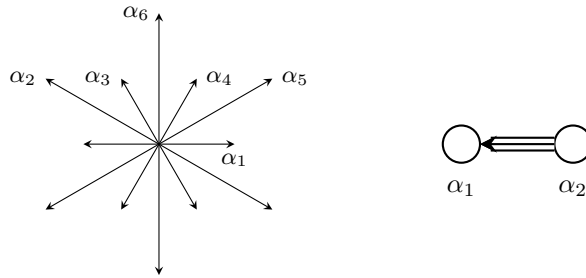
In [Bo, 3.5-3.6], it is shown that  $\overline{J(\mathfrak{l}_1, \mathcal{O}_1)}^\bullet = \bigcup_{J_2 \in \mathcal{J}} J_2$  where  $\mathcal{J}$  is the set of Jordan classes  $J_2$  having a datum of the form  $(\mathfrak{l}_2, \mathcal{O}_2)$ , with  $\mathfrak{l}_2 \supset \mathfrak{l}_1$  and  $\mathcal{O}_2$  is the nilpotent orbit of  $\mathfrak{l}_2$  parabolically induced from  $(\mathfrak{l}_1, \mathcal{O}_1)$ . Hence (v) follows from (i).  $\square$

A important consequence of (v) is that slice induction can be seen as a generalization of parabolic induction, fitting to the symmetric Lie algebra setting needs. Note in particular that (i), (ii), (iv) and (iii) are respective analogues of [Bo, 3.5, 3.8, 2.3] and [BK, 5.8.d].

The fact that a closure of a Jordan class is a union of Jordan classes has also been shown in [Le] when the ground field is precisely  $\mathbb{C}$ .

## 4 Sheets of $\mathfrak{g}$ in type $G_2$

In this section, we assume that  $\mathfrak{g}$  is a simple Lie algebra of type  $G_2$ . We adopt conventions and notations of [FH]. In particular, we fix a Cartan subalgebra  $h$  of  $\mathfrak{g}$ . In the corresponding root system, we fix a basis  $\{\alpha_1, \alpha_2\}$ , with  $\alpha_1$  a short root, and label the associated positive roots as pictured



We choose, as a Chevalley basis, the one given in [FH, p.346] and denote it by  $h_1, h_2, x_i (i \in \llbracket 1, 6 \rrbracket), y_i (i \in \llbracket 1, 6 \rrbracket)$ , with  $x_i \in \mathfrak{g}_{\alpha_i}, y_i \in \mathfrak{g}_{-\alpha_i}$  and  $(x_i, h_i, y_i)$  is an  $\mathfrak{sl}_2$ -triple for  $i = 1, 2$ .

The nilpotent cone of  $\mathfrak{g}$  consists of five nilpotent conjugacy classes. We denote the orbit of dimension  $2i$  ( $i \in \{0, 3, 4, 5, 6\}$ ) by  $\Omega_{2i}$ . Moreover, we choose

some particular representatives  $n_i$  (two of them when  $i = 5$ ) of these orbits as given in column 2 of Table 1.

$G$ -orbit $\Omega$	Representative	$K$ -orbit $\mathcal{O}$
$\Omega_0$	0	$\mathcal{O}_0$
$\Omega_6$	$n_3 := x_5$	$\mathcal{O}_3$
$\Omega_8$	$n_4 := x_4$	$\mathcal{O}_4$
$\Omega_{10}$	$n_{5a} := x_5 + y_3$	$\mathcal{O}_{5a}$
	$n_{5b} := x_2 + x_5$	$\mathcal{O}_{5b}$
$\Omega_{12}$	$n_6 := x_3 + x_5$	$\mathcal{O}_6$

Table 1: Representatives for  $\Omega_i$  and  $\mathcal{O}_i$

Since  $\mathfrak{g}$  is of rank two, the only non-regular non-nilpotent elements of  $\mathfrak{g}$  are semisimple subregular elements. This gives rise to two subregular sheets of  $\mathfrak{g}$  which can be described as union of  $G$ -Jordan classes as follows.

$$S_1^{\mathfrak{g}} := G.(\mathbb{k}^*h_1) \sqcup \Omega_{10}, \quad S_2^{\mathfrak{g}} := G.(\mathbb{k}^*h_2) \sqcup \Omega_{10}.$$

This follows from the fact that  $G.(\mathbb{k}^*h_1)$  and  $G.(\mathbb{k}^*h_2)$  are of the same dimension (here, 11) and that both must contain a nilpotent orbit in their regular closure (Proposition 2.7). A practical criterion to distinguish generic elements of  $S_1^{\mathfrak{g}}$  from those of  $S_2^{\mathfrak{g}}$  is that elements of the former lie in a Levi of type  $A_1$  while their centraliser is a Levi of type  $\widetilde{A}_1$  (i.e. an  $\mathfrak{sl}_2$  Levi associated to a long root).

It is known [Pe, Hi] that  $S_1^{\mathfrak{g}}$  is smooth at points of  $G.(\mathbb{k}^*h_1)$  and has triple singularities at  $\Omega_{10}$ . On the other hand,  $S_2^{\mathfrak{g}}$  is a smooth variety.

Letting

$$\mathfrak{k} := \mathfrak{h} \oplus \bigoplus_{i \in \{1,6\}} \mathfrak{g}_{\pm\alpha_i}, \quad \mathfrak{p} := \bigoplus_{i \in \{2,3,4,5\}} \mathfrak{g}_{\pm\alpha_i},$$

we construct a symmetric Lie algebra of type  $G_{2(2)}$ . It corresponds to the single non-compact form for an algebra of type  $G_2$  and we have  $\mathfrak{k} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ .

Let us describe the  $K$ -orbits of interest in this setting. First, the symmetric Lie algebra is of maximal rank, hence any  $G$ -orbit intersects  $\mathfrak{p}$  [An]. Since semisimple orbits intersect  $\mathfrak{p}$  into single orbits (see *e.g.* [Bu2, Proposition 6.6]), there are exactly two subregular semisimple ( $K$ -)Jordan classes:  $J_i := K.(\mathbb{k}^*\tilde{h}_i)$ , with  $\tilde{h}_i$  an element of  $G.h_i \cap \mathfrak{p}$ ,  $i = 1, 2$ . It turns out [Dj1] that  $\Omega_{2i} \cap \mathfrak{p}$  is a single  $K$ -orbit  $\mathcal{O}_i$  for  $i \in \{0, 3, 4, 6\}$  and is the union of two  $K$ -orbits  $\mathcal{O}_{5a}$  and  $\mathcal{O}_{5b}$  in the subregular case  $i = 5$  (respectively numbered by 3 and 4 in [Dj1]). We refer to Table 1 for representatives of these different  $K$ -orbits.

Note that, since sheets are union of Jordan classes, there are exactly two 6-dimensional subregular sheets. They are  $S_1 := \overline{J_1}$  and  $S_2 := \overline{J_2}$ . Even if each of these sheets must contain at least a nilpotent orbit among  $\mathcal{O}_{5a}$  and  $\mathcal{O}_{5b}$  (Proposition 2.7), it is a non-trivial problem to decide which orbit belong to a

given sheet. At this point, we may even not rule out the existence of a possible 5-dimensionnal subregular sheet of the form  $\mathcal{O}_{5x}$ .

## 4.1 Contribution of the Slodowy slice theory

We now apply partly the general theory of sections 2 and 3 to our special case. The main result of this subsection is the following

**Proposition 4.1.** (i) *The only sheets of subregular elements in  $\mathfrak{p}$  are  $S_1$  and  $S_2$ .*

(ii) *The decomposition of each sheet  $S_i$  ( $i = 1, 2$ ) as union of Jordan classes is*

$$S_i = J_i \sqcup \mathcal{O}_{5a} \sqcup \mathcal{O}_{5b} \quad (= S_i^{\mathfrak{g}} \cap \mathfrak{p}).$$

(iii) *The sheet  $S_1$  is smooth on  $J_1$  and  $\mathcal{O}_{5a}$ . At points of  $\mathcal{O}_{5b}$ , the singularities of  $S_1$  are smoothly equivalent to the intersection of three lines.*

(iv) *The sheet  $S_2$  is smooth.*

Recall that the only Jordan classes of subregular elements are  $J_1$ ,  $J_2$ ,  $\mathcal{O}_{5a}$  and  $\mathcal{O}_{5b}$ . So, since sheets are regular closures of Jordan classes, (ii) implies (i).

The statements (ii), (iii) and (iv) rely on computations on the Slodowy slices. In fact, embedding  $n_{5a}$  and  $n_{5b}$  into respective normal  $\mathfrak{sl}_2$ -triple,  $\mathcal{S}_{5a}$  and  $\mathcal{S}_{5b}$ , we claim that

**Lemma 4.2.** (i)  *$X(\mathfrak{p}^{(5)}, \mathcal{S}_{5a})$  is the union of two lines  $\mathbb{k}t_i$  ( $i = 1, 2$ ), with  $n_{5a} + t_1 \in J_1$  and  $n_{5a} + t_2 \in J_2$ .*

(ii)  *$X(\mathfrak{p}^{(5)}, \mathcal{S}_{5b})$  is the union of four lines  $\mathbb{k}t_i$  ( $i = 3, 4, 5, 6$ ), with  $n_{5b} + t_i \in J_1$  for  $i \in \{3, 4, 5\}$  and  $n_{5b} + t_6 \in J_2$ .*

In particular,  $X(J_i, \mathcal{S}_{5x}) \neq \emptyset$  so  $J_i$  slice induces  $\mathcal{O}_{5x}$  for  $i = 1, 2$  and  $x \in \{a, b\}$ . Hence Proposition 4.1 (ii) follows from Corollary 3.8 (or, in a more simple way, from Lemma 2.3).

The other consequence of Lemma 4.2 is that  $J_i \sqcup \mathcal{O}_{5a}$  ( $i = 1, 2$ ) (resp.  $J_2 \sqcup \mathcal{O}_{5b}$ ) is smoothly equivalent to the (smooth) affine line  $X(S_i, \mathcal{S}_{5a})$  (resp.  $X(S_2, \mathcal{S}_{5b})$ ), see Remark 2.9. On the other hand  $J_1 \sqcup \mathcal{O}_{5b}$  is smoothly equivalent to a union of three lines meeting at a point. This explains (iii) and (iv) of Proposition 4.1.

To sum up, the picture is the following. Only one branch in the neighborhood singularities of  $S_1^{\mathfrak{g}}$  at points of  $\mathcal{O}_{5a}$  is preserved under the intersection with  $\mathfrak{p}$ . On the contrary, the singularities at points of  $\mathcal{O}_{5b}$  are “intact” when intersected with  $\mathfrak{p}$ .

The remaining of this section is devoted to explain Lemma 4.2. For this we exhibit below some key details of the computations. Most of these computations



have been made by hand and then checked by [GAP] using W. de Graaf's package [SLA]<sup>2</sup>. First, we set  $m_{5a} := x_3 + y_5$  and  $m_{5b} := -\frac{4}{3}y_2 + \frac{2}{3}y_3 + \frac{2}{3}y_4 + \frac{4}{3}y_5$ . This turns  $\mathcal{S}_{5a} := (n_{5a}, -2h_2, m_{5a})$  and  $\mathcal{S}_{5b} := (n_{5b}, 2h_1 + 4h_2, m_{5b})$  into normal  $\mathfrak{sl}_2$ -triples. Then, a basis of  $\mathfrak{p}^{m_{5a}}$  (resp.  $\mathfrak{p}^{m_{5b}}$ ) is given by  $(z_a, z'_a, z''_a) := (x_2, x_3, y_5)$  (resp.  $(z_b, z'_b, z''_b) := (y_2 - \frac{2}{3}y_3, y_3 + y_4, y_4 + \frac{3}{2}y_5)$ ).

There remains to check whether an element of the form  $p := n + \beta_1 z + \beta_2 z' + \beta_3 z''$ , with  $(\beta_1, \beta_2, \beta_3) \in \mathbb{k}^3$ , is subregular or not. It follows from (2.5) that such an element is either regular or subregular. It is subregular if and only if  $\dim[\mathfrak{k}, p] < 6$ , that is, if and only if the linear map  $\begin{cases} \mathfrak{k} & \rightarrow \mathfrak{p} \\ k & \mapsto [k, p] \end{cases}$  is not of maximal rank. This amounts to check that all the minors of size 6 of the following matrices vanish (left for case  $a$  and right for case  $b$ ).

$$\begin{pmatrix} 3\beta_1 & -2\beta_1 & 0 & -3\beta_2 & \beta_3 & 0 \\ \beta_2 & -\beta_2 & -\beta_1 & 0 & 0 & 0 \\ 0 & 0 & -2\beta_2 & 1 & -1 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_3 & 2 & 0 & \beta_2 \\ 3\beta_3 & -\beta_3 & 0 & 0 & 0 & \beta_1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -3 & \frac{3}{2}\beta_3 & 0 \\ 1 & -1 & 0 & -2 & \beta_2 + \beta_3 & 0 \\ -1 & 0 & -2 & 0 & \frac{2}{3}\beta_1 - \beta_2 & 0 \\ 0 & 0 & 3 & 0 & -\beta_1 & 0 \\ -3\beta_1 & 2\beta_1 & -2\beta_1 + 3\beta_2 & 0 & 0 & 0 \\ \frac{2}{3}\beta_1 - \beta_2 & -\frac{2}{3}\beta_1 + \beta_2 & 2\beta_2 + 2\beta_3 & \beta_1 & 0 & -1 \\ \beta_2 + \beta_3 & 0 & -\frac{2}{3}\beta_3 & -\frac{4}{3}\beta_1 + 2\beta_2 & 0 & 1 \\ \frac{3}{2}\beta_3 & -\frac{3}{2}\beta_3 & 0 & -3\beta_2 - 3\beta_3 & 0 & 0 \end{pmatrix}$$

One can then check that these size 6 minors generate the ideals  $I_a := \langle \beta_1, (\beta_2 - \frac{1}{9}\beta_3)(\beta_2 - \beta_3) \rangle$  and  $I_b := \langle \beta_3(\beta_2 + \frac{1}{4}\beta_3), \beta_1(\beta_1 - 6\beta_2) \rangle$  in  $\mathbb{C}[\beta_1, \beta_2, \beta_3]$ . The statements of Lemma 4.2 on the number of lines then follows. For instance, we may choose  $t_1 := z'_a + 9z''_a$  and  $t_2 := z'_a + z''_a$ . The following table gives an explicit description of the  $t_i$  ( $i = \llbracket 1, 6 \rrbracket$ ) in terms of our Chevalley basis.

$t_1$	$x_3 + 9y_5$	$e_1$	$\sqrt{3}x_2 - \frac{1}{\sqrt{3}}y_2 + x_3 + y_3 + \frac{1}{\sqrt{3}}x_4 - \sqrt{3}y_4 - \frac{1}{3}x_5 - 3y_5$
$t_2$	$x_3 + y_5$	$e_2$	$3ix_2 + 3iy_2 + x_3 + y_3 + ix_4 + iy_4 - 3x_5 - 3y_5$
$t_3$	$y_3 + y_4$	$e_3$	$y_2 - x_5$
$t_4$	$6y_2 - 3y_3 + y_4$	$e_4$	$y_2 - x_5$
$t_5$	$y_3 - 3y_4 - 6y_5$	$e_5$	$x_2 + y_2 + x_3 - y_3 + x_4 + y_4 - x_5 + y_5$
$t_6$	$2y_2 - y_3 - y_4 - 2y_5$	$e_6$	$(-3 - i\sqrt{3})x_2 + 6y_2 + (-1 - i\sqrt{3})x_3 + (-3 + i\sqrt{3})y_3$ $+ (-1 - i\sqrt{3})x_4 + (3 - i\sqrt{3})y_4 + 2i\sqrt{3}x_5 + (3 - 3i\sqrt{3})y_5$

Table 2: Representatives of  $X(\mathfrak{p}^{(5)}, \mathcal{S}_{5x})$  and nilpotent elements of  $\mathfrak{g}^{n+t_i}$ .

The last thing to check is that the elements  $n + t_i$  belong to the prescribed Jordan classes (with  $n = n_{5a}$  for  $i \in \{1, 2\}$  and  $n = n_{5b}$  else). Since  $t_i \neq 0$ , we have  $\dim(F_t.(n + t_i))_{t \in \mathbb{k}^*} = 1$  (see (2.5)) and therefore cannot be nilpotent (see Proposition 2.4(ii)). These elements are non-nilpotent and subregular hence they necessarily belong either to  $J_1$  or  $J_2$ . In order to determine their class,

<sup>2</sup>These GAP computations are available at [http://perso.univ-st-etienne.fr/bm29130h/pageperso/pdf/gap\\_g2\\_v4.txt](http://perso.univ-st-etienne.fr/bm29130h/pageperso/pdf/gap_g2_v4.txt).

we find a non-zero nilpotent element  $e_i$  in  $[\mathfrak{g}^{n+t_i}, \mathfrak{g}^{n+t_i}] \cong \mathfrak{sl}_2$ . If  $\dim G.e_i = 6$  (resp. 8), then  $\mathfrak{g}^{n+t_i}$  is a Levi of type  $\tilde{A}_1$  (resp.  $A_1$ ) and  $n+t_i \in J_1$  (resp.  $J_2$ ). The elements  $e_i$  are also given in Table 2 and one can check that  $\dim[\mathfrak{g}, e_i] = 6$  (resp. 8) for  $i \in \{1, 3, 4, 5\}$  (resp.  $i \in \{2, 6\}$ ).

**Remarks 4.3.** - Note that, thanks to (2.5), our method also provides explicit one-parameter degenerations  $\lim_{\gamma \rightarrow 0} n_{5x} + \gamma t_i = n_{5x}$ .

- In the real case, we have no reason to think that Jordan classes closure inclusions always leaves footprints on the Slodowy slice level (see proof of the crucial statement Proposition 2.4(iii)). However, it is an interesting feature to note that our method may still provide real degeneration in some cases. Indeed, here, all the  $t_i$  belong to the obvious real form of  $\mathfrak{g}$ .
- The computations presented in this section can be improved in order to get similar results in higher rank. For example, the first author has checked that the sheets in the (ordinary) Lie algebra of type  $F_4$  are smooth to the exception of some sheets containing the nilpotent orbit labelled  $F_4(a_3)$  in Bala-Carter's classification.

## 4.2 4-ality and projections

We now use an other description of our algebra  $\mathfrak{g}$  of type  $G_2$  using 4-ality as defined in [LM, §3.4]. Namely, we choose four copies  $C_1, C_2, C_3, D$  of a 2-dimensional space equipped with a non-degenerate bilinear skew-symmetric form  $\omega$ . Then

$$\mathfrak{g}_0 := \mathfrak{sl}(C_1) \times \mathfrak{sl}(C_2) \times \mathfrak{sl}(C_3) \times \mathfrak{sl}(D) \oplus (C_1 \otimes C_2 \otimes C_3 \otimes D)$$

can be equipped with a Lie bracket in such a way that  $\mathfrak{g}_0 \cong \mathfrak{so}_8$ , see (4.1) for an identification. In this model,  $\mathfrak{k}_0 := \mathfrak{sl}(C_1) \times \mathfrak{sl}(C_2) \times \mathfrak{sl}(C_3) \times \mathfrak{sl}(D)$  is a Lie subalgebra of  $\mathfrak{g}_0$  and acts on  $\mathfrak{p}_0 := C_1 \otimes C_2 \otimes C_3 \otimes D$  in the usual way. We refer to [LM] for the definition of the bracket of two elements of  $\mathfrak{p}_0$  using  $\omega$  which, for instance, identifies  $C_1$  with its dual and hence  $\mathfrak{sl}_2(C_1)$  with  $S^2 C_1$ .

This presentation of  $\mathfrak{so}_8$  relies on the  $\mathfrak{S}_4$ -symmetry of  $\mathfrak{so}_8$  as can be seen on the extended Dynkin diagram  $\tilde{D}_4$ . We are interested in the  $\mathfrak{S}_3$ -action on  $\mathfrak{g}_0$  induced by permutations on the 3 spaces  $C_i$ . Its fixed point space  $\mathfrak{g}_2 \subset \mathfrak{g}_0$  is of type  $G_2$  and can be described as

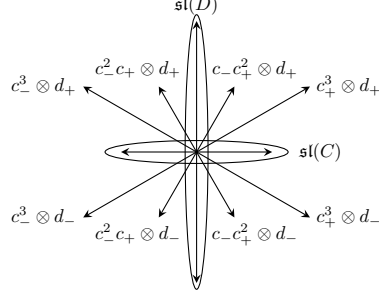
$$\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2 \text{ with } \begin{cases} \mathfrak{k}_2 := \mathfrak{sl}(C) \times \mathfrak{sl}(D) \\ \mathfrak{p}_2 := S^3 C \otimes D \end{cases}$$

where  $C = C_{123}$  is an other copy of our two-dimensional spaces with inclusion maps given by

$$\begin{array}{ccc} \mathfrak{sl}(C) & \hookrightarrow & \mathfrak{sl}(C_1) \times \mathfrak{sl}(C_2) \times \mathfrak{sl}(C_3) \\ x & \mapsto & (x, x, x) \end{array}, \quad \begin{array}{ccc} S^3 C & \hookrightarrow & C_1 \otimes C_2 \otimes C_3 \\ c_1 c_2 c_3 & \mapsto & \frac{1}{6} \sum_{\sigma \in \mathfrak{S}_3} c_{\sigma(1)} \otimes c_{\sigma(2)} \otimes c_{\sigma(3)} \end{array}$$

An important fact is that all the decompositions considered below are decompositions of  $\mathfrak{k}_2(= \mathfrak{sl}(C) \times \mathfrak{sl}(D))$ -module and all inclusions and projections considered are morphisms of  $\mathfrak{k}_2$ -modules.

Choosing a Cartan subalgebra of  $\mathfrak{g}_2$  in  $\mathfrak{sl}(C) \times \mathfrak{sl}(D)$  and weight vectors  $c_-, c_+$  (resp.  $d_-, d_+$ ) in  $C$  (resp.  $D$ ), we rediscover the combinatorics of the root spaces in  $\mathfrak{g}_2$  as pictured below.



As an intermediate construction, we will also consider the fixed point set under permutation of  $C_1$  with  $C_2$ . Let

$$\begin{cases} \mathfrak{k}_1 := \mathfrak{sl}(C') \times \mathfrak{sl}(C_3) \times \mathfrak{sl}(D), \\ \mathfrak{p}_1 := S^2 C' \otimes C_3 \otimes D. \end{cases}$$

with  $C' = C_{12}$  another copy with guessable associated inclusions. Then we get an algebra  $\mathfrak{g}_1$  (isomorphic to  $\mathfrak{so}_7$  see discussion below (4.1)) via:

$$\mathfrak{g}_1 := \mathfrak{k}_1 \oplus \mathfrak{p}_1$$

Moreover, we have projections  $\pi_{i,j} : \mathfrak{g}_i \rightarrow \mathfrak{g}_j$  ( $0 \leq i < j \leq 2$ )

$$\begin{aligned} \mathfrak{g}_0 &\rightarrow \mathfrak{g}_1 \\ \pi_{0,1} : \quad \mathfrak{k}_0 \ni (x_1, x_2, x_3, y) &\mapsto \left( \frac{1}{2}(x_1 + x_2), x_3, y \right), \\ \mathfrak{p}_0 \ni c_1 \otimes c_2 \otimes c_3 \otimes d &\mapsto c_1 c_2 \otimes c_3 \otimes d \end{aligned}$$

$$\begin{aligned} \mathfrak{g}_1 &\rightarrow \mathfrak{g}_2 \\ \pi_{1,2} : \quad \mathfrak{k}_1 \ni \left( \frac{1}{2}(x_1 + x_2), x_3, y \right) &\mapsto \left( \frac{1}{3} \sum_{i=1}^3 x_i, y \right), \\ \mathfrak{p}_1 \ni c_1 c_2 \otimes c_3 \otimes d &\mapsto c_1 c_2 c_3 \otimes d \\ \pi_{0,2} &:= \pi_{1,2} \circ \pi_{0,1}. \end{aligned}$$

There respective kernels are  $\mathfrak{sl}(C) \times \mathfrak{sl}(D)$ -modules:  $\text{Ker}(\pi_{0,1}) = \{(x, -x, 0, 0) | x \in \mathfrak{sl}(C')\} \oplus \Lambda^2 C' \otimes C_3 \otimes D$  and

$$\text{Ker}(\pi_{1,2}) = \{(x, x, -2x, 0) | x \in \mathfrak{sl}(C)\} \oplus \langle (-2c_1 c_2 \otimes c_3 + c_2 c_3 \otimes c_1 + c_3 c_1 \otimes c_2) \otimes D \rangle.$$

In particular, these maps are  $SL(C) \times SL(D)$ -equivariant.

Finally, we introduce faithful representations that allow us to manipulate more easily these algebras. Let  $V_8 := C_1 \otimes C_2 \oplus C_3 \otimes D$  with action of  $\mathfrak{p}_0$  given by

$$(c_1 \otimes c_2 \otimes c_3 \otimes d) \cdot (c'_1 \otimes c'_2 + c'_3 \otimes d) = \omega(c_1, c'_1) \omega(c_2, c'_2) c_3 \otimes d + \omega(c_3, c'_3) \omega(d, d') c_1 \otimes c_2$$

and standard action of  $\mathfrak{k}_0$ . It is one of the three inequivalent fundamental representations of  $\mathfrak{g}_0 \cong \mathfrak{so}_8$  of dimension 8 and  $\mathfrak{g}_0 \subseteq \mathfrak{gl}(V_8)$  is the subalgebra preserving the symmetric form

$$\omega(c_1 \otimes c_2 + c_3 \otimes d, c'_1 \otimes c'_2 + c'_3 \otimes d') = \omega(c_1, c'_1)\omega(c_2, c'_2) - \omega(c_3, c'_3)\omega(d, d'). \quad (4.1)$$

As an  $\mathfrak{sl}(C) \times \mathfrak{sl}(D)$ -module,  $V_8$  decomposes as  $V_1 \oplus V_a \oplus V_b$  where  $V_1 = \Lambda^2 C'$ ,  $V_a = S^2 C'$  and  $V_b = C_3 \otimes D$ . Setting  $V_7 := V_a \oplus V_b$ , one gets that  $\mathfrak{g}_1$  is the subalgebra of  $\mathfrak{so}_8$  stabilizing both subspaces  $V_1$  and  $V_7$  while  $\mathfrak{k}_1$  is the subalgebra of  $\mathfrak{g}_1$  stabilizing both subspaces  $V_a$  and  $V_b$ . In particular, we recover that  $\mathfrak{g}_1 \cong \mathfrak{so}_7 \oplus \mathfrak{so}_1 \cong \mathfrak{so}_7$  and that  $(\mathfrak{g}_1, \mathfrak{k}_1)$  is a symmetric Lie algebra isomorphic to  $(\mathfrak{so}_7, \mathfrak{so}_4 \oplus \mathfrak{so}_3)$  [PT].

**Remark 4.4.** This symmetric Lie algebra structure on  $\mathfrak{g}_1$  can be lift to the  $\mathfrak{g}_0$ -level in two inequivalent ways:  $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$  with  $\mathfrak{k}_0$  stabilizing  $(V_1 \oplus V_a)$  and  $V_b$  and another decomposition  $\mathfrak{g}_0 = \mathfrak{k}'_0 \oplus \mathfrak{p}'_0$  with  $\mathfrak{k}'_0$  stabilizing  $V_a$  and  $(V_b \oplus V_1)$ . It might be enlightening to note that  $\mathfrak{p}'_0 = \mathfrak{p}_1 \oplus \{(x, -x, 0, 0)\}$  while  $\mathfrak{p}_0 = \mathfrak{p}_1 \oplus \Lambda^2 C' \otimes C_3 \otimes D$ .

Let us turn our attention to orbits. We will denote by  $\Omega'_{10}$  the  $SO_7$ -orbit of dimension 10 in  $\mathfrak{g}_1$  and by  $\Omega_{min}$  the minimal  $SO_8$ -orbit in  $\mathfrak{g}_0$ . With respect

to our representations on  $V_7$  and  $V_8$ , their respective Young diagram are



. Moreover, we let  $\mathcal{T}$  be the set elements of  $\mathfrak{g}_1$  of rank at most 2 with respect to the representation on  $V_7$ . The closed set  $\mathcal{T}$  is nothing but the union  $SO_7.(\mathbb{k}^*t) \sqcup \overline{\Omega'_{10}}$  where  $t$  is any semisimple element whose centraliser is a Levi of type  $B_2$ .

One can extract the following information from [LSm, Kr, Hi], even if it would take us too far afield to actually show it in a rigorous way.

**Proposition 4.5.** (i)  $\pi_{0,1}$  induces a finite surjective map  $\overline{\Omega_{min}} \rightarrow \overline{\Omega'_{10}}$ .

(ii)  $\pi_{1,2}$  induces finite surjective maps  $\overline{\Omega'_{10}} \rightarrow \overline{\Omega_{10}}$  and  $\mathcal{T} \rightarrow \overline{S_1^g}$ .

(iii) The cardinalities of fibers are given by the following table:

$y \in$	$G_2.(\mathbb{k}^*h_1)$	$\Omega_{10}$	$\Omega_8$	$\Omega_6$	0
$\#\pi_{1,2}^{-1}(y) \cap \mathcal{T} =$	1	3	2	1	1
$\#\pi_{0,2}^{-1}(y) \cap \overline{\Omega_{min}} =$	2	6	3	1	1

(iv) The map  $\mathcal{T} \setminus \{0\} \rightarrow \overline{S_1^g} \setminus \{0\}$  induced by  $\pi_{1,2}$  is a desingularization.

**Remarks 4.6.** - In order to use results of [LSm] and [Kr], one should verify that our inclusions  $\mathfrak{g}_2 \subset \mathfrak{so}_7 \subset \mathfrak{so}_8$  and projection maps coincide with those used in these papers. This can be checked in the following way.

- First, one can see that there essentially exists a single way to embed  $\mathfrak{g}$  in  $\mathfrak{so}_7$  since there is a single non-trivial representation of  $\mathfrak{g}$  of dimension 7 and since this representation is irreducible. Moreover, the construction of  $\mathfrak{g}_2$  starting from  $\mathfrak{so}(8)$  and taking fixed point under  $\mathfrak{S}_3$ -action in [LSm] coincide with the one presented here.
- On the other hand, the projection maps  $\pi_{i,j}$  are  $\mathfrak{g}_j$ -equivariant, just as those used in [LSm, Kr]. Since  $\mathfrak{g}_j$  is simple and appear with multiplicity one in  $\mathfrak{g}_i$ , the considered maps differ only by a scalar multiplication (Note that the maps used in [LSm] are multiples of projection maps.)

We refer to [Oh2] for the classification of nilpotent orbits in  $\mathfrak{p}_1$ . In  $\mathfrak{g}_1$ , the  $SO_7$ -orbit  $\Omega'_{10}$  splits into two  $SO_3 \times SO_4$ -orbits  $\mathcal{O}'_{5a}$  and  $\mathcal{O}'_{5b}$  whose respective

$ab$ -diagrams are  $\begin{array}{|c|c|c|} \hline a & b & a \\ \hline a & & \\ \hline b & & \\ \hline b & & \\ \hline \end{array}$  and  $\begin{array}{|c|c|c|} \hline b & a & b \\ \hline a & & \\ \hline a & & \\ \hline b & & \\ \hline b & & \\ \hline \end{array}$ . We also define  $\mathcal{T}_{\mathfrak{p}_1} := \mathcal{T} \cap \mathfrak{p}_1$ . We can then state the following result

**Proposition 4.7.** (i)  $\pi_{1,2}$  induces a desingularization  $\mathcal{T}_{\mathfrak{p}_1} \setminus \{0\} \rightarrow \overline{S_1} \setminus \{0\}$

(ii) The cardinality of fibers is given as follows:

$y \in$	$J_1$	$\mathcal{O}_{5a}$	$\mathcal{O}_{5b}$	$\mathcal{O}_4$	$\mathcal{O}_3$	0
$\#\pi_{1,2}^{-1}(y) \cap \mathcal{T}_{\mathfrak{p}_1} =$	1	1	3	2	1	1

*Proof.* The desingularization property follows from  $\#\pi_{1,2}^{-1}(y) = 1$  for  $y \in J_1$  and the fact that  $\mathcal{T}_{\mathfrak{p}_1}$  is a smooth variety.

Let us check this last point. First it is an easy matter to check that, under the natural isomorphism  $\mathfrak{g}_1 \cong \Lambda^2 V_7$ , we have  $\mathfrak{p}_1 = V_a \wedge V_b$  and  $\mathcal{T}_{\mathfrak{p}_1} = \{v_a \wedge v_b | v_a \in V_a, v_b \in V_b\}$ . The smoothness is then a consequence of the transitivity of the action of  $GL(V_a) \times GL(V_b)$  on  $\mathcal{T}_{\mathfrak{p}_1} \setminus \{0\}$ .

An element of  $\mathfrak{p}_1$  acts on  $V_8$  in the following way:

$$\left( \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & p \\ \hline 0 & p^\vee & 0 \end{array} \right) \begin{array}{l} v_1 \\ V_a \\ V_b \end{array}$$

where  $p \in \text{Hom}(V_b, V_a)$  and  $p^\vee$  is the dual of  $p$  with respect to  $\omega|_{V_a \oplus V_b}$ . We are interested into rank 2 elements of this form and these are exactly elements for which  $p$  is of rank 1. We will make use of the 4-ality setting to describe such elements. An element of  $\mathfrak{p}_1 = (S^2 C') \otimes (C_3 \otimes D)$  induces a rank one element of  $\text{Hom}(S^2 C', C_3 \otimes D)$  (and, equivalently, of  $\text{Hom}(C_3 \otimes D, S^2 C')$ ) if and only if it is a pure tensor of the form

$$(\alpha c_+^2 + \beta c_+ c_- + \gamma c_-^2) \otimes (c_+ \otimes d_1 + c_- \otimes d_2)$$

with  $\alpha, \beta, \gamma \in \mathbb{k}$  and  $d_1, d_2 \in D$ . The projection of such an element on  $\mathfrak{g}_2$  is

$$c_+^3 \otimes \alpha d_1 + c_+^2 c_- \otimes (\alpha d_2 + \beta d_1) + c_+ c_-^2 \otimes (\beta d_2 + \gamma d_1) + c_-^3 \otimes \gamma d_2.$$

Then, computing the fiber  $\pi_{1,2}^{-1}(y) \cap \mathcal{T}_{\mathfrak{p}_1}$  for an element of  $\overline{S_1}$  amounts to solve some small system of equations which does not present any special difficulty. In Table 3, we give the result of these computations. The column *I* present the Jordan class of  $\overline{S_1}$  under consideration. In column *II*, we give a representative of the class, most of which are defined in Table 1. Up to easy  $SL(C) \times SL(D)$ -conjugation, one can turn our representative to the element  $y \in \mathfrak{p}_2$  in column *III*. Finally, column *IV* present all elements in the fiber  $\pi_{1,2}^{-1}(y) \cap \mathcal{T}_{\mathfrak{p}_1}$ .

<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>
$K.(\mathbb{k}^* \tilde{h}_1)$	$x_4 + y_4$	$c_+^2 c_- \otimes d_+ + c_+ c_-^2 \otimes d_-$	$c_+ c_- \otimes (c_+ \otimes d_+ + c_- \otimes d_-)$
$\mathcal{O}_{5a}$	$n_{5a}$	$c_+^3 \otimes d_+ + c_+^2 c_- \otimes d_-$	$c_+^2 \otimes (c_+ \otimes d_+ + c_- \otimes d_-)$
$\mathcal{O}_{5b}$	$n_{5b}$	$c_+^2 c_- \otimes d_+ + c_+ c_-^2 \otimes d_+$	$c_+ c_- \otimes (c_+ + c_-) \otimes d_+$
			$c_+(c_+ + c_-) \otimes c_- \otimes d_+$
			$(c_+ + c_-)c_- \otimes c_+ \otimes d_+$
$\mathcal{O}_4$	$n_4$	$c_+^2 c_- \otimes d_+$	$c_+^2 \otimes c_- \otimes d_+$
			$c_+ c_- \otimes c_+ \otimes d_+$
$\mathcal{O}_3$	$n_3$	$c_+^3 \otimes d_+$	$c_+^2 \otimes c_+ \otimes d_+$
0	0	0	0

Table 3: fibers of the form  $\pi_{1,2}^{-1}(y) \cap \mathcal{T}_{\mathfrak{p}_1}$

For instance, concerning  $\mathcal{O}_{5b}$ , one can check that  $SL(C) \times SL(D)$  acts transitively on  $\mathbb{k}^* x_3 + \mathbb{k}^* x_4$ . In order to get the elements in column IV, we then have to consider the system  $Q: \begin{cases} \alpha d_1 = 0, & \alpha d_2 + \beta d_1 = d_+ \\ \beta d_2 + \gamma d_1 = d_+, & \gamma d_2 = 0 \end{cases}$  whose resolution can be pictured as follows

$$Q: \begin{cases} \alpha = 0, \gamma = 0 \implies \beta d_1 = d_+, \beta d_2 = d_+ \implies \beta \neq 0, d_1 = \frac{d_+}{\beta} = d_2 & \square \\ \text{or} \\ \alpha = 0, d_2 = 0 \implies \beta d_1 = d_+, \gamma d_1 = d_+ \implies \beta \neq 0, \gamma = \beta, d_1 = \frac{d_+}{\beta} \\ \text{or} \\ \alpha = 0, d_1 = 0 \implies \alpha d_2 = d_+, \beta d_2 = d_+ \implies \beta \neq 0, \alpha = \beta, d_2 = \frac{d_+}{\beta} \end{cases}$$

**Remarks 4.8.** One can wonder where the 2 other preimages of  $n_{5a}$  did go. In fact they can be expressed in  $\mathfrak{sl}(C') \times \mathfrak{sl}(C_3) \times \mathfrak{sl}(D) \oplus S^2 C' \otimes C_3 \otimes D$  in the following form:

$$\pm \left( \frac{1}{2} u, -u, 0 \right) + (c_+ c_- \otimes c_+ \otimes d_- + c_+^2 \otimes c_+ \otimes d_+)$$

where  $u$  is some element of  $\mathfrak{sl}(C)$  such that  $\text{Ker}(u) = \langle c_+ \rangle$ . In particular, we see that these two preimages have a non-zero  $\mathfrak{k}_1$ -component.

One can also check that  $\pi_{1,2}$  induces finite surjective morphisms  $\overline{\mathcal{O}'_{5a}} \rightarrow \overline{\mathcal{O}_{5a}}$  and  $\overline{\mathcal{O}'_{5b}} \rightarrow \overline{\mathcal{O}_{5b}}$ . For instance, since  $c_+, c_-, d_+, d_-$  are isotropic vectors with respect to  $\omega$ , we can check that the pre-image  $z = c_+^2 \otimes (c_+ \otimes d_+ + c_- \otimes d_-)$  of  $n_{5a}$  sends  $c_-^2 \in V_a$  on a non-zero multiple of  $c_+ \otimes d_+ + c_- \otimes d_- \in V_b$  and this last one is sent on a non-zero multiple of  $c_+^2 \in V_a$ . Hence the  $ab$ -diagram of  $z$  is

$$\begin{array}{|c|c|c|} \hline a & b & a \\ \hline a & & \\ \hline b & & \\ \hline b & & \\ \hline \end{array}$$

and  $z \in \mathcal{O}'_{5a}$ .

On the  $\mathfrak{g}_0$ -level, one can also check that  $\pi_{0,2}$  induces a surjective morphism  $\overline{\mathcal{O}_{min} \cap \mathfrak{p}_0} \rightarrow \overline{\mathcal{O}_{5a}}$  and  $\overline{\mathcal{O}_{min} \cap \mathfrak{p}'_0} \rightarrow \overline{\mathcal{O}_{5b}}$ . However, in order to get preimages of  $\tilde{h}_1$ , one has to look for them in  $\mathfrak{p}_0 + \mathfrak{p}'_0$ . Such a description is less nice.

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