Max-Min optimization problem for Variable Annuities pricing

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Abstract

We study the valuation of variable annuities for an insurer. We concentrate on two types of these contracts that are the guaranteed minimum death benefits and the guaranteed minimum living benefits ones and that allow the insured to withdraw money from the associated account. As for many insurance contracts, the price of variable annuities consists in a fee, fixed at the beginning of the contract, that is continuously taken from the associated account. We use a utility indifference approach to determine this fee and, in particular, we consider the indifference fee rate in the worst case for the insurer i.e. when the insured makes the withdrawals that minimize the expected utility of the insurer. To compute this indifference fee rate, we link the utility maximization in the worst case for the insurer to a sequence of maximization and minimization problems that can be computed recursively. This allows to provide an optimal investment strategy for the insurer when the insured follows the worst withdrawals strategy and to compute the indifference fee. We finally explain how to approximate these quantities via the previous results and give numerical illustrations of parameter sensibility.

Keywords: Variable annuities, indifference pricing, backward stochastic differential equation, utility maximization, insurance.

MSC2000 subject classification: 60H99, 91B30, 93E20.

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1 Introduction

Variable annuities are insurance contracts that were introduced in the 1970s in the United States (see Sloane [16]). These insurance products provide, during a given period, deferred annuities that are fund linked. More precisely, the policyholder gives at the beginning an initial amount of money to the insurer who invests this amount in a reference portfolio. In return, the insurer provides annuities that depend on the performance of this reference portfolio.

In the 1990s, insurers add certain guarantees to these policies. They propose contracts with annuities that are at least greater than some guaranteed value. Nowadays, the most spread are Guaranteed Minimum Death Benefits (GMDB for short) and Guaranteed Minimum Living Benefits (GMLB for short) contracts. For a GMDB contract, the dependents of the insured obtain a guaranteed benefit if she dies before the maturity of the contract. On the contrary, the holder of a GMLB contract obtains a guaranteed benefit if she is still alive at the maturity of the contract. There are various ways to fix this guaranteed benefit and we refer to [2] for more details.

These new added guarantees developed the interest of the investors for these products and have made the variable annuities contracts highly demanded on financial markets. Therefore, their pricing and hedging have attracted a lot of interest in a growing literature. In their pioneering work, Boyle and Schwartz [6] used non-arbitrage models to extend the Black & Scholes framework to insurance issues. Then, Milevsky and Posner [12] applied risk neutral option pricing theory to value GMDB variable annuities. The case of withdrawal options is studied by Chu and Kwok in [8] and by Siu in [15], and a general framework to define variable annuities is presented by Bauer *et al.* in [2]. Milevsky and Salisbury studied in [13] the links between American put options and dynamic optimal withdrawal policies. In [3], Belanger *et al.* described the valuation of GMDB as an impulse control problem. They derive an HJB equation in a Markov framework and solve it numerically.

An important risk faced by the seller of a variable annuities contract concerns the characteristics of the buyer. The insurer has to take into account the behavior of the insured, i.e. her withdrawals, and her exit time from the contract, i.e. her death time,. In this paper, we study a valuation of GMDB and GMLB contracts that takes into account this uncertainty on the insured. We consider the worst case for the withdrawal strategy of the insured from the insurer point of view. Concerning the death time, we model it as a random time enlarging the initial filtration related to the market information. As such contracts are generally priced for a class of insured, we suppose that this random time corresponds to the death time of a representative agent in a specific class of clients that satisfy several conditions (age, job, wealth,...). We shall assume that such a class is small enough to be unable to affect the market. From a probabilistic point of view, this justifies that the well known assumption (H) holds true i.e. any martingale for the initial filtration remains a martingale for the initial filtration enlarged by the exit time.

Due to these risks coming from the insured, the market is incomplete and we have to choose a pricing definition. Following the approach of Chevalier *et al.* [7], we therefore consider the indifference utility valuation with an exponential criterion, which is commonly

admitted to be pertinent for the study of insurance products. We suppose that the price of a variable annuities contract is defined as a continuous time fee rate p that is taken from the fund related to the contract. The indifference pricing procedure consists in finding the fee rate p for which $V^0 = V(p)$ where

- V^0 is the maximal expected utility that the insurer can get when she invests the related fund on the market,
- $V(p)$ is the maximal expected utility that the insurer can get when she invests the related fund on the market and she sells a variable annuities contract at the price p.

The computation of V^0 is a classical problem that has been solved in [10] if the coefficients of the model do not depend on the death time and by [11] if these ones depend on the death time. However the computation of $V(p)$ is a challenging problem since it involves the behavior of the buyer of the variable annuities contract. Especially, we have to take into account the possible withdrawals that can be done by the insured. In [7], the insured withdraw strategy is assumed to be a given random process. We use a more robust pricing approach by considering the worst case for the insurer i.e. the insured chooses the withdrawal strategy that minimizes the expected utility of the insurer. This leads to study an optimization problem of max-min-type. Such a problem is in general difficult to solve due to the dependence of the maximizing strategy on the minimizing one. Here we take advantage of the multiplicative structure of the utility function to break this dependence. This allows us to transform the initial max-min problem into separated maximization and minimization problems on different time intervals. This decomposition allows then to give an optimal strategy for the insurer and the worst withdrawals from the insured for the insurer.

The rest of the paper is organized as follows. In Section 2, we present the probabilistic space and the financial market. In Section 3, we present the variable annuities products and their indifference princing valuation. We then show in Section 4 that the utility maximization in the worst case for the insurer can be reduced to a sequential utility maximization problem, and we provide the optimal strategy and worst withdrawals for the insurer. In Section 5 we explain how to solve numerically this indifference pricing and provide numerical examples. Finally, we relegate to the Appendix the proof of our main result (Theorem 4.1) and some additional results that are needed to complete this proof.

2 The model

2.1 Probability space

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion B and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the right continuous complete filtration generated by B. We consider on this space a random time τ, which represents the death time of a representative agent. The random time τ is not assumed to be an $\mathbb{F}$-stopping time. We therefore use the standard approach of filtration enlargement by considering the smallest right-continuous extension $\mathbb{G}$ of $\mathbb{F}$ that turns τ into

a $\mathbb{G}$-stopping time. More precisely $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is defined by

$$\mathcal{G}_t \;:=\; \bigcap_{\varepsilon>0} \tilde{\mathcal{G}}_{t+\varepsilon}\ ,$$

for any $t \geq 0$, where $\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(\mathbb{1}_{\tau\leq u}\ ,\ u \in [0,s])$, for any $s \geq 0$.

We denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable subsets of $\Omega\times\mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes.
We impose the following assumption, which is classical in the filtration enlargement theory.

Assumption 2.1. *The process B remains a $\mathbb{G}$-Brownian motion.*

The interpretation of the H-hypothesis is an assymetric dependance structure between B and τ. From a financial point of view, it means that the exit time τ may depend on the financial market randomness represented by B. On the contrary, the financial market does not depend on τ.

In the sequel N denotes the process $\mathbb{1}_{\tau\leq .}$.

Assumption 2.2. *The process N admits a $\mathbb{G}$-compensator of the form $\int_0^{.\wedge\tau}\lambda_t dt$, i.e. $N - \int_0^{.\wedge\tau}\lambda_t dt$ is a $\mathbb{G}$-martingale, where λ is a positive bounded $\mathcal{P}(\mathbb{F})$-measurable process.*

We denote by M the $\mathbb{G}$-martingale defined by

$$M_t \;:=\; N_t - \int_0^{t\wedge\tau}\lambda_s ds$$

for any $t \geq 0$.

2.2 Financial market

We consider that the insurer can invest in a financial market which is composed by two assets. The first one is a riskless bond $\hat{S}^0$ satisfying the following stochastic differential equation

$$d\hat{S}^0_t \;=\; r_t\hat{S}^0_t dt\ ,\quad t\geq 0\ ,\quad \hat{S}^0_0 = 1\ , \tag{2.1}$$

where r is a $\mathcal{P}(\mathbb{G})$-measurable process representing the riskless interest rate. The second asset $\hat{S}$ is a reference portfolio of risky assets underlying the variable annuities policy. $\hat{S}$ is assumed to be solution of the linear stochastic differential equation

$$d\hat{S}_t \;=\; \hat{S}_t(\mu_t dt + \sigma_t dB_t)\ ,\quad t\geq 0\ ,\quad \hat{S}_0 = s \geq 0\ , \tag{2.2}$$

where μ and σ are $\mathcal{P}(\mathbb{G})$-measurable processes. In our case the coefficients are $\mathcal{P}(\mathbb{F})$-measurable processes since we assume that the exit time τ does not affect the market. But to simplify the notations we assume that the coefficients are $\mathcal{P}(\mathbb{G})$-measurable processes. We refer to [7] for the case with $\mathcal{P}(\mathbb{F})$-measurable processes.
To ensure the existence and uniqueness of the processes $\hat{S}^0$ and $\hat{S}$, we make the following assumptions.

Assumption 2.3. *(i) The processes μ, σ and r are bounded.*

(ii) The process σ is lower bounded by a positive constant $\underline{\sigma}$.

We shall denote by S the discounted value of $\hat{S}$ and by θ the risk premium of $\hat{S}$:

$$S_t \;=\; e^{-\int_0^t r_s ds}\hat{S}_t \quad \text{and} \quad \theta_t \;=\; \frac{\mu_t - r_t}{\sigma_t}$$

for all $t \in [0,T]$.

A $\mathbb{G}$-predictable process $\pi = (\pi_t)_{0\le t\le T}$ is called a trading strategy if $\int \pi \frac{dS}{S}$ is well defined (for example if $\int_0^T |\pi_t\sigma_t|^2 dt < \infty\ \mathbb{P}-a.s.$). The process π describes the discounted amount of money invested in the portfolio of risky assets. Assuming that the investment strategy is self-financed and denoting by X_t^π the discounted value of the insurance portfolio with initial capital 0 and following the strategy π, we have

$$X_t^\pi \;=\; \int_0^t \pi_s(\mu_s - r_s)\, ds + \int_0^t \pi_s\sigma_s\, dB_s\ , \quad t \ge 0\ .$$

We also denote by $X_t^{s,\pi}$ the discounted wealth at time t when the initial capital at time s is 0 and the investment strategy is π.

We consider an insurance company (or an insurer) with preferences given by the utility function U defined by

$$U(y) \;=\; -e^{-\gamma y}\ , \quad y \in \mathbb{R}$$

where γ is a positive constant. Both theory and pratice have shown that it is appropriate to use this utility function. It is relevant since optimal controls do not depend on the initial wealth of the insurer. Moreover an appealing feature of decision making using this utility function is that decisions are based on comparisons between moment generating functions, which capture all the characteristics of the random outcomes being compared, so that comparisons are based on a wide range of features. We refer to [4] for more details about this choice.
In the following definition we precise its admissible strategies on a given random interval.

Definition 2.1. *Let ν_1 and ν_2 be two $\mathbb{G}$-stopping times such that $0 \le \nu_1 \le \nu_2 \le T$. The set $\mathcal{A}[\nu_1,\nu_2]$ of admissible strategies on the stochastic interval $[\nu_1,\nu_2]$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_t)_{0\le t\le T}$ which satisfy*

$$\mathbb{E}\Big[\int_{\nu_1}^{\nu_2} \big|\pi_t\big|^2 dt\Big] \;<\; \infty\ ,$$

and

$$\big\{\exp(-\gamma X_\nu^\pi),\ \nu \text{ is a stopping time such that } \nu_1 \le \nu \le \nu_2\big\}$$

is a uniformly integrable family.

3 Utility indifference pricing of variable annuities

3.1 Variable annuities

We consider a variable annuities product with a maturity $T > 0$. It consists in a deferred fund-linked annuity contract that we describe in the following lines.

Initial investment. The insured invests an initial capital, denoted by A_0, in the fund related to this product (also called insured account) at time $t = 0$.

Withdrawals. Let $\mathbb{T} := (t_i)_{0 \leq i \leq n}$ the set of policy anniversary dates, with $t_0 = 0$ and $t_n = T$. By convention we set $t_{n+1} = +\infty$.
At any date t_i, for $i \in \{1, \dots, n-1\}$, the insured, if she is still alive, is allowed to withdraw an amount of money. This should be lower than a bounded non-negative $\mathcal{G}_{t_i}$-measurable random variable $\hat{G}_i$ which may depend on previous withdrawals, on previous account values and on some guarantees determined in the policy.
We define $\hat{\mathcal{W}}$ as a finite subset of $[0, 1]$ which contains 0 and 1 and introduce the set of admissible withdraw policies

$$\begin{aligned} \hat{\mathcal{E}} &= \Big\{ (\alpha_i \hat{G}_i)_{1 \leq i \leq n-1} \ : \ \alpha_i \text{ is a } \mathcal{G}_{t_i}\text{-measurable random variable such that} \\ &\qquad \alpha_i \in \hat{\mathcal{W}} \quad \text{for all } i \in \{1, \dots, n-1\} \Big\} . \end{aligned}$$

For $\hat{\xi} \in \hat{\mathcal{E}}$ and $i \in \{1, \dots, n-1\}$, $\hat{\xi}_i$ is the withdrawal made by the insured at time t_i and we introduce the family $(\xi_i)_{1 \leq i \leq n-1}$ such that $\xi_i := e^{-\int_0^{t_i} r_s \, ds} \hat{\xi}_i$ is the discounted withdrawal made at time t_i. We define by $\mathcal{E}$ the admissible discounted withdraw policies with $\xi \in \mathcal{E}$ if and only if the vector $\hat{\xi} \in \hat{\mathcal{E}}$. For any $k \in \{0, \dots, n-2\}$ and $i \in \{1, \dots, n-k-1\}$, we also define the set $\mathcal{E}_k^i$ by

$$\mathcal{E}_k^i = \Big\{ \xi \in \mathcal{E} \text{ s.t. } \xi_j = 0 \text{ for all } j \notin \{k+1, \dots, k+i\} \Big\} .$$

$\mathcal{E}_k^i$ is the set of admissible withdraw policies such that all withdrawals are made between times t_{k+1} and t_{k+i}.

Dynamics of the related fund. We denote by A_t^p the discounted value at time t of the fund related to the variable annuities contract sold at fee rate p. If the insured follows the withdraw policy $\hat{\xi} \in \hat{\mathcal{E}}$, we have

$$\left\{ \begin{array}{rcll} dA_t^p &=& A_t^p \big[(\mu_t - r_t - p) dt + \sigma_t dB_t \big] , & \text{for } t \notin \mathbb{T} , \\ A_{t_i}^p &=& \big(A_{t_i^-}^p - f_i \big) \vee 0 , & \text{for } 1 \leq i \leq n-1 , \end{array} \right. \tag{3.3}$$

where f_i is a $\mathcal{G}_{t_i}$-measurable random variable greater than ξ_i for any $i \in \{1, \dots, n-1\}$ and depending on previous withdrawals, on previous account values and on some guarantees determined in the policy. The simplest case would be to have $f_i = \xi_i$ but variable annuities contracts may be more complex. For instance, for a given withdrawal $\hat{\xi}_i$ the insurer may

withdraw a larger amount of money from the insured account.

We now focus on the dependancies of f_i and $\hat{G}_i$. Let $\hat{g}$ be a bounded non-negative deterministic function, defined on $[0,T]\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$ such that for any $i\in\{1,\dots,n-1\}$ and $(t,x,e)\in[0,T]\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$, the function $y\mapsto\hat{g}(t,x,e_1,\dots,e_{i-1},y,e_{i+1},\dots,e_{n-1})$ is non-increasing and for any $j\in\{1,\dots,n+1\}$, the function $y\mapsto\hat{g}(t,x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_{n+1},e)$ is non-decreasing. We assume that, for any $i\in\{1,\dots,n-1\}$,

$$\hat{G}_i \;=\; \hat{g}(t_i,\hat{A}^p_{t_0},\dots,\hat{A}^p_{t_{i-1}},\hat{A}^p_{t_i^-},0,\dots,0,\hat{\xi}_1,\dots,\hat{\xi}_{i-1},0,\dots,0)\;,$$

where $\hat{A}^p_t = e^{\int_0^t r_s\,ds}A^p_t$ for all $t\in[0,T]$. In the same way, we define the random variables $(f_i)_{1\le i\le n-1}$. Let $\hat{f}$ be a deterministic function bounded and non-negative, defined on $\{1,\dots,n\}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$ such that for any $(j,x,e)\in\{1,\dots,n\}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$ and $i\in\{1,\dots,n+1\}$, $y\mapsto\hat{f}(j,x,e_1,\dots,e_{i-1},y,e_{i+1},\dots,e_{n-1})$ is non-decreasing and for any $i\in\{1,\dots,n+1\}$, the function $y\mapsto\hat{f}(j,x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_{n+1},e)$ is non-increasing. We assume that, for any $i\in\{1,\dots,n-1\}$,

$$\begin{aligned} f(i,\hat{A}^p_{t_0},\dots,\hat{A}^p_{t_{i-1}},\hat{A}^p_{t_i^-},0,\dots,0,\hat{\xi}_1,\dots,\hat{\xi}_i,0,\dots,0) \;&=\\ e^{\int_0^{t_i} r_s\,ds}\hat{f}(i,\hat{A}^p_{t_0},\dots,\hat{A}^p_{t_{i-1}},\hat{A}^p_{t_i^-},0,\dots,0,\hat{\xi}_1,\dots,\hat{\xi}_i,0,\dots,0)\;.& \end{aligned}$$

We give concrete examples of functions $\hat{g}$ and f in the next subsection.

Pay off contract. The last quantity to define is the pay-off of the variable annuities. Let $\hat{F}^L$ and $\hat{F}^D$ be bounded and non-negative functions defined on $[0,T]\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$ such that for any $Q\in\{L,D\}$, $i\in\{1,\dots,n+1\}$ and $(t,x,e)\in[0,T]\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$, the following function

$$y\mapsto\hat{F}^Q(t,x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_{n+1},e)$$

is non-decreasing and the function

$$y\mapsto\hat{F}^Q(t,x,e_1,\dots,e_{i-1},y,e_{i+1},\dots,e_{n-1})$$

is non-increasing. The pay-off is paid at time $T\wedge\tau$ to the insured or her dependents and is equal to the following random variable

$$\hat{F}(p,\hat{\xi}) \;:=\; \hat{F}^L(T,\hat{a}^p,\hat{\xi})\mathbb{1}_{\{T<\tau\}}+\hat{F}^D(\tau,\hat{a}^p,\hat{\xi})\mathbb{1}_{\{\tau\le T\}}\;, \tag{3.4}$$

where $\hat{a}^p=\big(\hat{A}^p_{t_i\wedge\tau}\big)_{0\le i\le n}$. $\hat{F}^L$ is the pay-off if the policyholder is alive at time T and $\hat{F}^D$ is the pay-off if the policyholder is dead at time τ. Notice that $\hat{F}(p,\hat{\xi})$ is $\mathcal{G}_{T\wedge\tau}$-measurable. In the following, we denote by $F(p,\hat{\xi})$ the discounted pay-off defined by

$$F(p,\hat{\xi}) \;=\; e^{-\int_0^{T\wedge\tau} r_s\,ds}\hat{F}(p,\hat{\xi})\;.$$

3.2 GMDB and GMLB contracts

Usual examples of variable annuities are GMDB and GMLB. In that case, there exist non-negative functions $\hat{G}^D$, $\hat{G}^L$ and $\hat{G}^W$ defined on $[0,T]\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$ such that for any $Q\in\{L,D,W\}$, $i\in\{1,\dots,n+1\}$ and $(t,x,e)\in[0,T]\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$, the function $y\mapsto\hat{G}^Q(t,x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_{n+1},e)$ is non-decreasing and the function $y\mapsto\hat{G}^Q(t,x_1,x,e_1,\dots,e_{i-1},y,e_{i+1},\dots,e_{n-1})$ is non-increasing . Moreover, for any $Q\in\{D,L\}$, on $[0,T]\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}$, we have

$$\hat{F}^Q(t,x,e) \;=\; x_{n+1}\vee\hat{G}^Q(t,x,e)\;,$$

and

$$\hat{g}(t,x,e) \;=\; \sum_{i=0}^{n}\Big[x_{i+1}\vee\hat{G}^W(t,x_0,\dots,x_{i+1},0,\dots,0,e_1,\dots,e_{i-1},0,\dots,0)\Big]\mathbb{1}_{\{t_i\le t<t_{i+1}\}}\;.$$

In that case, the penalty function f is often given by

$$f(i,x,e)=\left\{\begin{array}{ll} e_i & \text{if } e_i\le G_i\;,\\ G_i+\kappa(e_i-G_i) & \text{if } e_i>G_i\;,\end{array}\right.$$

where $\kappa>1$ and $G_i:=G^W(t_i,x_0,\dots,x_{i+1},0,\dots,0,e_1,\dots,e_{i-1},0,\dots,0)$. The insurer takes a fee if the insured withdraws more than the guarantee G_i, this fee is equal to $(\kappa-1)(e_i-G_i)$.

The usual guarantee functions used to define GMDB and GMLB are listed below (see [2] for more details).

- Constant guarantee. For $i\in\{0,\dots,n\}$ and $t_i\le t<t_{i+1}$, we set

$$\hat{G}^Q(t,x,e) \;=\; x_1-\sum_{k=1}^{i}\hat{f}(k,x,e)\quad\text{on } [0,T]\times\mathbb{R}^{n+1}\times\hat{\mathbb{R}}^{n-1}\;.$$

  Hence, following the withdraw strategy $\xi\in\mathcal{E}$, the insured will get

$$F(p,\hat{\xi}) \;=\; A^p_{T\wedge\tau}\vee\Big(e^{-\int_0^{T\wedge\tau}r_s\,ds}\sum_{i=0}^{n}\Big(A_0-\sum_{k=1}^{i}\hat{f}(k,\hat{a}^p,\hat{\xi})\Big)\mathbb{1}_{\{t_i\le T\wedge\tau<t_{i+1}\}}\Big)\;.$$

- Roll-up guarantee. For $\eta>0$, $i\in\{0,\dots,n\}$ and $t_i\le t<t_{i+1}$, we set

$$\hat{G}^Q(t,x,e) \;=\; x_1(1+\eta)^i-\sum_{k=1}^{i}\hat{f}(k,x,e)(1+\eta)^{i-k}\quad\text{on } [0,T]\times\mathbb{R}^{n+1}\times\mathbb{R}^{n-1}\;,$$

  and then if the insured follows the withdraw strategy $\xi\in\mathcal{E}$, she will get

$$F(p,\hat{\xi}) \;=\; A^p_{T\wedge\tau}\vee\Big(e^{-\int_0^{T\wedge\tau}r_s\,ds}\sum_{i=0}^{n}\Big(A_0(1+\eta)^i-\sum_{k=1}^{i}\hat{f}(k,\hat{a}^p,\hat{\xi})(1+\eta)^{i-k}\Big)\mathbb{1}_{\{t_i\le T\wedge\tau<t_{i+1}\}}\Big)\;.$$

- Ratchet guarantee. The guarantee depends on the path of A in the following way

$$\hat{G}^{Q}(t,x,e) \;=\; \sum_{i=0}^{n} \max\Big(x_1 - \sum_{k=1}^{i} \hat{f}(k,x,e), \dots, x_i - \hat{f}(i,x,e), x_{i+1}\Big) \mathbb{1}_{\{t_i \le T\wedge\tau < t_{i+1}\}} \,,$$

for any $(t,x,e) \in [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$. The insured will get

$$F(p,\hat{\xi}) \;=\; A^p_{T\wedge\tau} \vee \Big(e^{-\int_0^{T\wedge\tau} r_s\, ds} \sum_{i=0}^{n} \max\Big(\hat{a}^p_0 - \sum_{k=1}^{i} \hat{f}(k,\hat{a}^p,\hat{\xi}), \dots, \hat{a}^p_i\Big) \mathbb{1}_{\{t_i \le T\wedge\tau < t_{i+1}\}}\Big) \,.$$

Remark 3.1. We notice that such classical pay-offs are not bounded. Unfortunately, we need to suppose them to be bounded in our approach (see Remark A.1). From an economical point of view, the boundedness of the pay-offs can be justified by saying that the insurer can provide at the best an amount m which corresponds to her cash account. Therefore, the real pay-off that the insurer can provide is not $F(p,\hat{\xi})$ but $F(p,\hat{\xi}) \wedge m$.

3.3 Utility maximization and indifference pricing

Since the financial market is incomplete we propose to use an indifference pricing approach to determine the fee rate to fixe. We look for, if it exists, a fee rate p^* such that

- the insurer has better to sell the policy if the fee rate is greater than p^*,
- she has better not to sell the contract if the fee rate is smaller than p^*.

The optimal fee rate p^* is then the smallest p such that

$$\sup_{\pi\in\mathcal{A}[0,T]} \mathbb{E}\big[U\big(X^\pi_T\big)\big] \le \sup_{\pi\in\mathcal{A}[0,T]} \inf_{\xi\in\mathcal{E}} \mathbb{E}\Big[U\Big(A_0 + X^\pi_T - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i\le\tau} - F(p,\hat{\xi})\Big)\Big] \,. \tag{3.5}$$

A solution of inequality (3.5) will be called an indifference fee rate. Notice that, since the utility function is an exponential function, indifference fee rates will not depend on the initial wealth invested by the insurer but only on the initial deposit A_0 made by the insured. For this reason, we do not consider the initial wealth of the insurer and we assume w.l.o.g. that her initial endowment is zero.

In order to find the indifference fees, we shall compute the following quantities

$$V^0 \;:=\; \sup_{\pi\in\mathcal{A}[0,T]} \mathbb{E}\big[U\big(X^\pi_T\big)\big] \,, \tag{3.6}$$

and

$$\begin{aligned} V(p) \;&:=\; \sup_{\pi\in\mathcal{A}[0,T]} \inf_{\xi\in\mathcal{E}} \mathbb{E}\Big[U\Big(A_0 + X^\pi_T - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i\le\tau} - F(p,\hat{\xi})\Big)\Big] \\ &=\; -e^{-\gamma A_0} w(p) \,, \qquad p \in \mathbb{R} \,, \end{aligned} \tag{3.7}$$

where w is defined for any $p \in \mathbb{R}$ by

$$w(p) \quad := \quad \inf_{\pi \in \mathcal{A}[0,T]} \sup_{\xi \in \mathcal{E}} \mathbb{E}\Big[u\Big(X_T^\pi - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - F(p, \hat{\xi})\Big)\Big] \ , \tag{3.8}$$

with $u(y) = e^{-\gamma y}$ for all $y \in \mathbb{R}$.

The quantity V^0 corresponds to the maximal expected utility time T when the insurance company has not sold the variable annuities policy. We can characterize this value function V^0 and the optimal strategy π^* by mean of BSDEs as done by Kharroubi and Lim in [11]. To this end we define the following spaces.

- $S_{\mathbb{G}}^{\infty}$ is the set of càdlàg $\mathbb{G}$-adapted processes essentially bounded.
- $L_{\mathbb{G}}^2$ is the set of $\mathcal{P}(\mathbb{G})$-measurable processes z such that $\mathbb{E}\int_0^T |z_s|^2 ds < \infty$.
- $L^2(\lambda)$ is the set of $\mathcal{P}(\mathbb{G})$-measurable processes u such that $\mathbb{E}\int_0^{T\wedge\tau} |u_s|^2 \lambda_s ds < \infty$.

We then have the following result which is a consequence of Theorem 5.1 in [11].

Proposition 3.1. *The value function $V^0 := \sup_{\pi \in \mathcal{A}[0,T]} \mathbb{E}\big[U\big(X_T^\pi\big)\big]$ is given by*

$$V^0 \quad = \quad -\exp(\gamma y_0) \ ,$$

where (y, z, u) is the solution in $S_{\mathbb{G}}^{\infty} \times L_{\mathbb{G}}^2 \times L^2(\lambda)$ to the BSDE

$$\begin{cases} dy_t & = & \Big(\frac{\theta_t^2}{2\gamma} + \theta_t z_t - \lambda_t \frac{e^{\gamma u_t} - 1}{\gamma}\Big) dt + z_t dB_t + u_t dN_t \ , \\ y_T & = & 0 \ . \end{cases} \tag{3.9}$$

Moreover, the optimal strategy associated to this problem is defined by

$$\pi_t^* \quad := \quad \frac{\theta_t}{\gamma \sigma_t} + \frac{z_t}{\sigma_t} \ , \quad \forall t \in [0, T] \ .$$

Remark 3.2. *When the coefficients of the model, μ, r and σ, are $\mathbb{F}$-predictable we can prove that $u = 0$ and we then have a Brownian BSDEs to solve as in [10]..*

In the classical case, to determine the utility indfference price p of a contingent claim ξ we have to solve the equation

$$\exp(-\gamma p) \sup_{\pi} \mathbb{E}\big[-\exp(-\gamma(X_T^\pi - \xi))\big] \quad = \quad \sup_{\pi} \mathbb{E}\big[-\exp(-\gamma X_T^\pi)\big] \ .$$

Therefore, we can isolate p and we get a semi-explicit formula for the indifference price. A difficulty with our approach is that fees are continuously payed by the insured and that the fee rate p appears in the pay-off $F(p, \hat{\xi})$. Therefore, one cannot use algebraic properties of utility function to get semi-explicit formula for indifference fees. Nevertheless, we can prove some monotonic results on the value function V which will be used to prove that the indifference fee rate exists or not and to compute it.

Proposition 3.2. *The value function V is non-decreasing on $\mathbb{R}$.*

Proof. Let p_1 and p_2 two reals such as $p_1 < p_2$ and $\mathcal{E}^{p_1}$ (resp. $\mathcal{E}^{p_2}$) the set of admissible discounted withdraw policies for a fees p_1 (resp. p_2). It is obvious that $\mathcal{E}^{p_2} \subset \mathcal{E}^{p_1}$. Moreover, the function $p \mapsto \mathbb{E}\Big[u\big(X_T^\pi - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - F(p, \hat{\xi})\big)\Big]$ is non-increasing with respect to p because of the monotonicity properties of the functions $\hat{g}$, f and F^Q for any $Q \in \{L, D\}$. The result follows from the definition of V. □

This monotonic property of the function V allows to conclude about the existence of indifference fees.

- If $V(-\infty) < V^0 < V(+\infty)$, then there exists p^* such if $p < p^*$, the insurance company has no interest to sell the contract, and if $p \geq p^*$ then the company has interest to sell the contract.
- If $V(-\infty) > V^0$, the insurance should always sell the contract.
- If $V(+\infty) < V^0$, the insurance should never sell the contract.

The asymptotic behavior of V is then studied in the following subsection for usual guarantees.

3.4 Indifference fee for usual guarantees

In this part, we consider usual guarantees and study conditions for the existence of indifference fees.

Proposition 3.3 (Ratchet guarantee)**.** *Let $m > A_0$. We assume that*

$$F(p, \hat{\xi}) \;=\; m \wedge \Big[A^p_{T\wedge\tau} \vee \Big(e^{-\int_0^{T\wedge\tau} r_s\, ds} \sum_{i=0}^{n} \max\Big(\hat{a}^p_0 - \sum_{k=1}^{i} \hat{f}(k, \hat{a}^p, \hat{\xi}), \ldots, \hat{a}^p_i\Big) \mathbb{1}_{\{t_i \leq T\wedge\tau < t_{i+1}\}}\Big)\Big]$$

for $(p, \xi) \in \mathbb{R} \times \mathcal{E}$. Then, there exists $p^* \in \mathbb{R} \cup \{-\infty\}$ such that $V(p) \geq V^0$ for all $p \geq p^*$ and $V(p) < V^0$ for all $p < p^*$.

Proof. From Proposition 3.2, we just have to show that $\lim_{p\to+\infty} V(p) \geq V^0$.
It would follow from the monotonicity of V that there exists $p^* \in \mathbb{R} \cup \{-\infty\}$ such that $V(p) \geq V^0$ for $p \geq p^*$ and $V(p) < V^0$ for $p < p^*$. First, notice that we may deduce from Assumption 2.3 that there exists a positive constant C such that, for any $t \in [0, T]$, $\mathbb{E}[A^p_t] \leq Ce^{-pt}$. Therefore, as $A^p_t \geq 0$, we get

$$\lim_{p\to+\infty} A^p_t = 0 \quad \text{a.s. for any } t \in (0, T] \,.$$

Now, set $i \in \{1, \ldots, n-1\}$, as $f_i \geq \xi_i$ we obtain that

$$\begin{aligned} \lim_{p\to+\infty} \hat{G}_i \;&\leq\; \lim_{p\to+\infty} \max\Big(A_0 - \sum_{k=1}^{i-1} \hat{\xi}_k, \hat{A}^p_{t_1} - \sum_{k=2}^{i-1} \hat{\xi}_k, \ldots, \hat{A}^p_{t_i^-}\Big) \\ &=\; A_0 - \sum_{k=1}^{i-1} \hat{\xi}_k \,. \end{aligned}$$

In the same way, we get

$$\begin{aligned}\lim_{p\to+\infty} F(p,\hat{\xi}) &\leq \lim_{p\to+\infty} m \wedge \Big[A^p_{T\wedge\tau} \vee \Big(e^{-\int_0^{T\wedge\tau} r_s\,ds} \sum_{i=0}^{n} \max(\hat{a}^p_0 - \sum_{k=1}^{i} \hat{\xi}_k, \dots, \hat{a}^p_i) \mathbb{1}_{\{t_i\leq T\wedge\tau<t_{i+1}\}}\Big)\Big] \\ &\leq e^{-\int_0^{T\wedge\tau} r_s\,ds} \sum_{i=0}^{n} \big(A_0 - \sum_{k=1}^{i} \hat{\xi}_k\big) \mathbb{1}_{\{t_i\leq T\wedge\tau<t_{i+1}\}}\,.\end{aligned}$$

We now study the limit of V at $+\infty$. For all $\pi \in \mathcal{A}[0,T]$, there exists $\xi^{\pi,*} \in \mathcal{E}$ such that

$$\begin{aligned}V(p) &\geq \inf_{\xi\in\mathcal{E}} \mathbb{E}\Big[-\exp\Big(-\gamma\Big(X^\pi_T + A_0 - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i\leq\tau} - F(p,\hat{\xi})\Big)\Big)\Big] \\ &= \mathbb{E}\Big[-\exp\Big(-\gamma\Big(X^\pi_T + A_0 - \sum_{i=1}^{n-1} \hat{\xi}^{\pi,*}_i \mathbb{1}_{t_i\leq\tau} - F(p,\hat{\xi}^{\pi,*})\Big)\Big)\Big]\,.\end{aligned}$$

The last equality follows from the fact that $\mathcal{E}$ is a finite set and $\hat{\xi}_i \geq \xi_i$. We deduce from the monotone convergence theorem that

$$\begin{aligned}\lim_{p\to+\infty} V(p) &\geq \mathbb{E}\Big[-\exp\Big(-\gamma\big(X^\pi_T + A_0 - \sum_{i=1}^{n-1} \hat{\xi}^{\pi,*}_i \mathbb{1}_{t_i\leq\tau} - \lim_{p\to+\infty} F(p,0)\big)\Big)\Big] \\ &= \mathbb{E}\Big[-\exp\Big(-\gamma\Big(X^\pi_T + \sum_{i=0}^{n} \big(A_0 - \sum_{k=1}^{i} \hat{\xi}^{\pi,*}_k\big) \mathbb{1}_{\{t_i\leq T\wedge\tau<t_{i+1}\}} (1 - e^{-\int_0^{T\wedge\tau} rs\,ds})\Big)\Big)\Big] \\ &\geq \mathbb{E}\big[-\exp(-\gamma X^\pi_T)\big]\,.\end{aligned}$$

We recall that, from Proposition 3.1, there exists $\pi^* \in \mathcal{A}[0,T]$, such that $V^0 = \mathbb{E}[-\exp(-\gamma X^{\pi^*}_T)]$. Therefore, we obtain that $\lim_{p\to+\infty} V(p) \geq V^0$ which is the expected result.

□

Proposition 3.4 (Roll-up guarantee)**.** *Let $m > A_0$ and $\eta \geq 0$. Assume that*

$$F(p,\hat{\xi}) = m \wedge \Big[A^p_{T\wedge\tau} \vee \Big(e^{-\int_0^{T\wedge\tau} r_s\,ds} \sum_{i=0}^{n} \Big(A_0(1+\eta)^i - \sum_{k=1}^{i} \hat{f}(k,\hat{a}^p,\hat{\xi})(1+\eta)^{i-k}\Big) \mathbb{1}_{\{t_i\leq T\wedge\tau<t_{i+1}\}}\Big)\Big]\,,$$

for all $(p,\xi) \in \mathbb{R}\times\mathcal{E}$. There exists $\eta_* \geq 0$ such that for all $\eta \in [0,\eta_*]$, there exists $p^* \in \mathbb{R}\cup\{-\infty\}$ such that $V(p) \geq V^0$ for all $p \geq p^*$ and $V(p) < V^0$ for all $p < p^*$.

Proof. Let $\eta \geq 0$. Following the proof of Proposition 3.3, we can prove that, for $i \in \{1,\dots,n-1\}$, we have

$$\begin{aligned}\lim_{p\to+\infty} \hat{G}_i &\leq \lim_{p\to+\infty} \hat{A}^p_{t_i^-} \vee \Big(A_0(1+\eta)^i - \sum_{k=1}^{i-1} \hat{\xi}_k (1+\eta)^{i-k}\Big) \\ &= A_0(1+\eta)^i - \sum_{k=1}^{i-1} \hat{\xi}_k (1+\eta)^{i-k}\,.\end{aligned}$$

In the same way, we get

$$\begin{aligned}\lim_{p\to+\infty} F(p,\hat{\xi}) &\leq \lim_{p\to+\infty} m \wedge \Big[A^p_{T\wedge\tau} \vee \Big(e^{-\int_0^{T\wedge\tau} r_s\,ds}\sum_{i=0}^{n}\Big(A_0(1+\eta)^i - \sum_{k=1}^{i}\hat{\xi}_k(1+\eta)^{i-k}\Big)\mathbb{1}_{\{t_i\leq T\wedge\tau<t_{i+1}\}}\Big)\Big] \\ &\leq e^{-\int_0^{T\wedge\tau} r_s\,ds}\sum_{i=0}^{n}\Big(A_0(1+\eta)^i - \sum_{k=1}^{i}\hat{\xi}_k(1+\eta)^{i-k}\Big)\mathbb{1}_{\{t_i\leq T\wedge\tau<t_{i+1}\}}\,.\end{aligned}$$

From Proposition 3.1, there exists $\pi^* \in \mathcal{A}[0,T]$, such that $V^0 = \mathbb{E}[-\exp(-\gamma X_T^{\pi^*})]$. From the fact that $\mathcal{E}$ is a finite set, we deduce that

$$\begin{aligned}V(p) &\geq \inf_{\xi\in\mathcal{E}}\mathbb{E}\Big[-\exp\Big(-\gamma\Big(X_T^{\pi^*} + A_0 - \sum_{i=1}^{n-1}\xi_i\mathbb{1}_{t_i\leq\tau} - F(p,\hat{\xi})\Big)\Big)\Big] \\ &= \mathbb{E}\Big[-\exp\Big(-\gamma\Big(X_T^{\pi^*} + A_0 - \sum_{i=1}^{n-1}\xi_i^*\mathbb{1}_{t_i\leq\tau} - F(p,\hat{\xi}^*)\Big)\Big)\Big]\,.\end{aligned}$$

It follows from the monotone convergence theorem that

$$\begin{aligned}\lim_{p\to+\infty} V(p) &\geq \mathbb{E}\Big[-\exp\big(-\gamma\big(X_T^{\pi^*} + A_0 - \sum_{i=1}^{n-1}\xi_i\mathbb{1}_{t_i\leq\tau} - \lim_{p\to+\infty}F(p,0)\big)\big)\Big] \\ &\geq \Phi(\eta)\,,\end{aligned}$$

where we have set

$$\Phi(\eta) = \sum_{i=0}^{n}\mathbb{E}\Big[-\exp\big(-\gamma(X_T^{\pi^*} + \Phi_i(\eta))\big)\mathbb{1}_{\{t_i\leq T\wedge\tau<t_{i+1}\}}\Big]\,,$$

with, for $i\in\{1,\dots,n\}$,

$$\Phi_i(\eta) := A_0\Big(1 - e^{-\int_0^{T\wedge\tau} r_s\,ds}(1+\eta)^i\Big) - \sum_{k=1}^{i}\xi_k\Big(e^{\int_0^{t_k} r_s\,ds} - e^{-\int_0^{T\wedge\tau} r_s\,ds}(1+\eta)^{i-k}\Big)\,.$$

Obviously, Φ is continuous and non-increasing on $\mathbb{R}_+$. Moreover, we have

$$\Phi(0) \geq \mathbb{E}\big[-\exp(-\gamma X_T^{\pi^*})\big] = V^0 \qquad \text{and} \qquad \lim_{\eta\to+\infty}\Phi(\eta) = -\infty\,.$$

From the mean value theorem, we may define $\eta_* \geq 0$ as

$$\eta_* := \sup\{\eta\geq 0:\ \Phi(\eta) = V^0\}\,.$$

We conclude the proof by noticing that for $0\leq\eta\leq\eta_*$, we have

$$\lim_{p\to+\infty} V(p) \geq V^0\,.$$

□

4 Min-Max optimization problem

In this section we study the optimization problem (3.8). We determine the value function w using a sequential utility maximization.
In the sequel we use the following notations. For $x \in \mathbb{R}^n$ and $1 \leq k \leq n$ we denote by $x^{(k)}$ the vector of $\mathbb{R}^k$ defined by

$$x^{(k)} \quad := \quad (x_1, \ldots, x_k) \, .$$

For $y \in \mathbb{R}^k$ we denote by $\hat{y}$ the vector

$$\hat{y} \quad := \quad \big(y_1 e^{\int_0^{t_1} r_s ds}, \ldots, y_k e^{\int_0^{t_k} r_s ds}\big) \, .$$

4.1 Sequential utility maximization

The problem (3.8) is not classical for the following two reasons

- the terminal wealth is $\mathcal{G}_T$-measurable and the pay-off is $\mathcal{G}_{T\wedge\tau}$-measurable,
- there are a maximization w.r.t. the withdrawals ξ and a minimization w.r.t. the investment strategies π.

We first modify the problem to get a $\mathcal{G}_{T\wedge\tau}$-measurable wealth and a $\mathcal{G}_{T\wedge\tau}$-measurable pay-off.

Proposition 4.5 (Initialization)**.** *We have*

$$w(p) \quad = \quad \inf_{\pi \in \mathcal{A}[0,T\wedge\tau]} \sup_{\xi \in \mathcal{E}} \mathbb{E}\Big[u\big(X^{\pi}_{T\wedge\tau} - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - H(p, \hat{\xi})\big)\Big] \, ,$$

with

$$H(p,\hat{\xi}) \quad := \quad F(p,\hat{\xi}) + \frac{1}{\gamma} \log \Big[\operatorname*{ess\,inf}_{\pi \in \mathcal{A}[T\wedge\tau,T]} \mathbb{E}\big[u\big(X_T^{T\wedge\tau,\pi}\big)\big|\mathcal{G}_{T\wedge\tau}\big]\Big] \, ,$$

for any $p \in \mathbb{R}$.

Proof. We first prove the following inequality

$$w(p) \quad \geq \quad \inf_{\pi \in \mathcal{A}[0,T\wedge\tau]} \sup_{\xi \in \mathcal{E}} \mathbb{E}\Big[u\big(X^{\pi}_{T\wedge\tau} - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - H(p, \hat{\xi})\big)\Big] \, . \tag{4.10}$$

For any $\pi \in \mathcal{A}[0,T]$ and $\xi \in \mathcal{E}$, it follows from the fact that $u(x+y) = u(x)u(y)$ for any $x \in \mathbb{R}$ and $y \in \mathbb{R}$ that

$$\begin{aligned}
\mathbb{E}\Big[u\big(X^{\pi}_{T} - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - F(p, \hat{\xi})\big)\Big] \quad &= \quad \mathbb{E}\Big[u\big(X^{\pi}_{T\wedge\tau} - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - F(p, \hat{\xi})\big)\mathbb{E}\big[u(X_T^{T\wedge\tau,\pi})|\mathcal{G}_{T\wedge\tau}\big]\Big] \\
&\geq \quad \mathbb{E}\Big[u\big(X^{\pi}_{T\wedge\tau} - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - H(p, \hat{\xi})\big)\Big] \, .
\end{aligned}$$

Therefore, we can see that the inequality (4.10) holds.
We now prove the following inequality

$$w(p) \leq \inf_{\pi\in\mathcal{A}[0,T\wedge\tau]} \sup_{\xi\in\mathcal{E}} \mathbb{E}\Big[u\big(X^{\pi}_{T\wedge\tau} - \sum_{i=1}^{n-1}\xi_i \mathbb{1}_{t_i\leq\tau} - H(p,\hat{\xi})\big)\Big] .$$

From Lemma A.1 we know that there exists $\pi^{*,\tau} \in \mathcal{A}[T\wedge\tau, T]$ such that

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[T\wedge\tau,T]} \mathbb{E}\big[u(X_T^{T\wedge\tau,\pi})|\mathcal{G}_{T\wedge\tau}\big] \;=\; \mathbb{E}\big[u(X_T^{T\wedge\tau,\pi^{*,\tau}})|\mathcal{G}_{T\wedge\tau}\big] .$$

Then, we consider the subset $\mathcal{A}^*[0,T]$ of $\mathcal{A}[0,T]$ defined by $\mathcal{A}^*[0,T] := \{\pi_t \mathbb{1}_{t\leq T\wedge\tau} + \pi_t^{*,\tau}\mathbb{1}_{t>T\wedge\tau},\ \pi \in \mathcal{A}[0, T\wedge\tau]\}$. Since $\mathcal{A}^*[0,T] \subset \mathcal{A}[0,T]$, we get

$$\begin{aligned} w(p) &\leq \inf_{\pi\in\mathcal{A}^*[0,T]} \sup_{\xi\in\mathcal{E}} \mathbb{E}\Big[u\big(X^{\pi}_{T} - \sum_{i=1}^{n-1}\xi_i \mathbb{1}_{t_i\leq\tau} - F(p,\hat{\xi})\big)\Big] \\ &\leq \inf_{\pi\in\mathcal{A}[0,T\wedge\tau]} \sup_{\xi\in\mathcal{E}} \mathbb{E}\Big[u\big(X^{\pi}_{T\wedge\tau} - \sum_{i=1}^{n-1}\xi_i \mathbb{1}_{t_i\leq\tau} - H(p,\hat{\xi})\big)\Big] . \end{aligned}$$

Hence, we get the equality. □

Remark 4.3. *If F is a bounded random variable then H is also bounded since, from Lemma A.1, we have*

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[T\wedge\tau,T]} \mathbb{E}\big[u\big(X_T^{T\wedge\tau,\pi}\big)\big|\mathcal{G}_{T\wedge\tau}\big] = \exp\left(\gamma Y^{(n)}_{T\wedge\tau}\right) ,$$

and we know that $Y^{(n)}$ is bounded.

We now decompose the initial problem in n subproblems.

Theorem 4.1. *The value function w is given by*

$$w(p) \;=\; \inf_{\pi\in\mathcal{A}[0,t_1\wedge\tau]} \mathbb{E}\big[u\big(X^{\pi}_{t_1\wedge\tau}\big)v(1)\big] ,$$

where

- *$v(i,\xi^{(i-1)})$ is defined recursively for any $i \in \{2,\dots,n\}$ and $\xi \in \mathcal{E}$ by*

$$\begin{cases} v(n,\xi^{(n-1)}) &:= e^{\gamma H(p,\hat{\xi}^{(n-1)})} , \\ v(i,\xi^{(i-1)}) &:= \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_{i-1}} \operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_i\wedge\tau,t_{i+1}\wedge\tau]} J(i,\pi,\xi^{(i-1)},\zeta) , \end{cases}$$

  with for any $i \in \{1,\dots,n-1\}$, $\pi \in \mathcal{A}[t_i\wedge\tau, t_{i+1}\wedge\tau]$ and $\zeta \in \mathcal{E}^1_{i-1}$

$$J(i,\pi,\xi^{(i-1)},\zeta) \;:=\; \mathbb{E}\Big[u\Big(X^{t_i\wedge\tau,\pi}_{t_{i+1}\wedge\tau} - \zeta\mathbb{1}_{t_i<\tau}\Big)v\big(i+1,(\xi^{(i-1)},\zeta)\big)\Big|\mathcal{G}_{t_i\wedge\tau}\Big] ,$$

- *$v(1) := \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_0} \operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_1\wedge\tau,t_2\wedge\tau]} \mathbb{E}\Big[u\Big(X^{t_1\wedge\tau,\pi}_{t_2\wedge\tau} - \zeta\mathbb{1}_{t_1<\tau}\Big)v\big(2,\zeta\big)\Big|\mathcal{G}_{t_1\wedge\tau}\Big]$.*

The proof of this theorem is postponed to Subsection A.3 of the Appendix.

In the sequel, by abuse of notation, we write $v(1,\xi^0)$ for $v(1)$.

4.2 Optimal investment and worst withdrawals for the insurer

In the following result, we provide withdrawals ξ_i^* and investment strategies $\pi^{*,i}$ that attain the value functions $v(i,.)$. From Theorem 4.1, they correspond to the optimal investment strategy and the worst withdrawals for the insurer.

Proposition 4.6. *For any $i \in \{0,\dots,n-1\}$ there exists a strategy $\pi^{*,i} \in \mathcal{A}[t_i \wedge \tau, t_{i+1} \wedge \tau]$, a withdraw $\xi_i^* \in \mathcal{E}_{i-1}^1$ and a map $y^{(i),*}$ from $\hat{\mathcal{W}}^{i-1}$ to $L^\infty(\Omega, \mathcal{G}_{t_i\wedge\tau}, \mathbb{P})$ such that*

$$\begin{aligned} v(i,\xi^{(i-1)}) &= \mathbb{E}\Big[u\big(X_{t_{i+1}\wedge\tau}^{t_i\wedge\tau,\pi^{*,i}} - \xi_i^* \mathbb{1}_{t_i<\tau}\big) v\big(i+1,(\xi^{(i-1)},\xi_i^*)\big)\Big|\mathcal{G}_{t_i\wedge\tau}\Big] \\ &= \exp\big(\gamma y^{(i),*}(\hat{\xi}^{(i-1)})\big)\ , \end{aligned}$$

with $y^{(n)} = H$. Moreover the value function v of the initial problem (3.8) is given by

$$w(p) = \exp(y^{(0)})\ .$$

Proof. We prove by backward induction on $i \in \{1,\dots,n-2\}$ that

- the map H^i defined on $\hat{\mathcal{W}}^{i-1}$ by

$$H^i(x_1,\dots,x_{i-1}) = v\Big(i,\big(x_1 e^{\int_0^{t_1} r_s ds},\dots,x_i e^{\int_0^{t_{i-1}} r_s ds}\big)\Big)\ ,\quad x^{(i-1)} \in \hat{\mathcal{W}}^{i-1}\ ,$$

  is valued in $L^\infty(\Omega, \mathcal{G}_{t_i}\wedge\tau, \mathbb{P})$,

- there exists a strategy $\pi^{*,i} \in \mathcal{A}[t_i\wedge\tau, t_{i+1}\wedge\tau]$, a withdraw $\xi_i^* \in \mathcal{E}_{i-1}^1$ and a map $y^{(i),*}$ from $\hat{\mathcal{W}}^{i-1}$ to $L^\infty(\Omega,\mathcal{G}_{t_i\wedge\tau},\mathbb{P})$ such that

$$\begin{aligned} v(i,\xi^{(i-1)}) &= \mathbb{E}\Big[u\big(X_{t_{i+1}\wedge\tau}^{t_i\wedge\tau,\pi^{*,i}} - \xi_i^* \mathbb{1}_{t_i<\tau}\big) v\big(i+1,(\xi^{(i-1)},\xi_i^*)\big)\Big|\mathcal{G}_{t_i\wedge\tau}\Big] \\ &= \exp\big(\gamma y^{(i),*}(\hat{\xi}^{(i-1)})\big)\ . \end{aligned}$$

Fix $i = n-1$. Since H is valued in $L^\infty(\Omega,\mathcal{G}_{t_n}\wedge\tau,\mathbb{P})$, we can apply Lemma A.3 and we get a strategy $\pi^{*,n-1}\big(\xi^{(n-2)},\zeta\big) \in \mathcal{A}[t_{n-1}\wedge\tau, t_n\wedge\tau]$ and a map $y^{(n-1)}$ from $\hat{\mathcal{W}}^{n-1}$ to $L^\infty(\Omega,\mathcal{G}_{t_i\wedge\tau},\mathbb{P})$ such that

$$\begin{aligned} v(n-1,\xi^{(n-2)}) &= \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}_{n-2}^1} \mathbb{E}\Big[u\Big(X_{t_n\wedge\tau}^{t_{n-1}\wedge\tau,\pi^{*,n-1}(\xi^{(n-2)},\zeta)} - \zeta\mathbb{1}_{t_{n-1}<\tau}\Big) v\big(n,(\xi^{(n-2)},\zeta)\big)\Big|\mathcal{G}_{t_{n-1}\wedge\tau}\Big] \\ &= \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}_{n-2}^1} \exp\big(\gamma y^{(n-1)}(\hat{\xi}^{(n-2)},\zeta)\big)\ . \end{aligned}$$

We can then apply Lemma A.4 and we get a withdraw $\xi_{n-1}^* \in \mathcal{E}_{n-2}^1$ such that

$$v(n-1,\xi^{(n-2)}) = \exp\big(\gamma y^{(n-1)}(\hat{\xi}^{(i-2)},\xi_{n-1}^*)\big)\ .$$

Then $\xi_{n-1}^* \in \mathcal{E}_{n-2}^1$, $\pi^{*,n-1}(\xi^{(n-2)},\xi_{n-1}^*) \in \mathcal{A}[\tau\wedge t_{n-1}, \tau\wedge t_n]$ and the map $y^{(n-1)}$ defined by

$$y^{(n-1),*}(.) = y^{(n-1)}\big(.,\xi_{n-1}^* e^{\int_0^{t_{n-1}} r_s ds}\big)$$

satisfy the conditions. Moreover, since $y^{(n-1),*}$ is uniformly bounded, we get the same property for H^{n-1}.

We now suppose that the result holds for some $i \in \{2, \dots, n-1\}$ and we prove it for $i-1$. By definition we have

$$\begin{aligned} & v(i-1, \xi^{(i-2)}) \\ & \quad = \operatorname*{ess\,sup}_{\zeta \in \mathcal{E}_{i-2}^1} \operatorname*{ess\,inf}_{\pi \in \mathcal{A}[t_{i-1}\wedge\tau, t_i\wedge\tau]} \mathbb{E}\Big[u\Big(X_{t_i\wedge\tau}^{t_{i-1}\wedge\tau,\pi} - \zeta \mathbb{1}_{t_{i-1}<\tau}\Big) v\big(i, (\xi^{(i-2)}, \zeta)\big)\Big|\mathcal{G}_{t_{i-1}\wedge\tau}\Big] \\ & \quad = \operatorname*{ess\,sup}_{\zeta \in \mathcal{E}_{i-2}^1} \operatorname*{ess\,inf}_{\pi \in \mathcal{A}[t_{i-1}\wedge\tau, t_i\wedge\tau]} \mathbb{E}\Big[u\Big(X_{t_i\wedge\tau}^{t_{i-1}\wedge\tau,\pi} - \zeta \mathbb{1}_{t_{i-1}<\tau}\Big) H^i\big(\hat{\xi}^{(i-2)}, \zeta e^{\int_0^{t_{i-1}} r_s ds}\big)\Big|\mathcal{G}_{t_{i-1}\wedge\tau}\Big] \, . \end{aligned}$$

By the induction hypothesis H^i is valued in $L^\infty(\Omega, \mathcal{G}_{t_i\wedge\tau}, \mathbb{P})$. We can therefore apply Lemma A.3 and we get a strategy $\pi^{*,i-1}\big(\xi^{(i-2)}, \zeta\big) \in \mathcal{A}[t_{i-1}\wedge\tau, t_i\wedge\tau]$ and a map $y^{(i-1)}$ from $\hat{\mathcal{W}}^{i-1}$ to $L^\infty(\Omega, \mathcal{G}_{t_i\wedge\tau}, \mathbb{P})$ such that

$$\begin{aligned} v(i-1, \xi^{(i-2)}) &= \operatorname*{ess\,sup}_{\zeta \in \mathcal{E}_{i-2}^1} \mathbb{E}\Big[u\Big(X_{t_i\wedge\tau}^{t_{i-1}\wedge\tau,\pi^{*,i}(\xi^{(i-2)},\zeta)} - \zeta \mathbb{1}_{t_{i-1}<\tau}\Big) v\big(i, (\xi^{(i-2)}, \zeta)\big)\Big|\mathcal{G}_{t_{i-1}\wedge\tau}\Big] \\ &= \operatorname*{ess\,sup}_{\zeta \in \mathcal{E}_{i-2}^1} \exp\big(\gamma y^{(i-1)}(\hat{\xi}^{(i-2)}, \zeta)\big) \, . \end{aligned}$$

We can then apply Lemma A.4 and we get a withdraw $\xi_{i-1}^* \in \mathcal{E}_{i-2}^1$ such that

$$v(i-1, \xi^{(i-2)}) = \mathbb{E}\Big[u\Big(X_{t_i\wedge\tau}^{t_{i-1}\wedge\tau,\pi} - \xi_{i-1}^* \mathbb{1}_{t_{i-1}<\tau}\Big) v\big(i, (\xi^{(i-2)}, \xi_{i-1}^*)\big)\Big|\mathcal{G}_{t_{i-1}\wedge\tau}\Big] \, .$$

Then $\xi_{i-1}^* \in \mathcal{E}_{i-2}^1$, $\pi^{*,i-1}(\xi^{(i-2)}, \xi_{i-1}^*) \in \mathcal{A}[\tau \wedge t_{i-1}, \tau \wedge t_i]$ and the map $y^{(i-1)}$ defined by

$$y^{(i-1),*}(.) = y^{(i-1)}\big(., \xi_{i-1}^* e^{\int_0^{t_{i-1}} r_s ds}\big)$$

satisfy the conditions. Moreover, since $y^{(i-1),*}$ is uniformly bounded, we get the same property for H^{i-1}. □

5 Numerical resolution

5.1 Approximation procedure

Max-min problem. We first propose a scheme to solve the problem (3.7) by using Theorem 4.1. We describe the procedure in an induction way. We present the step 0 which corresponds to the initialization given by Proposition 4.5. Then the step i corresponds to the computation of the function $v(n-i, .)$ and the optimal strategy $\pi^{*,n-i}$ and the worst withdrawal ξ_{n-i}^* once the previous steps have been done.

Step 0: We solve the following problem

$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}[T\wedge\tau, T]} \mathbb{E}\big[u\big(X_T^{T\wedge\tau,\pi}\big)\big|\mathcal{G}_{T\wedge\tau}\big] \, .$$

From Lemma A.1 we know that

$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}[T\wedge\tau, T]} \mathbb{E}\big[u\big(X_T^{T\wedge\tau,\pi}\big)\big|\mathcal{G}_{T\wedge\tau}\big] = \exp(\gamma Y_{T\wedge\tau}^{(n)}) \, ,$$

where $Y^{(n)}$ is the solution to the linear BSDE

$$\left\{ \begin{array}{rcl} dY_t^{(n)} & = & \Big[\frac{|\theta_t|^2}{\gamma} + \theta_t Z_t^{(n)}\Big]dt + Z_t^{(n)} dW_t \ , \\ Y_T^{(n)} & = & 0 \ . \end{array} \right.$$

Therefore, we have

$$Y_{T\wedge\tau}^{(n)} \ = \ \mathbb{E}_{\mathbb{Q}}\Big[- \int_{T\wedge\tau}^{T} \frac{|\theta_t|^2}{\gamma} dt \Big| \mathcal{G}_{T\wedge\tau}\Big] \ ,$$

with $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{G}_t} = \mathcal{E}(-\int_0^\cdot \theta_s dW_s)_t$. This conditional expectation can be approximated by regression methods.

Step 1: We solve the following problem

$$\begin{array}{l} v(n-1, \xi^{(n-2)}) = \\ \underset{\zeta\in\mathcal{E}_{n-2}^1}{\operatorname{ess\,sup}} \underset{\pi\in\mathcal{A}[t_{n-1}\wedge\tau, t_n\wedge\tau]}{\operatorname{ess\,inf}} \mathbb{E}\Big[u\Big(X_{t_n\wedge\tau}^{t_{n-1}\wedge\tau,\pi} - \zeta \mathbb{1}_{t_{n-1}<\tau} - F(p, (\hat{\xi}^{(n-2)}, \hat{\zeta})) - Y_{T\wedge\tau}^{(n)}\Big)\Big| \mathcal{G}_{t_{n-1}\wedge\tau}\Big] \ . \end{array}$$

For that we first solve the infimum problem, and we know from Lemma A.3 that there exists a r.v. $y^{(n-1)}\big(\xi^{(n-2)}, \zeta\big)$ such that

$$\begin{array}{l} \underset{\pi\in\mathcal{A}[t_{n-1}\wedge\tau, t_n\wedge\tau]}{\operatorname{ess\,inf}} \mathbb{E}\Big[u\Big(X_{t_n\wedge\tau}^{t_{n-1}\wedge\tau,\pi} - \zeta \mathbb{1}_{t_{n-1}<\tau} - F(p, (\hat{\xi}^{(n-2)}, \hat{\zeta})) - Y_{T\wedge\tau}^{(n)}\Big)\Big| \mathcal{G}_{t_{n-1}\wedge\tau}\Big] \\ \qquad\qquad = \exp\big(\gamma y^{(n-1)}\big(\xi^{(n-2)}, \zeta\big)\big) \ . \end{array}$$

The r.v. $y^{(n-1)}\big(\xi^{(n-2)}, \zeta\big)$ is defined by $\zeta \mathbb{1}_{t_{n-1}<\tau} + Y_{t_{n-1}}^{(n-1)}\big(\xi^{(n-2)}, \zeta\big)$ where $Y^{(n-1)}\big(\xi^{(n-2)}, \zeta\big)$ is solution to the BSDE

$$\left\{ \begin{array}{rcl} -dY_t^{(n-1)} & = & \Big(\lambda_t \frac{e^{\gamma U_t^{(n-1)}} - 1}{\gamma} - \theta_t Z_t^{(n-1)} - \frac{|\theta_t|^2}{2\gamma}\Big)dt - Z_t^{(n-1)} dB_t - U_t^{(n-1)} dN_t \ , \\ Y_{T\wedge\tau}^{(n-1)} & = & F(p, (\hat{\xi}^{(n-2)}, \hat{\zeta})) + Y_{T\wedge\tau}^{(n)} \ . \end{array} \right.$$

We also get the optimal investment strategy $\pi^{*,n-1}$ by the formula

$$\pi^{*,n-1} \ = \ \frac{1}{\sigma}\Big[\frac{\theta}{\gamma} + Z^{(n-1)}\Big] \ .$$

We can now find $\zeta_{n-1}^* \in \mathcal{E}_{n-2}^1$ such that

$$\underset{\zeta\in\mathcal{E}_{n-2}^1}{\operatorname{ess\,sup}}\, y^{(n-1)}\big(\xi^{(n-2)}, \zeta\big) \ = \ y^{(n-1)}\big(\xi^{(n-2)}, \zeta_{n-1}^*\big) \ .$$

Step i: We now compute

$$\begin{array}{l} v(n-i, \xi^{(n-i-1)}) \\ = \underset{\zeta\in\mathcal{E}_{n-i-1}^1}{\operatorname{ess\,sup}} \underset{\pi\in\mathcal{A}[t_{n-i}\wedge\tau, t_{n-i+1}\wedge\tau]}{\operatorname{ess\,inf}} \mathbb{E}\Big[u\Big(X_{t_{n-i+1}\wedge\tau}^{t_{n-i}\wedge\tau,\pi} - \zeta \mathbb{1}_{t_{n-i}<\tau} - y^{(n-i+1)}(\xi^{(n-i-1)}, \zeta, \zeta_{n-i+1}^*)\Big)\Big| \mathcal{G}_{t_{n-i}\wedge\tau}\Big] \ . \end{array}$$

As previously, the infimum problem is solved using Lemma A.3. We have just adapted the terminal condition in the BSDE. Let $Y^{(n-i)}\big(\xi^{(n-i-1)},\zeta\big)$ is solution to the BSDE

$$\left\{\begin{array}{rcl} -dY_t^{(n-i)} & = & \Big(\lambda_t\frac{e^{\gamma U_t^{(n-i)}}-1}{\gamma}-\theta_t Z_t^{(n-i)}-\frac{|\theta_t|^2}{2\gamma}\Big)dt - Z_t^{(n-i)}dB_t - U_t^{(n-i)}dN_t\,, \\ Y_{t_{n-i}\wedge\tau}^{(n-i)} & = & y^{(n-i+1)}(\xi^{(n-i-1)},\zeta,\zeta^*_{n-i+1})\,. \end{array}\right.$$

Then the r.v. $y^{(n-i)}\big(\xi^{(n-i-1)},\zeta\big)$ defined by $\zeta\mathbb{1}_{t_{n-i}<\tau}+Y_{t_{n-i}}^{(n-i)}\big(\xi^{(n-i-1)},\zeta\big)$ satisfies

$$\begin{aligned}\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_{n-i}\wedge\tau,t_{n-i+1}\wedge\tau]}\mathbb{E}\Big[u\Big(X^{t_{n-i}\wedge\tau,\pi}_{t_{n-i+1}\wedge\tau}-\zeta\mathbb{1}_{t_{n-i}<\tau}-y^{(n-i+1)}(\xi^{(n-i-1)},\zeta,\zeta^*_{n-i+1})\Big)\Big|\mathcal{G}_{t_{n-i}\wedge\tau}\Big]&\\ =\exp\big(\gamma y^{(n-i)}\big(\xi^{(n-i-1)},\zeta\big)\big)\,.&\end{aligned}$$

We get the optimal investment strategy for the insurer by the formula

$$\pi^{*,n-i} \;=\; \frac{1}{\sigma}\Big[\frac{\theta}{\gamma}+Z^{(n-i)}\Big]\,.$$

Finally we get the worst withdrawal $\zeta^*_{n-i}\in\mathcal{E}^1_{n-i-1}$ such that

$$\operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_{n-i-1}} y^{(n-i)}\big(\xi^{(n-i-1)},\zeta\big) \;=\; y^{(n-i)}\big(\xi^{(n-i-1)},\zeta^*_{n-i}\big)\,.$$

Step n: We finish by solving the optimisation problem

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_0,t_1\wedge\tau]}\mathbb{E}\Big[u\Big(X^{t_0,\pi}_{t_1\wedge\tau}-y^{(1)}(\zeta^*_1)\Big)\Big]\,.$$

From Lemma A.3, there exists a r.v. $y^{(0)}$ such that

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_0,t_1\wedge\tau]}\mathbb{E}\Big[u\Big(X^{t_0,\pi}_{t_1\wedge\tau}-y^{(1)}(\zeta^*_1)\Big)\Big] \;=\; \exp\big(\gamma y^{(0)}\big)$$

where $y^{(0)}$ is the value at time 0 of the solution to the BSDE

$$\left\{\begin{array}{rcl} -dY_t^{(0)} & = & \Big(\lambda_t\frac{e^{\gamma U_t^{(0)}}-1}{\gamma}-\theta_t Z_t^{(0)}-\frac{|\theta_t|^2}{2\gamma}\Big)dt - Z_t^{(0)}dB_t - U_t^{(0)}dN_t\,, \\ Y_{t_1\wedge\tau}^{(0)} & = & y^{(1)}(\zeta^*_1)\,. \end{array}\right.$$

We get the optimal investment strategy for the insurer by the formula

$$\pi^{*,0} \;=\; \frac{1}{\sigma}\Big[\frac{\theta}{\gamma}+Z^{(0)}\Big]\,.$$

The value function associated to the optimization problem (3.7) is given by

$$-\exp(\gamma(y^{(0)}-A_0))\,,$$

and the worst withdrawals for the insurer are given by $(\zeta^*_1,\dots,\zeta^*_{n-1})$.

We can compute $(Y^{(n-i)},Z^{(n-i)},U^{(n-i)})$ for any $1\le i\le n$ by using discretization methods for BSDEs (see for example [7]).

Indifference price. We now know how to calculate V^0 and $V(p)$ for any $p \in \mathbb{R}$. The next step is to calculate the indifference fee rate p^* which is defined by

$$p^* \quad = \quad \inf\{p \in \mathbb{R}\ ,\ V_0 \leq V(p)\}\ .$$

From the previous results, we can rewrite this problem as follows

$$p^* \quad = \quad \inf\{p \in \mathbb{R}\ ,\ \ -\exp(\gamma y_0) \leq -\exp(\gamma(y^{(0)}(p) - A_0)\}\ ,$$

where $y^{(0)}(p)$ is defined by the step n.
Therefore p^* is defined by

$$p^* \quad = \quad \inf\{p \in \mathbb{R}\ ,\ y^{(0)}(p) \leq y_0 + A_0\}\ .$$

Then by dichotomy method we can approximate p^*.

5.2 Simulations

In this section we present numerical illustrations of parameters sensibility for indifference fee rates. We compute solutions for both optimization problems: V^0, the utility maximization problem without variable annuities, and $V^{(0)}(p)$, the utility maximization problem with variable annuities. We simulate the BSDEs involved, using the discretization scheme studied in [5]. For the computation of the conditional expectations, we use non-parametric regression method with the Gaussian function as kernel. Following a dichotomy method, we find p^* such that

$$p^* \quad = \quad \inf\{p \in \mathbb{R}\ ,\ V_0 \leq V(p)\}\ .$$

We consider that the insured can withdraw only every ten years. We shall give the following numerical values to parameters

$$\gamma = 1.3,\ T = 30,\ A_0 = 1,\ r_0 = 0.02,\ \mu_0 = 0.15,\ \sigma = 0.3\ .$$

We describe the dependence with respect to the market parameters: the initial interest rate, the initial drift and the volatility.

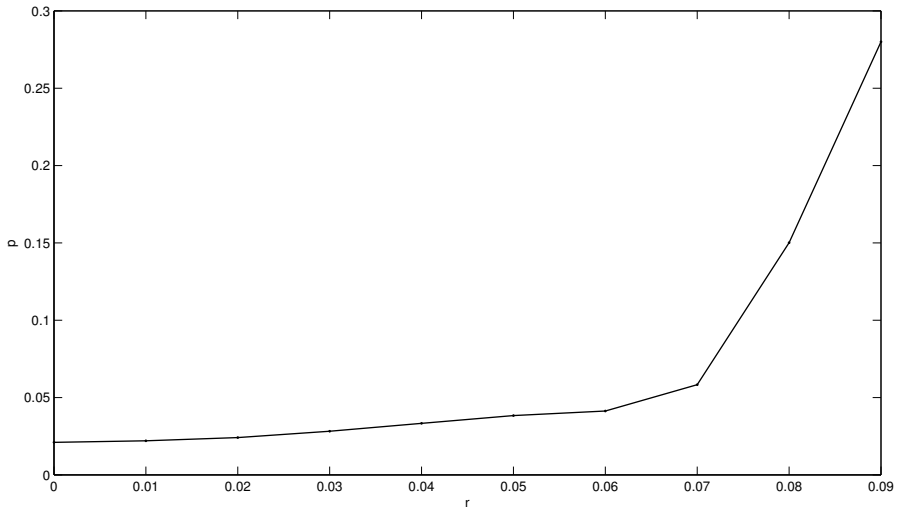

Figure 1: Indifference fee rate w.r.t. to r

We notice that indifference fee rates increase with interest rate. This is due to the guarantee structure of the product: a growth of interest rate will lead to a growth of the quantity $V^{(0)}(p)$ with respect to V^0 and to compensate this growth we will have to increase p, as $p \mapsto V^{(0)}(p)$ is non-increasing.

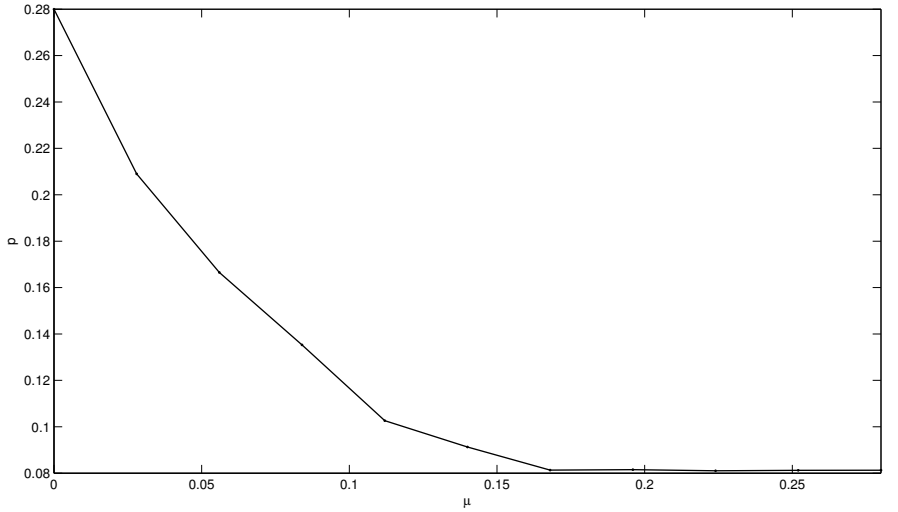

Figure 2: Indifference fee rate w.r.t. to μ

Notice that indifference fee rates decrease with respect to the drift. The bigger is the drift the less usefull are the guarantees, then the fees payed to get these guarantees have to decrease.

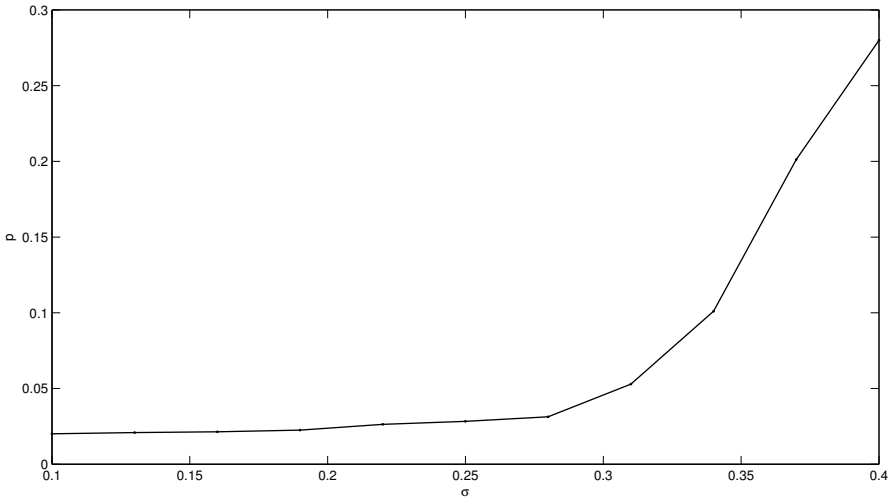

Figure 3: Indifference fee rate w.r.t. to σ

Once again, we can get a financial interpretation of the monotonicity of the fees w.r.t. market volatility. The bigger is the volatility the more usefull are the guarantees, then the fees payed to get these guarantees have to increase.

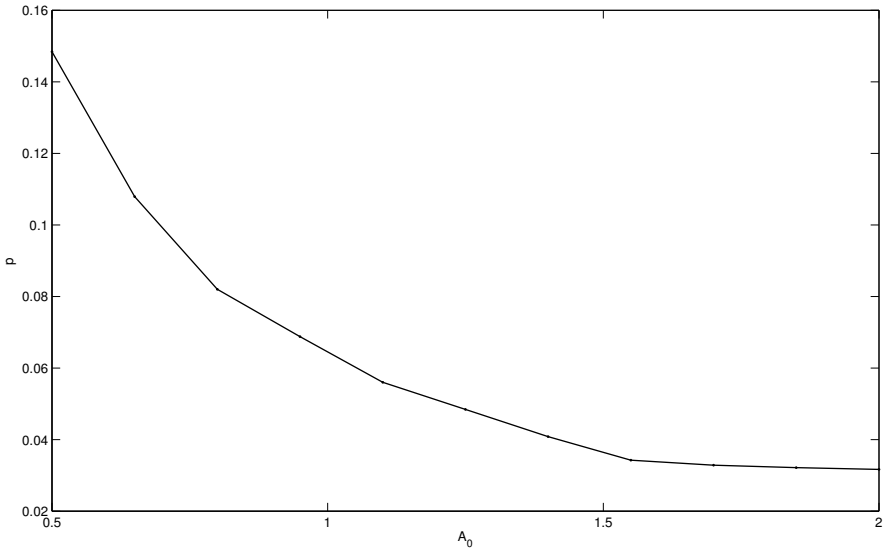

Figure 4: Indifference fee rate w.r.t. to A_0

A Appendix

A.1 Preliminary results

Lemma A.1. *There exists a strategy $\pi^{*,n} \in \mathcal{A}[T\wedge\tau, T]$ such that*

$$\inf_{\pi\in\mathcal{A}[T\wedge\tau,T]} \mathbb{E}\big[\exp(-\gamma X_T^{T\wedge\tau,\pi})|\mathcal{G}_{T\wedge\tau}\big] \;=\; \mathbb{E}\big[\exp(-\gamma X_T^{T\wedge\tau,\pi^{*,n}})|\mathcal{G}_{T\wedge\tau}\big]\,.$$

Moreover, there exists a process $Y^{(n)}$ such that

$$\inf_{\pi\in\mathcal{A}[T\wedge\tau,T]} \mathbb{E}\big[\exp(-\gamma X_T^{T\wedge\tau,\pi})|\mathcal{G}_{T\wedge\tau}\big] \;=\; \exp(\gamma Y^{(n)}_{T\wedge\tau})\,,$$

where $(Y^{(n)}, Z^{(n)})$ is solution of the BSDE

$$\begin{cases} dY_t^{(n)} &= \Big[\frac{|\theta_t|^2}{\gamma} + \theta_t Z_t^{(n)}\Big]dt + Z_t^{(n)}dW_t\,, \\ Y_T^{(n)} &= 0\,. \end{cases}$$

The optimal strategy $\pi^{*,n}$ is given by

$$\pi_t^{*,n} \;=\; \frac{1}{\sigma_t}\Big[\frac{\theta_t}{\gamma} + Z_t^{(n)}\Big]\,.$$

Proof. The proof of this lemma is similar to the proof of Theorem 7 in [10] therefore we only give a sketch of the proof. We look for a process $Y^{(n)}$ such that the family of processes $\{J^{(n)}(\pi), \pi \in \mathcal{A}[T\wedge\tau, T]\}$ defined for any $\pi \in \mathcal{A}[T\wedge\tau, T]$ by

$$J_t^{(n)}(\pi) \;:=\; \exp\big(-\gamma(X_t^{T\wedge\tau,\pi} - Y_t^{(n)})\big)\,, \quad \forall t \in [T\wedge\tau, T]\,,$$

satisfies the following conditions

(i) $J^{(n)}_{T\wedge\tau}(\pi)$ is a random variable $\mathcal{G}_{T\wedge\tau}$-measurable and independent of π.

(ii) $J_T^{(n)}(\pi) = \exp(-\gamma X_T^{T\wedge\tau,\pi})$.

(iii) $J^{(n)}(\pi)$ is a submartingale for any $\pi \in \mathcal{A}[T\wedge\tau, T]$ on the time interval $[T\wedge\tau, T]$.

(iv) There exists a strategy $\pi^{*,n} \in \mathcal{A}[T\wedge\tau, T]$ such that $J^{(n)}(\pi^{*,n})$ is a martingale on the time interval $[T\wedge\tau, T]$.

The process $Y^{(n)}$ is looked under the form

$$\begin{cases} -dY_t^{(n)} &= f(t, Y_t^{(n)}, Z_t^{(n)})dt - Z_t^{(n)}dW_t\,, \\ Y_T^{(n)} &= 0\,, \end{cases} \tag{A.1}$$

and we are bounded to choose the function f for which the family $\{J^{(n)}(\pi), \mathcal{A}[T\wedge\tau, T]\}$ satisfies the previous conditions. Classically we obtain that the function f is defined by

$$f(t, y, z) \;=\; -\frac{|\theta_t|^2}{\gamma} - \theta_t z\,,$$

and the candidate to be $\pi^{*,n}$ is given by

$$\pi_t^{*,n} \;=\; \frac{1}{\hat{\sigma}_t}\Big[\frac{\theta_t}{\gamma} + Z_t^{(n)}\Big]\,.$$

Since the generator f is Lipschitz, we know that there exists a unique solution $(Y^{(n)}, Z^{(n)})$ to the BSDE (A.1). To finish the proof we must prove that the family $\{J^{(n)}(\pi), \mathcal{A}[T\wedge\tau, T]\}$ satisfies the previous conditions, but it is identical to the proof given in [10]. □

Lemma A.2. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables valued in $\hat{\mathcal{W}}$. Then there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ and a random variable X_∞ such that*

$$X_{n_k} \;\longrightarrow\; X_\infty \quad \textit{as} \;\; k\to+\infty \;\; \mathbb{P}-a.s.$$

Proof. We first notice that since $\hat{\mathcal{W}}$ is finite, the set $\hat{\mathcal{W}}^{\mathbb{N}}$ is countable. For a sequence $w=(w_n)_{n\in\mathbb{N}}\in\hat{\mathcal{W}}^{\mathbb{N}}$, we define the subset Ω_w of Ω by

$$\Omega_w \;=\; \Big\{\omega\in\Omega \;:\; (X_n(\omega))_{n\in\mathbb{N}} = w\Big\}\,. \tag{A.2}$$

Then $\Omega_w\in\mathcal{G}$ for all $w\in\hat{\mathcal{W}}^{\mathbb{N}}$, $\Omega_w\cap\Omega_{w'}=\emptyset$ for $w,w'\in\hat{\mathcal{W}}^{\mathbb{N}}$ such that $w\neq w'$ and

$$\Omega \;=\; \bigcup_{w\in\mathcal{W}^{\mathbb{N}}}\Omega_w\,. \tag{A.3}$$

Since $\hat{\mathcal{W}}^{\mathbb{N}}$ is countable, we can enumerate its elements as

$$\hat{\mathcal{W}}^{\mathbb{N}} \;=\; \Big\{w^1, w^2, w^3, \dots\Big\} \;=\; \Big\{w^k,\; k\in\mathbb{N}\Big\}\,.$$

Since w^1 is valued in $\hat{\mathcal{W}}$ which is finite (and hence compact), there exists an increasing function $\varphi_1:\;\mathbb{N}\to\mathbb{N}$ such that $(w^1_{\varphi_1(n)})_{n\in\mathbb{N}}$ converges to some $w^1_\infty\in\hat{\mathcal{W}}$.
Consider now the sequence $(w^2_{\varphi_1(n)})_{n\in\mathbb{N}}$. It is also valued in the compact set $\hat{\mathcal{W}}$. We can therefore find an increasing function $\varphi_2:\;\mathbb{N}\to\mathbb{N}$ such that $(w^2_{\varphi_1\circ\varphi_2(n)})_{n\in\mathbb{N}}$ converges to some $w^2_\infty\in\hat{\mathcal{W}}$.
We proceed in the same way for the following integers to get a sequence $(\varphi^n)_{n\in\mathbb{N}}$ of non-decreasing functions from $\mathbb{N}$ to $\mathbb{N}$ such that $(w^k_{\varphi_1\circ\cdots\circ\varphi_k(n)})_{n\in\mathbb{N}}$ converges to some w^k_∞ for all $k\in\mathbb{N}$.
Next, we define the function $\varphi:\;\mathbb{N}\to\mathbb{N}$ by

$$\varphi(n) \;=\; \varphi_1\circ\cdots\circ\varphi_n(n)$$

for all $n\in\mathbb{N}$. From the construction of the functions $(\varphi_n)_{n\in\mathbb{N}}$ and the definition of φ we obtain that $(w^k_{\varphi(n)})_{n\in\mathbb{N}}$ converges to w^k_∞ for all $k\in\mathbb{N}$.
Using (A.2) and (A.3) we get that the sequence $(X_{\varphi(n)})_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.s. to the random variable X_∞ defined by

$$X_\infty(\omega) \;=\; w^k_\infty$$

for all $\omega\in\Omega_{w^k}$ and all $k\in\mathbb{N}$. □

A.2 Optimal strategies for the sub-period problems

In this part , we show that there exists optimal strategies for each sub-period problem. The two following lemmas give the optimal strategies in π (resp. in ξ). These two results will be used in the next subsection to prove Theorem 4.1.

Lemma A.3. *Fix $k \in \{0, \dots, n-1\}$. Let H_{k+1} be an application from $\hat{\mathcal{W}}^k$ to $L^\infty(\Omega, \mathcal{G}_{t_{k+1}\wedge\tau}, \mathbb{P})$. Then for any $\xi \in \mathcal{E}$ there exists a strategy $\pi^{*,k}(\xi^{(k)})$ and a random variable $y^{(k)}(\xi^{(k)}) \in L^\infty(\Omega, \mathcal{G}_{t_k\wedge\tau}, \mathbb{P})$ such that*

$$\begin{aligned}\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_k\wedge\tau, t_{k+1}\wedge\tau]} \mathbb{E}\Big[\exp\Big(-\gamma\big(X_{t_{k+1}\wedge\tau}^{t_k\wedge\tau,\pi} - H_{k+1}(\hat{\xi}^{(k)})\big)\Big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big] \\ = \mathbb{E}\Big[\exp\Big(-\gamma\big(X_{t_{k+1}\wedge\tau}^{t_k\wedge\tau,\pi^{*,k}(\xi^{(k)})} - H_{k+1}(\hat{\xi}^{(k)})\big)\Big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big] \\ = \exp\big(\gamma y^{(k)}\big(\xi^{(k)}\big)\big)\,.\end{aligned}$$

Proof. As for the proof of Lemma A.1 we look for a process $Y^{(k)}$ such that the family of processes $(R^{(k)}(\pi))_{\pi\in\mathcal{A}[t_k\wedge\tau, t_{k+1}\wedge\tau]}$, where $R^{(k)}(\pi)$ is defined by

$$R_t^{(k)}(\pi) \;=\; \exp\big(-\gamma(X_t^{t_k\wedge\tau,\pi} - Y_t^{(k)})\big)\,,$$

satisfies the following conditions

(i) $R_{t_k\wedge\tau}^{(k)}(\pi)$ is a random variable $\mathcal{G}_{t_k\wedge\tau}$-measurable and independent of π.

(ii) $R_{t_{k+1}\wedge\tau}^{(k)}(\pi) = \exp(-\gamma(X_{t_{k+1}}^{t_k\wedge\tau,\pi} - H_{k+1}(\hat{\xi}^{(k)})))$.

(iii) $R^{(k)}(\pi)$ is a submartingale for any $\pi \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]$ on the time interval $[t_k \wedge \tau, t_{k+1} \wedge \tau]$.

(iv) There exists a strategy $\pi^{*,k}(\xi^{(k)}) \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]$ such that $R^{(k)}(\pi^{*,k}(\xi^{(k)}))$ is a martingale on the time interval $[t_k \wedge \tau, t_{k+1} \wedge \tau]$.

We look for the process $Y^{(k)}$ under the following form

$$\left\{\begin{array}{rcl} -dY_t^{(k)} &=& f(t, Y_t^{(k)}, Z_t^{(k)}, U_t^{(k)})dt - Z_t^{(k)}dB_t - U_t^{(k)}dN_t\,, \\ Y_{t_{k+1}\wedge\tau}^{(k)} &=& H_{k+1}(\hat{\xi}^{(k)})\,. \end{array}\right. \tag{A.4}$$

After some calculus we get that the candidate $\pi^{*,k}(\xi^{(k)})$ is given by

$$\pi_t^{*,k}(\xi^{(k)}) \;=\; \frac{Z_t^{(k)}}{\sigma_t} + \frac{\theta_t}{\gamma\sigma_t}\,,$$

and

$$f(t, y, z, u) \;=\; \lambda_t\Big(\frac{e^{\gamma u} - 1}{\gamma}\Big) - \theta_t z - \frac{|\theta_t|^2}{2\gamma}\,.$$

To prove that the BSDE (A.4) admits a solution we use the result given in [7], and the end of the proof is similar to the verification theorem given in this companion paper. Therefore if we choose $y^{(k)}(\xi^{(k)}) = Y_{t_k\wedge\tau}^{(k)}$ we get the result. □

Remark A.1. *The hypothesis that $H_{k+1} \in L^\infty(\Omega, \mathcal{G}_{t_{k+1}\wedge\tau}, \mathbb{P})$ is crucial to obtain a solution to the BSDE* (A.4)*. It is for that we assume that the pay-off of the contract is bounded, else H_n is not bounded.*

Lemma A.4. *Fix $k \in \{0, \dots, n-1\}$. Let H_{k+1} be an application from $\hat{\mathcal{W}}^k$ to $L^\infty(\Omega, \mathcal{G}_{t_{k+1}\wedge\tau}, \mathbb{P})$. Then for any $\xi \in \mathcal{E}$ there exists $\zeta^* \in \mathcal{E}^1_{k-1}$ such that*

$$\operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_{k-1}} \mathbb{E}\Big[u(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau} - \zeta \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}))\Big|\mathcal{G}_{t_k\wedge\tau}\Big] = \mathbb{E}\Big[u(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau} - \zeta^* \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}^*))\Big|\mathcal{G}_{t_k\wedge\tau}\Big] ,$$

for all $\pi \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]$.

Proof. From a classical result on essential supremum of a family of random variables (see e.g. [14]) there exists a sequence $(\zeta^\ell)_{\ell\in\mathbb{N}}$ valued in $\mathcal{E}^1_{k-1}$ such that

$$u\Big(-\zeta^\ell \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}^\ell)\Big) \;\to\; \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_{k-1}} u\Big(-\zeta \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta})\Big) \quad \text{(A.5)}$$

$\mathbb{P}$-a.s. as ℓ goes to infinity. We now apply Lemma A.2 to the sequence $\big(\zeta^\ell e^{\int_0^{t_k} r_s ds}\big)_{\ell\in\mathbb{N}}$ and we obtain that, up to a subsequence, $(\zeta^\ell)_\ell$ converges $\mathbb{P}$-a.s. to some random variable ζ^*. From (A.5) we get

$$u\Big(-\zeta^* \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}^*)\Big) \;=\; \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_{k-1}} u\Big(-\zeta \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta})\Big) . \quad \text{(A.6)}$$

We then notice from the multiplicative structure of the function u that for any $\pi \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]$ and any $\zeta \in \mathcal{E}^1_{k-1}$ we have

$$\begin{aligned}
& \mathbb{E}\Big[u(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau} - \zeta \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}))\Big|\mathcal{G}_{t_k\wedge\tau}\Big] \\
= \;& \mathbb{E}\Big[u(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau}) u(-\zeta \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}))\Big|\mathcal{G}_{t_k\wedge\tau}\Big] \\
\leq \;& \mathbb{E}\Big[u(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau}) \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_{k-1}} u(-\zeta \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}))\Big|\mathcal{G}_{t_k\wedge\tau}\Big] \\
= \;& \mathbb{E}\Big[u(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau} - \zeta^* \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}^*))\Big|\mathcal{G}_{t_k\wedge\tau}\Big] .
\end{aligned}$$

Therefore we get

$$\mathbb{E}\Big[u(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau} - \zeta^* \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}^*))\Big|\mathcal{G}_{t_k\wedge\tau}\Big] = \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_{k-1}} \mathbb{E}\Big[u(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau} - \zeta \mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}))\Big|\mathcal{G}_{t_k\wedge\tau}\Big] ,$$

for all $\pi \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]$. □

A.3 Proof of Theorem 4.1

We shall now prove the result in two steps. For each step, we use method of induction (forward in the first, backward in the second).

First step. We first show that the following inequality holds

$$w(p) \quad \geq \quad \inf_{\pi \in \mathcal{A}[0,t_1\wedge\tau]} \mathbb{E}\Big[u\big(X^\pi_{t_1\wedge\tau}\big)v(1)\Big] \,. \tag{A.7}$$

We prove this inequality by induction on the number k of anniversary dates. More precisely, we show that for any $k \in \{1,\dots,n\}$ and any map H_k from $\hat{\mathcal{W}}^{k-1}$ to $L^\infty(\Omega, \mathcal{G}_{t_k\wedge\tau}, \mathbb{P})$, we have

$$\begin{aligned} v_{H_k} \quad &:= \quad \inf_{\pi\in\mathcal{A}[0,t_k\wedge\tau]} \sup_{\xi\in\mathcal{E}_0^{k-1}} \mathbb{E}\Big[u\big(X^\pi_{t_k\wedge\tau} - \sum_{i=1}^{k-1}\xi_i \mathbb{1}_{t_i\leq\tau} - H_k(\hat{\xi}^{(k-1)})\big)\Big] \\ &\geq \quad \inf_{\pi\in\mathcal{A}[0,t_1\wedge\tau]} \mathbb{E}\Big[u\big(X^\pi_{t_1\wedge\tau}\big)\hat{v}_{H_k}(1)\Big] \,, \end{aligned} \tag{A.8}$$

where $\hat{v}_{H_k}(i,\xi^{(i-1)})$ is defined recursively for any $i \in \{1,\dots,k\}$ and $\xi \in \mathcal{E}_0^{k-1}$ by

$$\left\{ \begin{array}{rcl} \hat{v}_{H_k}(k,\xi^{(k-1)}) & := & e^{\gamma H_k(\hat{\xi}^{(k-1)})}, \\ \hat{v}_{H_k}(i,\xi^{(i-1)}) & := & \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}_{i-1}^1} \operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_i\wedge\tau,t_{i+1}\wedge\tau]} \hat{J}(i,\pi,\xi^{(i-1)},\zeta) \,, \end{array} \right.$$

with

$$\hat{J}(i,\pi,\xi^{(i-1)},\zeta) \quad := \quad \mathbb{E}\Big[u\big(X^{t_i\wedge\tau,\pi}_{t_{i+1}\wedge\tau} - \zeta\mathbb{1}_{t_i<\tau}\big)\hat{v}_{H_k}(i+1,(\xi^{(i-1)},\zeta))\Big|\mathcal{G}_{t_i\wedge\tau}\Big] \,,$$

for any $i \in \{1,\dots,k-1\}$, $\pi \in \mathcal{A}[t_i\wedge\tau, t_{i+1}\wedge\tau]$ and $\zeta \in \mathcal{E}_{i-1}^1$.
By abuse of notation, we denote $\hat{v}_{H_k}(1,\xi^{(0)})$ for $\hat{v}_{H_k}(1)$.

For $k = 1$, the inequality (A.8) is obvious.
We now assume the result holds for some $k \in \{1,\dots,n-1\}$. Let H_{k+1} be a map from $\hat{\mathcal{W}}^k$ to $L^\infty(\Omega, \mathcal{G}_{t_{k+1}\wedge\tau}, \mathbb{P})$. We define

$$\begin{aligned} K(k+1,\pi,\xi) \quad &:= \quad \mathbb{E}\Big[u\big(X^\pi_{t_{k+1}\wedge\tau} - \sum_{i=1}^{k}\xi_i\mathbb{1}_{t_i\leq\tau} - H_{k+1}(\hat{\xi}^{(k)})\big)\Big] \\ &= \quad \mathbb{E}\Big[u\big(X^\pi_{t_k\wedge\tau} - \sum_{i=1}^{k-1}\xi_i\mathbb{1}_{t_i\leq\tau})\mathbb{E}\Big[u\big(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau} - \xi_k\mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k)})\big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big]\Big] \,, \end{aligned}$$

for any $\pi \in \mathcal{A}[0, t_{k+1}\wedge\tau]$ and $\xi \in \mathcal{E}_k^0$. From Lemma A.3 there exists $\pi^{*,k}(\xi) \in \mathcal{A}[t_k \wedge \tau, t_{k+1}\wedge\tau]$ such that

$$\begin{aligned} \operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_k\wedge\tau,t_{k+1}\wedge\tau]} &\mathbb{E}\Big[u\Big(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau} - \xi_k\mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi})\Big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big] \\ &= \mathbb{E}\Big[u\Big(X^{t_k\wedge\tau,\pi^{*,k}(\xi^{(k-1)},\xi_k)}_{t_{k+1}\wedge\tau} - \xi_k\mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi})\Big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big] \,. \end{aligned}$$

From Lemma A.4 and the previous equality, there exists $\zeta^* \in \mathcal{E}^1_{k-1}$ such that

$$\begin{aligned}\hat{v}_{H_{k+1}}(k,\xi^{(k-1)}) &= \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}^1_{k-1}} \mathbb{E}\Big[u\Big(X^{t_k\wedge\tau,\pi^{*,k}(\xi^{(k-1)},\zeta)}_{t_{k+1}\wedge\tau} - \zeta\mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)},\hat{\zeta})\Big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big] \\ &= \mathbb{E}\Big[u\Big(X^{t_k\wedge\tau,\pi^{*,k}(\xi^{(k-1)},\zeta^*)}_{t_{k+1}\wedge\tau} - \zeta^*\mathbb{1}_{t_k<\tau} - H_{k+1}(\hat{\xi}^{(k-1)},\hat{\zeta}^*)\Big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big]\,.\end{aligned}$$

By definition of $\pi^{*,k}$ we have

$$K(k+1,\pi,\xi) \;\geq\; K(k+1,\pi\mathbb{1}_{[0,t_k\wedge\tau]} + \pi^{*,k}(\xi^{(k)})\mathbb{1}_{[t_k\wedge\tau,t_{k+1}\wedge\tau]},\xi)\,,$$

for any $\pi \in \mathcal{A}[0,t_{k+1}\wedge\tau]$ and $\xi\in\mathcal{E}^k_0$. This implies

$$\begin{aligned}\sup_{\xi\in\mathcal{E}^k_0} K(k+1,\pi,\xi) &\geq \sup_{\xi\in\mathcal{E}^k_0} K\big(k+1,\pi\mathbb{1}_{[0,t_k\wedge\tau]} + \pi^{*,k}(\xi^{(k)})\mathbb{1}_{[t_k\wedge\tau,t_{k+1}\wedge\tau]},\xi\big) \\ &\geq \sup_{\xi\in\mathcal{E}^{k-1}_0} K\big(k+1,\pi\mathbb{1}_{[0,t_k\wedge\tau]} + \pi^{*,k}(\xi^{(k-1)},\zeta^*)\mathbb{1}_{[t_k\wedge\tau,t_{k+1}\wedge\tau]},(\xi^{(k-1)},\zeta^*)\big)\,.\end{aligned}$$

From the definition of K we have

$$\begin{aligned}& K(k+1,\pi\mathbb{1}_{[0,t_k\wedge\tau]} + \pi^{*,k}(\xi^{(k-1)},\zeta^*)\mathbb{1}_{[t_k\wedge\tau,t_{k+1}\wedge\tau]},(\xi^{(k-1)},\zeta^*)) \\ =\;& \mathbb{E}\Big[u\big(X^\pi_{t_k\wedge\tau} - \sum_{i=1}^{k-1}\xi_i\mathbb{1}_{t_i\leq\tau}\big)\mathbb{E}\Big[u\big(X^{t_k\wedge\tau,\pi^{*,k}(\xi^{(k-1)},\zeta^*)}_{t_{k+1}\wedge\tau} - \zeta^*\mathbb{1}_{t_k<\tau} - H_{k+1}(\xi^{(k-1)},\zeta^*)\big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big]\Big] \\ =\;& \mathbb{E}\Big[u\big(X^\pi_{t_k\wedge\tau} - \sum_{i=1}^{k-1}\xi_i\mathbb{1}_{t_i\leq\tau}\big)\hat{v}_{H_{k+1}}(k,\xi^{(k-1)})\Big]\,.\end{aligned}$$

Therefore we get

$$\sup_{\xi\in\mathcal{E}^k_0} K(k+1,\pi,\xi) \;\geq\; \sup_{\xi\in\mathcal{E}^{k-1}_0} \mathbb{E}\Big[u\big(X^\pi_{t_k\wedge\tau} - \sum_{i=1}^{k-1}\xi_i\mathbb{1}_{t_i\leq\tau}\big)\hat{v}_{H_{k+1}}(k,\xi^{(k-1)})\Big]\,. \tag{A.9}$$

We now define the application H_k by

$$H_k\big(x_1,\ldots,x_{k-1}\big) \;:=\; \frac{1}{\gamma}\log\big(\hat{v}_{H_{k+1}}(k,(x_1e^{-\int_0^{t_1}r_sds},\ldots,x_{k-1}e^{-\int_0^{t_{k-1}}r_sds})\big)\,,$$

for all $(x_1,\ldots,x_{k-1})\in\hat{\mathcal{W}}^{k-1}$. From Lemma A.3, we have

$$H_k\big(x_1,\ldots,x_{k-1}\big) \;=\; y^{(k)}\big(x_1,\ldots,x_{k-1},\zeta^*\big) \;\in\; L^\infty(\Omega,\mathcal{G}_{t_k\wedge\tau},\mathbb{P})\,.$$

We can then use the induction hypothesis and we get

$$\begin{aligned}\inf_{\pi\in\mathcal{A}[0,t_k\wedge\tau]}\sup_{\xi\in\mathcal{E}^{k-1}_0}\mathbb{E}\Big[u(X^\pi_{t_k\wedge\tau} - \sum_{i=1}^{k-1}\xi_i\mathbb{1}_{t_i\leq\tau} - H_k(\xi^{(k-1)})\Big] \\ \geq \inf_{\pi\in\mathcal{A}[0,t_1\wedge\tau]}\mathbb{E}\Big[u\big(X^\pi_{t_1\wedge\tau}\big)\hat{v}_{H_k}(1))\Big]\,.\end{aligned} \tag{A.10}$$

Taking the infimum in (A.9) we obtain

$$\inf_{\pi\in\mathcal{A}[0,t_{k+1}\wedge\tau]}\sup_{\xi\in\mathcal{E}_0^k}K(k+1,\pi,\xi) \;\geq\; \inf_{\pi\in\mathcal{A}[0,t_k\wedge\tau]}\sup_{\xi\in\mathcal{E}_0^{k-1}}\mathbb{E}\Big[u(X^\pi_{t_k\wedge\tau}-\sum_{i=1}^{k-1}\xi_i\mathbb{1}_{t_i\leq\tau}-H_k(\xi^{(k-1)})\Big]\,.$$

Using (A.10) we get

$$\inf_{\pi\in\mathcal{A}[0,t_{k+1}\wedge\tau]}\sup_{\xi\in\mathcal{E}_0^k}K(k+1,\pi,\xi) \;\geq\; \inf_{\pi\in\mathcal{A}[0,t_1\wedge\tau]}\mathbb{E}\Big[u\big(X^\pi_{t_1\wedge\tau}\big)\hat{v}_{H_k}(1))\Big]\,.$$

We conclude the proof of the inequality (A.7) by noticing that

$$\hat{v}_{H_k}(k,\xi^{(k-1)}) \;=\; \hat{v}_{H_{k+1}}(k,\xi^{(k-1)})\,.$$

Hence, we have $\hat{v}_{H_k}(1)=\hat{v}_{H_{k+1}}(1)$ and then the inequality (A.8) holds for any $k\in\{1,\dots,n\}$.

Second step. We now prove the following inequality

$$w(p) \;\leq\; \inf_{\pi\in\mathcal{A}[0,t_1\wedge\tau]}\mathbb{E}\Big[u\big(X^\pi_{t_1\wedge\tau}\big)v(1)\Big]\,. \tag{A.11}$$

To this end, we show by a backward induction on $k\in\{1,\dots,n-1\}$ that for $\xi^{(k)}\in\mathcal{E}_0^k$ and H a bounded $\mathcal{G}_{T\wedge\tau}$-measurable random variable, we have

$$\begin{aligned}&\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_k\wedge\tau,T\wedge\tau]}\operatorname*{ess\,sup}_{\zeta\in\mathcal{E}_k^{n-k-1}}\mathbb{E}\Big[u\Big(X^{t_k\wedge\tau,\pi}_{T\wedge\tau}-\xi_k\mathbb{1}_{t_k<\tau}-\sum_{j=1}^{n-k-1}\zeta_j\mathbb{1}_{t_{k+j}<\tau}-H\Big)\Big|\mathcal{G}_{t_k\wedge\tau}\Big]\\ &\qquad\leq\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_k\wedge\tau,t_{k+1}\wedge\tau]}\mathbb{E}\Big[u\Big(X^{t_k\wedge\tau,\pi}_{t_{k+1}\wedge\tau}-\xi_k\mathbb{1}_{t_k<\tau}\Big)v(k+1,\xi^{(k)})\Big|\mathcal{G}_{t_k\wedge\tau}\Big]\,.\end{aligned} \tag{A.12}$$

For $k=n-1$, the previous inequality obviously holds from the definition of $v(n,.)$.
We now assume that the inequality (A.12) holds for some $k\in\{2,\dots,n-1\}$ and we prove it for $k-1$.

We first write $\pi=(\pi^\ell,\dots,\pi^{n-1})$ for $\pi\in\mathcal{A}[t_\ell\wedge\tau,\dots,T\wedge\tau]$ with $\pi^k\in\mathcal{A}[t_k\wedge\tau,t_{k+1}\wedge\tau]$. We then define the map L by

$$L(\ell,\pi^\ell,\dots,\pi^{n-1},\xi^{(\ell)}) \;:=\; \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}_\ell^{n-\ell}}\mathbb{E}\Big[u\Big(X^{t_\ell\wedge\tau,\pi}_{T\wedge\tau}-\xi_\ell\mathbb{1}_{t_\ell<\tau}-\sum_{j=1}^{n-\ell}\zeta_{\ell+j}\mathbb{1}_{t_{\ell+j}<\tau}-H\Big)\Big|\mathcal{G}_{t_\ell\wedge\tau}\Big]\,,$$

for $\ell\in\{1,\dots,n-1\}$, $\xi\in\mathcal{E}$ and $\pi\in\mathcal{A}[0,T\wedge\tau]$.

Let H be a bounded $\mathcal{G}_{T\wedge\tau}$-measurable random variable, $\pi\in\mathcal{A}[t_{k-1}\wedge\tau,T\wedge\tau]$ and $\xi\in\mathcal{E}$. By definition of L we know that

$$L(k-1,\pi,\xi^{(k-1)})\leq\mathbb{E}\Big[u\big(X^{t_{k-1}\wedge\tau,\pi}_{t_k\wedge\tau}-\xi_{k-1}\mathbb{1}_{t_{k-1}<\tau}\big)\operatorname*{ess\,sup}_{\zeta_k\in\mathcal{E}^1_{k-1}}L(k,\pi^k,\dots,\pi^{n-1},(\xi^{(k-1)},\zeta_k))\Big|\mathcal{G}_{t_{k-1}\wedge\tau}\Big] \tag{A.13}$$

Using Lemma A.4, we can find $\zeta_k^*\in\mathcal{E}^1_{k-1}$ (depending only on $\pi^{k+1},\dots,\pi^{n-1}$) such that

$$\operatorname*{ess\,sup}_{\zeta_k\in\mathcal{E}^1_{k-1}}L\big(k,\pi^k,\pi^{k+1},\dots,\pi^{n-1},(\xi^{(k-1)},\zeta_k)\big)=L\big(k,\pi^k,\pi^{k+1},\dots,\pi^{n-1},(\xi^{(k-1)},\zeta_k^*)\big) \tag{A.14}$$

for all $\pi^k \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]$. We also know from Lemma A.3 that there exists $\pi^{k*} \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]$ (also depending on $\pi^{k+1}, \ldots, \pi^{n-1}$) such that

$$L\big(k, \pi^{k*}, \pi^{k+1}, \ldots, \pi^{n-1}, (\xi^{(k-1)}, \zeta_k^*)\big) = \operatorname*{ess\,inf}_{\pi^k \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]} L\big(k, \pi^k, \pi^{k+1}, \ldots, \pi^{n-1}, (\xi^{(k-1)}, \zeta_k^*)\big). \quad \text{(A.15)}$$

For $\pi \in \mathcal{A}[0, T \wedge \tau]$, define $\bar{\pi}$ by

$$\bar{\pi} \;:=\; \pi \mathbb{1}_{[0, t_k \wedge \tau]} + \pi^{k*} \mathbb{1}_{[t_k \wedge \tau, t_{k+1} \wedge \tau]} + \pi^{k+1} \mathbb{1}_{[t_{k+1} \wedge \tau, t_{k+2} \wedge \tau]} + \cdots + \pi^{n-1} \mathbb{1}_{[t_{n-1} \wedge \tau, T \wedge \tau]} .$$

From (A.13), (A.14) and (A.15) we deduce

$$\begin{aligned} &L(k-1, \bar{\pi}, \xi^{(k-1)}) \\ &\quad \leq \mathbb{E}\Big[u(X_{t_k \wedge \tau}^{t_{k-1} \wedge \tau, \pi} - \xi_{k-1} \mathbb{1}_{t_{k-1} < \tau}) \operatorname*{ess\,inf}_{\pi \in \mathcal{A}[t_k \wedge \tau, T \wedge \tau]} L(k, \pi, (\xi^{(k-1)}, \zeta_k^*), H) \Big| \mathcal{G}_{t_{k-1} \wedge \tau} \Big] . \quad \text{(A.16)} \end{aligned}$$

By taking the infimum we get

$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}[t_{k-1} \wedge \tau, T \wedge \tau]} L(k-1, \pi, \xi^{(k-1)}, H) \;\leq\; \operatorname*{ess\,inf}_{\pi \in \mathcal{A}[t_{k-1} \wedge \tau, t_k \wedge \tau]} L(k-1, \bar{\pi}, \xi^{(k-1)}, H) .$$

From this last inequality, (A.16) and the induction assumption we get

$$\begin{aligned} &\operatorname*{ess\,inf}_{\pi \in \mathcal{A}[t_{k-1} \wedge \tau, T \wedge \tau]} L(k-1, \pi, \xi^{(k-1)}, H) \\ &\qquad \leq \operatorname*{ess\,inf}_{\pi \in \mathcal{A}[t_{k-1} \wedge \tau, t_k \wedge \tau]} \mathbb{E}\Big[u\big(X_{t_k \wedge \tau}^{t_{k-1} \wedge \tau, \pi} - \xi_{k-1} \mathbb{1}_{t_{k-1} < \tau}\big) v(k, \xi^{(k-1)}) \Big| \mathcal{G}_{t_{k-1} \wedge \tau} \Big] , \end{aligned}$$

which gives the result for $k-1$ from the definition of the map L.

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