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# PERSISTENCE EXPONENT FOR RANDOM PROCESSES IN BROWNIAN SCENERY 

FABIENNE CASTELL, NADINE GUILLOTIN-PLANTARD, AND FRÉDÉRIQUE WATBLED


#### Abstract

In this paper we consider the persistence properties of random processes in Brownian scenery, which are examples of non-Markovian and non-Gaussian processes. More precisely we study the asymptotic behaviour for large $T$, of the probability $\mathbb{P}\left[\sup _{t \in[0, T]} \Delta_{t} \leq 1\right]$ where $\Delta_{t}=\int_{\mathbb{R}} L_{t}(x) d W(x)$. Here $W=\{W(x) ; x \in \mathbb{R}\}$ is a two-sided standard real Brownian motion and $\left\{L_{t}(x) ; x \in \mathbb{R}, t \geq 0\right\}$ is the local time of some self-similar random process $Y$, independent from the process $W$. We thus generalize the results of [7] where the increments of $Y$ were assumed to be independent.


## 1. Introduction

Let $W=\{W(x) ; x \in \mathbb{R}\}$ be a standard two-sided real Brownian motion and $Y=\{Y(t) ; t \geq 0\}$ be a real-valued self-similar process of index $\gamma \in(0,2)$ (i.e. for any $c>0,\{Y(c t) ; t \geq 0\} \stackrel{(d)}{=}$ $\left\{c^{\gamma} Y(t) ; t \geq 0\right\}$ ), with stationary increments. When it exists, we will denote by $\left\{L_{t}(x) ; x \in \mathbb{R}, t \geq 0\right\}$ a version of the local time of the process $\{Y(t) ; t \geq 0\}$. The process $L$ satisfies the occupation density formula: for each bounded measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$ and for each $t \geq 0$,

$$
\begin{equation*}
\int_{0}^{t} f(Y(s)) d s=\int_{\mathbb{R}} f(x) L_{t}(x) d x \tag{1}
\end{equation*}
$$

The processes $W$ and $Y$ are defined on the same probability space and are assumed to be independent. We consider the random process in Brownian scenery $\left\{\Delta_{t} ; t \geq 0\right\}$ defined as

$$
\Delta_{t}=\int_{\mathbb{R}} L_{t}(x) d W(x)
$$

The process $\Delta$ is itself a self-similar process of index $h$ with stationary increments, with

$$
h:=1-\frac{\gamma}{2} .
$$

This process can be seen as a mixture of Gaussian processes, but it is neither Gaussian nor Markovian. It appeared independently in the mathematical literature (see [14], [3]), and in the physics literature (see [18]) where it was originally introduced to model diffusion in layered white-noise velocity field. Note indeed that $\Delta_{t}$ can formally be written as $\int_{0}^{t} \dot{W}(Y(s)) d s$. Thus, it describes the horizontal motion of a tracer particle in a (1+1)-dimensional medium, where the motion of the particle along the vertical direction $y$ is described by the process $Y$, while along the horizontal direction, the particle is driven by the white noise velocity field $\dot{W}(y)$, that depends only on the vertical coordinate $y$. The process $\left\{\Delta_{t} ; t \geq 0\right\}$ provides a simple example of anomalous super-diffusion, being for instance of order $t^{3 / 4}$ when $Y$ is a Brownian motion. In

[^0]this paper, we are interested in the persistence probability of the process $\Delta$, i.e. the asymptotic behaviour of
$$
\mathcal{F}(T):=\mathbb{P}\left[\sup _{t \in[0, T]} \Delta_{t} \leq 1\right]
$$
as $T \rightarrow+\infty$. The study of the one-sided supremum process of random processes is a classical issue in probability. One usually gets polynomial decay of the persistence probability: $\mathcal{F}(T) \asymp$ $T^{-\theta}$ for a non-negative $\theta$, often called persistence exponent, or survival exponent. Classical examples, where this exponent can be computed, include random walks or Lévy processes, and we refer the reader to the recent survey papers [1, 5] for an account of models where the persistence exponent is known. However, there are relevant physical situations where this exponent remains unknown (see for instance $[16,5]$ ). The persistence exponent of the Brownian motion in Brownian scenery was studied by Redner [20, 21], and Majumdar [17]. Based on physical arguments, numerical simulations and analogy with fractional Brownian motion, Redner and Majumdar conjectured the value of the persistence exponent. In [7], their conjecture was proved up to logarithmic factors, when the process $Y$ is a stable Lévy process with index $\delta \in$ $(1,2]$. The proof of [7] depends heavily on the increments independence of the process $Y$ and the question raises if it is possible to compute the persistence exponent without it. The aim of this paper is to answer this question, and to provide assumptions on $Y$ allowing to compute the persistence exponent of the random process in Brownian scenery.
(H1): There exists a continuous version $\left\{L_{t}(x) ; x \in \mathbb{R}, t \geq 0\right\}$ of the local time of $Y$;
(H2): $Y$ is a self-similar process of index $\gamma \in(0,2)$;
(H3): For every $T>0,\{Y(T-t)-Y(T) ; t \in[0, T]\}$ has the same distribution as either $\{Y(t) ; t \in[0, T]\}$ or $\{-Y(t) ; t \in[0, T]\}$;
$(\mathbf{H 4})$ : Let $V_{1}:=\int L_{1}^{2}(x) d x$ be the self-intersection local time of $Y$. There exist a real number $\alpha>1$, and positive constants $C, c$ such that for any $x \geq 0$,
$$
\mathbb{P}\left[V_{1} \geq x\right] \leq C \exp \left(-c x^{\alpha}\right)
$$
(H5): There exist a real number $\beta>0$, and positive constants $C, c$ such that for any $x>0$,
$$
\mathbb{P}\left[V_{1} \leq x\right] \leq C \exp \left(-c x^{-\beta}\right)
$$

Our main result is the following one.
Theorem 1. Assume (H1) to (H5) hold. There exists a constant $c>0$, such that for large enough $T$,

$$
\begin{equation*}
T^{-\gamma / 2}(\ln T)^{-c} \leq \mathbb{P}\left[\sup _{t \in[0, T]} \Delta_{t} \leq 1\right] \leq T^{-\gamma / 2}(\ln T)^{+c} \tag{2}
\end{equation*}
$$

Our main achievement in this paper is to remove the increments independence assumption on $\{Y(t) ; t \geq 0\}$, which was crucial in [7]. Another interesting issue would be to obtain the persistence exponent for $\{W(x) ; x \in \mathbb{R}\}$ being a stable Lévy process. This seems delicate, our proof relying on the fact that the process $\{\Delta(t) ; t \geq 0\}$ is a Gaussian process, conditionally to $\{Y(t) ; t \geq 0\}$.

The paper is organized as follows. Section 2 provides three examples of processes satisfying (H1) to (H5), including the stable Lévy process with index $\delta \in(1,2]$, and thus generalizing the result of [7]. Section 3 states some useful properties of the process $\Delta$. Section 4 is devoted to the proof of Theorem 1.

## 2. Some examples

2.1. A sufficient condition for (H5). In this section we assume that (H1) holds and we provide a sufficient condition on $\{Y(t) ; t \geq 0\}$ allowing to check (H5). Let us first compare $V_{1}$ with $\max _{s \in[0,1]}|Y(s)|$. By noting that

$$
1=\int_{\left\{|x| \leq \max _{s \in[0,1]}|Y(s)|\right\}} L_{1}(x) d x \leq V_{1}^{1 / 2} \sqrt{2 \max _{s \in[0,1]}|Y(s)|},
$$

we deduce that

$$
\begin{equation*}
\frac{1}{2 \max _{s \in[0,1]}|Y(s)|} \leq V_{1} \tag{3}
\end{equation*}
$$

Lemma 2.1. Assume that (H1) holds and that there exist positive constants $C$ and $c$, and $a$ real number $\beta>0$ such that for all $x>0$,

$$
\mathbb{P}\left[\max _{s \in[0,1]}|Y(s)| \geq x\right] \leq C \exp \left(-c x^{\beta}\right)
$$

Then (H5) holds true.
Proof. It is a direct consequence of (3) that for all $x>0$,

$$
\mathbb{P}\left[V_{1} \leq x\right] \leq \mathbb{P}\left[\max _{s \in[0,1]}|Y(s)| \geq \frac{1}{2 x}\right] \leq C \exp \left(-c 2^{-\beta} x^{-\beta}\right)
$$

2.2. Stable Lévy processes. A process $Y=\{Y(t) ; t \geq 0\}$ is a strictly stable Lévy process with index $\delta \in(1,2]$ if it is a process with stationary and independent increments, such that $Y(0)=0$ and for any $t \geq 0$ and $u \in \mathbb{R}$,

$$
\begin{equation*}
\mathbb{E}\left[e^{i u Y(t)}\right]=\exp \left\{-c|u|^{\delta} t(1+i \zeta \operatorname{sgn}(u) \tan (\pi \delta / 2))\right\} \tag{4}
\end{equation*}
$$

where $\zeta \in[-1,1]$, and $c$ is a positive scale parameter.
Lemma 2.2. The stable Lévy process with index $\delta \in(1,2]$ satisfies (H1) to (H5) with $\gamma=\frac{1}{\delta}$, $\alpha=\delta$, and $\beta=\frac{\delta}{2 \delta-1}$.

Proof. It is immediate that the Lévy process is self-similar of index $1 / \delta$ and that for every $T>0$,

$$
\{Y(T-t)-Y(T) ; t \in[0, T]\} \stackrel{(d)}{=}\{-Y(t) ; t \in[0, T]\}
$$

then (H2) and (H3) are satisfied.
A continuous version of the local time exists, since $\delta>1$ (this was proved by Boylan [4]), so it satisfies (H1). Moreover, Corollary 5.6 in [15] states that there exist positive constants $C$ and $\xi$ s.t. for every $x>0$,

$$
\mathbb{P}\left[V_{1} \geq x\right] \leq C e^{-\xi x^{\delta}}
$$

which gives (H4). To prove (H5) we let $V_{t}:=\int L_{t}^{2}(x) d x$ be the self-intersection local time of $Y$ up to time $t$ for $t \geq 0$, and we show that there exists $l>0$ such that

$$
\begin{equation*}
\mathbb{P}\left[V_{t} \leq 1\right] \leq e^{-l t} \text { for } t \text { large enough. } \tag{5}
\end{equation*}
$$

As $\left\{V_{t} ; t \geq 0\right\}$ is self-similar of index $\frac{2 \delta-1}{\delta}$, (5) implies that for every positive $\varepsilon$ small enough,

$$
\mathbb{P}\left[V_{1} \leq \varepsilon\right]=\mathbb{P}\left[V_{\varepsilon-\frac{\delta}{2 \delta-1}} \leq 1\right] \leq e^{-l \varepsilon^{-\frac{\delta}{2 \delta-1}}}
$$

which gives (H5). To prove (5) we show that the function defined on $\mathbb{R}^{+}$by

$$
f(t)=\log \mathbb{P}\left[V_{t} \leq 1\right]
$$

is subadditive. Let us fix $s, t$ in $[0,+\infty)$. We consider the process $Y^{(s)}:=\left\{Y_{u}^{(s)} ; u \geq 0\right\}$ defined by

$$
Y_{u}^{(s)}=Y(u+s)-Y(s),
$$

its local time $\left\{L_{u}^{(s)}(x) ; x \in \mathbb{R}, u \geq 0\right\}$, and its self-intersection local time $\left\{V_{u}^{(s)} ; u \geq 0\right\}$. Note that

$$
L_{t+s}(x)=L_{s}(x)+L_{t}^{(s)}(x-Y(s))
$$

Hence

$$
V_{t+s} \geq V_{s}+V_{t}^{(s)}
$$

The process $Y$ being a Lévy process, $V_{t}^{(s)}$ and $V_{s}$ are independent and $V_{t}^{(s)}$ has the same law as $V_{t}$. Therefore

$$
\mathbb{P}\left[V_{t+s} \leq 1\right] \leq \mathbb{P}\left[V_{s} \leq 1 ; V_{t}^{(s)} \leq 1\right]=\mathbb{P}\left[V_{s} \leq 1\right] \mathbb{P}\left[V_{t} \leq 1\right] .
$$

Thus $f$ is subadditive, which implies that $\frac{f(t)}{t}$ converges, as $t \rightarrow+\infty$, towards a limit $-l=$ $\inf _{t>0} \frac{f(t)}{t}$ (see [12]). It remains to show that $l$ is strictly positive. As $-l$ is less than or equal to $f(1)$, it is enough to show that $\mathbb{P}\left[V_{1}>1\right]>0$. But thanks to (3) and Proposition 10.3 of [10],

$$
\mathbb{P}\left[V_{1}>1\right] \geq \mathbb{P}\left[\max _{s \in[0,1]}|Y(s)|<\frac{1}{2}\right]>0
$$

and this concludes the proof of (5).
2.3. Fractional Brownian motion. The fractional Brownian motion of Hurst index $H \in(0,1)$ is the real centered Gaussian process $\left\{B_{H}(t) ; t \geq 0\right\}$ with covariance function

$$
\mathbb{E}\left[B_{H}(t) B_{H}(s)\right]=\frac{1}{2}\left(t^{2 H}+s^{2 H}-|t-s|^{2 H}\right) .
$$

Lemma 2.3. The fractional Brownian motion with Hurst index $H \in(0,1)$ satisfies (H1) to (H5) with $\gamma=H, \alpha=1 / H, \beta=2$.

Proof. It follows readily from the definition that $B_{H}$ is self-similar with index $H$ and satisfies ( $H 3$ ). The existence of a jointly continuous version of its local time process for $H \in(0,1)$ is a classical fact (see for instance paragraph 22 in [11]). Theorem 1 of [13] asserts that there exists $\lambda_{0}>0$ such that for every $0<\lambda<\lambda_{0}$,

$$
\begin{equation*}
\mathbb{E}\left[e^{\lambda V_{1}^{1 / H}}\right]<\infty, \tag{6}
\end{equation*}
$$

which implies (H4) with $\alpha=1 / H$. Finally, (H5) follows from Lemma 2.1 and Fernique's estimation ([9], Theorem 4.1.1): there exists $c_{F}>0$ such that for any $x \geq \sqrt{5}$,

$$
\begin{equation*}
\mathbb{P}\left[\max _{s \in[0,1]}\left|B_{H}(s)\right| \geq c_{F} x\right] \leq 10 \int_{x}^{+\infty} e^{-\frac{v^{2}}{2}} d v \tag{7}
\end{equation*}
$$

This implies (H5) with $\beta=2$.
2.4. Iterated Brownian motion. Let $\{B(x) ; x \in \mathbb{R}\}$ be a two-sided real standard Brownian motion, and let $\{\tilde{B}(t) ; t \geq 0\}$ be a real standard Brownian motion, independent of $\{B(x) ; x \in \mathbb{R}\}$. The iterated Brownian motion is the process $\{Y(t) ; t \geq 0\}$ defined by

$$
Y(t)=B(\tilde{B}(t))
$$

Lemma 2.4. The iterated Brownian motion satisfies (H1) to (H5) with $\gamma=1 / 4$, and $\alpha=\beta=$ $4 / 3$.

Proof. The self-similarity of the iterated Brownian motion is a direct consequence of the selfsimilarity and independence of both Brownian motions. The assertion (H3) follows once again from the independence of $B$ and $\tilde{B}$, the increments stationarity of $B$ and the fact that $\tilde{B}$ satisfies (H3).

The existence and joint continuity of the local times of iterated Brownian motion were proved in $[6,8]$. Let us prove (H5). For $x>0$,

$$
\begin{align*}
\mathbb{P}\left[\max _{s \in[0,1]}|Y(s)| \geq x\right] & \leq \mathbb{P}\left[\max _{s \in[0,1]}|\tilde{B}(s)| \geq x^{2 / 3}\right]+\mathbb{P}\left[\max _{|u| \leq x^{2 / 3}}|B(u)| \geq x\right] \\
& \leq \mathbb{P}\left[\max _{s \in[0,1]}|\tilde{B}(s)| \geq x^{2 / 3}\right]+\mathbb{P}\left[\max _{u \in\left[0, x^{2 / 3}\right]}|B(u)| \geq x\right] \\
& \leq 2 \mathbb{P}\left[\max _{s \in[0,1]}|B(s)| \geq x^{2 / 3}\right] \\
& \leq 4 \mathbb{P}\left[\max _{s \in[0,1]} B(s) \geq x^{2 / 3}\right]=4 \mathbb{P}\left[|B(1)| \geq x^{2 / 3}\right] . \tag{8}
\end{align*}
$$

This proves (H5) with $\beta=4 / 3$ using Lemma 2.1.
To prove (H4), we use the uniform norm on the local times of iterated Brownian motion proved in Lemma 4 of [22]: there exists a constant $K>0$ such that for any even integer $n$,

$$
\begin{equation*}
\sup _{x \in \mathbb{R}} \mathbb{E}\left[\left(L_{1}(x)\right)^{n}\right] \leq K^{n}(n!)^{3 / 4} \tag{9}
\end{equation*}
$$

Since $\int_{\mathbb{R}} L_{1}(x) d x=1$, Hölder's inequality implies that for any integer $n$,

$$
V_{1}^{n}=\left(\int_{\mathbb{R}} L_{1}(x)^{2} d x\right)^{n} \leq \int_{\mathbb{R}} L_{1}(x)^{n+1} d x=\int_{-\max _{s \in[0,1]}|Y(s)|}^{\max _{s \in[0,1]}|Y(s)|} L_{1}(x)^{n+1} d x
$$

Therefore,

$$
\begin{align*}
\mathbb{E}\left[V_{1}^{n}\right] & \leq \int_{\mathbb{R}} \mathbb{E}\left[L_{1}(x)^{n+1} \mathbf{1}_{|x| \leq \max _{s \in[0,1]}|Y(s)|}\right] d x \\
& \leq \int_{\mathbb{R}} \sqrt{\mathbb{E}\left[L_{1}(x)^{2(n+1)}\right]} \sqrt{\mathbb{P}\left[\max _{s \in[0,1]}|Y(s)| \geq|x|\right]} d x \\
& \leq K^{n+1}((2 n+2)!)^{3 / 8} \int_{\mathbb{R}} \sqrt{\mathbb{P}\left[\max _{s \in[0,1]}|Y(s)| \geq|x|\right]} d x \tag{10}
\end{align*}
$$

where we used (9) in the last inequality. By (8), the integral in (10) is finite. We deduce then from Stirling's formula that there exists a constant $C>0$ such that for any integer $n$,

$$
\mathbb{E}\left[V_{1}^{n}\right] \leq C^{n} n^{\frac{3}{4} n}
$$

Hence, for any $x>0$, and any integer $n$,

$$
\mathbb{P}\left[V_{1} \geq x\right] \leq x^{-n} C^{n} n^{\frac{3}{4} n}
$$

Optimizing over the values of $n$ leads to take $n=\left\lceil e^{-1}\left(\frac{x}{C}\right)^{4 / 3}\right\rceil$, so that $x^{-n} C^{n} n^{\frac{3}{4} n} \simeq \exp \left(-\frac{3}{4 e}\left(\frac{x}{C}\right)^{4 / 3}\right)$ for large $x$. This proves that the iterated Brownian motion satisfies (H4) with $\alpha=4 / 3$.

## 3. Auxiliary statements on $\left(\Delta_{t} ; t \geq 0\right)$

For a certain class of stochastic processes $\left\{X_{t} ; t \geq 0\right\}$ (to be specified below), Molchan [19] proved that the asymptotic behavior of

$$
\mathbb{P}\left[\sup _{t \in[0, T]} X_{t} \leq 1\right]
$$

is related to the quantity

$$
I(T):=\mathbb{E}\left[\left(\int_{0}^{T} e^{X_{t}} d t\right)^{-1}\right]
$$

We refer to [2] where the relationship between both quantities is clearly explained as well as the heuristics. Moreover $I(T)$ can be estimated by the following result.

Theorem 3.1 (Statement 1, [19]). Let $\left\{X_{t} ; t \geq 0\right\}$ be a continuous process, self-similar with index $h>0$ such that for every $T>0$,

$$
\left\{X_{T-t}-X_{T} ; t \in[0, T]\right\} \stackrel{(d)}{=}\left\{X_{t} ; t \in[0, T]\right\}
$$

Moreover assume that for every $\theta>0$,

$$
\mathbb{E}\left[\exp \left(\theta \max _{t \in[0,1]}\left|X_{t}\right|\right)\right]<+\infty
$$

Then, as $T \rightarrow+\infty$,

$$
\mathbb{E}\left[\left(\int_{0}^{T} e^{X_{t}} d t\right)^{-1}\right]=h T^{-(1-h)}\left(\mathbb{E}\left[\max _{t \in[0,1]} X_{t}\right]+o(1)\right)
$$

Before applying this result to our random process $\Delta$ we first establish some useful facts concerning it. First we show that the process satisfies

$$
\left\{\Delta_{T-t}-\Delta_{T} ; t \in[0, T]\right\} \stackrel{(d)}{=}\left\{\Delta_{t} ; t \in[0, T]\right\}
$$

and has stationary increments (Lemma 3.2). Next we provide an exponential upper bound for the tail of $\Delta_{1}$, from which we deduce the Kolmogorov-Centsov continuity criterion (Lemma 3.3). Finally we show that the process satisfies a maximal inequality (Lemma 3.4).

Lemma 3.2. Assume $(H 1)$ and $(H 3)$. The process $\Delta$ satisfies for every $T>0$,

$$
\begin{equation*}
\left\{\Delta_{T-t}-\Delta_{T} ; t \in[0, T]\right\} \stackrel{(d)}{=}\left\{\Delta_{t} ; t \in[0, T]\right\} \tag{11}
\end{equation*}
$$

Therefore, it has stationary increments, i.e. for every $s>0,\left\{\Delta_{t+s}-\Delta_{s} ; t \geq 0\right\} \stackrel{(d)}{=}\left\{\Delta_{t} ; t \geq 0\right\}$.
Proof. Conditionally to $\{Y(t) ; t \geq 0\}$, the process $\left\{\Delta_{T-t}-\Delta_{T} ; t \in[0, T]\right\}$ is a centered Gaussian process with (random) covariance function:

$$
C_{T}(s, t)=\int_{\mathbb{R}}\left(L_{T}(x)-L_{T-t}(x)\right)\left(L_{T}(x)-L_{T-s}(x)\right) d x
$$

Note that $L_{T}(x)-L_{T-t}(x)=L_{t}^{(T)}(x-Y(T))$, where $\left\{L_{t}^{(T)}(x) ; t \in[0, T], x \in \mathbb{R}\right\}$ is the local time process of $Y_{t}^{(T)}:=Y(T-t)-Y(T)$. Hence, $C_{T}(s, t)=\int_{\mathbb{R}} L_{t}^{(T)}(x) L_{s}^{(T)}(x) d x$. Now, if $Y^{(T)}$ has
the same distribution as $Y$ (respectively $-Y$ ), then $\left\{L_{t}^{(T)}(x) ; t \in[0, T], x \in \mathbb{R}\right\}$ has the same law as $\left\{L_{t}(x) ; t \in[0, T], x \in \mathbb{R}\right\}$ (respectively $\left\{L_{t}(-x) ; t \in[0, T], x \in \mathbb{R}\right\}$ ). We deduce then that in both cases, $\left\{C_{T}(s, t) ; s, t \in[0, T]\right\}$ is distributed as the conditional covariance of $\left\{\Delta_{t} ; t \in[0, T]\right\}$ with respect to $Y$. Hence,

$$
\left\{\Delta_{T-t}-\Delta_{T} ; t \in[0, T]\right\} \stackrel{(d)}{=}\left\{\Delta_{t} ; t \in[0, T]\right\}
$$

Concerning the increments stationarity, fix $s>0$, and $0<t_{1}<\cdots<t_{n}$. Let $T>t_{n}+s$ be fixed.

$$
\begin{aligned}
\left(\Delta_{t_{1}+s}\right. & \left.-\Delta_{s}, \cdots, \Delta_{t_{n}+s}-\Delta_{s}\right) \\
& \stackrel{(d)}{=}\left(\Delta_{T-t_{1}-s}-\Delta_{T-s}, \cdots, \Delta_{T-t_{n}-s}-\Delta_{T-s}\right) \text { by time reversal at time } T \\
& \stackrel{(d)}{=}\left(\Delta_{t_{1}}, \cdots, \Delta_{t_{n}}\right) \text { by time reversal at time } T-s
\end{aligned}
$$

Lemma 3.3. Assume (H1) and (H4) hold. There exist $C>0$ and $\delta>0$ such that for any $x>0$,

$$
\begin{equation*}
\mathbb{P}\left[\Delta_{1} \geq x\right] \leq C \exp \left(-\delta x^{\frac{2 \alpha}{1+\alpha}}\right) \tag{12}
\end{equation*}
$$

Then, for every $a \geq 1$,

$$
\begin{equation*}
\left(\mathbb{E}\left|\Delta_{t}-\Delta_{s}\right|^{a}\right)^{1 / a}=C(a)|t-s|^{h}, t, s \geq 0 \tag{13}
\end{equation*}
$$

where $C(a) \leq c a^{\nu}$ with $\nu:=\frac{1}{2}\left(1+\frac{1}{\alpha}\right)$. In particular the process satisfies the KolmogorovCentsov criterion of continuity.

Proof. Conditionally to the process $Y$, the random variable $\Delta_{1}$ is a real centered Gaussian variable with variance $V_{1}$. For each $u \in \mathbb{R}$, let

$$
\Phi(u)=\frac{1}{\sqrt{2 \pi}} \int_{u}^{+\infty} e^{-\frac{s^{2}}{2}} d s
$$

Then for $x>0$ and $\theta>0$,

$$
\begin{aligned}
\mathbb{P}\left[\Delta_{1} \geq x\right]=\int_{0}^{+\infty} \Phi\left(x z^{-1 / 2}\right) \mathbb{P}_{V_{1}}(d z) & =\int_{0}^{x^{\theta}} \Phi\left(x z^{-1 / 2}\right) \mathbb{P}_{V_{1}}(d z)+\int_{x^{\theta}}^{+\infty} \Phi\left(x z^{-1 / 2}\right) \mathbb{P}_{V_{1}}(d z) \\
& \leq \Phi\left(x^{1-\frac{\theta}{2}}\right)+\mathbb{P}\left[V_{1} \geq x^{\theta}\right]
\end{aligned}
$$

where we used that the function $z \mapsto \Phi\left(x z^{-1 / 2}\right)$ is non decreasing with values in $[0,1]$. Using (H4) and the classical inequality

$$
\Phi(u) \leq \frac{1}{\sqrt{2 \pi} u} e^{-u^{2} / 2} \text { for every } u>0
$$

and choosing $\theta=\frac{2}{1+\alpha}$, we obtain that for $x$ large enough,

$$
\mathbb{P}\left[\Delta_{1} \geq x\right] \leq \frac{x^{-\alpha /(1+\alpha)}}{\sqrt{2 \pi}} e^{-\frac{1}{2} x^{2 \alpha /(1+\alpha)}}+C e^{-c x^{2 \alpha /(1+\alpha)}} \leq C e^{-\delta x^{2 \alpha /(1+\alpha)}}
$$

with $\delta=\min (c, 1 / 2)$. This implies (12) for every $x>0$.
Let us prove (13). The increments of the process $\Delta$ being stationary (see Lemma (3.2)), by self-similarity, we have for every $t, s \geq 0$,

$$
\mathbb{E}\left[\left|\Delta_{t}-\Delta_{s}\right|^{a}\right]=|t-s|^{h a} \mathbb{E}\left[\left|\Delta_{1}\right|^{a}\right]
$$

Now, using (12) and the symmetry of $\Delta_{1}$,

$$
\begin{align*}
\mathbb{E}\left[\left|\Delta_{1}\right|^{a}\right] & =\int_{0}^{+\infty} \mathbb{P}\left[\left|\Delta_{1}\right| \geq x^{1 / a}\right] d x \leq 2 \int_{0}^{+\infty} \mathbb{P}\left[\Delta_{1} \geq x^{1 / a}\right] d x \\
& \leq C \int_{0}^{+\infty} \exp \left(-\delta x^{\frac{2 \alpha}{a(1+\alpha)}}\right) d x=C a \delta^{-\frac{a(1+\alpha)}{2 \alpha}} \Gamma\left(\frac{a(1+\alpha)}{2 \alpha}\right) . \tag{14}
\end{align*}
$$

Hence, it follows from Stirling's formula that $C(a)=\mathbb{E}\left[\left|\Delta_{1}\right|^{a}\right]^{1 / a} \leq C a^{(1+\alpha) /(2 \alpha)}$.

Lemma 3.4. Assume (H1) and (H4) hold. Let $T, x \geq 0$. Then

$$
\begin{gather*}
\mathbb{P}\left[\max _{s \in[0, T]} \Delta_{s} \geq x \mid Y\right] \leq 2 \mathbb{P}\left[\Delta_{T} \geq x \mid Y\right],  \tag{15}\\
\mathbb{P}\left[\max _{s \in[0, T]} \Delta_{s} \geq x\right] \leq 2 \mathbb{P}\left[\Delta_{T} \geq x\right] . \tag{16}
\end{gather*}
$$

Proof. Conditionally to the process $Y$, the process $\left\{\Delta_{t} ; t \geq 0\right\}$ is a centered Gaussian process on $\mathbb{R}$ with covariance function

$$
\mathbb{E}\left[\Delta_{s} \Delta_{t} \mid Y\right]=\int_{\mathbb{R}} L_{s}(x) L_{t}(x) d x
$$

Moreover for any $t \geq s \geq 0$,

$$
\begin{aligned}
\mathbb{E}\left[\Delta_{t}^{2} \mid Y\right]-\mathbb{E}\left[\Delta_{s}^{2} \mid Y\right]-\mathbb{E}\left[\left(\Delta_{t}-\Delta_{s}\right)^{2} \mid Y\right] & =\int_{\mathbb{R}}\left(L_{t}^{2}(x)-L_{s}^{2}(x)-\left(L_{t}(x)-L_{s}(x)\right)^{2}\right) d x \\
& =\int_{\mathbb{R}} 2 L_{s}(x)\left(L_{t}(x)-L_{s}(x)\right) d x \geq 0,
\end{aligned}
$$

hence applying Proposition 2.2 in [15], we deduce the inequality (15). By integrating we obtain the maximal inequality (16).

We are now in position to use Theorem 3.1 and to state the main result of this section.
Proposition 3.5. Assume (H1) to (H4) hold. As $T \rightarrow+\infty$,

$$
\mathbb{E}\left[\left(\int_{0}^{T} e^{\Delta_{t}} d t\right)^{-1}\right]=\left(1-\frac{\gamma}{2}\right) T^{-\gamma / 2}\left(\mathbb{E}\left[\max _{t \in[0,1]} \Delta_{t}\right]+o(1)\right) .
$$

Proof. It follows easily from assumption (H2), that the process $\left\{\Delta_{t} ; t \geq 0\right\}$ is self-similar with index $h:=1-\frac{\gamma}{2}$. From Lemma 3.2, it satisfies (11). From (13) and Kolmogorov-Centsov continuity criterion, we can assume that it is continuous. Hence, to apply Theorem 3.1, it is enough to prove that for every $\theta>0$,

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(\theta \max _{t \in[0,1]}\left|\Delta_{t}\right|\right)\right]<+\infty \tag{17}
\end{equation*}
$$

Let $\theta>0$. We have

$$
\begin{aligned}
\mathbb{E}\left[\exp \left(\theta \max _{t \in[0,1]}\left|\Delta_{t}\right|\right)\right] & =\int_{0}^{\infty} \mathbb{P}\left[\exp \left(\theta \max _{t \in[0,1]}\left|\Delta_{t}\right|\right) \geq \lambda\right] d \lambda \\
& \leq 2+\int_{2}^{\infty} \mathbb{P}\left[\max _{t \in[0,1]}\left|\Delta_{t}\right| \geq \frac{\ln (\lambda)}{\theta}\right] d \lambda .
\end{aligned}
$$

Since the process $\left\{\Delta_{t} ; t \geq 0\right\}$ is symmetric and satisfies the maximal inequality (16) of Lemma 3.4,

$$
\begin{aligned}
\mathbb{P}\left[\max _{t \in[0,1]}\left|\Delta_{t}\right| \geq \frac{\ln (\lambda)}{\theta}\right] & \leq 2 \mathbb{P}\left[\max _{t \in[0,1]} \Delta_{t} \geq \frac{\ln (\lambda)}{\theta}\right] \\
& \leq 4 \mathbb{P}\left[\Delta_{1} \geq(\ln \lambda) / \theta\right]
\end{aligned}
$$

We apply the inequality (12) of Lemma 3.3, and since the function $\lambda \rightarrow \exp \left(-\delta((\ln \lambda) / \theta)^{2 \alpha /(1+\alpha)}\right)$ is integrable at infinity for any $\alpha>1$, the assertion (17) follows.

## 4. Proof of Theorem 1

In this section, we prove upper and lower bounds on the persistence probability $\mathbb{P}\left[\sup _{t \in[0, T)} \Delta(t) \leq 1\right]$.
In the case of fractional Brownian motion $\left\{B_{H}(t) ; t \geq 0\right\}$, Aurzada's proof of the lower bound (see [2]), rests on both following arguments: the fractional Brownian motion satisfies the hypothesis of Theorem 3.1, and it satisfies the equality (valid for $a$ large enough)

$$
\begin{equation*}
\left(\mathbb{E}\left|B_{H}(t)-B_{H}(s)\right|^{a}\right)^{1 / a}=C(a)|t-s|^{H}, t, s \geq 0 \tag{18}
\end{equation*}
$$

with $C(a) \leq c a^{\nu}$, for some $c$ and $\nu>0$. We showed in Lemmas 3.2 and 3.3 that these conditions are also satisfied by our random process $\Delta$. Therefore the proof of Aurzada [2] allows us to derive the lower bound in Theorem 1.

As in [19] and [2], the main idea in the proof of the upper bound in (2), is to bound $I(T)$ from below by restricting the expectation to a well-chosen set of paths.

Conditionally to $Y$, the process $\left\{\Delta_{t} ; t \geq 0\right\}$ is a centered Gaussian process such that for every $0 \leq t<s$,

$$
\begin{aligned}
\mathbb{E}\left[\Delta_{t} \Delta_{s} \mid Y\right] & =\int_{\mathbb{R}} L_{t}(x) L_{s}(x) d x \geq 0 \\
\mathbb{E}\left[\Delta_{t}\left(\Delta_{s}-\Delta_{t}\right) \mid Y\right] & =\int_{\mathbb{R}} L_{t}(x)\left(L_{s}(x)-L_{t}(x)\right) d x \geq 0,
\end{aligned}
$$

since $t \rightarrow L_{t}(x)$ is a.s. increasing for all $x \in \mathbb{R}$. It follows then from Slepian's lemma, that for every $0 \leq u<v<w$ and every real numbers $a, b$,

$$
\begin{array}{r}
\mathbb{P}\left[\sup _{t \in[u, v]} \Delta_{t} \leq a, \sup _{t \in[v, w]} \Delta_{t} \leq b \mid Y\right] \geq \mathbb{P}\left[\sup _{t \in[u, v]} \Delta_{t} \leq a \mid Y\right] \mathbb{P}\left[\sup _{t \in[v, w]} \Delta_{t} \leq b \mid Y\right] \\
\mathbb{P}\left[\sup _{t \in[u, v]} \Delta_{t} \leq a, \sup _{t \in[v, w]}\left(\Delta_{t}-\Delta_{v}\right) \leq b \mid Y\right] \geq \mathbb{P}\left[\sup _{t \in[u, v]} \Delta_{t} \leq a \mid Y\right] \mathbb{P}\left[\sup _{t \in[v, w]}\left(\Delta_{t}-\Delta_{v}\right) \leq b \mid Y\right] . \tag{20}
\end{array}
$$

Let $a_{T}=(\ln T)^{a}$, where $a>0$ will be chosen later; let $\alpha_{T}:=\frac{\ln T}{1+\ln T}$, which belongs to $(0,1)$ and set $\beta_{T}=\sqrt{V_{a_{T}}}$ where $V_{a_{T}}:=\int_{\mathbb{R}} L_{a_{T}}^{2}(x) d x$. Let us define the random function

$$
\phi(t):= \begin{cases}1 & \text { for } 0 \leq t<a_{T} \\ 1-\beta_{T} & \text { for } a_{T} \leq t \leq T\end{cases}
$$

which is $Y$-measurable. Clearly, we have

$$
\mathbb{E}\left[\left(\int_{0}^{T} e^{\Delta_{t}} d t\right)^{-\alpha_{T}} \mid Y\right] \geq\left(\int_{0}^{T} e^{\phi(t)} d t\right)^{-\alpha_{T}} \mathbb{P}\left[\forall t \in[0, T], \Delta_{t} \leq \phi(t) \mid Y\right]
$$

By Slepian's lemma (see (19)), we have

$$
\mathbb{P}\left[\forall t \in[0, T], \Delta_{t} \leq \phi(t) \mid Y\right] \geq \mathbb{P}\left[\forall t \in\left[0, a_{T}\right], \Delta_{t} \leq 1 \mid Y\right] \mathbb{P}\left[\forall t \in\left[a_{T}, T\right], \Delta_{t} \leq 1-\beta_{T} \mid Y\right]
$$

Remark that

$$
\begin{aligned}
\mathbb{P}\left[\forall t \in\left[a_{T}, T\right], \Delta_{t} \leq 1-\beta_{T} \mid Y\right] & \geq \mathbb{P}\left[\Delta_{a_{T}} \leq-\beta_{T} ; \forall t \in\left[a_{T}, T\right], \Delta_{t}-\Delta_{a_{T}} \leq 1 \mid Y\right] \\
& \geq \mathbb{P}\left[\Delta_{a_{T}} \leq-\beta_{T} \mid Y\right] \mathbb{P}\left[\forall t \in\left[a_{T}, T\right], \Delta_{t}-\Delta_{a_{T}} \leq 1 \mid Y\right],
\end{aligned}
$$

by Slepian's lemma (see (20)). Note that

$$
\mathbb{P}\left[\Delta_{a_{T}} \leq-\beta_{T} \mid Y\right]=\Phi\left(\beta_{T} V_{a_{T}}^{-1 / 2}\right)=\Phi(1)
$$

Moreover, it is easy to check that when $T$ goes to infinity,

$$
\int_{0}^{T} e^{\phi(t)} d t=O\left(a_{T}+T e^{-\beta_{T}}\right) .
$$

In the following $C$ is a constant whose value may change but does not depend on $T$. Then we can write that for $T$ large enough
$\mathbb{E}\left[\left(\int_{0}^{T} e^{\Delta_{t}} d t\right)^{-\alpha_{T}} \mid Y\right] \geq C\left(a_{T}+T e^{-\beta_{T}}\right)^{-\alpha_{T}} \mathbb{P}\left[\sup _{t \in\left[0, a_{T}\right]} \Delta_{t} \leq 1 \mid Y\right] \mathbb{P}\left[\sup _{t \in\left[a_{T}, T\right]}\left(\Delta_{t}-\Delta_{a_{T}}\right) \leq 1 \mid Y\right]$.
Next we use the maximal inequality (15) of Lemma 3.4 to write

$$
\begin{equation*}
\mathbb{P}\left[\sup _{t \in\left[0, a_{T}\right]} \Delta_{t} \leq 1 \mid Y\right]=1-\mathbb{P}\left[\sup _{t \in\left[0, a_{T}\right]} \Delta_{t} \geq 1 \mid Y\right] \geq 1-2 \mathbb{P}\left[\Delta_{a_{T}} \geq 1 \mid Y\right]=\mathbb{P}\left[|Z| \leq V_{a_{T}}^{-1 / 2} \mid Y\right] \tag{21}
\end{equation*}
$$

where $Z$ is a Gaussian variable $\mathcal{N}(0,1)$ independent of $Y$, from which we deduce that there exists a constant $c>0$ such that

$$
\begin{equation*}
\mathbb{P}\left[\sup _{t \in\left[0, a_{T}\right]} \Delta_{t} \leq 1 \mid Y\right] \geq c \min \left(V_{a_{T}}^{-1 / 2}, 1\right) . \tag{22}
\end{equation*}
$$

Injecting (22) into (21) we get that for $T$ large enough,

$$
\mathbb{P}\left[\sup _{t \in\left[a_{T}, T\right]}\left(\Delta_{t}-\Delta_{a_{T}}\right) \leq 1 \mid Y\right] \leq C \mathbb{E}\left[\left(\int_{0}^{T} e^{\Delta_{t}} d t\right)^{-\alpha_{T}} \mid Y\right]\left(a_{T}+T e^{-V_{a_{T}}^{1 / 2}}\right)^{\alpha_{T}} \max \left(V_{a_{T}}^{1 / 2}, 1\right) .
$$

By integrating and using successively Hölder's inequality with $p_{T}=\frac{1}{\alpha_{T}}, \frac{1}{q_{T}}+\frac{1}{p_{T}}=1$, Jensen's inequality, the inequality $(x+y)^{\alpha_{T}} \leq x^{\alpha_{T}}+y^{\alpha_{T}}$ for $x, y>0$, and Proposition 3.5, we get that for $T$ large enough,

$$
\begin{aligned}
\mathbb{P}\left[\sup _{t \in\left[a_{T}, T\right]}\right. & \left.\left(\Delta_{t}-\Delta_{a_{T}}\right) \leq 1\right] \\
& \leq C \mathbb{E}\left[\left(\int_{0}^{T} e^{\Delta_{t}} d t\right)^{-1}\right]^{1 / p_{T}} \mathbb{E}\left[\left(a_{T}^{\alpha_{T}} \max \left(V_{a_{T}}^{1 / 2}, 1\right)+T^{\alpha_{T}} e^{-\alpha_{T} V_{a_{T}}^{1 / 2}} \max \left(V_{a_{T}}^{1 / 2}, 1\right)\right)^{q_{T}}\right]^{1 / q_{T}} \\
& \leq C T^{-\frac{\gamma}{2 p_{T}}}\left\|f_{1}+f_{2}\right\|_{q_{T}} \leq C T^{-\frac{\gamma}{2 p_{T}}}\left(\left\|f_{1}\right\|_{q_{T}}+\left\|f_{2}\right\|_{q_{T}}\right),
\end{aligned}
$$

with

$$
f_{1}=a_{T}^{\alpha_{T}} \max \left(V_{a_{T}}^{1 / 2}, 1\right), \quad f_{2}=T^{\alpha_{T}} e^{-\alpha_{T} V_{a_{T}}^{1 / 2}} \max \left(V_{a_{T}}^{1 / 2}, 1\right)
$$

The lefthand term is greater than the quantity we want to bound from above, since by stationarity,

$$
\mathbb{P}\left[\sup _{t \in\left[a_{T}, T\right]}\left(\Delta_{t}-\Delta_{a_{T}}\right) \leq 1\right]=\mathbb{P}\left[\sup _{t \in\left[0, T-a_{T}\right]} \Delta_{t} \leq 1\right] \geq \mathbb{P}\left[\sup _{t \in[0, T]} \Delta_{t} \leq 1\right] .
$$

Concerning the righthand term, we recall that $\alpha_{T}=\frac{1}{p_{T}}=\frac{\ln T}{1+\ln T}$ and $\frac{1}{q_{T}}=1-\alpha_{T}=\frac{1}{1+\ln T}$. Hence, when $T$ goes to infinity, $T^{-\frac{\gamma}{2 p_{T}}} \leq C T^{-\frac{\gamma}{2}}$. Therefore,

$$
\begin{equation*}
\mathbb{P}\left[\sup _{t \in[0, T]} \Delta_{t} \leq 1\right] \leq C T^{-\frac{\gamma}{2}}\left(\left\|f_{1}\right\|_{q_{T}}+\left\|f_{2}\right\|_{q_{T}}\right) . \tag{23}
\end{equation*}
$$

It remains to prove that $\left\|f_{1}\right\|_{q_{T}}$ and $\left\|f_{2}\right\|_{q_{T}}$ are bounded by logarithmic terms.

$$
\left\|f_{1}\right\|_{q_{T}}=a_{T}^{\alpha_{T}} \mathbb{E}\left[V_{a_{T}}^{q_{T} / 2} \mathbf{1}_{V_{a_{T}} \geq 1}+\mathbf{1}_{V_{a_{T}} \leq 1}\right]^{1 / q_{T}} \leq a_{T}^{\alpha_{T}}\left(\mathbb{E}\left[V_{a_{T}}^{q_{T}}\right]^{1 /\left(2 q_{T}\right)}+1\right)
$$

By (H2), $V_{a_{T}} \stackrel{\mathcal{L}}{=} a_{T}^{2-\gamma} V_{1}$. Therefore,

$$
\mathbb{E}\left[V_{a_{T}}^{q_{T}}\right]^{1 /\left(2 q_{T}\right)}=a_{T}^{1-\frac{\gamma}{2}} \mathbb{E}\left[V_{1}^{q_{T}}\right]^{1 /\left(2 q_{T}\right)}
$$

As in (14), it follows from (H4) that

$$
\mathbb{E}\left[V_{1}^{m}\right] \leq C c^{-m / \alpha} \frac{m}{\alpha} \Gamma\left(\frac{m}{\alpha}\right) \text { for every } m \in \mathbb{N}
$$

so that using Stirling's formula, it is easy to show that for $T$ large enough

$$
\mathbb{E}\left[V_{1}^{q_{T}}\right]^{1 / 2 q_{T}} \leq C(\ln T)^{\frac{1}{2 \alpha}} .
$$

We conclude that for $T$ large enough

$$
\begin{equation*}
\left\|f_{1}\right\|_{q_{T}} \leq C(\ln T)^{a\left(2-\frac{\gamma}{2}\right)+\frac{1}{2 \alpha}} \tag{24}
\end{equation*}
$$

Let us now turn our attention to $\left\|f_{2}\right\|_{q_{T}}$.

$$
\begin{aligned}
\left\|f_{2}\right\|_{q_{T}} & \leq T^{\alpha_{T}} \mathbb{E}\left[e^{-2 q_{T} \alpha_{T} V_{a_{T}}^{1 / 2}}\right]^{1 / 2 q_{T}} \mathbb{E}\left[\max \left(V_{a_{T}}, 1\right)^{q_{T}}\right]^{1 / 2 q_{T}} \\
& \leq C T(\ln T)^{a\left(1-\frac{\gamma}{2}\right)+\frac{1}{2 \alpha}} \mathbb{E}\left[e^{-2 q_{T} \alpha_{T} a_{T}^{1-\frac{\gamma}{2}} \sqrt{V_{1}}}\right]^{1 / 2 q_{T}}
\end{aligned}
$$

Let us note $\lambda_{T}=2 q_{T} \alpha_{T} a_{T}^{1-\frac{\gamma}{2}}=2(\ln T)^{1+a\left(1-\frac{\gamma}{2}\right)}$. Then using (H5),

$$
\begin{aligned}
\mathbb{E}\left[e^{-\lambda_{T} V_{1}^{1 / 2}}\right] & \leq \int_{0}^{+\infty} \mathbb{P}\left[V_{1} \leq \frac{u^{2}}{\lambda_{T}^{2}}\right] e^{-u} d u \\
& \leq C \int_{0}^{+\infty} e^{-c \frac{\lambda_{T}^{\lambda^{2 \beta}}}{u^{2 \beta}}} e^{-u} d u
\end{aligned}
$$

for some constants $c$ and $C$. We perform the change of variable $u=\lambda_{T}^{\frac{2 \beta}{1+2 \beta}} v$ in the preceding integral. This yields

$$
\begin{aligned}
& \mathbb{E}\left[e^{-\lambda_{T} V_{1}^{1 / 2}}\right] \leq C \lambda_{T}^{2 \beta /(1+2 \beta)} \int_{0}^{+\infty} e^{-\lambda_{T}^{1+2 \beta}}\left(v+c v^{-2 \beta}\right) \\
& 10 \\
& \leq C \lambda_{T}^{2 \beta /(1+2 \beta)} \int_{1}^{+\infty} e^{-\lambda_{T}^{\frac{2 \beta}{1+2 \beta}} v} d v+C \lambda_{T}^{2 \beta /(1+2 \beta)} \int_{0}^{1} e^{-\lambda_{T}^{\frac{2 \beta}{1+2 \beta}} c v^{-2 \beta}} d v \\
& \leq C e^{-\lambda_{T}^{\frac{2 \beta}{1+2 \beta}}}+C e^{-c \lambda_{T}^{\frac{2 \beta}{1+2 \beta}}} \\
& \leq C e^{-c \lambda_{T}^{\frac{2 \beta}{1+2 \beta}}}
\end{aligned}
$$

for some other constants $c$ and $C$. This leads to

$$
\left\|f_{2}\right\|_{q_{T}} \leq C T(\ln T)^{a\left(1-\frac{\gamma}{2}\right)+\frac{1}{2 \alpha}} e^{-c(\ln T)^{\frac{2 \beta}{1+2 \beta}\left(1+a\left(1-\frac{\gamma}{2}\right)\right)-1}}
$$

We choose $a$ such that $a\left(1-\frac{\gamma}{2}\right)>1+\frac{1}{\beta}$. Then $T e^{-c(\ln T)^{\frac{2 \beta}{1+2 \beta}\left(1+a\left(1-\frac{\gamma}{2}\right)\right)-1}}$ remains bounded, and when $T$ goes to infinity, we get

$$
\begin{equation*}
\left\|f_{2}\right\|_{q_{T}} \leq C(\ln T)^{c} \tag{25}
\end{equation*}
$$

for some constant $c>1+\frac{1}{\beta}+\frac{1}{2 \alpha}$. From (23), (24) and (25) we deduce that for $T$ large enough,

$$
\mathbb{P}\left[\sup _{t \in[0, T]} \Delta_{t} \leq 1\right] \leq C T^{-\frac{\gamma}{2}}(\ln T)^{c}
$$

with $c>\frac{\beta+1}{\beta} \frac{4-\gamma}{2-\gamma}+\frac{1}{2 \alpha}$.
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