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PERSISTENCE EXPONENT FOR RANDOM PROCESSES IN BROWNIAN SCENERY

FABIENNE CASTELL, NADINE GUILLOTIN-PLANTARD, AND FRÉDÉRIQUE WATBLED

ABSTRACT. In this paper we consider the persistence properties of random processes in Brownian scenery, which are examples of non-Markovian and non-Gaussian processes. More precisely we study the asymptotic behaviour for large T , of the probability $\mathbb{P}[\sup_{t \in [0, T]} \Delta_t \leq 1]$ where $\Delta_t = \int_{\mathbb{R}} L_t(x) dW(x)$. Here $W = \{W(x); x \in \mathbb{R}\}$ is a two-sided standard real Brownian motion and $\{L_t(x); x \in \mathbb{R}, t \geq 0\}$ is the local time of some self-similar random process Y , independent from the process W . We thus generalize the results of [7] where the increments of Y were assumed to be independent.

1. INTRODUCTION

Let $W = \{W(x); x \in \mathbb{R}\}$ be a standard two-sided real Brownian motion and $Y = \{Y(t); t \geq 0\}$ be a real-valued self-similar process of index $\gamma \in (0, 2)$ (i.e. for any $c > 0$, $\{Y(ct); t \geq 0\} \stackrel{(d)}{=} \{c^\gamma Y(t); t \geq 0\}$), with stationary increments. When it exists, we will denote by $\{L_t(x); x \in \mathbb{R}, t \geq 0\}$ a version of the local time of the process $\{Y(t); t \geq 0\}$. The process L satisfies the occupation density formula: for each bounded measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ and for each $t \geq 0$,

$$\int_0^t f(Y(s)) ds = \int_{\mathbb{R}} f(x) L_t(x) dx. \quad (1)$$

The processes W and Y are defined on the same probability space and are assumed to be independent. We consider *the random process in Brownian scenery* $\{\Delta_t; t \geq 0\}$ defined as

$$\Delta_t = \int_{\mathbb{R}} L_t(x) dW(x).$$

The process Δ is itself a self-similar process of index h with stationary increments, with

$$h := 1 - \frac{\gamma}{2}.$$

This process can be seen as a mixture of Gaussian processes, but it is neither Gaussian nor Markovian. It appeared independently in the mathematical literature (see [14],[3]), and in the physics literature (see [18]) where it was originally introduced to model diffusion in layered white-noise velocity field. Note indeed that Δ_t can formally be written as $\int_0^t \dot{W}(Y(s)) ds$. Thus, it describes the horizontal motion of a tracer particle in a (1+1)-dimensional medium, where the motion of the particle along the vertical direction y is described by the process Y , while along the horizontal direction, the particle is driven by the white noise velocity field $\dot{W}(y)$, that depends only on the vertical coordinate y . The process $\{\Delta_t; t \geq 0\}$ provides a simple example of anomalous super-diffusion, being for instance of order $t^{3/4}$ when Y is a Brownian motion. In

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this paper, we are interested in the *persistence probability* of the process Δ , i.e. the asymptotic behaviour of

$$\mathcal{F}(T) := \mathbb{P} \left[\sup_{t \in [0, T]} \Delta_t \leq 1 \right]$$

as $T \rightarrow +\infty$. The study of the one-sided supremum process of random processes is a classical issue in probability. One usually gets polynomial decay of the persistence probability: $\mathcal{F}(T) \asymp T^{-\theta}$ for a non-negative θ , often called *persistence exponent*, or *survival exponent*. Classical examples, where this exponent can be computed, include random walks or Lévy processes, and we refer the reader to the recent survey papers [1, 5] for an account of models where the persistence exponent is known. However, there are relevant physical situations where this exponent remains unknown (see for instance [16, 5]). The persistence exponent of the Brownian motion in Brownian scenery was studied by Redner [20, 21], and Majumdar [17]. Based on physical arguments, numerical simulations and analogy with fractional Brownian motion, Redner and Majumdar conjectured the value of the persistence exponent. In [7], their conjecture was proved up to logarithmic factors, when the process Y is a stable Lévy process with index $\delta \in (1, 2]$. The proof of [7] depends heavily on the increments independence of the process Y and the question raises if it is possible to compute the persistence exponent without it. The aim of this paper is to answer this question, and to provide assumptions on Y allowing to compute the persistence exponent of the random process in Brownian scenery.

- (H1): There exists a continuous version $\{L_t(x); x \in \mathbb{R}, t \geq 0\}$ of the local time of Y ;
- (H2): Y is a self-similar process of index $\gamma \in (0, 2)$;
- (H3): For every $T > 0$, $\{Y(T-t) - Y(T); t \in [0, T]\}$ has the same distribution as either $\{Y(t); t \in [0, T]\}$ or $\{-Y(t); t \in [0, T]\}$;
- (H4): Let $V_1 := \int L_1^2(x) dx$ be the self-intersection local time of Y . There exist a real number $\alpha > 1$, and positive constants C, c such that for any $x \geq 0$,

$$\mathbb{P}[V_1 \geq x] \leq C \exp(-cx^\alpha);$$

- (H5): There exist a real number $\beta > 0$, and positive constants C, c such that for any $x > 0$,

$$\mathbb{P}[V_1 \leq x] \leq C \exp(-cx^{-\beta}).$$

Our main result is the following one.

Theorem 1. *Assume (H1) to (H5) hold. There exists a constant $c > 0$, such that for large enough T ,*

$$T^{-\gamma/2}(\ln T)^{-c} \leq \mathbb{P} \left[\sup_{t \in [0, T]} \Delta_t \leq 1 \right] \leq T^{-\gamma/2}(\ln T)^{+c}. \quad (2)$$

Our main achievement in this paper is to remove the increments independence assumption on $\{Y(t); t \geq 0\}$, which was crucial in [7]. Another interesting issue would be to obtain the persistence exponent for $\{W(x); x \in \mathbb{R}\}$ being a stable Lévy process. This seems delicate, our proof relying on the fact that the process $\{\Delta(t); t \geq 0\}$ is a Gaussian process, conditionally to $\{Y(t); t \geq 0\}$.

The paper is organized as follows. Section 2 provides three examples of processes satisfying (H1) to (H5), including the stable Lévy process with index $\delta \in (1, 2]$, and thus generalizing the result of [7]. Section 3 states some useful properties of the process Δ . Section 4 is devoted to the proof of Theorem 1.

2. SOME EXAMPLES

2.1. A sufficient condition for (H5). In this section we assume that (H1) holds and we provide a sufficient condition on $\{Y(t); t \geq 0\}$ allowing to check (H5). Let us first compare V_1 with $\max_{s \in [0,1]} |Y(s)|$. By noting that

$$1 = \int_{\{|x| \leq \max_{s \in [0,1]} |Y(s)|\}} L_1(x) dx \leq V_1^{1/2} \sqrt{2 \max_{s \in [0,1]} |Y(s)|},$$

we deduce that

$$\frac{1}{2 \max_{s \in [0,1]} |Y(s)|} \leq V_1. \quad (3)$$

Lemma 2.1. *Assume that (H1) holds and that there exist positive constants C and c , and a real number $\beta > 0$ such that for all $x > 0$,*

$$\mathbb{P} \left[\max_{s \in [0,1]} |Y(s)| \geq x \right] \leq C \exp(-cx^\beta).$$

Then (H5) holds true.

Proof. It is a direct consequence of (3) that for all $x > 0$,

$$\mathbb{P}[V_1 \leq x] \leq \mathbb{P} \left[\max_{s \in [0,1]} |Y(s)| \geq \frac{1}{2x} \right] \leq C \exp(-c2^{-\beta} x^{-\beta}).$$

□

2.2. Stable Lévy processes. A process $Y = \{Y(t); t \geq 0\}$ is a strictly stable Lévy process with index $\delta \in (1, 2]$ if it is a process with stationary and independent increments, such that $Y(0) = 0$ and for any $t \geq 0$ and $u \in \mathbb{R}$,

$$\mathbb{E}[e^{iuY(t)}] = \exp \left\{ -c|u|^\delta t \left(1 + i\zeta \operatorname{sgn}(u) \tan(\pi\delta/2) \right) \right\}, \quad (4)$$

where $\zeta \in [-1, 1]$, and c is a positive scale parameter.

Lemma 2.2. *The stable Lévy process with index $\delta \in (1, 2]$ satisfies (H1) to (H5) with $\gamma = \frac{1}{\delta}$, $\alpha = \delta$, and $\beta = \frac{\delta}{2\delta-1}$.*

Proof. It is immediate that the Lévy process is self-similar of index $1/\delta$ and that for every $T > 0$,

$$\{Y(T-t) - Y(T); t \in [0, T]\} \stackrel{(d)}{=} \{-Y(t); t \in [0, T]\}$$

then (H2) and (H3) are satisfied.

A continuous version of the local time exists, since $\delta > 1$ (this was proved by Boylan [4]), so it satisfies (H1). Moreover, Corollary 5.6 in [15] states that there exist positive constants C and ξ s.t. for every $x > 0$,

$$\mathbb{P}[V_1 \geq x] \leq C e^{-\xi x^\delta},$$

which gives (H4). To prove (H5) we let $V_t := \int L_t^2(x) dx$ be the self-intersection local time of Y up to time t for $t \geq 0$, and we show that there exists $l > 0$ such that

$$\mathbb{P}[V_t \leq 1] \leq e^{-lt} \text{ for } t \text{ large enough.} \quad (5)$$

As $\{V_t; t \geq 0\}$ is self-similar of index $\frac{2\delta-1}{\delta}$, (5) implies that for every positive ε small enough,

$$\mathbb{P}[V_1 \leq \varepsilon] = \mathbb{P}[V_{\varepsilon^{-\frac{\delta}{2\delta-1}}} \leq 1] \leq e^{-l\varepsilon^{-\frac{\delta}{2\delta-1}}},$$

which gives (H5). To prove (5) we show that the function defined on \mathbb{R}^+ by

$$f(t) = \log \mathbb{P}[V_t \leq 1]$$

is subadditive. Let us fix s, t in $[0, +\infty)$. We consider the process $Y^{(s)} := \{Y_u^{(s)}; u \geq 0\}$ defined by

$$Y_u^{(s)} = Y(u + s) - Y(s),$$

its local time $\{L_u^{(s)}(x); x \in \mathbb{R}, u \geq 0\}$, and its self-intersection local time $\{V_u^{(s)}; u \geq 0\}$. Note that

$$L_{t+s}(x) = L_s(x) + L_t^{(s)}(x - Y(s)).$$

Hence

$$V_{t+s} \geq V_s + V_t^{(s)}.$$

The process Y being a Lévy process, $V_t^{(s)}$ and V_s are independent and $V_t^{(s)}$ has the same law as V_t . Therefore

$$\mathbb{P}[V_{t+s} \leq 1] \leq \mathbb{P}[V_s \leq 1; V_t^{(s)} \leq 1] = \mathbb{P}[V_s \leq 1]\mathbb{P}[V_t \leq 1].$$

Thus f is subadditive, which implies that $\frac{f(t)}{t}$ converges, as $t \rightarrow +\infty$, towards a limit $-l = \inf_{t>0} \frac{f(t)}{t}$ (see [12]). It remains to show that l is strictly positive. As $-l$ is less than or equal to $f(1)$, it is enough to show that $\mathbb{P}[V_1 > 1] > 0$. But thanks to (3) and Proposition 10.3 of [10],

$$\mathbb{P}[V_1 > 1] \geq \mathbb{P}\left[\max_{s \in [0,1]} |Y(s)| < \frac{1}{2}\right] > 0,$$

and this concludes the proof of (5). \square

2.3. Fractional Brownian motion. The fractional Brownian motion of Hurst index $H \in (0, 1)$ is the real centered Gaussian process $\{B_H(t); t \geq 0\}$ with covariance function

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Lemma 2.3. *The fractional Brownian motion with Hurst index $H \in (0, 1)$ satisfies (H1) to (H5) with $\gamma = H$, $\alpha = 1/H$, $\beta = 2$.*

Proof. It follows readily from the definition that B_H is self-similar with index H and satisfies (H3). The existence of a jointly continuous version of its local time process for $H \in (0, 1)$ is a classical fact (see for instance paragraph 22 in [11]). Theorem 1 of [13] asserts that there exists $\lambda_0 > 0$ such that for every $0 < \lambda < \lambda_0$,

$$\mathbb{E}\left[e^{\lambda V_1^{1/H}}\right] < \infty, \tag{6}$$

which implies (H4) with $\alpha = 1/H$. Finally, (H5) follows from Lemma 2.1 and Fernique's estimation ([9], Theorem 4.1.1): there exists $c_F > 0$ such that for any $x \geq \sqrt{5}$,

$$\mathbb{P}\left[\max_{s \in [0,1]} |B_H(s)| \geq c_F x\right] \leq 10 \int_x^{+\infty} e^{-\frac{v^2}{2}} dv. \tag{7}$$

This implies (H5) with $\beta = 2$. \square

2.4. Iterated Brownian motion. Let $\{B(x); x \in \mathbb{R}\}$ be a two-sided real standard Brownian motion, and let $\{\tilde{B}(t); t \geq 0\}$ be a real standard Brownian motion, independent of $\{B(x); x \in \mathbb{R}\}$. The iterated Brownian motion is the process $\{Y(t); t \geq 0\}$ defined by

$$Y(t) = B(\tilde{B}(t)).$$

Lemma 2.4. *The iterated Brownian motion satisfies (H1) to (H5) with $\gamma = 1/4$, and $\alpha = \beta = 4/3$.*

Proof. The self-similarity of the iterated Brownian motion is a direct consequence of the self-similarity and independence of both Brownian motions. The assertion (H3) follows once again from the independence of B and \tilde{B} , the increments stationarity of B and the fact that \tilde{B} satisfies (H3).

The existence and joint continuity of the local times of iterated Brownian motion were proved in [6, 8]. Let us prove (H5). For $x > 0$,

$$\begin{aligned} \mathbb{P} \left[\max_{s \in [0,1]} |Y(s)| \geq x \right] &\leq \mathbb{P} \left[\max_{s \in [0,1]} |\tilde{B}(s)| \geq x^{2/3} \right] + \mathbb{P} \left[\max_{|u| \leq x^{2/3}} |B(u)| \geq x \right] \\ &\leq \mathbb{P} \left[\max_{s \in [0,1]} |\tilde{B}(s)| \geq x^{2/3} \right] + \mathbb{P} \left[\max_{u \in [0, x^{2/3}]} |B(u)| \geq x \right] \\ &\leq 2 \mathbb{P} \left[\max_{s \in [0,1]} |B(s)| \geq x^{2/3} \right] \\ &\leq 4 \mathbb{P} \left[\max_{s \in [0,1]} B(s) \geq x^{2/3} \right] = 4 \mathbb{P} \left[|B(1)| \geq x^{2/3} \right]. \end{aligned} \quad (8)$$

This proves (H5) with $\beta = 4/3$ using Lemma 2.1.

To prove (H4), we use the uniform norm on the local times of iterated Brownian motion proved in Lemma 4 of [22]: there exists a constant $K > 0$ such that for any even integer n ,

$$\sup_{x \in \mathbb{R}} \mathbb{E} [(L_1(x))^n] \leq K^n (n!)^{3/4}. \quad (9)$$

Since $\int_{\mathbb{R}} L_1(x) dx = 1$, Hölder's inequality implies that for any integer n ,

$$V_1^n = \left(\int_{\mathbb{R}} L_1(x)^2 dx \right)^n \leq \int_{\mathbb{R}} L_1(x)^{n+1} dx = \int_{-\max_{s \in [0,1]} |Y(s)|}^{\max_{s \in [0,1]} |Y(s)|} L_1(x)^{n+1} dx.$$

Therefore,

$$\begin{aligned} \mathbb{E} [V_1^n] &\leq \int_{\mathbb{R}} \mathbb{E} \left[L_1(x)^{n+1} \mathbf{1}_{|x| \leq \max_{s \in [0,1]} |Y(s)|} \right] dx \\ &\leq \int_{\mathbb{R}} \sqrt{\mathbb{E} [L_1(x)^{2(n+1)}]} \sqrt{\mathbb{P} \left[\max_{s \in [0,1]} |Y(s)| \geq |x| \right]} dx \\ &\leq K^{n+1} ((2n+2)!)^{3/8} \int_{\mathbb{R}} \sqrt{\mathbb{P} \left[\max_{s \in [0,1]} |Y(s)| \geq |x| \right]} dx, \end{aligned} \quad (10)$$

where we used (9) in the last inequality. By (8), the integral in (10) is finite. We deduce then from Stirling's formula that there exists a constant $C > 0$ such that for any integer n ,

$$\mathbb{E} [V_1^n] \leq C^n n^{\frac{3}{4}n}.$$

Hence, for any $x > 0$, and any integer n ,

$$\mathbb{P} [V_1 \geq x] \leq x^{-n} C^n n^{\frac{3}{4}n}.$$

Optimizing over the values of n leads to take $n = \left\lceil e^{-1} \left(\frac{x}{C}\right)^{4/3} \right\rceil$, so that $x^{-n} C^n n^{\frac{3}{4}n} \simeq \exp\left(-\frac{3}{4e} \left(\frac{x}{C}\right)^{4/3}\right)$ for large x . This proves that the iterated Brownian motion satisfies (H4) with $\alpha = 4/3$. \square

3. AUXILIARY STATEMENTS ON $(\Delta_t; t \geq 0)$

For a certain class of stochastic processes $\{X_t; t \geq 0\}$ (to be specified below), Molchan [19] proved that the asymptotic behavior of

$$\mathbb{P} \left[\sup_{t \in [0, T]} X_t \leq 1 \right]$$

is related to the quantity

$$I(T) := \mathbb{E} \left[\left(\int_0^T e^{X_t} dt \right)^{-1} \right].$$

We refer to [2] where the relationship between both quantities is clearly explained as well as the heuristics. Moreover $I(T)$ can be estimated by the following result.

Theorem 3.1 (Statement 1, [19]). *Let $\{X_t; t \geq 0\}$ be a continuous process, self-similar with index $h > 0$ such that for every $T > 0$,*

$$\{X_{T-t} - X_T; t \in [0, T]\} \stackrel{(d)}{=} \{X_t; t \in [0, T]\}.$$

Moreover assume that for every $\theta > 0$,

$$\mathbb{E} \left[\exp \left(\theta \max_{t \in [0, 1]} |X_t| \right) \right] < +\infty.$$

Then, as $T \rightarrow +\infty$,

$$\mathbb{E} \left[\left(\int_0^T e^{X_t} dt \right)^{-1} \right] = hT^{-(1-h)} \left(\mathbb{E} \left[\max_{t \in [0, 1]} X_t \right] + o(1) \right).$$

Before applying this result to our random process Δ we first establish some useful facts concerning it. First we show that the process satisfies

$$\{\Delta_{T-t} - \Delta_T; t \in [0, T]\} \stackrel{(d)}{=} \{\Delta_t; t \in [0, T]\}$$

and has stationary increments (Lemma 3.2). Next we provide an exponential upper bound for the tail of Δ_1 , from which we deduce the Kolmogorov-Centsov continuity criterion (Lemma 3.3). Finally we show that the process satisfies a maximal inequality (Lemma 3.4).

Lemma 3.2. *Assume (H1) and (H3). The process Δ satisfies for every $T > 0$,*

$$\{\Delta_{T-t} - \Delta_T; t \in [0, T]\} \stackrel{(d)}{=} \{\Delta_t; t \in [0, T]\}. \quad (11)$$

Therefore, it has stationary increments, i.e. for every $s > 0$, $\{\Delta_{t+s} - \Delta_s; t \geq 0\} \stackrel{(d)}{=} \{\Delta_t; t \geq 0\}$.

Proof. Conditionally to $\{Y(t); t \geq 0\}$, the process $\{\Delta_{T-t} - \Delta_T; t \in [0, T]\}$ is a centered Gaussian process with (random) covariance function:

$$C_T(s, t) = \int_{\mathbb{R}} (L_T(x) - L_{T-t}(x)) (L_T(x) - L_{T-s}(x)) dx.$$

Note that $L_T(x) - L_{T-t}(x) = L_t^{(T)}(x - Y(T))$, where $\{L_t^{(T)}(x); t \in [0, T], x \in \mathbb{R}\}$ is the local time process of $Y_t^{(T)} := Y(T - t) - Y(T)$. Hence, $C_T(s, t) = \int_{\mathbb{R}} L_t^{(T)}(x) L_s^{(T)}(x) dx$. Now, if $Y^{(T)}$ has

the same distribution as Y (respectively $-Y$), then $\{L_t^{(T)}(x); t \in [0, T], x \in \mathbb{R}\}$ has the same law as $\{L_t(x); t \in [0, T], x \in \mathbb{R}\}$ (respectively $\{L_t(-x); t \in [0, T], x \in \mathbb{R}\}$). We deduce then that in both cases, $\{C_T(s, t); s, t \in [0, T]\}$ is distributed as the conditional covariance of $\{\Delta_t; t \in [0, T]\}$ with respect to Y . Hence,

$$\{\Delta_{T-t} - \Delta_T; t \in [0, T]\} \stackrel{(d)}{=} \{\Delta_t; t \in [0, T]\}.$$

Concerning the increments stationarity, fix $s > 0$, and $0 < t_1 < \dots < t_n$. Let $T > t_n + s$ be fixed.

$$\begin{aligned} & (\Delta_{t_1+s} - \Delta_s, \dots, \Delta_{t_n+s} - \Delta_s) \\ & \stackrel{(d)}{=} (\Delta_{T-t_1-s} - \Delta_{T-s}, \dots, \Delta_{T-t_n-s} - \Delta_{T-s}) \text{ by time reversal at time } T, \\ & \stackrel{(d)}{=} (\Delta_{t_1}, \dots, \Delta_{t_n}) \text{ by time reversal at time } T - s. \end{aligned}$$

□

Lemma 3.3. *Assume (H1) and (H4) hold. There exist $C > 0$ and $\delta > 0$ such that for any $x > 0$,*

$$\mathbb{P}[\Delta_1 \geq x] \leq C \exp(-\delta x^{\frac{2\alpha}{1+\alpha}}). \quad (12)$$

Then, for every $a \geq 1$,

$$(\mathbb{E}|\Delta_t - \Delta_s|^a)^{1/a} = C(a)|t - s|^h, \quad t, s \geq 0 \quad (13)$$

where $C(a) \leq ca^\nu$ with $\nu := \frac{1}{2}(1 + \frac{1}{\alpha})$. In particular the process satisfies the Kolmogorov-Centsov criterion of continuity.

Proof. Conditionally to the process Y , the random variable Δ_1 is a real centered Gaussian variable with variance V_1 . For each $u \in \mathbb{R}$, let

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_u^{+\infty} e^{-\frac{s^2}{2}} ds.$$

Then for $x > 0$ and $\theta > 0$,

$$\begin{aligned} \mathbb{P}[\Delta_1 \geq x] &= \int_0^{+\infty} \Phi(xz^{-1/2}) \mathbb{P}_{V_1}(dz) = \int_0^{x^\theta} \Phi(xz^{-1/2}) \mathbb{P}_{V_1}(dz) + \int_{x^\theta}^{+\infty} \Phi(xz^{-1/2}) \mathbb{P}_{V_1}(dz) \\ &\leq \Phi(x^{1-\frac{\theta}{2}}) + \mathbb{P}[V_1 \geq x^\theta], \end{aligned}$$

where we used that the function $z \mapsto \Phi(xz^{-1/2})$ is non decreasing with values in $[0, 1]$. Using (H4) and the classical inequality

$$\Phi(u) \leq \frac{1}{\sqrt{2\pi}u} e^{-u^2/2} \text{ for every } u > 0,$$

and choosing $\theta = \frac{2}{1+\alpha}$, we obtain that for x large enough,

$$\mathbb{P}[\Delta_1 \geq x] \leq \frac{x^{-\alpha/(1+\alpha)}}{\sqrt{2\pi}} e^{-\frac{1}{2}x^{2\alpha/(1+\alpha)}} + C e^{-cx^{2\alpha/(1+\alpha)}} \leq C e^{-\delta x^{2\alpha/(1+\alpha)}},$$

with $\delta = \min(c, 1/2)$. This implies (12) for every $x > 0$.

Let us prove (13). The increments of the process Δ being stationary (see Lemma (3.2)), by self-similarity, we have for every $t, s \geq 0$,

$$\mathbb{E}[|\Delta_t - \Delta_s|^a] = |t - s|^{ha} \mathbb{E}[|\Delta_1|^a].$$

Now, using (12) and the symmetry of Δ_1 ,

$$\begin{aligned}\mathbb{E}[|\Delta_1|^a] &= \int_0^{+\infty} \mathbb{P}[|\Delta_1| \geq x^{1/a}] dx \leq 2 \int_0^{+\infty} \mathbb{P}[\Delta_1 \geq x^{1/a}] dx \\ &\leq C \int_0^{+\infty} \exp(-\delta x^{\frac{2\alpha}{a(1+\alpha)}}) dx = Ca\delta^{-\frac{a(1+\alpha)}{2\alpha}} \Gamma\left(\frac{a(1+\alpha)}{2\alpha}\right).\end{aligned}\quad (14)$$

Hence, it follows from Stirling's formula that $C(a) = \mathbb{E}[|\Delta_1|^a]^{1/a} \leq Ca^{(1+\alpha)/(2\alpha)}$.

□

Lemma 3.4. *Assume (H1) and (H4) hold. Let $T, x \geq 0$. Then*

$$\mathbb{P}\left[\max_{s \in [0, T]} \Delta_s \geq x | Y\right] \leq 2\mathbb{P}[\Delta_T \geq x | Y], \quad (15)$$

$$\mathbb{P}\left[\max_{s \in [0, T]} \Delta_s \geq x\right] \leq 2\mathbb{P}[\Delta_T \geq x]. \quad (16)$$

Proof. Conditionally to the process Y , the process $\{\Delta_t; t \geq 0\}$ is a centered Gaussian process on \mathbb{R} with covariance function

$$\mathbb{E}[\Delta_s \Delta_t | Y] = \int_{\mathbb{R}} L_s(x) L_t(x) dx.$$

Moreover for any $t \geq s \geq 0$,

$$\begin{aligned}\mathbb{E}[\Delta_t^2 | Y] - \mathbb{E}[\Delta_s^2 | Y] - \mathbb{E}[(\Delta_t - \Delta_s)^2 | Y] &= \int_{\mathbb{R}} (L_t^2(x) - L_s^2(x) - (L_t(x) - L_s(x))^2) dx \\ &= \int_{\mathbb{R}} 2L_s(x)(L_t(x) - L_s(x)) dx \geq 0,\end{aligned}$$

hence applying Proposition 2.2 in [15], we deduce the inequality (15). By integrating we obtain the maximal inequality (16). □

We are now in position to use Theorem 3.1 and to state the main result of this section.

Proposition 3.5. *Assume (H1) to (H4) hold. As $T \rightarrow +\infty$,*

$$\mathbb{E}\left[\left(\int_0^T e^{\Delta_t} dt\right)^{-1}\right] = \left(1 - \frac{\gamma}{2}\right) T^{-\gamma/2} \left(\mathbb{E}\left[\max_{t \in [0, 1]} \Delta_t\right] + o(1)\right).$$

Proof. It follows easily from assumption (H2), that the process $\{\Delta_t; t \geq 0\}$ is self-similar with index $h := 1 - \frac{\gamma}{2}$. From Lemma 3.2, it satisfies (11). From (13) and Kolmogorov-Centsov continuity criterion, we can assume that it is continuous. Hence, to apply Theorem 3.1, it is enough to prove that for every $\theta > 0$,

$$\mathbb{E}\left[\exp\left(\theta \max_{t \in [0, 1]} |\Delta_t|\right)\right] < +\infty. \quad (17)$$

Let $\theta > 0$. We have

$$\begin{aligned}\mathbb{E}\left[\exp\left(\theta \max_{t \in [0, 1]} |\Delta_t|\right)\right] &= \int_0^\infty \mathbb{P}\left[\exp\left(\theta \max_{t \in [0, 1]} |\Delta_t|\right) \geq \lambda\right] d\lambda \\ &\leq 2 + \int_2^\infty \mathbb{P}\left[\max_{t \in [0, 1]} |\Delta_t| \geq \frac{\ln(\lambda)}{\theta}\right] d\lambda.\end{aligned}$$

Since the process $\{\Delta_t; t \geq 0\}$ is symmetric and satisfies the maximal inequality (16) of Lemma 3.4,

$$\begin{aligned} \mathbb{P}\left[\max_{t \in [0,1]} |\Delta_t| \geq \frac{\ln(\lambda)}{\theta}\right] &\leq 2 \mathbb{P}\left[\max_{t \in [0,1]} \Delta_t \geq \frac{\ln(\lambda)}{\theta}\right] \\ &\leq 4 \mathbb{P}[\Delta_1 \geq (\ln \lambda)/\theta]. \end{aligned}$$

We apply the inequality (12) of Lemma 3.3, and since the function $\lambda \rightarrow \exp(-\delta((\ln \lambda)/\theta)^{2\alpha/(1+\alpha)})$ is integrable at infinity for any $\alpha > 1$, the assertion (17) follows. \square

4. PROOF OF THEOREM 1

In this section, we prove upper and lower bounds on the persistence probability $\mathbb{P}\left[\sup_{t \in [0,T]} \Delta(t) \leq 1\right]$.

In the case of fractional Brownian motion $\{B_H(t); t \geq 0\}$, Aurzada's proof of the lower bound (see [2]), rests on both following arguments: the fractional Brownian motion satisfies the hypothesis of Theorem 3.1, and it satisfies the equality (valid for a large enough)

$$(\mathbb{E}|B_H(t) - B_H(s)|^a)^{1/a} = C(a)|t - s|^H, \quad t, s \geq 0 \quad (18)$$

with $C(a) \leq ca^\nu$, for some c and $\nu > 0$. We showed in Lemmas 3.2 and 3.3 that these conditions are also satisfied by our random process Δ . Therefore the proof of Aurzada [2] allows us to derive the lower bound in Theorem 1.

As in [19] and [2], the main idea in the proof of the upper bound in (2), is to bound $I(T)$ from below by restricting the expectation to a well-chosen set of paths.

Conditionally to Y , the process $\{\Delta_t; t \geq 0\}$ is a centered Gaussian process such that for every $0 \leq t < s$,

$$\begin{aligned} \mathbb{E}[\Delta_t \Delta_s | Y] &= \int_{\mathbb{R}} L_t(x) L_s(x) dx \geq 0, \\ \mathbb{E}[\Delta_t (\Delta_s - \Delta_t) | Y] &= \int_{\mathbb{R}} L_t(x) (L_s(x) - L_t(x)) dx \geq 0, \end{aligned}$$

since $t \rightarrow L_t(x)$ is a.s. increasing for all $x \in \mathbb{R}$. It follows then from Slepian's lemma, that for every $0 \leq u < v < w$ and every real numbers a, b ,

$$\mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a, \sup_{t \in [v,w]} \Delta_t \leq b \mid Y\right] \geq \mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a \mid Y\right] \mathbb{P}\left[\sup_{t \in [v,w]} \Delta_t \leq b \mid Y\right] \quad (19)$$

$$\mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a, \sup_{t \in [v,w]} (\Delta_t - \Delta_v) \leq b \mid Y\right] \geq \mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a \mid Y\right] \mathbb{P}\left[\sup_{t \in [v,w]} (\Delta_t - \Delta_v) \leq b \mid Y\right]. \quad (20)$$

Let $a_T = (\ln T)^a$, where $a > 0$ will be chosen later; let $\alpha_T := \frac{\ln T}{1 + \ln T}$, which belongs to $(0, 1)$ and set $\beta_T = \sqrt{V_{a_T}}$ where $V_{a_T} := \int_{\mathbb{R}} L_{a_T}^2(x) dx$. Let us define the random function

$$\phi(t) := \begin{cases} 1 & \text{for } 0 \leq t < a_T, \\ 1 - \beta_T & \text{for } a_T \leq t \leq T, \end{cases}$$

which is Y -measurable. Clearly, we have

$$\mathbb{E}\left[\left(\int_0^T e^{\Delta_t} dt\right)^{-\alpha_T} \mid Y\right] \geq \left(\int_0^T e^{\phi(t)} dt\right)^{-\alpha_T} \mathbb{P}\left[\forall t \in [0, T], \Delta_t \leq \phi(t) \mid Y\right].$$

By Slepian's lemma (see (19)), we have

$$\mathbb{P}\left[\forall t \in [0, T], \Delta_t \leq \phi(t) \mid Y\right] \geq \mathbb{P}\left[\forall t \in [0, a_T], \Delta_t \leq 1 \mid Y\right] \mathbb{P}\left[\forall t \in [a_T, T], \Delta_t \leq 1 - \beta_T \mid Y\right].$$

Remark that

$$\begin{aligned} \mathbb{P}\left[\forall t \in [a_T, T], \Delta_t \leq 1 - \beta_T \mid Y\right] &\geq \mathbb{P}\left[\Delta_{a_T} \leq -\beta_T; \forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \mid Y\right] \\ &\geq \mathbb{P}\left[\Delta_{a_T} \leq -\beta_T \mid Y\right] \mathbb{P}\left[\forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \mid Y\right], \end{aligned}$$

by Slepian's lemma (see (20)). Note that

$$\mathbb{P}[\Delta_{a_T} \leq -\beta_T \mid Y] = \Phi(\beta_T V_{a_T}^{-1/2}) = \Phi(1).$$

Moreover, it is easy to check that when T goes to infinity,

$$\int_0^T e^{\phi(t)} dt = O(a_T + T e^{-\beta T}).$$

In the following C is a constant whose value may change but does not depend on T . Then we can write that for T large enough

$$\mathbb{E}\left[\left(\int_0^T e^{\Delta_t} dt\right)^{-\alpha_T} \mid Y\right] \geq C(a_T + T e^{-\beta T})^{-\alpha_T} \mathbb{P}\left[\sup_{t \in [0, a_T]} \Delta_t \leq 1 \mid Y\right] \mathbb{P}\left[\sup_{t \in [a_T, T]} (\Delta_t - \Delta_{a_T}) \leq 1 \mid Y\right]. \quad (21)$$

Next we use the maximal inequality (15) of Lemma 3.4 to write

$$\mathbb{P}\left[\sup_{t \in [0, a_T]} \Delta_t \leq 1 \mid Y\right] = 1 - \mathbb{P}\left[\sup_{t \in [0, a_T]} \Delta_t \geq 1 \mid Y\right] \geq 1 - 2\mathbb{P}[\Delta_{a_T} \geq 1 \mid Y] = \mathbb{P}[|Z| \leq V_{a_T}^{-1/2} \mid Y]$$

where Z is a Gaussian variable $\mathcal{N}(0, 1)$ independent of Y , from which we deduce that there exists a constant $c > 0$ such that

$$\mathbb{P}\left[\sup_{t \in [0, a_T]} \Delta_t \leq 1 \mid Y\right] \geq c \min(V_{a_T}^{-1/2}, 1). \quad (22)$$

Injecting (22) into (21) we get that for T large enough,

$$\mathbb{P}\left[\sup_{t \in [a_T, T]} (\Delta_t - \Delta_{a_T}) \leq 1 \mid Y\right] \leq C \mathbb{E}\left[\left(\int_0^T e^{\Delta_t} dt\right)^{-\alpha_T} \mid Y\right] (a_T + T e^{-V_{a_T}^{1/2}})^{\alpha_T} \max(V_{a_T}^{1/2}, 1).$$

By integrating and using successively Hölder's inequality with $p_T = \frac{1}{\alpha_T}$, $\frac{1}{q_T} + \frac{1}{p_T} = 1$, Jensen's inequality, the inequality $(x + y)^{\alpha_T} \leq x^{\alpha_T} + y^{\alpha_T}$ for $x, y > 0$, and Proposition 3.5, we get that for T large enough,

$$\begin{aligned} &\mathbb{P}\left[\sup_{t \in [a_T, T]} (\Delta_t - \Delta_{a_T}) \leq 1\right] \\ &\leq C \mathbb{E}\left[\left(\int_0^T e^{\Delta_t} dt\right)^{-1}\right]^{1/p_T} \mathbb{E}\left[\left(a_T^{\alpha_T} \max(V_{a_T}^{1/2}, 1) + T^{\alpha_T} e^{-\alpha_T V_{a_T}^{1/2}} \max(V_{a_T}^{1/2}, 1)\right)^{q_T}\right]^{1/q_T} \\ &\leq C T^{-\frac{\gamma}{2p_T}} \|f_1 + f_2\|_{q_T} \leq C T^{-\frac{\gamma}{2p_T}} (\|f_1\|_{q_T} + \|f_2\|_{q_T}), \end{aligned}$$

with

$$f_1 = a_T^{\alpha_T} \max(V_{a_T}^{1/2}, 1), \quad f_2 = T^{\alpha_T} e^{-\alpha_T V_{a_T}^{1/2}} \max(V_{a_T}^{1/2}, 1).$$

The lefthand term is greater than the quantity we want to bound from above, since by stationarity,

$$\mathbb{P}\left[\sup_{t \in [a_T, T]} (\Delta_t - \Delta_{a_T}) \leq 1\right] = \mathbb{P}\left[\sup_{t \in [0, T-a_T]} \Delta_t \leq 1\right] \geq \mathbb{P}\left[\sup_{t \in [0, T]} \Delta_t \leq 1\right].$$

Concerning the righthand term, we recall that $\alpha_T = \frac{1}{p_T} = \frac{\ln T}{1+\ln T}$ and $\frac{1}{q_T} = 1 - \alpha_T = \frac{1}{1+\ln T}$. Hence, when T goes to infinity, $T^{-\frac{\gamma}{2p_T}} \leq CT^{-\frac{\gamma}{2}}$. Therefore,

$$\mathbb{P}\left[\sup_{t \in [0, T]} \Delta_t \leq 1\right] \leq CT^{-\frac{\gamma}{2}}(\|f_1\|_{q_T} + \|f_2\|_{q_T}). \quad (23)$$

It remains to prove that $\|f_1\|_{q_T}$ and $\|f_2\|_{q_T}$ are bounded by logarithmic terms.

$$\|f_1\|_{q_T} = a_T^{\alpha_T} \mathbb{E}\left[V_{a_T}^{q_T/2} \mathbf{1}_{V_{a_T} \geq 1} + \mathbf{1}_{V_{a_T} \leq 1}\right]^{1/q_T} \leq a_T^{\alpha_T} \left(\mathbb{E}[V_{a_T}^{q_T}]^{1/(2q_T)} + 1\right).$$

By (H2), $V_{a_T} \stackrel{\mathcal{L}}{=} a_T^{2-\gamma} V_1$. Therefore,

$$\mathbb{E}[V_{a_T}^{q_T}]^{1/(2q_T)} = a_T^{1-\frac{\gamma}{2}} \mathbb{E}[V_1^{q_T}]^{1/(2q_T)}.$$

As in (14), it follows from (H4) that

$$\mathbb{E}[V_1^m] \leq Cc^{-m/\alpha} \frac{m}{\alpha} \Gamma\left(\frac{m}{\alpha}\right) \text{ for every } m \in \mathbb{N},$$

so that using Stirling's formula, it is easy to show that for T large enough

$$\mathbb{E}[V_1^{q_T}]^{1/2q_T} \leq C(\ln T)^{\frac{1}{2\alpha}}.$$

We conclude that for T large enough

$$\|f_1\|_{q_T} \leq C(\ln T)^{a(2-\frac{\gamma}{2})+\frac{1}{2\alpha}}. \quad (24)$$

Let us now turn our attention to $\|f_2\|_{q_T}$.

$$\begin{aligned} \|f_2\|_{q_T} &\leq T^{\alpha_T} \mathbb{E}\left[e^{-2q_T\alpha_T V_{a_T}^{1/2}}\right]^{1/2q_T} \mathbb{E}[\max(V_{a_T}, 1)^{q_T}]^{1/2q_T} \\ &\leq CT(\ln T)^{a(1-\frac{\gamma}{2})+\frac{1}{2\alpha}} \mathbb{E}\left[e^{-2q_T\alpha_T a_T^{1-\frac{\gamma}{2}} \sqrt{V_1}}\right]^{1/2q_T}. \end{aligned}$$

Let us note $\lambda_T = 2q_T\alpha_T a_T^{1-\frac{\gamma}{2}} = 2(\ln T)^{1+a(1-\frac{\gamma}{2})}$. Then using (H5),

$$\begin{aligned} \mathbb{E}\left[e^{-\lambda_T V_1^{1/2}}\right] &\leq \int_0^{+\infty} \mathbb{P}\left[V_1 \leq \frac{u^2}{\lambda_T^2}\right] e^{-u} du \\ &\leq C \int_0^{+\infty} e^{-c\frac{\lambda_T^{2\beta}}{u^{2\beta}}} e^{-u} du, \end{aligned}$$

for some constants c and C . We perform the change of variable $u = \lambda_T^{\frac{2\beta}{1+2\beta}} v$ in the preceding integral. This yields

$$\begin{aligned} \mathbb{E}\left[e^{-\lambda_T V_1^{1/2}}\right] &\leq C\lambda_T^{2\beta/(1+2\beta)} \int_0^{+\infty} e^{-\lambda_T^{\frac{2\beta}{1+2\beta}} (v+cv^{-2\beta})} dv \\ &\leq C\lambda_T^{2\beta/(1+2\beta)} \int_1^{+\infty} e^{-\lambda_T^{\frac{2\beta}{1+2\beta}} v} dv + C\lambda_T^{2\beta/(1+2\beta)} \int_0^1 e^{-\lambda_T^{\frac{2\beta}{1+2\beta}} cv^{-2\beta}} dv \\ &\leq Ce^{-\lambda_T^{\frac{2\beta}{1+2\beta}}} + Ce^{-c\lambda_T^{\frac{2\beta}{1+2\beta}}} \\ &\leq Ce^{-c\lambda_T^{\frac{2\beta}{1+2\beta}}}, \end{aligned}$$

for some other constants c and C . This leads to

$$\|f_2\|_{q_T} \leq CT(\ln T)^{a(1-\frac{\gamma}{2})+\frac{1}{2\alpha}} e^{-c(\ln T)^{\frac{2\beta}{1+2\beta}(1+a(1-\frac{\gamma}{2})) - 1}}.$$

We choose a such that $a(1 - \frac{\gamma}{2}) > 1 + \frac{1}{\beta}$. Then $T e^{-c(\ln T)^{\frac{2\beta}{1+2\beta}(1+a(1-\frac{\gamma}{2}))^{-1}}}$ remains bounded, and when T goes to infinity, we get

$$\|f_2\|_{q_T} \leq C(\ln T)^c, \quad (25)$$

for some constant $c > 1 + \frac{1}{\beta} + \frac{1}{2\alpha}$. From (23), (24) and (25) we deduce that for T large enough,

$$\mathbb{P}\left[\sup_{t \in [0, T]} \Delta_t \leq 1\right] \leq CT^{-\frac{\gamma}{2}}(\ln T)^c,$$

with $c > \frac{\beta+1}{\beta} \frac{4-\gamma}{2-\gamma} + \frac{1}{2\alpha}$.

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