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# PERSISTENCE EXPONENT FOR RANDOM PROCESSES IN BROWNIAN SCENERY

FABIENNE CASTELL, NADINE GUILLOTIN-PLANTARD, AND FRÉDÉRIQUE WATBLED

ABSTRACT. We consider the one-sided exit problem for random processes in Brownian scenery, that is the asymptotic behaviour for large  $T$ , of the probability

$$\mathbb{P}\left[\sup_{t \in [0, T]} \Delta_t \leq 1\right]$$

where

$$\Delta_t = \int_{\mathbb{R}} L_t(x) dW(x).$$

Here  $W = \{W(x); x \in \mathbb{R}\}$  is a two-sided standard real Brownian motion and  $\{L_t(x); x \in \mathbb{R}, t \geq 0\}$  is the local time of some self-similar random process with stationary increments, independent from the process  $W$ .

## 1. INTRODUCTION

Let  $W = \{W(x); x \in \mathbb{R}\}$  be a standard two-sided real Brownian motion and  $Y = \{Y(t), t \geq 0\}$  be a real-valued self-similar process of index  $\gamma \in (0, 2)$  (i.e. for any  $c > 0$ ,  $\{Y(ct), t \geq 0\} \stackrel{(d)}{=} \{c^\gamma Y(t), t \geq 0\}$ ), with stationary increments. When it exists, we will denote by  $\{L_t(x); x \in \mathbb{R}, t \geq 0\}$  a version of the local time of the process  $\{Y(t); t \geq 0\}$ . The process  $L$  satisfies the occupation density formula: for each bounded measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and for each  $t \geq 0$ ,

$$\int_0^t f(Y(s)) ds = \int_{\mathbb{R}} f(x) L_t(x) dx. \quad (1)$$

The processes  $W$  and  $Y$  are defined on the same probability space and are assumed to be independent. We consider *the random process in Brownian scenery*  $\{\Delta_t; t \geq 0\}$  defined as

$$\Delta_t = \int_{\mathbb{R}} L_t(x) dW(x).$$

The process  $\Delta$  is itself a  $h$ -self-similar process with stationary increments, with

$$h := 1 - \frac{\gamma}{2}.$$

This process can be seen as a mixture of Gaussian processes, but it is neither Gaussian nor Markovian. It appeared independently in the mathematical literature (see [13],[3]), and in the physics literature (see [17]) where it was originally introduced to model diffusion in layered white-noise velocity field. Note indeed that  $\Delta_t$  can formally be written as  $\int_0^t \dot{W}(Y(s)) ds$ . Thus, it describes the horizontal motion of a tracer particle in a (1+1)-dimensional medium, where the motion of the particle along the vertical direction  $y$  is described by the process  $Y$ , while along the horizontal direction, the particle is driven by the white noise velocity field  $\dot{W}(y)$ , that depends only on the vertical coordinate  $y$ . The process  $\{\Delta_t; t \geq 0\}$  provides a simple example

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of anomalous super-diffusion, being for instance of order  $t^{3/4}$  when  $Y$  is a Brownian motion. In this paper, we are interested in the *persistence probability* of the process  $\Delta$ , i.e. the asymptotic behaviour of

$$\mathcal{F}(T) := \mathbb{P} \left[ \sup_{t \in [0, T]} \Delta_t \leq 1 \right]$$

as  $T \rightarrow +\infty$ . The study of the one-sided supremum process of random processes is a classical issue in probability. One usually gets polynomial decay of the persistence probability:  $\mathcal{F}(T) \asymp T^{-\theta}$  for a non-negative  $\theta$ , often called *persistence exponent*, or *survival exponent*. Classical examples where this exponent can be computed, include random walks or Lévy processes, and we refer the reader to the recent survey paper [1] for an account of models where the persistence exponent is known. However, there are relevant physical situations where this exponent remains unknown (see for instance [15]). The persistence exponent of the Brownian motion in Brownian scenery was studied by Redner [19, 20], and Majumdar [16]. Based on physical arguments, numerical simulations and analogy with Fractional Brownian Motion, Redner and Majumdar conjectured the value of the persistence exponent. In [6], their conjecture was proved up to logarithmic factors, when the process  $Y$  is a stable Lévy process with index  $\delta \in (1, 2]$ . The proof of [6] depends heavily on the increments independence of the process  $Y$  and the question raises if it is possible to compute the persistence exponent without it. The aim of this paper is to answer this question, and to provide assumptions on  $Y$  allowing to compute the persistence exponent of the random process in Brownian scenery.

**(H1):** There exists a version  $\{L_t(x); x \in \mathbb{R}, t \geq 0\}$  of the local time of  $Y$ ;

**(H2):**  $Y$  is a self-similar process of index  $\gamma \in (0, 2)$ ;

**(H3):**  $Y$  has stationary increments;

**(H4):** Let  $V_1 := \int L_1^2(x) dx$  be the self-intersection local time of  $Y$ . There exist a real number  $\alpha > 1$ , and positive constants  $C, c$  such that for any  $t \geq 0$ ,

$$\mathbb{P}[V_1 \geq t] \leq C \exp(-ct^\alpha);$$

**(H5):** There exist a real number  $\beta > 0$ , and positive constants  $C, c$  such that for any  $t > 0$ ,

$$\mathbb{P}[V_1 \leq t] \leq C \exp(-ct^{-\beta}).$$

Our main result is the following one.

**Theorem 1.** *Assume (H1) to (H5) hold. There exists a constant  $c > 0$ , such that for large enough  $T$ ,*

$$T^{-\gamma/2}(\ln T)^{-c} \leq \mathbb{P} \left[ \sup_{t \in [0, T]} \Delta_t \leq 1 \right] \leq T^{-\gamma/2}(\ln T)^{+c}. \quad (2)$$

The paper is organized as follows. Section 2 provides three examples of processes satisfying (H1) to (H5), including the stable Lévy process with index  $\delta \in (1, 2]$ , and thus generalizing the result of [6]. Section 3 states some useful properties of the process  $\Delta$ . Section 4 is devoted to the proof of Theorem 1.

## 2. SOME EXAMPLES

**2.1. A sufficient condition for (H5).** In this section we assume that (H1) holds and we provide a sufficient condition on  $\{Y(t), t \geq 0\}$  allowing to check (H5). Let us first compare  $V_1$  with  $\max_{s \in [0, 1]} |Y(s)|$ . By noting that

$$1 = \int_{\{|x| \leq \max_{s \in [0, 1]} |Y(s)|\}} L_1(x) dx \leq V_1^{1/2} \sqrt{2 \max_{s \in [0, 1]} |Y(s)|},$$

we deduce that

$$\frac{1}{2 \max_{s \in [0,1]} |Y(s)|} \leq V_1. \quad (3)$$

**Lemma 2.1.** *Assume that (H1) holds and that there exist positive constants  $C$  and  $c$ , and a real number  $\beta > 0$  such that for all  $t > 0$ ,*

$$\mathbb{P} \left[ \max_{s \in [0,1]} |Y(s)| \geq t \right] \leq C \exp(-ct^\beta).$$

*Then, (H5) holds true.*

*Proof.* It is a direct consequence of (3) that for all  $t > 0$ ,

$$\mathbb{P}[V_1 \leq t] \leq \mathbb{P} \left[ \max_{s \in [0,1]} |Y(s)| \geq \frac{1}{2t} \right] \leq C \exp(-c2^{-\beta}t^{-\beta}).$$

□

**2.2. Stable Lévy processes.** A process  $Y = \{Y(t), t \geq 0\}$  is a strictly stable Lévy process with index  $\delta \in (1, 2]$  if it is a process with stationary and independent increments, such that  $Y(0) = 0$  and for any  $t \geq 0$  and  $u \in \mathbb{R}$ ,

$$\mathbb{E}[e^{iuY(t)}] = \exp \left\{ -c|u|^\delta t \left( 1 + i\zeta \operatorname{sgn}(u) \tan(\pi\delta/2) \right) \right\}, \quad (4)$$

where  $\zeta \in [-1, 1]$ , and  $c$  is a positive scale parameter.

**Lemma 2.2.** *The stable Lévy process with index  $\delta \in (1, 2]$  satisfies (H1) to (H5) with  $\gamma = \frac{1}{\delta}$ ,  $\alpha = \delta$ , and  $\beta = \frac{\delta}{2\delta-1}$ .*

*Proof.* It is immediate that the Lévy process satisfies (H2) and (H3). A continuous version of the local time exists since  $\delta > 1$  (this was proved by Boylan [4]), so it satisfies (H1). Moreover, Corollary 5.6 in [14] states that there exist positive constants  $C$  and  $\xi$  s.t. for every  $t > 0$ ,

$$\mathbb{P}[V_1 \geq t] \leq Ce^{-\xi t^\delta},$$

which gives (H4). To prove (H5) we let  $V_t := \int L_t^2(x) dx$  be the self-intersection local time of  $Y$  up to time  $t$  for  $t \geq 0$ , and we show that there exists  $l > 0$  such that

$$\mathbb{P}[V_t \leq 1] \leq e^{-lt} \text{ for } t \text{ large enough.} \quad (5)$$

As  $\{V_t, t \geq 0\}$  is self-similar of index  $\frac{2\delta-1}{\delta}$ , (5) implies that for every positive  $\varepsilon$  small enough,

$$\mathbb{P}[V_1 \leq \varepsilon] = \mathbb{P}[V_{\varepsilon^{-\frac{\delta}{2\delta-1}}} \leq 1] \leq e^{-l\varepsilon^{-\frac{\delta}{2\delta-1}}},$$

which gives (H5). To prove (5) we show that the function defined on  $\mathbb{R}^+$  by

$$f(t) = \log \mathbb{P}[V_t \leq 1]$$

is subadditive. Let us fix  $s, t$  in  $[0, +\infty)$ . We consider the process  $Y^{(s)} := \{Y_u^{(s)}, u \geq 0\}$  defined by

$$Y_u^{(s)} = Y_{u+s} - Y_s,$$

its local time  $\{L_u^{(s)}(x); x \in \mathbb{R}, u \geq 0\}$ , and its self-intersection local time  $\{V_u^{(s)}, u \geq 0\}$ . Note that

$$L_{t+s}(x) = L_s(x) + L_t^{(s)}(x - Y_s).$$

Hence

$$V_{t+s} \geq V_s + V_t^{(s)}.$$

The process  $Y$  being a Lévy process,  $V_t^{(s)}$  and  $V_s$  are independent and  $V_t^{(s)}$  has the same law as  $V_t$ . Therefore

$$\mathbb{P}[V_{t+s} \leq 1] \leq \mathbb{P}[V_s \leq 1; V_t^{(s)} \leq 1] = \mathbb{P}[V_s \leq 1]\mathbb{P}[V_t \leq 1].$$

Thus  $f$  is subadditive, which implies that  $\frac{f(t)}{t}$  converges, as  $t \rightarrow +\infty$ , towards a limit  $-l = \inf_{t>0} \frac{f(t)}{t}$  (see [11]). It remains to show that  $l$  is strictly positive. As  $-l$  is less or equal than  $f(1)$ , it is enough to show that  $\mathbb{P}[V_1 > 1] > 0$ . But thanks to (3) and Proposition 10.3 of [9],

$$\mathbb{P}[V_1 > 1] \geq \mathbb{P}\left[\max_{s \in [0,1]} |Y(s)| < \frac{1}{2}\right] > 0,$$

and this concludes the proof of (5).  $\square$

**2.3. Fractional Brownian motion.** The Fractional Brownian motion of Hurst index  $H \in (0, 1)$  is the real centered Gaussian process  $\{B_H(t), t \geq 0\}$  with covariance function

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

**Lemma 2.3.** *The Fractional Brownian motion with Hurst index  $H \in (0, 1)$  satisfies (H1) to (H5) with  $\gamma = H$ ,  $\alpha = 1/H$ ,  $\beta = 2$ .*

*Proof.* It follows readily from the definition that  $B_H$  is self-similar with index  $H$  and has stationary increments. The existence of a jointly continuous version of its local time process for  $H \in (0, 1)$  is a classical fact (see for instance paragraph 22 in [10]). Theorem 1 of [12] asserts that there exists  $\lambda_0 > 0$  such that for every  $0 < \lambda < \lambda_0$ ,

$$\mathbb{E}\left[e^{\lambda V_1^{1/H}}\right] < \infty, \quad (6)$$

which implies (H4) with  $\alpha = 1/H$ . Finally, (H5) follows from Lemma 2.1 and Fernique's estimation ([8], Theorem 4.1.1): there exists  $c_F > 0$  such that for any  $x \geq \sqrt{5}$ ,

$$\mathbb{P}\left[\max_{s \in [0,1]} |B_H(s)| \geq c_F x\right] \leq 10 \int_x^{+\infty} e^{-\frac{v^2}{2}} dv. \quad (7)$$

This implies (H5) with  $\beta = 2$ .  $\square$

**2.4. Iterated Brownian motion.** Let  $\{B(x), x \in \mathbb{R}\}$  be a two-sided real standard Brownian motion, and let  $\{\tilde{B}(t), t \geq 0\}$  be a real standard Brownian motion, independent of  $\{B(x), x \in \mathbb{R}\}$ . The iterated Brownian motion is the process  $\{Y(t), t \geq 0\}$  defined by

$$Y(t) = B(\tilde{B}(t)).$$

**Lemma 2.4.** *The iterated Brownian motion satisfies (H1) to (H5) with  $\gamma = 1/4$ , and  $\alpha = \beta = 4/3$ .*

*Proof.* The self-similarity and increments stationarity of the iterated Brownian motion are direct consequences of the self-similarity and increments independence of the Brownian motion. The existence and joint continuity of the local times of iterated Brownian motion were proved in

[5, 7]. Let us prove (H5). For  $x > 0$ ,

$$\begin{aligned}
\mathbb{P} \left[ \max_{s \in [0,1]} |Y(s)| \geq x \right] &\leq \mathbb{P} \left[ \max_{s \in [0,1]} |\tilde{B}(s)| \geq x^{2/3} \right] + \mathbb{P} \left[ \max_{|u| \leq x^{2/3}} |B(u)| \geq x \right] \\
&\leq \mathbb{P} \left[ \max_{s \in [0,1]} |\tilde{B}(s)| \geq x^{2/3} \right] + \mathbb{P} \left[ \max_{u \in [0, x^{2/3}]} |B(u)| \geq x \right] \\
&\leq 2 \mathbb{P} \left[ \max_{s \in [0,1]} |B(s)| \geq x^{2/3} \right] \\
&\leq 4 \mathbb{P} \left[ \max_{s \in [0,1]} B(s) \geq x^{2/3} \right] = 4 \mathbb{P} \left[ |B(1)| \geq x^{2/3} \right]. \tag{8}
\end{aligned}$$

This proves (H5) with  $\beta = 4/3$  using Lemma 2.1.

To prove (H4), we use the uniform norm on the local times of iterated Brownian motion proved in Lemma 4 of [21]: there exists a constant  $K > 0$  such that for any even integer  $n$ ,

$$\sup_{x \in \mathbb{R}} \mathbb{E} [(L_1(x))^n] \leq K^n (n!)^{3/4}. \tag{9}$$

Since  $\int_{\mathbb{R}} L_1(x) dx = 1$ , Hölder's inequality implies that for any integer  $n$ ,

$$V_1^n = \left( \int_{\mathbb{R}} L_1(x)^2 dx \right)^n \leq \int_{\mathbb{R}} L_1(x)^{n+1} dx = \int_{-\max_{s \in [0,1]} |Y(s)|}^{\max_{s \in [0,1]} |Y(s)|} L_1(x)^{n+1} dx.$$

Therefore,

$$\begin{aligned}
\mathbb{E} [V_1^n] &\leq \int_{\mathbb{R}} \mathbb{E} \left[ L_1(x)^{n+1} \mathbf{1}_{|x| \leq \max_{s \in [0,1]} |Y(s)|} \right] dx \\
&\leq \int_{\mathbb{R}} \sqrt{\mathbb{E} [L_1(x)^{2(n+1)}]} \sqrt{\mathbb{P} \left[ \max_{s \in [0,1]} |Y(s)| \geq |x| \right]} dx \\
&\leq K^{n+1} ((2n+2)!)^{3/8} \int_{\mathbb{R}} \sqrt{\mathbb{P} \left[ \max_{s \in [0,1]} |Y(s)| \geq |x| \right]} dx, \tag{10}
\end{aligned}$$

where we used (9) in the last inequality. By (8), the integral in (10) is finite. We deduce then from Stirling's formula that there exists a constant  $C > 0$  such that for any integer  $n$ ,

$$\mathbb{E} [V_1^n] \leq C^n n^{\frac{3}{4}n}.$$

Hence, for any  $t > 0$ , and any integer  $n$ ,

$$\mathbb{P} [V_1 \geq t] \leq t^{-n} C^n n^{\frac{3}{4}n}.$$

Optimizing over the values of  $n$  leads to take  $n = \left\lceil e^{-1} \left( \frac{t}{C} \right)^{4/3} \right\rceil$ , so that  $t^{-n} C^n n^{\frac{3}{4}n} \simeq \exp\left(-\frac{3}{4e} \left( \frac{t}{C} \right)^{4/3}\right)$  for large  $t$ . This proves that the iterated Brownian motion satisfies (H4) with  $\alpha = 4/3$ .  $\square$

### 3. AUXILIARY STATEMENTS ON $(\Delta_t, t \geq 0)$

For a certain class of stochastic processes  $\{X_t; t \geq 0\}$  (to be specified below), Molchan [18] proved that the asymptotic behavior of

$$\mathbb{P} \left[ \sup_{t \in [0, T]} X_t \leq 1 \right]$$

is related to the quantity

$$I(T) := \mathbb{E} \left[ \left( \int_0^T e^{X_t} dt \right)^{-1} \right].$$

We refer to [2] where the relationship between both quantities is clearly explained as well as the heuristics.

**Theorem 3.1** (Statement 1, [18]). *Let  $\{X_t; t \geq 0\}$  be a continuous process, self-similar with index  $\gamma > 0$ , with stationary increments s.t. for every  $\theta > 0$ ,*

$$\mathbb{E} \left[ \exp \left( \theta \max_{t \in [0,1]} |X_t| \right) \right] < +\infty.$$

Then, as  $T \rightarrow +\infty$ ,

$$\mathbb{E} \left[ \left( \int_0^T e^{X_t} dt \right)^{-1} \right] = HT^{-(1-\gamma)} \left( \mathbb{E} \left[ \max_{t \in [0,1]} X_t \right] + o(1) \right).$$

By applying this result to our random process  $\Delta$  we get

**Proposition 3.2.** *Assume (H1) to (H4) hold. For any  $\gamma \in (0, 2)$ , as  $T \rightarrow +\infty$ ,*

$$\mathbb{E} \left[ \left( \int_0^T e^{\Delta_t} dt \right)^{-1} \right] = \left( 1 - \frac{\gamma}{2} \right) T^{-\gamma/2} \left( \mathbb{E} \left[ \max_{t \in [0,1]} \Delta_t \right] + o(1) \right).$$

Before proving Proposition 3.2 we first establish three useful inequalities concerning the process  $\{\Delta_t, t \geq 0\}$ . First we show that the process satisfies the Kolmogorov-Centsov criterion, which allows us to assume that its sample paths are continuous (Lemma 3.3). Then we show that the process satisfies a maximal inequality (Lemma 3.4), and next we provide an exponential upper tail bound for  $\Delta_1$  (Lemma 3.5).

**Lemma 3.3.** *Assume (H1) to (H4) hold. Then for every  $a \geq 1$ ,*

$$\left( \mathbb{E} |\Delta_t - \Delta_s|^a \right)^{1/a} = C(a) |t - s|^h, \quad t, s \geq 0 \quad (11)$$

where  $C(a) \leq ca^\nu$  with  $\nu := \frac{1}{2} \left( 1 + \frac{1}{\alpha} \right)$ . In particular the process satisfies the Kolmogorov-Centsov criterion of continuity.

*Proof.* The increments of the process  $\Delta$  being stationary, by self-similarity, we have for every  $t, s \geq 0$ ,

$$\begin{aligned} \mathbb{E} [|\Delta_t - \Delta_s|^a] &= |t - s|^{ha} \mathbb{E} [|\Delta_1|^a] \\ &\leq |t - s|^{ha} \mathbb{E} [|\Delta_1|^{2([a]+1)a/(2([a]+1))}] \end{aligned}$$

From the formula of the even moments of a centered Gaussian law, we can derive the even moments of the random variable  $\Delta_1$ , namely, for any  $m \in \mathbb{N}$ ,

$$\mathbb{E} [\Delta_1^{2m}] = \mathbb{E} \left[ \mathbb{E} [\Delta_1^{2m} | Y] \right] = \mathbb{E} [V_1^m] \frac{(2m)!}{2^m m!}.$$

First of all, from Stirling's formula, for  $m$  large enough, we have

$$\frac{(2m)!}{2^m m!} \leq C \left( \frac{2}{e} \right)^m m^m$$

for some constant  $C > 0$ . Moreover, we deduce from (H4) that

$$\begin{aligned} \mathbb{E} [V_1^m] &= \int_0^{+\infty} mx^{m-1} \mathbb{P}[V_1 \geq x] dx \\ &\leq \int_0^{+\infty} Cmx^{m-1} \exp(-cx^\alpha) dx = \frac{Cm}{\alpha} c^{-m/\alpha} \Gamma \left( \frac{m}{\alpha} \right). \end{aligned} \quad (12)$$

It is now easy to derive (11). In particular taking  $a > \frac{1}{h}$  we get the Kolmogorov-Centsov criterion of continuity.  $\square$

**Lemma 3.4.** *Assume (H1) holds. Let  $T, x \geq 0$ . Then*

$$\mathbb{P}[\max_{s \in [0, T]} \Delta_s \geq x | Y] \leq 2\mathbb{P}[\Delta_T \geq x | Y], \quad (13)$$

$$\mathbb{P}[\max_{s \in [0, T]} \Delta_s \geq x] \leq 2\mathbb{P}[\Delta_T \geq x]. \quad (14)$$

*Proof.* Conditionally to the process  $Y$ , the process  $(\Delta_t)_{t \geq 0}$  is a centered Gaussian process on  $\mathbb{R}$  with covariance function

$$\mathbb{E}[\Delta_s \Delta_t | Y] = \int_{\mathbb{R}} L_s(x) L_t(x) dx.$$

Moreover for any  $t \geq s \geq 0$ ,

$$\begin{aligned} \mathbb{E}[\Delta_t^2 | Y] - \mathbb{E}[\Delta_s^2 | Y] - \mathbb{E}[(\Delta_t - \Delta_s)^2 | Y] &= \int_{\mathbb{R}} (L_t^2(x) - L_s^2(x) - (L_t(x) - L_s(x))^2) dx \\ &= \int_{\mathbb{R}} 2L_s(x)(L_t(x) - L_s(x)) dx \geq 0, \end{aligned}$$

hence applying Proposition 2.2 in [14], we deduce the inequality (13). By integrating we obtain the maximal inequality (14).  $\square$

**Lemma 3.5.** *Assume (H1) and (H4) hold. There exist  $C > 0$  and  $\delta > 0$  such that for any  $x > 0$ ,*

$$\mathbb{P}[\Delta_1 \geq x] \leq C \exp(-\delta x^{\frac{2\alpha}{1+\alpha}}). \quad (15)$$

*Proof.* Conditionally to the process  $Y$ , the random variable  $\Delta_1$  is a real centered Gaussian variable with variance  $V_1$ . For each  $u \in \mathbb{R}$ , let

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_u^{+\infty} e^{-\frac{s^2}{2}} ds.$$

Then for  $x > 0$  and  $\theta > 0$ ,

$$\begin{aligned} \mathbb{P}[\Delta_1 \geq x] &= \int_0^{+\infty} \Phi(xz^{-1/2}) \mathbb{P}_{V_1}(dz) = \int_0^{x^\theta} \Phi(xz^{-1/2}) \mathbb{P}_{V_1}(dz) + \int_{x^\theta}^{+\infty} \Phi(xz^{-1/2}) \mathbb{P}_{V_1}(dz) \\ &\leq \Phi(x^{1-\frac{\theta}{2}}) + \mathbb{P}[V_1 \geq x^\theta], \end{aligned}$$

where we used that the function  $z \mapsto \Phi(xz^{-1/2})$  is non decreasing with values in  $[0, 1]$ .

Using (H4) and the classical inequality

$$\Phi(u) \leq \frac{1}{\sqrt{2\pi}u} e^{-u^2/2} \text{ for every } u > 0,$$

and choosing  $\theta = \frac{2}{1+\alpha}$ , we obtain that for  $x$  large enough,

$$\mathbb{P}[\Delta_1 \geq x] \leq \frac{x^{-\alpha/(1+\alpha)}}{\sqrt{2\pi}} e^{-\frac{1}{2}x^{2\alpha/(1+\alpha)}} + C e^{-cx^{2\alpha/(1+\alpha)}} \leq C e^{-\delta x^{2\alpha/(1+\alpha)}},$$

with  $\delta = \min(c, 1/2)$ . This implies (15) for every  $x > 0$ .  $\square$

*Proof of Proposition 3.2.* It follows easily from assumptions (H1), (H2), (H3), that the process  $\{\Delta_t; t \geq 0\}$  is self-similar with index  $h := 1 - \frac{\gamma}{2}$ , with stationary increments. By Lemma 3.3 we can assume that it is continuous. Hence, it is enough to prove that for every  $\theta > 0$ ,

$$\mathbb{E} \left[ \exp \left( \theta \max_{t \in [0, 1]} |\Delta_t| \right) \right] < +\infty. \quad (16)$$



Let  $\theta > 0$ . We have

$$\begin{aligned} \mathbb{E}\left[\exp\left(\theta \max_{t \in [0,1]} |\Delta_t|\right)\right] &= \int_0^\infty \mathbb{P}\left[\exp\left(\theta \max_{t \in [0,1]} |\Delta_t|\right) \geq \lambda\right] d\lambda \\ &\leq 2 + \int_2^\infty \mathbb{P}\left[\max_{t \in [0,1]} |\Delta_t| \geq \frac{\ln(\lambda)}{\theta}\right] d\lambda. \end{aligned}$$

Since the process  $\{\Delta_t; t \geq 0\}$  is symmetric and satisfies the maximal inequality (14) of Lemma 3.4,

$$\begin{aligned} \mathbb{P}\left[\max_{t \in [0,1]} |\Delta_t| \geq \frac{\ln(\lambda)}{\theta}\right] &\leq 2 \mathbb{P}\left[\max_{t \in [0,1]} \Delta_t \geq \frac{\ln(\lambda)}{\theta}\right] \\ &\leq 4 \mathbb{P}[\Delta_1 \geq (\ln \lambda)/\theta]. \end{aligned}$$

We apply the inequality (15) of Lemma 3.5, and since the function  $\lambda \rightarrow \exp(-\delta((\ln \lambda)/\theta)^{2\alpha/(1+\alpha)})$  is integrable at infinity for any  $\alpha > 1$ , the assertion (16) follows.  $\square$

#### 4. PROOF OF THEOREM 1

Aurzada's proof of the lower bound of  $\mathbb{P}\left[\sup_{t \in [0,T]} B_H(t) \leq 1\right]$  (see [2]), rests on both following arguments: the Fractional Brownian motion satisfies the hypothesis of Theorem 3.1, and it satisfies the inequality (valid for  $a$  large enough)

$$(\mathbb{E}|B_H(t) - B_H(s)|^a)^{1/a} = C(a)|t - s|^H, \quad t, s \geq 0 \quad (17)$$

with  $C(a) \leq ca^\nu$ , for some  $c$  and  $\nu > 0$ . Our random process  $\Delta$  satisfies the hypothesis of Theorem 3.1, and we showed in Lemma 3.3 that it satisfies the inequality (17). Therefore the proof of Aurzada ([2]) allows us to derive the lower bound in Theorem 1.

As in [18] and [2], the main idea in the proof of the upper bound in (2), is to bound  $I(T)$  from below by restricting the expectation to a well-chosen set of paths.

Conditionally to  $Y$ , the process  $\{\Delta_t; t \geq 0\}$  is a centered Gaussian process such that for every  $0 \leq t < s$ ,

$$\begin{aligned} \mathbb{E}[\Delta_t \Delta_s | Y] &= \int_{\mathbb{R}} L_t(x) L_s(x) dx \geq 0, \\ \mathbb{E}[\Delta_t(\Delta_s - \Delta_t) | Y] &= \int_{\mathbb{R}} L_t(x)(L_s(x) - L_t(x)) dx \geq 0, \end{aligned}$$

since  $t \rightarrow L_t(x)$  is a.s. increasing for all  $x \in \mathbb{R}$ . It follows then from Slepian's lemma, that for every  $0 \leq u < v < w$  and every real numbers  $a, b$ ,

$$\mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a, \sup_{t \in [v,w]} \Delta_t \leq b \mid Y\right] \geq \mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a \mid Y\right] \mathbb{P}\left[\sup_{t \in [v,w]} \Delta_t \leq b \mid Y\right] \quad (18)$$

$$\mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a, \sup_{t \in [v,w]} (\Delta_t - \Delta_v) \leq b \mid Y\right] \geq \mathbb{P}\left[\sup_{t \in [u,v]} \Delta_t \leq a \mid Y\right] \mathbb{P}\left[\sup_{t \in [v,w]} (\Delta_t - \Delta_v) \leq b \mid Y\right]. \quad (19)$$

Let  $a_T = (\ln T)^a$ , where  $a > 0$  will be chosen later; let  $\alpha_T := \frac{\ln T}{1 + \ln T}$ , which belongs to  $(0, 1)$  and set  $\beta_T = \sqrt{V_{a_T}}$  where  $V_{a_T} := \int_{\mathbb{R}} L_{a_T}^2(x) dx$ . Let us define the random function

$$\phi(t) := \begin{cases} 1 & \text{for } 0 \leq t < a_T, \\ 1 - \beta_T & \text{for } a_T \leq t \leq T, \end{cases}$$

which is  $Y$ -measurable. Clearly, we have

$$\mathbb{E} \left[ \left( \int_0^T e^{\Delta_t} dt \right)^{-\alpha_T} \middle| Y \right] \geq \left( \int_0^T e^{\phi(t)} dt \right)^{-\alpha_T} \mathbb{P} \left[ \forall t \in [0, T], \Delta_t \leq \phi(t) \middle| Y \right].$$

By Slepian's lemma (see (18)), we have

$$\mathbb{P} \left[ \forall t \in [0, T], \Delta_t \leq \phi(t) \middle| Y \right] \geq \mathbb{P} \left[ \forall t \in [0, a_T], \Delta_t \leq 1 \middle| Y \right] \mathbb{P} \left[ \forall t \in [a_T, T], \Delta_t \leq 1 - \beta_T \middle| Y \right].$$

Remark that

$$\begin{aligned} \mathbb{P} \left[ \forall t \in [a_T, T], \Delta_t \leq 1 - \beta_T \middle| Y \right] &\geq \mathbb{P} \left[ \Delta_{a_T} \leq -\beta_T; \forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \middle| Y \right] \\ &\geq \mathbb{P} \left[ \Delta_{a_T} \leq -\beta_T \middle| Y \right] \mathbb{P} \left[ \forall t \in [a_T, T], \Delta_t - \Delta_{a_T} \leq 1 \middle| Y \right], \end{aligned}$$

by Slepian's lemma (see (19)). Note that

$$\mathbb{P} \left[ \Delta_{a_T} \leq -\beta_T \middle| Y \right] = \Phi(\beta_T V_{a_T}^{-1/2}) = \Phi(1).$$

Moreover, it is easy to check that when  $T$  goes to infinity,

$$\int_0^T e^{\phi(t)} dt = O(a_T + T e^{-\beta T}).$$

In the following  $C$  is a constant whose value may change but does not depend on  $T$ . Then we can write that for  $T$  large enough

$$\mathbb{E} \left[ \left( \int_0^T e^{\Delta_t} dt \right)^{-\alpha_T} \middle| Y \right] \geq C(a_T + T e^{-\beta T})^{-\alpha_T} \mathbb{P} \left[ \sup_{t \in [0, a_T]} \Delta_t \leq 1 \middle| Y \right] \mathbb{P} \left[ \sup_{t \in [a_T, T]} (\Delta_t - \Delta_{a_T}) \leq 1 \middle| Y \right]. \quad (20)$$

Next we use the maximal inequality (13) of Lemma 3.4 to write

$$\mathbb{P} \left[ \sup_{t \in [0, a_T]} \Delta_t \leq 1 \middle| Y \right] = 1 - \mathbb{P} \left[ \sup_{t \in [0, a_T]} \Delta_t \geq 1 \middle| Y \right] \geq 1 - 2\mathbb{P} \left[ \Delta_{a_T} \geq 1 \middle| Y \right] = \mathbb{P} \left[ |Z| \leq V_{a_T}^{-1/2} \middle| Y \right]$$

where  $Z$  is a Gaussian variable  $\mathcal{N}(0, 1)$  independent of  $Y$ , from which we deduce that there exists a constant  $c > 0$  such that

$$\mathbb{P} \left[ \sup_{t \in [0, a_T]} \Delta_t \leq 1 \middle| Y \right] \geq c \min(V_{a_T}^{-1/2}, 1). \quad (21)$$

Injecting (21) into (20) we get that for  $T$  large enough,

$$\mathbb{P} \left[ \sup_{t \in [a_T, T]} (\Delta_t - \Delta_{a_T}) \leq 1 \middle| Y \right] \leq C \mathbb{E} \left[ \left( \int_0^T e^{\Delta_t} dt \right)^{-\alpha_T} \middle| Y \right] (a_T + T e^{-V_{a_T}^{1/2}})^{\alpha_T} \max(V_{a_T}^{1/2}, 1).$$

By integrating and using successively Hölder's inequality with  $p_T = \frac{1}{\alpha_T}$ ,  $\frac{1}{q_T} + \frac{1}{p_T} = 1$ , Jensen's inequality, the inequality  $(x + y)^{\alpha_T} \leq x^{\alpha_T} + y^{\alpha_T}$  for  $x, y > 0$ , and Proposition 3.2, we get that for  $T$  large enough,

$$\begin{aligned} &\mathbb{P} \left[ \sup_{t \in [a_T, T]} (\Delta_t - \Delta_{a_T}) \leq 1 \right] \\ &\leq C \mathbb{E} \left[ \left( \int_0^T e^{\Delta_t} dt \right)^{-1} \right]^{1/p_T} \mathbb{E} \left[ \left( a_T^{\alpha_T} \max(V_{a_T}^{1/2}, 1) + T^{\alpha_T} e^{-\alpha_T V_{a_T}^{1/2}} \max(V_{a_T}^{1/2}, 1) \right)^{q_T} \right]^{1/q_T} \\ &\leq C T^{-\frac{\gamma}{2p_T}} \|f_1 + f_2\|_{q_T} \leq C T^{-\frac{\gamma}{2p_T}} (\|f_1\|_{q_T} + \|f_2\|_{q_T}), \end{aligned}$$

with

$$f_1 = a_T^{\alpha_T} \max(V_{a_T}^{1/2}, 1), \quad f_2 = T^{\alpha_T} e^{-\alpha_T V_{a_T}^{1/2}} \max(V_{a_T}^{1/2}, 1).$$

The lefthand term is greater than the quantity we want to bound from above, since by stationarity,

$$\mathbb{P}\left[\sup_{t \in [a_T, T]} (\Delta_t - \Delta_{a_T}) \leq 1\right] = \mathbb{P}\left[\sup_{t \in [0, T-a_T]} \Delta_t \leq 1\right] \geq \mathbb{P}\left[\sup_{t \in [0, T]} \Delta_t \leq 1\right].$$

Concerning the righthand term, we recall that  $\alpha_T = \frac{1}{p_T} = \frac{\ln T}{1+\ln T}$  and  $\frac{1}{q_T} = 1 - \alpha_T = \frac{1}{1+\ln T}$ . Hence, when  $T$  goes to infinity,  $T^{-\frac{\gamma}{2p_T}} \leq CT^{-\frac{\gamma}{2}}$ . Therefore,

$$\mathbb{P}\left[\sup_{t \in [0, T]} \Delta_t \leq 1\right] \leq CT^{-\frac{\gamma}{2}} (\|f_1\|_{q_T} + \|f_2\|_{q_T}). \quad (22)$$

It remains to prove that  $\|f_1\|_{q_T}$  and  $\|f_2\|_{q_T}$  are bounded by logarithmic terms.

$$\|f_1\|_{q_T} = a_T^{\alpha_T} \mathbb{E}\left[V_{a_T}^{q_T/2} \mathbf{1}_{V_{a_T} \geq 1} + \mathbf{1}_{V_{a_T} \leq 1}\right]^{1/q_T} \leq a_T^{\alpha_T} \left(\mathbb{E}[V_{a_T}^{q_T}]^{1/(2q_T)} + 1\right).$$

By (H2),  $V_{a_T} \stackrel{\mathcal{L}}{=} a_T^{2-\gamma} V_1$ . Therefore,

$$\mathbb{E}[V_{a_T}^{q_T}]^{1/(2q_T)} = a_T^{1-\frac{\gamma}{2}} \mathbb{E}[V_1^{q_T}]^{1/(2q_T)}.$$

Recall from (12) that ,

$$\mathbb{E}[V_1^m] \leq Cc^{-m/\alpha} \frac{m}{\alpha} \Gamma\left(\frac{m}{\alpha}\right) \text{ for every } m \in \mathbb{N},$$

so that using Stirling's formula, it is easy to show that for  $T$  large enough

$$\mathbb{E}[V_1^{q_T}]^{1/2q_T} \leq C(\ln T)^{\frac{1}{2\alpha}}.$$

We conclude that for  $T$  large enough

$$\|f_1\|_{q_T} \leq C(\ln T)^{a(2-\frac{\gamma}{2})+\frac{1}{2\alpha}}. \quad (23)$$

Let us now turn our attention to  $\|f_2\|_{q_T}$ .

$$\begin{aligned} \|f_2\|_{q_T} &\leq T^{\alpha_T} \mathbb{E}\left[e^{-2q_T\alpha_T V_{a_T}^{1/2}}\right]^{1/2q_T} \mathbb{E}[\max(V_{a_T}, 1)^{q_T}]^{1/2q_T} \\ &\leq CT(\ln T)^{a(1-\frac{\gamma}{2})+\frac{1}{2\alpha}} \mathbb{E}\left[e^{-2q_T\alpha_T a_T^{1-\frac{\gamma}{2}} \sqrt{V_1}}\right]^{1/2q_T}. \end{aligned}$$

Let us note  $\lambda_T = 2q_T\alpha_T a_T^{1-\frac{\gamma}{2}} = 2(\ln T)^{1+a(1-\frac{\gamma}{2})}$ . Then using (H5),

$$\begin{aligned} \mathbb{E}\left[e^{-\lambda_T V_1^{1/2}}\right] &\leq \int_0^{+\infty} \mathbb{P}\left[V_1 \leq \frac{u^2}{\lambda_T^2}\right] e^{-u} du \\ &\leq C \int_0^{+\infty} e^{-c\frac{\lambda_T^{2\beta}}{u^{2\beta}}} e^{-u} du, \end{aligned}$$

for some constants  $c$  and  $C$ . We perform the change of variable  $u = \lambda_T^{\frac{2\beta}{1+2\beta}} v$  in the preceding integral. This yields

$$\begin{aligned}
\mathbb{E} \left[ e^{-\lambda_T V_1^{1/2}} \right] &\leq C \lambda_T^{2\beta/(1+2\beta)} \int_0^{+\infty} e^{-\lambda_T^{\frac{2\beta}{1+2\beta}} (v+cv^{-2\beta})} dv \\
&\leq C \lambda_T^{2\beta/(1+2\beta)} \int_1^{+\infty} e^{-\lambda_T^{\frac{2\beta}{1+2\beta}} v} dv + C \lambda_T^{2\beta/(1+2\beta)} \int_0^1 e^{-\lambda_T^{\frac{2\beta}{1+2\beta}} cv^{-2\beta}} dv \\
&\leq C e^{-\lambda_T^{\frac{2\beta}{1+2\beta}}} + C e^{-c\lambda_T^{\frac{2\beta}{1+2\beta}}} \\
&\leq C e^{-c\lambda_T^{\frac{2\beta}{1+2\beta}}},
\end{aligned}$$

for some other constants  $c$  and  $C$ . This leads to

$$\|f_2\|_{q_T} \leq CT(\ln T)^{a(1-\frac{\gamma}{2})+\frac{1}{2\alpha}} e^{-c(\ln T)^{\frac{2\beta}{1+2\beta}(1+a(1-\frac{\gamma}{2}))^{-1}}}.$$

We choose  $a$  such that  $a(1-\frac{\gamma}{2}) > 1 + \frac{1}{\beta}$ . Then,  $T e^{-c(\ln T)^{\frac{2\beta}{1+2\beta}(1+a(1-\frac{\gamma}{2}))^{-1}}}$  remains bounded, and when  $T$  goes to infinity, we get

$$\|f_2\|_{q_T} \leq C(\ln T)^c, \quad (24)$$

for some constant  $c > 1 + \frac{1}{\beta} + \frac{1}{2\alpha}$ . From (22), (23) and (24) we deduce that for  $T$  large enough,

$$\mathbb{P} \left[ \sup_{t \in [0, T]} \Delta_t \leq 1 \right] \leq CT^{-\frac{\gamma}{2}} (\ln T)^c,$$

with  $c > \frac{\beta+1}{\beta} \frac{4-\gamma}{2-\gamma} + \frac{1}{2\alpha}$ .

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