Factorization of Cartesian Product Of Hypergraphs

Alain Bretto, Yannick Sylvestre

Factorization of Cartesian Products of Hypergraphs

Alain Bretto and Yannick Silvestre

Université de Caen, Département d'Informatique, GREYC CNRS UMR-6072,
Campus II, Bd Marechal Juin BP 5186, 14032 Caen cedex, France
alain.bretto@info.unicaen.fr, yannick.silvestre@info.unicaen.fr

Abstract. In this article we present the L2-section, a tool used to represent a hypergraph in terms of an "advanced graph" and results leading to first algorithm, in $O(nm)$, for a bounded-rank, bounded-degree hypergraph H, which factorizes H in prime factors. The paper puts a premium on the characterization of the prime factors of a hypergraph, by exploiting isomorphisms between the layers in the 2-section, as returned by a standard graph factorization algorithm, such as the one designed by Imrich and Peterin.

1 Introduction

Cartesian products of graphs have been studied since the 1960s by Vizing and Sabidussi. They independently showed ([Sab60]), that for every finite connected graph there is a unique (up to isomorphism) prime decomposition of the graph into factors. This fundamental theorem was the starting point for research concerning the relations between a Cartesian product and its factors [IPv97, Bre06]. Some of the questions raised are still open, as in the case of the Vizing's conjecture[1]. These relations are of particular interest as they allow us to break down problems by transfering algorithmic complexity from the product to the factors.

Some examples of problems that can be made easier by studying factors rather than the whole product include: classical problems, like the determination of the chromatic number, as the chromatic number of a Cartesian product is the maximum of the chromatic number of each factor; or the detemination of the independence number (see [IKR08]). More original numbers or properties are also investigated, especially in coloring theory: the antimagicness of a graph [ZS09] or the game chromatic number [Pet07], to quote a few. These numbers, easily computable thanks to Cartesian product operations, are graphical invariants. Graph products offer an interesting framework as soon as these graphical invariants are involved. This is the reason why they are studied with various applications; most of networks used in the context of parallel and distributed computation are Cartesian products: the hypercube, grid graphs, etc. In this context, the problem

[1] This conjecture expressed by Vizing in 1968 states that the domination number of the Cartesian product of graphs is greater than the product of the domination numbers of its factors.

of finding a "Cartesian" embedding of an interconnection network into another is of fundamental importance and thus has gained considerable attention. Cartesian products are also used in telecommunications [Ves02].

In 2006, Imrich and Peterin [IP07] gave an algorithm able to compute the prime factorization of connected graphs in linear time and space, making the use of Cartesian product decomposition particularly attractive.

Hypergraph theory has been introduced in the 1960s as a generalization of graph theory. A lot of applications of hypergraphs have been developed since (for a survey see [Bre04]). Cartesian products of hypergraphs can be defined in a same way as graphs, and similarly it is easier to study the hypergraph factors than the product. They also support graphical invariants (see [Bre06]), as it is the case for the linearity, conformity, transversal and matching number, Helly property, and it is generally possible to extend graphical invariants discovered on graphs to them.

Summary of the Results

In this paper we present an algorithm which gives the prime decomposition of a Cartesian product of hypergraphs. It is the first algorithm of recognition of Cartesian products of hypergraphs, up to our knowledge. This one is based on the algorithm of Imrich and Peterin [IP07] but it is easily adaptable to any algorithm which factorizes Cartesian products of graphs. Hypergraphs store more informations than graphs can (for fixed parameters); we explicit how a hypergraph can be seen as an "advanced" graph by introducing the *L2-section tool*. Some mathematical properties of the *L2-section* of a hypergraph help us then to design a recognition algorithm. By making an arrangement (called $\mathcal{R}^*$-*Cartesian joins*) on the factors returned by the recognition algorithm working on the 2-section, it finally releases the prime factors of a given hypergraph in $O(mn)$ time, when the rank and the degree of the hypergraph are constant.

Preliminaries

The cardinality of a set A is denoted by $|A|$. The set $\mathcal{P}_2(A)$ is the set of pairs $\{x, y\}$ such that $x, y \in A$ and $x \neq y$. A *hypergraph* H on a set of vertices V is a pair (V, E) where E is a set of non-empty subsets of V called *hyperedges* such that $\bigcup_{e \in E} e = V$. The set V may be written $\mathcal{V}(H)$. This implies in particular that every vertex is included in at least one hyperedge. If $\bigcup_{e \in E} e \neq V$, H is called a *pseudo-hypergraph*. A hypergraph is *simple* if no hyperedge is contained in another one.

In the sequel, we suppose that hypergraphs are simple and that no hyperedge is a *loop*, that is, the cardinality of a hyperedge is at least 2. Moreover we suppose that they are *connected*, a hypergraph being *connected* if there is a path between any pair of hyperedges. The number of hyperedges of a hypergraph H is denoted by $m(H)$ or simply m when unambiguous. It is convenient to define a *simple graph* as a particular case of simple hypergraph where every hyperedge is of size

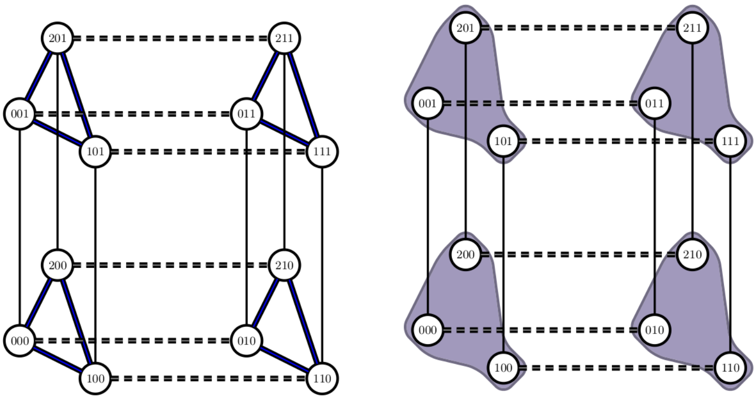

Fig. 1. To the left, the Cartesian product $T \square K_2 \square K_2 = T \square K_2^2$, where T stands for the triangle with vertex set $\{0,1,2\}$, and K_2 the graph $(\{0,1\},\{\{0,1\}\})$. Here we drew the layers with different dash patterns, and erased brackets and comas to name the nodes. For instance, T-layers are solid and thick, while K_2-layers are either dashed or doubled. We filled the nodes with their coordinates (without brackets). Notice that even if two isomorphic factors intervene in this product, their respective layers are considered as different as the coordinates of the nodes they are incident to only have one coordinate varying, but not the same. When considered isolated, "dashed" layers in the figure have their second coordinate varying while "thick" ones have their third coordinate varying. To the right the Cartesian product of hypergraphs $H = T' \square K_2^2$ where T' stands for the hypergraph $T' = (\{0,1,2\},\{\{0,1,2\}\})$.

2. So we suppose every graph $G = (V, E)$ to verify $\bigcup_{\{x,y\}\in E}\{x, y\} = V$. Given a graph $G = (V, E)$ and A a subset of E, we define $G(A) = (S, A)$ as a subgraph of G where $S = \{x \in a \in A\}$. For $E' \subseteq E$ the set $H' = (\bigcup_{e\in E'} e = V, E')$ is the *partial hypergraph* generated by E'. The *rank* of H is $r(H) = \max\{|e| : e \in E(H)\}$, written r when unambiguous.

The *2-section* $[H]_2$ of a hypergraph $H = (V, E)$ is the graph $G = (V, E')$ where $E' = \bigcup_{e\in E} \mathcal{P}_2(e)$, that is, two vertices are adjacent in G iff they belong to a same hyperedge. Notice that every hyperedge of H is a clique of $[H]_2$ and $\bigcup_{e\in E} e = V$ implies $\bigcup_{\{x,y\}\in E'}\{x, y\} = V$. Finally, an *isomorphism* from the hypergraph $H = (V, E)$ to the hypergraph $H' = (V', E')$ is a bijection f from V to V' such that, for every $e \subseteq V$, $e \in E$ iff $\forall x \in e$, $f(x) \in f(e)$. If H is isomorphic to H', we write $H \cong H'$. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. The *Cartesian product* of H_1 and H_2 is the hypergraph $H_1 \square H_2$ with set of vertices $V_1 \times V_2$ and set of edges:

$$E_1 \square E_2 = \underbrace{\{\{x\} \times e : x \in V_1 \text{ and } e \in E_2\}}_{A_1} \cup \underbrace{\{e \times \{u\} : e \in E_1 \text{ and } u \in V_2\}}_{A_2}.$$

Note that up to the isomorphism the Cartesian product is commutative and associative. That allows us to denote simply by $(v_1, \ldots, v_k)$ the vertices of $V_1 \times \ldots \times V_k$. The figure 1 illustrates the notion of Cartesian product of hypergraphs.

We conclude this section with well-known facts about Cartesian products:

Lemma 1. *$A_1 \cap A_2 = \emptyset$. Moreover, $|e \cap e'| \leq 1$ for any $e \in A_1$, $e' \in A_2$.*

Proposition 1. *Let H_1 and H_2 be two hypergraphs and H their Cartesian product. Then $[H_1]_2 \square [H_2]_2 = [H]_2$.*

2 Hypergraph Factorization Algorithm

In the sequel, we use some of the results from [IP07].
Cartesian coloring, layers, projections, coordinates, j-edges. In order to find prime factorizations of hypergraphs, we extend the algorithm given in [IP07]. This algorithm is based on a coloring of the edges of the graph G revealing the Cartesian structure of G. In the following, such a coloring will be adapted for hypergraphs and evoked as "*Cartesian coloring of the hyperedges*". Indeed, the hypergraph H to be factorized contains isomorphic copies of the factors which lay as proper subgraphs of H. These isomorphic copies are called *layers*. If $H = H_1 \square \ldots \square K_k$, a *$H_i$-layer* $(1 \leqslant i \leqslant k)$ can be defined more formally as a partial hypergraph $H_i^* = (V_i^*, E_i^*)$ such that E_i^* is a maximal set of edges where the coordinates of their endpoints are fixed except the i^{th} one, and $V_i^* = \bigcup_{e \in E_i^*} e$. By coloring with the same color these layers, we thus reveal what factors H is made of. In the sequel, the colorings we will deal with will mainly refer to these edge colorings. From this angle of view, layers can be identified to the colors covering their edges. For all $w \in V$, $H_i^w = (V_i^w, E_i^w)$ will stand for the H_i-layer incident to w. For a product $H = H_1 \square H_2 \square \ldots \square H_l$, we define the *projection on the i^{th} coordinate* $p_i : \mathcal{V}(H) \to \mathcal{V}(H_i)$ as the mapping $p_i : (x_1, x_2, \ldots, x_l) \mapsto x_i$. Note that $p_{i\,|\mathcal{V}(H_i^w)}$ is a hypergraph isomorphism, for all $w \in \mathcal{V}(H)$. It can be easily shown that every hyperedge of H is contained in exactly one layer. Moreover the hyperedge sets of the layers partition the hyperedge set of H. If we decide to assign edges colors corresponding to the layer they belong to, we may call j-edges edges laying in a H_j-layer, and may assign them the colour j.

Lemma 2 (Square lemma). *[IP07] Let G be a graph. If two edges of G are adjacent edges which belong to non-isomorphic layers, then these edges lay in a unique induced square.*

A straightforward consequence of the Square Lemma is that any triangle of G is necessarily contained in a single layer. Another consequence is that opposite edges in squares have the same color. In the sequel, we will say that the *square lemma is verified* when two adjacent edges with different colors lay in exactly one induced square. From the Square lemma we easily get the following result, also given in [IP07]: every clique in a graph is contained a single layer.
The extension to hypergraphs of the algorithm of [IP07] uses L2-sections.

Definition 1 ($L2$-section). *Let $H = (V, E)$ be a hypergraph, we define the $L2$-section $[H]_{L2}$ of H as the triple $\Gamma = (V, E', \mathcal{L})$ where (V, E') is the 2-section of H and $\mathcal{L} : E' \to \mathcal{P}(E)$ is defined by $\mathcal{L}(\{x, y\}) = \{e : x, y \in e \in E\}$. Conversely,*

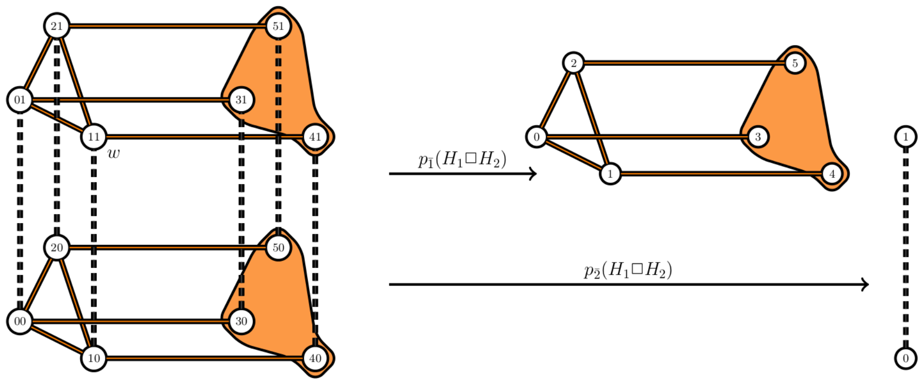

Fig. 2. We adopt the same conventions of representations of the vertices in the drawing as the ones available in figure 1. To the left, the hypergraph $H_1 \square H_2$ where $H_1 = (\{0,1,2,3,4,5\}, \{\{0,1\},\{0,2\},\{1,2\},\{0,3\},\{1,4\},\{2,5\},\{3,4,5\}\})$ and where $H_2 = K_2$, $\mathcal{V}(K_2) = \{0,1\}$. Dashed edges emphasize K_2-layers. The given decomposition is the prime decomposition of the given hypergraph. The 2-section of the given hypergraph, with its Cartesian coloring is shown in the figure 1 at the left (up to isomorphisms). You can notice that the colors of $H_1 \square H_2$ can be obtained by merging color classes in figure 1, left. If w stands for the vertex $(1,1)$ then H_1^w is the induced partial hypergraph with vertices $\{(0,1),(1,1),(2,1),(3,1)(4,1),(5,1)\}$. We note that, as expected, only the first coordinate vary for the vertices of this layer. The hypergraph H_2^w is simply $(\{(1,1),(1,0)\}, \{\{(1,1),(1,0)\}\})$. The hypergraph H_1 is represented to the left. Finally we observe that $H_1 = p_1(H) \cong H_1^w$ and $H_2 \cong H_2^w$, more precisely we have $p_1(H_1^w) = H_1$ and $p_2(H_2^w) = H_2$.

if $\Gamma = (V, E', \mathcal{L})$ is a triple where $\mathcal{L}$ is a function from E' to $\mathcal{P}(E)$, we define the inverse L2-section $[\Gamma]_{L2}^{-1} = (V, E)$ of Γ as the pseudo-hypergraph with $E = \bigcup_{e_i \in E'}(\mathcal{L}(e_i))$. The mapping $\mathcal{L}$ is also referred to as a labelling on edges.

It is easy to check that if Γ is a L2-section then $[\Gamma]_{L2}^{-1}$ is a hypergraph. Not surprisingly the following result comes from the definition.

Proposition 2. *For all hypergraphs H and L2-sections Γ we have $[[H]_{L2}]_{L2}^{-1} = H$ and $[[\Gamma]_{L2}^{-1}]_{L2} = \Gamma$.*

The Cartesian product operation is now extended to L2-sections.

Definition 2 (Cartesian product of $L2$-sections). *Let $\Gamma_1 = (V_1, E'_1, \mathcal{L}_1)$ and $\Gamma_2 = (V_2, E'_2, \mathcal{L}_2)$ be the L2-sections of $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$. We define their Cartesian product $\Gamma_1 \square \Gamma_2$ as the triple $(V, E', \mathcal{L}_1 \square \mathcal{L}_2)$ where:*

- *(V, E') is the Cartesian product of (V_1, E'_1) and (V_2, E'_2).*
- *$\mathcal{L}_1 \square \mathcal{L}_2$ is the map from $E' = E'_1 \square E'_2$ to $\mathcal{P}(E_1 \square E_2)$ defined by:*

$$\mathcal{L}_1 \square \mathcal{L}_2(\{(x,u),(y,v)\}) = \begin{cases} \{\, \{x\} \times e : e \in \mathcal{L}_2(\{u,v\}) \,\} \text{ if } x = y. \\ \{\, e \times \{u\} : e \in \mathcal{L}_1(\{x,y\}) \,\} \text{ if } u = v. \end{cases}$$

Lemma 3. *For all hypergraphs H_1, H_2 we have: $[H_1 \Box H_2]_{L2} = [H_1]_{L2} \Box [H_2]_{L2}$ and $[[H_1]_{L2} \Box [H_2]_{L2}]_{L2}^{-1} = [[H_1]_{L2}]_{L2}^{-1} \Box [[H_2]_{L2}]_{L2}^{-1}$.*

Definition 3 (Isomorphism of L2-sections). *An isomorphism between two L2-sections $\Gamma_1 = (V_1, E_1', \mathcal{L}_1)$ and $\Gamma_2 = (V_2, {E_2}', \mathcal{L}_2)$ is a bijection f from V_1 to V_2 such that:*

a) $\{x, y\} \in E_1'$ if and only if $\{f(x), f(y)\} \in E_2'$, for all $x, y \in V_1$.
b) $e \in \mathcal{L}_1(\{x, y\})$ if and only if $\{f(z) : z \in e\} \in \mathcal{L}_2(\{f(x), f(y)\})$, for all $x, y \in V_1$ and $e \subseteq V_1$.

To express that Γ_1 and Γ_2 are isomorphic, we write: $\Gamma_1 \cong \Gamma_2$

Lemma 4. *Let H and H' be two hypergraphs. Then $H \cong H'$ is equivalent to $[H]_{L2} \cong [H']_{L2}$. If moreover H and H' are conformal then these statements are equivalent to $[H]_2 \cong [H']_2$.*

Note that the Cartesian product is commutative and associative on L2-sections. That allows us to overlook parenthesis for Cartesian products of L2-sections. Recall that, given a hypergraph H and its "Cartesian" coloring, layers in its 2-section are isomorphic, what may not be the case of the layers on its L2-section (e.g. the labelled subgraphs are not isomorphic even if the non-labelled versions of these subgraphs are). This is why layers for the 2-section in figure 1 do not coincide with the ones of the hypergraph the 2-section comes from, given in figure 2.

In the sequel, the notions of H_i-layers, H_i^w etc. are naturally extended to L2-sections with similar notations (Γ_i-layers, Γ_i^w etc.) by adding the mapping $\mathcal{L}$ to the graph as defined in definition 1.

Definition 4 (G_j-adjacent layers and Γ_j-adjacent layers). *Let H be a hypergraph, with 2-section G and L2-section Γ. We will say that G_j (resp. Γ_j)-*adjacent layers *are vertex-disjoint layers G', G'' (resp. Γ', Γ'') such that there exists an edge $\{x, x'\}$ laying in a G_j- (resp. Γ_j-) layer where $x \in \mathcal{V}(G')$ and $y \in \mathcal{V}(G'')$.*

The following definition is intended to specify how layers are related one to the other: some colors connect the incident layers by maintaining a mapping between them which is an L2-isomorphism, although others do not. Clearly, when one color does not "define" an isomorphism between two adjacent layers, there is not the slightest risk that the involved layers can be considered in different labelled prime layers. Then this pattern can be envisaged as an obstruction for the colors to define prime hypergraph factors. This is precisely what expresses the $\mathcal{R}^*$ relation introduced below. It will be used further, to build the Cartesian joins, which can be seen as mergings of layers so that the new colors resulting from the mergings are good candidates for the definition of the prime factors.

Definition 5 (Induction of isomorphisms of Γ_i-layers by Γ_j-layers, relations $\mathcal{A}_{i,j}$ and $\mathcal{R}_{i,j}$). *We will say that* Γ_j-layers induce an isomorphism between Γ_i-layers *($i \neq j$) if the following property $\mathcal{A}_{i,j}$ is verified:* "For all $w, w' \in \mathcal{V}(\Gamma)$, let Γ_i^w and $\Gamma_i^{w'}$ be two Γ_j-adjacent layers. The graph isomorphism between G_i^w and $G_i^{w'}$ which maps x to x' such that $\{x, x'\}$ is a j-edge gives rise to an isomorphism of $L2$-sections between Γ_i^w to $\Gamma_i^{w'}$." *If this relation $\mathcal{A}_{i,j}$ is not verified and $i \neq j$, we will say that i is in relation with j and we will denote it by $i\mathcal{R}j$.*

In the sequel we will consider the reflexive, symmetric and transitive closure of $\mathcal{R}$, denoted by $\mathcal{R}^*$.

Definition 6 (Set Col of colors of a Cartesian coloring, equivalence classes colors $\bar{i} \in \mathbf{Col}/\mathcal{R}^*$, $\bar{i}$-edges). *The relation $\mathcal{R}^*$ is defined as the symmetric, reflexive and transtive closure of $\mathcal{R}$. Given Col the set of the colors involved in the Cartesian coloring of a graph G, the set of these $\mathcal{R}^*$-equivalence classes of colors will be denoted $Col/\mathcal{R}^*$. The equivalence class of colors of i, $i \in \{1, \ldots, k\}$ will be denoted by $\bar{i}$.*
We will talk of $\bar{i}$-edges for edges colored in one color $\mathcal{R}^*$-equivalent to i.

Definition 7 (Cartesian joins $\Gamma_{\bar{i}}^w = (V_{\bar{i}}^w, E_{\bar{i}}'^w, \mathcal{L}_{\bar{i}}^w)$, projection $p_{\bar{i}}$). *Let H be a hypergraph, $\Gamma = (V, E', \mathcal{L})$ its L2-section, $G = G_1 \Box \ldots \Box G_k$ its 2-section, and Col the set of the colors involved in the Cartesian coloring of G. Given w a vertex, we define $G_{\bar{i}}^w = (V_{\bar{i}}^w, E'^w_{\bar{i}})$ (resp. $\Gamma_{\bar{i}}^w = (V_{\bar{i}}^w, E'^w_{\bar{i}}, \mathcal{L}_{\bar{i}}^w)$) as the connected component of $\bar{i}$-edges adjacent to w in G (resp. Γ). The graph $G_{\bar{i}}^w$ and the labelled graphs $\Gamma_{\bar{i}}^w$ are called the $\mathcal{R}^*$-induced Cartesian joins; they will be referred to as (Cartesian) joins in the sequel. We also redefine the* projection $p_{\bar{i}}$ *as the mapping which associates to a vertex in the Cartesian product $\prod_{i \in Col/\mathcal{R}^*} \Gamma_{\bar{i}}$ the corresponding coordinate in the $\bar{i}$-class.*

These graphs can also be qualified as $G_{\bar{i}}$- (resp. $\Gamma_{\bar{i}}$-) layers; compared with traditional G_i- or Γ_i- layers, these ones are built on color classes, instead of single colors. One can also think of it as layers defined on colors resulting from the merging of elementary colors, the last ones being induced by the Cartesian coloring of the graph G. Notice, moreover, that Cartesian joins are L2-sections.

Proposition 3. *Let H be a hypergraph, $\Gamma = (V, E', \mathcal{L})$ its L2-section, G its 2-section and Col the set of the colors involved in the Cartesian coloring of G. For every w, for every $i \in \{1, \ldots, k\}$, we have: $\Gamma_{\bar{i}}^w \cong \Gamma_{\bar{i}}^{w'}$.*

Now that we know $\Gamma_{\bar{i}}$-layers are isomorphic, we can generalize the notion of induction of Γ_i-layers by Γ_j-layers, by simply considering classes of colors insted of single colors. It also allows us to define $\Gamma_{\bar{i}}$ as the L2-section $p_{\bar{i}}(\Gamma_{\bar{i}}^w)$, for all w.

Lemma 5. *Let $\Gamma = (V, E', \mathcal{L})$ be the Cartesianly colored L2-section of a hypergraph H, and let G_i be the layers of the 2-section G of H. Then, for all $\bar{i} \in Col/\mathcal{R}^*$, $G_{\bar{i}} \cong \prod_{k \in \bar{i}} G_k$.*

Algorithm 1. Hypergraph-prime decomposition

Require: A hypergraph $H = (V, E)$.
Return: The prime factors of H, that is $H_1, H_2, \ldots, H_l$

Compute $\Gamma = (V, E', \mathcal{L})$, the L2-section of H and $G = (V, E')$.
Run the algorithm of [IP07] on G and call $G_1 = (V_1, E'_1), \ldots, G_k = (V_k, E'_k)$ its prime factors.
Define $\mathcal{L}_i$ as the restriction of $\mathcal{L}$ to E'_i, $\Gamma_i = (V_i, E'_i, \mathcal{L}_i)$.
Define the graph $J = (\text{Col}=\{1, 2, \ldots, k\}, E_{\text{Col}} := \emptyset)$ whose connected components (CC) will "correspond to" the prime factors of H.
Define the set of the CCs as $S = \{C_1, \ldots, C_k\}$ with $C_i = \emptyset$ initially, $\forall i \in \{1, \ldots, k\}$.

Define c the investigated CC as 0, and T as an empty array connecting the colors to the CC they belong to.
Let $l = 0$ the number of the investigated CC.
For $i = 1$ to k **do**
  If $T[i]$ is defined **then**
    $c = T[i]$
  Else
    $l = l + 1$, $c = l$, $T[i] = c$
  End If
  For $j = 1$ to k, $j > i$, $j \notin C_c$ **do**
    If $i\mathcal{R}j$ OR $j\mathcal{R}i$ **then**
      $E_{Col} = E_{Col} \cup \{i, j\}$, $C_c = C_c \cup \{j\}$, $T[j] = c$
    End If
  EndFor
EndFor
Return $H_1 = [\Gamma_{C_1}]^{-1}_{L2}$, …, $H_l = [\Gamma_{C_l}]^{-1}_{L2}$.

Theorem 1. *Let $\Gamma = (V, E', \mathcal{L})$ be the L2-section of a hypergraph H. Let G be the 2-section of H. Suppose $G = G_1 \Box G_2 \Box \ldots \Box G_l$ where $G_i = (V_i, E'_i)$, $i \in \{1, \ldots, l\}$ are layers in G (up to isomorphisms). Define $\Gamma_i = (V_i, E'_i, \mathcal{L}_i)$, where $\mathcal{L}_i$ is the restriction of $\mathcal{L}$ to E'_i. Then*

$$H \cong \prod_{\bar{i} \in (Col/\mathcal{R}^*)} [\Gamma_{\bar{i}}]^{-1}_{L2} .$$

Theorem 2. *Let $\Gamma = (V, E', \mathcal{L})$ be the L2-section of a hypergraph H. Let G be the 2-section of H. Then $H \cong \prod_{\bar{i} \in (Col/\mathcal{R}^*)} [\Gamma_{\bar{i}}]^{-1}_{L2}$ is a prime decomposition of Γ.*

We introduce now an algorithm (algorithm 1) which gives the prime factorization of hypergraphs, derived from theorem 1. The idea is the following. From the connected hypergraph H it first builds the $L2$-section Γ of H. Then it runs the algorithm of Imrich and Peterin which colors the edges of the unlabelled underlying graph G. The color i is used for all edges of all layers that belong to the same factor G_i. When obtained the prime factorization of G, say $G_1, \ldots, G_k$ of G, we determine the $\mathcal{R}^*$-induced Cartesian joins thanks to $\mathcal{L}$. To determine the joins, we consider a graph whose vertices are the colors obtained from the prime decomposition of G (i.e. 1, …, k), and whose edges express an "obstruction" for the prime factors of G to be considered as labelled prime factors of Γ (this obstruction exists from the moment the colors are $\mathcal{R}$-related). Then the Cartesian joins are the connected components in this new graph.

Finally, by using the inverse L2-section operation, we build back the factors of H.

Theorem 3. *Algorithm 1 is correct.*

This is deduced from the previous results.

If we rely on special data structures, the overall complexity of this algorithm is in $O((\log_2 n)^2 r^3 m \Delta^2))$, where Δ stands for the maximal degree of H, and n stands for the number of vertices of H. This leads to a polynomial algorithm in $O(nm)$ when r, Δ are fixed.

References

[Ber89] Berge, C.: Hypergraphs. North Holland, Amsterdam (1989)

[Bre04] Bretto, A.: Introduction to hypergraph theory and its use in engineering and image processing. Advances in imaging and electron physics 131 (2004)

[Bre06] Bretto, A.: Hypergraphs and the helly property. Ars Combinatoria 78, 23–32 (2006)

[IKR08] Imrich, W., Klavžar, S., Rall, D.F.: Topics in graph theory. Graphs and their Cartesian product. A K Peters, Wellesley (2008)

[IP07] Imrich, W., Peterin, I.: Recognizing cartesian products in linear time. Discrete Mathematics 307, 472–483 (2007)

[IPv97] Imrich, W., Pisanski, T., Žerovnik, J.: Recognizing cartesian graph bundles. Discrete Mathematics 167-168, 393–403 (1997)

[Pet07] Peterin, I.: Game chromatic number of cartesian product graphs. Electronic Notes in Discrete Mathematics 29, 353–357 (2007)

[Sab60] Sabidussi, G.: Graph multiplication. Mathematische Zeitschrift 72 (1960)

[Ves02] Vesel, A.: Channel assignment with separation in the cartesian product of two cycles. In: Proceedings of the 24th International Conference on Information Technology Interfaces (2002)

[Viz63] Vizing, V.G.: The cartesian product of graphs. Vycisl. Sistemy 9, 30–43 (1963)

[ZS09] Zhang, Y., Sun, X.: The antimagicness of the cartesian product of graphs. Theor. Comput. Sci. 410(8-10), 727–735 (2009)