Spectral multiplier theorems and R-boundedness

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SPECTRAL MULTIPLIER THEOREMS AND AVERAGED R-BOUNDEDNESS

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ABSTRACT. Let A be a 0-sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\sigma \in (0,\pi)$, e.g. a Laplace type operator on $L^p(\Omega)$, $1 < p < \infty$, where Ω is a manifold or a graph. We show that A has a $\mathcal{H}^\alpha_2(\mathbb{R}_+)$ Hörmander functional calculus if and only if certain operator families derived from the resolvent $(\lambda - A)^{-1}$, the semigroup e^{-zA}, the wave operators e^{itA} or the imaginary powers A^{it} of A are R-bounded in an L^2-averaged sense. If X is an $L^p(\Omega)$ space with $1 \le p < \infty$, R-boundedness reduces to well-known estimates of square sums.

1. INTRODUCTION

Hörmander's Fourier multiplier theorem states that for a function $f \in \mathcal{H}^\alpha_2(\mathbb{R}_+)$ the operator $f(-\Delta)$, defined in terms of the functional calculus on $L^2(\mathbb{R}^d)$ can be extended to $L^p(\mathbb{R}^d)$ if $1 < p < \infty$ and $\alpha > \frac{d}{2}$. Here

$$\mathcal{H}^\alpha_2(\mathbb{R}_+) = \{f \in C(\mathbb{R}_+, \mathbb{C}) : \sup_{t>0} \|\phi f(t\cdot)\|_{W^\alpha_2(\mathbb{R}_+)} < \infty\}$$

where $\phi \in C^\infty(\mathbb{R})$ with compact $\operatorname{supp}\phi \subset (0,\infty)$ is a cut-off function and $W^\alpha_2(\mathbb{R}_+)$ is the usual Riesz-potential Sobolev space. For $\alpha \in \mathbb{N}$, an equivalent norm on $\mathcal{H}^\alpha_2$ is given by the "classical" Hörmander condition

$$\sup_{R>0,\, \beta=0,\dots,\alpha} \frac{1}{R} \int_R^{2R} |t^\beta D^\beta f(t)|^2 dt < \infty.$$

There is a large literature extending such a spectral multiplier result to more general self-adjoint operators on $L^p(\Omega)$, e.g. for Laplace type operators on manifolds, infinite graphs and fractals (see e.g. [1, 7, 10, 11, 26, 30] and the references therein). There are various approaches to the $\mathcal{H}^\alpha_2$ calculus using kernel estimates, maximal estimates or square function estimates for the resolvent $(\lambda - A)^{-1}$, the analytic semigroup e^{-zA} generated by $-A$ and their "boundary", the wave operators e^{itA}, or the imaginary powers A^{it} of A. Relevant are e.g. estimates on operator functions such as ($\alpha > \frac{1}{2}$, $m > \alpha - \frac{1}{2}$ are fixed)

- $T_\theta(t) = A^{\frac{1}{2}} e^{-e^{i\theta} tA}, \quad t \in \mathbb{R}_+,$
- $R_\theta(t) = A^{\frac{1}{2}} R(e^{i\theta} t, A), \quad t \in \mathbb{R}_+,$
- $W(s) = |s|^{-\alpha} A^{-\alpha+\frac{1}{2}} (e^{isA} - 1)^m, \quad s \in \mathbb{R},$

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- $I(t) = (1+|t|)^{-\alpha} A^{it}, \quad t \in \mathbb{R}$.

Many of these estimates imply or are closely related to square sum estimates of the following form

$$\left\| \left(\sum_i |S_i x_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq C \left\| \left(\sum_i |x_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \tag{1.1}$$

where $x_i \in L^p(\Omega)$ and the S_i are members of one of the families listed above (see e.g. [3, 32] for an early appearance of this square sum estimate in the context of spectral multiplier theorems). If (r_n) is a sequence of Rademacher functions on $[0,1]$ one can reformulate (1.1) equivalently as

$$\int_0^1 \| \sum_i r_i(\omega) S_i x_i \| d\omega \leq C \int_0^1 \| \sum_i r_i(\omega) x_i \| d\omega. \tag{1.2}$$

This statement makes sense in an arbitrary Banach space X and a set $\tau \subset B(X)$ is called R-bounded if (1.2) holds for all $S_i \in \tau$ and $x_i \in X$. Using R-boundedness in place of kernel estimates and the holomorphic $H^\infty(\Sigma_\sigma)$ calculus instead of the spectral theorem for selfadjoint operators, one can develop a theory of spectral multiplier theorems for 0-sectorial operators on Banach spaces (see [20, 21, 22, 23, 24]). Again, R-bounds for one of the operator families listed above are sufficient to secure $\mathcal{H}^\alpha_2(\mathbb{R}_+)$ spectral theorems for such operators A. However, neither in this general framework nor in the case of Laplace type operators on an $L^p(\Omega)$ space (see above), one obtains necessary and sufficient conditions in terms of R-bounds or kernel estimates. This is related to the (usually) difficult task of determining the optimal α for the $\mathcal{H}^\alpha_2(\mathbb{R}_+)$ spectral calculus of a given operator A. Thus the purpose of this paper is to give a characterization of the $\mathcal{H}^\alpha_2(\mathbb{R}_+)$ spectral multiplier theorem in terms of an L^2-averaged R-boundedness condition. More precisely, let $t \in J \mapsto N(t) \in B(X)$ be weakly square integrable on an interval J. Then $(N(t))_{t\in J}$ is called $R[L^2]$-bounded if for $h \in L^2(J)$ with $\|h\|_{L^2(J)} \leq 1$ the strong integrals

$$N_h x = \int_J h(t) N(t) x dt, \quad x \in X$$

define an R-bounded subset $\{N_h : \|h\|_{L^2(J)} \leq 1\}$ of $B(X)$. By $R[L^2(J)](N(t))$, we denote the R-bound of this set. In a Hilbert space X, $R[L^2(J)]$-boundedness reduces to the simple estimate

$$\left(\int_J |\langle N(t)x, y\rangle|^2 dt \right)^{\frac{1}{2}} \leq C\|x\| \, \|y\| \text{ for all } x, y \in H.$$

Assume now that A is a 0-sectorial operator with an $H^\infty(\Sigma_\sigma)$ calculus for some $\sigma \in (0,\pi)$ on a Banach space isomorphic to a subspace of an $L^p(\Omega)$ space with $1 \leq p < \infty$ (or more generally, let X have Pisier's property (α)). Then our main results, Theorems 6.1 and 6.4 show (among other statements), that the following conditions on the operator function above are essentially equivalent:

A has an R-bounded $\mathcal{H}^\alpha_2$ spectral calculus, i.e. $\{f(A) : \|f\|_{\mathcal{H}^\alpha_2(\mathbb{R}_+)} \leq 1\}$ is R-bounded in $B(X$

resolvents: $R[L^2(\mathbb{R}_+)](R_\theta(\cdot)) \leq C|\theta|^{-\alpha}$ for $\theta \to 0$

$$\text{semigroup: } R[L^2(\mathbb{R}_+)](T_\theta(\cdot)) \leq C(\frac{\pi}{2} - |\theta|)^{-\alpha} \text{ for } |\theta| \to \frac{\pi}{2}$$

$$\text{wave operators: } R[L^2(\mathbb{R})](W(\cdot)) < \infty$$

$$\text{imaginary powers: } R[L^2(\mathbb{R})](I(\cdot)) < \infty.$$

The estimates on $R_\theta(\cdot)$ (resp. on $T_\theta(\cdot)$) measure the growth of the resolvent (the analytic semigroup) as we approach the spectrum of A (resp. the "boundary" $i\mathbb{R}$ of $\mathbb{C}_+$) on rays in $\mathbb{C}\backslash\mathbb{R}_+$ (in $\mathbb{C}_+$). Clearly, the $R[L^2(J)]$-boundedness of $I(\cdot)$ measures the polynomial growth of the imaginary powers and $W(\cdot)$ the growth of the regularized wave operators e^{itA}. The latter regularization is necessary since outside Hilbert space the operators e^{itA} are usually unbounded. The equivalence of these statements shows in particular that estimates of resolvents, the semigroup, wave operators or imaginary powers are all equivalent ways to obtain the boundedness of $f(A)$ for arbitrary $f \in \mathcal{H}_2^\alpha(\mathbb{R}_+)$.

We end this introduction with an overview of the article. Section 2 contains the background on H^∞ functional calculus for a sectorial operator A, R-boundedness as well as the definition of relevant function spaces. In Section 3 we introduce the Hörmander and Mihlin function spaces and their functional calculus. In Section 4 as a preparation for the proof of Theorem 6.1, we relate the wave operators e^{isA} with imaginary powers A^{it} via the Mellin transform. In Section 5, we study the notion of averaged R-boundedness. We feel that it is worthwhile to introduce averaged R-boundedness also for other function spaces than $L^2(J)$ since these notions appeared already implicitly in the literature and have proven to be quite useful. Finally in Section 6 we state the main Theorem 6.1, which establishes equivalences between the smaller $\mathcal{W}_2^\alpha$ functional calculus and averaged R-boundedness of the operator families above (see Section 3 for the definition of this function space). However, most of the classical spectral multipliers (e.g. $f(\lambda) = \lambda^{it}$) belong to $\mathcal{H}_2^\alpha \backslash \mathcal{W}_2^\alpha$. Therefore we extend in Theorem 6.4 this calculus to $\mathcal{H}_2^\alpha$ by means of a localization procedure. Finally, in Section 7, we indicate how our main results can be transferred to bisectorial and strip-type operators.

2. Preliminaries

2.1. **0-sectorial operators.** We briefly recall standard notions on H^∞ calculus. For $\omega \in (0, \pi)$ we let $\Sigma_\omega = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \omega\}$ the sector around the positive axis of aperture angle 2ω. We further define $H^\infty(\Sigma_\omega)$ to be the space of bounded holomorphic functions on Σ_ω. This space is a Banach algebra when equipped with the norm $\|f\|_{\infty,\omega} = \sup_{\lambda\in\Sigma_\omega} |f(\lambda)|$.

A closed operator $A : D(A) \subset X \to X$ is called ω-sectorial, if the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$, $R(A)$ is dense in X and

$$\text{for all } \theta > \omega \text{ there is a } C_\theta > 0 \text{ such that } \|\lambda(\lambda - A)^{-1}\| \leq C_\theta \text{ for all } \lambda \in \overline{\Sigma_\theta}^c. \tag{2.1}$$

Note that $\overline{R(A)} = X$ along with (2.1) implies that A is injective. In the literature, in the definition of sectoriality, the condition $\overline{R(A)} = X$ is sometimes omitted. Note that if A satisfies the conditions defining ω-sectoriality except $\overline{R(A)} = X$ on $X = L^p(\Omega)$, $1 < p < \infty$ (or any reflexive space), then there is a canonical decomposition $X = \overline{R(A)} \oplus N(A)$, $x = x_1 \oplus x_2$, and $A = A_1 \oplus 0$, $x \mapsto Ax_1 \oplus 0$, such that A_1 is ω-sectorial on the space $\overline{R(A)}$ with domain $D(A_1) = \overline{R(A)} \cap D(A)$.

For an ω-sectorial operator A and a function $f \in H^\infty(\Sigma_\theta)$ for some $\theta \in (\omega, \pi)$ that satisfies moreover an estimate $|f(\lambda)| \leq C|\lambda|^\epsilon/|1+\lambda|^{2\epsilon}$, one defines the operator

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda, \tag{2.2}$$

where Γ is the boundary of a sector Σ_σ with $\sigma \in (\omega, \theta)$, oriented counterclockwise. By the estimate of f, the integral converges in norm and defines a bounded operator. If moreover there is an estimate $\|f(A)\| \leq C\|f\|_{\infty,\theta}$ with C uniform over all such functions, then A is said to have a bounded $H^\infty(\Sigma_\theta)$ calculus. In this case, there exists a bounded homomorphism $H^\infty(\Sigma_\theta) \to B(X)$, $f \mapsto f(A)$ extending the Cauchy integral formula (2.2).

We refer to [5] for details. We call A 0-sectorial if A is ω-sectorial for all $\omega > 0$.

For $\omega \in (0, \pi)$, define the algebras of functions $\mathrm{Hol}(\Sigma_\omega) = \{f : \Sigma_\omega \to \mathbb{C} : \exists\, n \in \mathbb{N} :\ \rho^n f \in H^\infty(\Sigma_\omega)\}$, where $\rho(\lambda) = \lambda(1+\lambda)^{-2}$. For a proof of the following lemma, we refer to [27, Section 15B] and [14, p. 91-96],[15],[16].

Lemma 2.1. Let A be a 0-sectorial operator. There exists a linear mapping, called the extended holomorphic calculus,

$$\bigcup_{0<\omega<\pi} \mathrm{Hol}(\Sigma_\omega) \to \{\text{closed and densely defined operators on } X\},\ f \mapsto f(A) \tag{2.3}$$

extending (2.2) such that for any $f, g \in \mathrm{Hol}(\Sigma_\omega)$, $f(A)g(A)x = (fg)(A)x$ for $x \in \{y \in D(g(A)) :\ g(A)y \in D(f(A))\} \subset D((fg)(A))$ and $D(f(A)) = \{x \in X :\ (\rho^n f)(A)x \in D(\rho(A)^{-n}) = D(A^n) \cap R(A^n)\}$, where $(\rho^n f)(A)$ is given by (2.2), i.e. $n \in \mathbb{N}$ is sufficiently large.

2.2. **Function spaces on the line and half-line.** In this subsection, we introduce several spaces of differentiable functions on $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}$. Let $\psi \in C_c^\infty(\mathbb{R})$. Assume that $\operatorname{supp}\psi \subset [-1, 1]$ and $\sum_{n=-\infty}^\infty \psi(t-n) = 1$ for all $t \in \mathbb{R}$. For $n \in \mathbb{Z}$, we put $\psi_n = \psi(\cdot - n)$ and call $(\psi_n)_{n\in\mathbb{Z}}$ an equidistant partition of unity. Let $\varphi \in C_c^\infty(\mathbb{R}_+)$. Assume that $\operatorname{supp}\varphi \subset [\frac{1}{2}, 2]$ and $\sum_{n=-\infty}^\infty \varphi(2^{-n}t) = 1$ for all $t > 0$. For $n \in \mathbb{Z}$, we put $\varphi_n = \varphi(2^{-n}\cdot)$ and call $(\varphi_n)_{n\in\mathbb{Z}}$ a dyadic partition of unity. Next let $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\phi_1 \subset [\frac{1}{2}, 2]$ and $\operatorname{supp}\phi_0 \subset [-1, 1]$. For $n \geq 2$, put $\phi_n = \phi_1(2^{1-n}\cdot)$, so that $\operatorname{supp}\phi_n \subset [2^{n-2}, 2^n]$. For $n \leq -1$, put $\phi_n = \phi_{-n}(-\cdot)$. We assume that $\sum_{n\in\mathbb{Z}} \phi_n(t) = 1$ for all $t \in \mathbb{R}$. Then we call $(\phi_n)_{n\in\mathbb{Z}}$ a dyadic partition of unity on $\mathbb{R}$, which we will exclusively use to decompose the Fourier image of a function. For the existence of such partitions, we refer to the idea in [2, Lemma 6.1.7]. We recall the following classical function spaces:

Notation 2.2. Let $m \in \mathbb{N}_0$ and $\alpha > 0$.

(1) $C_b^m = \{f : \mathbb{R} \to \mathbb{C} : f\ m\text{-times diff. and } f, f', \ldots, f^{(m)} \text{ uniformly cont. and bounded}\}$.
(2) $W_2^\alpha = \{f \in L^p(\mathbb{R}) :\ \|f\|_{W_2^\alpha} = \|(\hat{f}(t)(1+|t|)^\alpha)^\vee\|_2 < \infty\}$.
(3) $\mathcal{B}_{\infty,1}^\alpha$, the Besov space defined for example in [33, p. 45]: Let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity on $\mathbb{R}$. Then

$$\mathcal{B}^\alpha = \mathcal{B}_{\infty,1}^\alpha = \{f \in C_b^0 :\ \|f\|_{B_{\infty,1}^\alpha} = \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \|f * \check{\phi}_n\|_\infty < \infty\}.$$

The space W_2^α is a Banach algebra with respect to pointwise multiplication if $\alpha > \frac{1}{2}$, and the space $\mathcal{B}^\alpha$ is a Banach algebra for any $\alpha > 0$ [31, p. 222].

Further we also consider the local space

(4) $W^\alpha_{2,\mathrm{loc}} = \{f : \mathbb{R} \to \mathbb{C} : \ f\varphi \in W_2^\alpha \text{ for all } \varphi \in C_c^\infty\}$ for $\alpha > \frac{1}{2}$.

This space is closed under pointwise multiplication. Indeed, if $\varphi \in C_c^\infty$ is given, choose $\psi \in C_c^\infty$ such that $\psi\varphi = \varphi$. For $f, g \in W^\alpha_{2,\mathrm{loc}}$, we have $(fg)\varphi = (f\varphi)(g\psi) \in W^\alpha_{2,\mathrm{loc}}$.

2.3. **Rademachers, Gaussians and R-boundedness.** A classical theorem of Marcinkiewicz and Zygmund states that for elements $x_1, \dots, x_n \in L^p(U, \mu)$ we can express "square sums" in terms of random sums

$$\left\| \left(\sum_{j=1}^n |x_j(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(U)} \cong \left(\mathbb{E} \| \sum_{j=1}^n \epsilon_j x_j \|^q_{L^p(U)} \right)^{\frac{1}{q}} \cong \left(\mathbb{E} \| \sum_{j=1}^n \gamma_j x_j \|^q_{L^p(U)} \right)^{\frac{1}{q}}$$

with constants only depending on $p, q \in [1, \infty)$. Here $(\epsilon_j)_j$ is a sequence of independent Bernoulli random variables (with $P(\epsilon_j = 1) = P(\epsilon_j = -1) = \frac{1}{2}$) and $(\gamma_j)_j$ is a sequence of independent standard Gaussian random variables. Following [4] it has become standard by now to replace square functions in the theory of Banach space valued function spaces by such random sums (see e.g. [27]). Note however that Bernoulli sums and Gaussian sums for $x_1, \dots, x_n$ in a Banach space X are only equivalent if X has finite cotype (see [8, p. 218] for details).

Let τ be a subset of $B(X)$. We say that τ is R-bounded if there exists a $C < \infty$ such that

$$\mathbb{E} \Big\| \sum_{k=1}^n \epsilon_k T_k x_k \Big\| \leq C \mathbb{E} \Big\| \sum_{k=1}^n \epsilon_k x_k \Big\|$$

for any $n \in \mathbb{N}$, $T_1, \dots, T_n \in \tau$ and $x_1, \dots, x_n \in X$. The smallest admissible constant C is denoted by $R(\tau)$. We remark that one always has $R(\tau) \geq \sup_{T \in \tau} \|T\|$ and equality holds if X is a Hilbert space.

Recall that by definition, X has Pisier's property (α) if for any finite family $x_{k,l}$ in X, $(k, l) \in F$, where $F \subset \mathbb{Z} \times \mathbb{Z}$ is a finite array, we have a uniform equivalence

$$\mathbb{E}_\omega \mathbb{E}_{\omega'} \Big\| \sum_{(k,l) \in F} \epsilon_k(\omega) \epsilon_l(\omega') x_{k,l} \Big\|_X \cong \mathbb{E}_\omega \Big\| \sum_{(k,l) \in F} \epsilon_{k,l}(\omega) x_{k,l} \Big\|_X.$$

Note that property (α) is inherited by closed subspaces, and that an L^p space has property (α) provided $1 \leq p < \infty$ [27, Section 4].

3. Hörmander and Mihlin classes

Aside from the classical spaces in Notation 2.2 we introduce the following Mihlin class and Hörmander class.

Definition 3.1.

(1) Let $\alpha > 0$. We define the Mihlin class

$$\mathcal{M}^\alpha = \{f : \mathbb{R}_+ \to \mathbb{C} : \ f_e \in \mathcal{B}^\alpha\},$$

equipped with the norm $\|f\|_{\mathcal{M}^\alpha} = \|f_e\|_{\mathcal{B}^\alpha}$, where we write from now on

$$f_e : J \to \mathbb{C},\ z \mapsto f(e^z)$$

for a function $f : I \to \mathbb{C}$ such that $I \subset \mathbb{C}\backslash(-\infty, 0]$ and $J = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi,\ e^z \in I\}$. The space $\mathcal{M}^\alpha$ coincides with the space $\Lambda^\alpha_{\infty,1}(\mathbb{R}_+)$ in [5, p. 73].

(2) Let $\alpha > \frac{1}{2}$. We define

$$\mathcal{W}_2^\alpha = \{f : (0,\infty) \to \mathbb{C} :\ \|f\|_{\mathcal{W}_2^\alpha} = \|f_e\|_{W_2^\alpha} < \infty\}$$

and equip it with the norm $\|f\|_{\mathcal{W}_2^\alpha}$.

(3) Let $(\psi_n)_{n\in\mathbb{Z}}$ be an equidistant partition of unity and $\alpha > \frac{1}{2}$. We define the Hörmander class

$$\mathcal{H}_2^\alpha = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}_+) :\ \|f\|_{\mathcal{H}_2^\alpha} = \sup_{n\in\mathbb{Z}} \|\psi_n f_e\|_{W_2^\alpha} < \infty\}$$

and equip it with the norm $\|f\|_{\mathcal{H}_2^\alpha}$.

We have the following elementary properties of Mihlin and Hörmander spaces. Its proof may be found in [20, Propositions 4.8 and 4.9, Remark 4.16].

Lemma 3.2.

(1) The spaces $\mathcal{M}^\alpha, \mathcal{W}_2^\alpha$ and $\mathcal{H}_2^\alpha$ are Banach algebras.
(2) Different partitions of unity $(\psi_n)_n$ give the same space $\mathcal{H}_2^\alpha$ with equivalent norms.
(3) Let $\gamma > \alpha > \frac{1}{2}$, $\alpha > \beta + \frac{1}{2}$ and $\sigma \in (0, \pi)$. Then

$$H^\infty(\Sigma_\sigma) \hookrightarrow \mathcal{M}^\gamma \hookrightarrow \mathcal{H}_2^\alpha \hookrightarrow \mathcal{M}^\beta.$$

(4) For any $t > 0$, we have $\|f\|_{\mathcal{H}_2^\alpha} \cong \|f(t\cdot)\|_{\mathcal{H}_2^\alpha}$.

Remark 3.3. The names "Mihlin and Hörmander class" are justified by the following facts. The Mihlin condition for a β-times differentiable function $f : \mathbb{R}_+ \to \mathbb{C}$ is

$$\sup_{t>0, k=0,\dots,\beta} |t|^k |f^{(k)}(t)| < \infty \tag{3.1}$$

[10, (1)]. If f satisfies (3.1), then $f \in \mathcal{M}^\alpha$ for $\alpha < \beta$ [5, p. 73]. Conversely, if $f \in \mathcal{M}^\alpha$, then f satisfies (3.1) for $\alpha \geq \beta$. The proof of this can be found in [12, Theorem 3.1], where also the case $\beta \notin \mathbb{N}$ is considered.
The classical Hörmander condition with a parameter $\alpha_1 \in \mathbb{N}$ reads as follows [18, (7.9.8)]:

$$\sum_{k=0}^{\alpha_1} \sup_{R>0} \int_{R/2}^{2R} |R^k f^{(k)}(t)|^2 dt/R < \infty. \tag{3.2}$$

Furthermore, consider the following condition for some $\alpha > \frac{1}{2}$:

$$\sup_{t>0} \|\psi f(t\cdot)\|_{W_2^\alpha} < \infty, \tag{3.3}$$

where ψ is a fixed function in $C_c^\infty(\mathbb{R}_+)\backslash\{0\}$. This condition appears in several articles on Hörmander spectral multiplier theorems, we refer to [11] for an overview. One easily checks that (3.3) does not depend on the particular choice of ψ (see also [11, p. 445]).

By the following lemma which is proved in [20, Proposition 4.11], the norm $\|\cdot\|_{\mathcal{H}_2^\alpha}$ expresses condition (3.3) and generalizes the classical Hörmander condition (3.2).

Lemma 3.4. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Consider the conditions

(1) f satisfies (3.2),
(2) f satisfies (3.3),
(3) $\|f\|_{\mathcal{H}^\alpha_2} < \infty$.

Then (1) $\Rightarrow$ (2) if $\alpha_1 \geq \alpha$ and (2) $\Rightarrow$ (1) if $\alpha \geq \alpha_1$. Further, (2) $\Leftrightarrow$ (3).

3.1. **Functional calculus for 0-sectorial operators.** Let E be a Sobolev space or Besov space as in Notation 2.2. In this subsection we define an E functional calculus for a 0-sectorial operator A by tracing it back to the holomorphic functional calculus from Subsection 2.1. The following lemma which is proved in [20, Lemma 4.15] will be useful.

Lemma 3.5. Let $E \in \{\mathcal{M}^\alpha, \mathcal{W}^\beta_2\}$, where $\alpha > 0$ and $\beta > \frac{1}{2}$. Then $\bigcap_{0<\omega<\pi} H^\infty(\Sigma_\omega) \cap E$ is dense in E. More precisely, if $f \in E$, $\psi \in C^\infty_c$ such that $\psi(t) = 1$ for $|t| \leq 1$ and $\psi_n = \psi(2^{-n}(\cdot))$, then

$$(f_e * \check{\psi}_n) \circ \log \in \bigcap_{0<\omega<\pi} H^\infty(\Sigma_\omega) \cap E \text{ and } (f_e * \check{\psi}_n) \circ \log \to f \text{ in } E.$$

Thus if f happens to belong to several spaces E as above, then it can be simultaneously approximated by a holomorphic sequence in any of these spaces.

Lemma 3.5 enables to base the E calculus on the H^∞ calculus.

Definition 3.6. Let A be a 0-sectorial operator and $E \in \{\mathcal{M}^\alpha, \mathcal{W}^\beta_2\}$, where $\alpha > 0$ and $\beta > \frac{1}{2}$. We say that A has a (bounded) E calculus if there exists a constant $C > 0$ such that

$$\|f(A)\| \leq C\|f\|_E \quad (f \in \bigcap_{0<\omega<\pi} H^\infty(\Sigma_\omega) \cap E).$$

In this case, by the just proved density of $\bigcap_{0<\omega<\pi} H^\infty(\Sigma_\omega) \cap E$ in E, the algebra homomorphism $u : \bigcap_{0<\omega<\pi} H^\infty(\Sigma_\omega) \cap E \to B(X)$ given by $u(f) = f(A)$ can be continuously extended in a unique way to a bounded algebra homomorphism

$$u : E \to B(X),\ f \mapsto u(f).$$

We write again $f(A) = u(f)$ for any $f \in E$. Assume that E_1, $E_2 \in \{\mathcal{M}^\alpha, \mathcal{W}^\beta_2\}$ and that A has an E_1 calculus and an E_2 calculus. Then for $f \in E_1 \cap E_2$, $f(A)$ is defined twice by the above. However, the second part of Lemma 3.5 shows that these definitions coincide.

The following convergence property can be deduced from an extension of the well-known Convergence Lemma for the H^∞ calculus [5, Lemma 2.1]. For a proof we refer to [20, Corollary 4.20], see also [23].

Lemma 3.7. Let A be a 0-sectorial operator with bounded $\mathcal{M}^\alpha$ calculus for some $\alpha > 0$. If $(\varphi_n)_{n\in\mathbb{Z}}$ is a dyadic partition of unity, then for any $x \in X$,

$$x = \sum_{n\in\mathbb{Z}} \varphi_n(A)x \quad (\text{convergence in } X). \tag{3.4}$$

The following lemma gives a representation formula of the W_2^α calculus in terms of the C_0-group A^{it}. It can be proved with the Cauchy integral formula (2.2) in combination with the Fourier inversion formula [20, Proposition 4.22]. Here and below we use the short hand notation $\langle t\rangle = \sqrt{1+t^2}$.

Lemma 3.8. Let X be a Banach space with dual X'. Let $\alpha > \frac{1}{2}$, so that W_2^α is a Banach algebra. Let A be a 0-sectorial operator with imaginary powers $U(t) = A^{it}$.

(1) Assume that for some $C > 0$ and all $x \in X$, $x' \in X'$

$$\|\langle t\rangle^{-\alpha}\langle U(t)x, x'\rangle\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |\langle t\rangle^{-\alpha}\langle U(t)x, x'\rangle|^2 dt\right)^{1/2} \leq C\|x\|\,\|x'\|. \tag{3.5}$$

Then A has a bounded $\mathcal{W}_2^\alpha$ calculus. Moreover, for any $f \in \mathcal{W}_2^\alpha$, $f(A)$ is given by

$$\langle f(A)x, x'\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{(f_e)}(t)\langle U(t)x, x'\rangle dt \quad (x \in X,\ x' \in X'). \tag{3.6}$$

The above integral exists as a strong integral if moreover $\|\langle t\rangle^{-\alpha}U(t)x\|_{L^2(\mathbb{R})} < \infty$.

(2) Conversely, if A has a $\mathcal{W}_2^\alpha$ calculus, then (3.5) holds.

As for the H^∞ calculus, there is an extended $\mathcal{W}_2^\alpha$ calculus which is defined for $f_e \in W^\alpha_{2,\mathrm{loc}}$, as a counterpart of (2.3). Let $(\varphi_n)_n$ be a dyadic partition of unity and

$$D_A = \{x \in X :\ \exists\, N \in \mathbb{N} :\ \varphi_n(A)x = 0 \quad (|n| \geq N)\}. \tag{3.7}$$

Then D_A is a dense subset of X. Indeed, for any $x \in X$ let $x_N = \sum_{k=-N}^N \varphi_k(A)x$. Then for $|n| \geq N+1$, $\varphi_n(A)x_N = \sum_{k=-N}^N (\varphi_n\varphi_k)(A)x = 0$, so that x_N belongs to D_A. On the other hand, by (3.4), x_N converges to x for $N \to \infty$. Clearly, D_A is independent of the choice of $(\varphi_n)_n$. We call D_A the calculus core of A.

Definition 3.9. Assume that A has a $\mathcal{W}_2^\alpha$ calculus and a $\mathcal{M}^\beta$ calculus for some (possibly large) $\beta > 0$. Let $f_e \in W^\alpha_{2,\mathrm{loc}}$. We define the operator $f(A)$ to be the closure of

$$\begin{cases} D_A \subset X & \longrightarrow X \\ x & \longmapsto \sum_{n\in\mathbb{Z}} (f\varphi_n)(A)x \end{cases}.$$

Write $\widetilde{\varphi}_n = \varphi_{n-1} + \varphi_n + \varphi_{n+1}$. Since for $x \in D_A$ and large $|n|$, $(\varphi_n f)(A)x = (\widetilde{\varphi}_n\varphi_n f)(A)x = (\widetilde{\varphi}_n f)(A)(\varphi_n)(A)x = 0$, the above sum is finite.

Lemma 3.10. Let A and f be as above.

(a) The operator $f(A)$ is closed and densely defined with domain

$$D(f(A)) = \{x \in X :\ \sum_{k=-n}^{n} (f\cdot\varphi_k)(A)x \text{ converges in } X \text{ as } n \to \infty\},$$

and it is independent of the choice of the partition of unity $(\varphi_n)_n$. The sets D_A and $\{g(A)x :\ g \in C_c^\infty(\mathbb{R}_+),\ x \in X\}$ are both cores for $f(A)$.

(b) If furthermore $f \in \mathcal{W}_2^\alpha$ (resp. $f \in \mathcal{M}^\alpha$), then $f(A)$ coincides with the operator defined by the $\mathcal{W}_2^\alpha$ calculus (resp. $\mathcal{M}^\alpha$ calculus) of A. If $f \in \mathrm{Hol}(\Sigma_\omega)$ for some $\omega \in (0,\pi)$, then $f(A)$ coincides with the (unbounded) holomorphic calculus of A.

(c) Let g be a further function such that $g_e \in W^\alpha_{2,\mathrm{loc}}$. Then $f(A)g(A) \subset (fg)(A)$, where $f(A)g(A)$ is equipped with the natural domain $\{x \in D(g(A)) : g(A)x \in D(f(A))\}$. If $g(A)$ is a bounded operator, then even $f(A)g(A) = (fg)(A)$.

Lemma 3.10 essentially follows from the identity (3.4). The technical details can be found in [20, Proposition 4.25]. Note that the Hörmander class $\mathcal{H}^\alpha_2$ is contained in $\mathcal{W}^\alpha_{2,\mathrm{loc}}$. Thus the $\mathcal{W}^\alpha_{2,\mathrm{loc}}$ calculus in Lemma 3.10 enables us to define the $\mathcal{H}^\alpha_2$ calculus, whose boundedness is a main object of investigation in this article.

Definition 3.11. Let $\alpha > \frac{1}{2}$ and let A be a 0-sectorial operator. We say that A has a (bounded) $\mathcal{H}^\alpha_2$ calculus if there exists a constant $C > 0$ such that

$$\|f(A)\| \le C\|f\|_{\mathcal{H}^\alpha_2} \quad (f \in \bigcap_{\omega\in(0,\pi)} H^\infty(\Sigma_\omega) \cap \mathcal{H}^\alpha_2). \tag{3.8}$$

Let $\alpha > \frac{1}{2}$ and consider a 0-sectorial operator A having a $\mathcal{H}^\alpha_2$ calculus in the sense of Definition 3.11. Then A has a $\mathcal{W}^\alpha_2$ calculus and a $\mathcal{M}^\beta$ calculus for any $\beta > \alpha$. Thus we can apply Lemma 3.10 and consider the unbounded $\mathcal{W}^\alpha_{2,\mathrm{loc}}$ calculus of A, and in particular $f(A)$ is defined for $f \in \mathcal{H}^\alpha_2 \subset \mathcal{W}^\alpha_{2,\mathrm{loc}}$. Then condition (3.8) extends automatically to all $f \in \mathcal{H}^\alpha_2$.

4. Wave Operators and Bounded Imaginary Powers

In this section, we assume that A is a 0-sectorial operator. We relate wave operators with imaginary powers of A by means of the Mellin transform $M : L^2(\mathbb{R}_+, \frac{ds}{s}) \to L^2(\mathbb{R}, dt)$, $f \mapsto \int_0^\infty f(s)s^{it}\frac{ds}{s}$.

Proposition 4.1. Let $\alpha > \frac{1}{2}$ and $m \in \mathbb{N}$ such that $m > \alpha - \frac{1}{2}$. Assume that $A^{\frac{1}{2}-\alpha}(e^{\mp isA}-1)^m$ are well-defined closed operators for $s > 0$ with domain containing D_A from (3.7) (this is the case e.g. if A has a $\mathcal{M}^\gamma$ calculus for some (possibly large) $\gamma > 0$) and that

$$\|t \mapsto \langle t\rangle^{-\alpha}\langle A^{it}x, x'\rangle\|_{L^2(\mathbb{R},dt)} \le C\|x\|\,\|x'\|$$

or

$$\|s \mapsto \langle (sA)^{\frac{1}{2}-\alpha}(e^{\mp isA}-1)^m x, x'\rangle\|_{L^2(\mathbb{R}_+,\frac{ds}{s})} \le C\|x\|\,\|x'\|.$$

Then for any $x \in D_A$ from (3.7), and $x' \in X'$, we have the identity in $L^2(\mathbb{R}, dt)$:

$$M\left[\langle (sA)^{\frac{1}{2}-\alpha}(e^{\mp isA}-1)^m x, x'\rangle\right](t) = h_\mp(t)\langle A^{-it}x, x'\rangle,$$

where

$$h_\mp(t) = e^{\mp i\frac{\pi}{2}(\frac{1}{2}-\alpha)}e^{\pm\frac{\pi}{2}t}\Gamma(\frac{1}{2}-\alpha+it)f_m(\frac{1}{2}-\alpha+it) \tag{4.1}$$

with

$$f_m(z) = \sum_{k=1}^m \binom{m}{k}(-1)^{m-k}k^{-z} \tag{4.2}$$

and $h_\mp$ satisfies $|h_\mp(t)| \lesssim \langle t\rangle^{-\alpha}$.

The two following lemmas are devoted to the proof of Proposition 4.1.

Lemma 4.2. Let $m \in \mathbb{N}$ and $\operatorname{Re} z \in (-m, 0)$. Then

$$\int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s} = \Gamma(z) f_m(z),$$

with f_m given in (4.2). Note that $\Gamma(z) f_m(z)$ is a holomorphic function for $\operatorname{Re} z \in (-m, 0)$.

Proof. We proceed by induction over m. In the case $m = 1$, we obtain by integration by parts $\int_0^\infty s^z (e^{-s} - 1) \frac{ds}{s} = \left[\frac{1}{z} s^z (e^{-s} - 1)\right]_0^\infty + \int_0^\infty \frac{1}{z} s^z e^{-s} ds = 0 + \frac{1}{z}\Gamma(z+1) = \Gamma(z) = \Gamma(z) f_1(z)$. Next we claim that for $\operatorname{Re} z > -m$,

$$\int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} = \Gamma(z) \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} (k+1)^{-z}.$$

Note that the left hand side is well-defined and holomorphic for $\operatorname{Re} z > -m$ and the right hand side is meromorphic on $\mathbb{C}$. By the identity theorem for meromorphic functions, it suffices to show the claim for e.g. $\operatorname{Re} z > 0$. For these z in turn, we can develop

$$\int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \int_0^\infty s^z e^{-ks} e^{-s} \frac{ds}{s},$$

which gives the claim.

Assume now that the lemma holds for some m. Let first $\operatorname{Re} z \in (-m, 0)$. In the following calculation, we use both the claim and the induction hypothesis in the second equality, and the convention $\binom{m}{m+1} = 0$ in the third.

$$\begin{aligned}
\int_0^\infty s^z (e^{-s} - 1)^{m+1} \frac{ds}{s} &= \int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} - \int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s} \\
&= \Gamma(z) \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} (k+1)^{-z} - \Gamma(z) f_m(z) \\
&= \Gamma(z) \sum_{k=1}^{m+1} \binom{m}{k-1} (-1)^{m+1-k} k^{-z} + \Gamma(z) \sum_{k=1}^{m+1} \binom{m}{k} (-1)^{m+1-k} k^{-z} \\
&= \Gamma(z) \sum_{k=1}^{m+1} \left[\binom{m}{k-1} + \binom{m}{k}\right] (-1)^{m+1-k} k^{-z} \\
&= \Gamma(z) f_{m+1}(z).
\end{aligned}$$

Thus, the lemma holds for $m+1$ and $\operatorname{Re} z \in (-m, 0)$. For $\operatorname{Re} z \in (-(m+1), -m]$, we appeal again to the identity theorem. □

Lemma 4.3. Let $\operatorname{Re} z \in (-m, 0)$ and $\operatorname{Re} \lambda \geq 0$. Then

$$\int_0^\infty s^z (e^{-\lambda s} - 1)^m \frac{ds}{s} = \lambda^{-z} \int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s}.$$

Proof. This is an easy consequence of the Cauchy integral theorem. □

Proof of Proposition 4.1. Let $\mu > 0$ fixed. Combining Lemmas 4.2 and 4.3 with $\lambda = \pm i\mu$, we get $\int_0^\infty s^z (e^{\mp i\mu s} - 1)^m \frac{ds}{s} = (e^{\pm i\frac{\pi}{2}}\mu)^{-z}\Gamma(z) f_m(z)$. Put now $z = \frac{1}{2} - \alpha + it$ for $t \in \mathbb{R}$, so that $\operatorname{Re} z \in (-m, 0)$ by the assumptions of the proposition. Then $\int_0^\infty s^{it+\frac{1}{2}-\alpha}(e^{\mp i\mu s} - 1)^m \frac{ds}{s} = e^{\mp i\frac{\pi}{2}(\frac{1}{2}-\alpha+it)}\mu^{-it}\mu^{-(\frac{1}{2}-\alpha)}\Gamma(z) f_m(z)$, so that with $h_\mp(t)$ as in (4.1),

$$M\left[(s\mu)^{\frac{1}{2}-\alpha}(e^{\mp is\mu} - 1)^m\right](t) = h_\mp(t)\mu^{-it}. \tag{4.3}$$

The statement of the proposition was (4.3) with μ formally replaced by A (weak identity). It is easy to see that $\sup_{t\in\mathbb{R}} \left|f_m(\frac{1}{2} - \alpha + it)\right| < \infty$. Further, the Euler Gamma function has a development [28, p. 15], $\left|\Gamma(-\frac{1}{2} + \alpha + it)\right| \cong e^{-\frac{\pi}{2}|t|}|t|^{-\alpha}$ ($|t| \geq 1$), so that $|h_\mp(t)| \lesssim \langle t\rangle^{-\alpha}$. Thus by the assumption of the proposition, we have $t \mapsto h_\mp(t)\langle A^{-it}x, x'\rangle \in L^2(\mathbb{R}, dt)$, or $s \mapsto \langle (sA)^{\frac{1}{2}-\alpha}(e^{\mp isA} - 1)^m x, x'\rangle \in L^2(\mathbb{R}_+, \frac{ds}{s})$. Then the technicalities needed for the formal replacement are outlined in [20, Proposition 4.40]. □

A variant of the wave operator expression $(sA)^{\frac{1}{2}-\alpha}(e^{\mp isA} - 1)^m$ from Proposition 4.1 is given by the following proposition.

Proposition 4.4. Assume that $\alpha - \frac{1}{2} \notin \mathbb{N}_0$. Let $m \in \mathbb{N}_0$ such that $\alpha - \frac{1}{2} \in (m, m+1)$ and

$$w_\alpha(s) = |s|^{-\alpha}\left(e^{is} - \sum_{j=0}^{m-1} \frac{(is)^j}{j!}\right).$$

Then with M denoting again the Mellin transform, we have

$$M(\langle (sA)^{\frac{1}{2}} w_\alpha(sA)x, x'\rangle)(t) = i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha + \frac{1}{2} + it)\langle A^{-it}x, x'\rangle.$$

Proof. The proof is similar to that of Proposition 4.1. We determine the Mellin transform of $s^{\frac{1}{2}} w_\alpha(s)$: By a contour shift of the integral $s \rightsquigarrow is$,

$$\begin{aligned}\int_0^\infty s^{it} s^{\frac{1}{2}} w_\alpha(s)\frac{ds}{s} &= \int_0^\infty (is)^{it}(is)^{\frac{1}{2}} w_\alpha(is)\frac{ds}{s} \\ &= i^{-\alpha+\frac{1}{2}+it}\int_0^\infty s^{-\alpha+\frac{1}{2}+it}(e^{-s} - \sum_{j=0}^{m-1}\frac{(-s)^j}{j!})\frac{ds}{s}.\end{aligned}$$

Applying partial integration, one sees that this expression equals $i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha + \frac{1}{2} + it)$. Thus,

$$M(s^{\frac{1}{2}} w_\alpha(s))(t) = i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha + \frac{1}{2} + it),$$

and applying the functional calculus yields the proposition, see [20, Proposition 4.40] for details. □

5. Averaged R-boundedness

Let (Ω, μ) be a σ-finite measure space. Throughout the section, we consider spaces E which are subspaces of the space $\mathcal{L}$ of equivalence classes of measurable functions on (Ω, μ).

Here, equivalence classes refer to identity modulo μ-null sets. We require that the dual E' of E can be realized as the completion of

$$E'_0 = \{f \in \mathcal{L} : \exists C > 0 : |\langle f, g\rangle| = |\int_\Omega f(t)g(t)d\mu(t)| \leq C\|g\|_E\}$$

with respect to the norm $\|f\| = \sup_{\|g\|_E \leq 1} |\langle f, g\rangle|$. This is clearly the case in the following examples:

$$\begin{aligned} &E = L^p(\Omega, wd\mu) \text{ for } 1 \leq p \leq \infty \text{ and a weight } w, \\ &E = W_2^\alpha = W_2^\alpha(\mathbb{R}) \text{ for } \alpha > \frac{1}{2}, \\ &E = \mathcal{W}_2^\alpha. \end{aligned} \tag{5.1}$$

Definition 5.1. Let (Ω, μ) be a σ-finite measure space. Let E be a function space on (Ω, μ) as in (5.1). Let $(N(t) : t \in \Omega)$ be a family of closed operators on a Banach space X such that

(1) There exists a dense subspace $D_N \subset X$ which is contained in the domain of $N(t)$ for any $t \in \Omega$.
(2) For any $x \in D_N$, the mapping $\Omega \to X$, $t \mapsto N(t)x$ is measurable.
(3) For any $x \in D_N$, $x' \in X'$ and $f \in E$, $t \mapsto f(t)\langle N(t)x, x'\rangle$ belongs to $L^1(\Omega)$.

Then $(N(t) : t \in \Omega)$ is called R-bounded on the E-average or $R[E]$-bounded, if for any $f \in E$, there exists $N_f \in B(X)$ such that

$$\langle N_f x, x'\rangle = \int_\Omega f(t)\langle N(t)x, x'\rangle d\mu(t) \quad (x \in D_N,\, x' \in X') \tag{5.2}$$

and further

$$R[E](N(t) : t \in \Omega) := R(\{N_f : \|f\|_E \leq 1\}) < \infty.$$

A number of very useful criteria for R-bounded sets known in the literature can be restated in terms of $R[E]$-boundedness.

Example 5.2. Let (Ω, μ) be a σ-finite measure space and let $(N(t) : t \in \Omega)$ be a family of closed operators on X satisfying (1) and (2) of Definition 5.1.

a) $(E = L^1)$ If $\{N(t) : t \in \Omega\}$ is R-bounded in $B(X)$, then it is also $R[L^1(\Omega)]$-bounded, and

$$R[L^1(\Omega)](N(t) : t \in \Omega) \leq 2R(\{N(t) : t \in \Omega\}).$$

Conversely, assume in addition that Ω is a metric space, μ is a σ-finite strictly positive Borel measure and $t \mapsto N(t)$ is strongly continuous. If $(N(t) : t \in \Omega)$ is $R[L^1(\Omega)]$-bounded, then it is also R-bounded.

b) $(E = L^\infty)$ Assume that there exists $C > 0$ such that

$$\int_\Omega \|N(t)x\|d\mu(t) \leq C\|x\| \quad (x \in D_N).$$

Then $(N(t) : t \in \Omega)$ is $R[L^\infty(\Omega)]$-bounded with constant at most $2C$.

c) $(E = L^2)$ Assume that X is a reflexive $L^p(U)$ space. If

$$\left\| \left(\int_\Omega |(N(t)x)(\cdot)|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(U)} \leq C\|x\|_{L^p(U)}$$

for all $x \in D_N$, then $(N(t) : t \in \Omega)$ is $R[L^2(\Omega)]$-bounded and there exists a constant $C_0 = C_0(X)$ such that

$$R[L^2(\Omega)](N(t) : t \in \Omega) \leq C_0 C.$$

This can be generalized to spaces X with property (α) and the generalized square function spaces $l(\Omega, X)$ from [19].

d) $(E = L^{r'})$ Assume that X has type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Let $1 \leq r, r' < \infty$ with $\frac{1}{r} = 1 - \frac{1}{r'} > \frac{1}{p} - \frac{1}{q}$.

Assume that $N(t) \in B(X)$ for all $t \in \Omega$, that $t \mapsto N(t)$ is strongly measurable, and that

$$\|N(t)\|_{B(X)} \in L^r(\Omega).$$

Then $(N(t) : t \in \Omega)$ is $R[L^{r'}(\Omega)]$-bounded and there exists a constant $C_0 = C_0(r, p, q, X)$ such that

$$R[L^{r'}(\Omega)](N(t) : t \in \Omega) \leq C_0 C.$$

Proof. $(E = L^1)$ Assume that $(N(t) : t \in \Omega)$ is R-bounded. Then it follows from the Convex Hull Lemma [6, Lemma 3.2] that $R[L^1(\Omega)](N(t) : t \in \Omega) \leq 2R(\{N(t) : t \in \Omega\})$. Let us show the converse under the mentioned additional hypotheses. Suppose that $R(\{N(t) : t \in \Omega\}) = \infty$. We will deduce that also $R[L^1(\Omega)](N(t) : t \in \Omega) = \infty$. Choose for a given $N \in \mathbb{N}$ some $x_1, \ldots, x_n \in X \backslash \{0\}$ and $t_1, \ldots, t_n \in \Omega$ such that

$$\mathbb{E}\Big\| \sum_k \epsilon_k N(t_k) x_k \Big\|_X > N\mathbb{E}\Big\| \sum_k \epsilon_k x_k \Big\|_X.$$

It suffices to show that

$$\mathbb{E}\Big\| \sum_k \epsilon_k \int_\Omega f_k(t) N(t) x_k d\mu(t) \Big\|_X > N\mathbb{E}\Big\| \sum_k \epsilon_k x_k \Big\|_X \tag{5.3}$$

for appropriate $f_1, \ldots, f_n$. It is easy to see that by the strong continuity of N, (5.3) holds with $f_k = \frac{1}{\mu(B(t_k,\epsilon))} \chi_{B(t_k,\epsilon)}$ for ϵ small enough. Here the fact that μ is strictly positive and σ-finite guarantees that $\mu(B(t_k, \epsilon)) \in (0, \infty)$ for small ϵ.

$(E = L^\infty)$ By [27, Corollary 2.17],

$$\mathbb{E}\Big\| \sum_{k=1}^n \epsilon_k N_{f_k} x_k \Big\|_X \leq 2C\mathbb{E}\Big\| \sum_{k=1}^n \epsilon_k x_k \Big\|_X$$

for any finite family $N_{f_1}, \ldots, N_{f_n}$ from (5.2) such that $\|f_k\|_1 \leq 1$, and any finite family $x_1, \ldots, x_n \in D_N$. Since D_N is a dense subspace of X, we can deduce that $\{N_f : \|f\|_1 \leq 1\}$ is R-bounded.

($E = L^2$) For $x \in D_N$, set $\varphi(x) = N(\cdot)x \in L^p(U, L^2(\Omega))$. By assumption, φ extends to a bounded operator $L^p(U) \to L^p(U, L^2(\Omega))$. Then the assertion follows at once from [29, Proposition 3.3]. The case that X has property (α) follows from [13, Corollary 3.19].

($E = L^{r'}$) This is a result of Hytönen and Veraar, see [17, Proposition 4.1, Remark 4.2]. □

Proposition 5.3. If E is a space as in (5.1) and $R[E](N(t) : t \in \Omega) = C < \infty$, then

$$\|\langle N(\cdot)x, x'\rangle\|_{E'} \le C\|x\|\,\|x'\| \quad (x \in D_N,\, x' \in X'). \tag{5.4}$$

In particular, if $1 \le p, p' \le \infty$ are conjugated exponents and

$$R[L^{p'}(\Omega)](N(t) : t \in \Omega) = C < \infty,$$

then

$$\left(\int_\Omega |\langle N(t)x, x'\rangle|^p d\mu(t)\right)^{1/p} \le C\|x\|\,\|x'\| \quad (x \in D_N,\, x' \in X').$$

If X is a Hilbert space, then also the converse holds: Condition (5.4) implies that $(N(t) : t \in \Omega)$ is $R[E]$-bounded.

Proof. We have

$$\begin{aligned} &R[E](N(t) : t \in \Omega) \\ &\ge \sup\{\|N_f\|_{B(X)} : \|f\|_E \le 1\} \\ &= \sup\left\{\left|\int_\Omega f(t)\langle N(t)x, x'\rangle d\mu(t)\right| : \|f\|_E \le 1,\, x \in D_N, \|x\| \le 1,\, x' \in X', \|x'\| \le 1\right\} \\ &= \sup\left\{\|\langle N(\cdot)x, x'\rangle\|_{E'} : x \in D_N, \|x\| \le 1,\, x' \in X', \|x'\| \le 1\right\}. \end{aligned} \tag{5.5}$$

If X is a Hilbert space, then bounded subsets of $B(X)$ are R-bounded, and thus, "$\ge$" in (5.5) is in fact "=". □

An $R[E]$-bounded family yields a new averaged R-bounded family under a linear transformation in the function space variable.

Lemma 5.4. For $i = 1, 2$, let (Ω_i, μ_i) be a σ-finite measure space and E_i a function space on Ω_i as in (5.1), and $K \in B(E_1', E_2')$ such that its adjoint K' maps E_2 to E_1.

Let further $(N(t) : t \in \Omega_1)$ be an $R[E_1]$-bounded family of closed operators and D_N be a core for all $N(t)$. Assume that there exists a family $(M(t) : t \in \Omega_2)$ of closed operators with the same common core $D_M = D_N$ such that $t \mapsto M(t)x$ is measurable for all $x \in D_N$ and

$$\langle M(\cdot)x, x'\rangle = K(\langle N(\cdot)x, x'\rangle) \quad (x \in D_N, x' \in X').$$

Then $(M(t) : t \in \Omega_2)$ is $R[E_2]$-bounded and

$$R[E_2](M(t) : t \in \Omega_2) \le \|K\| R[E_1](N(t) : t \in \Omega_1).$$

Proof. Let $x \in D_N$ and $x' \in X'$. By (5.4) in Proposition 5.3, we have $\langle N(\cdot)x, x'\rangle \in E_1'$, and thus, $\langle M(\cdot)x, x'\rangle \in E_2'$. For any $f \in E_2$,

$$\int_{\Omega_2} \langle M(t)x, x'\rangle f(t) d\mu_2(t) = \int_{\Omega_1} \langle N(t)x, x'\rangle (K'f)(t) d\mu_1(t) = \langle N_{K'f}x, x'\rangle.$$

By assumption, the operator $N_{K'f}$ belongs to $B(X)$, and therefore also M_f belongs to $B(X)$. Furthermore,

$$\begin{aligned} R[E_2](M(t) : t \in \Omega_2) &= R(\{M_f : \|f\|_{E_2} \le 1\}) \\ &= R(\{N_{K'f} : \|f\|_{E_2} \le 1\}) \\ &\le \|K'\| R(\{N_{K'f} : \|K'f\|_{E_1} \le 1\}) \\ &\le \|K\| R(\{N_g : \|g\|_{E_1} \le 1\}) \\ &= \|K\| R[E_1](N(t) : t \in \Omega_1). \end{aligned}$$

□

In the following lemma, we collect some further simple manipulations of $R[E]$-boundedness. Its proof is immediate from Definition 5.1.

Lemma 5.5. Let (Ω, μ) be a σ-finite measure space, let E be as in (5.1) and let $(N(t) : t \in \Omega)$ satisfy (1) and (2) of Definition 5.1.

(1) Let $f \in L^\infty(\Omega)$ and $(N(t) : t \in \Omega)$ be $R[L^p(\Omega)]$-bounded for some $1 \le p \le \infty$. Then
$$R[L^p(\Omega)](f(t)N(t) : t \in \Omega) \le \|f\|_\infty R[L^p(\Omega)](N(t) : t \in \Omega).$$
In particular, $R[L^p(\Omega_1)](N(t) : t \in \Omega_1) \le R[L^p(\Omega)](N(t) : t \in \Omega)$ for any measurable subset $\Omega_1 \subset \Omega$.

(2) Let $w : \Omega \to (0, \infty)$ be measurable. Then for $1 \le p \le \infty$ and p' the conjugate exponent,
$$R[L^p(\Omega, w(t)d\mu(t))](N(t) : t \in \Omega) = R[L^p(\Omega, d\mu)](w(t)^{\frac{1}{p'}} N(t) : t \in \Omega).$$

(3) For $n \in \mathbb{N}$, let $\varphi_n : \Omega \to \mathbb{R}_+$ with $\sum_{n=1}^\infty \varphi_n(t) = 1$ for all $t \in \Omega$. Then
$$R[E](N(t) : t \in \Omega) \le \sum_{n=1}^\infty R[E](\varphi_n(t)N(t) : t \in \Omega).$$

We turn to applications to the functional calculus. That is, the R-bounded functional calculus yields $R[L^2]$-bounded sets by the following proposition. Here we may and do always choose the dense subset $D_N = D_A$, the calculus core from (3.7).

Definition 5.6. Let A be a 0-sectorial operator. Let $E \in \{\mathcal{H}_2^\alpha, \mathcal{M}^\alpha, \mathcal{W}_2^\alpha\}$. We say that A has an R-bounded E calculus if A has an E calculus, which is an R-bounded mapping in the sense of [25, Definition 2.7], i.e.
$$R(\{f(A) : \|f\|_E \le 1\}) < \infty.$$

In the next proposition we need the Mellin transform
$$M : L^2(\mathbb{R}_+, ds/s) \to L^2(\mathbb{R}, dt), f \mapsto (t \mapsto \int_0^\infty s^{it} f(s) ds/s).$$

Proposition 5.7. Let A be a 0-sectorial operator having an R-bounded $\mathcal{W}_2^\alpha$ calculus for some $\alpha > \frac{1}{2}$. Let $\phi \in W^\alpha_{2,\mathrm{loc}}(\mathbb{R}_+)$ such that $t \mapsto M\phi(t)\langle t\rangle^\alpha$ belongs to $L^\infty(\mathbb{R})$, where M denotes the Mellin transform. Then $(\phi(tA) : t > 0)$ is $R[L^2(\mathbb{R}_+, \frac{dt}{t})]$-bounded with bound $\le C\|M\phi(t)\langle t\rangle^\alpha\|_\infty$.

Proof. We have to show that $(\phi_e(t+\log(A)) : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded. For $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ with, say, compact support, we have

$$\int_{\mathbb{R}} h(-t)\phi_e(t+\log(A))x dt = (h * \phi_e) \circ \log(A)x \quad (x \in D_A). \tag{5.6}$$

Indeed, for fixed $x \in D_A$, there exists $\psi_0 \in C_c^\infty(\mathbb{R})$ such that $\psi_0 \circ \log(A)x = x$. Choose some $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(r) = 1$ for $r \in \operatorname{supp}\psi_0 - \operatorname{supp} h$, so that $\psi(t+\log(A))x = \psi(t+\log(A))\psi_0 \circ \log(A)x = \psi_0 \circ \log(A)x = x$ for any $-t \in \operatorname{supp} h$. Then for any $x' \in X'$,

$$\begin{aligned}
\int_{\mathbb{R}} h(-t)\langle \phi_e(t+\log(A))x, x'\rangle dt &= \int_{\mathbb{R}} h(-t)\langle (\phi_e\psi)(t+\log(A))x, x'\rangle dt \\
&= \int_{\mathbb{R}} h(-t)\frac{1}{2\pi}\int_{\mathbb{R}} (\phi_e\psi)\hat{}(s)e^{ist}\langle A^{is}x, x'\rangle ds dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(-t)e^{ist}(\phi_e\psi)\hat{}(s)dt\right)\langle A^{is}x, x'\rangle ds \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{h}(s)(\phi_e\psi)\hat{}(s)\langle A^{is}x, x'\rangle ds \\
&= \langle (h * (\phi_e\psi)) \circ \log(A)x, x'\rangle.
\end{aligned}$$

where we used $h \in L^1(\mathbb{R})$, $\phi_e\psi \in W_2^\alpha$ and $s \mapsto \langle s\rangle^{-\alpha}\langle A^{is}x, x'\rangle \in L^2(\mathbb{R})$, to apply Fubini in the third line. We also have $(h * (\phi_e\psi))\psi_0 = (h * \phi_e)\psi_0$ and (5.6) follows. Then the claim follows from $\|\phi_e * h\|_{W_2^\alpha} \le \|\hat{\phi}_e(t)\langle t\rangle^\alpha\|_{L^\infty(\mathbb{R})}\|h\|_{L^2(\mathbb{R})}$ and density of the above h in $L^2(\mathbb{R})$. □

Example 5.8. Consider $\phi(t) = t^\beta(e^{i\theta}-t)^{-1}$, where $\beta \in (0,1)$ and $|\theta| < \pi$ and A an operator as in the proposition above. Then $\phi(tA) = t^\beta A^\beta(e^{i\theta}-tA)^{-1} = t^{\beta-1}A^\beta(e^{i\theta}t^{-1}-A)^{-1}$ is an $R[L^2(\frac{dt}{t})]$-bounded family with bound $\lesssim \theta^{-\alpha}$. Indeed, $M\phi(t) = (-e^{i\theta})^{it+\beta-1}\frac{\pi}{\sin\pi(it+\beta-1)}$. As $|\sin\pi(it+\beta-1)| \cong \cosh(\pi t)$ for fixed β, we have $|M\phi(t)\langle t\rangle^\alpha| \cong e^{-(\theta-\pi)t}\langle t\rangle^\alpha\frac{1}{\cosh(\pi t)} \lesssim \theta^{-\alpha}$.

Theorem 6.1 will show that a converse to Proposition 5.7 holds, for many classical operator families including the above example, i.e. one can recover the R-bounded $\mathcal{W}_2^\alpha$ calculus from averaged R-boundedness conditions.

6. Main Results

We introduced the notion of $R[E]$-boundedness to give the following characterization of (R-bounded) $\mathcal{W}_2^\alpha$ calculus.

Theorem 6.1. Let A be a 0-sectorial operator on a Banach space X with a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \in (0,\pi)$. Let $\alpha > \frac{1}{2}$. Consider the following conditions.

Sobolev Calculus

(1) A has an R-bounded $\mathcal{W}_2^\alpha$ calculus.

Imaginary powers

(2) $(\langle t\rangle^{-\alpha}A^{it} : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded.

Resolvents

(3) For some/all $\beta \in (0,1)$ there exists $C > 0$ such that for all $\theta \in (-\pi,\pi)\backslash\{0\}$: $R[L^2(\mathbb{R}_+, dt/t)](t^\beta A^{1-\beta} R(e^{i\theta}t, A) : t > 0) \le C|\theta|^{-\alpha}$.
(4) For some/all $\beta \in (0,1)$ and $\theta_0 \in (0,\pi]$, $(|\theta|^{\alpha-\frac{1}{2}} t^\beta A^{1-\beta} R(e^{i\theta}t, A) : 0 < |\theta| \le \theta_0,\, t > 0)$ is $R[L^2((0,\infty)\times[-\theta_0,\theta_0]\backslash\{0\}, dt/td\theta)]$-bounded.

Analytic Semigroup $(T(z) = e^{-zA})$

(5) There exists $C > 0$ such that for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$: $R[L^2(\mathbb{R}_+)](A^{1/2}T(e^{i\theta}t) : t > 0) \le C(\frac{\pi}{2} - |\theta|)^{-\alpha}$.
(6) $(\langle \frac{x}{y}\rangle^\alpha |x|^{-\frac{1}{2}} A^{1/2} T(x+iy) : x > 0,\, y \in \mathbb{R})$ is $R[L^2(\mathbb{R}_+ \times \mathbb{R})]$-bounded.

Wave Operators

(7) The operators $A^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m$ are densely defined for some $m > \alpha - \frac{1}{2}$ and $(|s|^{-\alpha} A^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m : s \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded.
(8) The operators $A^{\frac{1}{2}-\alpha}\left(e^{isA} - \sum_{j=0}^{m-1} \frac{(isA)^j}{j!}\right)$ are densely defined and

$$\left(A^{\frac{1}{2}-\alpha}|s|^{-\alpha}\left(e^{isA} - \sum_{j=0}^{m-1}\frac{(isA)^j}{j!}\right) : s \in \mathbb{R}\right)$$

is $R[L^2(\mathbb{R})]$-bounded.

Then the following conditions are equivalent:

$$(1), (2), (4), (6), (7).$$

The condition (8) is also equivalent under the assumption that $\alpha - \frac{1}{2} \notin \mathbb{N}_0$ and $m \in \mathbb{N}_0$ such that $\alpha - \frac{1}{2} \in (m, m+1)$.

All these conditions imply the remaining ones (3) and (5). If X has property (α) then, conversely, these two conditions imply that A has an R-bounded $\mathcal{W}_2^{\alpha+\epsilon}$ calculus for any $\epsilon > 0$.

As a preparatory lemma for the proof of Theorem 6.1, we state

Lemma 6.2. Let $\beta \in \mathbb{R}$ and $f(t) = f_m(\beta + it)$ with f_m as in (4.2). Then there exist $C, \epsilon, \delta > 0$ such that for any interval $I \subset \mathbb{R}$ with $|I| \ge C$ there is a subinterval $J \subset I$ with $|J| \ge \delta$ so that $|f(t)| \ge \epsilon$ for $t \in J$. Consequently, for $N > C/\delta$,

$$\sum_{k=-N}^{N} |f(t + k\delta)| \gtrsim 1.$$

Proof. Suppose for a moment that

$$\exists\, C,\, \epsilon > 0\, \forall\, I \text{ interval with } |I| \ge C\, \exists\, t \in I : |f(t)| \ge \epsilon. \tag{6.1}$$

It is easy to see that $\sup_{t\in\mathbb{R}} |f'(t)| < \infty$, so that for such a t and $|s - t| \le \delta = \delta(\|f'\|_\infty, \epsilon)$, $|f(s)| \ge \epsilon/2$. Thus the lemma follows from (6.1) with $J = B(t, \delta/2)$.

It remains to show (6.1). Suppose that this is false. Then

$$\forall\, C,\, \epsilon > 0\, \exists\, I \text{ interval with } |I| \ge C : \forall\, t \in I : |f(t)| < \epsilon. \tag{6.2}$$

Since f'' is bounded and $\|f'\|_{L^\infty(I)} \leq \sqrt{8\|f\|_{L^\infty(I)}\|f''\|_{L^\infty(I)}}$, we deduce that (6.2) holds for f' in place of f, and successively also for $f^{(n)}$ for any n. But there is some $n \in \mathbb{N}$ such that $\inf_{t\in\mathbb{R}} |f^{(n)}(t)| > 0$. Indeed,

$$f^{(n)}(t) = \sum_{k=1}^{m} \alpha_k (-i \log k)^n e^{-it \log k},$$

with $\alpha_k = \binom{m}{k}(-1)^{m-k} k^{-\beta} \neq 0$, whence

$$|f^{(n)}(t)| \geq |\alpha_m| \, |\log m|^n - \sum_{k=1}^{m-1} |\alpha_k| \, |\log k|^n > 0$$

for n large enough. This contradicts (6.2), so that the lemma is proved. □

Proof of Theorem 6.1.

(1) ⇔ (2): By the $\mathcal{W}_2^\alpha$ representation formula (3.6), we have $R[L^2(\mathbb{R}, dt)](\langle t\rangle^{-\alpha} A^{it} : t \in \mathbb{R}) = R\left(\left\{\int_{\mathbb{R}} f(t)\langle t\rangle^{-\alpha} A^{it} dt : \|f\|_{L^2(\mathbb{R})} \leq 1\right\}\right) = R\left(\left\{2\pi f(A) : \|f\|_{\mathcal{W}_2^\alpha} \leq 1\right\}\right)$.

The strategy to show the stated remaining (almost) equivalences between (2) and (3) – (7) consists more or less in finding an integral transform K as in Lemma 5.4 mapping the imaginary powers A^{it} to resolvents, to the analytic semigroup and to the wave operators, and vice versa.

(2) ⇒ (7): By Example 5.2 (general E), we clearly have that $\|t \mapsto \langle t\rangle^{-\alpha}\langle A^{it}x, x'\rangle\|_{L^2(\mathbb{R},dt)} \leq C\|x\| \, \|x'\|$, provided (2) holds. Thus, by Proposition 4.1, and Lemmas 5.4 and 5.5 (1), (7) follows.

(7) ⇒ (2): Recall the function $h_\mp$ from Proposition 4.1. By the Euler Gamma function development [28, p. 15], we have the lower estimate

$$|h_\mp(t)| \gtrsim |f_m(\frac{1}{2} - \alpha + it)| e^{\frac{\pi}{2}(\pm t - |t|)} \langle t\rangle^{-\alpha}.$$

Thus, by Proposition 4.1, and Lemmas 5.4 and 5.5 (1),

$$\left(\langle t\rangle^{-\alpha} f_m(\frac{1}{2} - \alpha + it) A^{-it} : t \in \mathbb{R}\right) \text{ is } R[L^2(\mathbb{R})]\text{-bounded.} \tag{6.3}$$

To get rid of f_m in this expression, we apply Lemma 6.2. According to that lemma, we have $N \in \mathbb{N}$ and $\delta > 0$ such that $\sum_{k=-N}^{N} |f(t + k\delta)| \gtrsim 1$ for any $t \in \mathbb{R}$ and $f(t) = f_m(\frac{1}{2} - \alpha + it)$. Write

$$\sum_{k=-N}^{N} f(t + k\delta)\langle t\rangle^{-\alpha} A^{-it} = \sum_{k=-N}^{N} \left[\frac{\langle t + k\delta\rangle^\alpha}{\langle t\rangle^\alpha} A^{-ik\delta}\right] \left[f(t + k\delta)\langle t + k\delta\rangle^{-\alpha} A^{i(t+k\delta)}\right].$$

By (6.3), the term in the second brackets is $R[L^2(\mathbb{R})]$-bounded. The term in the first brackets is a bounded function times a bounded operator, due to the assumption that A has a bounded $H^\infty(\Sigma_\omega)$ calculus. Thus, the right hand side is $R[L^2(\mathbb{R})]$-bounded, and so the left hand side is. Now appeal once again to Lemma 5.5 (1) to deduce (2).

(2) $\Rightarrow$ (3): We fix $\theta \in (-\pi, \pi)$ and set

$$K_\theta : L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}, ds),\ f(s) \mapsto (\pi - |\theta|)^\alpha \frac{1}{\sin \pi(\beta + is)} e^{\theta s} \langle s \rangle^\alpha f(s).$$

We have

$$\sup_{|\theta|<\pi} \|K_\theta\| = \sup_{|\theta|<\pi,\, s\in\mathbb{R}} \langle s \rangle^\alpha (\pi - |\theta|)^\alpha \frac{e^{\theta s}}{|\sin \pi(\beta + is)|} \lesssim \sup_{\theta, s} \langle s(\pi - |\theta|) \rangle^\alpha e^{-|s|(\pi - |\theta|)} < \infty.$$

In [27, p. 228 and Theorem 15.18], the following formula is derived for $x \in A(D(A^2))$ and $|\theta| < \pi$:

$$\frac{\pi}{\sin \pi(\beta + is)} e^{\theta s} A^{is} x = \int_0^\infty t^{is} \left[t^\beta e^{i\theta\beta} A^{1-\beta} (e^{i\theta} t + A)^{-1} x \right] \frac{dt}{t} \tag{6.4}$$

for $|\theta| < \pi$ and $x \in A(D(A^2))$. Thus,

$$\begin{aligned} \sup_{0<|\theta|\le\pi} |\theta|^\alpha R[L^2(\mathbb{R}_+, dt/t)](t^\beta A^{1-\beta} R(te^{i\theta}, A)) &= \sup_{|\theta|<\pi} (\pi - |\theta|)^\alpha R[L^2(\mathbb{R}_+, dt/t)](t^\beta A^{1-\beta} (e^{i\theta} t + A)^{-1}) \\ &= \sup_{|\theta|<\pi} (\pi - |\theta|)^\alpha R[L^2(\mathbb{R}, ds)](\frac{\pi}{\sin \pi(\beta + is)} e^{\theta s} A^{is}) \\ &\lesssim R[L^2(\mathbb{R}, ds)](\langle s \rangle^{-\alpha} A^{is}). \end{aligned} \tag{6.5}$$

Next we claim that for any $\epsilon > 0$, (3) implies (2), where in (2), α is replaced by $\alpha + \epsilon$. First we consider $\langle s \rangle^{-(\alpha+\epsilon)} A^{is} x$ for $s \ge 1$. By Lemma 5.5 (3),

$$R[L^2([1, \infty), ds)](\langle s \rangle^{-(\alpha+\epsilon)} A^{is}) \le \sum_{n=0}^\infty 2^{-n\epsilon} R[L^2([2^n, 2^{n+1}])](\langle s \rangle^{-\alpha} A^{is}). \tag{6.6}$$

For $s \in [2^n, 2^{n+1}]$, we have

$$\langle s \rangle^{-\alpha} \lesssim 2^{-n\alpha} \lesssim 2^{-n\alpha} e^{-2^{-n} s} \lesssim (\pi - \theta_n)^\alpha \frac{e^{\theta_n s}}{\sin \pi(\beta + is)},$$

where $\theta_n = \pi - 2^{-n}$. Therefore

$$\begin{aligned} R[L^2([2^n, 2^{n+1}])](\langle s \rangle^{-\alpha} A^{is}) &\lesssim (\pi - \theta_n)^\alpha R[L^2(\mathbb{R}, ds)](\frac{\pi}{\sin \pi(\beta + is)} e^{\theta_n s} A^{is}) \\ &\overset{(6.5)}{\lesssim} \sup_{0<|\theta|\le\pi} |\theta|^\alpha R[L^2(\mathbb{R}_+, dt/t)](t^\beta A^{1-\beta} R(te^{i\theta}, A)) < \infty. \end{aligned}$$

Thus, the sum in (6.6) is finite.

The part $\langle s \rangle^{-(\alpha+\epsilon)} A^{is}$ for $s \le -1$ is treated similarly, whereas $R[L^2(-1, 1)](\langle s \rangle^{-\alpha} A^{is}) \cong R[L^2(-1, 1)](A^{is})$. It remains to show that the last expression is finite. We have assumed that X has property (α). Then the fact that A has an H^∞ calculus implies that $\{A^{is} : |s| < 1\}$ is R-bounded [27, Theorem 12.8]. For $f \in L^2(-1, 1)$, we have $\|f\|_1 \le C\|f\|_2$, and consequently,

$$\left\{ \int_{-1}^1 f(s) A^{is} ds :\ \|f\|_2 \le 1 \right\} \subset C \left\{ \int_{-1}^1 f(s) A^{is} ds :\ \|f\|_1 \le 1 \right\}.$$

In other words, $(A^{is} : |s| < 1)$ is $R[L^2]$-bounded.

(2) $\Longleftrightarrow$ (4):
Consider

$$K : L^2(\mathbb{R}, ds) \to L^2(\mathbb{R} \times (-\pi, \pi), ds d\theta),\ f(s) \mapsto (\pi - |\theta|)^{\alpha - \frac{1}{2}} \frac{1}{\sin \pi(\beta + is)} e^{\theta s} \langle s \rangle^{\alpha} f(s), \tag{6.7}$$

Note that $|\sin \pi(\beta + is)| \cong \cosh(\pi s)$ for $\beta \in (0,1)$ fixed. K is an isomorphic embedding. Indeed,

$$\|Kf\|_2^2 = \int_{\mathbb{R}} \int_{-\pi}^{\pi} \left((\pi - |\theta|)^{\alpha - \frac{1}{2}} e^{\theta s} \right)^2 d\theta \frac{1}{|\sin^2(\pi(\beta + is))|} \langle s \rangle^{2\alpha} |f(s)|^2 ds$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} (\pi - |\theta|)^{2\alpha - 1} e^{2\theta s} d\theta &\cong \int_0^{\pi} \theta^{2\alpha - 1} e^{2(\pi - \theta)|s|} d\theta \\ &\cong \cosh^2(\pi s) \int_0^{\pi} \theta^{2\alpha - 1} e^{-2\theta |s|} d\theta. \end{aligned}$$

For $|s| \geq 1$,

$$\int_0^{\pi} \theta^{2\alpha - 1} e^{-2\theta |s|} d\theta = (2|s|)^{-2\alpha} \int_0^{2|s|\pi} \theta^{2\alpha - 1} e^{-\theta} d\theta \cong |s|^{-2\alpha}.$$

This clearly implies that $\|Kf\|_2 \cong \|f\|_2$. Applying Lemma 5.4, we get

$$R[L^2(\mathbb{R}, ds)](\langle s \rangle^{-\alpha} A^{is}) \cong R[L^2(\mathbb{R} \times (-\pi, \pi), ds d\theta)]((\pi - |\theta|)^{\alpha - \frac{1}{2}} \frac{1}{\cosh(\pi s)} e^{\theta s} A^{is}).$$

Recall the formula (6.4), i.e.

$$\frac{\pi}{\sin \pi(\beta + is)} e^{\theta s} A^{is} x = \int_0^{\infty} t^{is} \left[t^{\beta} e^{i\theta\beta} A^{1-\beta} (e^{i\theta} t + A)^{-1} x \right] \frac{dt}{t}$$

for $|\theta| < \pi$ and $x \in A(D(A^2))$. Note that $A(D(A^2))$ is a dense subset of X. As the Mellin transform $f(s) \mapsto \int_0^{\infty} t^{is} f(s) \frac{ds}{s}$ is an isometry $L^2(\mathbb{R}_+, \frac{ds}{s}) \to L^2(\mathbb{R}, dt)$, we get by Lemma 5.4

$$\begin{aligned} R[L^2(\mathbb{R})](\langle s \rangle^{-\alpha} A^{is}) &\cong R[L^2(\mathbb{R}_+ \times (-\pi, \pi), \frac{dt}{t} d\theta)]((\pi - |\theta|)^{\alpha - \frac{1}{2}} t^{\beta} A^{1-\beta} (e^{i\theta} t + A)^{-1}) \\ &\cong R[L^2(\mathbb{R}_+ \times (0, 2\pi), dt/t d\theta)](|\theta|^{\alpha - \frac{1}{2}} t^{\beta} A^{1-\beta} R(e^{i\theta} t, A)). \end{aligned}$$

so that (2) $\Longleftrightarrow$ (4) for $\theta_0 = \pi$.

For a general $\theta_0 \in (0, \pi]$, consider K from (6.7) with restricted image, i.e.

$$K : L^2(\mathbb{R}, ds) \to L^2(\mathbb{R} \times (-\pi, -(\pi - \theta_0)] \cup [\pi - \theta_0, \pi), ds d\theta).$$

Then argue as in the case $\theta_0 = \pi$.

(4) $\Longleftrightarrow$ (6):
The proof of (2) $\Longleftrightarrow$ (4) above shows that condition (4) is independent of $\theta_0 \in (0, \pi]$ and $\beta \in (0,1)$. Put $\theta_0 = \pi$ and $\beta = \frac{1}{2}$. Apply Lemma 5.4 with

$$(e^{i\theta} \mu + it)^{-1} = K[\exp(-(\cdot) e^{i\theta} \mu) \chi_{(0,\infty)}(\cdot)](t),$$

where $K : L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}, dt)$ is the Fourier transform. This yields that (4) is equivalent to

$$R[L^2((0, \frac{\pi}{2}) \times \mathbb{R}_+, d\theta dt)](|\theta|^{\alpha-\frac{1}{2}} A^{\frac{1}{2}} T(\exp(\pm i(\frac{\pi}{2} - \theta))t)) < \infty.$$

Applying the change of variables $\theta \rightsquigarrow \frac{\pi}{2} \pm \theta$ and $dt \rightsquigarrow tdt$ shows that this is equivalent to

$$R[L^2((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}_+, d\theta dt)]((\frac{\pi}{2} - |\theta|)^{\alpha-\frac{1}{2}} t^{-\frac{1}{2}} A^{\frac{1}{2}} T(e^{i\theta}t)) < \infty.$$

Now the equivalence to (6) follows from the change of variables $a = t\cos\theta$, $b = t\sin\theta$, $t = |a + ib|$, $d\theta\, tdt = da\, db$.

(3) $\iff$ (5) for $\beta = \frac{1}{2}$: Use K and the first argument from the proof of (4) $\iff$ (6).

(2) $\iff$ (8): Recall the formula from Proposition 4.4, M denoting the Mellin transform,

$$M(\langle (sA)^{\frac{1}{2}} w_\alpha(sA)x, x'\rangle)(t) = i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha + \frac{1}{2} + it)\langle A^{-it}x, x'\rangle,$$

where $w_\alpha(s) = |s|^{-\alpha}\left(e^{is} - \sum_{j=0}^{m-1} \frac{(is)^j}{j!}\right)$. Then we have, since $\alpha - \frac{1}{2} \notin \mathbb{N}_0$,

$$|i^{-\alpha+\frac{1}{2}+it} \cdot \Gamma(-\alpha + \frac{1}{2} + it)| \cong e^{-\frac{\pi}{2}t} \cdot e^{-\frac{\pi}{2}|t|}\langle t\rangle^{-\alpha}$$

for $t \in \mathbb{R}$. Thus, with Lemmas 5.4 and 5.5 (2),

$$\begin{aligned} R[L^2(\mathbb{R}, dt)](\langle t\rangle^{-\alpha} A^{it}) < \infty &\iff R[L^2(\mathbb{R}_+, ds/s)]((sA)^{\frac{1}{2}} w_\alpha(\pm sA)) < \infty \\ &\iff R[L^2(\mathbb{R}, ds)](A^{\frac{1}{2}} w_\alpha(sA)) < \infty. \end{aligned}$$

□

Theorem 6.1 shows that averaged R-boundedness yields a good tool to describe $\mathcal{W}_2^\alpha$ functional calculus. However, many of the functions f that correspond to relevant spectral multipliers, as for example in (2) – (7) above, are not covered themselves by this calculus. To pass from the $\mathcal{W}_2^\alpha$ calculus to the $\mathcal{H}_2^\alpha$ calculus, which does cover all the spectral multipliers alluded to above, we shall use the spectral decomposition of Paley-Littlewood type in the following lemma, which is proved in [23].

Lemma 6.3. Let A be a 0-sectorial operator having a bounded $\mathcal{M}^\gamma$ calculus for some (possibly large) $\gamma > 0$. Let further $(\varphi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity. Then

$$\|x\| \cong \mathbb{E}\left\|\sum_{n\in\mathbb{Z}} \epsilon_n \varphi_n(A)x\right\|_X.$$

As a consequence we obtain

Theorem 6.4. Let A be a 0-sectorial operator on a Banach space X with property (α) having a bounded $H^\infty(\Sigma_\sigma)$ calculus for some $\sigma \in (0, \pi]$. Then the following are equivalent for $\alpha > \frac{1}{2}$.

(1) A has an R-bounded $\mathcal{W}_2^\alpha$ calculus.
(2) A has an R-bounded $\mathcal{H}_2^\alpha$ calculus.

Example 6.5. Consider the operator $A = -\Delta$ on $X = L^p(\mathbb{R}^d)$ for some $1 < p < \infty$ and $d \in \mathbb{N}$. Hörmander's classical result states that A has a bounded $\mathcal{H}^\alpha_2$ calculus for $\alpha > \frac{d}{2}$. In fact, a stronger result holds and A has an R-bounded $\mathcal{H}^\alpha_2$ calculus for the same range $\alpha > \frac{d}{2}$. This is proved in [21, Theorem 5.1], [22, Corollary 3.5].

Proof of Theorem 6.4. As $\mathcal{W}^\alpha_2 \subset \mathcal{H}^\alpha_2$, only the implication (1) $\Longrightarrow$ (2) has to be shown. Replacing A by a power if necessary, we may and do assume that $\sigma < \frac{\pi}{4}$. We first show that A has a bounded $\mathcal{M}^\gamma$ calculus for some $\gamma > 0$. For $\operatorname{Re} z > 0$, let $f_z(\lambda) = \exp(-z\lambda)$. We claim that for $\beta > \alpha$,

$$R(\{(\operatorname{Re} z/|z|)^\beta f_z(A) : \operatorname{Re} z > 0\}) < \infty. \tag{6.8}$$

Then by [24], (6.8) implies that A has a $\mathcal{M}^\gamma$ calculus for $\gamma > \beta + \frac{1}{2}$. Since X has property (α), the fact that A has a bounded H^∞ calculus extends by [27, Theorem 12.8] to

$$R(\{g(A) : \|g\|_{H^\infty(\Sigma_\theta)} \le 1\}) < \infty \tag{6.9}$$

for a $\theta < \frac{\pi}{4}$. By assumption, also

$$R(\{h(A) : \|h\|_{\mathcal{W}^\alpha_2} \le 1\}) < \infty. \tag{6.10}$$

By (6.9) and (6.10), it suffices to decompose $f_z = g_z + h_z$ such that $\|g_z\|_{H^\infty(\Sigma_\theta)} \lesssim (|z|/\operatorname{Re} z)^\beta$ and $\|h_z\|_{\mathcal{W}^\alpha_2} \lesssim (|z|/\operatorname{Re} z)^\beta$. By a simple scaling argument, we may assume that $|z| = 1$. We choose the decomposition

$$f_z(\lambda) = f_z(\lambda)e^{-\lambda} + f_z(\lambda)(1 - e^{-\lambda}).$$

Then $\|f_z(\lambda)e^{-\lambda}\|_{H^\infty(\Sigma_\theta)} = \|\exp(-(z+1)\lambda)\|_{H^\infty(\Sigma_\theta)} \lesssim 1$, since $\theta + |\arg(z+1)| \le \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. Further, it is a simple matter to check that $\|h_z\|_{\mathcal{W}^\alpha_2} \lesssim |\operatorname{Re} z|^{-\beta}$ for any $\beta > \alpha$. For example, if $\alpha = 1$, then

$$\begin{aligned}
\|h_z\|^2_{\mathcal{W}^1_2} &\cong \int_0^\infty |h_z(t)|^2 + |th_z'(t)|^2 \frac{dt}{t} \\
&\lesssim \int_0^\infty |e^{-zt}(1-e^{-t})|^2 + |t(-z)e^{-zt}(1-e^{-t})|^2 + |te^{-(z+1)t}|^2 \frac{dt}{t} \\
&\lesssim \int_0^1 e^{-2\operatorname{Re} zt}|1-e^{-t}|^2 + t^2|z|^2 e^{-2\operatorname{Re} zt}|1-e^{-t}|^2 + t^2 e^{-2(\operatorname{Re} z+1)t}\frac{dt}{t} \\
&\quad + \int_1^\infty e^{-2\operatorname{Re} zt}|1-e^{-t}|^2 + t^2|z|^2 e^{-2\operatorname{Re} zt}|1-e^{-t}|^2 + t^2 e^{-2(\operatorname{Re} z+1)t}\frac{dt}{t} \\
&\lesssim \int_0^1 t^2 \frac{dt}{t} + \int_1^\infty t^2 e^{-2\operatorname{Re} zt}\frac{dt}{t} \\
&\lesssim 1 + (\operatorname{Re} z)^{-2} \int_0^\infty t^2 e^{-t}\frac{dt}{t} \\
&\lesssim 1 + (\operatorname{Re} z)^{-2}.
\end{aligned}$$

Now we have established that A has a $\mathcal{M}^\gamma$ calculus and we can thus apply Lemma 6.3. Let now $f_1, \dots, f_K \in \mathcal{H}^\alpha_2$ of norm less than 1, $x_1, \dots, x_K \in X$, and $(\epsilon_k)_k$ and $(\epsilon'_n)_{n\in\mathbb{Z}}$ two independent Rademacher sequences. Note in the following calculation that $\|\varphi_n f_k\|_{\mathcal{W}^\alpha_2} \le$

$\|f_k\|_{\mathcal{H}^\alpha_2} \leq 1$. We have by property (α) and assumption (1) of the theorem, writing $\widetilde{\varphi}_n = \varphi_{n-1} + \varphi_n + \varphi_{n+1}$,

$$\begin{aligned}
\mathbb{E}\|\sum_{k=1}^K \epsilon_k f_k(A)x_k\| &\cong \mathbb{E}\mathbb{E}'\|\sum_{k=1}^K\sum_{n\in\mathbb{Z}} \epsilon_k\epsilon'_n\varphi_n(A)f_k(A)x_k\| \\
&= \mathbb{E}\mathbb{E}'\|\sum_{k=1}^K\sum_{n\in\mathbb{Z}} \epsilon_k\epsilon'_n\widetilde{\varphi}_n(A)\varphi_n(A)f_k(A)x_k\| \\
&\cong \mathbb{E}\|\sum_{k=1}^K\sum_{n\in\mathbb{Z}} \epsilon_{nk}(\varphi_n f_k)(A)\widetilde{\varphi}_n(A)x_k\| \\
&\leq R(\{(\varphi_n f_k)(A) :\ n\in\mathbb{Z}, k=1,\dots,K\})\mathbb{E}\|\sum_{k=1}^K\sum_{n\in\mathbb{Z}} \epsilon_{nk}\widetilde{\varphi}_n(A)x_k\| \\
&\lesssim \mathbb{E}\mathbb{E}'\|\sum_{k=1}^K\sum_{n\in\mathbb{Z}} \epsilon_k\epsilon'_n\widetilde{\varphi}_n(A)x_k\| \\
&\cong \mathbb{E}\|\sum_{k=1}^K \epsilon_k x_k\|.
\end{aligned}$$

This shows condition (2) of the theorem. □

7. Bisectorial operators and operators of strip type

7.1. **Bisectorial operators.** In this short subsection we indicate how to extend our results to bisectorial operators. An operator A with dense domain on a Banach space X is called bisectorial of angle $\omega \in [0, \frac{\pi}{2})$ if it is closed, its spectrum is contained in the closure of $S_\omega = \{z\in\mathbb{C} :\ |\arg(\pm z)| < \omega\}$, and one has the resolvent estimate

$$\|(I+\lambda A)^{-1}\|_{B(X)} \leq C_{\omega'},\ \forall\,\lambda\notin S_{\omega'},\ \omega' > \omega.$$

If X is reflexive, then for such an operator we have again a decomposition $X = N(A)\oplus\overline{R(A)}$, so that we may assume that A is injective. The $H^\infty(S_\omega)$ calculus is defined as in (2.2), but now we integrate over the boundary of the double sector S_ω. If A has a bounded $H^\infty(S_\omega)$ calculus, or more generally, if we have $\|Ax\| \cong \|(-A^2)^{\frac{1}{2}}x\|$ for $x\in D(A) = D((-A^2)^{\frac{1}{2}})$ (see e.g. [9]), then the spectral projections P_1, P_2 with respect to $\Sigma_1 = S_\omega\cap\mathbb{C}_+$, $\Sigma_2 = S_\omega\cap\mathbb{C}_-$ give a decomposition $X = X_1\oplus X_2$ of X into invariant subspaces for resolvents of A such that the part A_1 of A to X_1 and $-A_2$ of $-A$ to X_2 are sectorial operators with $\sigma(A_i)\subset\Sigma_i$. For $f\in H^\infty_0(S_\omega)$ we have

$$f(A)x = f|_{\Sigma_1}(A_1)P_1x + f|_{\Sigma_2}(A_2)P_2x. \tag{7.1}$$

We define the Hörmander class $\mathcal{H}^\alpha_2(\mathbb{R})$ on $\mathbb{R}$ by $f\in\mathcal{H}^\alpha_2(\mathbb{R})$ if $f\chi_{\mathbb{R}_+}\in\mathcal{H}^\alpha_2$ and $f(-\cdot)\chi_{\mathbb{R}_+}\in\mathcal{H}^\alpha_2$. Let A be a 0-bisectorial operator, i.e. A is ω-bisectorial for all $\omega>0$. Then A has an (R-bounded) $\mathcal{H}^\alpha_2(\mathbb{R})$ calculus if the set $\{f(A) :\ f\in\bigcap_{0<\omega<\pi}H^\infty(S_\omega)\cap\mathcal{H}^\alpha_2(\mathbb{R}),\ \|f\|_{\mathcal{H}^\alpha_2(\mathbb{R})}\leq 1\}$

is (R-)bounded. Clearly, A has an (R-bounded) $\mathcal{H}^\alpha_2(\mathbb{R})$ calculus if and only if A_1 and $-A_2$ have an (R-bounded) $\mathcal{H}^\alpha_2$ calculus and in this case (7.1) holds again.

Let $f_t(\lambda) = \begin{cases} \lambda^{it} : \operatorname{Re}\lambda > 0 \\ (-\lambda)^{it} : \operatorname{Re}\lambda < 0 \end{cases}$. Then $f_t \in H^\infty(S_\omega)$ for any $\omega \in (0, \frac{\pi}{2})$. Clearly, one has $f_t(A) = A_1^{it} \oplus (-A_2)^{it}$ on $X = X_1 \oplus X_2$. It is easy to show that $(\langle t\rangle^{-\alpha} f_t(A) : t \in \mathbb{R})$ is $R[L^2(\mathbb{R}, dt)]$-bounded if and only if $(\langle t\rangle^{-\alpha} A_1^{it} : t \in \mathbb{R})$ and $(\langle t\rangle^{-\alpha}(-A_2)^{it} : t \in \mathbb{R})$ are both $R[L^2(\mathbb{R}, dt)]$-bounded. Similarly, we have that

$$R[L^2(\mathbb{R}_+, dt/t)](t^\beta |A|^{1-\beta}(e^{i\theta}t - A)^{-1}) \lesssim (\min(|\theta|, \pi - |\theta|))^{-\alpha}$$

for $0 < |\theta| < \pi$ if and only if both of the following conditions hold:

$$R[L^2(\mathbb{R}_+, dt/t)](t^\beta A_1^{1-\beta}(e^{i\theta}t - A_1)^{-1}) \lesssim |\theta|^{-\alpha} \text{ and } R[L^2(\mathbb{R}_+, dt/t)](t^\beta (-A_2)^{1-\beta}(e^{i\theta}t + A_2)^{-1} \lesssim |\theta|^{-\alpha}$$

for $0 < |\theta| \le \frac{\pi}{2}$. Finally, we have that $|s|^{-\alpha}|A|^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m$ is $R[L^2(\mathbb{R})]$-bounded if and only if both $|s|^{-\alpha} A_1^{-\alpha+\frac{1}{2}}(e^{isA_1} - 1)^m$ and $|s|^{-\alpha}(-A_2)^{-\alpha+\frac{1}{2}}(e^{isA_2} - 1)^m)$ are $R[L^2(\mathbb{R})]$-bounded.

Note that if A has a bounded $\mathcal{M}^\gamma(\mathbb{R})$ calculus for some $\gamma > 0$, meaning that both A_1 and $-A_2$ have a $\mathcal{M}^\gamma$ calculus, then essentially by the same proof as for Lemma 6.3, we have a spectral decomposition

$$\|x\| \cong \mathbb{E}\left\|\sum_{n\in\mathbb{Z}} \epsilon_n \varphi_n(A_1)x\right\| + \mathbb{E}\left\|\sum_{n\in\mathbb{Z}} \epsilon_n \varphi_n(-A_2)x\right\|.$$

Then using the projections P_1 and P_2, it is clear how our main Theorems 6.1 and 6.4 extend to bisectorial operators.

7.2. **Strip-type operators.** For $\omega > 0$ we let $\mathrm{Str}_\omega = \{z \in \mathbb{C} : |\operatorname{Im} z| < \omega\}$ the horizontal strip of height 2ω. We further define $H^\infty(\mathrm{Str}_\omega)$ to be the space of bounded holomorphic functions on Str_ω, which is a Banach algebra equipped with the norm $\|f\|_{\infty,\omega} = \sup_{\lambda\in\mathrm{Str}_\omega} |f(\lambda)|$. A densely defined operator B is called ω-strip-type operator if $\sigma(B) \subset \overline{\mathrm{Str}_\omega}$ and for all $\theta > \omega$ there is a $C_\theta > 0$ such that $\|\lambda(\lambda - B)^{-1}\| \le C_\theta$ for all $\lambda \in \overline{\mathrm{Str}_\theta}^c$. Similarly to the sectorial case, one defines $f(B)$ for $f \in H^\infty(\mathrm{Str}_\theta)$ satisfying a decay for $|\operatorname{Re}\lambda| \to \infty$ by a Cauchy integral formula, and says that B has a bounded $H^\infty(\mathrm{Str}_\theta)$ calculus provided that $\|f(B)\| \le C\|f\|_{\infty,\theta}$, in which case $f \mapsto f(B)$ extends to a bounded homomorphism $H^\infty(\mathrm{Str}_\theta) \to B(X)$. We refer to [5] and [14, Chapter 4] for details. We call B 0-strip-type if B is ω-strip-type for all $\omega > 0$.

There is an analogous statement to Lemma 2.1 which holds for a 0-strip-type operator B and Str_ω in place of A and Σ_ω, and $\mathrm{Hol}(\mathrm{Str}_\omega) = \{f : \mathrm{Str}_\omega \to \mathbb{C} : \exists n \in \mathbb{N} : (\rho \circ \exp)^n f \in H^\infty(\mathrm{Str}_\omega)\}$, where $\rho(\lambda) = \lambda(1 + \lambda)^{-2}$.

In fact, 0-strip-type operators and 0-sectorial operators with bounded $H^\infty(\mathrm{Str}_\omega)$ and bounded $H^\infty(\Sigma_\omega)$ calculus are in one-one correspondence by the following lemma. For a proof we refer to [14, Proposition 5.3.3., Theorem 4.3.1 and Theorem 4.2.4, Lemma 3.5.1].

Lemma 7.1. Let B be a 0-strip-type operator and assume that there exists a 0-sectorial operator A such that $B = \log(A)$. This is the case if B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus

for some $\omega < \pi$. Then for any $f \in \bigcup_{0<\omega<\pi} \mathrm{Hol}(\mathrm{Str}_\omega)$ one has

$$f(B) = (f \circ \log)(A).$$

Note that the logarithm belongs to $\mathrm{Hol}(\Sigma_\omega)$ for any $\omega \in (0, \pi)$. Conversely, if A is a 0-sectorial operator that has a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \in (0, \pi)$, then $B = \log(A)$ is a 0-strip-type operator.

Let B be a 0-strip-type operator and $E = W_2^\alpha$ for some $\alpha > \frac{1}{2}$, or $E = \mathcal{B}^\alpha$ for some $\alpha > 0$. We say that B has a (bounded) E calculus if there exists a constant $C > 0$ such that

$$\|f(B)\| \leq C\|f\|_E \quad (f \in \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap E).$$

In this case, by density of $\bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap E$ in E, the definition of $f(B)$ can be continuously extended to $f \in E$.

Assume that B has an E calculus and a bounded $\mathcal{B}^\beta$ calculus for some $\beta > 0$. Let $f \in E_{\mathrm{loc}}$. We define the operator $f(B)$ to be the closure of

$$\begin{cases} D_B \subset X & \longrightarrow X \\ x & \longmapsto \sum_{n\in\mathbb{Z}} (\psi_n f)(B)x, \end{cases}$$

where $D_B = \{x \in X : \exists N \in \mathbb{N} : \ \psi_n(B)x = 0 \quad (|n| \geq N)\}$ and $(\psi_n)_{n\in\mathbb{Z}}$ is an equidistant partition of unity.

Then there holds an analogous version of Lemma 3.10, a proof of which can be found in [20, Proposition 4.25]. Let $\widetilde{\mathcal{H}}_2^\alpha = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}) : \ \|f\|_{\widetilde{\mathcal{H}}_2^\alpha} = \sup_{n\in\mathbb{Z}} \|\psi_n f\|_{W_2^\alpha} < \infty\}$. Note that $\widetilde{\mathcal{H}}_2^\alpha$ is contained in $W^\alpha_{2,\mathrm{loc}}$. Thus the $W^\alpha_{2,\mathrm{loc}}$ calculus for B enables us to define the $\widetilde{\mathcal{H}}_2^\alpha$ calculus: Let $\alpha > \frac{1}{2}$ and B be a 0-strip-type operator. We say that B has an (R-bounded) $\widetilde{\mathcal{H}}_2^\alpha$ calculus if there exists a constant $C > 0$ such that

$$\left\{ f(B) : \ f \in \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap \widetilde{\mathcal{H}}_2^\alpha, \ \|f\|_{\widetilde{\mathcal{H}}_2^\alpha} \leq 1 \right\} \text{ is } (R\text{-})\text{bounded.}$$

The strip-type version of the main Theorems 6.1 and 6.4 reads as follows.

Theorem 7.2. Let B be 0-strip-type operator with H^∞ calculus on some Banach space with property (α). Denote $U(t)$ the C_0-group generated by iB and $R(\lambda, B)$ the resolvents of B. For $\alpha > \frac{1}{2}$, consider the condition

$(C_2)_\alpha$ $\qquad$ B has an R-bounded $\widetilde{\mathcal{H}}_2^\alpha$ calculus.

Furthermore, we consider the conditions

- $(\mathrm{a})_\alpha$ The family $(\langle t\rangle^{-\alpha} U(t) : \ t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded.
- $(\mathrm{b})_\alpha$ The family $(R(t + ic, B) : \ t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded for any $c \neq 0$ and its bound grows at most like $|c|^{-\alpha}$ for $c \to 0$.

Then for all $\epsilon > 0$,

$$(C_2)_\alpha \iff (\mathrm{a})_\alpha \implies (\mathrm{b})_\alpha \implies (C_2)_{\alpha+\epsilon}$$

Proof. Consider the 0-sectorial operator $A = e^B$. Then $(C_2)_\alpha \iff (\mathrm{a})_\alpha$ follows from Theorems 6.1 and 6.4.
$(\mathrm{a})_\alpha \implies (\mathrm{b})_\alpha$: Let $R_c = |c|^\alpha R[L^2](R(t+ic,B) : t \in \mathbb{R})$. We have to show $\sup_{c\neq 0} R_c < \infty$. Applying Lemma 5.4 with K the Fourier transform and its inverse, we get

$$R_c = \begin{cases} R[L^2](c^\alpha e^{ct} U(t) : t < 0), & c > 0, \\ R[L^2](|c|^\alpha e^{ct} U(t) : t > 0), & c < 0. \end{cases}$$

For $t < 0$, $\sup_{c>0} c^\alpha e^{ct} = \sup_{c>0} \langle t\rangle^{-\alpha}\langle t\rangle^\alpha c^\alpha e^{-|ct|} \lesssim \langle t\rangle^{-\alpha}$. Thus, $\sup_{c>0} R[L^2](c^\alpha e^{ct}U(t) : t < 0) \lesssim R[L^2](\langle t\rangle^{-\alpha} U(t) : t < 0) < \infty$. The part $c < 0$ is estimated similarly.
$(\mathrm{b})_\alpha \implies (\mathrm{a})_{\alpha+\epsilon}$: Let R_c be as before. Split $\langle t\rangle^{-(\alpha+\epsilon)} U(t)$ into the parts $t \geq 1, t \leq -1, |t| < 1$, and further $t \geq 1$ into $t \in [2^n, 2^{n+1}], n \in \mathbb{N}_0$. Then $\langle t\rangle^{-\alpha} \lesssim 2^{-n\alpha} \lesssim 2^{-n\alpha} e^{-2^{-n}t}$, and by Lemma 5.5 (2),

$$\begin{aligned} R[L^2](\langle t\rangle^{-(\alpha+\epsilon)} U(t) : t \geq 1) &\leq \sum_{n=0}^\infty 2^{-n\epsilon} R[L^2](2^{-n\alpha} e^{-2^{-n}t} U(t) : t \in [2^n, 2^{n+1}]) \\ &\leq \sum_{n=0}^\infty 2^{-n\epsilon} \sup_{c<0} R_c < \infty. \end{aligned}$$

The estimate for $t \leq -1$ can be handled similarly. It remains to estimate $R[L^2](\langle s\rangle^{-(\alpha+\epsilon)} U(s) : |s| < 1) \cong R[L^2](U(s) : |s| < 1)$. We have assumed that X has property (α). Then the fact that B has an H^∞ calculus implies that $\{U(s) : |s| < 1\}$ is R-bounded [19, Corollary 6.6]. For $f \in L^2([-1,1])$, we have $\|f\|_1 \leq C\|f\|_2$, and consequently,

$$\left\{ \int_{-1}^1 f(s)U(s)ds : \|f\|_2 \leq 1 \right\} \subset C \left\{ \int_{-1}^1 f(s)U(s)ds : \|f\|_1 \leq 1 \right\}.$$

In other words, $(U(s) : |s| < 1)$ is $R[L^2]$-bounded. □

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