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SPECTRAL MULTIPLIER THEOREMS AND AVERAGED R-BOUNDEDNESS

CHRISTOPH KRIEGLER AND LUTZ WEIS

ABSTRACT. Let A be a 0-sectorial operator with a bounded $H^\infty(\Sigma_\sigma)$ -calculus for some $\sigma \in (0, \pi)$, e.g. a Laplace type operator on $L^p(\Omega)$, $1 < p < \infty$, where Ω is a manifold or a graph. We show that A has a $\mathcal{H}_2^\alpha(\mathbb{R}_+)$ Hörmander functional calculus if and only if certain operator families derived from the resolvent $(\lambda - A)^{-1}$, the semigroup e^{-zA} , the wave operators e^{itA} or the imaginary powers A^{it} of A are R -bounded in an L^2 -averaged sense. If X is an $L^p(\Omega)$ space with $1 \leq p < \infty$, R -boundedness reduces to well-known estimates of square sums.

1. INTRODUCTION

Hörmander's Fourier multiplier theorem states that for a function $f \in \mathcal{H}_2^\alpha(\mathbb{R}_+)$ the operator $f(-\Delta)$, defined in terms of the functional calculus on $L^2(\mathbb{R}^d)$ can be extended to $L^p(\mathbb{R}^d)$ if $1 < p < \infty$ and $\alpha > \frac{d}{2}$. Here

$$\mathcal{H}_2^\alpha(\mathbb{R}_+) = \{f \in C(\mathbb{R}_+, \mathbb{C}) : \sup_{t>0} \|\phi f(t \cdot)\|_{W_2^\alpha(\mathbb{R}_+)} < \infty\}$$

where $\phi \in C^\infty(\mathbb{R})$ with compact support $\text{supp } \phi \subset (0, \infty)$ is a cut-off function and $W_2^\alpha(\mathbb{R}_+)$ is the usual Riesz-potential Sobolev space. For $\alpha \in \mathbb{N}$, an equivalent norm on \mathcal{H}_2^α is given by the "classical" Hörmander condition

$$\sup_{R>0, \beta=0, \dots, \alpha} \frac{1}{R} \int_R^{2R} |t^\beta D^\beta f(t)|^2 dt < \infty.$$

There is a large literature extending such a spectral multiplier result to more general self-adjoint operators on $L^p(\Omega)$, e.g. for Laplace type operators on manifolds, infinite graphs and fractals (see e.g. [1, 7, 10, 11, 27, 31] and the references therein). There are various approaches to the \mathcal{H}_2^α calculus using kernel estimates, maximal estimates or square function estimates for the resolvent $(\lambda - A)^{-1}$, the analytic semigroup e^{-zA} generated by $-A$ and their "boundary", the wave operators e^{itA} , or the imaginary powers A^{it} of A . Relevant are e.g. estimates on operator functions such as ($\alpha > \frac{1}{2}$, $m > \alpha - \frac{1}{2}$ are fixed)

- $T_\theta(t) = A^{\frac{1}{2}} e^{-e^{i\theta} t A}$, $t \in \mathbb{R}_+$,
- $R_\theta(t) = A^{\frac{1}{2}} R(e^{i\theta} t, A)$, $t \in \mathbb{R}_+$,

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- $W(s) = |s|^{-\alpha} A^{-\alpha + \frac{1}{2}} (e^{isA} - 1)^m, \quad s \in \mathbb{R},$
- $I(t) = (1 + |t|)^{-\alpha} A^{it}, \quad t \in \mathbb{R}.$

Many of these estimates imply or are closely related to square sum estimates of the following form

$$(1.1) \quad \left\| \left(\sum_i |S_i x_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \leq C_1 \left\| \left(\sum_i |x_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

where $x_i \in L^p(\Omega)$ and the S_i are members of one of the families listed above (see e.g. [3, 34] for an early appearance of this square sum estimate in the context of spectral multiplier theorems). If (r_n) is a sequence of Rademacher functions on $[0, 1]$ one can reformulate (1.1) equivalently as

$$(1.2) \quad \int_0^1 \left\| \sum_i r_i(\omega) S_i x_i \right\| d\omega \leq C_2 \int_0^1 \left\| \sum_i r_i(\omega) x_i \right\| d\omega.$$

This statement makes sense in an arbitrary Banach space X and a set $\tau \subset B(X)$ is called R -bounded if (1.2) holds for all $S_i \in \tau$ and $x_i \in X$. Using R -boundedness in place of kernel estimates and the holomorphic $H^\infty(\Sigma_\sigma)$ calculus instead of the spectral theorem for selfadjoint operators, one can develop a theory of spectral multiplier theorems for 0-sectorial operators on Banach spaces (see [21, 22, 23, 24, 25]). Again, R -bounds for one of the operator families listed above are sufficient to secure $\mathcal{H}_2^\alpha(\mathbb{R}_+)$ spectral theorems for such operators A . However, neither in this general framework nor in the case of Laplace type operators on an $L^p(\Omega)$ space (see above), one obtains necessary and sufficient conditions in terms of R -bounds or kernel estimates. This is related to the (usually) difficult task of determining the optimal α for the $\mathcal{H}_2^\alpha(\mathbb{R}_+)$ spectral calculus of a given operator A . Thus the purpose of this paper is to give a characterization of the $\mathcal{H}_2^\alpha(\mathbb{R}_+)$ spectral multiplier theorem in terms of an L^2 -averaged R -boundedness condition. More precisely, let $t \in J \mapsto N(t) \in B(X)$ be weakly square integrable on an interval J . Then $(N(t))_{t \in J}$ is called $R[L^2]$ -bounded if for $h \in L^2(J)$ with $\|h\|_{L^2(J)} \leq 1$ the strong integrals

$$N_h x = \int_J h(t) N(t) x dt, \quad x \in X$$

define an R -bounded subset $\{N_h : \|h\|_{L^2(J)} \leq 1\}$ of $B(X)$. By $R[L^2(J)](N(t))$, we denote the R -bound of this set. In a Hilbert space X , $R[L^2(J)]$ -boundedness reduces to the simple estimate

$$\left(\int_J |\langle N(t)x, y \rangle|^2 dt \right)^{\frac{1}{2}} \leq C \|x\| \|y\| \text{ for all } x, y \in H.$$

Assume now that A is a 0-sectorial operator with an $H^\infty(\Sigma_\sigma)$ calculus for some $\sigma \in (0, \pi)$ on a Banach space isomorphic to a subspace of an $L^p(\Omega)$ space with $1 \leq p < \infty$ (or more generally, let X have Pisier's property (α)). Then our main results, Theorems 6.1 and 6.4 show (among other statements), that the following conditions on the operator function above are essentially equivalent:

1. A has an R -bounded \mathcal{H}_2^α spectral calculus, i.e.

- $\{f(A) : \|f\|_{\mathcal{H}_2^\alpha(\mathbb{R}_+)} \leq 1\}$ is R -bounded in $B(X)$.
- 2. resolvents: $R[L^2(\mathbb{R}_+)](R_\theta(\cdot)) \leq C|\theta|^{-\alpha}$ for $\theta \rightarrow 0$
- 3. semigroup: $R[L^2(\mathbb{R}_+)](T_\theta(\cdot)) \leq C(\frac{\pi}{2} - |\theta|)^{-\alpha}$ for $|\theta| \rightarrow \frac{\pi}{2}$
- 4. wave operators: $R[L^2(\mathbb{R})](W(\cdot)) < \infty$
- 5. imaginary powers: $R[L^2(\mathbb{R})](I(\cdot)) < \infty$.

The estimates on $R_\theta(\cdot)$ (resp. on $T_\theta(\cdot)$) measure the growth of the resolvent (the analytic semigroup) as we approach the spectrum of A (resp. the ‘‘boundary’’ $i\mathbb{R}$ of \mathbb{C}_+) on rays in $\mathbb{C} \setminus \mathbb{R}_+$ (in \mathbb{C}_+). Clearly, the $R[L^2(J)]$ -boundedness of $I(\cdot)$ measures the polynomial growth of the imaginary powers and $W(\cdot)$ the growth of the regularized wave operators e^{itA} . The latter regularization is necessary since outside Hilbert space the operators e^{itA} are usually unbounded. The equivalence of these statements shows in particular that estimates of resolvents, the semigroup, wave operators or imaginary powers are all equivalent ways to obtain the boundedness of $f(A)$ for arbitrary $f \in \mathcal{H}_2^\alpha(\mathbb{R}_+)$.

We end this introduction with an overview of the article. Section 2 contains the background on H^∞ functional calculus for a sectorial operator A , R -boundedness as well as the definition of relevant function spaces. In Section 3 we introduce the Hörmander function spaces and their functional calculus. In Section 4 as a preparation for the proof of Theorem 6.1, we relate the wave operators e^{isA} with imaginary powers A^{it} via the Mellin transform. In Section 5, we study the notion of averaged R -boundedness. We feel that it is worthwhile to introduce averaged R -boundedness also for other function spaces than $L^2(J)$ since these notions appeared already implicitly in the literature and have proven to be quite useful [18, Proposition 4.1, Remark 4.2], [28, Corollary 2.14], [13, Corollary 3.19]. Finally in Section 6 we state the main Theorem 6.1, which establishes equivalences between the smaller \mathcal{W}_2^α functional calculus and averaged R -boundedness of the operator families above (see Section 3 for the definition of this function space). However, most of the classical spectral multipliers (e.g. $f(\lambda) = \lambda^{it}$) belong to $\mathcal{H}_2^\alpha \setminus \mathcal{W}_2^\alpha$. Therefore we extend in Theorem 6.4 this calculus to \mathcal{H}_2^α by means of a localization procedure. Finally, in Section 7, we indicate how our main results can be transferred to bisectorial and strip-type operators.

2. PRELIMINARIES

2.1. 0-sectorial operators. We briefly recall standard notions on H^∞ calculus. For $\omega \in (0, \pi)$ we let $\Sigma_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\}$ be the sector around the positive axis of aperture angle 2ω . We further define $H^\infty(\Sigma_\omega)$ to be the space of bounded holomorphic functions on Σ_ω . This space is a Banach algebra when equipped with the norm $\|f\|_{\infty, \omega} = \sup_{\lambda \in \Sigma_\omega} |f(\lambda)|$.

A closed operator $A : D(A) \subset X \rightarrow X$ is called ω -sectorial, if the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$, $R(A)$ is dense in X and

$$(2.1) \quad \text{for all } \theta > \omega \text{ there is a } C_\theta > 0 \text{ such that } \|\lambda(\lambda - A)^{-1}\| \leq C_\theta \text{ for all } \lambda \in \overline{\Sigma_\theta}^c.$$

Note that $\overline{R(A)} = X$ along with (2.1) implies that A is injective. In the literature, in the definition of sectoriality, the condition $\overline{R(A)} = X$ is sometimes omitted. Note that if A

satisfies the conditions defining ω -sectoriality except $\overline{R(A)} = X$ on $X = L^p(\Omega)$, $1 < p < \infty$ (or any reflexive space), then there is a canonical decomposition $X = \overline{R(A)} \oplus N(A)$, $x = x_1 \oplus x_2$, and $A = \overline{A_1} \oplus 0$, $x \mapsto Ax_1 \oplus 0$, such that A_1 is ω -sectorial on the space $\overline{R(A)}$ with domain $D(A_1) = \overline{R(A)} \cap D(A)$.

For an ω -sectorial operator A and a function $f \in H^\infty(\Sigma_\theta)$ for some $\theta \in (\omega, \pi)$ that satisfies moreover an estimate $|f(\lambda)| \leq C|\lambda|^\epsilon/|1 + \lambda|^{2\epsilon}$, one defines the operator

$$(2.2) \quad f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda,$$

where Γ is the boundary of a sector Σ_σ with $\sigma \in (\omega, \theta)$, oriented counterclockwise. By the estimate of f , the integral converges in norm and defines a bounded operator. If moreover there is an estimate $\|f(A)\| \leq C\|f\|_{\infty, \theta}$ with C uniform over all such functions, then A is said to have a bounded $H^\infty(\Sigma_\theta)$ calculus. In this case, there exists a bounded homomorphism $H^\infty(\Sigma_\theta) \rightarrow B(X)$, $f \mapsto f(A)$ extending the Cauchy integral formula (2.2).

We refer to [5] for details. We call A 0-sectorial if A is ω -sectorial for all $\omega > 0$.

For $\omega \in (0, \pi)$, define the algebras of functions $\text{Hol}(\Sigma_\omega) = \{f : \Sigma_\omega \rightarrow \mathbb{C} : \exists n \in \mathbb{N} : \rho^n f \in H^\infty(\Sigma_\omega)\}$, where $\rho(\lambda) = \lambda(1 + \lambda)^{-2}$. For a proof of the following lemma, we refer to [28, Section 15B] and [16],[17].

Lemma 2.1. Let A be a 0-sectorial operator. There exists a linear mapping, called the extended holomorphic calculus,

$$(2.3) \quad \bigcup_{0 < \omega < \pi} \text{Hol}(\Sigma_\omega) \rightarrow \{\text{closed and densely defined operators on } X\}, f \mapsto f(A)$$

extending (2.2) such that for any $f, g \in \text{Hol}(\Sigma_\omega)$, $f(A)g(A)x = (fg)(A)x$ for $x \in \{y \in D(g(A)) : g(A)y \in D(f(A))\} \subset D((fg)(A))$ and $D(f(A)) = \{x \in X : (\rho^n f)(A)x \in D(\rho(A)^{-n}) = D(A^n) \cap R(A^n)\}$, where $(\rho^n f)(A)$ is given by (2.2), i.e. $n \in \mathbb{N}$ is sufficiently large.

2.2. Function spaces on the line and half-line. In this subsection, we introduce several spaces of differentiable functions on $\mathbb{R}_+ = (0, \infty)$ and \mathbb{R} . Let $\psi \in C_c^\infty(\mathbb{R})$. Assume that $\text{supp } \psi \subset [-1, 1]$ and $\sum_{n=-\infty}^\infty \psi(t - n) = 1$ for all $t \in \mathbb{R}$. For $n \in \mathbb{Z}$, we put $\psi_n = \psi(\cdot - n)$ and call $(\psi_n)_{n \in \mathbb{Z}}$ an equidistant partition of unity. Let $\varphi \in C_c^\infty(\mathbb{R}_+)$. Assume that $\text{supp } \varphi \subset [\frac{1}{2}, 2]$ and $\sum_{n=-\infty}^\infty \varphi(2^{-n}t) = 1$ for all $t > 0$. For $n \in \mathbb{Z}$, we put $\varphi_n = \varphi(2^{-n}\cdot)$ and call $(\varphi_n)_{n \in \mathbb{Z}}$ a dyadic partition of unity. Next let $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R})$ such that $\text{supp } \phi_1 \subset [\frac{1}{2}, 2]$ and $\text{supp } \phi_0 \subset [-1, 1]$. For $n \geq 2$, put $\phi_n = \phi_1(2^{1-n}\cdot)$, so that $\text{supp } \phi_n \subset [2^{n-2}, 2^n]$. For $n \leq -1$, put $\phi_n = \phi_{-n}(-\cdot)$. We assume that $\sum_{n \in \mathbb{Z}} \phi_n(t) = 1$ for all $t \in \mathbb{R}$. Then we call $(\phi_n)_{n \in \mathbb{Z}}$ a dyadic partition of unity on \mathbb{R} , which we will exclusively use to decompose the Fourier image of a function. For the existence of such partitions, we refer to the idea in [2, Lemma 6.1.7]. We recall the following classical function spaces:

Notation 2.2. Let $m \in \mathbb{N}_0$ and $\alpha > 0$.

- (1) $C_b^m = \{f : \mathbb{R} \rightarrow \mathbb{C} : f \text{ } m\text{-times diff. and } f, f', \dots, f^{(m)} \text{ uniformly cont. and bounded}\}$.
- (2) $W_2^\alpha = \{f \in L^2(\mathbb{R}) : \|f\|_{W_2^\alpha} = \|(\hat{f}(t)(1 + |t|)^\alpha)\check{}\|_2 < \infty\}$.

The space W_2^α is a Banach algebra with respect to pointwise multiplication if $\alpha > \frac{1}{2}$, [33, p. 222].

Further we also consider the local space

$$(3) \quad W_{2,\text{loc}}^\alpha = \{f : \mathbb{R} \rightarrow \mathbb{C} : f\varphi \in W_2^\alpha \text{ for all } \varphi \in C_c^\infty\} \text{ for } \alpha > \frac{1}{2}.$$

This space is closed under pointwise multiplication. Indeed, if $\varphi \in C_c^\infty$ is given, choose $\psi \in C_c^\infty$ such that $\psi\varphi = \varphi$. For $f, g \in W_{2,\text{loc}}^\alpha$, we have $(fg)\varphi = (f\varphi)(g\psi) \in W_{2,\text{loc}}^\alpha$.

2.3. Rademachers, Gaussians and R -boundedness. A classical theorem of Marcinkiewicz and Zygmund states that for elements $x_1, \dots, x_n \in L^p(U, \mu)$ we can express “square sums” in terms of random sums

$$\left\| \left(\sum_{j=1}^n |x_j(\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(U)} \cong \left(\mathbb{E} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|_{L^p(U)}^q \right)^{\frac{1}{q}} \cong \left(\mathbb{E} \left\| \sum_{j=1}^n \gamma_j x_j \right\|_{L^p(U)}^q \right)^{\frac{1}{q}}$$

with constants only depending on $p, q \in [1, \infty)$. Here $(\epsilon_j)_j$ is a sequence of independent Bernoulli random variables (with $P(\epsilon_j = 1) = P(\epsilon_j = -1) = \frac{1}{2}$) and $(\gamma_j)_j$ is a sequence of independent standard Gaussian random variables. Following [4] it has become standard by now to replace square functions in the theory of Banach space valued function spaces by such random sums (see e.g. [28]). Note however that Bernoulli sums and Gaussian sums for x_1, \dots, x_n in a Banach space X are only equivalent if X has finite cotype (see [8, p. 218] for details).

Let τ be a subset of $B(X)$. We say that τ is R -bounded if there exists a $C < \infty$ such that

$$\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k T_k x_k \right\| \leq C \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|$$

for any $n \in \mathbb{N}$, $T_1, \dots, T_n \in \tau$ and $x_1, \dots, x_n \in X$. The smallest admissible constant C is denoted by $R(\tau)$. We remark that one always has $R(\tau) \geq \sup_{T \in \tau} \|T\|$ and equality holds if X is a Hilbert space.

Recall that by definition, X has Pisier’s property (α) if for any finite family $x_{k,l}$ in X , $(k, l) \in F$, where $F \subset \mathbb{Z} \times \mathbb{Z}$ is a finite array, we have a uniform equivalence

$$\mathbb{E}_\omega \mathbb{E}_{\omega'} \left\| \sum_{(k,l) \in F} \epsilon_k(\omega) \epsilon_l(\omega') x_{k,l} \right\|_X \cong \mathbb{E}_\omega \left\| \sum_{(k,l) \in F} \epsilon_{k,l}(\omega) x_{k,l} \right\|_X.$$

Note that property (α) is inherited by closed subspaces, and that an L^p space has property (α) provided $1 \leq p < \infty$ [28, Section 4].

3. HÖRMANDER CLASSES

Aside from the classical spaces in Notation 2.2 we introduce the following Hörmander class. We write from now on

$$f_e : J \rightarrow \mathbb{C}, \quad z \mapsto f(e^z)$$

for a function $f : I \rightarrow \mathbb{C}$ such that $I \subset \mathbb{C} \setminus (-\infty, 0]$ and $J = \{z \in \mathbb{C} : |\text{Im } z| < \pi, e^z \in I\}$.

Definition 3.1.

(1) Let $\alpha > \frac{1}{2}$. We define

$$\mathcal{W}_2^\alpha = \{f : (0, \infty) \rightarrow \mathbb{C} : \|f\|_{\mathcal{W}_2^\alpha} = \|f_e\|_{W_2^\alpha} < \infty\}$$

and equip it with the norm $\|f\|_{\mathcal{W}_2^\alpha}$.

(2) Let $(\psi_n)_{n \in \mathbb{Z}}$ be an equidistant partition of unity and $\alpha > \frac{1}{2}$. We define the Hörmander class

$$\mathcal{H}_2^\alpha = \{f \in L_{\text{loc}}^2(\mathbb{R}_+) : \|f\|_{\mathcal{H}_2^\alpha} = \sup_{n \in \mathbb{Z}} \|\psi_n f_e\|_{W_2^\alpha} < \infty\}$$

and equip it with the norm $\|f\|_{\mathcal{H}_2^\alpha}$.

We have the following elementary properties of Hörmander spaces. Its proof may be found in [21, Propositions 4.8 and 4.9, Remark 4.16].

Lemma 3.2.

- (1) The spaces \mathcal{W}_2^α and \mathcal{H}_2^α are Banach algebras.
- (2) Different partitions of unity $(\psi_n)_n$ give the same space \mathcal{H}_2^α with equivalent norms.
- (3) Let $\alpha > \frac{1}{2}$ and $\sigma \in (0, \pi)$. Then

$$H^\infty(\Sigma_\sigma) \hookrightarrow \mathcal{H}_2^\alpha.$$

- (4) For any $t > 0$, we have $\|f\|_{\mathcal{H}_2^\alpha} \cong \|f(t \cdot)\|_{\mathcal{H}_2^\alpha}$.

Remark 3.3. The name ‘‘Hörmander class’’ is justified by the following facts. The classical Hörmander condition with a parameter $\alpha_1 \in \mathbb{N}$ reads as follows [19, (7.9.8)]:

$$(3.1) \quad \sum_{k=0}^{\alpha_1} \sup_{R>0} \int_{R/2}^{2R} |R^k f^{(k)}(t)|^2 dt / R < \infty.$$

Furthermore, consider the following condition for some $\alpha > \frac{1}{2}$:

$$(3.2) \quad \sup_{t>0} \|\psi f(t \cdot)\|_{W_2^\alpha} < \infty,$$

where ψ is a fixed function in $C_c^\infty(\mathbb{R}_+) \setminus \{0\}$. This condition appears in several articles on Hörmander spectral multiplier theorems, we refer to [11] for an overview. One easily checks that (3.2) does not depend on the particular choice of ψ (see also [11, p. 445]).

By the following lemma which is proved in [21, Proposition 4.11], the norm $\|\cdot\|_{\mathcal{H}_2^\alpha}$ expresses condition (3.2) and generalizes the classical Hörmander condition (3.1).

Lemma 3.4. Let $f \in L_{\text{loc}}^1(\mathbb{R}_+)$, $\alpha_1 \in \mathbb{N}$ and $\alpha > \frac{1}{2}$. Consider the conditions

- (1) f satisfies (3.1),
- (2) f satisfies (3.2),
- (3) $\|f\|_{\mathcal{H}_2^\alpha} < \infty$.

Then (1) \Rightarrow (2) if $\alpha_1 \geq \alpha$ and (2) \Rightarrow (1) if $\alpha \geq \alpha_1$. Further, (2) \Leftrightarrow (3).

Let E be a Sobolev space as in Notation 2.2. In this subsection we define an E functional calculus for a 0-sectorial operator A by tracing it back to the holomorphic functional calculus from Subsection 2.1. The following lemma which is proved in [21, Lemma 4.15] will be useful.

Lemma 3.5. Let $\beta > \frac{1}{2}$. Then $\bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_2^\beta$ is dense in \mathcal{W}_2^β . More precisely, if $f \in \mathcal{W}_2^\beta$, $\psi \in C_c^\infty$ such that $\psi(t) = 1$ for $|t| \leq 1$ and $\psi_n = \psi(2^{-n}(\cdot))$, then

$$(f_e * \check{\psi}_n) \circ \log \in \bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_2^\beta \text{ and } (f_e * \check{\psi}_n) \circ \log \rightarrow f \text{ in } \mathcal{W}_2^\beta.$$

Thus if f happens to belong to several spaces \mathcal{W}_2^β for different β as above, then it can be simultaneously approximated by a holomorphic sequence in any of these spaces.

Lemma 3.5 enables to base the \mathcal{W}_2^β calculus on the H^∞ calculus.

Definition 3.6. Let A be a 0-sectorial operator and $\beta > \frac{1}{2}$. We say that A has a (bounded) \mathcal{W}_2^β calculus if there exists a constant $C > 0$ such that

$$\|f(A)\| \leq C \|f\|_{\mathcal{W}_2^\beta} \quad (f \in \bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_2^\beta).$$

In this case, by the just proved density of $\bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_2^\beta$ in \mathcal{W}_2^β , the algebra homomorphism $u : \bigcap_{0 < \omega < \pi} H^\infty(\Sigma_\omega) \cap \mathcal{W}_2^\beta \rightarrow B(X)$ given by $u(f) = f(A)$ can be continuously extended in a unique way to a bounded algebra homomorphism

$$u : \mathcal{W}_2^\beta \rightarrow B(X), f \mapsto u(f).$$

We write again $f(A) = u(f)$ for any $f \in \mathcal{W}_2^\beta$. Assume that $\beta_1, \beta_2 > \frac{1}{2}$ and that A has a $\mathcal{W}_2^{\beta_1}$ calculus and a $\mathcal{W}_2^{\beta_2}$ calculus. Then for $f \in \mathcal{W}_2^{\beta_1} \cap \mathcal{W}_2^{\beta_2}$, $f(A)$ is defined twice by the above. However, the second part of Lemma 3.5 shows that these definitions coincide.

The following lemma gives a representation formula of the \mathcal{W}_2^α calculus in terms of the C_0 -group A^{it} . It can be proved with the Cauchy integral formula (2.2) in combination with the Fourier inversion formula [21, Proposition 4.22]. Here and below we use the short hand notation $\langle t \rangle = \sqrt{1+t^2}$.

Lemma 3.7. Let X be a Banach space with dual X' . Let $\alpha > \frac{1}{2}$, so that \mathcal{W}_2^α is a Banach algebra. Let A be a 0-sectorial operator with bounded imaginary powers $U(t) = A^{it}$.

(1) Assume that for some $C > 0$ and all $x \in X$, $x' \in X'$

$$(3.3) \quad \|\langle t \rangle^{-\alpha} \langle U(t)x, x' \rangle\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |\langle t \rangle^{-\alpha} \langle U(t)x, x' \rangle|^2 dt \right)^{1/2} \leq C \|x\| \|x'\|.$$

Then A has a bounded \mathcal{W}_2^α calculus. Moreover, for any $f \in \mathcal{W}_2^\alpha$, $f(A)$ is given by

$$(3.4) \quad \langle f(A)x, x' \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} (\widehat{f_e})(t) \langle U(t)x, x' \rangle dt \quad (x \in X, x' \in X').$$

The above integral exists as a strong integral if moreover $\|\langle t \rangle^{-\alpha} U(t)x\|_{L^2(\mathbb{R})} < \infty$.

(2) Conversely, if A has a \mathcal{W}_2^α calculus, then (3.3) holds.

Proof. (1): For $f \in \mathcal{W}_2^\alpha$, $x \in X$, and $x' \in X'$, set

$$\langle \Phi(f)x, x' \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} (\widehat{f_e})(t) \langle U(t)x, x' \rangle dt.$$

We have

$$\begin{aligned}
|\langle \Phi(f)x, x' \rangle| &\lesssim \int_{\mathbb{R}} |(f_e)^\wedge(t) \langle U(t)x, x' \rangle| dt = \int_{\mathbb{R}} |\langle t \rangle^\alpha (f_e)^\wedge(t) \langle t \rangle^{-\alpha} \langle U(t)x, x' \rangle| dt \\
&\leq \|\langle t \rangle^\alpha (f_e)^\wedge(t)\|_2 \|\langle t \rangle^{-\alpha} \langle U(t)x, x' \rangle\|_2 \stackrel{(3.3)}{\lesssim} \|\langle t \rangle^\alpha (f_e)^\wedge(t)\|_2 \|x\| \|x'\| \\
(3.5) \quad &\leq \|f\|_{\mathcal{W}_2^\alpha} \|x\| \|x'\|,
\end{aligned}$$

so that Φ defines a bounded operator $\mathcal{W}_2^\alpha \rightarrow B(X, X'')$. Let

$$K = \bigcap_{\omega > 0} \{f \in H^\infty(\Sigma_\omega) : \exists C > 0 \forall z \in \Sigma_\omega : |f_e(z)| \leq C(1 + |\operatorname{Re} z|)^{-2} \text{ and } (f_e)^\wedge \text{ has comp. supp.}\}$$

We have that K is a dense subset of \mathcal{W}_2^α .

Indeed, by the Cauchy integral formula, $K \subset \mathcal{W}_2^\alpha$. We now approximate a given $f \in \mathcal{W}_2^\alpha$ by elements of K . Since $C_c^\infty(0, \infty)$ is dense in \mathcal{W}_2^α , we can assume $f \in C_c^\infty(0, \infty)$. Let ψ and ψ_n be as in the Density Lemma 3.5 and put $f_n = (f_e * \check{\psi}_n) \circ \log$. Then $(f_{n,e})^\wedge = (f_e)^\wedge \psi_n$ has compact support. Further, the estimate $|f_{n,e}(z)| \leq C(1 + |\operatorname{Re} z|)^{-2}$ for z in a given strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \omega\}$ follows from the Paley-Wiener theorem and the fact that $(f_{n,e})^\wedge = (f_e)^\wedge \psi_n \in C_c^\infty(\mathbb{R})$. Thus, $(f_e * \check{\psi}_n) \circ \log \in K$, and K is dense in \mathcal{W}_2^α .

Assume for a moment that

$$(3.6) \quad \Phi(f) = f(A) \quad (f \in K).$$

Then by (3.5), there exists $C > 0$ such that for any $f \in K$, $\|f(A)\| \lesssim \|f\|_{\mathcal{W}_2^\alpha}$. By the density of K in \mathcal{W}_2^α , A has a \mathcal{W}_2^α calculus. Then for any $f \in \mathcal{W}_2^\alpha$ and $(f_n)_n$ a sequence in K such that $f = \lim_n f_n$,

$$f(A) = \lim_n f_n(A) = \lim_n \Phi(f_n) = \Phi(f),$$

where limits are in $B(X, X'')$. Thus, $f(A) = \Phi(f)$ for arbitrary f , and (3.4) follows.

We show (3.6). Let $f \in K$. Denote iB the generator of the group $U(t)$. We argue as in [14, Lemma 2.2]. Choose some $\omega > 0$. According to the representation formula

$$(3.7) \quad R(\lambda, B)x = -\operatorname{sgn}(\operatorname{Im} \lambda) i \int_0^\infty e^{i \operatorname{sgn}(\operatorname{Im} \lambda) \lambda t} U(-\operatorname{sgn}(\operatorname{Im} \lambda)t) x dt,$$

we have by the composition rule [15, Theorem 4.2.4]

$$\begin{aligned}
\langle f(A)x, x' \rangle &= \frac{1}{2\pi i} \int_{\mathbb{R}} f_e(s - i\omega) \langle R(s - i\omega, B)x, x' \rangle ds - \frac{1}{2\pi i} \int_{\mathbb{R}} f_e(s + i\omega) \langle R(s + i\omega, B)x, x' \rangle ds \\
&= \frac{1}{2\pi i} \left[\int_{\mathbb{R}} f_e(s - i\omega) \cdot i \int_0^\infty e^{-i(s-i\omega)t} \langle U(t)x, x' \rangle dt ds \right. \\
&\quad \left. + \int_{\mathbb{R}} f_e(s + i\omega) \cdot i \int_{-\infty}^0 e^{-i(s+i\omega)t} \langle U(t)x, x' \rangle dt ds \right] \\
&\stackrel{(*)}{=} \frac{1}{2\pi} \left[\int_0^\infty \left(\int_{\mathbb{R}} f_e(s - i\omega) e^{-i(s-i\omega)t} ds \right) \langle U(t)x, x' \rangle dt \right. \\
&\quad \left. + \int_{-\infty}^0 \left(\int_{\mathbb{R}} f_e(s + i\omega) e^{-i(s+i\omega)t} ds \right) \langle U(t)x, x' \rangle dt \right]
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(**)}{=} \frac{1}{2\pi} \left[\int_0^\infty \int_{\mathbb{R}} f_e(s) e^{-ist} ds \langle U(t)x, x' \rangle dt + \int_{-\infty}^0 \int_{\mathbb{R}} f_e(s) e^{-ist} ds \langle U(t)x, x' \rangle dt \right] \\
& = \frac{1}{2\pi} \int_{\mathbb{R}} (\widehat{f_e})(t) \langle U(t)x, x' \rangle dt \\
(3.8) \quad & = \langle \Phi(f)x, x' \rangle.
\end{aligned}$$

As $f \in K$, we could apply Fubini's theorem in (*) and shift the contour of the integral in (**). Hence (3.6) follows.

The last sentence of part (1) is now clear.

(2): If A has a \mathcal{W}_2^α calculus, then (3.8) still holds, and thus

$$\langle f(A)x, x' \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} (\widehat{f_e})(t) \langle U(t)x, x' \rangle dt \quad (f \in K).$$

Therefore, by the density of K in \mathcal{W}_2^α ,

$$\begin{aligned}
\|\langle t \rangle^{-\alpha} \langle U(t)x, x' \rangle\|_2 &= 2\pi \sup\{|\langle f(A)x, x' \rangle| : f \in K, \|\langle t \rangle^\alpha (\widehat{f_e})(t)\|_2 \leq 1\} \\
&\lesssim \|x\| \|x'\|.
\end{aligned}$$

□

In order to state some of our main results for a general class of operators A , we introduce now an auxiliary functional calculus which allows to define $f(A)$ as a closed, not necessarily bounded operator for f_e in (a subclass) of $W_{2,\text{loc}}^\alpha$. Let A be a 0-sectorial operator on some Banach space X . For $\theta > 0$, we let $D(\theta) = D(A^\theta) \cap R(A^\theta)$, which is a Banach space with the norm $\|x\|_{D(\theta)} = \|\rho^{-\theta}(A)x\|_X$. $D(\theta)$ forms a decreasing scale of spaces when θ grows. Recall that $\rho(\lambda) = \lambda(1+\lambda)^{-2}$ and its powers ρ^θ belong to $H_0^\infty(\Sigma_\omega)$ for any $\omega \in (0, \pi)$, and $R(\rho^\theta(A)) = D(A^\theta) \cap R(A^\theta)$. Note that $D(\theta)$ is dense in X (see [28, 9.4 Proposition (c)]). Assume that A satisfies one of the following assumptions for some $\theta > 0$ and $\beta > 0$.

(3.9)

$$\int_{\mathbb{R}} |\langle t \rangle^{-\beta} \langle A^{it}x, x' \rangle|^2 dt \leq C \|x\|_{D(\theta)}^2 \|x'\|_{X'}^2,$$

(3.10)

$$\int_{\mathbb{R}} |\langle s \rangle^{-\beta} \langle e^{isA}x, x' \rangle|^2 ds \leq C \|x\|_{D(\theta)}^2 \|x'\|_{X'}^2,$$

(3.11)

$$\int_0^\infty |\langle \exp(-e^{i\omega}tA)x, x' \rangle|^2 dt \leq C \left(\frac{\pi}{2} - |\omega|\right)^{-2\beta} \|x\|_{D(\theta)}^2 \|x'\|_{X'}^2, \text{ for } |\omega| < \frac{\pi}{2},$$

(3.12)

$$\int_0^\infty |t^\gamma \langle R(e^{i\omega}t, A)x, x' \rangle|^2 \frac{dt}{t} \leq C |\omega|^{-2\beta} \|x\|_{D(\theta)}^2 \|x'\|_{X'}^2, \text{ for some fixed } \gamma \in (0, 1) \text{ and any } \omega \in (-\pi, \pi) \setminus \{0\}.$$

Here in (3.10), $\langle e^{isA}x, x' \rangle$ is understood to be the limit of $\langle e^{(is-t)A}x, x' \rangle$ as $t \rightarrow 0+$. If A satisfies (3.9) with $\beta = \alpha > \frac{1}{2}$, then one can show as in Lemma 3.7 (1) that there is a

bounded linear mapping

$$\Phi_A : \mathcal{W}_2^\alpha \rightarrow B(D(\theta), X), f \mapsto \Phi_A(f)$$

with $\langle \Phi_A(f)x, x' \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{(f_e)}(t) \langle A^{it}x, x' \rangle dt$ for $x \in D(\theta)$ and $x' \in X'$. Moreover, $\Phi_A(f)x = f(A)x$ for $x \in D(\theta)$ and $f \in H^\infty(\Sigma_\omega) \cap \mathcal{W}_2^\alpha$, where the right hand side is defined by the holomorphic functional calculus from Lemma 2.1. Note that $\Phi_A(f)$ is a closeable operator on X . Indeed, if $x_n \rightarrow 0$ in X with $x_n \in D(\theta)$ and $\Phi_A(f)x_n \rightarrow y$ for some $y \in X$, then $\Phi_A(f)\rho^\theta(A) = \rho^\theta(A)\Phi_A(f)$ is a bounded operator in $B(X)$, so that $\rho^\theta(A)\Phi_A(f)x_n$ converges to both 0 and $\rho^\theta(A)y$. By injectivity of $\rho^\theta(A)$, it follows that $y = 0$, and thus, Φ_A is closeable. Therefore we can denote without ambiguity its closure by $f(A)$. We will show later in Proposition 6.3 that the conditions (3.10), (3.11) and (3.12) imply (3.9) for some $\alpha > \frac{1}{2}$ and $\theta' > 0$ and thus also imply the auxiliary functional calculus Φ_A with the properties above.

Lemma 3.8. Let A be a 0-sectorial operator with auxiliary functional calculus Φ_A as above. Let $\omega \in (0, \pi)$.

- (1) If $f \in \mathcal{W}_2^\alpha$, $g \in H^\infty(\Sigma_\omega)$ and $x \in D(\theta)$, then $f(A)g(A)x = (fg)(A)x$.
- (2) If $f \in \mathcal{W}_2^\alpha$, $g \in H_0^\infty(\Sigma_\omega)$ and $x \in D(\theta)$, then $g(A)f(A)x = (gf)(A)x$.
- (3) If $f, g \in \mathcal{W}_2^\alpha$ and $x \in D(\theta)$, then $g(A)x \in D(f(A))$ and $f(A)g(A)x = (fg)(A)x$.

Proof. (1) Let $f_n \in \mathcal{W}_2^\alpha \cap H^\infty(\Sigma_\omega)$ with $f_n \rightarrow f$ in \mathcal{W}_2^α . Then on the one hand, since $g(A)$ commutes with $\rho^\theta(A)$, and thus, $g(A)x$ belongs to $D(\theta)$, we have $f_n(A)g(A)x \rightarrow f(A)g(A)x$. On the other hand, $f_n(A)g(A)x = (f_n g)(A)x \rightarrow (fg)(A)x$, since $f_n g \rightarrow fg$ in \mathcal{W}_2^α .

(2) Let again $f_n \in \mathcal{W}_2^\alpha \cap H^\infty(\Sigma_\omega)$ with $f_n \rightarrow f$ in \mathcal{W}_2^α . Then $f_n(A)x \rightarrow f(A)x$, so $g(A)f_n(A)x \rightarrow g(A)f(A)x$. On the other hand, $g(A)f_n(A)x = (gf_n)(A)x \rightarrow (gf)(A)x$, since $gf_n \rightarrow gf$ in \mathcal{W}_2^α .

(3) We first show that if $x \in D(2\theta)$, then $g(A)x \in D(\theta)$ and $f(A)g(A)x = (fg)(A)x$. Note that by (1) and (2), $g(A)$ commutes with $\rho^\theta(A)$, so that $g(A)$ maps $D(2\theta)$ into $D(\theta)$. This implies $g(A)x \in D(\theta)$. Moreover, we have for $x' \in X'$,

$$\begin{aligned} \langle (fg)(A)x, x' \rangle &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{(f_e g_e)}(t) \langle A^{it}x, x' \rangle dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{2\pi} \widehat{(f_e)} * \widehat{(g_e)}(t) \langle A^{it}x, x' \rangle dt \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{(f_e)}(s) \widehat{(g_e)}(t-s) \langle A^{i(t-s)} A^{is}x, x' \rangle dt ds \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \widehat{(f_e)}(s) \int_{\mathbb{R}} \widehat{(g_e)}(t) \langle A^{it} A^{is}x, x' \rangle dt ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{(f_e)}(s) \langle g(A) A^{is}x, x' \rangle ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{(f_e)}(s) \langle A^{is} g(A)x, x' \rangle ds \\ &= \langle f(A)g(A)x, x' \rangle. \end{aligned}$$

Now for general $x \in D(\theta)$, we let $x_n \in D(2\theta)$ such that $x_n \rightarrow x$ in $D(\theta)$. Then $g(A)x_n \rightarrow g(A)x$ in X . By the above, $f(A)g(A)x_n = (fg)(A)x_n$, which converges to $(fg)(A)x$ in X . By closedness of $f(A)$, this implies $g(A)x \in D(f(A))$ and $f(A)g(A)x = (fg)(A)x$. \square

For a 0-sectorial operator A with auxiliary functional calculus Φ_A as above, we define the following subset D_A of X . Let $(\varphi_n)_{n \in \mathbb{Z}}$ be a dyadic partition of unity and θ be given as in (3.9).

$$(3.13) \quad D_A = \left\{ \sum_{n=-N}^N \varphi_n(A)x : N \in \mathbb{N}, x \in D(\theta) \right\}.$$

We call D_A the calculus core of A .

Lemma 3.9. Let A and D_A be as above. Then D_A is dense in X .

Proof. For $n \in \mathbb{Z}$, let $\phi_n(t) = \exp(-2^n t) - \exp(-2^{n+1} t)$. Then by a telescopic sum argument, $\sum_{n \in \mathbb{Z}} \phi_n(A)x = \lim_{t \rightarrow 0} e^{-tA}x - \lim_{t \rightarrow \infty} e^{-tA}x = x$ for any $x \in X$, due to the property $N(A) = \{0\}$. It thus suffices to show that $\sum_{m \in \mathbb{Z}} \varphi_m(A)\phi_n(A)x$ converges to $\phi_n(A)x$ for $x \in D(2\theta)$, since $\sum_{m=-N}^N \varphi_m(A)\phi_n(A)x$ belongs to D_A , so that then $\phi_n(A)x \in \overline{D_A}$ and by the above $x \in \overline{D_A}$. Thus, $D(2\theta) \subset \overline{D_A}$, and we conclude since $D(2\theta)$ is dense in X . We note that by Lemma 3.8, $\|\varphi_m(A)\phi_n(A)x\| = \|(\varphi_m\phi_n)(A)x\| \lesssim \|\varphi_m\phi_n\|_{\mathcal{W}_2^\alpha} \lesssim 2^{-|n+m|}$. Indeed, the last inequality can be seen as follows. Let $M > \alpha$ be a natural number. Then

$$\begin{aligned} \|\varphi_m\phi_n\|_{\mathcal{W}_2^\alpha} &= \|\varphi_{m+n}\phi_0\|_{\mathcal{W}_2^\alpha} \\ &\lesssim \|\varphi_0(2^{(\cdot)})\|_{C_b^M[m+n-1, m+n+1]} \\ &= \|\exp(-2^{(\cdot)}) - \exp(-2 \cdot 2^{(\cdot)})\|_{C_b^M[m+n-1, m+n+1]}. \end{aligned}$$

If $m+n \geq 0$, then this can be estimated by $\leq \|\exp(-2^{(\cdot)})\|_{C_b^M[m+n-1, m+n+1]} + \|\exp(-2 \cdot 2^{(\cdot)})\|_{C_b^M[m+n-1, m+n+1]} \lesssim \exp(-2^{m+n-1}) \lesssim 2^{-(m+n)}$. If $m+n \leq 0$, we use that $\exp(-2^{(\cdot)}) - \exp(-2 \cdot 2^{(\cdot)})$ is holomorphic and in absolute value less than $C|2^{(\cdot)}|$, for $\operatorname{Re}(\cdot) \leq 0$. Then the above quantity can be estimated by $\leq C2^{m+n}$.

In all, $\sum_{m \in \mathbb{Z}} \varphi_m(A)\phi_n(A)x$ converges absolutely in X . Therefore, $\sum_{m=-N}^N (\varphi_m\phi_n)(A)x - \phi_n(A)x = \sum_{|m| \geq N+1} (\varphi_m\phi_n)(A)x \rightarrow 0$ in X , and the claim follows. \square

As for the H^∞ calculus, there is an extended \mathcal{W}_2^α calculus which is defined for $f_e \in W_{2, \text{loc}}^\alpha$ with $f\rho^\nu = f(\cdot)(\cdot)^\nu(1 + (\cdot))^{-2\nu} \in \mathcal{W}_2^\alpha$ for some $\nu > 0$, as a counterpart of (2.3).

Definition 3.10. Let A satisfy one of the conditions (3.9) - (3.12), so that there is an auxiliary calculus Φ_A . Let $f_e \in W_{2, \text{loc}}^\alpha$ with $f\rho^\nu \in \mathcal{W}_2^\alpha$ for some $\nu > 0$. We define the operator $f(A)$ on D_A by

$$f(A) \left(\sum_{n=-N}^N \varphi_n(A)x \right) = \sum_{n=-N}^N (f\varphi_n)(A)x.$$

Note that this definition does not depend on the representation $\sum_{n=-N}^N \varphi_n(A)x$ of the element in D_A .

Lemma 3.11. Let A and f be as in Definition 3.10 and g a further function with same assumptions as f .

- (a) The operator $f(A)$ is closeable in X , we denote the closure by slight abuse of notation again by $f(A)$.
- (b) If furthermore $f \in \mathcal{W}_2^\alpha$ then $f(A)$ coincides with the operator defined by the calculus Φ_A . If $f \in \text{Hol}(\Sigma_\omega)$ for some $\omega \in (0, \pi)$, then $f(A)$ coincides with the (unbounded) holomorphic calculus of A .
- (c) For any $x \in D_A$, we have $g(A)x \in D(f(A))$ and $f(A)g(A)x = (fg)(A)x$.

Proof. (a) Let $x_n \in D_A$ with $x_n \rightarrow 0$ in X such that $f(A)x_n \rightarrow y$ for some $y \in X$. It is easy to check that $\rho^\nu(A)f(A)x_n = (f\rho^\nu)(A)x_n$. Then $\rho^{\theta+\nu}(A)f(A)x_n = \rho^\theta(A)(f\rho^\nu)(A)x_n$ converges to 0 on the one hand, and to $\rho^\theta(A)\rho^\nu(A)y$ on the other hand. By injectivity of $\rho^{\theta+\nu}(A)$, it follows that $y = 0$, and thus, $f(A)$ is closeable.

(b) For the statement for $f \in \mathcal{W}_2^\alpha$, this is easy to check on D_A . Moreover, D_A is dense in $D(f(A))$ by (a) and also in $D(\Phi_A(f))$ which is checked as in Lemma 3.9. Now pass to the closures of $f(A)$ and $\Phi_A(f)$.

For the statement for $f \in \text{Hol}(\Sigma_\omega)$, argue similarly with D_A replaced by $\{\sum_{n=-N}^N \varphi_n(A)x : N \in \mathbb{N}, x \in D(\theta + \mu)\}$ and $\mu > 0$ such that $f\rho^\mu \in H_0^\infty(\Sigma_\omega)$.

(c) We first check that for $x \in D(\theta)$ and $n \in \mathbb{Z}$, $g(A)\varphi_n(A)x$ belongs to $D(f(A))$ and $f(A)g(A)\varphi_n(A)x = (fg\varphi_n)(A)x$. To this end, let $x_m \in D(2\theta)$ with $x_m \rightarrow x$ in $D(\theta)$ as $m \rightarrow \infty$. Then $\varphi_n(A)x_m \rightarrow \varphi_n(A)x$ in X . Moreover, $g(A)\varphi_n(A)x_m = (g\varphi_n)(A)x_m \rightarrow (g\varphi_n)(A)x$. By Lemma 3.8, we have $(g\varphi_n)(A)x_m = \varphi_n(A)(g\tilde{\varphi}_n)(A)x_m$, where $\tilde{\varphi}_n = \varphi_{n-1} + \varphi_n + \varphi_{n+1}$ satisfies $\tilde{\varphi}_n\varphi_n = \varphi_n$. Since $(g\tilde{\varphi}_n)(A)x_m \in D(\theta)$, $(g\varphi_n)(A)x_m$ belongs to D_A . By Lemma 3.8, $f(A)(g\varphi_n)(A)x_m = f(A)\varphi_n(A)(g\tilde{\varphi}_n)(A)x_m = (f\varphi_n)(A)(g\tilde{\varphi}_n)(A)x_m = (fg\varphi_n)(A)x_m \rightarrow (fg\varphi_n)(A)x$ in X . By closedness of $f(A)$, $g(A)\varphi_n(A)x = (g\varphi_n)(A)x \in D(f(A))$ and $f(A)g(A)\varphi_n(A)x = (fg\varphi_n)(A)x$ for any $x \in D(\theta)$. We infer

$$f(A)g(A) \sum_{n=-N}^N \varphi_n(A)x = \sum_{n=-N}^N (fg\varphi_n)(A)x = (fg)(A) \sum_{n=-N}^N \varphi_n(A)x$$

for any element $\sum_{n=-N}^N \varphi_n(A)x \in D_A$. □

Note that the Hörmander class \mathcal{H}_2^α is contained in $\mathcal{W}_{2,\text{loc}}^\alpha$. Thus the $\mathcal{W}_{2,\text{loc}}^\alpha$ calculus in Lemma 3.11 enables us to define the \mathcal{H}_2^α calculus, whose boundedness is a main object of investigation in this article.

Definition 3.12. Let $\alpha > \frac{1}{2}$ and let A be a 0-sectorial operator having a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \in (0, \pi)$. We say that A has a (bounded) \mathcal{H}_2^α calculus if there exists a constant $C > 0$ such that

$$(3.14) \quad \|f(A)\| \leq C\|f\|_{\mathcal{H}_2^\alpha} \quad (f \in \bigcap_{\omega \in (0, \pi)} H^\infty(\Sigma_\omega) \cap \mathcal{H}_2^\alpha).$$

Let $\alpha > \frac{1}{2}$ and consider a 0-sectorial operator A having a \mathcal{H}_2^α calculus in the sense of Definition 3.12. Then A has a \mathcal{W}_2^α calculus and thus, we can apply Lemma 3.11 (with

$\theta = 0$) and consider the unbounded $\mathcal{W}_{2,\text{loc}}^\alpha$ calculus of A , and in particular $f(A)$ is defined for $f \in \mathcal{H}_2^\alpha \subset \mathcal{W}_{2,\text{loc}}^\alpha$. Then condition (3.14) extends automatically to all $f \in \mathcal{H}_2^\alpha$.

4. WAVE OPERATORS AND BOUNDED IMAGINARY POWERS

In this section, we assume that A is a 0-sectorial operator. We relate wave operators with imaginary powers of A by means of the Mellin transform $M : L^2(\mathbb{R}_+, \frac{ds}{s}) \rightarrow L^2(\mathbb{R}, dt)$, $f \mapsto \int_0^\infty f(s) s^{it} \frac{ds}{s}$.

Proposition 4.1. Let $\alpha > \frac{1}{2}$ and $m \in \mathbb{N}$ such that $m > \alpha - \frac{1}{2}$. Assume that A satisfies one of the assumptions (3.9), (3.10), (3.11) or (3.12) for some β and θ . Since $\lambda \mapsto \lambda^{\frac{1}{2}-\alpha} (e^{\mp is\lambda} - 1)^m$ belongs to $\mathcal{W}_{2,\text{loc}}^\alpha$ with polynomial growth at 0 and ∞ , this entails that $A^{\frac{1}{2}-\alpha} (e^{\mp isA} - 1)^m$ are well-defined closed operators for $s > 0$ with domain containing D_A from (3.13). Assume moreover that

$$\|t \mapsto \langle t \rangle^{-\alpha} \langle A^{it} x, x' \rangle\|_{L^2(\mathbb{R}, dt)} \leq C \|x\|_X \|x'\|_{X'} \quad (x \in D_A, x' \in X')$$

or

$$\|s \mapsto \langle (sA)^{\frac{1}{2}-\alpha} (e^{\mp isA} - 1)^m x, x' \rangle\|_{L^2(\mathbb{R}_+, \frac{ds}{s})} \leq C \|x\|_X \|x'\|_{X'} \quad (x \in D_A, x' \in X').$$

Then for any $x \in D_A$ from (3.13), and $x' \in X'$, we have the identity in $L^2(\mathbb{R}, dt)$:

$$M \left[\langle (sA)^{\frac{1}{2}-\alpha} (e^{\mp isA} - 1)^m x, x' \rangle \right] (t) = h_{\mp}(t) \langle A^{-it} x, x' \rangle,$$

where

$$(4.1) \quad h_{\mp}(t) = e^{\mp i \frac{\pi}{2} (\frac{1}{2}-\alpha)} e^{\pm \frac{\pi}{2} t} \Gamma\left(\frac{1}{2} - \alpha + it\right) f_m\left(\frac{1}{2} - \alpha + it\right)$$

with

$$(4.2) \quad f_m(z) = \sum_{k=1}^m \binom{m}{k} (-1)^{m-k} k^{-z}$$

and h_{\mp} satisfies $|h_{\mp}(t)| \lesssim \langle t \rangle^{-\alpha}$.

The two following lemmas are devoted to the proof of Proposition 4.1.

Lemma 4.2. Let $m \in \mathbb{N}$ and $\text{Re } z \in (-m, 0)$. Then

$$\int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s} = \Gamma(z) f_m(z),$$

with f_m given in (4.2). Note that $\Gamma(z) f_m(z)$ is a holomorphic function for $\text{Re } z \in (-m, 0)$.

Proof. We proceed by induction over m . In the case $m = 1$, we obtain by integration by parts $\int_0^\infty s^z (e^{-s} - 1) \frac{ds}{s} = \left[\frac{1}{z} s^z (e^{-s} - 1) \right]_0^\infty + \int_0^\infty \frac{1}{z} s^z e^{-s} ds = 0 + \frac{1}{z} \Gamma(z+1) = \Gamma(z) = \Gamma(z) f_1(z)$. Next we claim that for $\text{Re } z > -m$,

$$\int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} = \Gamma(z) \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} (k+1)^{-z}.$$

Note that the left hand side is well-defined and holomorphic for $\operatorname{Re} z > -m$ and the right hand side is meromorphic on \mathbb{C} . By the identity theorem for meromorphic functions, it suffices to show the claim for e.g. $\operatorname{Re} z > 0$. For these z in turn, we can develop

$$\int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \int_0^\infty s^z e^{-ks} e^{-s} \frac{ds}{s},$$

which gives the claim.

Assume now that the lemma holds for some m . Let first $\operatorname{Re} z \in (-m, 0)$. In the following calculation, we use both the claim and the induction hypothesis in the second equality, and the convention $\binom{m}{m+1} = 0$ in the third.

$$\begin{aligned} \int_0^\infty s^z (e^{-s} - 1)^{m+1} \frac{ds}{s} &= \int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} - \int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s} \\ &= \Gamma(z) \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} (k+1)^{-z} - \Gamma(z) f_m(z) \\ &= \Gamma(z) \sum_{k=1}^{m+1} \binom{m}{k-1} (-1)^{m+1-k} k^{-z} + \Gamma(z) \sum_{k=1}^{m+1} \binom{m}{k} (-1)^{m+1-k} k^{-z} \\ &= \Gamma(z) \sum_{k=1}^{m+1} \left[\binom{m}{k-1} + \binom{m}{k} \right] (-1)^{m+1-k} k^{-z} \\ &= \Gamma(z) f_{m+1}(z). \end{aligned}$$

Thus, the lemma holds for $m+1$ and $\operatorname{Re} z \in (-m, 0)$. For $\operatorname{Re} z \in (-(m+1), -m]$, we appeal again to the identity theorem. \square

Lemma 4.3. Let $\operatorname{Re} z \in (-m, 0)$ and $\operatorname{Re} \lambda \geq 0$. Then

$$\int_0^\infty s^z (e^{-\lambda s} - 1)^m \frac{ds}{s} = \lambda^{-z} \int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s}.$$

Proof. This is an easy consequence of the Cauchy integral theorem. \square

Proof of Proposition 4.1. Let $\mu > 0$ fixed. Combining Lemmas 4.2 and 4.3 with $\lambda = \pm i\mu$, we get $\int_0^\infty s^z (e^{\mp i\mu s} - 1)^m \frac{ds}{s} = (e^{\pm i\frac{\pi}{2}} \mu)^{-z} \Gamma(z) f_m(z)$. Put now $z = \frac{1}{2} - \alpha + it$ for $t \in \mathbb{R}$, so that $\operatorname{Re} z \in (-m, 0)$ by the assumptions of the proposition. Then $\int_0^\infty s^{it+\frac{1}{2}-\alpha} (e^{\mp i\mu s} - 1)^m \frac{ds}{s} = e^{\mp i\frac{\pi}{2}(\frac{1}{2}-\alpha+it)} \mu^{-it} \mu^{-(\frac{1}{2}-\alpha)} \Gamma(z) f_m(z)$, so that with $h_{\mp}(t)$ as in (4.1),

$$(4.3) \quad M \left[(s\mu)^{\frac{1}{2}-\alpha} (e^{\mp is\mu} - 1)^m \right] (t) = h_{\mp}(t) \mu^{-it}.$$

The statement of the proposition was (4.3) with μ formally replaced by A (weak identity). It is easy to see that $\sup_{t \in \mathbb{R}} |f_m(\frac{1}{2} - \alpha + it)| < \infty$. Further, the Euler Gamma function has a development [29, p. 15], $|\Gamma(-\frac{1}{2} + \alpha + it)| \cong e^{-\frac{\pi}{2}|t|} |t|^{-\alpha}$ ($|t| \geq 1$), so that $|h_{\mp}(t)| \lesssim \langle t \rangle^{-\alpha}$. Thus by the assumption of the proposition, we have $t \mapsto h_{\mp}(t) \langle A^{-it} x, x' \rangle \in L^2(\mathbb{R}, dt)$, or $s \mapsto \langle (sA)^{\frac{1}{2}-\alpha} (e^{\mp isA} - 1)^m x, x' \rangle \in L^2(\mathbb{R}_+, \frac{ds}{s})$. It remains to show that in (4.3), one can replace μ by A in the weak sense. To this end, let f belong to the class $D = \operatorname{span}\{t^n e^{-t^2/2} : n \in \mathbb{N}_0\}$,

which is dense in $L^2(\mathbb{R})$. Moreover, we shall use that $\max(s^N, s^{-N})M'f(s) \rightarrow 0$ for any $N \in \mathbb{N}$ as $s \rightarrow 0$ and $s \rightarrow \infty$ for such functions f and M' the adjoint mapping of the Mellin transform. Denote $g(s, \lambda) = (s\lambda)^{\frac{1}{2}-\alpha}(e^{\mp is\lambda} - 1)^m$. Let $y = \sum_{n=-N}^N \varphi_n(A)x \in D_A$ and $x' \in X'$, and write $\tilde{g}(s, \lambda) = g(s, \lambda) \sum_{n=-N}^N \varphi_n(\lambda)$. We have with $F\tilde{g}(s, \lambda)$ denoting the Mellin transform of \tilde{g} in the variable λ and M' acting in the variable s ,

$$\begin{aligned} \int_{\mathbb{R}} M[\langle g(\cdot, A)y, x' \rangle](t)f(t)dt &= \int_{\mathbb{R}_+} \langle g(s, A)y, x' \rangle M'f(s) \frac{ds}{s} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} F\tilde{g}(s, \lambda) \langle A^{i\lambda}x, x' \rangle d\lambda M'f(s) \frac{ds}{s} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}_+} F\tilde{g}(s, \lambda) \langle A^{i\lambda}x, x' \rangle M'f(s) \frac{ds}{s} d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} MF\tilde{g}(t, \lambda) \langle A^{i\lambda}x, x' \rangle f(t) dt d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} FM\tilde{g}(t, \lambda) \langle A^{i\lambda}x, x' \rangle f(t) dt d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} FM\tilde{g}(t, \lambda) \langle A^{i\lambda}x, x' \rangle d\lambda f(t) dt \\ &= \int_{\mathbb{R}} \langle (Mg)(t, A)y, x' \rangle f(t) dt. \end{aligned}$$

Hereby, the (at most) polynomial growth of $\|\tilde{g}(s, \cdot)\|_{\mathcal{W}_2^\alpha}$ at $s \rightarrow 0$ and $s \rightarrow \infty$, the (at most) polynomial growth of $\|M\tilde{g}(t, \cdot)\|_{\mathcal{W}_2^\alpha}$ at $t \rightarrow -\infty$ and $t \rightarrow \infty$, and the decay of f and $M'f$ allowed us to apply two times Fubini's theorem. By density of such functions f in $L^2(\mathbb{R})$, the proposition follows. \square

A variant of the wave operator expression $(sA)^{\frac{1}{2}-\alpha}(e^{\mp isA} - 1)^m$ from Proposition 4.1 is given by the following proposition.

Proposition 4.4. Let A be a 0-sectorial operator. Assume that A satisfies one of the assumptions (3.9), (3.10), (3.11) or (3.12) for some β and θ . Let $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, let $m \in \mathbb{N}_0$ such that $\alpha - \frac{1}{2} \in (m, m+1)$ and

$$w_\alpha(s) = s^{-\alpha} \left(e^{is} - \sum_{j=0}^{m-1} \frac{(is)^j}{j!} \right).$$

Then $w_\alpha(sA)$ is a closed densely defined operator, and with M denoting again the Mellin transform, we have

$$M(\langle (sA)^{\frac{1}{2}}w_\alpha(sA)x, x' \rangle)(t) = i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha + \frac{1}{2} + it)\langle A^{-it}x, x' \rangle \quad (x \in D_A, x' \in X').$$

Proof. The proof is similar to that of Proposition 4.1. We determine the Mellin transform of $s^{\frac{1}{2}}w_\alpha(s)$: By a contour shift of the integral $s \rightsquigarrow is$,

$$\int_0^\infty s^{it} s^{\frac{1}{2}}w_\alpha(s) \frac{ds}{s} = \int_0^\infty (is)^{it} (is)^{\frac{1}{2}}w_\alpha(is) \frac{ds}{s}$$

$$= i^{-\alpha+\frac{1}{2}+it} \int_0^\infty s^{-\alpha+\frac{1}{2}+it} (e^{-s} - \sum_{j=0}^{m-1} \frac{(-s)^j}{j!}) \frac{ds}{s}.$$

Applying partial integration, one sees that this expression equals $i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha + \frac{1}{2} + it)$. Thus,

$$M(s^{\frac{1}{2}}w_\alpha(s))(t) = i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha + \frac{1}{2} + it),$$

and applying the functional calculus yields the proposition, see the end of the proof of Proposition 4.1 for details. \square

5. AVERAGED R -BOUNDEDNESS

Let (Ω, μ) be a σ -finite measure space. Throughout the section, we consider spaces E which are subspaces of the space \mathcal{L} of equivalence classes of measurable functions on (Ω, μ) . Here, equivalence classes refer to identity modulo μ -null sets. We require that a subspace E'_0 of the dual E' of E is given by

$$E'_0 = \{f \in \mathcal{L} : \exists C > 0 \forall g \in E : |\langle f, g \rangle| = |\int_\Omega f(t)g(t)d\mu(t)| \leq C\|g\|_E\}$$

with duality bracket $\langle f, g \rangle = \int_\Omega f(t)g(t)d\mu(t)$ and that this space is norming for E , i.e. $\|g\|_E \cong \sup_{\|f\|_{E'} \leq 1, f \in E'_0} |\langle f, g \rangle|$. This is clearly the case in the following examples:

$$(5.1) \quad \begin{aligned} E &= L^p(\Omega, w d\mu) \text{ for } 1 \leq p \leq \infty \text{ and a weight } w, \\ E &= W_2^\alpha = W_2^\alpha(\mathbb{R}) \text{ for } \alpha > \frac{1}{2}, \\ E &= \mathcal{W}_2^\alpha. \end{aligned}$$

Definition 5.1. Let (Ω, μ) be a σ -finite measure space. Let E be a function space on (Ω, μ) as in (5.1). Let $(N(t) : t \in \Omega)$ be a family of closed operators on a Banach space X such that

- (1) There exists a dense subspace $D_N \subset X$ which is contained in the domain of $N(t)$ for any $t \in \Omega$.
- (2) For any $x \in D_N$, the mapping $\Omega \rightarrow X, t \mapsto N(t)x$ is measurable.
- (3) For any $x \in D_N, x' \in X'$ and $f \in E, t \mapsto f(t)\langle N(t)x, x' \rangle$ belongs to $L^1(\Omega)$.

Then $(N(t) : t \in \Omega)$ is called R -bounded on the E -average or $R[E]$ -bounded, if for any $f \in E$, there exists $N_f \in B(X)$ such that

$$(5.2) \quad \langle N_f x, x' \rangle = \int_\Omega f(t)\langle N(t)x, x' \rangle d\mu(t) \quad (x \in D_N, x' \in X')$$

and further

$$R[E](N(t) : t \in \Omega) := R(\{N_f : \|f\|_E \leq 1\}) < \infty.$$

A number of very useful criteria for R -bounded sets known in the literature can be restated in terms of $R[E]$ -boundedness.

Example 5.2. Let (Ω, μ) be a σ -finite measure space and let $(N(t) : t \in \Omega)$ be a family of closed operators on X satisfying (1) and (2) of Definition 5.1.

- a) ($E = L^1$) Assume that the $N(t)$ are bounded operators. If $\{N(t) : t \in \Omega\}$ is R -bounded in $B(X)$, then it is also $R[L^1(\Omega)]$ -bounded, and

$$R[L^1(\Omega)](N(t) : t \in \Omega) \leq 2R(\{N(t) : t \in \Omega\}).$$

Conversely, assume in addition that Ω is a metric space, μ is a σ -finite strictly positive Borel measure and $t \mapsto N(t)$ is strongly continuous. If $(N(t) : t \in \Omega)$ is $R[L^1(\Omega)]$ -bounded, then it is also R -bounded.

- b) ($E = L^\infty$) Assume that there exists $C > 0$ such that

$$\int_{\Omega} \|N(t)x\| d\mu(t) \leq C\|x\| \quad (x \in D_N).$$

Then $(N(t) : t \in \Omega)$ is $R[L^\infty(\Omega)]$ -bounded with constant at most $2C$.

- c) ($E = L^2$) Assume that X is a reflexive $L^p(U)$ space. If

$$\left\| \left(\int_{\Omega} |(N(t)x)(\cdot)|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(U)} \leq C\|x\|_{L^p(U)}$$

for all $x \in D_N$, then $(N(t) : t \in \Omega)$ is $R[L^2(\Omega)]$ -bounded and there exists a constant $C_0 = C_0(X)$ such that

$$R[L^2(\Omega)](N(t) : t \in \Omega) \leq C_0C.$$

This can be generalized to spaces X with property (α) and the generalized square function spaces $l(\Omega, X)$ from [20].

- d) ($E = L^{r'}$) Assume that X has type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Let $1 \leq r, r' < \infty$ with $\frac{1}{r} = 1 - \frac{1}{r'} > \frac{1}{p} - \frac{1}{q}$.

Assume that $N(t) \in B(X)$ for all $t \in \Omega$, that $t \mapsto N(t)$ is strongly measurable, and that

$$\|N(t)\|_{B(X)} \in L^r(\Omega).$$

Then $(N(t) : t \in \Omega)$ is $R[L^{r'}(\Omega)]$ -bounded and there exists a constant $C_0 = C_0(r, p, q, X)$ such that

$$R[L^{r'}(\Omega)](N(t) : t \in \Omega) \leq C_0C.$$

Proof. ($E = L^1$) Assume that $(N(t) : t \in \Omega)$ is R -bounded. Then it follows from the Convex Hull Lemma [6, Lemma 3.2] that $R[L^1(\Omega)](N(t) : t \in \Omega) \leq 2R(\{N(t) : t \in \Omega\})$. Let us show the converse under the mentioned additional hypotheses. Suppose that $R(\{N(t) : t \in \Omega\}) = \infty$. We will deduce that also $R[L^1(\Omega)](N(t) : t \in \Omega) = \infty$. Choose for a given $N \in \mathbb{N}$ some $x_1, \dots, x_n \in X \setminus \{0\}$ and $t_1, \dots, t_n \in \Omega$ such that

$$\mathbb{E} \left\| \sum_k \epsilon_k N(t_k) x_k \right\|_X > N \mathbb{E} \left\| \sum_k \epsilon_k x_k \right\|_X.$$

It suffices to show that

$$(5.3) \quad \mathbb{E} \left\| \sum_k \epsilon_k \int_{\Omega} f_k(t) N(t) x_k d\mu(t) \right\|_X > N \mathbb{E} \left\| \sum_k \epsilon_k x_k \right\|_X$$

for appropriate f_1, \dots, f_n . It is easy to see that by the strong continuity of N , (5.3) holds with $f_k = \frac{1}{\mu(B(t_k, \epsilon))} \chi_{B(t_k, \epsilon)}$ for ϵ small enough. Here the fact that μ is strictly positive and σ -finite guarantees that $\mu(B(t_k, \epsilon)) \in (0, \infty)$ for small ϵ .

($E = L^\infty$) By [28, the proof of Corollary 2.17] with $Y = X$ there,

$$\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k N_{f_k} x_k \right\|_X \leq 2C \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k x_k \right\|_X$$

for any finite family N_{f_1}, \dots, N_{f_n} from (5.2) such that $\|f_k\|_\infty \leq 1$, and any finite family $x_1, \dots, x_n \in D_N$. Since D_N is a dense subspace of X , we can deduce that $\{N_f : \|f\|_\infty \leq 1\}$ is R -bounded.

($E = L^2$) For $x \in D_N$, set $\varphi(x) = N(\cdot)x \in L^p(U, L^2(\Omega))$. By assumption, φ extends to a bounded operator $L^p(U) \rightarrow L^p(U, L^2(\Omega))$. Then the assertion follows at once from [30, Proposition 3.3] in the case that Ω is an interval. The general case that Ω is a measure space and X has property (α) follows from [13, Corollary 3.19].

($E = L^{r'}$) This is a result of Hytönen and Veraar, see [18, Proposition 4.1, Remark 4.2]. \square

Proposition 5.3. If E is a space as in (5.1) and $R[E](N(t) : t \in \Omega) = C < \infty$, then

$$(5.4) \quad \|\langle N(\cdot)x, x' \rangle\|_{E'} \leq C \|x\| \|x'\| \quad (x \in D_N, x' \in X').$$

In particular, if $1 \leq p, p' \leq \infty$ are conjugated exponents and

$$R[L^{p'}(\Omega)](N(t) : t \in \Omega) = C < \infty,$$

then

$$\left(\int_{\Omega} |\langle N(t)x, x' \rangle|^p d\mu(t) \right)^{1/p} \leq C \|x\| \|x'\| \quad (x \in D_N, x' \in X').$$

If X is a Hilbert space, then also the converse holds: Condition (5.4) implies that $(N(t) : t \in \Omega)$ is $R[E]$ -bounded.

Proof. We have

$$(5.5) \quad \begin{aligned} & R[E](N(t) : t \in \Omega) \\ & \geq \sup \{ \|N_f\|_{B(X)} : \|f\|_E \leq 1 \} \\ & = \sup \left\{ \left| \int_{\Omega} f(t) \langle N(t)x, x' \rangle d\mu(t) \right| : \|f\|_E \leq 1, x \in D_N, \|x\| \leq 1, x' \in X', \|x'\| \leq 1 \right\} \\ & = \sup \{ \|\langle N(\cdot)x, x' \rangle\|_{E'} : x \in D_N, \|x\| \leq 1, x' \in X', \|x'\| \leq 1 \}. \end{aligned}$$

If X is a Hilbert space, then bounded subsets of $B(X)$ are R -bounded, and thus, “ \geq ” in (5.5) is in fact “ $=$ ”. \square

An $R[E]$ -bounded family yields a new averaged R -bounded family under a linear transformation in the function space variable.

Lemma 5.4. For $i = 1, 2$, let (Ω_i, μ_i) be a σ -finite measure space and E_i a function space on Ω_i as in (5.1), and $K \in B(E'_1, E'_2)$ such that its adjoint K' maps E_2 to E_1 .

Let further $(N(t) : t \in \Omega_1)$ be an $R[E_1]$ -bounded family of closed operators and D_N be a core for all $N(t)$. Assume that there exists a family $(M(t) : t \in \Omega_2)$ of closed operators with the same common core $D_M = D_N$ such that $t \mapsto M(t)x$ is measurable for all $x \in D_N$ and

$$\langle M(\cdot)x, x' \rangle = K(\langle N(\cdot)x, x' \rangle) \quad (x \in D_N, x' \in X').$$

Then $(M(t) : t \in \Omega_2)$ is $R[E_2]$ -bounded and

$$R[E_2](M(t) : t \in \Omega_2) \leq \|K\|R[E_1](N(t) : t \in \Omega_1).$$

Proof. Let $x \in D_N$ and $x' \in X'$. By (5.4) in Proposition 5.3, we have $\langle N(\cdot)x, x' \rangle \in E'_1$, and thus, $\langle M(\cdot)x, x' \rangle \in E'_2$. For any $f \in E_2$,

$$\int_{\Omega_2} \langle M(t)x, x' \rangle f(t) d\mu_2(t) = \int_{\Omega_1} \langle N(t)x, x' \rangle (K'f)(t) d\mu_1(t) = \langle N_{K'f}x, x' \rangle.$$

By assumption, the operator $N_{K'f}$ belongs to $B(X)$, and therefore also M_f belongs to $B(X)$. Furthermore,

$$\begin{aligned} R[E_2](M(t) : t \in \Omega_2) &= R(\{M_f : \|f\|_{E_2} \leq 1\}) \\ &= R(\{N_{K'f} : \|f\|_{E_2} \leq 1\}) \\ &\leq \|K'\|R(\{N_{K'f} : \|K'f\|_{E_1} \leq 1\}) \\ &\leq \|K\|R(\{N_g : \|g\|_{E_1} \leq 1\}) \\ &= \|K\|R[E_1](N(t) : t \in \Omega_1). \end{aligned}$$

□

In the following lemma, we collect some further simple manipulations of $R[E]$ -boundedness. Its proof is immediate from Definition 5.1.

Lemma 5.5. Let (Ω, μ) be a σ -finite measure space, let E be as in (5.1) and let $(N(t) : t \in \Omega)$ satisfy (1) and (2) of Definition 5.1.

(1) Let $f \in L^\infty(\Omega)$ and $(N(t) : t \in \Omega)$ be $R[L^p(\Omega)]$ -bounded for some $1 \leq p \leq \infty$. Then

$$R[L^p(\Omega)](f(t)N(t) : t \in \Omega) \leq \|f\|_\infty R[L^p(\Omega)](N(t) : t \in \Omega).$$

In particular, $R[L^p(\Omega_1)](N(t) : t \in \Omega_1) \leq R[L^p(\Omega)](N(t) : t \in \Omega)$ for any measurable subset $\Omega_1 \subset \Omega$.

(2) Let $w : \Omega \rightarrow (0, \infty)$ be measurable. Then for $1 \leq p \leq \infty$ and p' the conjugate exponent,

$$R[L^p(\Omega, w(t)d\mu(t))](N(t) : t \in \Omega) = R[L^p(\Omega, d\mu)](w(t)^{\frac{1}{p'}} N(t) : t \in \Omega).$$

(3) For $n \in \mathbb{N}$, let $\varphi_n : \Omega \rightarrow \mathbb{R}_+$ with $\sum_{n=1}^\infty \varphi_n(t) = 1$ for all $t \in \Omega$. Then

$$R[E](N(t) : t \in \Omega) \leq \sum_{n=1}^\infty R[E](\varphi_n(t)N(t) : t \in \Omega).$$

We turn to applications to the functional calculus. That is, the R -bounded functional calculus yields $R[L^2]$ -bounded sets by the following proposition. Here we may and do always choose the dense subset $D_N = D_A$, the calculus core from (3.13).

Definition 5.6. Let A be a 0-sectorial operator. Let $E \in \{\mathcal{H}_2^\alpha, \mathcal{W}_2^\alpha\}$. We say that A has an R -bounded E calculus if A has an E calculus, which is an R -bounded mapping in the sense of [26, Definition 2.7], i.e.

$$R(\{f(A) : \|f\|_E \leq 1\}) < \infty.$$

In the next proposition we need the Mellin transform

$$(5.6) \quad M : L^2(\mathbb{R}_+, ds/s) \rightarrow L^2(\mathbb{R}, dt), f \mapsto (t \mapsto \int_0^\infty s^{it} f(s) ds/s)$$

which is an isometry.

Proposition 5.7. Let A be a 0-sectorial operator having an R -bounded \mathcal{W}_2^α calculus for some $\alpha > \frac{1}{2}$. Let $\phi \in W_{2,\text{loc}}^\alpha(\mathbb{R}_+)$ such that $t \mapsto M\phi(t)\langle t \rangle^\alpha$ belongs to $L^\infty(\mathbb{R})$, where M denotes the Mellin transform. Then $(\phi(tA) : t > 0)$ is $R[L^2(\mathbb{R}_+, \frac{dt}{t})]$ -bounded with bound $\leq C\|M\phi(t)\langle t \rangle^\alpha\|_\infty$.

Proof. Since $\phi_e(t + \log(s)) = \phi(se^t)$, we have to show that $(\phi_e(t + \log(A)) : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$ -bounded with the stated bound. Then the proposition follows using the isometry $L^2(\mathbb{R}_+, \frac{dt}{t}) \rightarrow L^2(\mathbb{R})$, $f \mapsto f(e^{\cdot})$ and Lemma 5.4. For $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ with, say, compact support, we have

$$(5.7) \quad \int_{\mathbb{R}} h(-t)\phi_e(t + \log(A))x dt = (h * \phi_e) \circ \log(A)x \quad (x \in D_A).$$

Indeed, for fixed $x \in D_A$, there exists $\psi_0 \in C_c^\infty(\mathbb{R})$ such that $\psi_0 \circ \log(A)x = x$. Choose some $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(r) = 1$ for $r \in \text{supp } \psi_0 - \text{supp } h$, so that $\psi(t + \log(A))x = \psi(t + \log(A))\psi_0 \circ \log(A)x = \psi_0 \circ \log(A)x = x$ for any $-t \in \text{supp } h$. Then for any $x' \in X'$,

$$\begin{aligned} \int_{\mathbb{R}} h(-t)\langle \phi_e(t + \log(A))x, x' \rangle dt &= \int_{\mathbb{R}} h(-t)\langle (\phi_e\psi)(t + \log(A))x, x' \rangle dt \\ &= \int_{\mathbb{R}} h(-t) \frac{1}{2\pi} \int_{\mathbb{R}} (\phi_e\psi)^\wedge(s) e^{ist} \langle A^{is}x, x' \rangle ds dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(-t) e^{ist} (\phi_e\psi)^\wedge(s) dt \right) \langle A^{is}x, x' \rangle ds \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}(s) (\phi_e\psi)^\wedge(s) \langle A^{is}x, x' \rangle ds \\ &= \langle (h * (\phi_e\psi)) \circ \log(A)x, x' \rangle. \end{aligned}$$

where we used $h \in L^1(\mathbb{R})$, $\phi_e\psi \in W_2^\alpha$ and $s \mapsto \langle s \rangle^{-\alpha} \langle A^{is}x, x' \rangle \in L^2(\mathbb{R})$, to apply Fubini in the third line, and Lemma 3.7 in the last step. We also have $(h * (\phi_e\psi))\psi_0 = (h * \phi_e)\psi_0$ and (5.7) follows. Then the claim follows from $\|\phi_e * h\|_{W_2^\alpha} \leq \|\hat{\phi}_e(t)\langle t \rangle^\alpha\|_{L^\infty(\mathbb{R})} \|h\|_{L^2(\mathbb{R})}$ and density of the above h in $L^2(\mathbb{R})$. \square

Example 5.8. Consider $\phi(t) = t^\beta(e^{i\theta} - t)^{-1}$, where $\beta \in (0, 1)$ and $|\theta| < \pi$ and A an operator as in the proposition above. Then $\phi(tA) = t^\beta A^\beta(e^{i\theta} - tA)^{-1} = t^{\beta-1} A^\beta(e^{i\theta} t^{-1} - A)^{-1}$ is an $R[L^2(\frac{dt}{t})]$ -bounded family with bound $\lesssim \theta^{-\alpha}$. Indeed, $M\phi(t) = e^{i(\theta-\pi)(it+\beta-1)} \frac{\pi}{\sin \pi(it+\beta-1)}$. As $|\sin \pi(it + \beta - 1)| \cong \cosh(\pi t)$ for fixed β , we have $|M\phi(t)\langle t \rangle^\alpha| \lesssim e^{-(\theta-\pi)|t|} \langle t \rangle^\alpha \frac{1}{\cosh(\pi t)} \lesssim \theta^{-\alpha}$.

Theorem 6.1 will show that a converse to Proposition 5.7 holds, for many classical operator families including the above example, i.e. one can recover the R -bounded \mathcal{W}_2^α calculus from averaged R -boundedness conditions.

6. MAIN RESULTS

We introduced the notion of $R[E]$ -boundedness to give the following characterization of (R -bounded) \mathcal{W}_2^α calculus.

Theorem 6.1. Let A be a 0-sectorial operator on a Banach space X with a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \in (0, \pi)$. Let $\alpha > \frac{1}{2}$. Consider the following conditions.

Sobolev Calculus

- (1) A has an R -bounded \mathcal{W}_2^α calculus.

Imaginary powers

- (2) $(\langle t \rangle^{-\alpha} A^{it} : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$ -bounded.

Resolvents

- (3) For some/all $\beta \in (0, 1)$ there exists $C > 0$ such that for all $\theta \in (-\pi, \pi) \setminus \{0\}$:
 $R[L^2(\mathbb{R}_+, dt/t)](t^\beta A^{1-\beta} R(e^{i\theta} t, A) : t > 0) \leq C|\theta|^{-\alpha}$.
(4) For some/all $\beta \in (0, 1)$ and $\theta_0 \in (0, \pi]$, $(|\theta|^{\alpha-\frac{1}{2}} t^\beta A^{1-\beta} R(e^{i\theta} t, A) : 0 < |\theta| \leq \theta_0, t > 0)$ is $R[L^2((0, \infty) \times [-\theta_0, \theta_0] \setminus \{0\}, dt/t d\theta)]$ -bounded.

Analytic Semigroup ($T(z) = e^{-zA}$)

- (5) There exists $C > 0$ such that for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$: $R[L^2(\mathbb{R}_+)](A^{1/2} T(e^{i\theta} t) : t > 0) \leq C(\frac{\pi}{2} - |\theta|)^{-\alpha}$.
(6) $(\langle \frac{x}{y} \rangle^\alpha |x|^{-\frac{1}{2}} A^{1/2} T(x + iy) : x > 0, y \in \mathbb{R})$ is $R[L^2(\mathbb{R}_+ \times \mathbb{R})]$ -bounded.

Wave Operators

- (7) A has the auxiliary functional calculus $\Phi_A : \mathcal{W}_2^\gamma \rightarrow B(D(\theta), X)$ from Section 3 for some (possibly large) $\gamma > 0$ and $\theta > 0$ so that in particular, the operators $A^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m$ are densely defined for some $m > \alpha - \frac{1}{2}$. Assume moreover that $(|s|^{-\alpha} A^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m : s \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$ -bounded.
(8) A has the auxiliary functional calculus $\Phi_A : \mathcal{W}_2^\gamma \rightarrow B(D(\theta), X)$ from Section 3 for some (possibly large) $\gamma > 0$ and $\theta > 0$ so that in particular, the operators $A^{\frac{1}{2}-\alpha} \left(e^{isA} - \sum_{j=0}^{m-1} \frac{(isA)^j}{j!} \right)$ are densely defined. Assume moreover that

$$\left(A^{\frac{1}{2}-\alpha} |s|^{-\alpha} \left(e^{isA} - \sum_{j=0}^{m-1} \frac{(isA)^j}{j!} \right) : s \in \mathbb{R} \right)$$

is $R[L^2(\mathbb{R})]$ -bounded.

Then the following conditions are equivalent:

$$(1), (2), (4), (6), (7).$$

The condition (8) is also equivalent under the assumption that $\alpha - \frac{1}{2} \notin \mathbb{N}_0$ and $m \in \mathbb{N}_0$ such that $\alpha - \frac{1}{2} \in (m, m + 1)$.

All these conditions imply the remaining ones (3) and (5). If X has property (α) then, conversely, these two conditions imply that A has an R -bounded $\mathcal{W}_2^{\alpha+\epsilon}$ calculus for any $\epsilon > 0$.

As a preparatory lemma for the proof of Theorem 6.1, we state

Lemma 6.2. Let $\beta \in \mathbb{R}$ and $f(t) = f_m(\beta + it)$ with f_m as in (4.2). Then there exist $C, \epsilon, \delta > 0$ such that for any interval $I \subset \mathbb{R}$ with $|I| \geq C$ there is a subinterval $J \subset I$ with $|J| \geq \delta$ so that $|f(t)| \geq \epsilon$ for $t \in J$. Consequently, for $N > C/\delta$,

$$\sum_{k=-N}^N |f(t + k\delta)| \gtrsim 1 \quad (t \in J).$$

Proof. Suppose for a moment that

$$(6.1) \quad \exists C, \epsilon > 0 \forall I \text{ interval with } |I| \geq C \exists t \in I : |f(t)| \geq \epsilon.$$

It is easy to see that $\sup_{t \in \mathbb{R}} |f'(t)| < \infty$, so that for such a t and $|s - t| \leq \delta = \delta(\|f'\|_\infty, \epsilon)$, $|f(s)| \geq \epsilon/2$. Thus the lemma follows from (6.1) with $J = B(t, \delta/2)$.

It remains to show (6.1). Suppose that this is false. Then

$$(6.2) \quad \forall C, \epsilon > 0 \exists I \text{ interval with } |I| \geq C : \forall t \in I : |f(t)| < \epsilon.$$

Since f'' is bounded and $\|f'\|_{L^\infty(I)} \leq \sqrt{4\|f\|_{L^\infty(I)} \max(\|f''\|_{L^\infty(I)}, 1)}$, see [32, p.115 Exercice 15], we deduce that (6.2) holds for f' in place of f , and successively also for $f^{(n)}$ for any n . But there is some $n \in \mathbb{N}$ such that $\inf_{t \in \mathbb{R}} |f^{(n)}(t)| > 0$. Indeed,

$$f^{(n)}(t) = \sum_{k=1}^m \alpha_k (-i \log k)^n e^{-it \log k},$$

with $\alpha_k = \binom{m}{k} (-1)^{m-k} k^{-\beta} \neq 0$, whence

$$|f^{(n)}(t)| \geq |\alpha_m| |\log m|^n - \sum_{k=1}^{m-1} |\alpha_k| |\log k|^n > 0$$

for n large enough. This contradicts (6.2), so that the lemma is proved. \square

Proof of Theorem 6.1.

(1) \Leftrightarrow (2): By the \mathcal{W}_2^α representation formula (3.4), we have $R[L^2(\mathbb{R}, dt)](\langle t \rangle^{-\alpha} A^{it} : t \in \mathbb{R}) = R(\{\int_{\mathbb{R}} f(t) \langle t \rangle^{-\alpha} A^{it} dt : \|f\|_{L^2(\mathbb{R})} \leq 1\}) = R(\{2\pi f(A) : \|f\|_{\mathcal{W}_2^\alpha} \leq 1\})$.

The strategy to show the stated remaining (almost) equivalences between (2) and (3) – (7) consists more or less in finding an integral transform K as in Lemma 5.4 mapping the imaginary powers A^{it} to resolvents, to the analytic semigroup and to the wave operators, and vice versa.

(2) \Rightarrow (7): By the above shown equivalence of (1) and (2), A has an R -bounded \mathcal{W}_2^α calculus, thus in particular an auxiliary calculus $\Phi_A : \mathcal{W}_2^\gamma \rightarrow B(D(\theta), X)$ with $\gamma = \alpha$ and $\theta = 0$. By

Proposition 5.3, we clearly have that $\|t \mapsto \langle t \rangle^{-\alpha} \langle A^{it} x, x' \rangle\|_{L^2(\mathbb{R}, dt)} \leq C \|x\| \|x'\|$, provided (2) holds. Thus, by Proposition 4.1, and Lemmas 5.4 and 5.5 (1), using the fact that the Mellin transform from (5.6) is an isometry, (7) follows.

(7) \Rightarrow (2): Recall the function h_{\mp} from Proposition 4.1. By the Euler Gamma function development [29, p. 15], we have the lower estimate

$$|h_{\mp}(t)| \gtrsim |f_m(\frac{1}{2} - \alpha + it)| e^{\frac{\pi}{2}(\pm t - |t|)} \langle t \rangle^{-\alpha}.$$

Thus, by Proposition 4.1, and Lemmas 5.4 and 5.5 (1),

$$(6.3) \quad \left(\langle t \rangle^{-\alpha} f_m(\frac{1}{2} - \alpha + it) A^{-it} : t \in \mathbb{R} \right) \text{ is } R[L^2(\mathbb{R})]\text{-bounded.}$$

To get rid of f_m in this expression, we apply Lemma 6.2. According to that lemma, we have $N \in \mathbb{N}$ and $\delta > 0$ such that $\sum_{k=-N}^N |f(t + k\delta)| \gtrsim 1$ for any $t \in \mathbb{R}$ and $f(t) = f_m(\frac{1}{2} - \alpha + it)$. Write

$$\sum_{k=-N}^N f(t + k\delta) \langle t \rangle^{-\alpha} A^{-it} = \sum_{k=-N}^N \left[\frac{\langle t + k\delta \rangle^\alpha}{\langle t \rangle^\alpha} A^{ik\delta} \right] [f(t + k\delta) \langle t + k\delta \rangle^{-\alpha} A^{-i(t+k\delta)}].$$

By (6.3), the term in the second brackets is $R[L^2(\mathbb{R})]$ -bounded. The term in the first brackets is a bounded function times a bounded operator, due to the assumption that A has a bounded $H^\infty(\Sigma_\omega)$ calculus. Thus, the right hand side is $R[L^2(\mathbb{R})]$ -bounded, and so the left hand side is. Now appeal once again to Lemma 5.5 (1) to deduce (2).

(2) \Rightarrow (3): We fix $\theta \in (-\pi, \pi)$ and set

$$K_\theta : L^2(\mathbb{R}, ds) \rightarrow L^2(\mathbb{R}, ds), f(s) \mapsto (\pi - |\theta|)^\alpha \frac{1}{\sin \pi(\beta + is)} e^{\theta s} \langle s \rangle^\alpha f(s).$$

We have

$$\sup_{|\theta| < \pi} \|K_\theta\| = \sup_{|\theta| < \pi, s \in \mathbb{R}} \langle s \rangle^\alpha (\pi - |\theta|)^\alpha \frac{e^{\theta s}}{|\sin \pi(\beta + is)|} \lesssim \sup_{\theta, s} \langle s (\pi - |\theta|) \rangle^\alpha e^{-|s|(\pi - |\theta|)} < \infty.$$

In [28, p. 228 and Theorem 15.18], the following formula is derived for $x \in A(D(A^2))$ and $|\theta| < \pi$:

$$(6.4) \quad \frac{\pi}{\sin \pi(\beta + is)} e^{\theta s} A^{is} x = \int_0^\infty t^{is} [t^\beta e^{i\theta\beta} A^{1-\beta} (e^{i\theta} t + A)^{-1} x] \frac{dt}{t}.$$

Thus, with $R(\lambda, A) = (\lambda - A)^{-1}$,

$$(6.5) \quad \begin{aligned} \sup_{0 < |\theta| \leq \pi} |\theta|^\alpha R[L^2(\mathbb{R}_+, dt/t)](t^\beta A^{1-\beta} R(te^{i\theta}, A)) &= \sup_{|\theta| < \pi} (\pi - |\theta|)^\alpha R[L^2(\mathbb{R}_+, dt/t)](t^\beta A^{1-\beta} (e^{i\theta} t + A)^{-1}) \\ &= \sup_{|\theta| < \pi} (\pi - |\theta|)^\alpha R[L^2(\mathbb{R}, ds)] \left(\frac{\pi}{\sin \pi(\beta + is)} e^{\theta s} A^{is} \right) \\ &\lesssim R[L^2(\mathbb{R}, ds)](\langle s \rangle^{-\alpha} A^{is}). \end{aligned}$$

Next we claim that for any $\epsilon > 0$, (3) implies (2), where in (2), α is replaced by $\alpha + \epsilon$. First we consider $\langle s \rangle^{-(\alpha+\epsilon)} A^{is} x$ for $s \geq 1$. By Lemma 5.5 (3),

$$(6.6) \quad \begin{aligned} R[L^2([1, \infty), ds)](\langle s \rangle^{-(\alpha+\epsilon)} A^{is}) &\leq \sum_{n=0}^{\infty} R[L^2([2^n, 2^{n+1}])](\langle s \rangle^{-\epsilon} \langle s \rangle^{-\alpha} A^{is}) \\ &\leq \sum_{n=0}^{\infty} 2^{-n\epsilon} R[L^2([2^n, 2^{n+1}])](\langle s \rangle^{-\alpha} A^{is}). \end{aligned}$$

For $s \in [2^n, 2^{n+1}]$, we have

$$\langle s \rangle^{-\alpha} \lesssim 2^{-n\alpha} \lesssim 2^{-n\alpha} e^{-2^{-n}s} \lesssim (\pi - \theta_n)^\alpha \frac{e^{\theta_n s}}{\sin \pi(\beta + is)},$$

where $\theta_n = \pi - 2^{-n}$. Therefore

$$(6.5) \quad \begin{aligned} R[L^2([2^n, 2^{n+1}])](\langle s \rangle^{-\alpha} A^{is}) &\lesssim (\pi - \theta_n)^\alpha R[L^2(\mathbb{R}, ds)]\left(\frac{\pi}{\sin \pi(\beta + is)} e^{\theta_n s} A^{is}\right) \\ &\lesssim \sup_{0 < |\theta| \leq \pi} |\theta|^\alpha R[L^2(\mathbb{R}_+, dt/t)](t^\beta A^{1-\beta} R(te^{i\theta}, A)) < \infty. \end{aligned}$$

Thus, the sum in (6.6) is finite.

The part $\langle s \rangle^{-(\alpha+\epsilon)} A^{is}$ for $s \leq -1$ is treated similarly, whereas $R[L^2(-1, 1)](\langle s \rangle^{-\alpha} A^{is}) \cong R[L^2(-1, 1)](A^{is})$. It remains to show that the last expression is finite. We have assumed that X has property (α) . Then the fact that A has an H^∞ calculus implies that $\{A^{is} : |s| < 1\}$ is R -bounded [28, Theorem 12.8], so by Example 5.2 a), it is $R[L^1(-1, 1)]$ -bounded. For $f \in L^2(-1, 1)$, we have $\|f\|_1 \leq C\|f\|_2$, and consequently,

$$\left\{ \int_{-1}^1 f(s) A^{is} ds : \|f\|_2 \leq 1 \right\} \subset C \left\{ \int_{-1}^1 f(s) A^{is} ds : \|f\|_1 \leq 1 \right\}.$$

In other words, $(A^{is} : |s| < 1)$ is $R[L^2]$ -bounded.

(2) \iff (4):

Consider

$$(6.7) \quad K : L^2(\mathbb{R}, ds) \rightarrow L^2(\mathbb{R} \times (-\pi, \pi), dsd\theta), \quad f(s) \mapsto (\pi - |\theta|)^{\alpha - \frac{1}{2}} \frac{1}{\sin \pi(\beta + is)} e^{\theta s} \langle s \rangle^\alpha f(s),$$

Note that $|\sin \pi(\beta + is)| \cong \cosh(\pi s)$ for $\beta \in (0, 1)$ fixed. K is an isomorphic embedding. Indeed,

$$\|Kf\|_2^2 = \int_{\mathbb{R}} \int_{-\pi}^{\pi} \left((\pi - |\theta|)^{\alpha - \frac{1}{2}} e^{\theta s} \right)^2 d\theta \frac{1}{|\sin^2(\pi(\beta + is))|} \langle s \rangle^{2\alpha} |f(s)|^2 ds$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} (\pi - |\theta|)^{2\alpha - 1} e^{2\theta s} d\theta &\cong \int_0^{\pi} \theta^{2\alpha - 1} e^{2(\pi - \theta)|s|} d\theta \\ &\cong \cosh^2(\pi s) \int_0^{\pi} \theta^{2\alpha - 1} e^{-2\theta|s|} d\theta. \end{aligned}$$

For $|s| \geq 1$,

$$\int_0^\pi \theta^{2\alpha-1} e^{-2\theta|s|} d\theta = (2|s|)^{-2\alpha} \int_0^{2|s|\pi} \theta^{2\alpha-1} e^{-\theta} d\theta \cong |s|^{-2\alpha}.$$

This clearly implies that $\|Kf\|_2 \cong \|f\|_2$. We now apply Lemma 5.4 for K and for the mapping $L : \text{Im } K \oplus (\text{Im } K)^\perp \rightarrow L^2(\mathbb{R})$, $x \oplus y \mapsto K^{-1}x$. Note that $\text{Im } K$ is closed since K is an isomorphic embedding, so that $\text{Im } K \oplus (\text{Im } K)^\perp = L^2(\mathbb{R} \times (-\pi, \pi))$ and L is bounded since $\|L(x \oplus y)\| = \|K^{-1}x\| \cong \|x\| \leq \|x \oplus y\|$. We deduce

$$R[L^2(\mathbb{R}, ds)](\langle s \rangle^{-\alpha} A^{is}) \cong R[L^2(\mathbb{R} \times (-\pi, \pi), dsd\theta)]((\pi - |\theta|)^{\alpha-\frac{1}{2}} \frac{1}{\cosh(\pi s)} e^{\theta s} A^{is}).$$

Recall the formula (6.4), i.e.

$$\frac{\pi}{\sin \pi(\beta + is)} e^{\theta s} A^{is} x = \int_0^\infty t^{is} [t^\beta e^{i\theta\beta} A^{1-\beta} (e^{i\theta} t + A)^{-1} x] \frac{dt}{t}$$

for $|\theta| < \pi$ and $x \in A(D(A^2))$. Note that $A(D(A^2))$ is a dense subset of X . As the Mellin transform $f(s) \mapsto \int_0^\infty t^{is} f(s) \frac{ds}{s}$ is an isometry $L^2(\mathbb{R}_+, \frac{ds}{s}) \rightarrow L^2(\mathbb{R}, dt)$, we get by Lemma 5.4

$$\begin{aligned} R[L^2(\mathbb{R})](\langle s \rangle^{-\alpha} A^{is}) &\cong R[L^2(\mathbb{R}_+ \times (-\pi, \pi), \frac{dt}{t} d\theta)]((\pi - |\theta|)^{\alpha-\frac{1}{2}} t^\beta A^{1-\beta} (e^{i\theta} t + A)^{-1}) \\ &\cong R[L^2(\mathbb{R}_+ \times (0, 2\pi), dt/t d\theta)](|\theta|^{\alpha-\frac{1}{2}} t^\beta A^{1-\beta} R(e^{i\theta} t, A)). \end{aligned}$$

so that (2) \iff (4) for $\theta_0 = \pi$. Here we used that $(e^{i\theta} t + A)^{-1} = -(e^{i(\pm\pi+\theta)} t - A)^{-1} = -R(e^{i(\pm\pi+\theta)} t, A)$.

For a general $\theta_0 \in (0, \pi]$, consider K from (6.7) with restricted image, i.e.

$$K : L^2(\mathbb{R}, ds) \rightarrow L^2(\mathbb{R} \times (-\pi, -(\pi - \theta_0)] \cup [\pi - \theta_0, \pi), dsd\theta).$$

Then argue as in the case $\theta_0 = \pi$.

(4) \iff (6):

The proof of (2) \iff (4) above shows that condition (4) is independent of $\theta_0 \in (0, \pi]$ and $\beta \in (0, 1)$. Put $\theta_0 = \pi$ and $\beta = \frac{1}{2}$. Apply Lemma 5.4 with

$$(e^{i\theta} \mu + it)^{-1} = K[\exp(-(\cdot) e^{i\theta} \mu) \chi_{(0, \infty)}(\cdot)](t),$$

where $K : L^2(\mathbb{R}, ds) \rightarrow L^2(\mathbb{R}, dt)$ is the Fourier transform. This yields that (4) is equivalent to

$$R[L^2((0, \frac{\pi}{2}) \times \mathbb{R}_+, d\theta dt)](|\theta|^{\alpha-\frac{1}{2}} A^{\frac{1}{2}} T(\exp(\pm i(\frac{\pi}{2} - \theta)) t)) < \infty.$$

Applying the change of variables $\theta \rightsquigarrow \frac{\pi}{2} \pm \theta$ and $dt \rightsquigarrow t dt$ shows that this is equivalent to

$$R[L^2((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}_+, d\theta t dt)]((\frac{\pi}{2} - |\theta|)^{\alpha-\frac{1}{2}} t^{-\frac{1}{2}} A^{\frac{1}{2}} T(e^{i\theta} t)) < \infty.$$

Now the equivalence to (6) follows from the change of variables $a = t \cos \theta$, $b = t \sin \theta$, $t = |a + ib|$, $d\theta t dt = da db$.

(3) \iff (5) for $\beta = \frac{1}{2}$: Use K and the first argument from the proof of (4) \iff (6).

(2) \iff (8): Recall the formula from Proposition 4.4, M denoting the Mellin transform,

$$M(\langle (sA)^{\frac{1}{2}} w_\alpha(sA)x, x' \rangle)(t) = i^{-\alpha + \frac{1}{2} + it} \Gamma(-\alpha + \frac{1}{2} + it) \langle A^{-it}x, x' \rangle,$$

where $w_\alpha(s) = |s|^{-\alpha} \left(e^{is} - \sum_{j=0}^{m-1} \frac{(is)^j}{j!} \right)$. Then we have according to [29, p. 15], since $\alpha - \frac{1}{2} \notin \mathbb{N}_0$,

$$|i^{-\alpha + \frac{1}{2} + it} \cdot \Gamma(-\alpha + \frac{1}{2} + it)| \cong e^{-\frac{\pi}{2}t} \cdot e^{-\frac{\pi}{2}|t|} \langle t \rangle^{-\alpha}$$

for $t \in \mathbb{R}$. Thus, with Lemmas 5.4 and 5.5 (2),

$$\begin{aligned} R[L^2(\mathbb{R}, dt)](\langle t \rangle^{-\alpha} A^{it}) < \infty &\iff R[L^2(\mathbb{R}_+, ds/s)]((sA)^{\frac{1}{2}} w_\alpha(\pm sA)) < \infty \\ &\iff R[L^2(\mathbb{R}, ds)](A^{\frac{1}{2}} w_\alpha(sA)) < \infty. \end{aligned}$$

□

We can now complete the proof of a claim in Section 3.

Proposition 6.3. Let A be a 0-sectorial operator and $\beta, \theta > 0$. Assume one of the following conditions.

(6.8)

$$\int_{\mathbb{R}} |\langle s \rangle^{-\beta} \langle e^{isA}x, x' \rangle|^2 ds \leq C \|x\|_{D(\theta)}^2 \|x'\|_{X'}^2,$$

(6.9)

$$\int_0^\infty |\langle \exp(-e^{i\omega t}A)x, x' \rangle|^2 dt \leq C \left(\frac{\pi}{2} - |\omega| \right)^{-2\beta} \|x\|_{D(\theta)}^2 \|x'\|_{X'}^2, \text{ for some fixed } C > 0 \text{ and any } \frac{\pi}{4} \leq |\omega| < \frac{\pi}{2},$$

(6.10)

$$\int_0^\infty |t^\gamma \langle R(e^{i\omega t}, A)x, x' \rangle|^2 \frac{dt}{t} \leq C |\omega|^{-2\beta} \|x\|_{D(\theta)}^2 \|x'\|_{X'}^2, \text{ for some fixed } \gamma \in (0, 1) \text{ and any } \omega \in (-\pi, \pi) \setminus \{0\}.$$

Then A satisfies

$$(6.11) \quad \int_{\mathbb{R}} |\langle t \rangle^{-\alpha} \langle A^{it}x, x' \rangle|^2 dt \leq C \|x\|_{D(\theta')}^2 \|x'\|_{X'}^2$$

for some $\alpha, \theta' > 0$.

Proof. Assume that (6.8) holds. We have with $e^{i\omega t} = r + is$,

$$\begin{aligned} \left(\frac{r}{|s|} \right)^\alpha \exp(-(r + is)A) &= \left[\left(\frac{r}{|s|} \right)^\alpha (1 + rA)^{-\alpha} (1 + |s|)^\alpha (1 + A)^\alpha \right] \times \\ &\quad \times [(1 + |s|)^{-\alpha} (1 + A)^{-\alpha} e^{-isA}] [(1 + rA)^\alpha \exp(-rA)]. \end{aligned}$$

We handle each of the three brackets separately. For the first bracket, note that the $H^\infty(\Sigma_\sigma)$ norm of $\lambda \mapsto \left(\frac{r}{|s|} \right)^\alpha (1 + r\lambda)^{-\alpha} (1 + |s|)^\alpha (1 + \lambda)^\alpha \rho^\alpha(\lambda)$ (where we recall $\rho(\lambda) = \lambda(1 + \lambda)^{-2}$) is uniformly bounded for $|s| \geq r > 0$, thus the first bracket defines a bounded operator $D(\alpha + 1) \rightarrow X$ with uniform norm bound in s, r . The third bracket is also uniformly bounded $X \rightarrow X$. Thus, we deduce (6.9) with the same β and θ replaced by $\theta + \alpha + 1$.

If (6.9) in turn holds, then by the boundedness of $A^{\frac{1}{2}} : D(\frac{1}{2}) \rightarrow X$, we also have

$$(6.12) \quad \int_0^\infty |\langle A^{\frac{1}{2}} \exp(-e^{i\omega} t A) x, x' \rangle|^2 dt \leq C \left(\frac{\pi}{2} - |\omega|\right)^{-2\beta} \|x\|_{D(\theta+\frac{1}{2})}^2 \|x'\|_{X'}^2$$

for $|\omega| < \frac{\pi}{2}$.

By essentially the same proof as Theorem 6.1, i.e. using the integral transforms from “(4) \iff (6)” and “(3) \implies (2)” to go forth and back between the different operator families, and using that $\|A^{it}\|_{B(D(\sigma), X)} \leq C$ for $|t| \leq 1$ and any $\sigma > 0$, one can show that this implies (6.11) with $\alpha > \beta$ and $\theta' = \theta$.

Finally, again arguing as in the proof of Theorem 6.1 and using that $A^{\frac{1}{2}}$ is bounded $D(\frac{1}{2}) \rightarrow X$, one can show that (6.10) implies (6.12), which in turn implies (6.11). \square

Theorem 6.1 shows that averaged R -boundedness yields a good tool to describe \mathcal{W}_2^α functional calculus. However, many of the functions f that correspond to relevant spectral multipliers, as for example in (2) – (7) above, are not covered themselves by this calculus. To pass from the \mathcal{W}_2^α calculus to the \mathcal{H}_2^α calculus, which does cover all the spectral multipliers alluded to above, we shall use the spectral decomposition of Paley-Littlewood type used in the proof of the following Theorem.

Theorem 6.4. Let A be a 0-sectorial operator on a Banach space X with property (a) having a bounded $H^\infty(\Sigma_\sigma)$ calculus for some $\sigma \in (0, \pi]$. Then the following are equivalent for $\alpha > \frac{1}{2}$.

- (1) A has an R -bounded \mathcal{W}_2^α calculus.
- (2) A has an R -bounded \mathcal{H}_2^α calculus.

Example 6.5. Consider the operator $A = -\Delta$ on $X = L^p(\mathbb{R}^d)$ for some $1 < p < \infty$ and $d \in \mathbb{N}$. Hörmander’s classical result states that A has a bounded \mathcal{H}_2^α calculus for $\alpha > \frac{d}{2}$. In fact, a stronger result holds and A has an R -bounded \mathcal{H}_2^α calculus for the same range $\alpha > \frac{d}{2}$. This is proved in [22, Theorem 5.1], [23, Beginning of Section 4].

Proof of Theorem 6.4. As $\mathcal{W}_2^\alpha \subset \mathcal{H}_2^\alpha$, only the implication (1) \implies (2) has to be shown. Consider a function $\phi \in H_0^\infty(\Sigma_\nu)$ such that $\sum_{n \in \mathbb{Z}} \phi^3(2^{-n} \lambda) = 1$ for any $\lambda \in \Sigma_\nu$ and some $\nu > \sigma$. Furthermore, consider a function $\eta \in C_c^\infty$ with $\text{supp } \eta \subset [\frac{1}{2}, 2]$ such that $\sum_{n \in \mathbb{Z}} \eta(2^{-n} t) = 1$ for any $t > 0$. Let $f_1, \dots, f_N \in C^\infty(\mathbb{R}_+)$ with $\|f_j\|_{\mathcal{H}_2^\alpha} \leq 1$ for $j = 1, \dots, N$. Then for x_1, \dots, x_N belonging to the dense set $D_A \subseteq D(f_j(A))$ from (3.13),

$$\begin{aligned} \mathbb{E} \left\| \sum_{j=1}^N \epsilon_j f_j(A) x_j \right\| &= \mathbb{E} \left\| \sum_{j=1}^N \sum_{k \in \mathbb{Z}} \epsilon_j \eta(2^{-k} A) f_j(A) x_j \right\| \\ &\cong \mathbb{E} \mathbb{E}' \left\| \sum_{j=1}^N \sum_{n, k \in \mathbb{Z}} \epsilon_j \epsilon'_n \phi^2(2^{-n} A) \eta(2^{-k} A) f_j(A) x_j \right\| \\ &\leq \sum_{l \in \mathbb{Z}} \mathbb{E} \mathbb{E}' \left\| \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \epsilon_j \epsilon'_n [\phi(2^{-n} A) \eta(2^{-n-l} A) f_j(A)] \phi(2^{-n} A) x_j \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{l \in \mathbb{Z}} C_l \right) \mathbb{E} \mathbb{E}' \left\| \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \epsilon_j \epsilon'_n \phi(2^{-n} A) x_j \right\| \\
&\lesssim \left(\sum_{l \in \mathbb{Z}} C_l \right) \mathbb{E} \left\| \sum_{j=1}^N \epsilon_j x_j \right\|,
\end{aligned}$$

where we have used that $\|x\| \cong \mathbb{E} \left\| \sum_{n \in \mathbb{Z}} \epsilon_n \phi^2(2^{-n} A) x \right\| \cong \mathbb{E} \left\| \sum_{n \in \mathbb{Z}} \epsilon_n \phi(2^{-n} A) x \right\|$. Indeed, the second expression is estimated by the third one, since $\{\phi(2^{-n} A) : n \in \mathbb{Z}\}$ is R -bounded by the R -boundedness of the $H^\infty(\Sigma_\nu)$ calculus [28, 12.8 Theorem]. The third expression is estimated by the first one according to [28, 12.2 Theorem and 12.3 Remark]. Finally the first expression is estimated by the second one again by [28, 12.2 Theorem and 12.3 Remark] and $|\langle x, x' \rangle| = |\mathbb{E} \langle \sum_{n \in \mathbb{Z}} \epsilon_n \phi^2(2^{-n} A) x, \sum_{k \in \mathbb{Z}} \epsilon_k \phi(2^{-k} A) x' \rangle| \leq \mathbb{E} \left\| \sum_{n \in \mathbb{Z}} \epsilon_n \phi^2(2^{-n} A) x \right\| \mathbb{E} \left\| \sum_{k \in \mathbb{Z}} \epsilon_k \phi(2^{-k} A) x' \right\| \lesssim \mathbb{E} \left\| \sum_{n \in \mathbb{Z}} \epsilon_n \phi^2(2^{-n} A) x \right\| \|x'\|$.

Furthermore, we used property (α) in the fourth line, and $C_l = R(\{\phi(2^{-n} A) \eta(2^{-n-l} A) f_j(A) : n \in \mathbb{Z}, j = 1, \dots, N\})$ and

$$\begin{aligned}
C_l &\lesssim \sup_{j=1, \dots, N} \sup_{n \in \mathbb{Z}} \|\phi(2^l \cdot) \eta f_j(2^{n+l} \cdot)\|_{\mathcal{W}_2^\alpha} \\
&\lesssim \sup_{j=1, \dots, N} \sup_{k \in \mathbb{Z}} \|\eta f_j(2^k \cdot)\|_{\mathcal{W}_2^\alpha} \sup_{m=0, \dots, [\alpha]+1} \sup_{t \in [\frac{1}{2}, 2]} t^m \left| \frac{d^m}{dt^m} \phi(2^l \cdot)(t) \right| \\
&\lesssim \sup_{j=1, \dots, N} \|f_j\|_{\mathcal{H}_2^\alpha} 2^{-\epsilon|l|}, \\
&\leq 2^{-\epsilon|l|}
\end{aligned}$$

where $\epsilon > 0$ and we used the fact that $\phi \in H_0^\infty(\Sigma_\nu)$. Hence $\sum_{l \in \mathbb{Z}} C_l \lesssim \sup_{j=1, \dots, N} \|f_j\|_{\mathcal{H}_2^\alpha} < \infty$. We have shown that

$$(6.13) \quad \{f(A) : f \in C^\infty(\mathbb{R}_+), \|f\|_{\mathcal{H}_2^\alpha} \leq 1\}$$

is R -bounded. In particular, since $C^\infty(\mathbb{R}_+) \supset \bigcap_{\omega > 0} H^\infty(\Sigma_\omega)$, A has a bounded \mathcal{H}_2^α calculus in the sense of Definition 3.12, and by taking the closure of (6.13), this calculus is R -bounded. \square

7. BISECTORIAL OPERATORS AND OPERATORS OF STRIP TYPE

7.1. Bisectorial operators. In this short subsection we indicate how to extend our results to bisectorial operators. An operator A with dense domain on a Banach space X is called bisectorial of angle $\omega \in [0, \frac{\pi}{2})$ if it is closed, its spectrum is contained in the closure of $S_\omega = \{z \in \mathbb{C} : |\arg(\pm z)| < \omega\}$, and one has the resolvent estimate

$$\|(I + \lambda A)^{-1}\|_{B(X)} \leq C_{\omega'}, \quad \forall \lambda \notin S_{\omega'}, \quad \omega' > \omega.$$

If X is reflexive, then for such an operator we have again a decomposition $X = N(A) \oplus \overline{R(A)}$, so that we may assume that A is injective. The $H^\infty(S_\omega)$ calculus is defined as in (2.2), but now we integrate over the boundary of the double sector S_ω . If A has a bounded $H^\infty(S_\omega)$ calculus, or more generally, if we have $\|Ax\| \cong \|(-A^2)^{\frac{1}{2}} x\|$ for $x \in D(A) = D((-A^2)^{\frac{1}{2}})$ (see e.g. [9]), then the spectral projections P_1, P_2 with respect to $\Sigma_1 = S_\omega \cap \mathbb{C}_+, \Sigma_2 = S_\omega \cap \mathbb{C}_-$

give a decomposition $X = X_1 \oplus X_2$ of X into invariant subspaces for resolvents of A such that the part A_1 of A to X_1 and $-A_2$ of $-A$ to X_2 are sectorial operators with $\sigma(A_i) \subset \Sigma_i$. For $f \in H_0^\infty(S_\omega)$ we have

$$(7.1) \quad f(A)x = f|_{\Sigma_1}(A_1)P_1x + f|_{\Sigma_2}(A_2)P_2x.$$

We define the Hörmander class $\mathcal{H}_2^\alpha(\mathbb{R})$ on \mathbb{R} by $f \in \mathcal{H}_2^\alpha(\mathbb{R})$ if $f\chi_{\mathbb{R}_+} \in \mathcal{H}_2^\alpha$ and $f(-\cdot)\chi_{\mathbb{R}_+} \in \mathcal{H}_2^\alpha$. Let A be a 0-bisectorial operator, i.e. A is ω -bisectorial for all $\omega > 0$. Assume that A has a bounded $H^\infty(S_\omega)$ calculus for some $\omega \in (0, \pi/2)$. Then A has an (R -bounded) $\mathcal{H}_2^\alpha(\mathbb{R})$ calculus if the set $\{f(A) : f \in \bigcap_{0 < \omega < \pi/2} H^\infty(S_\omega) \cap \mathcal{H}_2^\alpha(\mathbb{R}), \|f\|_{\mathcal{H}_2^\alpha(\mathbb{R})} \leq 1\}$ is (R -)bounded. Clearly, A has an (R -bounded) $\mathcal{H}_2^\alpha(\mathbb{R})$ calculus if and only if A_1 and $-A_2$ have an (R -bounded) \mathcal{H}_2^α calculus and in this case (7.1) holds again.

Let $f_t(\lambda) = \begin{cases} \lambda^{it} : \operatorname{Re} \lambda > 0 \\ (-\lambda)^{it} : \operatorname{Re} \lambda < 0 \end{cases}$. Then $f_t \in H^\infty(S_\omega)$ for any $\omega \in (0, \frac{\pi}{2})$. Clearly, one

has $f_t(A) = A_1^{it} \oplus (-A_2)^{it}$ on $X = X_1 \oplus X_2$. It is easy to show that $(\langle t \rangle^{-\alpha} f_t(A) : t \in \mathbb{R})$ is $R[L^2(\mathbb{R}, dt)]$ -bounded if and only if $(\langle t \rangle^{-\alpha} A_1^{it} : t \in \mathbb{R})$ and $(\langle t \rangle^{-\alpha} (-A_2)^{it} : t \in \mathbb{R})$ are both $R[L^2(\mathbb{R}, dt)]$ -bounded. Let $|A| = f(A)$ with $f(z) = 1$ for $\operatorname{Re} z > 0$ and $f(z) = -1$ for $\operatorname{Re} z < 0$. Then similarly, we have that

$$R[L^2(\mathbb{R}_+, dt/t)](t^\beta |A|^{1-\beta} (e^{i\theta} t - A)^{-1}) \lesssim (\min(|\theta|, \pi - |\theta|))^{-\alpha}$$

for $0 < |\theta| < \pi$ if and only if both of the following conditions hold:

$$R[L^2(\mathbb{R}_+, dt/t)](t^\beta A_1^{1-\beta} (e^{i\theta} t - A_1)^{-1}) \lesssim |\theta|^{-\alpha} \text{ and } R[L^2(\mathbb{R}_+, dt/t)](t^\beta (-A_2)^{1-\beta} (e^{i\theta} t + A_2)^{-1}) \lesssim |\theta|^{-\alpha}$$

for $0 < |\theta| \leq \frac{\pi}{2}$. Finally, we have that $|s|^{-\alpha} |A|^{-\alpha + \frac{1}{2}} (e^{isA} - 1)^m$ is $R[L^2(\mathbb{R})]$ -bounded if and only if both $|s|^{-\alpha} A_1^{-\alpha + \frac{1}{2}} (e^{isA_1} - 1)^m$ and $|s|^{-\alpha} (-A_2)^{-\alpha + \frac{1}{2}} (e^{isA_2} - 1)^m$ are $R[L^2(\mathbb{R})]$ -bounded.

Then using the projections P_1 and P_2 , it is clear how our main Theorems 6.1 and 6.4 extend to bisectorial operators.

7.2. Strip-type operators. For $\omega > 0$ we let $\operatorname{Str}_\omega = \{z \in \mathbb{C} : |\operatorname{Im} z| < \omega\}$ the horizontal strip of height 2ω . We further define $H^\infty(\operatorname{Str}_\omega)$ to be the space of bounded holomorphic functions on $\operatorname{Str}_\omega$, which is a Banach algebra equipped with the norm $\|f\|_{\infty, \omega} = \sup_{\lambda \in \operatorname{Str}_\omega} |f(\lambda)|$. A densely defined operator B is called ω -strip-type operator if $\sigma(B) \subset \overline{\operatorname{Str}_\omega}$ and for all $\theta > \omega$ there is a $C_\theta > 0$ such that $\|\lambda(\lambda - B)^{-1}\| \leq C_\theta$ for all $\lambda \in \overline{\operatorname{Str}_\theta}^c$. Similarly to the sectorial case, one defines $f(B)$ for $f \in H^\infty(\operatorname{Str}_\theta)$ satisfying a decay for $|\operatorname{Re} \lambda| \rightarrow \infty$ by a Cauchy integral formula, and says that B has a bounded $H^\infty(\operatorname{Str}_\theta)$ calculus provided that $\|f(B)\| \leq C\|f\|_{\infty, \theta}$, in which case $f \mapsto f(B)$ extends to a bounded homomorphism $H^\infty(\operatorname{Str}_\theta) \rightarrow B(X)$. We refer to [5] and [15, Chapter 4] for details. We call B 0-strip-type if B is ω -strip-type for all $\omega > 0$.

There is an analogous statement to Lemma 2.1 which holds for a 0-strip-type operator B and $\operatorname{Str}_\omega$ in place of A and Σ_ω , and $\operatorname{Hol}(\operatorname{Str}_\omega) = \{f : \operatorname{Str}_\omega \rightarrow \mathbb{C} : \exists n \in \mathbb{N} : (\rho \circ \exp)^n f \in H^\infty(\operatorname{Str}_\omega)\}$, where $\rho(\lambda) = \lambda(1 + \lambda)^{-2}$, see [15, p. 91-96].

In fact, 0-strip-type operators and 0-sectorial operators with bounded $H^\infty(\operatorname{Str}_\omega)$ and bounded $H^\infty(\Sigma_\omega)$ calculus are in one-one correspondence by the following lemma. For a proof we refer to [15, Proposition 5.3.3., Theorem 4.3.1 and Theorem 4.2.4, Lemma 3.5.1].

Lemma 7.1. Let B be a 0-strip-type operator and assume that there exists a 0-sectorial operator A such that $B = \log(A)$. This is the case if B has a bounded $H^\infty(\text{Str}_\omega)$ calculus for some $\omega < \pi$. Then for any $f \in \bigcup_{0 < \omega < \pi} \text{Hol}(\text{Str}_\omega)$ one has

$$f(B) = (f \circ \log)(A).$$

Note that the logarithm belongs to $\text{Hol}(\Sigma_\omega)$ for any $\omega \in (0, \pi)$. Conversely, if A is a 0-sectorial operator that has a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \in (0, \pi)$, then $B = \log(A)$ is a 0-strip-type operator.

Let B be a 0-strip-type operator and $\alpha > \frac{1}{2}$. We say that B has a (bounded) W_2^α calculus if there exists a constant $C > 0$ such that

$$\|f(B)\| \leq C \|f\|_{W_2^\alpha} \quad (f \in \bigcap_{\omega > 0} H^\infty(\text{Str}_\omega) \cap W_2^\alpha).$$

In this case, by density of $\bigcap_{\omega > 0} H^\infty(\text{Str}_\omega) \cap W_2^\alpha$ in W_2^α , the definition of $f(B)$ can be continuously extended to $f \in W_2^\alpha$.

Assume that B has a W_2^α calculus. Let $f \in W_{2,\text{loc}}^\alpha$. We define the operator $f(B)$ to be the closure of

$$\begin{cases} D_B \subset X & \longrightarrow X \\ x & \longmapsto \sum_{n \in \mathbb{Z}} (\psi_n f)(B)x, \end{cases}$$

where $D_B = \{x \in X : \exists N \in \mathbb{N} : \psi_n(B)x = 0 \quad (|n| \geq N)\}$ and $(\psi_n)_{n \in \mathbb{Z}}$ is an equidistant partition of unity.

Then there holds an analogous version of Lemma 3.11. Let $\tilde{\mathcal{H}}_2^\alpha = \{f \in L_{\text{loc}}^2(\mathbb{R}) : \|f\|_{\tilde{\mathcal{H}}_2^\alpha} = \sup_{n \in \mathbb{Z}} \|\psi_n f\|_{W_2^\alpha} < \infty\}$. Note that $\tilde{\mathcal{H}}_2^\alpha$ is contained in $W_{2,\text{loc}}^\alpha$. Thus the $W_{2,\text{loc}}^\alpha$ calculus for B enables us to define the $\tilde{\mathcal{H}}_2^\alpha$ calculus: Let $\alpha > \frac{1}{2}$ and B be a 0-strip-type operator. We say that B has an (R -bounded) $\tilde{\mathcal{H}}_2^\alpha$ calculus if there exists a constant $C > 0$ such that

$$\left\{ f(B) : f \in \bigcap_{\omega > 0} H^\infty(\text{Str}_\omega) \cap \tilde{\mathcal{H}}_2^\alpha, \|f\|_{\tilde{\mathcal{H}}_2^\alpha} \leq 1 \right\} \text{ is } (R)\text{-bounded.}$$

The strip-type version of the main Theorems 6.1 and 6.4 reads as follows.

Theorem 7.2. Let B be 0-strip-type operator with H^∞ calculus on some Banach space with property (α) . Denote $U(t)$ the C_0 -group generated by iB and $R(\lambda, B)$ the resolvents of B . For $\alpha > \frac{1}{2}$, consider the condition

$$(C_2)_\alpha \quad B \text{ has an } R\text{-bounded } \tilde{\mathcal{H}}_2^\alpha \text{ calculus.}$$

Furthermore, we consider the conditions

- (a) $_\alpha$ The family $(\langle t \rangle^{-\alpha} U(t) : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$ -bounded.
- (b) $_\alpha$ The family $(R(t + ic, B) : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$ -bounded for any $c \neq 0$ and its bound grows at most like $|c|^{-\alpha}$ for $c \rightarrow 0$.

Then for all $\epsilon > 0$,

$$(C_2)_\alpha \iff (a)_\alpha \implies (b)_\alpha \implies (C_2)_{\alpha+\epsilon}$$

Proof. Consider the 0-sectorial operator $A = e^B$. Then $(C_2)_\alpha \iff (a)_\alpha$ follows from Theorems 6.1 and 6.4.

$(a)_\alpha \implies (b)_\alpha$: Let $R_c = |c|^\alpha R[L^2](R(t + ic, B) : t \in \mathbb{R})$. We have to show $\sup_{c \neq 0} R_c < \infty$. Applying Lemma 5.4 with K the Fourier transform and its inverse, we get

$$R_c = \begin{cases} R[L^2](c^\alpha e^{ct}U(t) : t < 0), & c > 0, \\ R[L^2](|c|^\alpha e^{ct}U(t) : t > 0), & c < 0. \end{cases}$$

For $t < 0$, $\sup_{c > 0} c^\alpha e^{ct} = \sup_{c > 0} \langle t \rangle^{-\alpha} \langle t \rangle^\alpha c^\alpha e^{-|ct|} \lesssim \langle t \rangle^{-\alpha}$. Thus, $\sup_{c > 0} R[L^2](c^\alpha e^{ct}U(t) : t < 0) \lesssim R[L^2](\langle t \rangle^{-\alpha}U(t) : t < 0) < \infty$. The part $c < 0$ is estimated similarly.

$(b)_\alpha \implies (a)_{\alpha+\epsilon}$: Let R_c be as before. Split $\langle t \rangle^{-(\alpha+\epsilon)}U(t)$ into the parts $t \geq 1, t \leq -1, |t| < 1$, and further $t \geq 1$ into $t \in [2^n, 2^{n+1}], n \in \mathbb{N}_0$. Then $\langle t \rangle^{-\alpha} \lesssim 2^{-n\alpha} \lesssim 2^{-n\alpha} e^{-2^{-n}t}$, and by Lemma 5.5 (2),

$$\begin{aligned} R[L^2](\langle t \rangle^{-(\alpha+\epsilon)}U(t) : t \geq 1) &\leq \sum_{n=0}^{\infty} 2^{-n\epsilon} R[L^2](2^{-n\alpha} e^{-2^{-n}t}U(t) : t \in [2^n, 2^{n+1}]) \\ &\leq \sum_{n=0}^{\infty} 2^{-n\epsilon} \sup_{c < 0} R_c < \infty. \end{aligned}$$

The estimate for $t \leq -1$ can be handled similarly. It remains to estimate $R[L^2](\langle s \rangle^{-(\alpha+\epsilon)}U(s) : |s| < 1) \cong R[L^2](U(s) : |s| < 1)$. We have assumed that X has property (α) . Then the fact that B has an H^∞ calculus implies that $\{U(s) : |s| < 1\}$ is R -bounded [20, Corollary 6.6]. For $f \in L^2([-1, 1])$, we have $\|f\|_1 \leq C\|f\|_2$, and consequently,

$$\left\{ \int_{-1}^1 f(s)U(s)ds : \|f\|_2 \leq 1 \right\} \subset C \left\{ \int_{-1}^1 f(s)U(s)ds : \|f\|_1 \leq 1 \right\}.$$

In other words, $(U(s) : |s| < 1)$ is $R[L^2]$ -bounded. \square

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