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ISOMETRISABLE GROUP ACTIONS

ITAI BEN YAACOV AND JULIEN MELLERAY

ABSTRACT. Given a separable metrisable space X , and a group G of homeomorphisms of X , we introduce a topological property of the action $G \curvearrowright X$ which is equivalent to the existence of a G -invariant compatible metric on X . This extends a result of Marjanović obtained under the additional assumption that X is locally compact.

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INTRODUCTION

This paper grew out of the following question: given a metrisable topological space X , and a homeomorphism g of X , how can one determine whether there exists a distance inducing the topology of X and for which g is an isometry? More generally, it is interesting to determine when there exists a compatible invariant distance for an action by homeomorphisms of some group G on X . When this happens we say that the action $G \curvearrowright X$ is *isometrisable*.

When X is compact, this problem is well understood, and various characterisations are available – for instance, in that case an action $G \curvearrowright X$ is isometrisable if and only if it is equicontinuous, in the sense that for any open $U \subseteq X \times X$ containing the diagonal Δ_X , there exists an open $V \subseteq X \times X$ containing Δ_X and such that for all $g \in G$ one has $(g \times g)V \subseteq U$. One could equivalently formalise this by saying that G generates a relatively compact subgroup of the group of homeomorphisms of X , endowed with the topology of uniform convergence for some compatible distance on X .

Beyond that, only the locally compact case seems to be addressed in the literature. Marjanović [Mar69] appears to be the first with a significant result in this direction. In order to formulate it, we recall that, if \mathcal{F} is a family of continuous maps from a topological space X to a topological space Y , \mathcal{F} is said to be *evenly continuous* if for all $x \in X$, all $y \in Y$ and all open $V \ni y$, there exists an open $U \ni x$ and an open W with $y \in W \subseteq V$ and such that

$$\forall f \in \mathcal{F} \ f(x) \in W \Rightarrow f(U) \subseteq V.$$

Theorem (Marjanović [Mar69]). *Let X be a locally compact separable metrisable space, and f be homeomorphism of X . Then there is a compatible distance for which f is an isometry if, and only if, the family $\{f^n : n \in \mathbf{Z}\}$ is evenly continuous from X to its Alexandrov compactification.*

This result was slightly extended by Borges [Bor71] and Kiang [Kia73]; it follows from Kiang’s work that Marjanović’s result extends to arbitrary groups acting on locally compact separable metrisable spaces (though that fact is not explicitly formulated in [Kia73] and could also be deduced from Marjanović’s argument, it is a direct consequence of the main theorem of [Kia73]). One obstacle to extend these results beyond the locally compact case is the presence of the Alexandrov compactification in the statement; another is that

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Marjanović's and Kiang's arguments rely heavily on compactness. To address the first issue, one might try considering a stronger property than even continuity.

Definition (Royden [Roy63]). The action $G \curvearrowright X$ is *topologically equicontinuous* if, for any $x, y \in X$ and any open subset $V \ni y$, there exists open subsets $U \ni x$ and $W \subseteq V$ such that

$$\forall g \in G \quad (gU \cap W \neq \emptyset) \Rightarrow gU \subseteq V .$$

It is obvious that, if $G \curvearrowright X$ is isometrisable, then it is topologically equicontinuous. It is also not hard to check that, when X is locally compact, even continuity of G as a family of maps from X to its Alexandrov compactification is equivalent to topological equicontinuity of G as a family of maps from X to itself. Topological equicontinuity is a strong assumption, and we discuss some consequences in the second section. It appears not to be sufficient for isometrisability of the action $G \curvearrowright X$, leading us to consider an even stronger property.

Definition. We say that $G \curvearrowright X$ is *uniformly topologically equicontinuous* if, for any $y \in X$ and any open $V \ni y$, there exists an open U with $y \in U \subseteq V$ and such that for all $x \in X$ there exists an open neighborhood W of x satisfying

$$\forall g \in G \quad (gW \cap U \neq \emptyset) \Rightarrow gW \subseteq V .$$

Our main result is the following.

Theorem. *Let X be a second countable metrisable space, and G be a group acting on X by homeomorphisms. Then the action $G \curvearrowright X$ is isometrisable if, and only if, it is uniformly topologically equicontinuous.*

1. TOPOLOGICAL EQUICONTINUITY

Throughout the text X stands for a separable metrisable space, and G is a group of homeomorphisms of X .

Lemma 1.1. *Assume that $G \curvearrowright X$ is topologically equicontinuous. Assume that $x_n \rightarrow x$ and $y_n \rightarrow y$ in X , and let $(g_n) \in G^{\mathbb{N}}$ be such $g_n x_n \rightarrow y$. Then $g_n^{-1} y_n \rightarrow x$.*

In particular $g_n x \rightarrow y$ if and only if $g_n^{-1} y \rightarrow x$, so a topologically equicontinuous action is minimal if and only if it is topologically transitive.

Proof. Fix U open containing x , and find V open contained in U and containing x , W open containing y such that $(gW \cap V \neq \emptyset) \Rightarrow gW \subseteq U$ for any $g \in G$. For n large enough $x_n \in V$ and $y_n, g_n x_n \in W$, so $g_n^{-1} W \subseteq U$, and in particular $g_n^{-1} y_n \in U$, as desired. ■

Proposition 1.2. *Assume that $G \curvearrowright X$ is minimal. Then $G \curvearrowright X$ is isometrisable if, and only if, it is topologically equicontinuous.*

Proof. One implication is clear. For the other, assume that $G \curvearrowright X$ is topologically equicontinuous, and denote by τ the topology of X . Consider the family of sets of the form $G \cdot U^2 \subseteq X^2$ where U varies over all nonempty open sets in X .

Since the action is minimal, $G \cdot U^2$ contains the diagonal. Given such a set $G \cdot U^2$, find an open $\emptyset \neq V \subseteq U$ such that $gV \cap V \neq \emptyset \Rightarrow gV \subseteq U$. Assume now that $(x, y), (y, z) \in G \cdot V^2$, say $gx, gy, hy, hz \in V$. Then $gh^{-1}V \cap V \neq \emptyset$, so $gh^{-1}V \subseteq U$. Thus $gz \in U$ and $(x, z) \in G \cdot U^2$.

Thus the sets $G \cdot U^2$ form a basis of entourages for a uniformity, which is metrisable by a G -invariant distance d since it is countably generated by G -invariant entourages, and we claim that it is compatible with the topology on X . Since each $G \cdot U^2$ is open, d is continuous. Conversely, assume that $x_n \rightarrow^d x$. For every neighbourhood U of x we have $(x_n, x) \in G \cdot U^2$ for all n large enough, giving rise to a sequence (g_n) such that $g_n x_n \rightarrow x$ and $g_n x \rightarrow x$. By **Lemma 1.1** we have $x_n \rightarrow x$. Therefore d is a compatible metric. ■

When the action $G \curvearrowright X$ is assumed to be transitive, the above result appears as an exercise in Royden [Roy63].

Next we introduce the ‘‘topological ergodic decomposition’’ associated to $G \curvearrowright X$.

Definition 1.3. For $x, y \in X$, let $[x] = \overline{Gx}$ and say that $x \sim y$ if $x \in [y]$.

Lemma 1.4. *Assume that $G \curvearrowright X$ is topologically equicontinuous. Then the relation \sim is a closed equivalence relation on X (i.e., it is an equivalence relation which is closed as a subset of $X \times X$).*

In particular, $[x] = \{y : x \sim y\}$ and $x \sim y$ if and only if $[x] \cap [y] \neq \emptyset$.

Proof. It is clear that \sim is transitive and reflexive, and when $G \curvearrowright X$ is topologically equicontinuous it is also symmetric, by [Lemma 1.1](#). In order to see that it is closed, assume that $(x_n, y_n) \rightarrow (x, y)$ in X^2 , where $x_n \sim y_n$ for all n . If $U \ni x$ is open, for n large enough we have $x_n \in U$, and since $x_n \sim y_n$, where $g_n \in G$ such that $g_n y_n \in U$. We thus construct a sequence (g_n) with $g_n y_n \rightarrow x$. By [Lemma 1.1](#) we have $g_n^{-1} x \rightarrow y$ and $y \sim x$, as desired. ■

Consequently the quotient space which we denote by $X // G$ is Hausdorff. If the action $G \curvearrowright X$ is isometrisable then this quotient must be metrisable.

Lemma 1.5. *Assume that $G \curvearrowright X$ is topologically equicontinuous. Then the projection map $\pi: X \rightarrow X // G$ is open.*

Proof. Let $U \subseteq X$ be open, $x \in U$ and $y \sim x$. Then $Gy \cap U \neq \emptyset$, or equivalently, $y \in GU$. It follows that the open set GU is the \sim -saturation of U , so πU is open. ■

2. UNIFORM TOPOLOGICAL EQUICONTINUITY AND ISOMETRISABILITY

Metrisability of $X // G$ is obviously a necessary condition for the action $G \curvearrowright X$ to be isometrisable. Outside the realm of locally compact spaces, this seems to require a stronger hypothesis than mere topological equicontinuity.

Definition 2.1. We say that $G \curvearrowright X$ is *uniformly topologically equicontinuous* if for any $x \in X$ and any open $V \ni x$ there exists an open U with $x \in U \subseteq V$ such that for all $y \in X$ there exists an open $W_y \ni y$ satisfying

$$\forall g \in G (gW_y \cap U \neq \emptyset) \Rightarrow gW_y \subseteq V.$$

When the conditions above are satisfied, we say that U witnesses uniform topological equicontinuity for x, V .

This definition is obtained by inverting two quantifiers in the definition of topological equicontinuity, and is still a necessary condition for isometrisability of $G \curvearrowright X$.

Proposition 2.2. *Assume that $G \curvearrowright X$ is uniformly topologically equicontinuous. Then $X // G$ is metrisable.*

Proof. Since X is second countable so is $X // G$, and it will suffice to prove that $X // G$ is regular. In other words, we need to prove that given a closed G -invariant $F \subseteq X$ and $x \notin F$, there exist open sets $U \ni x$ and $W \supseteq F$ such that $U \cap GW = \emptyset$. We choose U which witnesses uniform topological equicontinuity for $x, X \setminus F$, and for each $y \in F$ we let $W_y \ni y$ be the corresponding neighbourhood. If there existed $y \in F$ and $g \in G$ such that $gW_y \cap U \neq \emptyset$ then $gW_y \subseteq X \setminus F$ and in particular $gy \notin F$, a contradiction. Therefore $U \cap \bigcup_{y \in F} GW_y = \emptyset$, which is enough. ■

Given Marjanović's result recalled in the introduction, the following fact is worth mentioning. (If one merely wishes to prove that $X // G$ is metrisable when X is locally compact and the action is topologically equicontinuous, a much shorter argument exists.)

Proposition 2.3. *Let X be a locally compact separable metrisable space, and G a group acting on X by homeomorphisms. The following conditions are equivalent.*

- (i) $G \curvearrowright X$ is uniformly topologically equicontinuous.
- (ii) $G \curvearrowright X$ is topologically equicontinuous.
- (iii) G , seen as a family of maps from X to its Alexandrov compactification X^* , is evenly continuous.

Proof. Note that (iii) is equivalent to saying that, for all $x \in X$ and $y \in X^*$, if (x_i) converges to x and $(g_i x)$ converges to y then $(g_i x_i)$ also converges to y .

The implication (i) \Rightarrow (ii) is by definition. To see that (ii) implies (iii), assume that there exists $x \in X$ and a compact $K \subseteq X$ such that for all open $U \ni x$ and for all compact $L \supseteq K$ there is $g \in G$ such that $g(x) \notin L$ and $g(U) \cap K \neq \emptyset$. From this we may build a sequence (x_i) converging to x and elements $g_i \in G$ such that $g_i(x) \rightarrow \infty$ and $g_i(x_i) \rightarrow k \in K$. This is incompatible with (iii).

It remains to prove that (iii) \Rightarrow (i). We again proceed by contradiction and assume that $G \curvearrowright X$ is not uniformly topologically equicontinuous but G is an evenly continuous family of maps from X to X^* . By assumption, there exists $y \in X$ and an open $V \ni y$ such that for any open U with $y \in U \subseteq V$ there exists $x \in X$ such that for all open $W \ni x$ there exists $g \in G$ with both $gW \cap U \neq \emptyset$ and $gW \not\subseteq V$. Letting U vary over a basis of open neighborhoods of y , we obtain a sequence (x_i) witnessing the above condition; up to extractions we see that there are two cases to consider:

- (x_i) converges to some $x \in X$. Then there exists sequences (g_i) and $(y_i), (z_i)$ converging to x such that $g_i y_i$ converges to y and $g_i z_i$ lives outside V . Up to some extraction, we may assume that $g_i x$ and $g_i z_i$ both converge in X^* , and the fact that $g_i y_i$ and $g_i z_i$ have different limits shows that even continuity must be violated at x .
- (x_i) converges to ∞ , and for all compact K there exists I such that for all $i \geq I$ and all g one has $g x_i \notin K$ (otherwise, replacing x_i by some $g_i x_i$ and going to a subsequence we would be in the situation of the first case above). Letting U be a relatively compact neighborhood of y , we see that for i large enough we have $\overline{G x_i} \cap \overline{U} = \emptyset$. Then the even continuity of G implies that there must exist some neighborhood W of x_i such that $GW \cap \overline{U} = \emptyset$ (by essentially the same argument as above), which contradicts the choice of x_i . ■

Definition 2.4. Let \mathcal{U} be an open cover of X . We say that it is *G-invariant* if for any $U \in \mathcal{U}$ one has $gU \in \mathcal{U}$.

A *basis* of a G -invariant open cover \mathcal{U} is a subset \mathcal{B} such that all elements of \mathcal{U} are of the form gB for some $B \in \mathcal{B}$.

We say that a G -invariant open cover is *G-locally finite* if it admits a basis \mathcal{B} such that for any $x \in X$ there exists a neighborhood A of x such that $\{B \in \mathcal{B} : \exists g \in G \ gB \cap A \neq \emptyset\}$ is finite.

Lemma 2.5. *Assume that $G \curvearrowright X$ is uniformly topologically equicontinuous. Then any G-invariant open cover admits a G-locally finite open refinement.*

Proof. Let \mathcal{U} be a G -invariant open cover. Let also $\pi: X \rightarrow X // G$ denote the open quotient map. Since $X // G$ is metrisable it is paracompact, so we can find a locally finite refinement \mathcal{V} of $\pi\mathcal{U}$. For any $V \in \mathcal{V}$, pick some $U_V \in \mathcal{U}$ such that $\pi(U_V) \supseteq V$, and set $W_V = U_V \cap \pi^{-1}(V)$. Let \mathcal{W} be the G -invariant open cover with basis $\{W_V : V \in \mathcal{V}\}$. By construction \mathcal{W} is an open cover and refines \mathcal{U} .

Now pick any $x \in X$. There is an open neighborhood O of $\pi(x)$ which meets only finitely many elements of \mathcal{V} . If W_V is such that $gW_V \cap \pi^{-1}(O) \neq \emptyset$ for some $g \in G$, then $V \cap O \neq \emptyset$, so \mathcal{W} is G -locally finite. ■

Notation 2.6. To an open cover \mathcal{U} of X we associate an entourage $\mathcal{E}(\mathcal{U}) = \bigcup_{U \in \mathcal{U}} U^2 \subseteq X^2$.

Lemma 2.7. *Assume that $G \curvearrowright X$ is uniformly topologically equicontinuous. Let \mathcal{U} be a G-invariant open cover of X . Then there exists a G-invariant open refinement \mathcal{V} of \mathcal{U} with the property that for all $x, y, z \in X$, if $(x, y), (y, z) \in \mathcal{E}(\mathcal{V})$ then $(x, z) \in \mathcal{E}(\mathcal{U})$.*

Proof. Uniform topological equicontinuity of the action enables us to find a G -invariant open refinement \mathcal{W} of \mathcal{U} with the property that for all $W \in \mathcal{W}$ there exists $U(W) \in \mathcal{U}$ containing W such that for all $y \in X$ there exists an open $C_{W,y} \ni y$ satisfying $gC_{W,y} \cap W \neq \emptyset \Rightarrow gC_{W,y} \subseteq U(W)$. Using [Lemma 2.5](#) we may assume that \mathcal{W} is G -locally finite and \mathcal{B} is a basis of \mathcal{W} witnessing that property. We let \mathcal{V} consist of all open sets V such that for all $g \in G$ and $B \in \mathcal{B}$:

$$gV \cap B \neq \emptyset \Rightarrow gV \subseteq U(B).$$

Given $x \in X$ there exists an open $A \ni x$ such that $\mathcal{B}_A = \{B \in \mathcal{B} : gB \cap A \neq \emptyset\}$ is finite, so $x \in A \cap \bigcap_{B \in \mathcal{B}_A} C_{B,x} \in \mathcal{V}$. Thus \mathcal{V} is a cover, and it is clearly G -invariant and refines \mathcal{U} .

Assume now that $(x, y), (y, z) \in \mathcal{E}(\mathcal{V})$, say $x, y \in V_1$ and $y, z \in V_2$ where $V_i \in \mathcal{V}$. There exist some $B \in \mathcal{B}$ and $g \in G$ such that $gy \in B$, so $gV_1 \cup gV_2 \subseteq U(B)$ and $x, z \in g^{-1}U(B) \in \mathcal{U}$. ■

Lemma 2.8. *Assume that $G \curvearrowright X$ is uniformly topologically equicontinuous, and fix $x \in X$. For any G-invariant open cover \mathcal{U} there exists a G-invariant open refinement of \mathcal{U} with basis \mathcal{B} and $B \in \mathcal{B}$ such that for any $A \in \mathcal{B}$ different from B and any $g \in G$ one has $x \notin gA$.*

Proof. Pick $U \in \mathcal{U}$ such that $x \in U$. Using uniform topological equicontinuity, choose an open neighborhood V of x such that for any $y \in X \setminus GU$ there exists an open set W_y satisfying $gW_y \cap V = \emptyset$ for all $g \in G$. Refining if necessary, we may assume that each W_y is contained in some element of \mathcal{U} ; then $\{W_y : y \in X \setminus GU\} \cup \{U\}$ form a basis for a G -invariant open refinement of \mathcal{U} with the desired property. ■

Lemma 2.9. *Assume that $G \curvearrowright X$ is uniformly topologically equicontinuous. Then for any $x \in X$ there exists a continuous G -invariant pseudometric d_x such that $d_x(x_i, x)$ converges to 0 if and only if (x_i) converges to x .*

Proof. Fix $x \in X$. Using lemmas 2.8 and 2.7, we can build a sequence of G -invariant coverings \mathcal{U}_n of X with basis \mathcal{B}_n with the following properties:

- For each n there exists a unique $B_n \in \mathcal{B}_n$ such that $x \in GB_n$, and $\{B_n\}_n$ forms a basis of neighbourhoods of x .
- For all n , if $(y, z), (z, t) \in \mathcal{E}_{n+1}$ then $(y, t) \in \mathcal{E}_n$, where $\mathcal{E}_n = \mathcal{E}(\mathcal{U}_n)$.

As in the proof of Proposition 1.2, the family of entourages \mathcal{E}_n gives rise to a uniformity which is moreover metrisable by G -invariant pseudo-metric d_x . Since all entourages are open, d_x is continuous.

Assuming that $d_x(x_i, x) \rightarrow 0$, for all n there must exist some $U_n \in \mathcal{U}_n$ such that $x_i, x \in U_n$ for all i large enough, and U_n must be of the form $g_n B_n$. It follows that there exists a sequence h_i with $h_i x_i \rightarrow x$ and $h_i x \rightarrow x$, so $x_i \rightarrow x$ by Lemma 1.1. ■

Theorem 2.10. *Let X be a second-countable metrisable space, and let G act on X by homeomorphisms. Then $G \curvearrowright X$ is isometrisable if and only if it is uniformly topologically equicontinuous.*

Proof. One direction is clear, so we prove the other. Applying Lemma 2.9, we obtain a family of continuous G -invariant pseudometrics $(d_x)_{x \in X}$. Since $d_x(x, x_i) \rightarrow 0$ if and only if $x_i \rightarrow x$, for any open subset U of X and any $x \in U$, there exists $\varepsilon > 0$ such that $d_x(x, y) < \varepsilon \Rightarrow y \in U$. By the Lindelöf property, we obtain that for any open $U \subseteq X$ there exists a countable subset $A \subseteq U$ and a family $(\varepsilon_a)_{a \in A}$ such that

$$U = \bigcup_{a \in A} \{x : d_a(x, a) < \varepsilon_a\}.$$

Applying this to a countable basis for the topology of X , we obtain a countable family of G -invariant pseudometrics which generate the topology, and this countable family may be subsumed into a single G -invariant metric. ■

3. COMPLETE AND INCOMPLETE METRICS

In this section we assume that $G \curvearrowright X$ is isometrisable, and X admits a compatible complete metric. A natural question is then: must $G \curvearrowright X$ admit a complete invariant metric?

The following observation is immediate (and well-known in the case $G \curvearrowright X$ is the left-translation action of G on itself).

Proposition 3.1. *Assume that $G \curvearrowright X$ is minimal. Then there exists a complete compatible G -invariant distance on X if and only if all compatible G -invariant distances are complete.*

Proof. Assume that a compatible complete G -invariant distance exists, fix $x \in X$, and let d be another G -invariant compatible distance. Then a sequence $(g_i x)$ is d -Cauchy if and only if for any neighborhood V of x there exists N such that $g_i^{-1} g_j x \in V$ for all $i, j \geq N$. This property does not depend on d but only on the topology of X , so any d -Cauchy sequence of the form $(g_i x)$ must converge.

Given any d -Cauchy sequence (x_i) , the minimality of $G \curvearrowright X$ enables us to find g_i such that $d(g_i x, x_i) < 2^{-i}$ for all i . Then $(g_i x)$ is also d -Cauchy, hence convergent, and so is (x_i) . ■

The following simple example was suggested by C. Rosendal.

Example 3.2. There exists a Polish space X and a \mathbf{Z} -action on X which is isometrisable but which admits no complete invariant distance.

Proof. Let r be an irrational rotation of the unit circle \mathbf{S} , and let $X = \mathbf{S} \setminus \{r^i(1) : i \in \mathbf{Z}\}$. Then X is a G_δ subset of \mathbf{S} , hence Polish, and the restriction of r to X generates an isometrisable \mathbf{Z} -action; the metric on X induced from the usual metric on \mathbf{S} is both invariant and not complete, so there cannot exist an invariant complete metric on X . ■

As it turns out, the minimal case contains essentially all the obstructions to the existence of a complete invariant metric.

Theorem 3.3. *Assume that X is completely metrisable and the action $G \curvearrowright X$ is isometrisable. Then there exists a compatible complete G -invariant distance on X if and only if there exists such a distance on the closure of each G -orbit.*

Proof. The condition is clearly necessary. Now assume that there exists a compatible complete metric on the closure of each G -orbit, and let d be a G -invariant metric on X . By [Proposition 3.1](#) the restriction of d to each $[x] = \overline{Gx}$ is complete. Also, since the projection map $X \rightarrow X // G$ is open and open maps with range in a metrisable space preserve complete metrisability, there exists a complete distance ρ on $X // G$. Consider the new metric d' defined by

$$d'(x, y) = d(x, y) + \rho([x], [y]).$$

Clearly d' is G -invariant and compatible with the topology of X . Assume now that (x_n) is d' -Cauchy. Since ρ is complete, $[x_n]$ must converge to $[x]$ for some $x \in X$, i.e., there exists a sequence (g_n) such that $g_n x_n \rightarrow x$. By invariance, $d(x_n, g_n^{-1}x) \rightarrow 0$, so $(g_n^{-1}x)$ is a d -Cauchy sequence in $[x]$ which must converge to some y . Therefore $x_n \rightarrow y$ as well, concluding the proof. ■

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