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ISOMORPHISMS OF TENSOR ALGEBRAS OF TOPOLOGICAL GRAPHS

KENNETH R. DAVIDSON AND JEAN ROYDOR

ABSTRACT. We show that if two tensor algebras of topological graphs are algebraically isomorphic, then the graphs are locally conjugate. Conversely, if the base space is at most one dimensional and the edge space is compact, then locally conjugate topological graphs yield completely isometrically isomorphic tensor algebras.

1. INTRODUCTION

In [7], Katsura defined the notion of a topological graph which unifies countable directed graphs and dynamical systems. This enabled him to introduce a new class of C^* -algebras generalizing the classes of graph C^* -algebras and homeomorphism C^* -algebras. These C^* -algebras arise as Cuntz-Pimsner algebras over C^* -correspondences, a construction due to Pimsner [13]. Muhly and Solel have a long series of papers which consider the non-selfadjoint tensor algebra associated to a C^* -correspondence (see [10, 11] for a start) which develop a detailed theory, making this a very tractable, yet rich, family of operator algebras.

This class of tensor algebras contains two well-known classes of operator algebras as special cases: tensor algebras of countable directed graphs and operator algebras associated to dynamical systems in one or several variables. One important direction is the attempt to classify these algebras, and in particular to decide to what extent one can recover the graph or dynamical system from the operator algebra. We continue this program in this paper in the context of topological graphs.

This was successfully done for countable directed graphs, showing that the graph can be completely recovered from the operator algebra. This was accomplished in increasing generality by Kribs and Power [9], Solel [16] and Katsoulis and Kribs [6].

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Operator algebras of dynamical systems has a very long history. The story for nonself-adjoint algebras begins with the work of Arveson [1, 2] where an operator algebra is associated to a dynamical system determined by a single map from a compact Hausdorff space to itself subject to certain additional hypotheses. Peters [12] showed how to define an abstract operator algebra, known as a semicrossed product, to any such discrete dynamical system, and classified them with weaker constraints. Further results by Hadwin and Hoover [5], Power [14] and Davidson and Katsoulis [3] eventually showed that the dynamical system can be completely recovered from the operator algebra up to conjugacy for any proper continuous map of a locally compact Hausdorff space to itself.

In the case of multivariable dynamics, Davidson and Katsoulis [4] define a universal operator algebra that encodes n proper continuous maps of a locally compact Hausdorff space X to itself. This algebra is the tensor algebra of a C*-correspondence. It is shown that isomorphic algebras have dynamical systems which are piecewise conjugate, which is a local version of conjugacy that allows different permutations of the maps in different neighbourhoods. A strong converse, showing that piecewise conjugate systems determine completely isometrically isomorphic algebras, was established if $\dim X \leq 1$ or if the number of maps was at most three.

In the case of topological graphs, it is necessary to formulate an appropriate notion of piecewise conjugacy, that we call local conjugacy. In this paper, we show that if two tensor algebras of topological graphs are algebraically isomorphic, then the topological graphs are locally conjugate. The proof of this result uses the same ingredients as in the multivariable dynamics case: characters and nest representations of the tensor algebras. However the converse proves to be considerably harder. This is because the topological graphs that arise in [4] are topologically much simpler than the general case. We prove that the strong converse holds when the edge space is compact and the base space has dimension at most one.

2. PRELIMINARIES

A topological graph $E = (E^0, E^1, r, s)$ consists of two locally compact spaces E^0, E^1 , a continuous proper mapping $r : E^1 \rightarrow E^0$ and a local homeomorphism $s : E^1 \rightarrow E^0$. The set E^0 is called the base space and E^1 the edge space. A topological graph is called compact if both of these spaces are compact Hausdorff spaces. An important class of examples of topological graphs are multivariable dynamical systems. More precisely, if X is a locally compact Hausdorff space endowed

with n proper continuous selfmaps $(\sigma_1, \dots, \sigma_n)$, then let $E^0 = X$, $E^1 = \{1, \dots, n\} \times X$, let s be the natural projection onto E^0 , and $r = \cup_i \sigma_i$ the union the maps σ_i . This defines a topological graph. More examples of topological graphs can be found in [8].

Following [7], we consider the associated C*-correspondence $\mathcal{X}(E)$ over $C_0(E^0)$. We recall that the right and left actions of $C_0(E^0)$ on $C_c(E^1)$ are given by

$$(f \cdot x \cdot g)(e) = f(r(e))x(e)g(s(e))$$

for $x \in C_c(E^1)$, $f, g \in C_0(E^0)$ and $e \in E^1$. The inner product is defined for $x, y \in C_c(E^1)$ by

$$\langle x, y \rangle(v) = \sum_{e \in s^{-1}(v)} \overline{x(e)}y(e) \quad \text{for } v \in E^0.$$

Finally, $\mathcal{X}(E)$ denotes the completion of $C_c(E^1)$ for the norm

$$\|x\| = \sup_{v \in E^0} \langle x, x \rangle(v)^{1/2}.$$

See [15] for more details.

Looking at the C*-correspondence in [4], it is easy to see that the tensor algebra of a multivariable dynamical system $(X; \sigma_1, \dots, \sigma_n)$ coincides with the tensor algebra of the topological graph described above.

In general, the tensor algebra of a C*-correspondence can be defined by a certain universal property [10]. In our case, this tensor algebra is the universal algebra for pairs of representations (π, t, \mathcal{H}) where $\pi : C_0(E^0) \rightarrow \mathcal{B}(\mathcal{H})$ is a *-homomorphism and $t : \mathcal{X}(E) \rightarrow \mathcal{B}(\mathcal{H})$ is a completely contractive map satisfying *covariance relations*

$$t(f \cdot x) = \pi(f)t(x) \quad \text{and} \quad t(x \cdot f) = t(x)\pi(f).$$

More precisely, $\mathcal{T}(E)_+$ is the unique operator algebra containing $C_0(E^0)$ and $\mathcal{X}(E)$ such that every such pair of representations (π, t, \mathcal{H}) extends uniquely to a completely contractive representation of $\mathcal{T}(E)_+$. We will use repeatedly this universal property to construct characters and nest representations of $\mathcal{T}(E)_+$.

The tensor algebra associated to a C*-correspondence can also be realized concretely as a subalgebra of the adjointable operators on the related Fock space (see [7]). We recall briefly this construction in our case. Let $E = (E^0, E^1, r, s)$ be a topological graph and let $\mathcal{X}(E)$ be the associated C*-correspondence over $C_0(E^0)$. We define

$$\mathcal{F}(E) = C_0(E^0) \oplus \bigoplus_{n \geq 1} \mathcal{X}(E)^n$$

to be the Fock space of E , where $\mathcal{X}(E)^n$ denotes the n -fold tensor product of $\mathcal{X}(E)$. Let $\mathcal{L}(\mathcal{F}(E))$ be the C^* -algebra of adjointable operators on $\mathcal{F}(E)$. The Fock representation (π_∞, t_∞) is defined in the following way: for $f \in C_0(E^0)$ and $x_1 \otimes \cdots \otimes x_n \in \mathcal{X}(E)^n$, set

$$\pi_\infty(f)(x_1 \otimes \cdots \otimes x_n) = (f \cdot x_1) \otimes \cdots \otimes x_n.$$

This is a $*$ -isomorphism from $C_0(E^0)$ into $\mathcal{L}(\mathcal{F}(E))$. Also for $x \in \mathcal{X}(E)$ and $x_1 \otimes \cdots \otimes x_n \in \mathcal{X}(E)^n$, let

$$t_\infty(x)(x_1 \otimes \cdots \otimes x_n) = x \otimes x_1 \otimes \cdots \otimes x_n.$$

This is a completely contractive mapping from $\mathcal{X}(E)$ into $\mathcal{L}(\mathcal{F}(E))$. It is not difficult to check that the pair (π_∞, t_∞) satisfies the covariance relations. The tensor algebra associated to the topological graph E is

$$\mathcal{T}(E)_+ = \overline{\text{span}}\{t_\infty^n(x) : n \geq 0, x \in \mathcal{X}(E)^n\} \subset \mathcal{L}(\mathcal{F}(E))$$

with the conventions $\mathcal{X}(E)^0 = C_0(E^0)$, $t_\infty^0 = \pi_\infty$ and t_∞^n denotes the n -fold tensor product of t_∞ .

3. IDENTIFYING THE BASE SPACE

Following [4], we wish to identify the character space $\mathcal{M}(E)$ of $\mathcal{T}(E)_+$ and the maximal analytic subsets. Recall that an analytic subset of $\mathcal{M}(E)$ is the image of an injective function Φ from a domain Ω in \mathbb{C}^k into $\mathcal{M}(E)$ such that for each $A \in \mathcal{T}(E)_+$, the function $\Phi(z)(A)$ is analytic on Ω . It is a maximal analytic set if it is not contained in any larger analytic set.

Since $\mathcal{T}(E)_+$ contains $C_0(E^0)$, the restriction of any character to this subalgebra yields a point evaluation δ_v for some $v \in E^0$. Let $\mathcal{M}(E)_v$ denote those characters which extend δ_v . Recall that there is a canonical conditional expectation Ξ of $\mathcal{T}(E)_+$ onto $C_0(E^0)$ obtained by integrating the gauge automorphisms. This map is a homomorphism, and thus $\theta_{v,0} := \delta_v \Xi$ is always a character. So $\mathcal{M}(E)_v$ is never empty.

For each $v \in E^0$, we denote

$$E_v^1 = \{e \in E^1 : r(e) = v = s(e)\}.$$

This is a compact set because r is proper, and is discrete because s is a local homeomorphism. Hence it is finite.

Lemma 3.1. *Let $x \in \mathcal{X}(E)$ such that $\text{supp}(x) \cap E_v^1 = \emptyset$. Then $\theta(x) = 0$ for every $\theta \in \mathcal{M}(E)_v$.*

Proof. Note that $\text{supp}(x) \cap s^{-1}(v)$ is finite; and so its image under r is compact and disjoint from v . Thus there is a neighbourhood U_1 of $\text{supp}(x) \cap s^{-1}(v)$ with compact closure so that $r(\overline{U_1})$ is disjoint from v .

Therefore there is a positive function g_1 in $C_0(E^0)$ so that $g_1 r|_{\overline{U_1}} = 1$ and $g_1(v) = 0$. Let $x_1 = g_1 \cdot x$ and $x_2 = x - x_1$.

Observe that $\text{supp}(x_2) \cap U_1 = \emptyset$. So $s(\text{supp}(x_2))$ is compact and disjoint from v . Therefore there is a positive function g_2 in $C_0(E^0)$ so that $g_2 s|_{\text{supp}(x_2)} = 1$ and $g_2(v) = 0$. Note that $x_2 = x_2 \cdot g_2$.

For any $\theta \in \mathcal{M}(E)_v$,

$$\theta(x) = \theta(g_1 \cdot x_1) + \theta(x_2 \cdot g_2) = \delta_v(g_1)\theta(x_1) + \theta(x_2)\delta_v(g_2) = 0. \quad \blacksquare$$

Theorem 3.2. *For $v \in E^0$, let n denote the (finite) cardinality of E_v^1 . If $n = 0$, then $\mathcal{M}(E)_v = \{\theta_{v,0}\}$. If $n \geq 1$, then $\mathcal{M}(E)_v$ is homeomorphic to $\overline{\mathbb{B}}_n$, the closed unit ball of \mathbb{C}^n , and the image of the open ball \mathbb{B}_n is a maximal analytic set. Moreover, these are the only maximal analytic sets.*

Proof. Let $E_v^1 = \{e_i : 1 \leq i \leq n\}$. Since E_v^1 is a finite set, we can select disjoint neighbourhoods $e_i \in V_i \subset \overline{V_i} \subset U_i$ of e_i such that $\overline{U_i}$ is compact and such that s maps each U_i homeomorphically onto a neighbourhood U of v . Pick contractions $\mathbf{t}_i \in C_c(E^1) \subset \mathcal{X}(E)$ with compact support in U_i such that $\mathbf{t}_i|_{V_i} = 1$. Observe that the row operator $\mathbf{t} = [\mathbf{t}_1 \ \dots \ \mathbf{t}_n]$ is a contraction. Consider the map κ defined on $\mathcal{M}(E)_v$ by

$$\kappa(\theta) = (\theta(\mathbf{t}_1), \dots, \theta(\mathbf{t}_n)).$$

As a character θ is completely contractive, the range of κ is contained in $\overline{\mathbb{B}}_n$.

To prove that κ is surjective, let $z = (z_1, \dots, z_n) \in \overline{\mathbb{B}}_n$, and consider the complete contraction t_z defined on $\mathcal{X}(E)$ by

$$t_z(x) = \sum_{i=1}^n z_i x(e_i).$$

Then $(\delta_v, t_z, \mathbb{C})$ is covariant representation. By the universal property of $\mathcal{T}(E)_+$, it extends uniquely to a character $\theta_{v,z} \in \mathcal{M}(E)_v$. Clearly $\kappa(\theta_{v,z}) = z$.

For the injectivity, let $\theta_1, \theta_2 \in \mathcal{M}(E)_v$ such that $\kappa(\theta_1) = \kappa(\theta_2)$. As $\mathcal{T}(E)_+$ is generated by $C_0(E^0)$ and $\mathcal{X}(E)$, it suffices to prove that θ_1 and θ_2 agree on $\mathcal{X}(E)$. Let $x \in \mathcal{X}(E)$ and pick an open set U of E^1 with compact closure such that $E_v^1 \cap \overline{U} = \emptyset$ and $\{U, V_0, V_1, \dots, V_n\}$ forms an open cover of $\text{supp}(x)$. Let $\{p, p_i : 1 \leq i \leq n\}$ be a partition of the unity in $C_c(E^1) \subset \mathcal{X}(E)$ associated to this open cover. Then $x = xp + xp_1 + \dots + xp_n$. By Lemma 3.1,

$$\theta_1(xp) = \theta_2(xp) = 0.$$

Moreover for each $i \geq 1$, $\text{supp}(xp_i) \subset V_i$ so $xp_i = \mathbf{t}_i \cdot b_i$, where b_i is a function in $C_0(E^0)$ extending $xp_i \circ s|_{V_i}^{-1}$. Hence

$$\theta_1(xp_i) = \theta_1(\mathbf{t}_i)b_i(v) = \theta_2(\mathbf{t}_i)\theta_2(b_i) = \theta_2(xp_i).$$

This is true for each i , so $\theta_1 = \theta_2$.

Thus κ is a bijection. We denote the character $\kappa^{-1}(z)$ as $\theta_{v,z}$. Now for $x \in \mathcal{X}(E)$, $\theta_{v,z}(x) = t_z(x)$ is a linear function. Thus the extension to $\sum_{k \geq 0} t_\infty^k (\mathcal{X}(E)^k)$ takes polynomial values. A sequence of such elements converging to $T \in \mathcal{T}(E)_+$ converges uniformly on $\mathcal{M}(E)$ because characters are (completely) contractive. It follows that the function $\hat{T}(z) := \theta_{v,z}(T)$ is analytic on the relative interior \mathcal{M}_v^0 of \mathcal{M}_v , which is identified with the open ball \mathbb{B}_n . Thus this is an analytic set.

Finally if U is an analytic subset of $\mathcal{M}(E)$, then the restriction to $C_0(E^0)$ is analytic, which forces it to lie in some \mathcal{M}_v . Thus it is clear that it is a subset of \mathcal{M}_v^0 . So we have identified the maximal analytic sets. \blacksquare

As in [4], we observe that the quotient space of $\mathcal{M}(E)$ obtained by mapping the closure of each maximal analytic set to a point is homeomorphic to E^0 . Hence we deduce the following important corollary.

Corollary 3.3. *The character space $\mathcal{M}(E)$ determines E^0 up to homeomorphism.*

4. LOCAL CONJUGACY

For v, w in E^0 , we denote

$$E_{v,w}^1 = \{e \in E^1 : r(e) = v \text{ and } s(e) = w\} = r^{-1}(v) \cap s^{-1}(w).$$

This is a finite set because $r^{-1}(v)$ is compact, and $s^{-1}(w)$ is a discrete set. Denote its cardinality by $n_{v,w}$.

Recall that a nest representation is a homomorphism of the algebra into the 2×2 upper triangular matrices with respect to a basis e_1, e_2 such that the only proper invariant subspace is $\mathbb{C}e_1$. The two diagonal entries are characters, and the 1, 2 entry is a derivation. Let $\text{Rep}_{\mathcal{T}_2}(E)$ denote the set of all 2-dimensional nest representations of $\mathcal{T}(E)_+$. Let $\text{Rep}_{v,w}(E)$ be the set of 2-dimensional nest representations of $\mathcal{T}(E)_+$ such that the 1, 1 entry belongs to $\mathcal{M}(E)_v$ and the 2, 2 entry belongs to $\mathcal{M}(E)_w$. We also denote $\text{Rep}_{v,w}^d(E)$ the subset of $\text{Rep}_{v,w}(E)$ of representations which are diagonal when restricted to $C_0(E^0)$.

Lemma 4.1. *$\text{Rep}_{v,w}(E)$ is non-empty if and only if $E_{v,w}^1 \neq \emptyset$. Moreover, if $x \in \mathcal{X}(E)$ is such that $\text{supp}(x) \cap E_{v,w}^1 = \emptyset$, then $\rho(x)$ is diagonal for every $\rho \in \text{Rep}_{v,w}^d(E)$.*

Proof. First assume that $E_{v,w}^1$ is non-empty. Let λ_e for $e \in E_{v,w}^1$ be complex scalars such that $0 < \sum |\lambda_e| \leq 1$. Define a $*$ -homomorphism $\pi : C_0(E^0) \rightarrow \mathbb{M}_2$ by

$$\pi(f) = \begin{bmatrix} f(v) & 0 \\ 0 & f(w) \end{bmatrix}$$

and a map $t : \mathcal{X}(E) \rightarrow \mathbb{M}_2$ by

$$t(x) = \begin{bmatrix} 0 & \sum_{e \in E_{v,w}^1} \lambda_e x(e) \\ 0 & 0 \end{bmatrix}$$

This is contractive (as it is a convex combination of contractive maps). Thus t is completely contractive because it's 1-dimensional.

Clearly

$$t(f \cdot x) = \sum \lambda_e f(r(e))x(e) = \pi(f)t(x)$$

and

$$t(x \cdot f) = \sum \lambda_e x(e)f(s(e)) = t(x)\pi(f).$$

Hence by the universal property of the tensor algebra, this pair of representations extends to a completely contractive representation ρ of $\mathcal{T}(E)_+$. The range of ρ is the entire upper triangular 2×2 matrices because some λ_e is non-zero. If $x \in \mathcal{X}(E)$ satisfies $\text{supp}(x) \cap E_{v,w}^1 = \emptyset$, then ρ is diagonal. So ρ belongs to $\text{Rep}_{v,w}^d(E)$.

Conversely suppose that $\text{Rep}_{v,w}(E)$ is non-empty. Then applying a similarity (as in Lemma 4.2 below) yields an element of $\text{Rep}_{v,w}^d(E)$. Let $x \in \mathcal{X}(E)$ such that $\text{supp}(x) \cap E_{v,w}^1 = \emptyset$. We will prove that $\rho(x)$ is diagonal for every ρ in $\text{Rep}_{v,w}^d(E)$. Arguing as in Lemma 3.1, we can split $x = g_1 \cdot x_1 + x_2 \cdot g_2$ so that $g_1(v) = 0$ and $g_2(w) = 0$. Thus $\rho(x)$ has the form

$$\begin{aligned} \rho(x) &= \rho(g_1)\rho(x_1) + \rho(x_2)\rho(g_2) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} + \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \end{aligned}$$

If $E_{v,w}^1$ were empty and $\rho \in \text{Rep}_{v,w}^d(E)$, then $\rho(x)$ is diagonal for every $x \in \mathcal{X}(E)$. Thus ρ is not a nest representation, contrary to assumption. Hence $E_{v,w}^1$ is not empty. \blacksquare

We also need a technical result which is a minor modification of [4, Lemma 3.13] using $\sigma = r_{E^0} s_{E^0}^{-1}$.

Lemma 4.2. *Let $E = (E^0, E^1, s, r)$ be a topological graph. Let $L \subset E^1$, $K \subset E^0$ two compact subsets such that $s_E|_L$ maps L onto K homeomorphically. Set $\sigma = r_E s_E|_L^{-1}$. Let Ω be a domain in \mathbb{C}^d . Suppose that there exists a map ρ from $K \times \Omega$ to $\text{Rep } \mathcal{T}(E)_+$ such that*

- (a) $\rho(x, z) \in \text{Rep}_{\sigma(x), x}$,
- (b) ρ is continuous in the point-norm topology,
- (c) for each $x \in K$, $\rho(x, z)$ is analytic in $z \in \Omega$.

Then there exists a map A from $K \times \Omega$ onto the group of invertible upper triangular matrices, so that

- (1) $A(x, z)\rho(x, z)(\cdot)A(x, z)^{-1} \in \text{Rep}_{\sigma(x), x}^d$,
- (2) A is continuous on $K \setminus \{x : \sigma(x) = x\} \times \Omega$,
- (3) $A(x, z) = I_2$ when $\sigma(x) = x$,
- (4) for each $x \in K$, $A(x, z)$ is analytic in $z \in \Omega$,
- (5) $\max \{ \|A(x, z)\|, \|A(x, z)^{-1}\| \} \leq 1 + \|\rho(x, z)\|$.

Now we define a notion which extends the notion of piecewise conjugacy introduced in [4]. To paraphrase, local conjugacy means that the base spaces are homeomorphic and there is an open cover of the base spaces which lift to local intertwining of the two graphs. From now on, we will need to assume that the edge space is compact.

Definition 4.3. Let $E = (E^0, E^1, s_E, r_E)$ and $F = (F^0, F^1, s_F, r_F)$ be two compact topological graphs. They are said to be *locally conjugate* if there exists a homeomorphism $\tau : E^0 \rightarrow F^0$ such that for every $u \in E^0$, there is a neighbourhood U of u , a homeomorphism γ of $s_E^{-1}(U)$ onto $s_F^{-1}(\tau U)$ such that $s_F \gamma = \tau s_E|_{s_E^{-1}(U)}$ and $r_F \gamma = \tau r_E|_{s_E^{-1}(U)}$.

Remark 4.4. While we will only deal with compact topological graphs in this paper, if E^1 and F^1 are not compact, we would modify the definition of local conjugacy to say: there exists a homeomorphism $\tau : E^0 \rightarrow F^0$ such that for every $v \in E^0$, and for every open set O in E^0 such that \overline{O} is compact, there is a neighbourhood V of v , a homeomorphism h_V from $r_E^{-1}(O) \cap s_E^{-1}(V)$ onto $r_F^{-1}(\tau(O)) \cap s_F^{-1}(\tau(V))$ such that $s_F h_V = \tau s_E$ and $r_F h_V = \tau r_E$.

To prove the classification result, we need an equivalence relation between edges. Two edges $e, e' \in E^1$ are said to be equivalent if there exist U neighbourhood of e , U' neighbourhood of e' such that s_E restricted to both U and U' is a homeomorphism and $s_E(U) = s_E(U')$ and for every $v \in s(U)$, $r_E s_E|_{U^{-1}(v)} = r_E s_E|_{U'^{-1}(v)}$. In this case, we denote $e \sim e'$. When there is a homeomorphism $\tau : E^0 \rightarrow F^0$, this equivalence relation naturally extends to an equivalence relation between edges of E^1 and F^1 (relative to τ).

The proof follows the lines of [4, Theorem 3.22] with some additional technicalities.

Theorem 4.5. *Let $E = (E^0, E^1, s_E, r_E)$ and $F = (F^0, F^1, s_F, r_F)$ be two compact topological graphs. If $\mathcal{T}(E)_+$ and $\mathcal{T}(F)_+$ are algebraically isomorphic, then E and F are locally conjugate.*

Proof. Let γ be an isomorphism of $\mathcal{T}(E)_+$ onto $\mathcal{T}(F)_+$. This induces a canonical map γ_c from $\mathcal{M}(E)$ onto $\mathcal{M}(F)$ between the character spaces. It also induces a map γ_r from $\text{Rep}_{\mathcal{T}_2}(E)$ onto $\text{Rep}_{\mathcal{T}_2}(F)$.

Since $\mathcal{M}(E)$ is endowed with the weak- $*$ topology, it is easy to see that γ_c is continuous. Indeed, if θ_α is a net in $\mathcal{M}(E)$ converging to θ and $B \in \mathcal{T}(F)_+$, then

$$\lim_{\alpha} \gamma_c \theta_\alpha(B) = \lim_{\alpha} \theta_\alpha(\gamma^{-1}(B)) = \theta(\gamma^{-1}(B)) = \gamma_c \theta(B).$$

The same holds for γ_c^{-1} . So γ_c is a homeomorphism.

Observe that γ_c carries analytic sets to analytic sets. Indeed, if Θ is an analytic function of a domain Ω into $\mathcal{M}(E)$, then

$$\gamma_c \Theta(z)(B) = \Theta(z)(\gamma^{-1}(B))$$

is analytic for every $B \in \mathcal{T}(F)_+$; and thus $\gamma_c \Theta$ is analytic. Since the same holds for γ^{-1} , it follows that γ_c takes maximal analytic sets to maximal analytic sets. Thus it carries their closures, $\mathcal{M}(E)_u$, onto sets the corresponding sets $\mathcal{M}(F)_v$.

By Corollary 3.3, γ_c induces a homeomorphism τ of E^0 onto F^0 . For simplicity of notation, we identify E^0 and F^0 via τ , so that $F^0 = E^0$ and $\tau = \text{id}$.

Fix $v_0 \in E^0$. Since s_E and s_F are local homeomorphisms and E^1 and F^1 are compact, both $s_E^{-1}(v_0) = \{e_1, \dots, e_p\}$ and $s_F^{-1}(v_0) = \{f_1, \dots, f_q\}$ are finite sets. Let \mathcal{G} denote the equivalence class of e_1 among $\{e_1, \dots, e_p, f_1, \dots, f_q\}$. After relabeling

$$\mathcal{G} = \{e_1, \dots, e_k, f_1, \dots, f_l : e_1 \sim \dots \sim e_k \sim f_1 \sim \dots \sim f_l\}.$$

By exchanging E and F if necessary, we can suppose that $k \geq l$. Our purpose is to prove that $k = l$. So, suppose that $k > l$.

As in [7, Lemma 1.4], there exists a neighbourhood W of v_0 , U_i disjoint neighbourhoods of e_i ($1 \leq i \leq p$), V_j disjoint neighbourhoods of f_j ($1 \leq j \leq q$) such that $s_E(U_i) = W = s_F(V_j)$ homeomorphically, and

$$s_E^{-1}(W) = \bigcup_{i=1}^p U_i \quad \text{and} \quad s_F^{-1}(W) = \bigcup_{j=1}^q V_j$$

and

$$\sigma := r_E s_E|_{U_1}^{-1} = r_E s_E|_{U_i}^{-1} = r_F s_F|_{V_j}^{-1} \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq l.$$

If $\sigma(v_0) \neq v_0$, we also choose W so that $\sigma(\overline{W}) \cap \overline{W} = \emptyset$.

For each $v \in W$ and $z = (z_1, \dots, z_k)$ in the \mathbb{C}^k , define a representation in $\text{Rep}_{w,v}^d(E)$ by

$$\rho_{v,z} = \begin{bmatrix} \delta_{\sigma(v)} & \sum_{i=1}^k z_i \delta_{e_i} \\ 0 & \delta_v \end{bmatrix}$$

This is a continuous family of nest representations which is analytic in the second variable. Here $\rho_{v,z}$ is completely contractive if $\|z\|_2 \leq 1$ and bounded by $1 + \|z\|_2$ in general.

Let $\eta_{v,z} := \gamma_r \rho_{v,z} = \rho_{v,z} \circ \gamma^{-1}$. This is a nest representation in $\text{Rep}_{\sigma(v),v}(F)$ with $\|\eta(x,z)\| \leq (1 + \|z\|_2)\|\gamma^{-1}\|$. Apply Lemma 4.2 to obtain a matrix valued function $A(v,z)$ such that

$$\Phi(v,z) = A(v,z)\eta_{v,z}A(v,z)^{-1} \in \text{Rep}_{w,v}^d(F)$$

which is continuous on $W \setminus \{x : \sigma(x) = x\} \times \mathbb{C}^k$, analytic in the second variable, and such that

$$\max \{ \|A(x,z)\|, \|A(x,z)^{-1}\| \} \leq 1 + \|\eta(x,z)\|.$$

Now for $1 \leq j \leq q$, pick a contraction $\mathbf{t}_j \in \mathcal{C}_c(F^1)$ with $\text{supp}(\mathbf{t}_j) \subset V_j$ and a compact neighbourhood $\tilde{V}_j \subset V_j$ of f_j such that $\mathbf{t}_j|_{\tilde{V}_j} = 1$. Define $\psi_j(z)$ to be the $(1,2)$ entry of $\Phi(v_0,z)(\mathbf{t}_j)$, and let

$$\Psi(z) = (\psi_1(z), \dots, \psi_q(z)).$$

This is an analytic function from unit ball of \mathbb{C}^q into \mathbb{C}^q .

We claim that $\psi_j(z) = 0$ for $j > l$. If $j > l$, then f_j is not in \mathcal{G} . Hence there is a net $(v_\lambda) \in W$ converging to v_0 such that

$$r_F s_F|_{V_j}^{-1}(v_\lambda) \neq \sigma(v_\lambda).$$

Note that

$$\text{supp}(\mathbf{t}_j) \cap s_F^{-1}(v_\lambda) \subset V_j \cap s_F^{-1}(v_\lambda) = \{s_F|_{V_j}^{-1}(v_\lambda)\}.$$

Therefore, $\text{supp}(\mathbf{t}_j) \cap F_{\sigma(v_\lambda),v_\lambda}^1 = \emptyset$. Hence by Lemma 4.1, $\Phi(v_\lambda,z)(\mathbf{t}_j)$ is diagonal for all λ and all $z \in \mathbb{C}^k$.

Now there are two cases. If $\sigma(v_0) \neq v_0$, then A is continuous (since $\sigma(\overline{W}) \cap \overline{W} = \emptyset$). Hence $\Phi(v,z)$ is point-norm continuous. Taking a limit in λ , we obtain that $\Phi(v_0,z)(\mathbf{t}_j)$ is diagonal, whence $\psi_j(z) = 0$.

In the second case, $\sigma(v_0) = v_0$. Thus the diagonal elements of the range of $\Phi(v_0,z)$ are scalars. For z fixed, the function $A(v,z)$ and its inverse are bounded. Therefore we can pass to subnet so that

$A(z) := \lim_{\lambda} A(v_{\lambda}, z)$ exists in the the group of invertible upper triangular matrices. As $\eta_{v,z} = \rho_{v,z} \circ \gamma^{-1}$ is point-norm continuous and $A(v_0, z) = I_2$, passing to the limit we deduce that $A(z)\eta_{v,z}(\mathbf{t}_j)A(z)^{-1}$ is diagonal, hence scalar. So $\Phi(v_0, z)(\mathbf{t}_j)$ is scalar, which means that $\psi_j(z) = 0$.

We may now consider Ψ as a function from \mathbb{C}^k into \mathbb{C}^l . We have $\Psi(0) = 0$. As in [4, Proposition 3.21], it follows that this cannot be an isolated zero when $l < k$. Thus there is some $z_0 \neq 0$ such that $\Psi(z_0) = 0$. As in the proof of Theorem 3.2, every $x \in \mathcal{X}(F)$ can be written as a sum $x = x_0 + \sum_{j=1}^q \mathbf{t}_j \cdot f_j$ where x_0 vanishes on a neighbourhood of $s_F^{-1}(v_0)$ and $f_j \in C_0(F^0)$. It follows from Lemma 4.1 that $\Phi(v_0, z_0)(x_0)$ is diagonal. As each $\Phi(v_0, z_0)(\mathbf{t}_j)\Phi(v_0, z_0)(f_j)$ is also diagonal, we deduce that $\Phi(v_0, z_0)$ is diagonal—and hence is not a nest representation. This contradiction shows that $l = k$.

Therefore we can partition $\{e_1, \dots, e_p\}$ so they agree on a common neighbourhood W of v_0 , and these equivalence classes will be paired with a corresponding partition of $\{f_1, \dots, f_q\}$. In particular, $q = p$. After reordering, we may suppose that $e_i \sim f_i$ for $1 \leq i \leq p$. Thus there are disjoint neighbourhoods U_i of e_i and V_i of f_i for $1 \leq i \leq p$ such that $s_E(U_i) = W = s_F(V_j)$ homeomorphically,

$$s_E^{-1}(W) \subset \bigcup_{i=1}^p U_i \quad \text{and} \quad s_F^{-1}(W) \subset \bigcup_{j=1}^q V_j,$$

and $r_E s_E|_{U_i}^{-1} = r_F s_F|_{V_i}^{-1}$. The map $\gamma_i = s_F|_{V_i}^{-1} s_E|_{U_i}$ is a homeomorphism of U_i onto V_i so that

$$s_F \gamma_i = s_E|_{U_i} \quad \text{and} \quad r_F \gamma_i = r_E|_{U_i}.$$

Hence the map $\gamma = \bigcup_{i=1}^p \gamma_i$ is the desired homeomorphism of $s_E^{-1}(W)$ onto $s_F^{-1}(W)$ establishing local equivalence of E and F . \blacksquare

5. THE CONVERSE

In this section, we discuss the converse of Theorem 4.5, namely, if two topological graphs E and F are locally conjugate then are their associated tensor algebras isomorphic (as Banach algebras)? We are able to prove this converse when E^0 or F^0 has topological dimension less or equal to 1 and E^1 and F^1 are compact. In this case, the tensor algebras are completely isometrically isomorphic.

We begin by constructing a particularly good open cover of E .

Definition 5.1. Let E and F be two locally conjugate compact topological graphs with $\dim(E^0) \leq 1$ via a homeomorphism τ . An open cover $\mathcal{U} = (U_i)_{i \leq n}$ of E^0 is called *admissible* if:

- (C1) For each i , there is an integer $m(i)$ and disjoint open sets U_{ik} of E^1 for $1 \leq k \leq m(i)$ such that $s_E^{-1}(U_i) = \bigcup_{k=1}^{m(i)} U_{ik}$ and s_E maps each U_{ik} homeomorphically onto U_i .
- (C2) For each $1 \leq i \leq n$, there is a homeomorphism γ_i from $s_E^{-1}(U_i)$ onto $s_F^{-1}(\tau U_i)$ such that $s_F \gamma_i = \tau s_E|_{s_E^{-1}(U_i)}$ and $r_F \gamma_i = \tau r_E|_{s_E^{-1}(U_i)}$.
- (C3) $U_i \cap U_j \cap U_k = \emptyset$, whenever $i \neq j \neq k \neq i$.
- (C4) $\overline{U_i} \setminus U_j$ and $\overline{U_j} \setminus U_i$ are disjoint for $i \neq j$.
- (C5) If $U_i \cap U_j \neq \emptyset$, then there is a permutation $\pi \in S_{m(i)}$ so that $U_{ik} \cap U_{j\pi(k)} \neq \emptyset$ if and only if $l = \pi(k)$.
- (C6) The corresponding open cover of F^0 and F^1 obtained by setting $V_i = \tau(U_i)$ and $V_{ik} = \gamma_i(U_{ik})$ also satisfies (C5).

Note that $\{U_{ik} : 1 \leq i \leq n, 1 \leq k \leq m(i)\}$ is an open cover of E^1 . The property (C1) is easily achieved in general. It implies that $m(i) = m(j)$ if U_i and U_j intersect. Thus this is a locally constant function.

(C2) is just a restatement of local equivalence. The fact that γ_i intertwines the source maps s_E and s_F and takes U_{ik} onto V_{ik} means that

$$\gamma_i|_{U_{ik}} = (s_F|_{V_{ik}})^{-1} s_E|_{U_{ik}} \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq k \leq m(i).$$

Property (C3) is an expression of $\dim(E^0) \leq 1$. Properties (C4) and (C5) may be achievable in general, but our proof will use the low dimension in a key way.

Property (C5) means that U_{ik} intersects exactly one of the $U_{j\pi(k)}$, not some more complicated mixing. This seems essential in order to be able to understand what is happening. If the sets $U_i \cap U_j$ were connected, then this would be automatic. But in the dimension 0 case, that can't happen.

The open cover of F^0 and F^1 obtained as defined in (C6) is easily seen to satisfy (C1,3,4). However property (C5) is not automatically transferred.

Theorem 5.2. *Suppose that E is compact topological graph where the covering dimension of E^0 is at most 1 and that E is locally conjugate to another topological graph F via a homeomorphism τ . Then there exists an admissible open cover $(U_i)_{1 \leq i \leq n}$ of E^0 .*

Proof. The definition of local equivalence shows that for each $u \in E^0$, there is an open neighbourhood U and a homeomorphism γ from $s_E^{-1}(U)$ onto $s_F^{-1}(\tau U)$ such that $s_F \gamma = \tau s_E|_{s_E^{-1}(U)}$ and $r_F \gamma = \tau r_E|_{s_E^{-1}(U)}$.

Now $s_E^{-1}(u)$ is finite, say $\{e_j : 1 \leq j \leq m\}$, and s_E is a local homeomorphism. Thus there is an open set $\tilde{U} \subset U$ containing u and disjoint open sets V_j containing e_j so that $s_E|_{V_j}$ is a homeomorphism onto \tilde{U} . Let $U' = \tilde{U} \setminus s_E(E^1 \setminus \bigcup_{j=1}^m V_j)$. This is open, contains u , and $s_E^{-1}(U') = \bigcup_{j=1}^m V'_j$ where $V'_j = s_E^{-1}(U') \cap V_j$, and s_E maps each V'_j homeomorphically onto U' . The restriction of γ to $s_E^{-1}(U')$ is the desired intertwining map.

By the compactness of E^0 , there is a finite subcover $(U'_i)_{i \leq n}$ of sets constructed in the previous paragraph. Thus this cover satisfies (C1) and (C2). Since E^0 has covering dimension at most 1, this cover can be further refined so that (C3) holds as well.

Now we make another modification to obtain (C4). Let f_i be a partition of unity in $C(E^0)$ such that $\text{supp}(f_i) \subset U'_i$. Observe that at most two of the $f_i(u)$ can be non-zero for any $u \in E^0$. Thus for each $u \in E^0$, there is some i with $f_i(u) > 1/3$. Set $U_i = \{u : f_i(u) > 1/3\}$. This is a finer cover of E^0 with $\overline{U}_i \subset U'_i$.

Fix $i \neq j$, and consider $u \in (\overline{U}_i \setminus U_j) \cap (\overline{U}_j \setminus U_i)$. Then

$$\frac{1}{3} \leq f_i(u) \leq \frac{1}{3} \quad \text{and} \quad \frac{1}{3} \leq f_j(u) \leq \frac{1}{3}.$$

So $f_i(u) = f_j(u) = 1/3$ and $f_p(u) = 0$ for $i \neq p \neq j$. This contradicts $\sum f_i = 1$, so the intersection is empty and (C4) holds and properties (C1-3) are not affected.

Suppose that $U_i \cap U_j \neq \emptyset$. By Tietze's extension theorem, there is a continuous function f in $C(E^0, [0, 1])$ such that $f|_{\overline{U}_i \setminus U_j} = 1$ and $f|_{\overline{U}_j \setminus U_i} = 0$.

For each $1 \leq k, l \leq m(i)$, let $W_{kl} = U_{ik} \cap U_{jl}$. Observe that for fixed k , $U_{ik} \cap s_E^{-1}(U_i \cap U_j)$ is the disjoint union of W_{kl} for $1 \leq l \leq m(i)$. So $\{s_E(W_{kl}) : 1 \leq l \leq m(i)\}$ partitions $U_i \cap U_j$ into disjoint open sets. Thus $\{s_E(W_{kl}) : 1 \leq k, l \leq m(i)\}$ partitions $U_i \cap U_j$ into disjoint open sets. Likewise the sets $V_{ik} = \gamma_i(U_{ik})$ and $V_{jl} = \gamma_j(U_{jl})$ determine open sets $X_{kl} = V_{ik} \cap V_{jl}$. Thus we obtain a partition of $U_i \cap U_j$ into disjoint open sets $\{Y_r : 1 \leq r \leq r(i, j)\}$ determined by

$$\{s_E(W_{kl}), \tau^{-1}s_F(X_{kl}) : 1 \leq k, l \leq m(i)\}.$$

Using the function f constructed above, define

$$U'_i = \{u \in U_i : f(u) > .6\} \quad \text{and} \quad U'_j = \{u \in U_j : f(u) < .4\},$$

and set

$$Y'_r = \{u \in Y_r : .3 < f(u) < .7\} \text{ for } 1 \leq r \leq r(i, j).$$

Replace U_i and U_j in the cover by these new sets. For each $r \leq r(i, j)$ and $k \leq m(i)$, set $Y_{rk} = s_E^{-1}(Y'_r) \cap U_{ik}$. Note that if $i \neq p \neq j$, then

$U_p \cap U'_i = U_p \cap U_i$, $U_p \cap U'_j = U_p \cap U_j$ and $U_p \cap Y'_r = \emptyset$. In addition, $U'_i \cap U'_j = \emptyset$. It is clear by construction that $U_{ik} \cap W_{rl}$ is empty except for $l = k$. Likewise $W_{rk} \cap U_{jl}$ is empty except for a single choice of $l = \pi(k)$. The same argument shows that the corresponding cover of F satisfies (C5).

Treat all intersecting pairs U_i and U_j from the original cover in this manner, and the new cover constructed satisfies (C5). This does not affect (C1-3). If necessary, go through the construction to get (C4) again. Reducing each open set does not affect (C1-3,5). So we are done. \blacksquare

It is clear that $m(i) = m(j)$ if U_i and U_j intersect. So E^0 splits into a finite number of clopen subsets on which $m(i)$ is constant. Clearly it suffices to deal with each set separately. So we may suppose that $m(i) = m$ is constant.

Let $\mathcal{U} = (U_i)_{1 \leq i \leq n}$ be an admissible open cover of E^0 . Note that if $U_i \cap U_j \neq \emptyset$, then there is a unique permutation $\pi_{i,j}^E \in S_m$ defined by $\pi_{i,j}^E(k) = l$ if and only if $U_{ik} \cap U_{jl} \neq \emptyset$. We denote $\Pi_{E,\mathcal{U}}$ the function defined on a subset of $\{1, \dots, n\} \times \{1, \dots, n\}$ by $\Pi_{E,\mathcal{U}}(i, j) = \pi_{i,j}^E$ when U_i and U_j intersect. Obviously, $\pi_{j,i}^E = (\pi_{i,j}^E)^{-1}$.

First we record an easy lemma to set the stage.

Lemma 5.3. *Suppose that E and F are two locally conjugate compact topological graphs, and that E^0 has dimension at most 1. Let $\mathcal{U} = (U_i)_{1 \leq i \leq n}$ denote an admissible open cover of E^0 with $m(i) = m$ for $1 \leq i \leq n$, and let \mathcal{V} be the corresponding cover for F . We assume that $\Pi_{E,\mathcal{U}} = \Pi_{F,\mathcal{V}}$. Then $\mathcal{X}(E)$ and $\mathcal{X}(F)$ are unitarily equivalent as C^* -correspondences; and $\mathcal{T}(E)_+$ and $\mathcal{T}(F)_+$ are completely isometrically isomorphic.*

Proof. The map $\gamma = \bigcup_{1 \leq i \leq n} \gamma_i$ is a well defined homeomorphism of E^1 onto F^1 because the assumption $\Pi_{E,\mathcal{U}} = \Pi_{F,\mathcal{U}}$ means that γ_i and γ_j agree on $s_E^{-1}(U_i \cap U_j)$. Moreover $s_F \gamma = \tau s_E$ and $r_F \gamma = \tau r_E$. This induces a unitary isomorphism Γ from $\mathcal{X}(E)$ onto $\mathcal{X}(F)$ given by $\Gamma(x) = x \circ \gamma^{-1}$. To see that Γ is a bimodule map, note that τ induces a $*$ -isomorphism ρ of $C(E^0)$ onto $C(F^0)$ by $\rho(f) = f \circ \tau^{-1}$. Moreover, $\tau^{-1} s_F = s_E \gamma_i^{-1}$ and $\tau^{-1} r_F = r_E \gamma_i^{-1}$. Thus if $g \in C(E^0)$ and $x \in \mathcal{X}(E)$, then

$$\begin{aligned} \Gamma(x \cdot g) &= (x \circ \gamma_i^{-1})(g \circ s_E \gamma_i^{-1}) \\ &= \Gamma(x)(g \circ \tau^{-1} s_F) = \Gamma(x) \cdot \rho(g). \end{aligned}$$

Similarly, $\Gamma(g \cdot x) = \rho(g) \cdot \Gamma(x)$. Finally this map is isometric because if $v \in E^0$ and $s_E^{-1}(v) = \{e_i : 1 \leq i \leq m\}$, then

$$s_F^{-1}(\tau v) = \{f_i = \gamma(e_i) : 1 \leq i \leq m\}.$$

So

$$\begin{aligned} \langle \Gamma(x), \Gamma(x) \rangle(\tau v) &= \sum_{i=1}^m |\Gamma(x)(f_i)|^2 \\ &= \sum_{i=1}^m |x(e_i)|^2 = \langle x, x \rangle(v) \end{aligned}$$

Thus $\langle \Gamma(x), \Gamma(x) \rangle = \rho(\langle x, x \rangle)$. It follows that this is a unitary equivalence of C^* -correspondences. Hence $\mathcal{T}(E)_+$ and $\mathcal{T}(F)_+$ are completely isometrically isomorphic. \blacksquare

Now we modify this proof to get the key step in the proof of the converse.

Lemma 5.4. *Suppose that E and F are two locally conjugate compact topological graphs, and that E^0 has dimension at most 1. Let $\mathcal{U} = (U_i)_{1 \leq i \leq n}$ denote an admissible open cover of E^0 with $m(i) = m$ for $1 \leq i \leq n$, and let \mathcal{V} be the corresponding cover for F . We assume that $\Pi_{E, \mathcal{U}} = \Pi_{F, \mathcal{V}}$ where they're defined, except for a single pair i_0, j_0 at which the permutations differ by a single transposition $(k_0 l_0)$:*

$$\pi_{i_0, j_0}^E = \pi_{i_0, j_0}^F \circ (k_0 l_0).$$

Then $\mathcal{X}(E)$ and $\mathcal{X}(F)$ are unitarily equivalent as C^ -correspondences.*

Proof. Let $\mathcal{V} = \{V_i = \tau U_i : 1 \leq i \leq n\}$ be the corresponding cover of F^0 , and let $V_{ik} = \gamma_i(U_{ik})$ for $1 \leq i \leq n$ and $1 \leq k \leq m$. Since the labelling is arbitrary, we can suppose without loss of generality that $i_0 = k_0 = 1$ and $j_0 = l_0 = 2$ and that $\pi_{1,2}^E = id$ and so $\pi_{1,2}^F = (12)$; while $\pi_{i,j}^F = \pi_{i,j}^E$ when $\{i, j\} \neq \{1, 2\}$. Thus the sets $U_{11} \cap U_{21}$, $U_{12} \cap U_{22}$, $V_{11} \cap V_{22}$ and $V_{12} \cap V_{21}$ are all non-empty.

Except for this one flip, we have that $V_{ik} \cap V_{jl} \neq \emptyset$ if and only if $U_{ik} \cap U_{jl} \neq \emptyset$ if and only if $\pi_{i,j}^E(k) = l$. Thus

$$\gamma_i|_{U_{ik} \cap U_{jl}} = (s_F|_{V_{ik} \cap V_{jl}})^{-1} s_E|_{U_{ik} \cap U_{jl}} = \gamma_j|_{U_{ik} \cap U_{jl}}.$$

To describe what happens on the remaining four intersections, let

$$\sigma = (s_F|_{V_{12} \cap V_{21}})^{-1} s_F|_{V_{11} \cap V_{22}}$$

be the canonical homeomorphism of $V_{11} \cap V_{22}$ onto $V_{12} \cap V_{21}$. Then

$$\gamma_1|_{U_{11} \cap U_{21}} = (s_F|_{V_{11} \cap V_{22}})^{-1} s_E|_{U_{11} \cap U_{21}}$$

whereas

$$\gamma_2|_{U_{11} \cap U_{21}} = (s_F|_{V_{12} \cap V_{21}})^{-1} s_E|_{U_{11} \cap U_{21}} = \sigma \gamma_1|_{U_{11} \cap U_{21}}.$$

Similarly, $\gamma_1|_{U_{12} \cap U_{22}}$ is a homeomorphism onto $V_{12} \cap V_{21}$, while

$$\gamma_2|_{U_{12} \cap U_{22}} = \sigma^{-1} \gamma_1|_{U_{12} \cap U_{22}}$$

is a homeomorphism onto $V_{11} \cap V_{22}$.

Using property (C4) and Tietze's extension theorem, there exists a continuous function $g \in C(F^0, [0, \pi/2])$ such that $g|_{\overline{V_1} \setminus V_2} = 0$ and $g|_{\overline{V_2} \setminus V_1} = \pi/2$. Let $h = g \circ s_F$. For $x \in \mathcal{X}(F)$ and $a \in F^1$, we define $\Gamma(x) \in \mathcal{X}(F)$ by setting $\Gamma(x)(a)$ to be

$$\begin{aligned} x\gamma_i^{-1}(a) & & a \in V_{ik}, \ a \notin V_{11} \cap V_{22} \text{ or } V_{12} \cap V_{21} \\ \cos(h(a))x\gamma_1^{-1}(a) + \sin(h(a))x\gamma_2^{-1}(a) & & a \in V_{11} \cap V_{22} \\ e^{2ih(a)}(\cos(h(a))x\gamma_1^{-1}(a) - \sin(h(a))x\gamma_2^{-1}(a)) & & a \in V_{12} \cap V_{21} \end{aligned}$$

By the calculations in the previous paragraph, $\Gamma(x)$ is well defined on F^1 .

It is easy to verify that $\Gamma(x)$ is a continuous function on F^1 . As before, there is an induced *-isomorphism ρ between $C(E^0)$ and $C(F^0)$ given by sending $f \in C(E^0)$ to $\rho(f) = f \circ \tau^{-1}$. The relations $\tau^{-1}s_F = s_E\gamma_i^{-1}$ and $\tau^{-1}r_F = r_E\gamma_i^{-1}$ allow us to show that Γ is a bimodule map: if $f \in C(E^0)$, $x \in \mathcal{X}(E)$ and $a \in F^1 \setminus ((V_{11} \cap V_{22}) \cap (V_{12} \cap V_{21}))$, then the argument is as in Lemma 5.3. While if $a \in V_{11} \cap V_{22}$, then

$$\begin{aligned} \Gamma(x \cdot f)(a) &= (\cos(h(a))x\gamma_1^{-1}(a) + \sin(h(a))x\gamma_2^{-1}(a))f(s_E\gamma_i^{-1}(a)) \\ &= \Gamma(x)(a)f(\tau^{-1}s_F(a)) = \Gamma(x) \cdot \rho(f)(a). \end{aligned}$$

A similar calculation works on $V_{12} \cap V_{21}$, so Γ is a right module map. Similarly, it is a left module map.

Finally we show that Γ is isometric. We will prove that for every $v \in E^0$, $\langle \Gamma(x), \Gamma(x) \rangle(\tau v) = \langle x, x \rangle(v)$. It is clear from the definition of Γ and the argument in Lemma 5.3 that the only place a problem might arise is when $v \in U_1 \cap U_2$. If $s_E^{-1}(v) = \{e_i : 1 \leq i \leq m\}$, then $s_F^{-1}(\tau v) = \{f_i := \gamma_1(e_i) : 1 \leq i \leq m\}$. So

$$\langle x, x \rangle(v) = \sum_{i=1}^m |x(e_i)|^2$$

and

$$\begin{aligned} \langle \Gamma(x), \Gamma(x) \rangle(\tau v) &= \sum_{i=1}^m |\Gamma(x)(\gamma_1(e_i))|^2 \\ &= \sum_{i=1}^2 |\Gamma(x)(f_i)|^2 + \sum_{i=3}^m |x(e_i)|^2. \end{aligned}$$

Thus we can ignore the terms $3 \leq i \leq m$. We compute

$$\begin{aligned} &|\Gamma(x)(f_1)|^2 + |\Gamma(x)(f_2)|^2 = \\ &= \left| \cos(h(f_1))x\gamma_1^{-1}(f_1) + \sin(h(f_1))x\gamma_2^{-1}(f_1) \right|^2 \\ &\quad + \left| e^{2ih(f_2)} (\cos(h(f_2))x\gamma_1^{-1}(f_2) - \sin(h(f_2))x\gamma_2^{-1}(f_2)) \right|^2 \\ &= \cos^2(g(\tau v))|x(e_1)|^2 + \sin(2g(\tau v)) \operatorname{Re} \overline{x(e_1)}x(e_2) + \sin^2(g(\tau x))|x(e_2)|^2 \\ &\quad + \cos^2(g(\tau v))|x(e_2)|^2 - \sin(2g(\tau v)) \operatorname{Re} \overline{x(e_1)}x(e_2) + \sin^2(g(\tau x))|x(e_1)|^2 \\ &= |x(e_1)|^2 + |x(e_2)|^2 \end{aligned}$$

Hence Γ is a unitary.

It follows that $\mathcal{X}(E)$ and $\mathcal{X}(F)$ are unitarily equivalent as C^* -correspondences. \blacksquare

Now we make the routine extension to the general result.

Theorem 5.5. *Let $E = (E^0, E^1, s_E, r_E)$ and $F = (F^0, F^1, s_F, r_F)$ be two compact topological graphs with $\dim E^0 \leq 1$. Then the following are equivalent:*

- (i) $\mathcal{T}(E)_+$ and $\mathcal{T}(F)_+$ are algebraically isomorphic,
- (ii) E and F are locally conjugate,
- (iii) $\mathcal{T}(E)_+$ and $\mathcal{T}(F)_+$ are completely isometrically isomorphic.

Proof. We established (i) implies (ii) in Theorem 4.5. As (iii) clearly implies (i), it remains to establish (ii) implies (iii).

Let \mathcal{U} be an admissible cover. Let us write $\sigma_{i,j} = (\pi_{i,j}^F)^{-1}\pi_{i,j}^E$ for $1 \leq i, j \leq n$. Fix k and set $\pi_{i,j}^E(k) = l$, $k' = \pi_{i,j}^{F^{-1}}(l)$ and $l' = \pi_{i,j}^E(k')$; so that $\sigma_{ij}(k) = k'$. Then γ_j carries $U_{ik} \cap U_{il}$ onto $V_{ik'} \cap V_{jl}$, which is carried by γ_i^{-1} onto $U_{ik'} \cap U_{j'l'}$. By the local equivalence (C2), it follows that

$$s_E \gamma_i^{-1} \gamma_j|_{U_{ik} \cap U_{il}} = s_E|_{U_{ik} \cap U_{il}} \quad \text{and} \quad r_E \gamma_i^{-1} \gamma_j|_{U_{ik} \cap U_{il}} = r_E|_{U_{ik} \cap U_{il}}.$$

It follows that we can build a new topological graph by changing π_{ij}^E by the transposition (l, l') and leaving the rest the same. That is, we declare that U_{ik} intersects $U_{j'l'}$ and $U_{ik'}$ intersects U_{jl} , leaving all other intersections unchanged. This defines a new topological graph,

which by Lemma 5.4 determines a C^* -correspondence which is unitarily equivalent to $\mathcal{X}(E)$.

Repeated use of this procedure a finite number of times yields an equivalent topological graph \tilde{E} for which $\Pi_{\tilde{E},\mathcal{M}} = \Pi_{F,\nu}$. By Lemma 5.3, the two C^* -correspondences $\mathcal{X}(\tilde{E})$ and $\mathcal{X}(F)$ are unitarily equivalent. It follows that $\mathcal{X}(E)$ and $\mathcal{X}(F)$ are unitarily equivalent, whence $\mathcal{T}(E)_+$ and $\mathcal{T}(F)_+$ are completely isometrically isomorphic. ■

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