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# Regular propagators of bilinear quantum systems

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## Abstract

The present analysis deals with the regularity of solutions of bilinear control systems of the type  $x' = (A + u(t)B)x$  where the state  $x$  belongs to some complex infinite dimensional Hilbert space, the (possibly unbounded) linear operators  $A$  and  $B$  are skew-adjoint and the control  $u$  is a real valued function. Such systems arise, for instance, in quantum control with the bilinear Schrödinger equation. For the sake of the regularity analysis, we consider a more general framework where  $A$  and  $B$  are generators of contraction semi-groups.

Under some hypotheses on the commutator of the operators  $A$  and  $B$ , it is possible to extend the definition of solution for controls in the set of Radon measures to obtain precise *a priori* energy estimates on the solutions, leading to a natural extension of the celebrated noncontrollability result of Ball, Marsden, and Slemrod in 1982.

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Elementary obstructions to controllability in a Banach space . . . . .	3
1.1.1	Conservation of the norm . . . . .	3
1.1.2	Continuity of the propagators . . . . .	4
1.1.3	Invariance of the domain . . . . .	4
1.2	Attainable sets in quantum control . . . . .	5
1.3	Main results . . . . .	5
1.3.1	Upper bound for attainable sets of bilinear control systems . . . . .	5
1.3.2	Higher regularity . . . . .	6
1.3.3	Applications to the bilinear Schrödinger equation . . . . .	7
1.4	Organization of our analysis . . . . .	8
1.5	Notations and Definitions . . . . .	8
<b>2</b>	<b>Well-posedness and continuity for BV controls</b>	<b>10</b>
2.1	Abstract framework: definitions and notations . . . . .	10
2.2	Propagators . . . . .	12
2.3	Continuity . . . . .	14
<b>3</b>	<b>Interaction framework</b>	<b>18</b>
3.1	Generalized propagators . . . . .	18
3.2	The attainable set . . . . .	23
<b>4</b>	<b>Higher order norm estimates for mildly coupled systems</b>	<b>24</b>
4.1	The mild coupling . . . . .	24
4.2	Higher regularity . . . . .	29
4.3	Extension to Radon measures . . . . .	31
<b>5</b>	<b>Bounded control potentials</b>	<b>32</b>
5.1	Dyson expansion solutions . . . . .	32
5.2	On the notion of solution in the Radon framework . . . . .	37
5.3	Noninvariance of the domain . . . . .	37
<b>6</b>	<b>Examples</b>	<b>38</b>
6.1	Quantum systems with smooth potentials on compact manifolds . . . . .	38
6.2	Potential well with dipolar interaction . . . . .	39
6.3	Quantum harmonic oscillator . . . . .	42
<b>A</b>	<b>Interpolation</b>	<b>43</b>
A.1	Convergence of sequences . . . . .	43
A.2	Interpolation of fractional powers of operators . . . . .	44
<b>B</b>	<b>Sufficient conditions for approximate controllability with bounded variation controls</b>	<b>46</b>
<b>C</b>	<b>Analytical perturbations</b>	<b>47</b>

# 1 Introduction

A bilinear control system in a Banach space  $\mathcal{X}$  is given by an evolution equation

$$\frac{d}{dt}x(t) = (A + u(t)B)x(t) \quad (1.1)$$

where  $A$  and  $B$  are two (possibly unbounded) linear operators on  $\mathcal{X}$  and  $u$  is a real-valued function, the control. Well-posedness of bilinear evolution equations of type (1.1) for a given control  $u$  is usually a difficult question. If  $K$  is a subset of  $\mathbf{R}$ , we define  $PC(K)$  the set of right-continuous piecewise constant taking value in  $K$ .

If  $K$ ,  $A$  and  $B$  are such that for every  $u$  in  $K$ ,  $A + uB$  generates a  $C^0$  semi-group  $t \mapsto e^{t(A+uB)}$ , then for every  $T \geq 0$  and every  $u$  in  $PC(K)$ , the restriction of  $u$  on  $[0, T)$  writes

$$u = \sum_{j=1}^p u_j \mathbb{I}_{[\tau_j, \tau_{j+1})} \quad (1.2)$$

with  $p \in \mathbf{N}$ ,  $u_1, \dots, u_p \in K$  and  $\tau_1 < \tau_2 < \dots < \tau_{p+1} = T$ , and one defines the associated propagator of (1.1) by

$$\Upsilon_{t, \tau_1}^u = e^{(t-\tau_j)(A+u_jB)} \circ e^{(\tau_j-\tau_{j-1})(A+u_{j-1}B)} \circ \dots \circ e^{(\tau_2-\tau_1)(A+u_1B)},$$

for every  $t$  in  $(\tau_j, \tau_{j+1})$ . The solution of (1.1) with initial value  $x_0$  at time  $\tau_1$  is  $t \mapsto \Upsilon_{t, \tau_1}^u \psi_0$ . When  $\tau_1 = 0$ , we denote  $\Upsilon_t^u := \Upsilon_{t, 0}^u$ .

It is of particular interest in the applications to study the set of points that can be attained in finite time from a given initial datum  $\psi_0$  using a set of admissible controls  $\mathcal{Z}$

$$\text{Att}_{\mathcal{Z}}(\psi_0) = \cup_{t \geq 0} \{ \Upsilon_t^u \psi_0 \mid u \in \mathcal{Z} \}$$

where  $\mathcal{Z}$  is a subset of  $PC(K)$  or, possibly, a larger set (provided that a suitable extension of  $\Upsilon$  to  $\mathcal{Z}$  makes sense). The set  $\text{Att}_{\mathcal{Z}}(\psi_0)$  is called attainable set from  $\psi_0$  with controls  $\mathcal{Z}$ .

The precise description of the propagators is, in principle, a hard task. To guarantee the controllability of (1.1), i.e. to bound the attainable set from below, is a challenging issue as well. One could try to use the regularity of the solutions of (1.1) to provide upper bounds of these attainable sets of bilinear systems. This will provide obstructions to the controllability of (1.1). This is the purpose of the present analysis.

## 1.1 Elementary obstructions to controllability in a Banach space

There are several upper bounds on the attainable sets that can be obtained by natural properties of the system. We list below some of them.

### 1.1.1 Conservation of the norm

In the Hilbertian case, in which  $\mathcal{X}$  is a Hilbert space, the propagator  $t \mapsto \Upsilon_t^u$  is unitary as soon as  $A + uB$  is essentially skew-adjoint for every  $u$  in  $K$ . If  $PC(K)$  is endowed with a topology for which  $u \mapsto \Upsilon_T^u \psi_0$  is continuous for every  $T > 0$  and every  $\psi_0$  in  $\mathcal{X}$ , then any continuous extension of the mapping  $u \mapsto \Upsilon_T^u \psi_0$  to a subset of the closure of  $PC(K)$  takes value in the sphere of radius  $\|\psi_0\|$ .

### 1.1.2 Continuity of the propagators

In the general case in which  $\mathcal{X}$  is a Banach space, assume that  $PC(K)$  is endowed with a topology for which  $u \mapsto \Upsilon_T^u \psi_0$  is continuous for every  $T > 0$  and every  $\psi_0$  in  $\mathcal{X}$ , and that  $u \mapsto \Upsilon_T^u \psi_0$  admits a (necessarily unique) continuous extension to  $\mathcal{Z} \supset PC(K)$ . If  $\mathcal{Z}_0 \subset \mathcal{Z}$ , endowed with a topology finer than the one induced by  $\mathcal{Z}$ , is sequentially compact (for its own topology), then for every  $\psi_0$  in  $\mathcal{X}$ , for every  $T > 0$ , the attainable set at time  $T$  from  $\psi_0$  with controls in  $\mathcal{Z}_0$ ,  $\{\Upsilon_T^u \psi_0 | u \in \mathcal{Z}_0\}$  is compact.

If  $(\mathcal{Z}_i)_{i \in \mathbf{N}}$  is a countable covering of  $\mathcal{Z}$ ,  $\mathcal{Z} = \cup_{i \in \mathbf{N}} \mathcal{Z}_i$ ,  $\mathcal{Z}_i$  is sequentially compact for every  $i$ , and the topology of  $\mathcal{Z}_i$  is finer than the topology induced by  $\mathcal{Z}$ , then the attainable set at time  $T$  from  $\psi_0$  with controls in  $\mathcal{Z}$ ,  $\{\Upsilon_T^u \psi_0 | u \in \mathcal{Z}\} = \cup_{i \in \mathbf{N}} \{\Upsilon_T^u \psi_0 | u \in \mathcal{Z}_i\}$  is a countable union of compact sets in  $\mathcal{X}$  (hence is a meager set in the sense of Baire as soon as  $\mathcal{X}$  is infinite dimensional).

Notice that if the input-output mapping  $PC(K) \ni u \mapsto \Upsilon^u \psi_0 \in C^0([0, T], \mathcal{X})$  is continuous, then the above results can be generalized to show that the attainable set from  $\psi_0$  at time less than  $T$   $\cup_{0 \leq t \leq T} \{\Upsilon_t^u \psi_0 | u \in \mathcal{Z}\} = \cup_{i \in \mathbf{N}} \cup_{0 \leq t \leq T} \{\Upsilon_t^u \psi_0 | u \in \mathcal{Z}_i\}$  is an union of compact sets.

This principle is an abstraction of the proof of the following result by Ball, Marsden, and Slemrod.

**Theorem** (Theorem 3.6 in [BMS82]). Let  $\mathcal{X}$  be an infinite dimensional Banach space,  $A$  generate a  $C^0$  semi-group of bounded linear operators on  $\mathcal{X}$ , and  $B$  be a bounded linear operator on  $\mathcal{X}$ . Then for any  $T \geq 0$ , the input-output mapping  $u \mapsto \Upsilon_T^u$  admits a unique continuous extension to  $L^1([0, T], \mathbf{R})$  and the attainable set

$$\bigcup_{r > 1} \bigcup_{T \geq 0} \bigcup_{u \in L^r([0, T], \mathbf{R})} \{\Upsilon_t^u \psi_0, t \in [0, T]\} \quad (1.3)$$

is contained in a countable union of compact subsets of  $\mathcal{X}$ , and, in particular, has dense complement.

In this case, for any  $T \geq 0$   $\mathcal{Z} = \cup_{r > 1} L^r([0, T], \mathbf{R})$  endowed with weak-\* topology,  $\mathcal{Z}_{i,j} = \cup_{r \geq 1 + \frac{1}{j}} \{f \in L^r([0, T], \mathbf{R}), \|f\|_{L^r([0, T])} \leq i\}$  and the sequential-compactness of  $\mathcal{Z}_{i,j}$  is granted by Banach–Alaoglu–Bourbaki Theorem. The main difficulty in [BMS82] is to prove the continuity of the input-output mapping  $u \mapsto \Upsilon^u \psi_0$  for the weak-\* topology.

**Remark 1.** The above argument does not hold anymore if one considers controls in  $L^1$ , since  $L^1$  is not a reflexive space. This is the content of [BMS82, Remark 3.8], where the question of possible extensions of the above result to  $r = 1$  is left open except in the so-called (see [Sle84]) diagonal case, see [BMS82, Theorem 5.5].

Another example of the same obstruction is given below in Corollary 9 with  $\mathcal{Z}$  equal to the set of functions with bounded variations. In this case, the sequential compactness in  $\mathcal{Z}$  is given by Helly's selection theorem.

### 1.1.3 Invariance of the domain

In the case in which  $A$  and  $B$  are bounded operators on  $\mathcal{X}$ , if  $\mathcal{F}$  is a closed subspace of  $\mathcal{X}$  left invariant by  $A + uB$  for every  $u$  in  $K$ , then for every  $u$ , the  $C^0$  semi-group generated by  $A + uB$  leaves  $\mathcal{F}$  invariant. Thus, for every  $u$  in  $PC(K)$  and every  $t \geq 0$ ,  $\Upsilon_t^u$  leaves  $\mathcal{F}$  invariant. If, moreover, the dynamics is time-reversible, then for every  $\psi_0$  in  $\mathcal{X}$ , for every  $u$  in  $PC(K)$ , for every  $t > 0$ ,  $\Upsilon_t^u \psi_0 \in \mathcal{F}$  if and only if  $\psi_0 \in \mathcal{F}$ .

Even in the unbounded case, the same conclusion holds if  $\mathcal{F}$  a subspace of  $\mathcal{X}$  left invariant by the dynamics  $\Upsilon_t^u$  and its time-reverse dynamics (when it makes sense).

We will see in Section 4.2 below that these invariance properties remain true in the Hilbert case when  $\mathcal{F}$  is the domain of a power of  $A$  left invariant by  $B$ .

## 1.2 Attainable sets in quantum control

The main motivation for the present analysis is due to questions on the controllability of closed quantum systems. The state of a quantum system evolving on a finite dimensional Riemannian manifold  $\Omega$ , with associated measure  $\mu$ , is described by its *wave function*, represented as a point in the unit sphere of  $L^2(\Omega, \mathbf{C})$ . In the absence of interactions with the environment and neglecting the relativistic effects, the time evolution of the wave function is given by the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + V(x) \psi(x, t),$$

where  $\Delta$  is the Laplace-Beltrami operator on  $\Omega$  and  $V : \Omega \rightarrow \mathbf{R}$  is a real function (usually called potential) accounting for the physical properties of the system. When submitted to an excitation by an external field (e.g. a laser), the Schrödinger equation reads

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + V(x) \psi(x, t) + u(t) W(x) \psi(x, t), \quad (1.4)$$

where  $W : \Omega \rightarrow \mathbf{R}$  is a real function accounting for the physical properties of the external field and  $u$  is a real function of time accounting for the intensity of it.

In the last decades, many efforts have been made to describe the attainable set of (1.4). In [Tur00], Turinici adapted the Ball, Marsden and Slemrod [BMS82, Theorem 3.6] to (1.4) with a measurable bounded  $W$ . The first positive result known, up to now, has been obtained by Beauchard in [Bea05], and improved in [BL10, BM14], for  $\Omega = (0, \pi)$  with Dirichlet boundary conditions,  $V = 0$  and  $W : x \mapsto x^2$ , see Section 6.2 for more details. In the case of the quantum harmonic oscillator:  $\Omega = \mathbf{R}$ ,  $V(x) = x^2$  and  $W : x \mapsto x$ , the attainable set is finite dimensional due to symmetries of the system, see Rouchon and Mirrahimi in [MR04] and Section 6.3. Instead of lower bounds of the attainable sets, many works have considered lower bounds of its closure (in different natural norms) which is sufficient from a physical point of view. We can, for instance, cite the work by Nersesyan [Ner09], in Sobolev spaces by means of Lyapunov technics for bounded domains and potentials. Concerning unbounded domains but with bounded potentials, we can cite [Mir09] with Lyapunov technics as well. Geometrical methods has been used to prove the density of the attainable set in  $L^2$ -norm when the spectrum is purely discrete and nonresonance conditions are satisfied, see [CMSB09, MS10, BCCS12, Cha12, BCS14, CS17]. The present work, similarly to [Ner09], considers the question of the regularity of a solution of (1.1) but in a more general way, following the spirit of [BCC13].

## 1.3 Main results

### 1.3.1 Upper bound for attainable sets of bilinear control systems

Our aim is to give upper bounds for attainable set of bilinear control systems. The main result is the following.

**Theorem 1.** *Let  $\mathcal{H}$  be an infinite dimensional Hilbert space,  $A$  be a maximal dissipative operator on  $\mathcal{H}$  with domain  $D(A)$ , and  $B$  be an operator on  $\mathcal{H}$  such that  $B - c$  and  $-B - c'$  generate contraction semi-groups leaving  $D(A)$  invariant for some real constants  $c \geq 0$  and  $c' \geq 0$ . Assume that  $A + uB$  is maximal dissipative with domain  $D(A)$  for every  $u$  in  $\mathbf{R}$  and that the map  $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$  is locally Lipschitz. Then, for every  $T > 0$ , there exists a unique continuous extension to  $L^1([0, T], \mathbf{R})$  of the input-output mapping  $u \mapsto \Upsilon_T^u \in L(\mathcal{H}, \mathcal{H})$  of (1.1), and for every  $\psi_0$  in  $H$ , the set*

$$\text{Att}_{L^1}(\psi_0) := \bigcup_{T \geq 0} \bigcup_{u \in L^1([0, T], \mathbf{R})} \{ \Upsilon_{t,0}^u \psi_0, t \in [0, T] \}$$

*is contained in a countable union of compact subsets of  $\mathcal{H}$ .*

*Proof.* See Section 3.2. □

As a consequence of Theorem 1,

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in L^1([0, T], \mathbf{R})} \{\alpha \Upsilon_{t,0}^u \psi_0, t \in [0, T]\}$$

is a meager set in  $\mathcal{H}$  and hence it has dense complement.

In the special case where the control operator  $B$  is bounded, using a different construction, we obtain a simplified statement similar to the one of [BMS82] and dealing with  $L^1$  controls.

**Proposition 2.** *Let  $\mathcal{X}$  be an infinite dimensional Banach space,  $A$  generate a  $C^0$  semi-group of bounded linear operators on  $\mathcal{X}$ , and  $B$  be a bounded linear operator on  $\mathcal{X}$ . Then for every  $T > 0$ , there exists a unique continuous extension to  $L^1([0, T], \mathbf{R})$  of the input-output mapping  $u \mapsto \Upsilon_T^u \in L(\mathcal{H}, \mathcal{H})$  of (1.1) and, for every  $\psi_0$  in  $\mathcal{H}$ ,*

$$\text{Att}_{L^1}(\psi_0) := \bigcup_{T \geq 0} \bigcup_{u \in L^1([0, T], \mathbf{R})} \{\Upsilon_t^u \psi_0, t \in [0, T]\}$$

is contained in a countable union of compact subsets of  $\mathcal{X}$  and, in particular, has dense complement.

*Proof.* See Section 5. □

These results set the open question by Ball, Marsden, and Slemrod in [BMS82, Remark 3.8]. The scheme of the proofs of Theorem 1 and Proposition 2 follows the structure of the proof of [BMS82, Theorem 3.6]. The lack of reflectiveness of  $L^1$  leads us to consider Radon measures as controls, the weak-compactness of bounded sequences is ensured by Helly's Selection Theorem. The main difficulty is to define a continuous input-output mapping associated with (1.1) in such a way to guarantee compactness properties for the attainable sets.

**Remark 2.** Theorem 1 still holds true for Radon measures controls, as stated in Corollary 16 below. Here the result is presented in term of  $L^1$  controls for the sake of readability, indeed the definition of the propagator associated with Radon measures requires preliminary notions presented in Section 3.1. The hypotheses of Theorem 1 are needed in order to prove continuity of the propagators after a particular change of variable (the *interaction framework* presented in Section 3). The key result in the proof of the continuity is an adaptation of a classical result by Kato [Kat53] (see Proposition 7).

### 1.3.2 Higher regularity

The Lipschitz assumption on the map  $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$  in Theorem 1 is crucial for our analysis when  $B$  is unbounded, however it may be hard to check in practice. For bilinear systems encountered in quantum physics, one can take advantage of the skew-adjointness of the operators to make the analysis simpler. For instance, it is possible to replace the Lipschitz assumption of Theorem 1 by a hypothesis of boundedness of the commutator of operators  $A$  and  $B$  as stated in the following result.

**Theorem 3.** *Let  $\mathcal{H}$  be an infinite dimensional Hilbert space,  $k$  a positive number,  $A$  and  $B$  be two skew-adjoint operators such that:*

- (i)  $A$  is invertible with bounded inverse from  $D(A)$  to  $\mathcal{H}$ ,
- (ii) for any  $t \in \mathbf{R}$ ,  $e^{tB} D(|A|^{k/2}) \subset D(|A|^{k/2})$ ,

- (iii) there exists  $c \geq 0$  and  $c' \geq 0$  such that  $B - c$  and  $-B - c'$  generate contraction semi-groups on  $D(|A|^{k/2})$  for the norm  $\|\cdot\|_{k/2}$ ,
- (iv)  $B$  is  $A$ -bounded with  $\|B\|_A = 0$  (see (2.2) below for the precise definition).

Then, for every  $T > 0$ , there exists a unique strongly continuous extension to  $BV([0, T], \mathbf{R})$ , endowed with the  $\|\cdot\|_{L^1} + \text{TV}(\cdot, ([0, T], \mathbf{R}))$ -norm, of the end-point mapping  $u \mapsto \Upsilon_T^u$  of (1.1). Moreover, for every  $\psi_0$  in  $D(|A|^{k/2})$ , the set

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in BV([0, T], \mathbf{R})} \{\alpha \Upsilon_t^u \psi_0, t \in [0, T]\},$$

is contained in a countable union of compact subsets of  $D(|A|^{k/2})$ .

*Proof.* See Section 4.2. □

**Remark 3.** Theorem 3 is a reformulation of Theorem 1 in the smaller functional framework of conservative dynamics. Theorem 3 is an immediate consequence of Corollary 24 below in the case of bounded variation controls. For extension of this result to Radon measures controls we refer to Section 4.3.

Notice that Corollary 24 allows an extension of Theorem 3 from  $D(|A|^{k/2})$  to  $D(|A|^{k/2+1-\varepsilon})$  if  $\psi_0$  is in  $D(|A|^{k/2+1-\varepsilon})$ , for  $\varepsilon \in (0, 1)$ .

**Remark 4.** A simple checkable condition for a pair of skew-adjoint operators  $(A, B)$  to satisfy assumptions (i) – (iii) in Theorem 3 is to be weakly coupled in the sense of [BCC13, Definition 1]. See Lemma 19 below.

**Remark 5.** Recall that there exists  $c \geq 0$  and  $c' \geq 0$  such that  $B - c$  and  $-B - c'$  generate contraction semi-groups on  $D(|A|^{k/2})$  if and only if these operators are maximal dissipative in the functional space  $D(|A|^{k/2})$ . Assumption (iii) in Theorem 3 is, in some sense, an assumption on the commutator of  $A$  and  $B$ , see Section 4.

### 1.3.3 Applications to the bilinear Schrödinger equation

Here we consider the motion of a nonrelativistic quantum charged particle trapped in an infinite square potential well excited by an external electric field. That is the dynamics governed by a Schrödinger equation on the interval  $(0, 1)$  with a control potential  $W : (0, 1) \rightarrow \mathbf{R}$ , which writes

$$\begin{cases} i \frac{\partial \psi}{\partial t}(t, x) = -\frac{\partial^2 \psi}{\partial x^2}(t, x) - u(t)W(x)\psi(t, x), & x \in (0, 1), t \in (0, T), \\ \psi(t, 0) = \psi(t, 1) = 0. \end{cases} \quad (1.5)$$

We denote by  $H_{(0)}^s((0, 1), \mathbf{C})$  the domain of  $|A|^{s/2}$  where  $A$  is the Laplace–Dirichlet operator on  $(0, 1)$ , and by  $\varphi_k$ ,  $k \in \mathbf{N}$  its (normalized) eigenvectors associated respectively to  $\lambda_k$ ,  $k \in \mathbf{N}$  its increasing sequence of eigenvalues (which are known to be simple). Let us recall the main result of [BL10].

**Theorem** (Theorem 1 in [BL10]). Let  $T > 0$  and  $W \in H^3((0, 1), \mathbf{R})$  be such that there exists  $c > 0$  verifying  $\frac{c}{k^3} \leq |\langle W \varphi_1, \varphi_k \rangle|$ , for all  $k \in \mathbf{N}$ . There exists  $\delta > 0$  and a  $C^1$  map  $\Gamma : \mathcal{V}_T \rightarrow L^2((0, T), \mathbf{R})$  where

$$\mathcal{V}_T := \{\psi_f \in H_{(0)}^3((0, 1), \mathbf{C}) \mid \|\psi_f\| = 1, \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

such that,  $\Gamma(\psi_1(T)) = 0$  and for every  $\psi_f \in \mathcal{V}_T$ , the solution of (1.5) with initial condition  $\psi(0) = \varphi_1$  and control  $u = \Gamma(\psi_f)$  satisfies  $\psi(T) = \psi_f$ .



The above result applies for instance to  $W : x \mapsto x^2$ . The techniques introduced in this paper provide estimates from above and from below for the attainable set when using different classes of admissible controls:

**Proposition 4.** *Let  $W : x \mapsto x^2$ . Then, for every  $T > 0$ , the input-output mapping  $u \mapsto \Upsilon_T^u$  of (1.5) admits a unique continuous extension to  $L^1([0, T], \mathbf{R})$ .*

*The attainable set from  $\varphi_1$  with  $L^1$  controls,*

$$\text{Att}_{L^1}(\varphi_1) = \bigcup_{T \geq 0} \bigcup_{u \in L^1([0, T], \mathbf{R})} \{\Upsilon_t^u \varphi_1 \mid 0 \leq t \leq T\},$$

*satisfies  $\text{Att}_{L^1}(\varphi_1) \subset \bigcap_{s < 5/2} H_{(0)}^s((0, 1), \mathbf{C})$ .*

*The attainable set from  $\varphi_1$  with bounded variation (BV) controls,*

$$\text{Att}_{BV}(\varphi_1) = \bigcup_{T \geq 0} \bigcup_{u \in BV([0, T], \mathbf{R})} \{\Upsilon_t^u \varphi_1 \mid 0 \leq t \leq T\},$$

*is a  $H^s$ -dense subset of  $\{\psi \in L^2((0, 1), \mathbf{C}) \mid \|\psi\| = 1\} \cap H_{(0)}^s((0, 1), \mathbf{C})$  for every  $s < 9/2$ .*

## 1.4 Organization of our analysis

In Section 2 we consider bilinear evolution equations (not necessarily conservative) from an abstract point of view and we define the solution for controls with bounded variations. We also prove the well-posedness within this framework and prove the continuity of the propagators with respect to the control parameters.

In Section 3, we use a reparametrization, inspired by physics, the interaction framework, to extend the results of Section 2 to the case where the control is a Radon measure. This provides a proof of Theorem 1.

When considering closed quantum systems, the operators  $A$  and  $B$  appearing in (1.1) are skew-adjoint. Section 4 is devoted to the regularity analysis of the solution obtained so far when further assumptions are made on the control potential and to the proof of Theorem 3.

Section 5 is dedicated to the case where  $B$  is bounded and to the proof of Proposition 2.

Section 6 presents various examples.

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## 1.5 Notations and Definitions

Throughout this analysis,  $T$  will be a positive real and  $I$  a bounded interval of  $\mathbf{R}$ .

**Bounded operators space** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two Banach spaces,  $L(\mathcal{X}, \mathcal{Y})$  is the space of linear bounded operator acting on  $\mathcal{X}$  with values in  $\mathcal{Y}$ . If  $\mathcal{X} = \mathcal{Y}$  we write  $L(\mathcal{X}) := L(\mathcal{X}, \mathcal{Y})$ .

**Weak and strong topology** Let  $(A_n)_{n \in \mathbf{N}}$  a sequence in  $L(\mathcal{X}, \mathcal{Y})$ , let  $A$  in  $L(\mathcal{X}, \mathcal{Y})$ . We say that  $A_n$  converges to  $A$  in the strong sense, or strongly, if for any  $\psi$  in  $\mathcal{X}$ ,  $(A_n \psi)_{n \in \mathbf{N}}$  converges to  $A\psi$  in  $\mathcal{Y}$ . We say that  $A_n$  converges to  $A$  in the weak sense, or weakly, if for any  $\psi$  in  $\mathcal{X}$  and  $\phi$  in  $\mathcal{Y}^*$ , the topological dual of  $\mathcal{Y}$ ,  $(\phi(A_n \psi))_{n \in \mathbf{N}}$  converges to  $\phi(A\psi)$  in  $\mathbf{C}$ .

**Maximal dissipative operators on Hilbert spaces** An operator  $A$  on a Hilbert space  $\mathcal{H}$  is dissipative if for any  $\phi \in D(A)$ ,  $\Re \langle \phi, A\phi \rangle \leq 0$ . It is maximal dissipative if it has no proper dissipative extension.

**Graph topology** Consider an operator  $A$  on a Hilbert space  $\mathcal{H}$  with domain  $D(A)$ , the graph topology on  $D(A)$  is the topology associated with the norm  $\psi \in D(A) \mapsto \|\psi\|_{\mathcal{H}} + \|A\psi\|_{\mathcal{H}} \in [0, \infty)$ .

**Bounded variation functions** Let  $E \subset \mathcal{X}$  for  $\mathcal{X}$  Banach space. A family  $t \in I \mapsto u(t) \in E$  is in  $BV(I, E)$ , i.e. is a bounded variation function from  $I$  to  $E$ , if there exists  $N \geq 0$  such that

$$\sum_{j=1}^n \|u(t_j) - u(t_{j-1})\|_{\mathcal{X}} \leq N,$$

for any partition  $(t_i)_{i=0}^n$  of the interval  $I$ . The mapping

$$u \in BV(I, E) \mapsto \sup_{(t_i)_i} \sum_{j=1}^n \|u(t_j) - u(t_{j-1})\|_{\mathcal{X}}$$

is a semi-norm on  $BV(I, E)$  denoted by  $\text{TV}(\cdot, (I, E))$  and it is called *total variation*.

The space  $BV(I, E)$  endowed with the norm  $\|\cdot\|_{BV(I)} := \|\cdot\|_{L^1} + \text{TV}(\cdot, (I, E))$  is a Banach space.

In what follows we consider, on  $BV(I, E)$ , the topology associated with the convergence given below:  $(u_n)_{n \in \mathbf{N}} \in BV(I, E)$  converges to  $u \in BV(I, E)$  if  $(u_n)_{n \in \mathbf{N}}$  is a bounded sequence in  $BV(I, E)$  pointwise convergent to  $u \in BV(I, E)$ .

Notice that convergence in the norm  $\|\cdot\|_{BV(I)}$  implies pointwise convergence.

The Jordan Decomposition Theorem provides that any bounded variation function is the difference of two nondecreasing bounded functions. This fact, together with Helly's Theorem provides the well-known Helly's Selection Theorem (see for example [Hel12, Nat55]).

**Theorem** (Helly's Selection Theorem). Let  $I$  be compact and  $(f_n)_{n \in \mathbf{N}}$  be a sequence in  $BV(I, \mathbf{R})$ . If

- (i) there exists  $M > 0$  such that for all  $n \in \mathbf{N}$ ,  $\text{TV}(f_n, (I, \mathbf{R})) < M$ ,
- (ii) there exists  $x_0 \in I$  such that  $(f_n(x_0))_{n \in \mathbf{N}}$  is bounded.

Then  $(f_n)_{n \in \mathbf{N}}$  has a pointwise convergent subsequence.

**Radon measures** We consider the space  $\mathcal{R}(I)$  of (signed) Radon measures on  $I$ . Recall that a positive Radon measure is a Borel measure which is locally finite and inner regular. Using Hahn decomposition [Dos80] any signed Radon measure  $\mu$  is the difference  $\mu = \mu^+ - \mu^-$  of two positive Radon measures  $\mu^+$  and  $\mu^-$  (at least one being finite) with disjoint support. We denote the total variation of  $\mu$  by  $|\mu|(I)$ , where  $|\mu| = \mu^+ + \mu^-$ . In the rest of the work we consider only Radon measure with bounded total variation. In particular both  $\mu^+$  and  $\mu^-$  are finite.

Here we only consider finite measures on  $I$  so the inner regularity requirement in the definition can be dropped. In the more general  $\sigma$ -finite case, this requirement can be dropped as well. In the first

case, a positive Radon measure is a finite Borel measure, while in the second case a positive Radon measure is a locally finite Borel measure. Note that, sometimes Borel measures are by definition locally finite. Sometimes the outer regularity is added to the definition of Radon measures, which again is redundant for finite measures.

We say that  $(\mu_n)_{n \in \mathbf{N}} \in \mathcal{R}([0, T])$  converges to  $\mu \in \mathcal{R}([0, T])$  if  $\sup_n |\mu_n|([0, T]) < +\infty$  (i.e.  $(\mu_n)_{n \in \mathbf{N}}$  has uniformly bounded total variations) and  $\mu_n((0, t]) \rightarrow \mu((0, t])$  for every  $t \in (0, T]$  as  $n$  tends to  $\infty$ . Note that this notion of convergence is *not* the same as the topology induced by the norm of total variation, see also Remark 14 below.

The cumulative function  $u(t) = \mu((0, t])$  of a Radon measure  $\mu$  is locally of bounded variation and the associated total variation (which does not depend on the choice of the cumulative function) coincides with the total variation of the Radon measure.

Every function in  $u \in L^1_{\text{loc}}(I, \mathbf{R})$  can be seen as the density of an absolutely continuous Radon measure  $\mu$ , namely  $\mu(J) = \int_J u f \lambda$  for every  $J \subset I$  borelian. When it does not create ambiguity we identify the function  $u$  with the associated Radon measure  $\mu$ . Moreover we have the following convergence.

**Lemma.** Let  $(u_n)_{n \in \mathbf{N}} \subset L^1(I, \mathbf{R})$  and  $u \in L^1(I, \mathbf{R})$  such that  $u_n \rightarrow u$  in  $L^1(I, \mathbf{R})$  as  $n$  tends to  $\infty$ . Let  $(\mu_n)_{n \in \mathbf{N}} \subset \mathcal{R}(I)$  and  $\mu \in \mathcal{R}(I)$  be the associated Radon measures. Then  $(\mu_n)_{n \in \mathbf{N}}$  converges to  $\mu$  in  $\mathcal{R}(I)$ .

Note that for  $u$  in  $L^1(I, \mathbf{R})$  the total variation of the associated Radon measure is the  $L^1$ -norm of  $u$  and hence  $L^1(I, \mathbf{R})$  is closed for the total variation topology.

The topology induced by our definition of convergence on  $BV$  is stronger than the weak topology on measures, see [EG92, Section 1.9] and weaker than the strong or total variation topology. It is also stronger than the narrow topology (also called weak topology in [Bil99, Kle14, Mat95]). For instance, the sequence  $(\delta_{\frac{1}{n}})_{n \in \mathbf{N}}$  converges narrowly to  $\delta_0$  but is not convergent according to our definition.

**Other notations** For any interval  $I \subset \mathbf{R}$ , we define

$$\Delta_I := \{(s, t) \in I^2 \mid s \leq t\}.$$

In a metric space  $E$ , the notation  $B_E(v_0, r)$  stands for the open ball of radius  $r$  and center  $v_0$  in  $E$ . For a densely defined operator  $B$  on a Hilbert space,  $B^*$  stands for its adjoint. Recall that  $B^*$  is densely defined if and only if  $B$  is closable, in such a case  $B^*$  is closed.

The set  $C^1_0(I, \mathcal{X})$  is the set of functions from an interval  $I$  to a Banach  $\mathcal{X}$  of class  $C^1$  with compact support in the interior of  $I$ .

## 2 Well-posedness and continuity for BV controls

In this section, we present global well-posedness results for a class of nonautonomous perturbations of a maximal dissipative linear Cauchy problem as well as a continuity criterion for a convergence problem.

### 2.1 Abstract framework: definitions and notations

Here we consider a general framework for bilinear dynamics in Hilbert spaces. Classical definitions and tools in this context can be found in [RS75, Section X.8], as well as the associated notes and problems. Notice that however we consider an opposite sign for the generators. Thus, following [Phi59], we use the word *dissipative* instead of *accretive* (see also [RS75, Notes of Section X.8]). As we restrict our analysis to the Hilbert space framework, generators of contraction (Lipschitz maps with Lipschitz

constant less than one) semi-groups and maximal dissipative operators coincide (see [Phi59, Theorem 1.1.3]). The equivalence between these two notions is important in our analysis at many levels, in particular, for what concerns mild coupling in Section 4.

Let  $\mathcal{H}$  be a Hilbert space (possibly infinite dimensional) with scalar product  $\langle \cdot, \cdot \rangle$  and corresponding norm  $\| \cdot \|$ . Let  $A, B$  be two (possibly unbounded) dissipative operators on  $\mathcal{H}$ . We consider the formal bilinear control system

$$\frac{d}{dt}\psi(t) = A\psi(t) + u(t)B\psi(t), \quad (2.1)$$

where the scalar control  $u$  is to be chosen in a set of real functions.

In general, given an initial data  $\psi(0) = \psi_0 \in \mathcal{H}$ , the solution of system (2.1) may not be well-defined. Indeed, even the definition of  $A + B$  is not obvious when  $A$  and  $B$  are unbounded. To this aim it is usually assumed that the operators  $A$  and  $B$  satisfy the following condition.

**Definition 1.** Let  $(A, B)$  be a couple of operators acting on  $\mathcal{H}$ . Then  $B$  is said *relatively bounded* with respect to  $A$ , or  $A$ -bounded, if  $D(A) \subset D(B)$  and there exist  $a, b > 0$  such that for every  $\psi$  in  $D(A)$ ,  $\|B\psi\| \leq a\|A\psi\| + b\|\psi\|$ .

It is well-known that if  $A$  is skew-adjoint and  $B$  skew-symmetric, from Kato–Rellich Theorem, (see for example [RS75, Theorem X.12]), if  $B$  is relatively bounded with respect to  $A$ , then for every real constant  $u$  such that  $|u| < 1/a$  (with  $a$  from Definition 1),  $A + uB$  is skew-adjoint with domain  $D(A)$  and generates a group of unitary operators. System (2.1) is then well-posed for every initial condition. From [RS75, Corollary to Theorem X.50],  $A + uB$  is maximal dissipative with domain  $D(A)$  and generates a contraction semi-group when  $A$  is maximal dissipative,  $B$  is dissipative,  $B$  is  $A$ -bounded and  $0 \leq u < 1/a$  (again  $a$  is from Definition 1).

In most of the examples in Section 6, we consider the skew-adjoint case and  $a$  arbitrary small, so that we can define the solutions of (2.1) for every piecewise constant control  $u$  with real values.

In the general case, we will refer to the following assumptions.

**Assumption 1.**  $(A, B, K)$  is a triple where  $A$  is a maximal dissipative operator on  $\mathcal{H}$ ,  $B$  is an operator on  $\mathcal{H}$  with  $D(A) \subset D(B)$ , and  $K$  a real interval containing 0, such that for any  $u \in K$ ,  $A + uB$  is a maximal dissipative operator on  $\mathcal{H}$  with domain  $D(A)$ .

Assumption 1 implies that the operator  $B$  is  $A$ -bounded from  $D(A)$  to  $\mathcal{H}$  and allows us to define

$$\|B\|_A := \inf_{\lambda > 0} \|B(\lambda - A)^{-1}\|. \quad (2.2)$$

The number  $\|B\|_A$  is the lower bound of all possible constants  $a$  in Definition 1 and in principle it can be zero. We also have,

$$\|B\|_A = \liminf_{\lambda \rightarrow +\infty} \|B(\lambda - A)^{-1}\|. \quad (2.3)$$

We consider also the following assumption in order to extend the definition of propagator to the case of Radon measures controls (see Section 3.1).

**Assumption 2.**  $(A, B, K)$  is a triple where  $A$  is a maximal dissipative operator on  $\mathcal{H}$ ,  $K$  a real interval containing 0, and

(A2.1) there exists  $c \geq 0$  and  $c' \geq 0$  such that  $B - c$  and  $-B - c'$  generate contraction semi-groups on  $\mathcal{H}$  leaving  $D(A)$  invariant,

(A2.2) for every  $u \in \mathcal{R}([0, T])$ , with  $u((0, t]) \in K$  for any  $t \in [0, T]$ ,

$$t \in [0, T] \mapsto \mathcal{A}(t) := e^{u((0, t])B} A e^{-u((0, t])B}$$

is a family of maximal dissipative operators with common domain  $D(A)$  such that :

- $\sup_{t \in [0, T]} \|(1 - \mathcal{A}(t))^{-1}\|_{L(\mathcal{H}, D(A))} < +\infty$ ,
- $\mathcal{A}$  has finite total variation from  $[0, T]$  to  $L(D(A), \mathcal{H})$ .

**Remark 6.** From Assumption 2,  $B$  et  $-B$ , with same domains, are generators of continuous semi-groups. We can prove  $e^{-tB} = (e^{tB})^{-1}$ , for any real  $t$ , and thus  $B$  generates a continuous group.

The triple  $(A, B, K)$  satisfies Assumption 2 for any interval  $K$  containing 0 if the pair  $(A, B)$  satisfies the following one.

**Assumption 3.**  $(A, B)$  is a pair such that

- (A3.1)  $A$  is a maximal dissipative operator on  $\mathcal{H}$  with domain  $D(A)$ ,
- (A3.2) there exists  $c \geq 0$  and  $c' \geq 0$  such that  $B - c$  and  $-B - c'$  generate contraction semi-groups on  $\mathcal{H}$  leaving  $D(A)$  invariant,
- (A3.3) the map  $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$  is locally Lipschitz.

**Remark 7.** Assumption (A3.3) is a strong assumption on the regularity of  $B$  with respect to the scale of  $A$ . Indeed it implies that  $B$  is the generator of a strongly continuous semi-group on  $D(A)$  since the semi-groups generated by  $B$  or  $-B$  are continuous on  $\mathcal{H}$  from Assumption (A3.2) and

$$\begin{aligned} \|Ae^{-tB}\psi - A\psi\| &\leq e^{ct} \|e^{tB} A e^{-tB}\psi - e^{tB} A\psi\| \\ &\leq e^{ct} \|e^{tB} A e^{-tB} - A\| \|\psi\| + \|e^{tB} A\psi - A\psi\|, \end{aligned}$$

for  $t > 0$  and  $\psi \in D(A)$ , which provides the continuity on  $D(A)$ . In Section 4, we consider higher regularity assumptions in the skew-adjoint case and operators on  $D(|A|^k)$  with  $k > 1$ .

## 2.2 Propagators

Since the problem (2.1) is nonautonomous, the notion of semi-group is replaced by the following

**Definition 2** (Propagator on a Hilbert space). A family  $(s, t) \in \Delta_I \mapsto X(s, t)$  of linear contractions on a Hilbert space  $\mathcal{H}$ , strongly continuous in  $t$  and  $s$  and such that

- (i)  $X(t, s) = X(t, r)X(r, s)$ , for any  $s < r < t$ ,
- (ii)  $X(t, t) = I_{\mathcal{H}}$ ,

is called a *contraction propagator* on  $\mathcal{H}$ .

**Remark 8.** In Section 3 below, we introduce a generalized notion of propagators, see Definition 4, with relaxed assumptions on the continuity of  $(s, t) \mapsto X(s, t)$  in order to extend to the framework to Radon measure controls.

Following [Kat53] in the construction of propagators, we introduce the following

**Assumption 4.** Let  $\mathcal{D}$  be a dense subset of  $\mathcal{H}$

- (A4.1)  $A(t)$  is a maximal dissipative operator on  $\mathcal{H}$  with domain  $\mathcal{D}$  for every  $t \in I$ ,
- (A4.2)  $t \mapsto A(t)$  has bounded variation from  $I$  to  $L(\mathcal{D}, \mathcal{H})$ , where  $\mathcal{D}$  is endowed with the graph topology associated with  $A(a)$  for some  $a \in I$ ,
- (A4.3)  $M := \sup_{t \in I} \|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < \infty$ .

In the following Assumption 4 will apply mainly to the family of operators  $A(t) = A + u(t)B$  or  $A(t) = e^{-u((0,t)B)}Ae^{u((0,t)B)}$ .

**Remark 9.** In Assumption (A4.2), the bounded variation of  $t \mapsto A(t)$  ensures that any choice of  $a \in I$  will be equivalent.

**Remark 10.** As  $A(t)$  is a maximal dissipative operator, that is the generator of a contraction semi-group, its resolvent set contains the positive half line and from Hille–Yosida Theorem [RS75, Theorem X.47a] (see also Proposition 17 below) any generator of a contraction semi-group satisfies

$$\sup_{t \in I} \|(1 - A(t))^{-1}\|_{L(\mathcal{H})} < \infty.$$

Note that  $\|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < +\infty$  for every  $t \in I$  and the essence of Assumption (A4.3) is that  $\|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})}$  is uniformly bounded with respect to  $t \in I$ .

We do not assume  $t \mapsto A(t)$  to be continuous. However, as a consequence of Assumption (A4.2) (see [Edw57, Theorem 3]) it admits right and left limit in  $L(\mathcal{D}, \mathcal{H})$ , denoted by  $A(t-0) := \lim_{\varepsilon \rightarrow 0^+} A(t - \varepsilon)$  and  $A(t+0) := \lim_{\varepsilon \rightarrow 0^+} A(t + \varepsilon)$ , for all  $t \in I$ , and  $A(t-0) = A(t+0)$  for all  $t \in I$  except, at most a countable set.

The core of our analysis is the following result due to Kato (see [Kat53, Theorem 2 and Theorem 3]) providing sufficient conditions for the well-posedness of system (2.1).

**Theorem 5.** *If  $t \in I \mapsto A(t)$  satisfies Assumption 4, then there exists a unique contraction propagator  $X : \Delta_I \rightarrow L(\mathcal{H})$  such that if  $\psi_0 \in \mathcal{D}$  then  $X(t, s)\psi_0 \in \mathcal{D}$  and is strongly right differentiable in  $t$  with derivative  $A(t+0)X(t, s)\psi_0$ .*

Moreover, with  $M$  from Assumption (A4.3),

$$\|A(t)X(t, s)\psi_0\| \leq Me^{MTV(A, (I, L(\mathcal{D}, \mathcal{H})))} \|A(s)\psi_0\|, \quad \text{for } (t, s) \in \Delta_I \text{ and } \psi_0 \in \mathcal{D},$$

and  $X(t, s)\psi_0$  is left differentiable in  $s$  with derivative  $-A(s-0)\psi_0$  when  $t = s$ .

In the case in which  $t \mapsto A(t)$  is continuous and skew-adjoint, if  $\psi_0 \in \mathcal{D}$  then  $t \in (s, +\infty) \mapsto X(t, s)\psi_0$  is strongly continuously differentiable in  $\mathcal{H}$  with derivative  $A(t)X(t, s)\psi_0$ .

*Proof.* The statement of this theorem is obtained by gathering statements of [Kat53]. The point which may not be stated clearly with respect to [Kat53] is the existence of  $C > 0$  such that

$$\|A(t)X(t, s)\psi_0\| \leq C \|A(s)\psi_0\|,$$

for  $(t, s) \in \Delta_I$  and for any  $\psi_0 \in \mathcal{D}$ . This is in [Kat53, §3.10] with  $C = M \exp(MN)$  and

$$M = \sup_{t \in I} \|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} \quad \text{and} \quad N = TV(A, (I, L(\mathcal{D}, \mathcal{H}))). \quad \square$$

We call  $t \mapsto X(t, s)\phi_0$  a “mild” solution in  $\mathcal{H}$  of

$$\begin{cases} \frac{d}{dt} \phi(t) = A(t)\phi(t), \\ \phi(s) = \phi_0, \end{cases} \quad (2.4)$$

even if, in general, it is not differentiable.

**Remark 11.** If  $(A, B, K)$  satisfies Assumption 2, the operator  $t \in [0, T] \mapsto \mathcal{A}(t) := e^{u((0,t)B)}Ae^{-u((0,t)B)}$  defined in Assumption (A2.2) satisfies Assumption 4 for any Radon measure  $u$  on  $(0, T)$  with  $u((0, t]) \in K$  for any  $t \in (0, T]$ . If  $(A, B)$  satisfies Assumption 3 then  $(A, B, \mathbf{R})$  satisfies Assumption 2.

The fact that Assumption 1 is stronger, in some sense, than Assumption 4 is the content of the following lemma.

**Lemma 6.** *If  $(A, B, K)$  satisfies Assumption 1 and  $u : [0, T] \mapsto K$  has bounded total variation such that  $\overline{u([0, T])} \subset K$  then  $A(t) := A + u(t)B$  satisfies Assumption 4 with  $I = [0, T]$ .*

*Proof.* The only point to verify is Assumption (A4.3). First the set  $C := \overline{u([0, T])}$  is a bounded closed subset of  $K$  and thus is a compact of  $K$ . Then the map

$$u \mapsto (1 - A)(1 - A - uB)^{-1},$$

is continuous from  $K$  to  $L(\mathcal{H})$ . Indeed

$$\begin{aligned} (1 - A)(1 - A - uB)^{-1} - (1 - A)(1 - A - vB)^{-1} \\ &= (1 - A) \left( (1 - A - uB)^{-1} - (1 - A - vB)^{-1} \right) \\ &= (v - u)(1 - A) \left( (1 - A - uB)^{-1} B (1 - A - vB)^{-1} \right) \\ &= (v - u)(1 - A) \left( (1 - A - uB)^{-1} B (1 - A)^{-1} (1 - A)(1 - A - vB)^{-1} \right) \end{aligned}$$

so that

$$\begin{aligned} (1 - A)(1 - A - uB)^{-1} - (1 - A)(1 - A - vB)^{-1} \\ &- (v - u)(1 - A)(1 - A - uB)^{-1} B (1 - A)^{-1} \left( (1 - A)(1 - A - vB)^{-1} - (1 - A)(1 - A - uB)^{-1} \right) \\ &= (v - u)(1 - A) \left( (1 - A - uB)^{-1} B (1 - A)^{-1} (1 - A)(1 - A - uB)^{-1} \right) \end{aligned}$$

Define

$$L(u) = \|(1 - A)(1 - A - uB)^{-1}\|_{L(\mathcal{H})} \quad \text{and} \quad b = \|B(1 - A)^{-1}\|$$

so that

$$(1 - |v - u|bL(u)) \|(1 - A)(1 - A - uB)^{-1} - (1 - A)(1 - A - vB)^{-1}\| \leq |v - u|L(u)^2b, \quad (2.5)$$

which provides the desired continuity. Then as  $|u(t) - u(0)| \leq \|u\|_{BV(I)}$  for any  $t \in I$ ,  $u(t)$  is in  $C$  a compact subset of  $K$  for all  $t \in I$  thus the closure of its image is compact and

$$t \in I \mapsto \|(1 - A - u(t)B)^{-1}\|_{L(\mathcal{H}, \mathcal{D})}$$

is bounded. □

### 2.3 Continuity

In this section we focus on the continuity of the propagators with respect to the control  $u$ . The main tool is a consequence of the work by Kato [Kat53], Proposition 7 below.

**Definition 3.** Let  $(A_n)_n$  be a family of generators of contraction semi-groups and  $A$  a generator of a contraction semi-group. The family  $(A_n)_n$  tends to  $A$  in the *strong resolvent sense* if

$$(\lambda - A_n)^{-1}\phi \rightarrow (\lambda - A)^{-1}\phi \quad \text{as } n \rightarrow \infty,$$

for every  $\phi$  in  $\mathcal{H}$  and for some  $\lambda \geq 0$  (and hence all  $\lambda$ , see [RS72, Section VIII.7]).

**Proposition 7.** *Let  $(A_n)_{n \in \mathbb{N}}$  and  $A$  satisfy Assumption 4. Let  $(\mathcal{D}_n)_{n \in \mathbb{N}}$  and  $\mathcal{D}$  be their respective domains (for any  $t \in I$ ). Let  $X_n$  (respectively  $X$ ) be the contraction propagator associated with  $A_n$  (respectively  $A$ ).*

*Assume that*

- (i)  $\sup_{n \in \mathbf{N}} \sup_{t \in I} \|(1 - A_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D}_n)} < +\infty$ ,
- (ii)  $A_n(\tau)$  converges to  $A(\tau)$  in the strong resolvent sense for almost every  $\tau \in I$  as  $n$  tends to infinity,
- (iii)  $\sup_{n \in \mathbf{N}} \text{TV}(A_n, (I, L(\mathcal{D}_n, \mathcal{H}))) < +\infty$ ,
- (iv) For every  $\phi \in \mathcal{H}$ ,  $\delta > 0$ ,  $n \in \mathbf{N}$  there exists  $\psi^n \in \mathcal{D}_n$  with  $\|\phi - \psi^n\| < \delta$  such that  $\sup_{n \in \mathbf{N}} \|A_n(a)\psi^n\| < +\infty$  for some  $a \in I$ .

Then  $X_n(t, s)$  tends strongly to  $X(t, s)$  locally uniformly in  $s, t \in \Delta_I$ .

*Proof.* Following [Kat53, §3.8] it is sufficient to prove the statement for piecewise constant controls (i.e. replacing  $X_n$  and  $X$  by any of their Riemann products). Let  $\Delta := \{s = t_0 < t_1 < \dots < t_n = t\}$  be a partition of the interval  $(t, s)$  and  $X_n(\Delta)$  be the propagator associated with  $\sum_{j=1}^n A_n(t_{j-1})\chi_{[t_{j-1}, t_j]}$ .

Then, for every  $n$ ,

$$\|(X_n(t, s; \Delta) - X_n(t, s))\phi\| \leq \overline{M} e^{\overline{M}\overline{N}} \overline{N} |\Delta| \|A_n(a)\phi\|, \text{ for every } \phi \in \mathcal{D}_n$$

where

$$\begin{aligned} \overline{M} &= \max\{\sup_{t \in I} \sup_{n \in \mathbf{N}} \|(1 - A_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D}_n)}, \sup_{t \in I} \|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})}\}, \\ \overline{N} &= \max\{\sup_{n \in \mathbf{N}} \text{TV}(A_n, (I, L(\mathcal{D}_n, \mathcal{H}))), \text{TV}(A, (I, L(\mathcal{D}, \mathcal{H})))\}, \end{aligned}$$

and  $|\Delta| = \sup_{1 \leq j \leq n} |t_j - t_{j-1}|$ . Similarly we define  $X(\Delta)$  as the propagator associated with

$$\sum_{j=1}^n A(t_{j-1})\chi_{[t_{j-1}, t_j]}.$$

We have

$$\|(X(t, s; \Delta) - X(t, s))\phi\| \leq \overline{M} e^{\overline{M}\overline{N}} \overline{N} |\Delta| \|A(a)\phi\|, \text{ for every } \phi \in \mathcal{D}.$$

Following the proof of [RS75, Theorem X.47a (Hille–Yosida)] (see also Proposition 17 below), we have that

$$\left\| e^{tA_n(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi \right\| \leq t \left\| A_n(\tau)\phi - A_n^\lambda(\tau)\phi \right\|, \text{ for every } \phi \in \mathcal{D}_n,$$

with  $A_n^\lambda(\tau) := \lambda(\lambda - A_n(\tau))^{-1}A_n(\tau)$ , for  $\lambda > 0$ , and

$$\left\| e^{tA(\tau)}\phi - e^{tA^\lambda(\tau)}\phi \right\| \leq t \left\| A(\tau)\phi - A^\lambda(\tau)\phi \right\|, \text{ for every } \phi \in \mathcal{D},$$

with  $A^\lambda(\tau) := \lambda(\lambda - A(\tau))^{-1}A(\tau)$ .

Since  $A_n$  are generators of contraction semi-groups, then  $\|\lambda(\lambda - A_n(\tau))^{-1}\| \leq 1$  for every  $\lambda > 0$ , in particular it is uniformly bounded in  $n$  and  $\tau$ .

By assumption (iv) for every  $\phi \in \mathcal{H}$  and  $\delta > 0$  there exist  $\psi \in \mathcal{D}$  and  $\psi^n \in \mathcal{D}_n$  such that

$$\|\phi - \psi\| \leq \delta \quad \text{and} \quad \|\phi - \psi^n\| \leq \delta,$$

and  $\sup_{n \in \mathbf{N}} \|A_n(a)\psi^n\| < +\infty$  for  $a \in I$ . We deduce that  $\lambda(\lambda - A_n(\tau))^{-1}\psi^n$  tends to  $\psi^n$  as  $\lambda \rightarrow \infty$  uniformly in  $n$  and  $\tau$ . Similarly  $A^\lambda(\tau)\psi$  tends strongly to  $A(\tau)\psi$  uniformly in  $\tau$  as  $\lambda \rightarrow \infty$ . So that

$$\begin{aligned} \left\| e^{tA(\tau)}\phi - e^{tA_n(\tau)}\phi \right\| &\leq 4\delta + \left\| e^{tA(\tau)}\psi - e^{tA^\lambda(\tau)}\psi \right\| + \left\| e^{tA^\lambda(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi \right\| + \left\| e^{tA_n^\lambda(\tau)}\psi^n - e^{tA_n(\tau)}\psi^n \right\| \\ &\leq 4\delta + t \left\| A(\tau)\psi - A^\lambda(\tau)\psi \right\| + \left\| e^{tA^\lambda(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi \right\| + t \left\| A_n(\tau)\psi^n - A_n^\lambda(\tau)\psi^n \right\|. \end{aligned}$$



It suffices to show convergence of  $\left\| e^{tA^\lambda(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi \right\|$  as  $n \rightarrow \infty$  in order to prove the Proposition. Since  $e^{tA_n^\lambda(\tau)} = e^{-\lambda t} e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}$  and  $e^{tA^\lambda(\tau)} = e^{-\lambda t} e^{t\lambda^2(\lambda - A(\tau))^{-1}}$  (see [RS75, Theorem X.47a (Hille-Yosida)]), we have that

$$\begin{aligned} \left\| e^{tA^\lambda(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi \right\| &= \left\| e^{-\lambda t} e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}\phi - e^{-\lambda t} e^{t\lambda^2(\lambda - A(\tau))^{-1}}\phi \right\| \\ &= e^{-\lambda t} \left\| e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}\phi - e^{t\lambda^2(\lambda - A(\tau))^{-1}}\phi \right\|. \end{aligned}$$

Recall that  $\|(\lambda - A_n(\tau))^{-1}\| \leq \frac{1}{\lambda}$  (see Proposition 17 with  $\omega = 0$ ) and hence  $\|e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}\| \leq e^{\lambda t}$ . Duhamel's identity writes, for  $0 \leq t \leq T$ ,

$$\begin{aligned} &\left\| e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}\phi - e^{t\lambda^2(\lambda - A(\tau))^{-1}}\phi \right\| \\ &= \left\| \int_0^t \lambda^2 e^{(t-s)\lambda^2(\lambda - A_n(\tau))^{-1}} \{(\lambda - A_n(\tau))^{-1} - (\lambda - A(\tau))^{-1}\} e^{s\lambda^2(\lambda - A(\tau))^{-1}}\phi \, ds \right\| \quad (2.6) \\ &\leq \lambda^2 e^{T\lambda} \int_0^T \left\| \{(\lambda - A_n(\tau))^{-1} - (\lambda - A(\tau))^{-1}\} e^{s\lambda^2(\lambda - A(\tau))^{-1}}\phi \right\| \, ds. \end{aligned}$$

The result follows from Lebesgue Dominated Convergence Theorem, using the convergence of  $A_n(\tau)$  to  $A(\tau)$  in the strong resolvent sense for almost every  $\tau \in I$  as  $n$  tends to infinity.  $\square$

**Remark 12.** In the case in which  $\mathcal{D}_n = \mathcal{D}$ , for all  $n \in \mathbf{N}$ , the assumptions of Proposition 7 are verified whenever:

- (i)'  $\sup_{n \in \mathbf{N}} \sup_{t \in I} \|(1 - A_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < +\infty$ ,
- (ii)'  $A_n(\tau)$  converges to  $A(\tau)$  in the strong sense in  $\mathcal{D}$  for almost every  $\tau \in I$  as  $n \rightarrow \infty$ ,
- (iii)'  $\sup_{n \in \mathbf{N}} \text{TV}(A_n, (I, L(\mathcal{D}, \mathcal{H}))) < +\infty$ .

This can be proved by adapting [RS72, Theorem VIII.25] to the maximal dissipative case and using Banach–Steinhaus Theorem to prove that assumption (iv) of Proposition 7 follows from (ii)'.

**Corollary 8.** *Let  $(A, B, K)$  satisfy Assumption 1. Let  $(u_n)_{n \in \mathbf{N}}$  be a sequence in  $BV(I, K)$  converging to  $u \in BV(I, K)$ . Let  $A_n(t) = A + u_n(t)B$ ,  $A(t) = A + u(t)B$  and let  $X_n$  (respectively  $X$ ) be the contraction propagators associated with  $A_n$  (respectively  $A$ ). If  $\overline{\cup_{n \in \mathbf{N}} u_n([0, T])} \subset K$ , then  $X_n(t, s)$  tends strongly to  $X(t, s)$  locally uniformly in  $(s, t) \in \Delta_I$ .*

*Proof.* The proof consists in verifying that the hypotheses of Proposition 7 are satisfied. To this aim, we just have to check points (i)', (ii)' and (iii)' of Remark 12.

Point (i)': the mapping  $L : s \in K \mapsto \|(1 - A)(1 - sB)^{-1}\|$  has been defined in the proof of Lemma 6 where it is shown to be continuous. By hypothesis, there exists a compact set  $K_1 \subset K$  such that for every  $n$  in  $\mathbf{N}$  and every  $t$  in  $[0, T]$ ,  $u_n(t) \in K_1$ . Hence,  $\sup_{n \in \mathbf{N}} \sup_{y \in [0, T]} \|(1 - A)(1 - A - u_n(t)B)^{-1}\| \leq \sup C(K_1) < +\infty$  which proves point (i)'

Point (ii)' is immediate from the assumption that  $(u_n)_{n \in \mathbf{N}}$  converges toward  $u$ .

Point (iii)': for every  $n$  in  $\mathbf{N}$ ,

$$\text{TV}(A_n, (I, L(\mathcal{D}, \mathcal{H}))) = \text{TV}(u_n B, (I, L(\mathcal{D}, \mathcal{H}))) = \|B\|_{L(\mathcal{D}, \mathcal{H})} \text{TV}(u_n, (I, \mathbf{R})).$$

This last quantity is bounded as  $n$  tends to infinity since  $(u_n)_{n \in \mathbf{N}}$  converges to  $u$ .  $\square$

**Corollary 9.** *Assume that  $(A, B, K)$  satisfy Assumption 1. Let  $\psi_0 \in \mathcal{H}$ . Then*

$$\{\Upsilon_t^u(\psi_0) \mid u \in BV([0, \infty), K), t \geq 0\}$$

*is contained in a countable union of compact subsets of  $\mathcal{H}$ .*

*Proof.* We follow the principle presented in Section 1.1.2. We first introduce a nondecreasing sequence  $(K_i)_{i \in \mathbf{N}}$  of compact subsets of  $K$  such that  $K = \cup_{i \in \mathbf{N}} K_i$ , and the subset

$$\mathcal{Z}_{i,j,n} = \{u \in BV([0, \infty), K_i), \text{TV}(u, ([0, n], K_i)) \leq j\}$$

of the set of functions with bounded variations. By Helly's selection Theorem,  $\mathcal{Z}_{i,j,n}$  is sequentially compact. By Corollary 8, the set  $\{\Upsilon_t^u(\psi_0) \mid u \in \mathcal{Z}_{i,j,n}\}$  is compact in  $\mathcal{H}$  for every  $(n, i, j)$  in  $\mathbf{N}^3$ . Hence

$$\begin{aligned} & \{\Upsilon_t^u(\psi_0) \mid u \in BV([0, \infty), K), t \geq 0\} \subset \\ & \cup_{n \in \mathbf{N}} \cup_{i \in \mathbf{N}} \cup_{j \in \mathbf{N}} \{\Upsilon_t^u(\psi_0) \mid u \in \mathcal{Z}_{i,j,n}, 0 \leq t \leq n\} \end{aligned}$$

is contained in a countable union of compact sets of  $\mathcal{H}$ .  $\square$

Recall that  $I$  is bounded and that convergence for a sequence of Radon measures is in the sense given in Section 1.5.

**Corollary 10.** *Let  $(A, B, K)$  satisfy Assumption 2. Let  $I = [0, T]$  for some  $T > 0$ . Let  $(v_n)_{n \in \mathbf{N}}$  be a sequence in  $\mathcal{R}(I)$  converging to  $v \in \mathcal{R}(I)$ . Assume that  $v_n((0, t]) \in K$  and  $v((0, t]) \in K$  for every  $t \in (0, T]$  and  $n \in \mathbf{N}$ . Let  $\mathcal{A}_n(t) = e^{-v_n((0,t])B} A e^{v_n((0,t])B}$  and  $\mathcal{A}(t) = e^{-v((0,t])B} A e^{v((0,t])B}$  and let  $X_n$  (respectively  $X$ ) be the contraction propagators associated with  $\mathcal{A}_n$  (respectively  $\mathcal{A}$ ). If  $\sup_{n \in \mathbf{N}} \text{TV}(\mathcal{A}_n, (I, L(D(A), \mathcal{H}))) < +\infty$ , then  $X_n(t, s)$  tends strongly to  $X(t, s)$  locally uniformly in  $(s, t) \in \Delta_I$ .*

*Proof.* The proof consists in checking that the assumptions of Proposition 7 are fulfilled. Here  $\mathcal{D} = D(A)$ .

(i) We have  $\sup_{n \in \mathbf{N}} \sup_{t \in I} \|(1 - \mathcal{A}_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < \infty$ . Indeed

$$\begin{aligned} & \|(1 - A)(1 - \mathcal{A}_n(t))^{-1}\|_{L(\mathcal{H})} \\ &= \|(1 - A)e^{v_n((0,t])B}(1 - A)^{-1}e^{-v_n((0,t])B}\|_{L(\mathcal{H})} \\ &\leq \|e^{v_n((0,t])B}\|_{L(\mathcal{H})} \|e^{-v_n((0,t])B}(1 - A)e^{v_n((0,t])B}(1 - A)^{-1}\|_{L(\mathcal{H})} \|e^{-v_n((0,t])B}\|_{L(\mathcal{H})} \\ &= \|e^{v_n((0,t])B}\|_{L(\mathcal{H})} \|(1 - \mathcal{A}_n(t))(1 - A)^{-1}\|_{L(\mathcal{H})} \|e^{-v_n((0,t])B}\|_{L(\mathcal{H})} \\ &\leq \|e^{v_n((0,t])B}\|_{L(\mathcal{H})} (\|\mathcal{A}_n(t) - \mathcal{A}_n(0)\|_{L(\mathcal{D}, \mathcal{H})} + \|1 - A\|_{L(\mathcal{D}, \mathcal{H})}) \|e^{-v_n((0,t])B}\|_{L(\mathcal{H})} \\ &\leq \|e^{v_n((0,t])B}\|_{L(\mathcal{H})} (\text{TV}(\mathcal{A}_n, (I, L(\mathcal{D}, \mathcal{H}))) + 1) \|e^{-v_n((0,t])B}\|_{L(\mathcal{H})}. \end{aligned} \tag{2.7}$$

Notice that since  $(v_n)_{n \in \mathbf{N}}$  converges to  $v$  then by definition  $v_n((0, t])$  is uniformly bounded in  $n \in \mathbf{N}$  and  $t \in [0, T]$ . Then from Assumption (A2.1), there exists  $\omega \in \mathbf{R}$  such that

$$\|e^{vB}\|_{L(\mathcal{H})} \leq e^{\omega|v|}, \quad \text{for every } v \in \mathbf{R}, \tag{2.8}$$

which provides the desired boundedness.

(ii) The sequence  $\mathcal{A}_n(t)$  tends to  $\mathcal{A}(t)$  in the strong resolvent sense for all  $t \in [0, T]$  as  $n \rightarrow \infty$ . Indeed from

$$(1 - \mathcal{A}_n(t))^{-1} - (1 - \mathcal{A}(t))^{-1} = e^{-v_n((0,t)B)}(1 - A)^{-1}e^{v_n((0,t)B)} - e^{-v((0,t)B)}(1 - A)^{-1}e^{-v((0,t)B)}$$

we have

$$(1 - \mathcal{A}_n(t))^{-1} - (1 - \mathcal{A}(t))^{-1} = (e^{-v_n((0,t)B)} - e^{-v((0,t)B)})(1 - A)^{-1}e^{v_n((0,t)B)} + e^{-v((0,t)B)}(1 - A)^{-1}(e^{v_n((0,t)B)} - e^{v((0,t)B)})$$

then using (2.8) the boundedness of the sequence  $(v_n)$  and the strong continuity of  $t \in \mathbf{R} \mapsto e^{tB}$ , we conclude the strong resolvent convergence.

(iii) By Assumption (A2.2) we have  $\sup_{n \in \mathbf{N}} \text{TV}(\mathcal{A}_n, (I, L(D(A), \mathcal{H}))) < +\infty$ .

(iv) Assumption (iv) of Proposition 7 follows from  $\mathcal{A}_n(0) = A$  and the fact that the domain  $\mathcal{D}$  of  $A$  is dense in  $\mathcal{H}$ .  $\square$

**Remark 13.** The last assumption of Corollary 10, namely  $\sup_{n \in \mathbf{N}} \text{TV}(\mathcal{A}_n, (I, L(\mathcal{D}, \mathcal{H}))) < +\infty$  for  $\mathcal{A}_n(t) = e^{-u_n((0,t)B)}Ae^{u_n((0,t)B)}$ , is a consequence of Assumption (A3.3) since this provides the existence of a real constant  $L_I(A, B)$  such that for every  $s, t \in I$ ,

$$\|e^{-tB}Ae^{tB} - e^{-sB}Ae^{sB}\|_{L(\mathcal{D}, \mathcal{H})} \leq L_I(A, B)|t - s|. \quad (2.9)$$

We also notice that with  $s = 0$  it provides

$$\|e^{-tB}Ae^{tB}\|_{L(\mathcal{D}, \mathcal{H})} \leq L_I(A, B)|t| + 1 \quad (2.10)$$

as  $\|A\|_{L(\mathcal{D}, \mathcal{H})} \leq 1$ .

### 3 Interaction framework

In this section we consider the framework of Assumptions 2 or 3. We show that these assumptions lead to a notion of weak solution for (2.1) when the control is integrable and we provide the proofs of Theorem 1 and Proposition 2 in the general Radon measure case.

#### 3.1 Generalized propagators

In this section, we explain the link between Assumptions 1 and 2 and thus emphasize the fact that (2.1) admits solutions associated with a Radon measure  $u$ .

We use the following result of approximation of Radon measures by piecewise constant functions.

**Lemma 11.** *For every  $u \in \mathcal{R}([0, T])$  there exists a sequence  $(u_n)_n$  of piecewise constant functions such that  $\int_0^t u_n$  tends to  $u((0, t])$  and  $\int_0^t |u_n|$  tends to  $|u|((0, t])$  for all  $t$  in  $[0, T]$  as  $n$  tends to infinity with  $\int_0^T |u_n| \leq |u|((0, T])$ . If  $u$  is positive, the sequence  $(u_n)_n$  can be chosen in such a way that  $t \mapsto \int_0^t u_n(\tau) d\tau$  is nondecreasing for every  $n$ . If  $t \mapsto u((0, t])$  is  $M$ -Lipschitz continuous on  $[0, T]$  then  $(u_n)_n$  can be chosen in such a way that  $|u_n| \leq M$ .*

*Proof.* It is not restrictive to prove the statement for positive Radon measures since by Hahn–Jordan decomposition, any Radon measure  $u$  is the difference of two nonnegative Radon measures with disjoint supports.

Let us assume  $u$  positive. Then  $U : t \in (0, T] \mapsto u((0, t])$  is an increasing function (with bounded variation). Except on an at most countable set,  $U$  is continuous. So  $U$  is the sum of an increasing step function, possibly with an infinite number of steps, and an increasing continuous function. Both can be approximated by an increasing sequences of increasing continuous piecewise affine functions.

The last statement follows by considering approximation of Lipschitz continuous functions by continuous piecewise affine ones.  $\square$

**Remark 14.** This lemma explains why we excluded the total variation topology while it seems more natural at first sight.

Note that, for a positive  $u \in \mathcal{R}([0, T])$  the sequence  $(u_n)_n$  of piecewise constant functions such that  $\int_0^t u_n$  tends to  $u((0, t])$  pointwise is, in our construction, an increasing sequence. Since each  $t \mapsto \int_0^t u_n$  is continuous, if  $t \mapsto u((0, t])$  is not continuous then the same result in the total variation topology is excluded. Indeed the convergence in total variation of measures sequences implies the uniform convergence of the corresponding sequence of cumulative functions.

Notice as well that, it is not possible to associate with a general positive  $u \in \mathcal{R}([0, T])$  a sequence  $(u_n)_{n \in \mathbf{N}}$  of piecewise constant functions such that  $t \mapsto u((0, t]) - \int_0^t u_n$  be both positive and increasing in  $t$ . Otherwise, we would have  $|u((0, t]) - \int_0^t u_n| \leq |u((0, T]) - \int_0^T u_n|$  for any  $t$ , implying convergence in the total variation topology.

**Definition 4.** Let  $(A, B, K)$  satisfy Assumption 2. Let  $u \in \mathcal{R}([0, T])$ . For any  $v \in BV([0, T], K)$  with distributional derivative  $u$  let  $t \mapsto Y_t^u$  be the contraction propagator with initial time  $s = 0$  associated with  $\mathcal{A}_v(t) := e^{-v(t)B} A e^{v(t)B}$ . We define the *generalized propagator* associated with  $A + u(t)B$  with initial time zero, to be  $\Upsilon_{t,0}^{dv} = e^{v(t)B} Y_t^u$  for every  $t$  in  $[0, T]$  and  $v$  in  $BV([0, T], K)$  such that  $v' = u$  in the distributional sense.

**Remark 15.** Let  $u \in \mathcal{R}([0, T])$  and define  $v_0(t) = u((0, t])$  the associated right-continuous cumulative function and let  $v \in BV([0, T], \mathbf{R})$  be such that  $v' = u$ . Then  $v - v_0$  is in  $BV([0, T], \mathbf{R})$  and it is almost everywhere 0 since it is supported on the at most countable set where  $v$  is not right-continuous. A somehow pathological example could be  $u = 0$  and  $v$  any characteristic function of a negligible set.

The propagator  $Y_t^u$  will not depend on the choice of  $v$  being right-continuous, or not, at its discontinuities. Indeed the latter is a negligible set and a Duhamel formula provides the equality of the propagators. On the other hand, the factor  $e^{vB}$  depends crucially on this choice. This explains the notation  $\Upsilon^{dv}$  instead of  $\Upsilon^u$ .

The reason for introducing the notion of generalized propagator is that imposing any extra requirement on the choice of  $v$  will lead to loss of the compactness provided by Helly's Selection Theorem. This will for instance make more complicate the presentation of the principle exposed in Section 1.1.2.

Notice that for any  $v_1$  and  $v_2$  in  $BV([0, T], K)$  with the same distributional derivative one has

$$\Upsilon^{dv_1} = e^{(v_1 - v_2)B} \Upsilon^{dv_2}.$$

**Lemma 12.** Let  $(A, B, K_1)$  satisfy Assumption 1 and  $(A, B, K_2)$  satisfy Assumption 2. Let  $u : [0, T] \mapsto K_1$  be of bounded total variation with  $\overline{u([0, T])} \subset K_1$  and such that  $\int_0^T |u(s)| ds \in K_2$ . Let  $\Upsilon_t^u$  be the propagator associated with  $A + u(t)B$  with initial time  $s = 0$ . Let  $U(t) := \int_0^t u(s) ds$  and let  $t \mapsto Y_t^u$  be the contraction propagator associated with  $\mathcal{A}(t) := e^{-U(t)B} A e^{U(t)B}$  with initial time  $s = 0$ . Then

$$\Upsilon_t^u = e^{U(t)B} \Upsilon^{u(t)} \left( = \Upsilon_t^{dU(t)} \right),$$

for every  $t \in [0, T]$ .

*Proof.* Let  $\psi_0 \in D(A)$  and define the continuous function  $\Psi : t \mapsto e^{-\int_0^t u(s) ds B} \Upsilon_t^u(\psi_0)$ . By Theorem 5,  $\Psi(t) \in D(A)$  is strongly right differentiable in  $t$  with derivative

$$-u(t+0)B\Psi(t) + e^{-U(t)B}(A + u(t+0)B)\Upsilon_t^u(\psi_0) = e^{-U(t)B}Ae^{U(t)B}\Psi(t).$$

By uniqueness, see Theorem 5,  $\Psi(t) = Y_t^u\psi_0$  for every  $t \in [0, T]$ .  $\square$

**Proposition 13.** *Let  $(A, B, [0, +\infty))$  satisfy Assumption 1 and  $(A, B, K)$  satisfy Assumption 2. Then for every  $\psi_0 \in \mathcal{H}$  and  $t \in [0, T]$  the map  $\Upsilon_t(\psi_0) : u \mapsto \Upsilon_t^u(\psi_0) \in \mathcal{H}$  admits a unique continuous extension on  $\{u \in \mathcal{R}([0, T]) \text{ positive} \mid u([0, T]) \in K\}$  denoted by  $\Upsilon_t(\psi_0)$  which satisfies*

$$\Upsilon_t^u(\psi_0) = e^{u(0,t)B}Y_t^u(\psi_0), \quad \text{for every } t \in [0, T]. \quad (3.1)$$

*Proof.* For every  $u \in \mathcal{R}([0, T])$  positive with  $u([0, T]) \in K$  let  $(u_n)_{n \in \mathbf{N}}$  be a sequence of (right-continuous) positive piecewise constant functions on  $[0, T]$  such that  $\int_0^T |u_n| \in K$  converging to  $u$  and which existence is given by Lemma 11.

From Remark 13 and Corollary 10, for every  $\psi_0 \in \mathcal{H}$ ,  $Y_t^{u_n}(\psi_0)$  tends to  $Y_t^u(\psi_0)$  as  $n$  tends to  $\infty$ . We set  $\Upsilon_t^u(\psi_0) = e^{u(0,t)B}Y_t^u(\psi_0)$ . Then  $\Upsilon_t^{u_n}(\psi_0)$  tends to  $\Upsilon_t^u(\psi_0)$  as  $n$  tends to  $\infty$ . The uniqueness of the extension is guaranteed by Lemma 12.  $\square$

**Remark 16.** With respect to Definition 4, Proposition 13 fixes the choice of antiderivative of  $u$  to the right-continuous one, accordingly to the arbitrary choice of the notion of convergence made for Radon measures in Section 1.5. Other choices would have led to another choice of antiderivative for  $u$ . Qualitatively speaking, any choice would provide the same results in the sequel.

**Remark 17.** The definition of propagator associated with positive Radon measures given in (3.1) can be extended to signed Radon measures provided that  $(A, B, \mathbf{R})$  satisfies Assumption 1. Notice, however, that if  $(A, B, \mathbf{R})$  satisfies Assumption 1 then  $B$  is necessarily symmetric.

In the case in which  $B$  is not symmetric the definition of propagator can be extended to signed Radon measures provided that  $(A, B, K)$  satisfies Assumption 2. The uniqueness of the continuous extension can be obtained if  $(A, B - c, [0, \infty))$  and  $(A, -B - c', [0, \infty))$  satisfy Assumption 1. Indeed consider  $u \in BV([0, T], \mathbf{R})$  and split  $u$  in the difference of positive part  $u^+ := \max\{u, 0\}$  and negative part  $u^- := \max\{-u, 0\}$ . Then  $A(t) = A + u^+(t)(B - c) + u^-(t)(-B - c')$  satisfies Assumption 4.

**Proposition 14.** *Let  $(A, B)$  satisfy Assumption 3 and  $D(A) \subset D(B)$ . Then for every  $\psi_0$  in  $D(A)$ , for every  $u \in L^1([0, T], \mathbf{R})$ , the map  $t \mapsto \Upsilon_t^u(\psi_0)$  satisfies*

$$\int_{[0, T]} \langle f'(t), \Upsilon_t^u(\psi_0) \rangle dt = \int_{[0, T]} \langle f(t), A\Upsilon_t^u(\psi_0) \rangle dt + \int_{[0, T]} \langle f(t), B\Upsilon_t^u(\psi_0) \rangle u(t) dt, \quad (3.2)$$

for every  $f \in C_0^1([0, T], \mathcal{H})$ .

A mapping  $t \mapsto \Upsilon_t^u(\psi_0)$  satisfying (3.2) is called *weak solution* of (2.1) with initial condition  $\psi_0$ .

*Proof.* For every  $u \in L^1([0, T])$  let  $(u_n)_{n \in \mathbf{N}}$  be a sequence of piecewise constant functions on  $[0, T]$  that converges to  $u$  for the topology of  $\mathcal{R}([0, T])$ .

For every  $f \in C_0^1([0, T], \mathcal{H})$ ,

$$- \int_{[0, T]} \langle f'(t), \Upsilon_t^{u_n}(\psi_0) \rangle dt = \int_{[0, T]} \langle f(t), A\Upsilon_t^{u_n}(\psi_0) \rangle dt + \int_{[0, T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0) \rangle u_n(t) dt$$

since from Theorem 5,  $Y_t^{u_n}(\psi_0) \in D(A)$  for any  $t \in [0, T]$ .

Recall that the adjoint of  $A$  is also maximal dissipative as soon as  $A$  is maximal dissipative [TW09, Chapter 3.1]. We can restrict to  $f \in C_0^1([0, T], D(A^*))$  by replacing  $f$  with  $\lambda(\lambda - A^*)^{-1}f$ , where  $\lambda$  is a large positive real.

We have the following convergences

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{[0, T]} \langle f'(t), \Upsilon_t^{u_n}(\psi_0) \rangle dt &= \int_{[0, T]} \langle f'(t), \Upsilon_t^u(\psi_0) \rangle dt, \\ \lim_{n \rightarrow \infty} \int_{[0, T]} \langle f(t), A\Upsilon_t^{u_n}(\psi_0) \rangle dt &= \int_{[0, T]} \langle f(t), A\Upsilon_t^u(\psi_0) \rangle dt, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \int_{[0, T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0) \rangle u_n(t) dt = \int_{[0, T]} \langle f(t), B\Upsilon_t^u(\psi_0) \rangle u(t) dt.$$

Indeed, for this last term we have

$$\begin{aligned} &\int_{[0, T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0) \rangle u_n(t) dt - \int_{[0, T]} \langle f(t), B\Upsilon_t^u(\psi_0) \rangle u(t) dt \\ &= \int_{[0, T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0) \rangle (u_n - u)(t) dt + \int_{[0, T]} (\langle f(t), B\Upsilon_t^{u_n}(\psi_0) \rangle - \langle f(t), B\Upsilon_t^u(\psi_0) \rangle) u(t) dt. \end{aligned} \quad (3.3)$$

The convergence for the second term of the right-hand side of the last identity can be proved similarly as in the other terms, for instance,

$$\lim_{n \rightarrow \infty} \int_{[0, T]} \langle f'(t), \Upsilon_t^{u_n}(\psi_0) \rangle dt = \int_{[0, T]} \langle f'(t), \Upsilon_t^u(\psi_0) \rangle dt,$$

by using Lebesgue Dominated Convergence Theorem being the integrand uniformly bounded. Finally, from Theorem 5, estimates (2.7), (2.8) and (2.9), there exists  $C > 0$  and  $\omega > 0$  depending on  $A$  and  $B$  only, such that

$$\begin{aligned} &\left| \int_{[0, T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0) \rangle (u - u_n)(t) dt \right| \\ &\leq C \sup_{t \in [0, T]} \|f(t)\| \|B\|_A (1 + CL_{[0, \|u\|_1]}(A, B) \|u\|_1) e^{2\omega \|u\|_1} \times \\ &\times e^{(1 + CL_{[0, \|u\|_1]}(A, B) \|u\|_1) e^{2\omega \|u\|_1} CL_{[0, \|u\|_1]}(A, B) \|u\|_1} \|\psi_0\|_{D(A)} \|u - u_n\|_1 \\ &\rightarrow_{n \rightarrow \infty} 0 \end{aligned}$$

since  $f(0) = 0$  and using Lemma 11 and  $|u - u_n| = |u^+ - u_n^+| + |u^- - u_n^-|$  the sequence  $(u_n)$  converges to  $u$  in  $L^1$ -norm.  $\square$

**Remark 18.** An interesting question would be to understand the relation between the assumptions associated with the two constructions of propagators considered in this section. For example, on what extent does Assumption 3 ensure that  $A + uB$  has a maximal dissipative closure for  $u \in \mathbf{R}$ ?

This seems to be a hard question. However in the skew-adjoint case, the following considerations are in place. Let  $A$  and  $B$  be skew-adjoint with  $D(A) \subset D(B)$ . For any  $v \in \mathcal{H}$ , any  $u \in D(A)$  the map

$$t \in K \mapsto \langle (1 - \varepsilon A)^{-1}v, e^{tB} A e^{-tB} (1 - \varepsilon A)^{-1}u \rangle$$

is Lipschitz, its distributional derivative is bounded uniformly in  $\varepsilon$  by the Banach-Steinhaus theorem. So that  $[A, B] \in L(D(A) \cap D(B), (D(A) \cap D(B))^*)$  extends to an operator such that

$$[A, B] \in L(D(A), \mathcal{H}).$$

with a slight abuse of notation and hence for any  $u \in \mathbf{R}$

$$[A, A + uB] \in L(D(A), \mathcal{H}).$$

The Nelson commutator theorem, see [RS75, Section X.5], gives that  $A+uB$  is essentially skew-adjoint for any  $u \in \mathbf{R}$ .

**Remark 19.** Considering Definition 2,  $X(t, s) = e^{v(t)B}Y_{t,s}^u e^{-v(s)B}$  defines a propagator when  $v$  is continuous, that is when  $u$  has no atoms. Otherwise, we no longer require any continuity keeping in mind that  $v_0$  the right-continuous cumulative function of  $u$  will lead to a right-continuous propagator which is compatible with the requirements on the initial conditions.

From Proposition 14, when  $v$  is absolutely continuous,  $X(t, s) = e^{v(t)B}Y_{t,s}^u e^{-v(s)B}$  defines a weak solution of (2.1). The question of the extension of this proposition to Radon measures is then natural. If one considers  $A = 0$  and  $B$  bounded, as in Section 5.2 below, the solution of (2.1) is  $1 + H(t)B$ , where  $H$  is a Heaviside function jumping at 0. This is different from  $e^{H(t)B}$  provided by our analysis.

Proposition 14 can be extended to measures with singular continuous part. Indeed any Radon measure is in the closure of the set of absolute continuous measures for the topology we imposed. Notice that in Lemma 11 the sequence is also narrow convergent. Since the propagators associated with absolute continuous and singular continuous are bounded continuous, the first term in (3.3) will tend to 0.

**Nonexistence of bounded solution propagators for unbounded control potentials in the skew-adjoint case** We comment in the possible extension of Proposition 14. For  $(A, B)$  a couple of skew-adjoint operators. We exhibit here an example of system (2.1) with a Radon measure control for which it is not possible to construct a strong solution which is obtained by a bounded propagator applied to the initial condition (even if it is in the domain of the generator).

Let  $\psi_0 \in D(A)$  and  $\psi_1 \in D(B)$  with  $B\psi_1 \in D(A)$  then for any solution of (2.1) for  $u = \delta_{T/2}$  with initial condition  $\psi_0$  at  $t = 0$  the jump at  $T/2$  is exactly  $B\psi(T/2)$  (after integration of (2.1) around  $T/2$ ). So setting  $\psi(T/2) = \psi_1$ , we have

$$\Upsilon_t^u(\psi_0) = \begin{cases} e^{tA}\psi_0 & \text{for } t \in [0, \frac{T}{2}), \\ \psi_1 & \text{for } t = \frac{T}{2}, \\ e^{tA}\psi_0 + e^{(t-\frac{T}{2})A}B\psi_1 & \text{for } t \in (\frac{T}{2}, T]. \end{cases}$$

The determination of  $\psi_1$  leads to a uniqueness issue and a modelling interpretation. A way to overcome this issue is to impose a continuity at  $t = T/2$ . Note that the left-continuity leads to

$$\psi_1 = e^{\frac{T}{2}A}\psi_0 + B\psi_1.$$

So if 1 is in the spectrum of  $B$  then for some  $\psi_0$  this is not solvable. Note that with  $u = \alpha\delta_{T/2}$  the problem is the same when  $\alpha$  is in the spectrum of  $B$ . This excludes the possibility to construct a left-continuous propagator.

A natural requirement seems to be the right-continuity and thus  $\psi_1 = e^{\frac{T}{2}A}\psi_0$ . Then, when  $B$  is unbounded, a issue is to extend continuously the propagator from  $D(A)$  to  $\mathcal{H}$  as for some  $\psi_0 \in \mathcal{H}$ ,  $e^{\frac{T}{2}A}\psi_0 \notin D(B)$ .

By convexity, a linear combination of left and right continuity will lead to the same kind of contradictions.

In conclusion, when  $B$  is unbounded one cannot expect to construct bounded solutions with Radon controls in the skew-adjoint case. In Section 5, we prove that when  $B$  is bounded, there exists

a strongly regulated propagator defining a weak solution of (2.1). This propagator is not necessarily a contraction.

Similar questions arise for ODEs. We refer for instance to [PD82].

Nonetheless, with absolutely continuous Radon measures, we have built proper propagators and the extension to Radon measures presented in this work has consequences in the analysis of the attainable sets that we present in the sequel.

### 3.2 The attainable set

The key result in the proof of Theorem 1 is given by the following proposition.

**Proposition 15.** *Let  $T > 0$ . Let  $\psi_0 \in \mathcal{H}$ . Let  $(A, B)$  satisfy Assumption 3. Then, for every  $L > 0$ , the set*

$$\{\Upsilon_t^u(\psi_0) : u \in \mathcal{R}([0, T]), |u|((0, T]) \leq L, t \in [0, T]\}$$

*is relatively compact in  $\mathcal{H}$ .*

*Proof.* For every  $u \in \mathcal{R}([0, T])$  consider  $v(t) = u((0, t])$ . Then  $v \in BV([0, T])$  and  $\text{TV}(v, ([0, T]), \mathbf{R}) \leq L$ . Note that  $\|v\|_{L^\infty} = \sup_{t \in [0, T]} |v(t) - v(0)| \leq \text{TV}(v, ([0, T]), \mathbf{R}) \leq L$  since  $v(0) = u(\emptyset) = 0$ . Consider a sequence  $(u_n)_{n \in \mathbf{N}} \subset \mathcal{R}([0, T])$  such that  $|u_n|((0, T]) \leq L$  for every  $n$ . By Helly's Selection Theorem the sequence  $v_n : t \mapsto u_n((0, t])$ , has a subsequence pointwise converging to some  $v \in BV([0, T])$ ,  $\|v\| \leq L$ . Let  $u$  the Radon measure associated with  $v$ . Thus the sequence  $(u_n)_{n \in \mathbf{N}}$  has a subsequence converging to  $u \in \mathcal{R}([0, T])$ . We relabel this convergent subsequence by  $(u_n)_{n \in \mathbf{N}}$ .

From (2.9) we have that  $\mathcal{A}_n(t) = e^{-u_n((0, t])B} A e^{u_n((0, t])B}$  is uniformly bounded in  $BV([0, T], L(\mathcal{D}, \mathcal{H}))$  by  $L\|A, B\|_{L(\mathcal{D}, \mathcal{H})}$ . By Corollary 10,  $t \mapsto Y_t^{u_n}(\psi_0)$  converges uniformly for  $t \in [0, T]$  to  $t \mapsto Y_t^u(\psi_0)$  as  $n \rightarrow \infty$ . For any sequence  $(t_n)_n$ ,  $(v_n((0, t_n]))_n$  is a bounded sequence and so is  $(e^{v_n((0, t_n])B})_n$ . In particular, it has a strongly convergent subsequence. Finally  $t \mapsto \Upsilon_t^{u_n}(\psi_0)$  converges pointwise in  $t \in [0, T]$  to  $t \mapsto \Upsilon_t^v(\psi_0)$ .  $\square$

**Remark 20.** Note that the set  $\{\Upsilon^u(\psi_0) : u \in L^1([0, T], \mathbf{R}), \|u\|_{L^1} \leq L\}$  is relatively compact in  $L^\infty([0, T], \mathcal{H})$ . However, despite the compactness of  $[0, T]$ , the set  $\{\Upsilon^u(\psi_0) : u \in \mathcal{R}([0, T]), |u|((0, T]) \leq L\}$  may be not relatively compact in  $L^\infty([0, T], \mathcal{H})$ . Indeed, if this set were relatively compact, then the *generalized propagator* associated with  $A + u(t)B$  would be strongly continuous, due to the pointwise density of solutions of (2.1) which are continuous. This is not the case in general due to the factor  $e^{u((0, t])B}$  in (3.1).

From the above result the attainable set is a countable union of totally bounded sets.

**Corollary 16.** *Let  $\psi_0 \in \mathcal{H}$ . If  $(A, B)$  satisfies Assumption 3 then*

$$\text{Att}_{\mathcal{R}}(\psi_0) := \{\Upsilon_t^u(\psi_0), u \in \mathcal{R}([0, +\infty)), t \geq 0\}$$

*is contained in a countable union of compact subsets of  $\mathcal{H}$ .*

*Proof.* The attainable set can be rewritten as

$$\bigcup_{L, T > 0} \{\Upsilon_t^u(\psi_0), u \in \mathcal{R}([0, T]), |u|((0, T]) \leq L, t \in [0, T]\}$$

and this union can be, in fact, restricted to  $L, T$  in a countable set, for instance  $\mathbf{N}^2$ . Then Proposition 15 tells that each set of the union is relatively compact in  $\mathcal{H}$  and thus with empty interior.  $\square$

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* The well-posedness result for  $L^1$  controls is a consequence of Proposition 14 proved for Radon controls. The conclusion on the attainable set for  $L^1$  controls is a consequence of Corollary 16 proved for Radon controls.  $\square$



## 4 Higher order norm estimates for mildly coupled systems

In the following we will restrict our analysis to the skew-adjoint case. The motivation for this assumption is twofold. On the one hand this is the case for most of the mathematical objects appearing in quantum mechanics and, on the other hand, the restriction to self-adjoint operators makes the analysis simpler avoiding technicalities.

Our aim in this section is to analyze under which conditions the solution built in the previous sections are smoother in the scale of  $A$ . This is indeed the case of Assumptions 1, 2 or 3 are stated in  $D(|A|^{k/2})$  instead of  $\mathcal{H}$ . Our aim is to provide a somewhat simpler criteria showing that the extension of assumptions on  $B$  will be sufficient. To this aim, the  $A$ -boundedness of  $B$  as operators acting on  $D(|A|^{k/2})$  is crucial and it is stated in Lemma 21 which is the cornerstone of the analysis of this section. This is especially important if we want to obtain the regularity of propagators in the scale of  $A$  up to the order  $k/2$ . For lower orders, a simple interpolation argument provides the desired results. The criteria will be used in a perturbative framework (Kato-Rellich type argument) and we will not consider the whole of  $K$  for the values of  $u$ , unless we assume that the domain of powers of  $A + uB$  are the same for any  $u \in K$ . We recall that in the dissipative framework in order to use Kato-Rellich criterion  $u$  has to be nonnegative when  $B$  is dissipative, below we assume that both  $B$  and  $-B$  have dissipativity properties (up to a shift by a constant as in Assumption (A2.1) or (A3.2)) so that the sign of  $u$  does not play any role.

This shows that for time reversible systems, the input-output mapping does not change the regularity with respect to  $A$  in the spirit of Section 1.1.3. Since eigenvectors belong to any  $D(|A|^k)$  this shows that exact controllability clearly relies on the regularity of  $B$  in the scale of  $A$ .

### 4.1 The mild coupling

Given a skew-adjoint operator  $A$  and  $k \geq 0$ ,  $k \in \mathbf{R}$ , we define

$$\|\psi\|_{k/2} = \sqrt{\langle |A|^k \psi, \psi \rangle}.$$

**Definition 5** (mild coupling). Let  $k$  be a nonnegative real. A pair of *skew-adjoint* operators  $(A, B)$  is *k-mildly coupled* if

- (i)  $A$  is invertible with bounded inverse from  $D(A)$  to  $\mathcal{H}$ ,
- (ii) for any real  $t$ ,  $e^{tB}D(|A|^{k/2}) \subset D(|A|^{k/2})$ ,
- (iii) there exists  $c \geq 0$  and  $c' \geq 0$  such that  $B - c$  and  $-B - c'$  generate contraction semi-groups on  $D(|A|^{k/2})$  for the norm  $\|\cdot\|_{k/2}$ .

The *optimal exponential growth* is defined by

$$c_k(A, B) := \sup_{t \in \mathbf{R}} \frac{\log \|e^{tB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))}}{|t|}. \quad (4.1)$$

**Remark 21.** As in Section 5, if

$$t \mapsto e^{tB}$$

is a strongly continuous semi-group on  $D(|A|^{k/2})$  then there exists  $\omega > 0$  and  $C > 0$  such that

$$\|e^{tB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))} \leq C e^{\omega t}, \quad \text{for all } t > 0.$$

Consider the equivalent norm

$$\sup_{t > 0} \|e^{t(B-\omega)}\psi\|_{k/2}$$

then one can assume  $C = 1$ . However with this change of norm one may lose the hilbertian structure and the skew-adjoint character of  $A$ .

**Remark 22.** Notice that the content of Definition 5.(iii) is that the restriction to  $D(|A|^{k/2})$  of the group generated by  $B - c$ , originally acting on  $\mathcal{H}$ , defines a contraction semi-group. We thus have two generators and due to Definition 5.(ii) the domain of the second is included in the first and the smallest domain is dense in  $\mathcal{H}$ . The same comment can be made for  $-B - c'$ . Neglecting the constant  $c$  and  $c'$  and the minus sign, we identify these three operators (which are closed in  $\mathcal{H}$  and  $D(|A|^{k/2})$  respectively) and with an abuse of notation we denote them by the same symbol  $B$  as they are restrictions of  $B$ . The domains of  $B - c$  and  $-B - c'$  acting on  $D(|A|^{k/2})$  are actually both equal to  $\{\phi \in D(|A|^{k/2}) \cap D(B) \mid B\phi \in D(|A|^{k/2})\}$ . They both contain this set as, if  $\phi$  is in this set, then

$$e^{\pm tB}\phi - \phi \mp tB\phi = \int_0^t (e^{\pm sB} - 1)B\phi ds = o(|t|),$$

in  $\mathcal{H}$  or  $D(|A|^{k/2})$ . They are obviously contained in this set since if  $t \mapsto e^{\pm tB}\phi$  is differentiable in  $D(|A|^{k/2})$  then it is also differentiable in  $\mathcal{H}$ .

The invertibility of  $A$  is needed to ensure that  $\|\cdot\|_{k/2}$  is a norm equivalent to the graph norm of  $D(|A|^{k/2})$ . The use of the associated norm is due to the interpolation criterion handful in Lemma 20.

**Remark 23.** The quantity  $c_k(A, B)$  is also the growth abscissa of  $B$  in  $D(|A|^{k/2})$ , see [Prü84, Section 3]. The link between the growth abscissa and the spectral radius of a semigroup on a Hilbert space is considered by Prüss in [Prü84, Section 3].

**Remark 24.** For many systems encountered in the physics literature, the operator  $A$  is skew-adjoint with a spectral gap. Hence the invertibility of  $A$  can be obtained by replacing  $A$  by  $A - \lambda i$  for a suitable  $\lambda$  in  $\mathbf{R}$ . Notice that this translation on  $A$  only induces a global phase shift on the propagator that is physically irrelevant (i.e., undetectable by observations).

The following proposition gives another characterization of the mild coupling using Hille–Yosida Theorem.

**Proposition 17.** *Let  $k$  be a nonnegative real. A pair of skew-adjoint operators  $(A, B)$  with  $A$  invertible is  $k$ -mildly coupled if and only if  $B$  is closed in  $D(|A|^{k/2})$ , and there exists  $\omega$  such that*

$$\|(\lambda I - B)^{-1}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))} \leq \frac{1}{|\lambda| - \omega}, \quad (4.2)$$

for every real  $\lambda$ ,  $|\lambda| > \omega$  in the resolvent set of  $B$ .

Moreover the smallest  $\omega$  satisfying (4.2) is  $c_k(A, B)$  given by (4.1).

*Proof.* If  $(A, B)$  be  $k$ -mildly coupled then  $B - c_k(A, B)$  is the generator of a contraction semi-group in  $D(|A|^{k/2})$ . From Hille–Yosida Theorem, we deduce the equivalence with Definition 5.  $\square$

The following proposition gives an equivalent definition which may be easier to check in practice.

**Proposition 18.** *Let  $k$  be a nonnegative real. A pair of skew-adjoint operators  $(A, B)$  with  $A$  invertible is  $k$ -mildly coupled, if and only if for some  $\omega > 0$ ,*

$$(\omega \pm B)^{-1}D(|A|^{k/2}) \subset D(|A|^{k/2})$$

and for any  $\psi \in (\omega - B)^{-1}D(|A|^{k/2}) = (\omega + B)^{-1}D(|A|^{k/2})$ , one has

$$|\Re\langle |A|^k\psi, B\psi \rangle| \leq \omega \|\psi\|_{D(|A|^{k/2})}^2. \quad (4.3)$$

Moreover the smallest  $\omega$  satisfying (4.3) is  $c_k(A, B)$  given by (4.1).

*Proof.* We first notice that, for any  $\omega$  in the resolvent sets of  $B$  and  $-B$ ,

$$(\omega \pm B)^{-1}D(|A|^{k/2}) \subset D(|A|^{k/2})$$

implies  $(\omega - B)^{-1}D(|A|^{k/2}) = (\omega + B)^{-1}D(|A|^{k/2})$ . Indeed, from the resolvent identity, we deduce

$$(\omega - B)^{-1}D(|A|^{k/2}) \subset (\omega + B)^{-1}D(|A|^{k/2}).$$

Assume that  $(A, B)$  is  $k$ -mildly coupled, then since  $B - c_k(A, B)$  and  $-B - c_k(A, B)$  are generator of contraction semi-groups on  $D(|A|^{k/2})$ , they are closed and maximal dissipative on  $D(|A|^{k/2})$ , their respective resolvent sets contains positive half lines (by means of Hille–Yosida theorem) and their domain, by definition of resolvent, is  $(\omega \pm B)^{-1}D(|A|^{k/2})$  for any  $\omega > c_k(A, B)$ . Since they are maximal dissipative, we have that

$$|\Re\langle |A|^k \psi, B\psi \rangle| \leq c_k(A, B) \|\psi\|_{D(|A|^{k/2})}^2.$$

for any  $\psi \in (\omega - B)^{-1}D(|A|^{k/2}) = (\omega + B)^{-1}D(|A|^{k/2})$ .

Reciprocally,  $B + \omega$  and  $-B + \omega$  are closed as operators on  $\mathcal{H}$  and so they are closed on  $D(|A|^{k/2})$ . Since  $B + \omega$  and  $-B + \omega$  are dissipative on  $D(|A|^{k/2})$ , they are generators of a contractions semi-groups if they are surjective. So they are since  $(\pm B + \omega)^{-1}f \in D(|A|^{k/2})$  for any  $f \in D(|A|^{k/2})$ .  $\square$

The notion of mild coupling is related to the notion of “weak coupling” introduced in [BCC13]. The relation between these two definitions is given by the following lemma.

**Lemma 19.** *Let  $(A, B)$  be a pair of linear operators such that  $A$  is invertible and skew-adjoint with domain  $D(A)$ ,  $B$  is skew-symmetric with  $D(A) \subset D(B)$ ,  $A + uB$  (seen as an operator acting on  $\mathcal{H}$ ) is essentially skew-adjoint on  $D(A)$  for every  $u$  in  $\mathbf{R}$ ,  $D(|A + uB|^{k/2}) = D(|A|^{k/2})$  for some  $k \geq 1$  and for any real  $u$ , and there exists a constant  $C$  such that for every  $\psi$  in  $D(|A|^k)$ ,*

$$|\Re\langle |A|^k \psi, B\psi \rangle| \leq C |\langle |A|^k \psi, \psi \rangle|.$$

*Then  $(A, B)$  is  $k$ -mildly coupled and  $c_k(A, B)$  is the best possible constant  $C$  in the above inequality.*

*Proof.* The assumption that there exists  $k \geq 1$  and a constant  $C$  such that for every  $\psi$  in  $D(|A|^k)$ ,

$$|\Re\langle |A|^k \psi, B\psi \rangle| \leq C |\langle |A|^k \psi, \psi \rangle|$$

and the Nelson Commutator Theorem, see [RS75, Section X.5], imply that  $B$  is essentially skew-adjoint on the domain  $D(|A|^{k/2})$ . Therefore  $B$  is essentially skew-adjoint on  $D(A)$ . Then Trotter Product Formula, see [RS72, Theorem VIII.31], implies that

$$\left( e^{\frac{t}{n}(A+uB)} e^{-\frac{t}{n}A} \right)^n \rightarrow e^{tuB}$$

in the strong sense as  $n$  goes to infinity. Since each of the term of the above sequence is bounded on  $D(|A|^{k/2})$  with a bound  $e^{C|t||u|}$ , see [BCC13, Proposition 2], we conclude that  $e^{tB}$  is bounded on  $D(|A|^{k/2})$  with the same bound  $e^{C|t||u|}$ . Then  $(A, B)$  is  $k$ -mildly coupled.  $\square$

**Remark 25.** In general,  $(A, B)$  can be  $k$ -mildly coupled without being weakly coupled (in the sense of [BCC13, Definition 1]) or without satisfying the assumption of Lemma 19. Indeed for any invertible skew-adjoint unbounded operator  $(A, iA^2)$  is 2-mildly coupled and  $D(A) \not\subset D(iA^2)$  or  $D(A + iA^2) = D(A^2) \neq D(A)$ .

Let us state state an interpolation result.

**Lemma 20.** *Let  $k$  be a positive real. If  $(A, B)$  is  $k$ -mildly coupled then  $(A, B)$  is  $s$ -mildly coupled for any  $s \in [0, k]$  and*

$$c_s(A, B) \leq \frac{s}{k} c_k(A, B).$$

*Proof.* We will consider  $s \in (0, k)$ . Indeed, for  $s = k$  this is obvious and  $s = 0$  there is nothing to prove since  $B$  is skew-adjoint by assumption.

Moreover since  $B$  is skew-adjoint, for every  $\psi$  in  $D(|A|^{\frac{k}{2}})$ ,

$$\|e^{tB}\psi\|_{D(|A|^{k/2})} = \| |A|^{k/2} e^{tB}\psi \| = \| |\mathcal{A}(t)|^{k/2} \psi \|^2.$$

where  $\mathcal{A}(t) = e^{-tB} A e^{tB}$  (which is skew-adjoint with domain  $D(A)$ ).

Since  $(A, B)$  is  $k$ -mildly coupled we deduce

$$\frac{1}{\|A^{-1}\|^k} \leq |\mathcal{A}(t)|^k \leq e^{2c|t|} |A|^k.$$

which from Proposition 36 in Appendix A yields

$$|\mathcal{A}(t)|^s \leq e^{2cs|t|/k} |A|^s,$$

that concludes the proof.  $\square$

A corollary of this interpolation result is the following result which is crucial in our analysis. It shows that if  $(A, B)$  is  $k$ -mildly coupled the  $A$ -boundedness of  $B$  extends naturally to  $D(|A|^{k/2})$ . Hence, from now on, we will work in  $D(|A|^{k/2})$ , that is we consider  $\mathcal{H} = D(|A|^{k/2})$ .

**Lemma 21.** *Let  $k$  be a nonnegative real. Let  $(A, B)$  be  $k$ -mildly coupled and such that  $B$  is  $A$ -bounded. Then*

$$\inf_{\lambda > 0} \|B(A - \lambda)^{-1}\|_{L(D(|A|^{\frac{k}{2}}), D(|A|^{\frac{k}{2}}))} \leq \|B\|_A$$

*Proof.* Note that for any  $s$  in  $[0, k]$  due to Lemma 20,  $B$  is  $s$ -mildly coupled. The proof follows [RS75, Section X.5]. The commutator  $[|A|^k, B] = |A|^k B - B|A|^k$  is defined from  $D(|A|^{k+1})$  to  $D(|A|^k)^*$  which we identify with  $D(|A|^{-k})$  and since  $B$  is  $k$ -mildly coupled, for  $\psi \in D(|A|^{k+1})$ , we have

$$\left| \left\langle \psi, (|A|^k B - B|A|^k)\psi \right\rangle \right| = 2 \left| \Re \left\langle B\psi, |A|^k \psi \right\rangle \right| \leq 2c_k(A, B) \| |A|^{k/2} \psi \|^2.$$

This provides, after polarization, the boundedness of  $[|A|^k, B]$  from  $D(|A|^{k/2})$  to  $D(|A|^{-k/2})$ .

Recall  $B$  is  $A$ -bounded, that is  $B$  bounded from  $D(A)$  to  $\mathcal{H}$ . As  $-B^*$  is an extension of  $B$  and is bounded from  $\mathcal{H}$  to  $D(A)^*$ , by interpolation, see Appendix A,  $B$  is bounded from  $D(|A|^{1+s})$  to  $D(|A|^s)$  for any  $s \in [-1, 0]$ . We now provide a bound on the norm of  $B$  as an operator from  $D(|A|^{1+s})$  to  $D(|A|^s)$ ,  $s \in [-1, 0]$ . We have for any  $\epsilon > 0$  the existence of  $\lambda_\epsilon > 0$  such that for any  $\psi \in D(A)$

$$\|B\psi\| \leq (\|B\|_A + \epsilon) \| |A + \lambda_\epsilon| \psi \|^2$$

and for  $\psi \in \mathcal{H}$

$$\| |A + \lambda_\epsilon|^{-1} B\psi \| \leq (\|B\|_A + \epsilon) \|\psi\|$$

and by interpolation for any  $s \in [0, 1]$  and any  $\psi \in D(|A|^{1-s})$

$$\| |A + \lambda_\epsilon|^{-s} B\psi \| \leq (\|B\|_A + \epsilon) \| |A + \lambda_\epsilon|^{1-s} \psi \|^2.$$

Then since for any  $\delta > 0$  there exist  $C > 0$  such that for  $x > 1$

$$|x|^{-s} \leq C|x^2 + \lambda_\epsilon^2|^{-1/2} + \delta|x^2 + \lambda_\epsilon^2|^{-s/2}$$

and

$$|x^2 + \lambda_\epsilon^2|^{-s/2} \leq C|x^2 + \lambda_\epsilon^2|^{-1/2} + \delta|x|^{-s/2}$$

we deduce, for  $\psi \in D(A)$ , that there exists  $\Lambda_\epsilon$  such that

$$\||A|^{-s}B\psi\| \leq (\|B\|_A + \epsilon)\||A|^{-s}|A + \Lambda_\epsilon|\psi\|.$$

Then, we deduce by density of  $D(A)$  in  $D(|A|^{-s})$

$$\inf_{\lambda>0} \|B(A - \lambda)^{-1}\|_{L(D(|A|^{-s}), D(|A|^{-s}))} \leq \|B\|_A.$$

Hence if  $k \leq 2$ ,  $B|A|^k$  is bounded from  $D(|A|^{k/2+1})$  to  $D(|A|^{-k/2})$ . As  $|A|^k B = B|A|^k + [|A|^k, B]$ ,  $|A|^k B$  extends as a bounded operator from  $D(|A|^{k/2+1})$  to  $D(|A|^{-k/2})$  and  $B$  is bounded from  $D(|A|^{k/2+1})$  to  $D(|A|^{k/2})$ . Hence  $B$  is bounded from  $D(|A|^{1+s})$  to  $D(|A|^s)$  for any  $s \in [-1, k/2]$ . By duality  $B$  is bounded from  $D(|A|^{1+s})$  to  $D(|A|^s)$  for any  $s \in [-1 - k/2, 0]$  and by interpolation for any  $s \in [-1 - k/2, k/2]$ . As for the norm, for any  $\epsilon$  there exists  $L_\epsilon$  such that, for any  $\psi \in D(|A|^{k/2+1})$ ,

$$\begin{aligned} \|B\psi\|_{k/2} &= \||A|^{-k/2}|A|^k B\psi\| \\ &\leq \||A|^{-k/2}B|A|^k\psi\| + \||A|^{-k/2}[|A|^k, B]\psi\| \\ &\leq (\|B\|_A + \epsilon)\||A|^{-k/2}|A|^k|A + \Lambda_\epsilon|\psi\| + 2c_k(A, B)\||A|^{k/2}\psi\| \\ &\leq (\|B\|_A + \epsilon)\||A|^{k/2}|A + L_\epsilon|\psi\|. \end{aligned}$$

Then, we deduce

$$\inf_{\lambda>0} \|B(A - \lambda)^{-1}\|_{L(D(|A|^{-k/2}), D(|A|^{-k/2}))} \leq \|B\|_A,$$

or, by interpolation, as done previously, we deduce

$$\inf_{\lambda>0} \|B(A - \lambda)^{-1}\|_{L(D(|A|^{-s}), D(|A|^{-s}))} \leq \|B\|_A,$$

for any  $s \in [-1 - k/2, k/2]$ .

Hence the Lemma is proved for  $k$  in  $[0, 2n]$  with  $n = 1$ , we now extend it by an induction  $n$ . Assume for  $0 \leq k \leq 2n$ , for some integer  $n$ , that  $B$  is bounded from  $D(|A|^{1-k/2})$  to  $D(|A|^{-k/2})$ . Then as  $|A|^k B = B|A|^k + [|A|^k, B]$  and  $B|A|^k$  is bounded from  $D(|A|^{k/2+1})$  to  $D(|A|^{k/2})$ , we obtain  $|A|^k B$  extends as a bounded operator from  $D(|A|^{k/2+1})$  to  $D(|A|^{-k/2})$  and  $B$  is bounded from  $D(|A|^{k/2+1})$  to  $D(|A|^{k/2})$ . By duality  $B$  is bounded from  $D(|A|^{-k/2})$  to  $D(|A|^{-k/2-1})$ . Hence by interpolation  $B$  is bounded from  $D(|A|^{s+1})$  to  $D(|A|^s)$  for any  $s \in [-1 - k/2, k/2]$ . For instance  $B$  is bounded from  $D(|A|^{1-(k+2)/2})$  to  $D(|A|^{-(k+2)/2})$  and thus we can extend the boundedness of  $B$  from  $D(|A|^{k/2+1})$  to  $D(|A|^{k/2})$  with  $0 \leq k \leq 2(n+1)$ . The norm will follow as in the initialisation.  $\square$

The following result shows sufficient conditions to have  $D(|A + uB|^{k/2}) = D(|A|^{k/2})$  as required in Lemma 19. Otherwise, in general, checking this property in practice is a difficult task. Recall that as  $D(A) \subset D(B)$ ,  $A + uB$  is self-adjoint with  $D(A + uB) = D(A)$  for sufficiently small  $u$  by Kato–Rellich theorem.

**Lemma 22.** *Let  $k$  be a positive real,  $(A, B)$  be  $k$ -mildly coupled, and  $u \in \mathbf{R}$  such that  $|u| < 1/\|B\|_A$ . Then  $D(|A|^s) = D(|A + uB|^s)$  for every  $s \in [0, k/2 + 1]$ .*

*Proof.* We proceed by induction on  $j$  to prove  $D(|A|^{k/2 - [k/2] + j}) = D(|A + uB|^{k/2 - [k/2] + j})$  for  $j \leq [k/2] + 1$ . By Kato–Rellich theorem,  $D(A) = D(A + uB)$  for every  $u$  in  $(-1/\|B\|_A, 1/\|B\|_A)$ . By interpolation, see Corollary 40 in Appendix A,  $D(|A|^s) = D(|A + uB|^s)$  for  $0 \leq s \leq 1$  and in particular for  $s = \frac{k}{2} - [\frac{k}{2}]$ . This initializes the induction for  $j = 0$ .

Let us assume that  $D(A^\ell) = D((A + uB)^\ell)$  for some  $\ell \leq \lfloor k/2 \rfloor$ . By definition,

$$D(A^{\ell+1}) = \{f \in D(A^\ell) \mid Af \in D(A^\ell)\},$$

and, using the inductive hypothesis,

$$D(|A+uB|^{\ell+1}) = \{f \in D(|A+uB|^\ell) \mid |A+uB|f \in D(|A+uB|^\ell)\} = \{f \in D(|A|^\ell) \mid (A+uB)f \in D(|A|^\ell)\}.$$

So that  $D(|A + uB|^{\ell+1})$  is the domain of  $A + uB$  as an operator acting on  $D(|A|^\ell)$ . The domain of  $A$  as an operator acting on  $D(|A|^\ell)$  is  $D(|A|^{\ell+1})$ . Since  $A$  is skew adjoint on  $D(|A|^\ell)$  and  $B - c$  (or  $-B - c'$ ) is dissipative, since  $\ell \leq k/2$ , in  $D(|A|^\ell)$  due to Proposition 18, using Lemma 21 and Kato-Rellich theorem we conclude that  $A + uB - c'$  with domain  $D(|A|^{\ell+1})$  is maximal dissipative in  $D(|A|^\ell)$  for some constant  $c'$  sufficiently large. This implies  $D(|A + uB|^{\ell+1}) = D(|A|^{\ell+1})$ .

This completes the iteration and provides the conclusion.  $\square$

**Remark 26.** Notice that if  $D(|A + uB|^s) = D(|A|^s)$  for some positive real  $s$  then the associated norms are equivalent as the operators are closed.

## 4.2 Higher regularity

From Lemma 6 and Proposition 7, we deduce the following statement.

**Proposition 23.** *Let  $k$  be a nonnegative real,  $(A, B)$  be  $k$ -mildly coupled,  $B$  be  $A$ -bounded, and  $K = [-1/(2\|B\|_A), 1/(2\|B\|_A)]$ . For any  $u \in BV([0, T], K)$ , the family of contraction propagators  $\Upsilon^u$  obtained in Theorem 5 with Lemma 6, satisfies  $\Upsilon_{t,s}^u(D(|A|^{k/2})) \subset D(|A|^{k/2})$ , for any  $(s, t) \in \Delta_{[0,T]}$ , and:*

(i) for any  $t \in [0, T]$  and for any  $\psi_0 \in D(|A|^{k/2})$

$$\|\Upsilon_t^u(\psi_0)\|_{k/2} \leq e^{c_k(A,B) \int_0^t |u|} \|\psi_0\|_{k/2}.$$

(ii) for any  $t \in [0, T]$  and for any  $\psi_0 \in D(|A|^{1+k/2})$  there exists  $M$  (depending only on  $A, B$ , and  $\|u\|_{L^\infty([0,T])}$ )

$$\|\Upsilon_t^u(\psi_0)\|_{1+k/2} \leq M e^{MTV(u, ([0,t], K))} e^{c_k(A,B) \int_0^t |u|} \|\psi_0\|_{1+k/2}$$

Moreover, for every  $\varepsilon$  in  $(0, 1 + k/2)$ , for every  $\psi_0$  in  $D(|A|^{k/2+1-\varepsilon})$ , the end-point mapping

$$\begin{aligned} \Upsilon_T(\psi_0) &: BV([0, T], K) \rightarrow D(|A|^{k/2+1-\varepsilon}) \\ u &\mapsto \Upsilon_T^u(\psi_0) \end{aligned}$$

is continuous.

*Proof.* We prove the existence of propagator in  $D(|A|^{k/2})$ . By Lemma 20, this is also valid for any  $s \in [0, k]$ . This provides several propagators for different exponents  $s$ . All of them are restrictions of the one for  $s = 0$  due to the uniqueness statement of Theorem 5 as  $D(|A|^{k/2}) \subset \mathcal{H}$ .

Below, we will obtain the first part of the statement, (i) and (ii), as a consequence of Theorem 5 and the continuity of the end-point mapping as a consequence both of Corollary 8 in  $\mathcal{H}$  (for  $k = 0$ ) and an interpolation argument (Lemma 35) with statement (ii) that we prove below.

Let us begin with the case  $k = 0$ . By hypothesis,  $(A, B, K)$  satisfies Assumption 1 and, by Lemma 6,  $t \mapsto A + u(t)B$  satisfies 4 for every  $u$  in  $BV(I, K)$ . The statements (i) and (ii) for  $k = 0$  follows from Theorem 5. The continuity of the end-point mapping with value in  $\mathcal{H}$  follows from Corollary 8.

We now consider the case  $k > 0$ . If  $c_k(A, B) = 0$ , Lemma 21 ensures that the triple  $(A, B, K)$  satisfies Assumption 1 in  $D(|A|^{\frac{k}{2}})$  and, for any  $u$  in  $BV(I, K)$ , the mapping  $t \mapsto A + u(t)B$  satisfies Assumption 4 in  $D(|A|^{\frac{k}{2}})$ . Statements (i) and (ii) follows from Theorem 5. In the case where  $c_k(A, B) > 0$ , in order to obtain contraction semi-groups, we consider  $A(t) = A + u(t)B - c_k(A, B)|u(t)|$ . This induces minor technical variations in the proof to check that  $t \mapsto A(t)$  satisfies Assumption 4. For the reader's sake, we detail them below.

Using  $(A, B)$  to be  $k$ -mildly coupled, in Lemma 21 and Kato–Rellich theorem for dissipative operators (see [RS78, Corollary of Theorem X.50]) provides that  $A(t)$  satisfies Assumption (A4.1) with  $I = [0, T]$  and  $D(|A|^{\frac{k}{2}})$  instead of  $\mathcal{H}$ . Notice that indeed the domain of  $A$  as an operator acting on  $D(|A|^{\frac{k}{2}})$  is  $D(|A|^{1+\frac{k}{2}})$ . In the following, we check Assumptions (A4.2) and (A4.3).

From Lemma 21, if  $a \in (\|B\|_A, 2\|B\|_A)$  we deduce that there exists  $b_a$  such that for any  $\psi \in D(|A|^{1+k/2})$

$$\begin{aligned} \|(1 - A - uB + c_k(A, B)|u|)\psi\|_{k/2} &\geq \|(1 - A)\psi\|_{k/2} - |u|\|B\psi\|_{k/2} - c_k(A, B)|u|\|\psi\|_{k/2} \\ &\geq (1 - a|u|)\|(1 - A)\psi\|_{k/2} - |u|(b_a + c_k(A, B))\|\psi\|_{k/2} \end{aligned}$$

or

$$\|(1 - A - uB + c_k(A, B)|u|)\psi\|_{k/2} + |u|(b_a + c_k(A, B))\|\psi\|_{k/2} \geq (1 - a|u|)\|(1 - A)\psi\|_{k/2}.$$

Note that for the choice of  $K$  and  $a$  we have that  $a|u| < 1$ . Therefore

$$\|B\psi\|_{k/2} \leq a\|(1 - A)\psi\|_{k/2} \leq \frac{a}{1 - a|u|}\|(1 - A - uB + c_k(A, B)|u|)\psi\|_{k/2} + \frac{b_a + c_k(A, B)}{1 - a|u|}\|\psi\|_{k/2}.$$

Hence,

$$\begin{aligned} \|(1 - A(t))^{-1}\|_{L(D(|A|^{k/2}), D(|A|^{1+k/2}))} &= \|A(1 - A - u(t)B + c_k(A, B)|u(t)|)^{-1}\|_{k/2} \\ &\leq \|(A + u(t)B - c_k(A, B)|u(t)|)(1 - A - u(t)B + c_k(A, B)|u(t)|)^{-1}\|_{k/2} \\ &\quad + |u(t)|\|B(1 - A - u(t)B + c_k(A, B)|u(t)|)^{-1}\|_{k/2} \\ &\leq 2 + \frac{|u(t)|}{1 - a|u(t)|} (a + b_a + c_k(A, B)). \end{aligned}$$

Recall that, by assumption,  $\sup_{t \in [0, T]} a|u(t)| \leq \frac{a}{2\|B\|_A} < 1$ . Taking the supremum on  $t \in [0, T]$  leads to

$$\sup_{t \in [0, T]} \|(1 - A(t))^{-1}\|_{L(D(|A|^{k/2}), D(|A|^{1+k/2}))} \leq 2 + \frac{a}{2\|B\|_A - a} (a + b_a + c_k(A, B)). \quad (4.4)$$

As moreover for  $A_n(t) = A + u_n(t)B - |u_n(t)|c_k(A, B)$  and  $A(t) = A + u(t)B - |u(t)|c_k(A, B)$  and  $\lambda$  sufficiently large such that  $\text{TV}(A_n, ([0, T], L(D(A), \mathcal{H}))) \leq \text{TV}(u_n, ([0, T], K))(\|B\|_{L(D(A), \mathcal{H})} + c_k(A, B))$ ,  $\|A_n(0)\|_{L(D(A), \mathcal{H})} \leq 1 + |u_n(0)|(\|B\|_{L(D(A), \mathcal{H})} + c_k(A, B))$ , and

$$\begin{aligned} (A_n(t) - \lambda)^{-1} - (A(t) - \lambda)^{-1} &= (u_n(t) - u(t))(A_n(t) - \lambda)^{-1}B(A(t) - \lambda)^{-1} \\ &\quad + (|u_n(t)| - |u(t)|)(A_n(t) - \lambda)^{-1}c_k(A, B)(A(t) - \lambda)^{-1} \end{aligned}$$

so that the strong resolvent convergence of  $A_n$  to  $A$  turns to be a consequence of the convergence of  $u_n$  to  $u$  in  $BV([0, T], K)$ .

For any  $k$ , the proof of the continuity of the end-point mapping in  $D(|A|^{\frac{k}{2}+1-\varepsilon})$  is, as already said, a consequence of Corollary 8 in  $\mathcal{H}$  and an interpolation argument with statement (ii) (Lemma 35).  $\square$

**Remark 27.** The bound on the control  $|u| \leq 1/(2\|B\|_A)$  in Proposition 23 is technical. We could enlarge the set of admissible control and consider  $K = [-1/\|B\|_A + \varepsilon, 1/\|B\|_A - \varepsilon]$  for some  $\varepsilon > 0$ . In this case the constant  $a$  in the proof would be in the open interval  $(\|B\|_A, \|B\|_A/(1 - \varepsilon\|B\|_A))$ , the bound (4.4) would depend on  $\varepsilon$ , and would tend to infinity as  $\varepsilon$  goes to 0.

We now state another version of Corollary 16.

**Corollary 24.** *Let  $k$  be a nonnegative real. Let  $(A, B)$  be  $k$ -mildly coupled,  $B$  be  $A$ -bounded, and  $K = (-1/(2\|B\|_A), 1/(2\|B\|_A))$ . Then, for every  $\varepsilon$  in  $(0, 1 + k/2)$  and every  $\psi_0$  in  $D(|A|^{1+k/2-\varepsilon})$   $\{ \Upsilon_t^u(\psi_0), u \in BV([0, +\infty), K), t \geq 0 \}$  is a countable union of relatively compact subsets in  $D(|A|^{\frac{k}{2}+1-\varepsilon})$ .*

*Proof.* The proof follows step-by-step the principle exposed in Section 1.1.2 and the proof of Corollary 9.  $\square$

*Proof of Theorem 3.* Theorem 3 is consequence of Corollary 24 when the  $\|B\|_A$  vanishes.  $\square$

**A comment on the exact controllability associated with the time reversibility.** Let  $(A, B, K)$  satisfies Assumption 1 (or Assumptions 2) with  $A$  skew-adjoint and  $B$  skew-symmetric then  $(-A, -B, K)$  satisfies Assumption 1 (or Assumptions 2). If  $(A, B)$  is  $k$ -mildly coupled then  $(-A, -B)$  is  $k$ -mildly coupled.

For  $u$ , a bounded variation function (or a Radon measure) on  $(0, T]$  with value in  $K$  and  $\Upsilon^u$  the associated contraction propagator. For any  $(t, s) \in \Delta_{[0, T]}$ ,  $\Upsilon_{t, s}^u$  is unitary and its inverse coincides with  $\Upsilon_{T-s, T-t}^{u(T-\cdot)}$  where  $u(T-\cdot)$  denotes  $t \in [0, T] \mapsto u(T-t)$  in the framework of Assumption 1 (or  $t \in [0, T] \mapsto u((0, T]) - u((0, t]) = u([t, T])$ ) in the framework of Assumption 2).

### 4.3 Extension to Radon measures

The conclusion of Proposition 15 can be extended to  $D(|A|^{k/2})$  if Assumption (A3.3) is true in  $D(|A|^{\frac{k}{2}})$  instead of  $\mathcal{H}$ . This is indeed the only missing assumption needed in order to apply Corollary 10 with  $D(|A|^{\frac{k}{2}})$  instead of  $\mathcal{H}$ . Without this assumption the following result together with the interpolation result of Lemma 35 gives an interesting extension.

**Proposition 25.** *Let  $k$  be a positive real. Let  $(A, B)$  satisfy Assumption 3 and be  $k$ -mildly coupled. Then, for every  $s \in [0, k]$ ,  $\psi_0 \in D(|A|^{s/2})$ , for every  $T \geq 0$ , one has  $\Upsilon_T^{\partial v}(\psi_0) \in D(|A|^{s/2})$  and*

$$\|\Upsilon_T^{\partial v}(\psi_0)\|_{s/2} \leq e^{\frac{s}{k}c_k(A, B)|u|([0, T])} \|\psi_0\|_{s/2}$$

for every  $v$  in  $BV([0, T], K)$  with derivative  $v' = u \in \mathcal{R}([0, T])$ .

*Proof.* We give the proof for  $s = k$ , then by Lemma 20 the proof easily extend to the case  $s < k$ .

Consider a sequence  $v_n$  of piecewise constant functions converging to  $v$  pointwise with  $\|v_n\|_{BV([0, T])} \leq K$ . Then  $v_n$  is the cumulative function of  $v'_n$ , a discrete sum of Dirac delta functions and, from (3.1),  $\Upsilon_t^{\partial v_n}$  is a product of unitary operators of the form

$$e^{v_n B} e^{-v_n B} e^{t A} e^{v_n B} = e^{t A} e^{v_n B}.$$

So that, for every  $\psi$  in  $D(|A|^{k/2})$ ,

$$\|e^{v_n B} e^{-v_n B} e^{t A} e^{v_n B} \psi\|_{k/2} = \|e^{t A} \psi\|_{k/2} \leq M(v) \|\psi\|_{k/2}$$

where  $M(v) := \|e^{v B}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))}$ . From Definition 5, equation (4.1), and  $M(v_1 + v_2) \leq M(v_1)M(v_2)$  for any pair  $(v_1, v_2)$  in  $[0, \delta]^2$  imply

$$M(v) \leq e^{c_k(A, B)|v|}, \quad \text{for all } v \in \mathbf{R}.$$



Hence, for every  $n$ ,

$$\|\Upsilon_t^{v_n}(\psi_0)\|_{k/2} \leq e^{c_k(A,B)K} \|\psi_0\|_{k/2}.$$

For every  $f$  in  $D(|A|^k)$ ,

$$|\langle |A|^k f, \Upsilon_t^{v_n} \psi_0 \rangle| \leq \|f\|_{k/2} \|\psi_0\|_{k/2} e^{c_k(A,B)K}.$$

Because of the continuity result (Proposition 7, Corollary 10 and Remark 13), the left hand side tends to  $|\langle |A|^k f, \Upsilon_t^u \psi_0 \rangle|$  as  $n$  tends to infinity. Hence, for every  $f$  in  $D(|A|^k)$

$$|\langle |A|^k f, \Upsilon_t^u \psi_0 \rangle| \leq \|f\|_{k/2} \|\psi_0\|_{k/2} e^{c_k(A,B)K}.$$

As a consequence,  $\Upsilon_t^u \psi_0$  belongs to  $D((|A|^{k/2})^*) = D(|A|^{k/2})$  and

$$\||A|^{k/2} \Upsilon_t^u \psi_0\| \leq \|\psi_0\|_{k/2} e^{c_k(A,B)K}. \quad \square$$

**Remark 28.** Assumption (A3.3) implies that  $(A, B)$  is 2-mildly coupled. Indeed, if  $(A, B)$  is a pair of skew-adjoint operators operators satisfying Assumption 3, then Assumption (A3.3) implies, see Remark 13, for small  $|t|$  that, for every  $\psi$  in  $D(A)$ ,

$$\begin{aligned} \||A|e^{-tB}\psi\| &= \|Ae^{-tB}\psi\| = \|e^{tB}Ae^{-tB}\psi\| \leq \|e^{tB}Ae^{-tB}\psi - A\psi\| + \|A\psi\| \\ &\leq (1 + L|t|)\|A\psi\| \leq e^{L|t|}\|A\psi\| = e^{L|t|}\||A|\psi\| \end{aligned}$$

as the map  $t \in \mathbf{R} \mapsto e^{tB}Ae^{-tB} \in L(D(A), \mathcal{H})$  is locally Lipschitz with constant  $L$ . Thus  $(A, B)$  is 2-mildly coupled. We also refer to Remark 18.

As a consequence of Corollary 10 and Lemma 35 we have the following proposition.

**Proposition 26.** *Let  $k$  be a positive real, let  $(A, B)$  satisfy Assumption 3, and let  $(A, B)$  be  $k$ -mildly coupled. Then for any  $s \in [0, k)$ , for every  $\psi_0$  in  $D(|A|^{s/2})$ , the end-point mapping*

$$\begin{aligned} \Upsilon(\psi_0) : BV([0, T], \mathbf{R}) &\rightarrow D(|A|^{s/2}) \\ v &\mapsto \Upsilon_T^{dv}(\psi_0) \end{aligned}$$

is continuous.

*Proof.* Let  $(v_n)_{n \in \mathbf{N}}$  be a converging sequence in  $BV([0, T], \mathbf{R})$  to some  $v$  in  $BV([0, T], \mathbf{R})$ . Then  $\Upsilon_T^{dv_n}(\psi_0) - \Upsilon_T^{dv}(\psi_0)$  is uniformly bounded in  $D(|A|^{k/2})$  (by Proposition 25) and converges to 0 in  $\mathcal{H}$  (by Proposition 7, Corollary 10 and Remark 13). By Lemma 35 it converges to 0 in  $D(|A|^{s/2})$  for  $s < k$ .  $\square$

**Remark 29.** One can notice that under the assumptions of Proposition 26 both Proposition 15 and Corollary 16 extend to  $D(|A|^{s/2})$  for  $s \in [0, k)$ .

## 5 Bounded control potentials

### 5.1 Dyson expansion solutions

In the Hilbert setting, if  $A$  is maximal dissipative and  $B$  stabilizes  $D(A)$  Corollary 16 provides an extension of [BMS82, Theorem 3.6] to  $L^1$  controls. This can be extended to the Banach framework with  $A$  generator of strongly continuous semi-group since the assumption on  $B$  allows integration by part in the Dyson expansion for the propagator (see (5.2) below) and thus reduces the analysis to absolute continuous function. Hence the extra regularity in  $B$  compensates the lack of reflexivity of

$L^1$ . Below we extend Corollary 16 with bounded control potentials, for  $L^1$  controls, in the Banach framework without any other assumption in  $B$ .

Troughout this section only, we consider a Banach space  $\mathcal{X}$  and we assume that  $A$ , acting on  $\mathcal{X}$ , is the generator of a strongly continuous semi-group with domain  $D(A)$  and  $B$  is bounded. Then for every  $u$  in  $\mathbf{R}$ ,  $A + uB$  is also a generator of a strongly continuous semi-group with domain  $D(A)$ . This can be deduced from an analysis of the Dyson expansion.

Since  $A$  generates a strongly continuous semi-group there exist  $C_A > 0$  and  $\omega \in \mathbf{R}$  such that

$$\|e^{tA}\| \leq C_A e^{\omega t}, \quad \forall t > 0. \quad (5.1)$$

For the equivalent norm

$$N(\psi) = \sup_{t>0} \|e^{t(A-\omega)}\psi\|,$$

we have that  $A - \omega$  is the generator of a contraction semi-group. An operator  $B \in L(\mathcal{X})$  is bounded for the norm  $N$  and let  $\|B\|_N$  be its norm. Now for every  $u \in BV([0, T], [-R, R])$  we consider the family of operators  $A - \omega + u(t)B - R\|B\|_N$  which satisfies the assumptions of [Kat53] in the Banach space structure associated with the norm  $N$ . So that in this case the results of § 2.2 are still valid.

It is classical (see [BMS82]) that the input-output mapping  $\Upsilon$  admits a unique continuous extension to  $L^1(\mathbf{R}, \mathbf{R})$ . We consider below the extension to Radon measures. Recall that  $C_A$  is defined in (5.1).

**Theorem 27.** *Let  $A$ , with domain  $D(A)$ , be the generator of a strongly continuous semi-group on  $\mathcal{X}$  and let  $B$  be bounded on  $\mathcal{X}$ . Then, for every  $\psi_0$  in  $\mathcal{X}$ , for any  $u \in \mathcal{R}([0, T])$ , the Cauchy problem (2.1) with initial condition  $\psi_0$ , admits a unique mild solution  $t \mapsto \Xi_{t,s}^u \psi_0$  bounded in  $\mathcal{X}$  uniformly on  $[0, T]$ , that is for every  $(s, t) \in \Delta_{[0, T]}$*

$$\begin{aligned} \Xi_{t,s}^u \psi_0 &= e^{(t-s)A} \psi_0 + \int_{(s,t]} e^{(t-s_1)A} B \Xi_{s_1,s}^u \psi_0 du(s_1) \\ \sup_{(s,t) \in \Delta_{[0, T]}} \|\Xi_{t,s}^u \psi_0\| &< \infty. \end{aligned}$$

Moreover

- (i)  $\Xi^u(t, t) = I_{\mathcal{X}}$ ,
- (ii)  $\Xi_{t,s}^u = \Xi^u(t, r) \Xi^u(r, s)$ , for any  $s < r < t$ ,
- (iii) if  $u$  has bounded variation on  $[0, T]$ , for any  $\psi_0 \in \mathcal{X}$ ,  $(s, t) \in \Delta_{[0, T]} \mapsto \Xi_{t,s}^u \psi_0$  is strongly continuous in  $s$  and  $t$  and if  $\psi_0 \in D(A)$  then it is strongly right differentiable in  $t$  with derivative  $(A + u(t+0)B) \Upsilon^u(t, s) \psi_0$ ,
- (iv) for any  $u \in \mathcal{R}([0, T])$ ,  $\Xi^u$  satisfies

$$\|\Xi_{t,s}^u\|_{L(\mathcal{X})} \leq C_A e^{\omega|t-s| + |u|([s,t]) C_A \|B\|},$$

- (v) for any  $r > 0$ ,  $R > 0$ ,  $\psi_0 \in \mathcal{X}$  with  $\|\psi_0\| = r > 0$ , the set

$$\{\Xi_{t,s}^u \psi_0, u \in \mathcal{R}([0, T]), |u|((0, T]) \leq R, (s, t) \in \Delta_{[0, T]}\}$$

is relatively compact.

*Proof.* Let  $u \in L^1((0, T), \mathbf{R})$ . Let  $\psi_0$  in  $\mathcal{X}$ . The associated propagator  $t \mapsto \Xi_{t,s}^u \psi_0$  is absolutely continuous and satisfies, see [BMS82]

$$\Xi_{t,s}^u \psi_0 = e^{(t-s)A} \psi_0 + \int_{(s,t]} e^{(t-s_1)A} u(s_1) B \Xi_{s_1,s}^u \psi_0 ds_1,$$

and replacing iteratively  $\Xi_{t,s}^u \psi_0$  by its expression  $p$  times, we get the formal expansion

$$\begin{aligned} \Xi_{t,s}^u \psi_0 &= e^{(t-s)A} \psi_0 + \sum_{n=1}^{\infty} \int_{s < s_1 < s_2 < \dots < s_n < t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \circ \dots \\ &\dots \circ B e^{(s_2-s_1)A} B e^{(s_1-s)A} \psi_0 u(s_1) u(s_2) \dots u(s_n) ds_1 \dots ds_n. \end{aligned}$$

This allows us to extend the propagator to Radon measures. Namely let  $u \in \mathcal{R}([0, T])$ , define for every  $n \in \mathbf{N}$  the linear operator

$$W_{(n)}^u(t, s) \psi_0 := \int_{s < s_1 < s_2 < \dots < s_n < t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \dots B e^{(s_2-s_1)A} B e^{(s_1-s)A} \psi_0 du(s_1) du(s_2) \dots du(s_n).$$

Notice that

$$W_{(0)}^u(t, s) \psi_0 = e^{(t-s)A} \psi_0, \quad \text{and} \quad W_{(n+1)}^u(t, s) \psi_0 = \int_{(s,t]} e^{(t-\tau)A} B W_{(n)}^u(\tau, s) \psi_0 du(\tau).$$

As  $B$  is bounded,

$$\|W_{(n)}^u(t, s) \psi_0\| \leq e^{\omega(t-s)} C_A^{n+1} \|B\|^n \|\psi_0\| \int_{s < s_1 < s_2 < \dots < s_n < t} d|u|(s_1) d|u|(s_2) \dots d|u|(s_n),$$

and since  $(s, t)^n$  contains the disjoint union of  $\{s < s_{\sigma(1)} < s_{\sigma(2)} < \dots < s_{\sigma(n)} < t\}$  over all permutations  $\sigma$  of  $\{1, 2, \dots, n\}$

$$\|W_{(n)}^u(t, s) \psi_0\| \leq e^{\omega(t-s)} C_A^{n+1} \|B\|^n \|\psi_0\| \frac{|u|((s, t))^n}{n!}.$$

Note that

$$\Xi_{t,s}^u = \sum_{n=0}^{\infty} W_{(n)}^u(t, s), \tag{5.2}$$

converges in norm in the set  $L(\mathcal{X})$  of the bounded operators of  $\mathcal{X}$ . This also provides

$$\|\Xi_{t,s}^u\|_{L(\mathcal{X})} \leq C_A e^{\omega|t-s| + |u|((s,t)) C_A \|B\|}.$$

This provides a solution  $F$  of

$$F(t, s) = e^{(t-s)A} \psi_0 + \int_{(s,t]} e^{(t-s_1)A} B F(s_1, s) du(s_1).$$

Let us now consider the uniqueness, which amounts at proving

$$F(t, s) = \int_{(s,t]} e^{(t-s_1)A} B F(s_1, s) du(s_1) \implies F \equiv 0.$$

Note that

$$F(t, s) = \int_{(s,t]} e^{(t-s_1)A} B F(s_1, s) du(s_1) \implies \|F(t, s)\| \leq e^{\omega(t-s)} C_A \|B\| |u|((s, t)) \sup_{s_1 \in (s,t)} \|F(s_1, s)\|.$$

so that  $e^{\omega(t-s)}C_A\|B\|\|u\|((s,t)) < 1$  implies  $F \equiv 0$ . Alternatively, iterating the fixed point equation, we end up with the estimate

$$\|F(t,s)\| \leq e^{\omega(t-s)}C_A^n\|B\|^n \frac{|u|((s,t))^n}{n!} \sup_{s_1 \in (s,t)} \|F(s_1,s)\|.$$

As for  $n$  large enough  $e^{\omega(t-s)}C_A^n\|B\|^n \frac{|u|((s,t))^n}{n!} < 1$  the same conclusion holds.

Then we have that

$$\begin{aligned} \Xi_{t,s}^u \psi_0 &= e^{(t-s)A} \psi_0 + \int_{(s,t]} e^{(t-s_1)A} B \Xi_{s_1,s}^u \psi_0 du(s_1) \\ &= e^{(t-s)A} \psi_0 + \int_{(s,r]} e^{(t-s_1)A} B \Xi_{s_1,s}^u \psi_0 du(s_1) \\ &\quad + \int_{(r,t]} e^{(t-s_1)A} B \Xi_{s_1,s}^u \psi_0 du(s_1) \\ &= e^{(t-r)A} \Xi_{r,s}^u \psi_0 + \int_{(r,t]} e^{(t-s_1)A} B \Xi_{s_1,s}^u \psi_0 du(s_1) \\ &= \Xi_{t,r}^u \Xi_{r,s}^u \psi_0 \end{aligned}$$

where we used the uniqueness in the last identity.

The differentiability properties in the bounded variation case are due to [Kat53, Theorem 1] since  $A - \omega$  is the generator of a contraction semi-group and since  $B$  in  $L(\mathcal{X})$  then  $B$  is bounded for the norm  $N$ . So that  $A - \omega + u(t)B - R\|B\|_N$  for any  $R > |u|_\infty$  satisfies the assumptions of [Kat53, Theorem 1] in the Banach space  $\mathcal{X}$  with norm  $N$ .

We now consider the compactness property in the last statement. Without loss of generality by linearity and up to scaling  $B$ , we can assume  $r = R = 1$ . Let us prove that, for  $\|\psi_0\| = 1$ ,

$$\{\Xi_{t,s}^u \psi_0, u \in \mathcal{R}([0, T]), |u|((0, T]) \leq 1, (s, t) \in \Delta_{[0, T]}\}$$

is totally bounded for the  $\mathcal{X}$  topology. Then its closure will be totally bounded and complete and thus compact.

Let us consider a radius  $\epsilon > 0$ . In place of  $\Xi_{t,s}^u \psi_0$ , due to its norm convergence we can consider one of the truncated series in (5.2), namely

$$\sum_{n=0}^{n_\epsilon} W_{(n)}^u(t, s),$$

for some  $n_\epsilon \in \mathbf{N}$  such that

$$\|\Xi_{t,s}^u - \sum_{n=0}^{n_\epsilon} W_{(n)}^u(t, s)\| \leq \sum_{n=n_\epsilon+1}^{\infty} e^{\omega T} C_A^{n+1} \|B\|^n \|\psi_0\| \frac{1}{n!} \leq \epsilon,$$

Since we consider a finite number of  $W_{(n)}(\cdot, \cdot)$ , namely  $n_\epsilon$  of them, it is then enough to prove that

$$\mathcal{W}_n^T := \{W_n^u(t, s) \psi_0, u \in \mathcal{R}([0, T]), |u|((0, T]) \leq 1, (s, t) \in \Delta_{[0, T]}\}$$

is totally bounded for the  $\mathcal{X}$  topology for any integer  $n$ . This will be done by iteration on  $n \in \mathbf{N} \cup \{0\}$ :

- For  $n = 0$ ,  $W_0^u(t, s) \psi_0 = e^{(t-s)A} \psi_0$  and since  $\Delta_{[0, T]}$  is compact, the strong continuity provides the compactness of  $\mathcal{W}_0^T$ .

- For any integer  $n$ , we now assume  $\mathcal{W}_n^T$  is totally bounded. The map

$$(\tau, t, \psi) \in \Delta_{[0,T]} \times \mathcal{X} \mapsto e^{(t-\tau)A} B\psi \in \mathcal{X}$$

is continuous. So

$$\mathcal{Z}_n^T := \left\{ e^{(t-\tau)A} B W_n^u(\tau, s) \psi_0, u \in \mathcal{R}([0, T]), |u|((0, T]) \leq 1, (s, \tau) \in \Delta_{[0,T]}, (\tau, t) \in \Delta_{[0,T]} \right\}$$

is totally bounded.

For any  $\delta > 0$ , there exist  $\psi_1, \dots, \psi_{N_\delta}$  in  $\mathcal{X}$  such that

$$\mathcal{Z}_n^T \subset \cup_{j=1}^{N_\delta} B\mathcal{X}(\psi_j, \delta).$$

Let  $\phi_1, \dots, \phi_{N_\delta}$  be a partition of the unity in  $\mathcal{Z}_n^T$  such that  $\text{supp } \phi_j \subset \overline{B\mathcal{X}(\psi_j, 2\delta)}$  and  $\pi : \psi \in \mathcal{X} \mapsto \sum_{j=1}^{N_\delta} \psi_j \phi_j(x)$ .

Define  $p_n^u(t, \tau, s) := \pi(e^{(t-\tau)A} B W_n^u(\tau, s) \psi_0)$  and  $\phi_{n,j}^u(t, \tau, s) := \phi_j(e^{(t-\tau)A} B W_n^u(\tau, s) \psi_0)$ , then

$$p_n^u(t, \tau, s) = \sum_{j=1}^{N_\delta} \psi_j \phi_{n,j}^u(t, \tau, s)$$

and

$$\|e^{(t-\tau)A} B W_n^u(\tau, s) \psi_0 - p_n^u(t, \tau, s)\| \leq 2\delta.$$

Thus  $\mathcal{W}_{n+1}^T$  is totally bounded if

$$\mathcal{P}_n^T := \left\{ \int_{(s,t)} p_n^u(t, \tau, s) u(\tau) d\tau, u \in \mathcal{R}([0, T]), |u|((0, T]) \leq 1, (s, \tau) \in \Delta_{[0,T]}, (\tau, t) \in \Delta_{[0,T]} \right\}$$

is totally bounded. Since for  $u \in \mathcal{R}([0, T])$ ,  $|u|((0, T]) \leq 1$ ,  $(s, \tau) \in \Delta_{[0,T]}$  and  $(\tau, t) \in \Delta_{[0,T]}$

$$\int_{(s,t)} p_n^u(t, \tau, s) du(\tau) = \sum_{j=1}^{N_\delta} \psi_j \int_{(s,t)} \phi_{n,j}^u(t, \tau, s) du(\tau)$$

and

$$\left| \int_{(s,t)} \phi_{n,j}^u(t, \tau, s) du(\tau) \right| \leq |u|((0, T]) \leq 1$$

this implies  $\mathcal{P}_n^T$  is relatively compact (and thus totally bounded).

This concludes the iteration. We thus have the relative compactness of

$$\{\Xi_{t,s}^u \psi_0, u \in \mathcal{R}([0, T]), |u|((0, T]) \leq 1, (s, t) \in \Delta_{[0,T]}\}. \quad \square$$

**Corollary 28.** *Let  $A$  be the generator of a strongly continuous semi-group on  $\mathcal{X}$ , let  $B$  be bounded and denote with  $\Xi$  the propagator defined in Theorem 27. Then for every  $\psi_0$  in  $\mathcal{X}$ , the set*

$$\text{Att}_{\mathcal{R}}^{\Xi}(\psi_0) := \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0, T])} \{\Xi_{t,0}^u \psi_0 | t \in [0, T]\}$$

is contained in a countable union of compact subsets of  $\mathcal{X}$ .

*Proof.* Due to Theorem 27, the proof is similar to the one of Corollary 16.  $\square$

We are now ready to prove Proposition 2.

*Proof of Proposition 2.* As already mentioned, the well-posedness result is classical (see [BMS82] for instance), while the property of the attainable set with  $L^1$  controls follows from Corollary 28.  $\square$

**Remark 30.** If  $\mathcal{X}$  is an Hilbert space  $\mathcal{H}$ ,  $A$  is skew-adjoint,  $B$  is bounded in  $D(|A|^{k/2})$ , then  $D(|A|^{k/2})$  can be considered in place of  $\mathcal{X}$  in all the analysis of the present section. These leads to result similar to Section 4 on the mild coupling theory in a simpler way.

## 5.2 On the notion of solution in the Radon framework

The previous theorem does not state the continuity with respect to the control  $u$  for the  $\mathcal{R}([0, 1])$  topology. With a Dirac measure  $\delta_{t_0}$ ,  $t_0 \in (0, T]$ , it turns out that the solution built here is

$$\Xi_{t,s}^u \psi_0 = e^{(t-s)A} \psi_0 + e^{(t-t_0)A} B e^{(t_0-s)A} \psi_0 \mathbb{I}_{(s,t]}(t_0). \quad (5.3)$$

This does not coincide with the generalized propagator in Definition 4 even if the framework is set similarly for instance, as in Remark 19, when  $A = 0$  as the latter is

$$\Upsilon_{t,s}^u \psi_0 = e^{B \mathbb{I}_{(s,t]}(t_0)} \psi_0$$

Both expansions coincide only up to the first order term in the control. This discrepancy is due to the noncontinuity of the cumulative function of the control.

If we restrict the analysis to controls with continuous cumulative functions and set the topology to the one of the total variation, the continuity is restored and both constructions thus coincide.

Consequently the propagator in Theorem 27 is not continuous in  $u$  and is not the extension of the corresponding propagator for, say, controls with continuous cumulative functions when the topology is the one we choose for  $\mathcal{R}([0, T])$ . This also induce that the accumulations points of the compact set

$$\{\Xi_{t,s}^u \psi_0, u \in \mathcal{R}([0, T]), t \mapsto u((0, t]) \text{ is continuous}, |u|((0, T]) \leq 1, (s, t) \in \Delta_{[0, T]}\}$$

are not given by values of the propagator in Theorem 27 and thus actual solutions but the propagator in Definition 4.

## 5.3 Noninvariance of the domain

In this section, we consider the invariance of the domain of  $A$ , in the framework of Theorem 27, by  $\Xi^u$  when  $u$  is in  $L^1([0, T], \mathbf{R})$ . Notice that if  $u$  is in  $L^1([0, T], \mathbf{R})$ ,  $\Upsilon^u$  and  $\Xi^u$  coincide if Assumption 2 is fulfilled and, hence, the invariance of the domain is a consequence of Theorem 5. The question is whether this is still true when  $B$  is bounded but the corresponding  $C^0$ -semi-group does not preserve  $D(A)$ . The answer is negative and we provide a counter-example.

Let  $\mathcal{X} = L^2(\mathbf{R})$ ,  $A = \partial_x$  with  $D(A) = H^1(\mathbf{R})$  and  $B = iw$  for some bounded measurable function  $w$ . This provides a controlled transport equation and the corresponding solution of (1.1) with  $u \in L^1(\mathbf{R})$  is given by

$$\Xi_t^u(\psi_0)(x) = e^{\frac{i}{2} \int_{-t-x}^{t-x} u(\frac{t-x+\tau}{2}) w(\frac{t+x-\tau}{2}) d\tau} \psi_0(t+x)$$

which rewrites as

$$\Xi_t^u(\psi_0)(x) = e^{i \int_x^{t+x} u(t-s) w(s) ds} \psi_0(t+x).$$

Let us set  $w = \mathbb{I}_{[0,+\infty)}$  and get for  $t \geq 0$  and  $x \geq 0$

$$\Xi_t^u(\psi_0)(x) = e^{i \int_{[-x, t-x]} u(s) ds} \psi_0(t+x).$$

For fixed time  $t$ , the function  $x \mapsto e^{i \int_{[-x, t-x]} u(s) ds}$  is absolutely continuous and the distributional derivative of  $x \mapsto \Xi_t^u(\psi_0)(x)$  is given by

$$\Xi_t^u(\psi_0)(x) = e^{i \int_{[-x, t-x]} u(s) ds} (\psi_0'(t+x) + i(u(-x) - u(t-x))\psi_0(t+x))$$

for  $t > 0$  and  $x > 0$ .

If  $\psi_0$  is in  $H^1(\mathbf{R})$  then  $\Xi_t^u(\psi_0)$  is in  $H^1(\mathbf{R})$  if and only if

$$v : x \mapsto (u(-x) - u(t-x))\psi_0(t+x)$$

is in  $L^2(\mathbf{R})$ .

Set  $u : t \mapsto |1-t|^{-1/2}$ , which is integrable but not square integrable, and  $\psi_0$  a smooth compactly supported function equal to 1 in  $[1-\varepsilon, 1+\varepsilon]$ , for some  $\varepsilon \in (0, 1/2)$ . Consider  $t \in [1-\varepsilon/2, 1+\varepsilon/2]$ ,  $x \mapsto \psi_0(t+x)$  is equal to 1 on  $[1-t-\varepsilon, 1-t+\varepsilon] \subset [-\frac{3}{2}\varepsilon, \frac{3}{2}\varepsilon] \subset [-3/4, 3/4]$ . Hence  $-1 \notin [1-t-\varepsilon, 1-t+\varepsilon]$ . While  $[-\varepsilon/2, \varepsilon/2] \subset [1-t-\varepsilon, 1-t+\varepsilon]$  and  $x = t-1 \in [-\varepsilon/2, \varepsilon/2]$ . This implies that  $v$  is not square integrable on  $[1-t-\varepsilon, 1-t+\varepsilon]$  for any  $t \in [1-\varepsilon/2, 1+\varepsilon/2]$ .

## 6 Examples

Most of the examples of bilinear control systems (2.1) encountered in the literature, also without any relation to quantum control, deal with bounded control operator  $B$ . Proposition 2 applies and allows, for instance, to complete the studies of the rod equation with clamped ends made in [BMS82, Section 6, Example 4] and [Bea08]. In the following, we concentrate on examples in relation with quantum control.

### 6.1 Quantum systems with smooth potentials on compact manifolds

This example motivated the present analysis because of its physical importance. We consider  $\Omega$  a compact Riemannian manifold endowed with the associated Laplace-Beltrami operator  $\Delta$  and the associated measure  $\mu$ . For  $r$  a positive real,  $D(|\Delta|^{\frac{r}{2}}) = H^r(\Omega, \mathbf{C})$ . Since  $\Omega$  is compact, for  $r > r'$ ,  $D(|\Delta|^{\frac{r}{2}}) \subset D(|\Delta|^{\frac{r'}{2}})$  is a compact embedding.

Let  $k \in \mathbf{N}$ . Let  $V, W : \Omega \rightarrow \mathbf{R}$  two functions of class  $C^{2(k-1)}$ , and the bilinear quantum system

$$i \frac{\partial \psi}{\partial t} = \Delta \psi + V \psi + u(t)W \psi. \tag{6.1}$$

With the notations of Section 2,  $\mathcal{H} = L^2(\Omega, \mathbf{C})$  endowed with the Hilbert product  $\langle f, g \rangle = \int_{\Omega} \bar{f}g d\mu$ ,  $A = -i(\Delta + V)$  and  $B = -iW$ . As  $V$  is continuous and so bounded,  $A$  has a spectral gap. Up to subtracting a sufficiently large constant, we can assume  $A$  is positive and invertible.

For  $r$  a positive real with  $r \leq 2k$ ,  $D(|A|^{\frac{r}{2}}) = H^r(\Omega, \mathbf{C})$ . Since  $B$  is bounded from  $D(|A|^{\frac{s}{2}})$  to  $D(|A|^{\frac{s}{2}})$ , for  $s$  a positive real with  $s \leq 2(k-1)$ ,  $(A, B, \mathbf{R})$  satisfies Assumption 1 and  $(A, B)$  is  $s$ -mildly coupled by Proposition 18.

In particular, the two notions of propagators  $\Upsilon$  and  $\Xi$  defined in Proposition 13 and Theorem 27 respectively can be used and we have the following statement.

**Proposition 29.** For every  $T > 0$ , for every  $\psi_0$  in  $H^{2(k-1)}(\Omega, \mathbf{C})$ , the sets

$$\begin{aligned} & \bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0, T])} \{\alpha \Upsilon_t^u \psi_0, t \in [0, T]\} \\ & \bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0, T])} \{\alpha \Xi_t^u \psi_0, t \in [0, T]\} \end{aligned}$$

are contained in countable unions of compact subsets of  $H^{2(k-1)}(\Omega, \mathbf{C})$  and, in particular, they have dense complement in  $H^{2(k-1)}$ .

For any  $\varepsilon \in (0, 1)$ , if  $\psi_0$  in  $H^{2(k-\varepsilon)}(\Omega, \mathbf{C})$ , the set

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in BV([0, T], \mathbf{R})} \{\alpha \Upsilon_t^u \psi_0, t \in [0, T]\}$$

contained in a countable union of compact subsets of  $H^{2(k-\varepsilon)}(\Omega, \mathbf{C})$  and, in particular, it has dense complement in  $H^{2(k-\varepsilon)}(\Omega, \mathbf{C})$ .

*Proof.* The first statement is an adaptation of Proposition 15 and Corollary 16, see Remark 29.

The second statement follows from Corollary 28.

The last statement is a consequence of Corollary 24.  $\square$

Notice that from the compactness of the Sobolev embeddings and the conservation of the regularity we can deduce less optimal result such as

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0, T])} \{\alpha \Upsilon_t^u \psi_0, t \in [0, T]\}$$

are contained in a countable union of totally bounded sets of  $H^{2(k-1)-\delta}$  for any  $\delta \in (0, 1)$  whenever  $\psi_0$  in  $H^{2(k-1)}$ .

## 6.2 Potential well with dipolar interaction

In this example,  $\Omega = (0, \pi)$  endowed with the standard Lebesgue measure,  $V$  is the constant zero function and  $W$  is some function of class  $C^k$ , for some integer  $k$ . This academic example is a simplification of the harmonic oscillator, presented in Section 6.3, in the sense that  $\Omega$  is bounded. It has been thoroughly studied by K. Beauchard in [Bea05, BL10]. These works give the first (and, at this time, almost the only one) satisfying description of the reachable set with  $L^2$  controls from the first eigenvector for systems of the type of (1.4). Using Lyapunov techniques, V. Nersessyan [Ner10] gave practical algorithms for approximate controllability.

Equation (1.4) writes

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - u(t)W(x)\psi \quad (6.2)$$

with boundary conditions  $\psi(0) = \psi(\pi) = 0$ .

The linear operators  $A = \frac{1}{2}\Delta$  defined on  $D(A) = (H^2 \cap H_0^1)((0, \pi), \mathbf{C})$  and  $B : \psi \mapsto iW\psi$  are skew symmetric in the Hilbert space  $H = L^2(\Omega, \mathbf{C})$  endowed with the hermitian product  $L^2(\Omega, \mathbf{C})$ ,

$$\langle f, g \rangle = \int_0^\pi \overline{f(x)}g(x)dx.$$

Defining, for every  $k$  in  $\mathbf{N}$ ,

$$\phi_k : x \mapsto \sqrt{\frac{2}{\pi}} \sin(kx)$$



the family  $\Phi = (\phi_k)_{k \in \mathbf{N}}$  is an orthonormal basis of  $\mathcal{H}$  made of eigenvectors of  $A$ .

The triple  $(A, B, \mathbf{R})$  satisfies Assumption 1.

Classical results of interpolation [LM68, Chapter 1] allow to find the domain of fractional derivative operators. In particular, for any  $k$  in  $\mathbf{N}$  and  $0 \leq s < 1$ , we get following ([NOS16]):

$$\begin{aligned} D(|A|^k) &= \{\psi \in H^{2k} \mid \psi^{[2l]}(0) = \psi^{[2l]}(\pi) = 0, l = 0 \dots k-1\} && \text{for } k \in \mathbf{N} \\ D(|A|^{k+s}) &= D(|A|^k) \cap H^{2s} && \text{for } s < 1/4 \\ D(|A|^{k+\frac{1}{4}}) &= \{\psi \in D(|A|^k) \mid |A|^k \psi \in H_{00}^{\frac{1}{2}}\} \\ D(|A|^{k+s}) &= \{\psi \in D(|A|^k) \mid |A|^k \psi \in H_0^{2s}\} && \text{for } 1/4 < s < 1/2 \\ D(|A|^{k+s}) &= \{\psi \in D(|A|^k) \mid |A|^k \psi \in H_0^{2s} \cap H_0^1\} && \text{for } 1/2 \leq s \leq 1 \end{aligned}$$

where

$$H_{00}^{\frac{1}{2}} = \left\{ \psi \in H^{\frac{1}{2}} \mid \int_0^\pi \psi^2(x) \frac{dx}{\sin(x)} < +\infty \right\}$$

is the Lions-Magenes space.

**Lemma 30.** *Let  $k$  and  $p$  in  $\mathbf{N}$ ,  $W : [0, \pi] \rightarrow \mathbf{R}$  be  $C^{2k+1}$  such that  $W^{[2l+1]}(0) = W^{[2l+1]}(\pi) = 0$  for  $l = 0 \dots p-1$ . Then  $B$  is bounded from  $D(|A|^a)$  to  $D(|A|^a)$  for every  $a < p+1 + \frac{1}{4}$ .*

*Proof.* Since  $W$  is  $C^{2k+1}$ ,  $B$  leaves invariant  $H^s$  for every  $s \leq 2k+1$ . If  $a$  is an integer, the result follows from the Leibniz rule, using the vanishing of the derivatives of odd orders of  $W$  on the boundary of  $[0, \pi]$ . For  $a - [a] < 1/4$ , the interpolation result given above states that there is no additional boundary conditions to check.  $\square$

Theorem 3.6 in [BMS82] by Ball, Marsden and Slemrod implies (see [Tur00]) that equation (1.4) is not controllable in (the Hilbert unit sphere of)  $L^2(\Omega)$  when  $\psi \mapsto W\psi$  is bounded in  $L^2(\Omega)$ . Moreover, in the case in which  $\Omega$  is a domain of  $\mathbf{R}^n$  and  $W : \Omega \rightarrow \mathbf{R}$  is  $C^2$ , if the control  $u$  belongs to  $L^p([0, +\infty), \mathbf{R})$  with  $p > 1$ , then equation (1.4) is neither controllable in the Hilbert sphere  $\mathbf{S}$  of  $L^2(\Omega)$  nor in the natural functional space where the problem is formulated, namely the intersection of  $\mathbf{S}$  with the Sobolev spaces  $H^2(\Omega)$  and  $H_0^1(\Omega)$ .

The fact that the present system is not more than 5/2-mildly coupled is the purpose of the following lemmas.

**Lemma 31.** *Let  $k \in \mathbf{N} \cup \{0\}$ . Let  $F : [0, \pi] \rightarrow \mathbf{R}$  of class  $C^{2k+3}$ . If  $F^{(2j+1)}(0) = F^{(2j+1)}(\pi) = 0$  for  $j = 0, \dots, k-1$  and  $|F^{(2k+1)}(\pi)| + |F^{(2k+1)}(0)| \neq 0$  then  $F\phi_1$  is not in  $D(|A|^a)$  if  $a \geq k + \frac{5}{4}$ .*

*Proof.* Consider, for any integer  $n$ , the following quantity

$$I_n(F) := \frac{\pi}{2} \langle F\phi_1, \phi_n \rangle = \int_0^\pi F(x) \sin(x) \sin(nx) dx.$$

Then we have the following  $I_n(F) = \frac{1}{2}(J_{n-1}(F) - J_{n+1}(F))$  with

$$\begin{aligned} J_\ell(F) &:= \int_0^\pi F(x) \cos(\ell x) dx = -\frac{1}{\ell} \int_0^\pi F'(x) \sin(\ell x) dx \\ &= \frac{1}{\ell^2} \left( (-1)^\ell F'(\pi) - F'(0) \right) - \frac{1}{\ell^2} J_\ell(F'') \end{aligned}$$

Now assume  $F^{(2j+1)}(0) = F^{(2j+1)}(\pi) = 0$  for  $j = 0, \dots, k-1$ , we deduce:

$$J_\ell(F) = \frac{1}{\ell^{2k+2}} \left( (-1)^\ell F^{(2k+1)}(\pi) - F^{(2k+1)}(0) \right) - \frac{1}{\ell^{2k+2}} J_\ell(F^{(2k+2)}).$$

It follows that

$$\begin{aligned}
I_n(F) &= \frac{1}{2} \left( \frac{1}{(n-1)^{2k+2}} - \frac{1}{(n+1)^{2k+2}} \right) \left( -(-1)^n F^{(2k+1)}(\pi) - F^{(2k+1)}(0) \right) \\
&\quad - \frac{1}{2} \frac{1}{(n-1)^{2k+2}} J_{n-1}(F^{(2k+2)}) + \frac{1}{2} \frac{1}{(n+1)^{2k+2}} J_{n+1}(F^{(2k+2)}) \\
&= \frac{1}{2} \left( \frac{1}{(n-1)^{2k+2}} - \frac{1}{(n+1)^{2k+2}} \right) \left( -(-1)^n F^{(2k+1)}(\pi) - F^{(2k+1)}(0) \right) \\
&\quad + \frac{1}{2} \frac{1}{(n-1)^{2k+3}} \int_0^\pi F^{(2k+3)}(x) \sin((n-1)x) dx - \frac{1}{2} \frac{1}{(n+1)^{2k+3}} \int_0^\pi F^{(2k+3)}(x) \sin((n+1)x) dx
\end{aligned}$$

As

$$\frac{1}{(n-1)^{2k+2}} - \frac{1}{(n+1)^{2k+2}} = \frac{(n+1)^{2k+2}}{(n^2-1)^{2k+2}} \left( 1 - \left( \frac{n-1}{n+1} \right)^{2k+2} \right) \underset{n \rightarrow \infty}{\sim} \frac{4k+4}{n^{2k+3}}$$

If  $|F^{(2k+1)}(\pi)| + |F^{(2k+1)}(0)| \neq 0$ , then either  $F^{(2k+1)}(\pi) - F^{(2k+1)}(0) \neq 0$  or  $F^{(2k+1)}(\pi) + F^{(2k+1)}(0) \neq 0$  and due to Riemann-Lebesgues Lemma,

- if  $F^{(2k+1)}(\pi) + F^{(2k+1)}(0) \neq 0$  then

$$I_{2n}(F) \underset{n \rightarrow \infty}{\sim} -\frac{2k+2}{(2n)^{2k+3}} \left( F^{(2k+1)}(\pi) + F^{(2k+1)}(0) \right)$$

and hence,  $(n^{2a} I_n(F))_{n \in \mathbf{N}}$  is not square integrable if  $2a - 2k - 3 \geq -\frac{1}{2}$  and consequently  $F\phi_1$  is not in  $D(|A|^a)$  if  $a \geq k + \frac{5}{4}$

- if  $F^{(2k+1)}(\pi) - F^{(2k+1)}(0) \neq 0$  then

$$I_{2n+1}(F) \underset{n \rightarrow \infty}{\sim} \frac{2k+2}{(2n+1)^{2k+3}} \left( F^{(2k+1)}(\pi) - F^{(2k+1)}(0) \right)$$

and similarly  $F\phi_1$  is not in  $D(|A|^a)$  if  $a \geq k + \frac{5}{4}$ .  $\square$

**Lemma 32.** Let  $k \in \mathbf{N} \cup \{0\}$ . Let  $W : [0, \pi] \rightarrow \mathbf{R}$  of class  $C^{2k+3}$  with  $W^{(2j+1)}(0) = W^{(2j+1)}(\pi) = 0$  for  $j = 0, \dots, k-1$  and  $|W^{(2k+1)}(\pi)| + |W^{(2k+1)}(0)| \neq 0$ .

Then for every  $a$  in  $(0, +\infty)$ ,  $e^{\pi B} \phi_1 \in D(|A|^a) \Leftrightarrow a < \frac{5}{4} + k$ .

*Proof.* Set  $F = e^{iW}$  and recall Faà di Bruno formula

$$(e^{iW})^{(n)}(x) = \sum \frac{n!}{m_1! 1!^{m_1} m_2! 2!^{m_2} \dots m_n! n!^{m_n}} e^{iW(x)} \prod_{j=1}^n \left( (iW)^{(j)}(x) \right)^{m_j},$$

where the sums is over the  $n$ -uplets  $(m_1, \dots, m_n)$  in  $\mathbf{N} \cup \{0\}$  such that:  $1m_1 + 2m_2 + 3m_3 + \dots + nm_n = n$ .

If  $n$  is odd and  $(m_1, \dots, m_n)$  is an  $n$ -uplets of  $\mathbf{N} \cup \{0\}$  such that  $1m_1 + 2m_2 + 3m_3 + \dots + nm_n = n$  there exists  $\ell$  such that  $2\ell + 1 \leq n$  and  $m_{2\ell+1} \neq 0$ . It follows that  $F : [0, \pi] \rightarrow \mathbf{R}$  is of class  $C^{2k+3}$  with  $F^{(2j+1)}(0) = F^{(2j+1)}(\pi) = 0$  for  $j = 0, \dots, k-1$  and  $|F^{(2k+1)}(\pi)| + |F^{(2k+1)}(0)| \neq 0$ .

Then Lemmas 31 and 30 provide the conclusion.  $\square$

We sum up our results in the following

**Proposition 33.** *Let  $k \in \mathbf{N} \cup \{0\}$ . Let  $W : [0, \pi] \rightarrow \mathbf{R}$  of class  $C^{2k+3}$  with  $W^{(2j+1)}(0) = W^{(2j+1)}(\pi) = 0$  for  $j = 0, \dots, k-1$  and  $|W^{(2k+1)}(\pi)| + |W^{(2k+1)}(0)| \neq 0$ .*

*Then*

$$\begin{aligned} \text{Att}_{\mathcal{R}}(\phi_1) &= \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0, T])} \{\Upsilon_{t,0}^u \phi_1 \mid 0 \leq t \leq T\} \subset \bigcap_{s < \frac{5}{4} + k} D(|A|^s) \\ \text{Att}_{\overline{\mathcal{R}}}(\phi_1) &= \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0, T])} \{\Xi_{t,0}^u \phi_1 \mid 0 \leq t \leq T\} \subset \bigcap_{s < \frac{5}{4} + k} D(|A|^s) \end{aligned}$$

*and both attainable sets are contained in a countable union of relatively compact subsets of  $D(|A|^s)$ , for any  $s < \frac{5}{4} + k$ .*

*Moreover, we have*

$$\text{Att}_{\mathcal{R}}(\phi_1) \not\subset D(|A|^{\frac{5}{4} + k}) \quad \text{and} \quad \text{Att}_{\overline{\mathcal{R}}}(\phi_1) \not\subset D(|A|^{\frac{5}{4} + k}).$$

Recall that  $\Upsilon$  is defined in Proposition 13 and  $\Xi$  in Theorem 27.

*Proof.* From Lemma 31,  $B$  is bounded from  $D(|A|^a)$  to  $D(|A|^a)$  for every  $a < p + \frac{5}{4}$  and hence  $(A, B)$  is  $a$ -mildly coupled, for every  $a < p + \frac{5}{4}$ , by Proposition 18. Then Proposition 25 provides the first statement. While following Remark 30, Theorem 27 provides the second statement.

The relative compactnesses follow from Proposition 26 (similarly to Proposition 15) and Theorem 27, respectively.

From (3.1), with  $u = \pi \delta_{t_0}$  for some  $t_0 > 0$  and Lemma 32 we deduce the first noninclusion statement. From (5.3) and Lemma 31, we deduce the last assertion.  $\square$

**Remark 31.** Notice that [BL10, Theorem 2] states the exact controllability of (6.2) in  $D(|A|^{\frac{5}{2}})$  with  $H_0^1$  controls and  $W : x \mapsto x$ . While Proposition 23 states no exact controllability of (6.2) in  $D(|A|^s)$ ,  $s < \frac{9}{4}$ , with BV controls for example with  $W : x \mapsto x$ . Whether this 1/4 discrepancy is optimal is still an open question.

Similarly [BL10, Theorem 1] states the exact controllability of (6.2) in  $D(|A|^{\frac{3}{2}})$  with  $L^2$  controls and  $W : x \mapsto x$ . While Proposition 26 states the nonexact controllability of (6.2) in  $D(|A|^s)$ ,  $s < \frac{5}{4}$ , with Radon controls and  $W : x \mapsto x$ . But this time, the above statement states that the 1/4 discrepancy is optimal.

From [BCCS12], we know that  $\{(k, k+1) \mid k \in \mathbf{N}\}$  is a nondegenerate chain of connectedness for  $(A + \eta B, B)$  for almost every real  $\eta$ . Hence Proposition 42 guarantees the approximate controllability of the system (6.2) from  $\phi_1$  in  $D(|A + \eta B|^r) = D(|A|^r)$ , for  $\frac{3}{2} < r < \frac{5}{4} + 1$ . The global exact controllability in  $D(|A|^{\frac{3}{2}})$  (inside the unit sphere) with explicit controls follows from Proposition 42, in order to reach a neighborhood of the target in  $D(|A|^r)$ , for  $\frac{3}{2} < r < \frac{5}{4} + 1$  (see for instance [BCC12]). It is then enough to concatenate the dynamics with  $L^2$  controls given by [BL10] for exact local controllability. This explicit construction provides estimates on control time and norms, see [Duc17].

### 6.3 Quantum harmonic oscillator

In this section, we present an example of  $s$ -mildly coupled, system, for any  $s > 0$ , with an unbounded control potential on contrast with the previous examples.

The quantum harmonic oscillator with angular frequency  $\omega$  describes the oscillations of a particle of mass  $m$  subject to the potential  $V(x) = \frac{1}{2}m\omega x^2$ . The corresponding uncontrolled Schrödinger equation is

$$i \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi(x, t) + \frac{1}{2}m\omega x^2 \psi(x, t).$$

With a suitable choice of units, it becomes

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta\psi(x, t) + \frac{1}{2}x^2\psi(x, t)$$

The operator  $A = \frac{i}{2}\Delta - \frac{x^2}{2}$  is self-adjoint on  $L^2(\mathbf{R}, \mathbf{C})$ ,  $A$  has a pure discrete spectrum. The  $k^{\text{th}}$  eigenvalue (corresponding to the  $k^{\text{th}}$  energy level) is equal to  $\frac{2k+1}{2}i$  and is associated with the eigenstate

$$\phi_k : x \mapsto \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} \exp\left(-\frac{x^2}{2}\right) H_k(x)$$

where  $H_k$  is the  $k^{\text{th}}$  Hermite polynomial, namely  $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} (e^{-x^2})$ .

When considering the classical dipolar interaction, the control potential  $W$  takes the form  $W(x) = x$  for every  $x$  in  $\mathbf{R}$ . It is well known (see [MR04] and references therein) that the resulting control system (1.4) is not controllable in any reasonable sense. Indeed the system splits in two uncoupled subsystems. The first one is a finite dimensional classical harmonic oscillator which is controllable. The second one is a free (that is, without control) quantum harmonic oscillator, whose evolution does not depend on the control and is therefore not controllable.

In [BCC13, Section IV], we show that  $(i(-\Delta + V), iW)$  is  $s$ -mildly coupled for every  $s > 0$ . The proof given in [MR04, ILT06] (and especially the decomposition of the system in two decoupled systems) does not require more to the control than to be the derivative of a derivable function. Using the continuity in Proposition 13, it can be extended to Radon measures.

**Proposition 34.** *The system (1.4) with  $\Omega = \mathbf{R}$ ,  $V : x \mapsto x^2$  and  $W : x \mapsto x$  is not approximately controllable by means of Radon measures.*

Although this example is not approximately controllable, an arbitrarily small perturbation of  $W$  by some smooth localized function  $W_2$  restores this feature, see [CMSB09, Proposition 6.4]. Nonetheless, the approximate controllability in arbitrarily small time is not possible, see [BCT14], recently extended in [BCT16]. This does not affect the mild coupling at any order as  $(A, iW_2)$  is also mildly coupled at any order and  $W_2$  commutes with  $W$  which ensures that  $(A, i(W + W_2))$  is  $s$ -mildly coupled for every  $s > 0$ .

Note that existence of the dynamics is obtained in [Fuj79] for measurable in time and and locally bounded in space-time control potentials. It can be extended to Radon measures controls using Section 3.1. Note that in the case of Radon measures without atoms, for instance  $L^1$ -controls, the resulting propagator is a weak solution of (1.5), see Proposition 14 and Remark 19.

## A Interpolation

### A.1 Convergence of sequences

Through the present analysis, the following simple interpolation lemma was useful.

**Lemma 35.** *Let  $A$  be a skew-adjoint operator, let  $S$  be a set and  $(u_n)_{n \in \mathbf{N}}$  take value in the set of functions from  $S$  to  $D(|A|^k)$ , such that  $(u_n)_{n \in \mathbf{N}}$  is uniformly bounded in  $S$  for the norm of  $D(|A|^k)$ ,  $k > 0$ . If  $(u_n)_{n \in \mathbf{N}}$  tends to zero in  $\mathcal{H}$  uniformly in  $S$ , then  $(u_n)_{n \in \mathbf{N}}$  tends to zero in  $D(|A|^l)$ , uniformly in  $S$  for every  $l < k$ .*

*Proof.* The proof follows from the logarithmic convexity of  $l \in [0, k] \mapsto \| |A|^l u \|$ . Indeed

$$\| |A|^{\frac{l+j}{2}} u \| = \sqrt{\langle |A|^l u, |A|^j u \rangle} \leq \| |A|^l u \|^{1/2} \| |A|^j u \|^{1/2}.$$

If  $l < k$  then

$$\| |A|^l u_n \| \leq \| u_n \|^{k-l} \| |A|^k u_n \|^{l/k}.$$

Let  $C = \sup_{n \in \mathbf{N}} \| |A|^k u_n \|^2$  and  $N > 0$  such that for any  $n > N$ ,  $\| u_n \|^2 \leq \varepsilon$  we obtain

$$n > N \implies \| |A|^l u_n \|^2 \leq \varepsilon^{\frac{k-l}{k}} C^{\frac{l}{k}},$$

which provides the lemma. □

## A.2 Interpolation of fractional powers of operators

Let us now state a more sophisticated result. The following result can also be deduced from the content of [ABG96, Section 2.8] and its proof is an extension to the unbounded case of the result by [Ped72].

**Proposition 36.** *Let  $A$  and  $B$  be two self-adjoint positive operators in  $\mathcal{H}$  such that there exists  $c > 0$  with*

$$c \leq B \leq A$$

*in the form sense. Then for any  $\alpha \in (0, 1)$ , the following is true*

$$c^\alpha \leq B^\alpha \leq A^\alpha.$$

*Proof.* The proof follows from the following series of lemma.

For a selfadjoint operator  $\mathcal{H}$  and  $z \in \mathbf{C} \setminus \mathbf{R}$ , the functional calculus is the extension of the mapping:

$$\{x \in \mathbf{R} \mapsto (x - z)^{-1}\} \in B(\mathbf{R}) \rightarrow (A - z)^{-1} \in B(\mathcal{H})$$

as a strong continuous  $*$ -algebra homomorphisme map on bounded borelian functions on the real line with the bounded pointwise topology :  $B(\mathbf{R})$ .

Let us recall the following functional calculus identity based on the Poisson formula, see [ABG96, Lemma 6.1.1].

**Lemma 37.** *Let  $A$  be a selfadjoint operator in  $\mathcal{H}$ . Let  $f$  be a bounded borelian function. Then*

$$f(A) := w - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2i\pi} \int_{\mathbf{R}} f(\lambda) \Im(A - \lambda - i\varepsilon)^{-1} d\lambda.$$

The notations  $w - \lim$  refers to the weak limit.

We also recall the formula for  $\alpha \in (0, 1)$  and  $x > 0$

$$x^{-\alpha} = \frac{\pi}{\sin(\pi\alpha)} \int_0^\infty \frac{w^{-\alpha}}{x+w} dw.$$

and then the Fubini theorem with Lemma 37 we obtain the

**Lemma 38.** *Let  $A$  be a positive selfadjoint operator in  $\mathcal{H}$ . Then for  $\alpha \in (0, 1)$*

$$A^\alpha = \frac{\pi}{\sin(\pi\alpha)} \int_0^\infty \frac{w^{-1+\alpha} A}{A+w} dw$$

*on  $D(A)$ .*

The domain of validity of the above identity can be extended to any core of  $A^\alpha$  that makes the integral strongly convergent.

With this identity and the following lemma we have the proof of the proposition.

**Lemma 39.** *Let  $A$  and  $B$  be two self-adjoint positive operators in  $\mathcal{H}$  such that there exists  $c > 0$  with*

$$c \leq B \leq A.$$

Then

$$A^{-1} \leq B^{-1}.$$

*Proof.* First notice that both  $A$  and  $B$  are invertible from their domains to  $\mathcal{H}$  as well as their square roots. Then from

$$\sqrt{c}\|u\| \leq \|\sqrt{B}u\| \leq \|\sqrt{A}u\|,$$

we deduce that  $\sqrt{B}\sqrt{A}^{-1}$  is a bounded operator with norm at most 1.

In the other hand the operator  $\sqrt{A}^{-1}\sqrt{B}$  defined on  $D(\sqrt{B})$  extends as the adjoint of  $\sqrt{B}\sqrt{A}^{-1}$  to a closed operator on  $\mathcal{H}$  and hence is bounded with norm at most 1 and

$$\|\sqrt{A}^{-1}\sqrt{B}u\| \leq \|u\|, \forall u \in D(\sqrt{B})$$

and thus

$$\|\sqrt{A}^{-1}u\| \leq \|\sqrt{B}^{-1}u\|.$$

and the result follows. □

The proof of Proposition 36 then follows as

$$c \leq B \leq A$$

implies for any  $w > 0$ ,

$$1 - w(B + w)^{-1} \leq 1 - w(A + w)^{-1}.$$

and thus

$$\frac{w^{-1+\alpha}B}{B+w} \leq \frac{w^{-1+\alpha}A}{A+w}$$

integrating on  $w > 0$  (first restricted top  $D(A) \times D(A)$ ) gives the desired inequality by density. □

The above result can be extend to the case  $c = 0$  by replacing  $A$  and  $B$  by  $A + \epsilon$  and  $B + \epsilon$  as in [Ped72], we obtain

$$0 \leq B^\alpha \leq (B + \epsilon)^\alpha \leq (A + \epsilon)^\alpha.$$

The second inequality is immediate. We obtain

$$0 \leq (A + \epsilon)^{-\alpha/2} B^\alpha (A + \epsilon)^{-\alpha/2} \leq 1$$

so that taking  $\epsilon$  to 0 giving

$$0 \leq B^\alpha \leq A^\alpha.$$

We immediately deduce the following corollary.

**Corollary 40.** *Let  $A$  and  $B$  be two positive self-adjoint operators sharing the same domains. For any  $\alpha \in (0, 1)$ , we have :*

$$D(A^\alpha) = D(B^\alpha)$$

*Proof.* As  $B$  is closed it is a bounded operator from  $D(A)$  to  $\mathcal{H}$ . Thus

$$\exists c > 0, \forall \phi \in D(A), \|B\phi\| \leq c\|A\phi\|.$$

Hence

$$B^2 \leq c^2 A^2$$

from the above that  $B^{2\alpha/2}$  is bounded from  $D(A^{2\alpha/2})$  to  $\mathcal{H}$ .

The proof being now symmetric in  $A$  and  $B$  we can conclude  $\square$

## B Sufficient conditions for approximate controllability with bounded variation controls

The aim of this Section is to recall approximate controllability results obtained in other contexts and how this results may be adapted in our context.

We first recall the following definitions from [CMSB09].

**Definition 6.** Let  $(A, B, \mathbf{R})$  satisfy Assumptions 1 such that  $A$  and  $B$  are skew-symmetric. Let  $\Phi = (\phi_k)_k$  be a Hilbert basis of  $\mathcal{H}$  made of eigenvectors of  $A$ ,  $A\phi_k = i\lambda_k\phi_k$  for every  $k$  in  $\mathbf{N}$ . A pair  $(j, k)$  of integers is a *nondegenerate transition* of  $(A, B, \Phi)$  if (i)  $\langle \phi_j, B\phi_k \rangle \neq 0$  and (ii) for every  $(l, m)$  in  $\mathbf{N}^2$ ,  $|\lambda_j - \lambda_k| = |\lambda_l - \lambda_m|$  implies  $(j, k) = (l, m)$  or  $\langle \phi_l, B\phi_m \rangle = 0$  or  $\{j, k\} \cap \{l, m\} = \emptyset$ .

**Definition 7.** Let  $(A, B, \mathbf{R})$  satisfy Assumptions 1 such that  $A$  and  $B$  are skew-symmetric. Let  $\Phi = (\phi_k)_k$  be a Hilbert basis of  $\mathcal{H}$  made of eigenvectors of  $A$ ,  $A\phi_k = i\lambda_k\phi_k$  for every  $k$  in  $\mathbf{N}$ . A subset  $S$  of  $\mathbf{N}^2$  is a *nondegenerate chain of connectedness* of  $(A, B, \Phi)$  if (i) for every  $(j, k)$  in  $S$ ,  $(j, k)$  is a nondegenerate transition of  $(A, B)$  and (ii) for every  $r_a, r_b$  in  $\mathbf{N}$ , there exists a finite sequence  $r_a = r_0, r_1, \dots, r_p = r_b$  in  $\mathbf{N}$  such that, for every  $j \leq p-1$ ,  $(r_j, r_{j+1})$  belongs to  $S$ .

**Proposition 41.** Let  $(A, B, \mathbf{R})$  satisfy Assumptions 1 such that  $A$  and  $B$  are skew-symmetric. Let  $\Phi = (\phi_k)_k$  be a Hilbert basis of  $\mathcal{H}$  made of eigenvectors of  $A$ ,  $A\phi_k = i\lambda_k\phi_k$  for every  $k$  in  $\mathbf{N}$ . Let  $S$  be a nondegenerate chain of connectedness of  $(A, B)$ . Then, for every  $\eta > 0$ ,  $(A, B)$  is simultaneously approximately controllable in  $D(|A|^{1-\eta})$ .

*Proof.* First of all, it is enough to prove the result for target propagators  $\hat{\Upsilon}$  leaving invariant the space of co-dimension 2 spanned by  $(\phi_j, \phi_k)$  for  $(j, k)$  in  $S$

$$\hat{\Upsilon} = e^{i\nu_l}(\cos(\theta)\phi_l^*\phi_l + \sin(\theta)\phi_l^*\phi_k) + e^{i\nu_k}(-\sin(\theta)\phi_k^*\phi_l + \cos(\theta)\phi_l^*\phi_k)$$

The result in  $\mathcal{H}$ -norm is a consequence of [Cha12, Theorem 1]: for every piecewise constant  $u^* : \mathbf{R} \rightarrow \mathbf{R}$ ,  $2\pi/|\lambda_j - \lambda_k|$ -periodic such that

$$\int_0^{\frac{2\pi}{|\lambda_j - \lambda_k|}} u^*(\tau) e^{i(\lambda_j - \lambda_k)\tau} d\tau \neq 0$$

and

$$\int_0^{\frac{2\pi}{|\lambda_j - \lambda_k|}} u^*(\tau) e^{i(\lambda_l - \lambda_m)\tau} d\tau = 0$$

for every  $l, m$  such that  $(\lambda_l - \lambda_m) \in \mathbf{Z}(\lambda_j - \lambda_k)$  and  $b_{l,m} \neq 0$ , there exists  $T^*$  such that  $\Upsilon^{u^*/n}(nT^*, 0)$  tends to  $\hat{\Upsilon}$  as  $n$  tends to infinity.

The conclusion follows using Lemma 35 and the estimate in  $A$ -norm of Theorem 5.  $\square$

Let us just mention the following result in case of higher regularity.

**Proposition 42.** *Let  $k$  be a positive real. Let  $(A, B, \mathbf{R})$  satisfy Assumptions 1 such that  $(A, B)$  is  $k$ -mildly coupled. Let  $\Phi = (\phi_k)_k$  be a Hilbert basis of  $\mathcal{H}$  made of eigenvectors of  $A$ ,  $A\phi_k = i\lambda_k\phi_k$  for every  $k$  in  $\mathbf{N}$ . Let  $S$  be a nondegenerate chain of connectedness of  $(A, B)$  such that, for every  $(j, k)$  in  $S$ , the set  $\{(l, m) \in \mathbf{N}^2 | (\lambda_l - \lambda_m) \in \mathbf{Z}(\lambda_j - \lambda_k) \text{ and } \langle \phi_l, B\phi_m \rangle \neq 0\}$  is finite. Then, for every  $\eta > 0$ ,  $(A, B)$  is simultaneously approximately controllable in  $D(|A|^{k/2+1-\eta})$ .*

*Proof.* The proof differs from the previous for the interpolation step and for the use of Proposition 23.  $\square$

## C Analytical perturbations

To apply our sufficient condition for approximate controllability (Proposition 42), we need to find a nonresonant chain of connectedness, which may require some work on practical examples. A classical idea we already used in this study is to introduce a new control  $\tilde{u} = u - \bar{u}$  and to consider the system  $x' = (A + \bar{u}B) + (u - \bar{u})B$  for a suitably chosen constant  $\bar{u}$ .

We have the following results by Kato [Kat66, Section VII.2].

**Definition 8.** Let  $D_0$  be a domain of the complex plane, a family  $(T(z))_{z \in D_0}$  of closed operators from a Banach space  $X$  to a Banach space  $Y$  is said to be a *holomorphic family of type (A)* if

1.  $D(T(z)) = D$  is independent of  $z$ ,
2.  $T(z)u$  is holomorphic for  $z$  in  $D_0$  for every  $u$  in  $D$ .

**Theorem 43** ([Kat66, Theorem VII.3.9]). *Let  $T(z)$  be a selfadjoint holomorphic family of type (A) defined for  $z$  in a neighborhood of an interval  $I_0$  of the real axis such that  $T(z)^* = T(\bar{z})$ . Furthermore, let  $T(z)$  have a compact resolvent. Then all eigenvalues of  $T(z)$  can be represented by functions which are holomorphic in  $I_0^1$ .*

*More precisely, there is a sequence of scalar-valued functions  $(z \mapsto \lambda_n(z))_{n \in \mathbf{N}}$  and operator-valued functions  $(z \mapsto \phi_n(z))_{n \in \mathbf{N}}$ , all holomorphic on  $I_0$ , such that for  $z$  in  $I_0$ , the sequence  $(\lambda_n(z))_{n \in \mathbf{N}}$  represents all the repeated eigenvalues of  $T(z)$  and  $(\phi_n(z))_{n \in \mathbf{N}}$  forms a complete orthonormal family of the associated eigenvectors of  $T(z)$ .*

**Proposition 44.** *If  $(A, B, K)$  satisfies Assumptions 1 then the family  $i(A + zB)_{z \in \mathbf{C}, |z| < 1/\|B\|_A}$  is holomorphic of type (A).*

*Proof.* The question of domain is solved by the Kato–Rellich Theorem. The holomorphy is immediate as the family  $i(A + zB)$  is affine in  $z$ .  $\square$

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<sup>1</sup>Each of them is holomorphic in some neighborhood of  $I_0$  but possibly different for each in such a way that their intersection is just  $I_0$ .



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