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Nabile Boussaid, Marco Caponigro, Thomas Chambrion. Regular propagators of bilinear quantum systems. 2014. hal-01016299v1

HAL Id: hal-01016299

<https://hal.science/hal-01016299v1>

Preprint submitted on 29 Jun 2014 (v1), last revised 26 Sep 2019 (v3)

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Regular propagators of bilinear quantum systems

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June 29, 2014

Abstract

This present analysis deals with the regularity of the solutions of bilinear control systems of the type $x' = (A + u(t)B)x$ where the state x belongs to some complex infinite dimensional Hilbert space, the (possibly unbounded) linear operators A and B are skew-adjoint and the control u is a real valued function. Such systems arise for instance in quantum control with the bilinear Schrödinger equation.

Under some hypotheses on the commutator of the operators A and B , it is possible to define the solution of the system for controls in the set of Radon measures and to obtain precise a priori energy estimates of the solutions. This leads to a natural extension of a celebrated non-controllability result of Ball, Marsden and Slemrod.

Moreover, upper bounds on the error made by replacing the original systems by its finite dimensional Galerkin approximation are obtain from the energy estimates. This allows to use geometric (finite dimensional) techniques to obtain approximate controllability results for the original infinite dimensional problem.

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1 Introduction

1.1 Physical context

The state of a quantum system evolving on a finite dimensional Riemannian manifold Ω , with associated measure μ , is described by its wave function, modeled as a point in the unit sphere of $L^2(\Omega, \mathbf{C})$. A physical quantity or observable \mathcal{O} (such as energy or position), is in turn modeled by a Hermitian operator $O : D(O) \subset L^2(\Omega, \mathbf{C}) \rightarrow L^2(\Omega, \mathbf{C})$ such that the expected value of quantity \mathcal{O} for a system with wave function ψ in $D(O)$ is $\int_{\Omega} \bar{\psi} O \psi d\mu$. For instance, a system with wave function ψ will be in a subset ω of Ω with the probability $\int_{\omega} |\psi|^2 d\mu$.

In the absence of interaction with the environment and neglecting the relativistic effects, the time evolution of the wave function is given by the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + V(x) \psi(x, t),$$

where Δ is the Laplace-Beltrami operator on Ω and $V : \Omega \rightarrow \mathbf{R}$ is a real function (usually called potential) accounting for the physical properties of the system. When submitted to an excitation by an external electric field (*e.g.* a laser), the Schrödinger equation reads

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + V(x) \psi(x, t) + u(t) W(x) \psi(x, t), \quad (1.1)$$

where $W : \Omega \rightarrow \mathbf{R}$ is a real function accounting for the physical properties of the laser and u is a real function of the time accounting for the intensity of the laser.

For skew-symmetry reasons, the unit sphere \mathbf{S} of $L^2(\Omega, \mathbf{C})$ is invariant by the dynamics of (1.1). A natural question, with many practical implications, is whether it is possible or not to find a control u that steers the quantum system from a given initial wave function in \mathbf{S} to a given target wave function in \mathbf{S} .

1.2 State of the art

If $\Omega = (-1, 1)$, $V = 0$, and $W : x \mapsto x$, with Dirichlet boundary conditions, then the system (1.1) is exactly controllable (see [BL10]) in $H_{(0)}^3(\Omega) \cap \mathbf{S}$. The results extends to control potentials W sufficiently irregular, *i.e.*, such that the sequence $\left(\int_{-1}^1 W(x) \cos(kx) \cos(x) dx \right)_{k \in \mathbf{N}}$ converges toward zero slowly enough (see also [NN12]).

Our point of view in the present analysis was somehow the opposite. Starting with smooth potentials we tried to give upper bounds on the attainable set. To the best of our knowledge, no description of the attainable set is currently available for systems that do not satisfy the hypotheses of [BL10], for instance when Ω is a compact manifold, $V = 0$ and W is smooth (C^∞) function. The regularity of W is a no-go for the linear test (see [Cor07]) used in [BL10]. Smooth potentials are the most relevant from the physical point of view, see [CTLD73, Figure 7a, page 35] and physicists pay a particular attention to potentials producing smooth wave function, see Cohen-Tannoudji et al., [CTLD73, Section II-A-1, page 94] :

“From a physical point of view, it is clear that the set $L^2(\Omega, \mathbf{C})$ is too wide in scope: given the meaning attributed to $|\psi(x, t)|^2$, the wave functions which are actually used possess certain properties of regularity. We can only retain the functions $\psi(x, t)$ which are everywhere defined, continuous, and infinitely differentiable”

As a consequence it seems that relevant models have to include smooth potentials.

Equation (1.1) belongs to a class called *bilinear Schrödinger equation* in the control community, the term *bilinear* referring to the linear dependence with respect to ψ and the affine dependence with respect to u . A (single input) bilinear control system in a vector space \mathcal{H} is given by an evolution equation

$$\frac{d}{dt}x = Ax + u(t)Bx \quad (1.2)$$

where A and B are two (possibly unbounded) linear operators on \mathcal{H} and u is a real-valued function usually called control. Well-posedness of bilinear evolution equations of type (1.2) for a given control u is usually a difficult question. If K is a subset of \mathbf{R} such that for every u in K , $A + uB$ generates a contraction semi-group $t \mapsto e^{t(A+uB)}$, then for every piecewise constant function of the form $u = \sum_{j=1}^p u_j \mathbf{1}_{[\tau_j, \tau_{j+1})}$ where $u_1, \dots, u_p \in K$ and $\tau_1 < \tau_2 < \dots < \tau_{p+1}$ one defines the associated propagator of (1.2) by

$$\Upsilon_{t, \tau_1}^u = e^{(t-\tau_j)(A+u_j B)} \circ e^{(\tau_j-\tau_{j-1})(A+u_{j-1} B)} \circ \dots \circ e^{(\tau_2-\tau_1)(A+u_1 B)}$$

for every t in (τ_j, τ_{j+1}) . The solution of (1.2) with initial value x_0 at time τ_1 is $t \mapsto \Upsilon_{t, \tau_1}^u \psi_0$. When $\tau_1 = 0$, we will denote $\Upsilon_{t, 0}^u = \Upsilon_t^u$.

Since their precise description seems out of reach, one could try to use the regularity of the solutions of (1.1) to bound the attainable sets of bilinear systems.

Ball, Marsden, and Slemrod proved the general result

Theorem 1 (Theorem 3.6 in [BMS82]). *Let \mathcal{X} be an infinite dimensional Banach space, A generates a C^0 semi-group of bounded linear operators on \mathcal{X} and B be a bounded linear operator on \mathcal{X} . Then for any $T \geq 0$, the input-output mapping $u \mapsto \Upsilon_{T, 0}^u$ admits a unique continuous extension to the set of integrable functions on $[0, T]$ endowed with the L^1 topology, and the attainable set*

$$\bigcup_{r>1} \bigcup_{T \geq 0} \bigcup_{u \in L^r([0, T], \mathbf{R})} \{\Upsilon_{t, 0}^u \psi_0, t \in [0, T]\} \quad (1.3)$$

is a meager set in \mathcal{X} and hence has dense complement.

Theorem 1 implies (see [Tur00]) that equation (1.1) is not controllable in (the Hilbert unit sphere of) $L^2(\Omega)$ when $\psi \mapsto W\psi$ is bounded in $L^2(\Omega)$. Moreover, in the case in which Ω is a domain of \mathbf{R}^n and $W : \Omega \rightarrow \mathbf{R}$ is C^2 , if the control u is restrained to $L^p([0, +\infty), \mathbf{R})$ with $p > 1$, then equation (1.1) is neither controllable in the Hilbert sphere \mathbf{S} of $L^2(\Omega)$ nor in the natural functional space where the problem is formulated, namely the intersection of \mathbf{S} with the Sobolev spaces $H^2(\Omega)$ and $H_0^1(\Omega)$.

The main argument of [BMS82] is as follows. If a sequence $(u_n)_{n \in \mathbf{N}}$ converges weakly in L^p toward some function u in L^p , then for every ψ_0 in \mathcal{X} , for every $t \geq 0$, $(\Upsilon_{t, 0}^{u_n} \psi_0)_{n \in \mathbf{N}}$ converges toward $\Upsilon_{t, 0}^u \psi_0$ in \mathcal{X} . By Alaoglu-Bourbaki theorem, one can extract from any bounded sequence in L^p a subsequence that weakly converges in the (topological) dual of the (topological) dual of L^p , that is L^p provided $p > 1$. Hence the attainable set (1.3) of (1.1) from ψ_0 with L^p , $p > 1$, controls can be written as a countable union of relatively compact subsets of \mathcal{X} . This union is a meager set, hence has empty interior in \mathcal{X} by Baire theorem.

Remark 1. The above argument does not hold anymore if one considers controls in L^1 , since L^1 is not a reflexive space. This is the content of [BMS82, Remark 3.8], where the question of the possible extension of the above result to $r = 1$ is qualified as open except in the so called ([Sle84]) diagonal case, see [BMS82, Theorem 5.5].

1.3 Main results

1.3.1 Upper bound for attainable set of bilinear control systems

Our aim is to give upper bounds for attainable set of bilinear control systems. Our main result is the following

Theorem 2 (Proved in Section 3.2). *Let \mathcal{H} be an infinite dimensional separable Hilbert space, A be a maximal dissipative operator on \mathcal{H} with domain $D(A)$ and B an operator on \mathcal{H} such that $B - c$ and $-B - c'$ generate contraction semi-groups leaving $D(A)$ invariant for some real constants c and c' . If $A + uB$ is maximal dissipative with domain $D(A)$ for every u in \mathbf{R} and if the map $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$ is locally Lipschitz, then, (i) for every $T > 0$, there exists a unique continuous extension to $L^1([0, T], \mathbf{R})$ of the input-output mapping of (1.2) $u \mapsto \Upsilon_{T,0}^u \in L(\mathcal{H}, \mathcal{H})$, and (ii) for every ψ_0 in H , the set*

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in L^1([0, T], \mathbf{R})} \{\alpha \Upsilon_{t,0}^u \psi_0, t \in [0, T]\}$$

is a meager set in \mathcal{H} and hence has dense complement.

In the special case where the control operator B is bounded, we obtain a simplified statement similar to the one of [BMS82] and dealing with L^1 controls:

Proposition 3 (Proved in Section 3.4). *Let \mathcal{H} be an infinite dimensional separable Hilbert space, A generate a C^0 semi-group of bounded linear operators on \mathcal{H} and B be a bounded linear operator on \mathcal{H} . Then for every $T > 0$, there exists a unique continuous extension to $L^1([0, T], \mathbf{R})$ of the input-output mapping of (1.2) $u \mapsto \Upsilon_{T,0}^u \in L(\mathcal{H}, \mathcal{H})$ and, for every ψ_0 in \mathcal{H} ,*

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in L^1([0, T], \mathbf{R})} \{\alpha \Upsilon_{t,0}^u \psi_0, t \in [0, T]\}$$

is a meager set in \mathcal{H} and hence has dense complement.

These results settle the open question by Ball, Marsden and Slemrod mentioned in Remark 1.

The Lipschitz hypothesis in Theorem 2 is crucial for our analysis when B is unbounded but may be difficult to check in practice. For bilinear systems encountered in quantum physics, one can take advantage of the skew-adjointness of the operators to make the analysis simpler. For instance, it is possible to replace the Lipschitz assumption of Theorem 2 by a hypothesis of boundedness of the commutator of operators A and B :

Theorem 4 (Proved in Section 4.2). *Let \mathcal{H} be an infinite dimensional separable Hilbert space, k a positive number, A and B be two skew-adjoint operators such that (i) A is invertible, (ii) B is A -bounded with $\|B\|_A = 0$, see (2.2), (iii) for every u in \mathbf{R} , $|A + uB|^{k/2}$ is self-adjoint with domain $D(|A|^{k/2})$ and (iv) there exists a constant $c_k(A, B)$ such that, for every ψ in $D(|A|^k)$, $|\Re \langle |A|^k \psi, B \psi \rangle| \leq c_k(A, B) |\langle |A|^k \psi, \psi \rangle|$. Then, for every $T > 0$, there exists a unique continuous extension to $BV([0, T], \mathbf{R})$ of the input-output mapping $u \mapsto \Upsilon_T^u$. Moreover, for every $s < 1 + k/2$ and every ψ_0 in $D(|A|^s)$, the set*

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in BV([0, T], \mathbf{R})} \{\alpha \Upsilon_{t,0}^u \psi_0, t \in [0, T]\}$$

has dense complement in $D(|A|^s)$.

The key of our analysis is a set of estimates of the energy of the solution of (1.1) in terms of the L^1 norm of the control u . From these estimates, one can deduce an upper bound of the distance between the solutions of the infinite dimensional system (1.1) and the solutions of finite dimensional approximations of this PDE. To this end, we use the notion of weak-coupling (see Definition 6). This notion has already been introduced in [BCC13], with a slightly narrower sense and proved its effectiveness both for theoretical analysis ([BCC12b], [BCC12c]) and for numerical simulations ([BCC12a]). Here, we apply results of geometric control theory on the (finite-dimensional) Galerkin approximations of the system to prove:

Theorem 5 (Proved in Section 5.5). *Let Ω be a Riemannian compact manifold, r in \mathbf{N} , $r \geq 2$, $V, W : \Omega \rightarrow \mathbf{R}$ be two functions of class C^r , $a \neq b$ two real numbers, and ψ_0 be an eigenvector of $-\Delta + V$. Then the closure for the $H^r(\Omega, \mathbf{C})$ topology of the attainable set of (1.1)*

$$\bigcup_{T \geq 0} \bigcup_{u \in \mathcal{U}} \{\Upsilon_{t,0}^u \psi_0 | t \in [0, T]\}$$

is the same if the set of admissible controls \mathcal{U} is equal to L^1 controls or to the set of piecewise constant controls taking value in $\{a, b\}$.

1.3.2 Higher regularity

Under some regularity assumptions on the control potential with respect to the scale of the uncontrolled operator, we are able to extend the results of the previous section. For instance we can replace the ambient Hilbert space by any iterated domain of the uncontrolled part up to the maximal regularity allowed by the control potential. The immediate outcome of this extension is that the exact controllability is linked to the lack of regularity of the uncontrolled potential.

Let us recall the main result of [BL10]. Below $H_{(0)}^s((0, 1), \mathbf{C})$ will denote the domain of $|A|^{s/2}$ where A is the Laplace-Dirichlet operator on $(0, 1)$, and ϕ_1 its first eigenvector normalized.

Theorem. Let $T > 0$ and $\mu \in H^3((0, 1), \mathbf{R})$ be such that exists $c > 0$ such that $\frac{c}{k^3} \leq |\langle \mu \phi_1, \varphi_k \rangle|$, for all $k \in \mathbf{N}^*$. There exists $\delta > 0$ and a C^1 map $\Gamma : \mathcal{V}_T \rightarrow L^2((0, T), \mathbf{R})$ where

$$\mathcal{V}_T := \{\psi_f \in H_{(0)}^3((0, 1), \mathbf{C}); \|\psi_f\| = 1, \mathbf{C}; \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of

$$\begin{cases} i \frac{\partial \psi}{\partial t}(t, x) = -\frac{\partial^2 \psi}{\partial x^2}(t, x) - u(t) \mu(x) \psi(t, x), x \in (0, 1), t \in (0, T), \\ \psi(t, 0) = \psi(t, 1) = 0, \end{cases} \quad (1.4)$$

with initial condition $\phi(0) = \phi_1$ and control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

The above result applies for instance when $\mu : x \mapsto x$ is the identity function. The techniques introduced in this paper provide estimates from above and from below for the attainable set when using different classes of admissible controls:

Proposition 6 (Proved in Section 6.2). *Let $\mu : x \mapsto x$ be the identity function on $(0, 1)$. Then, for every $T > 0$, the input-output mapping $u \mapsto \Upsilon_T^u$ associated with (1.4) admits a unique continuous extension to the set $\mathcal{R}((0, T])$ of the Radon measures.*

The attainable set from ϕ_1 with Radon controls, $\text{Att}_{\mathcal{R}}(\phi_1) = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}((0, T])} \{\Upsilon_{t,0}^u \phi_1 | 0 \leq t \leq T\}$, sat-

isfies $\text{Att}_{\mathcal{R}}(\phi_1) \subset \bigcap_{s < 5/2} H_{(0)}^s((0, 1), \mathbf{C})$ and $\text{Att}_{\mathcal{R}}(\phi_1) \not\subset H_{(0)}^{5/2}((0, 1), \mathbf{C})$.

The attainable set from ϕ_1 with BV controls, $\text{Att}_{BV}(\phi_1) = \bigcup_{T \geq 0} \bigcup_{u \in BV(0, T]} \{\Upsilon_{t,0}^u \phi_1 \mid 0 \leq t \leq T\}$, is a H^s -dense subset of the intersection of the L^2 unit sphere and $H_{(0)}^s((0, 1), \mathbb{C})$ for every $s < 9/2$.

1.4 Organization of our analysis

The scheme of the proofs of Theorem 2 and Proposition 3 follows the structure of the proof of Theorem 3.6 in [BMS82]. The lack of reflectiveness of L^1 leads us to consider controls as Radon measures, the weak-compactness of bounded sequences is ensured by Helly selection theorem. The main difficulty is to define a continuous input-output mapping associated with (1.2) in such a way to guarantee compactness properties of the attainable sets.

In Section 2 we consider bilinear evolution equations (that are not necessarily conservative) from an abstract point of view and we define the solution for controls with bounded variations. We also prove the well-posedness within this framework and prove the continuity of the propagators with respect to the control parameters.

In Section 3, we use a reparametrization, inspired by physics, which we call the interacting framework, to extend the results of Section 2 to the case where the control is a Radon measure. This provides a proof of Theorem 2. A special paragraph (Section 3.4) is dedicated to the case where B is bounded and to the proof of Proposition 3.

When considering closed quantum systems, the operators A and B appearing in (1.2) are skew-adjoint. Section 4 is devoted to the regularity analysis of the solution obtained so far when further assumption are made on the control potential. The energy estimates we obtain are then used in Section 5 which is dedicated to finite dimensional approximation of our solutions. Some consequences for control are expanded in Section 5.5 which contains a proof of Theorem 5. These results are applied in Section 6 to various examples.

1.5 Notations

All along this analysis, T is a positive real.

Bounded operators space Let \mathcal{X} and \mathcal{Y} be two Banach spaces, $L(\mathcal{X}, \mathcal{Y})$ is the space of linear bounded operator acting on \mathcal{X} with values in \mathcal{Y} . If $\mathcal{X} = \mathcal{Y}$ we write $L(\mathcal{X})$.

Maximal dissipative operators An operator A on \mathcal{H} is dissipative if for any $\phi \in D(A)$, $\Re \langle \phi, A\phi \rangle \leq 0$. It is maximal dissipative if it has no proper dissipative extension.

Weak and strong topology Let $(A_n)_{n \in \mathbf{N}}$ a sequence in $L(\mathcal{X}, \mathcal{Y})$, let A in $L(\mathcal{X}, \mathcal{Y})$. Then A_n converges to A in the strong sense if for any ψ in \mathcal{X} , $(A_n \psi)_{n \in \mathbf{N}}$ converges to $A\psi$ in \mathcal{Y} and A_n converges to A in the weak sense if for any ψ in \mathcal{X} and ϕ in \mathcal{Y}^* , the topological dual of \mathcal{Y} , $(\phi(A_n \psi))_{n \in \mathbf{N}}$ converges to $\phi(A\psi)$ in \mathbb{C} .

Graph topology Consider a self-adjoint operator A acting on a Hilbert space \mathcal{H} with domain $D(A)$, the graph topology on $D(A)$ is the topology associated to the norm $\psi \in D(A) \mapsto \|\psi\|_{\mathcal{H}} + \|A\psi\|_{\mathcal{H}} \in \mathbf{R}^+$.

Bounded variation function Let $I \subset \mathbf{R}$ be an interval. A family $t \in I \mapsto u(t) \in E$, E a subset of a Banach space \mathcal{X} , is in $BV(I, E)$ if there exists $N \geq 0$ such that

$$\sum_{j=1}^n \|u(t_j) - u(t_{j-1})\|_{\mathcal{X}} \leq N$$

for any partition $a = t_0 < t_1 < \dots < t_n = b$ of the interval (a, b) . The mapping

$$u \in BV(I, E) \mapsto \sup_{a=t_0 < t_1 < \dots < t_n=b} \sum_{j=1}^n \|u(t_j) - u(t_{j-1})\|_{\mathcal{X}}$$

is a semi-norm on $BV(I, E)$ that we denote with $\|\cdot\|_{BV(I, E)}$. The semi-norm in $BV(I, E)$ is also called total variation.

We say that $(u_n)_{n \in \mathbf{N}} \in BV(I, E)$ converges to $u \in BV(I, E)$ if $(u_n)_{n \in \mathbf{N}}$ is a bounded sequence in $BV(I, E)$ pointwise convergent to $u \in BV(I, E)$.

The Jordan decomposition theorem provides that any bounded variation function is the difference of two nondecreasing bounded functions. This coupled to Helly's theorem provides the famous Helly's selection theorem :

Theorem. Let $(f_n)_{n \in \mathbf{N}}$ be a sequence in $BV(I, \mathbf{R})$, where I is a compact interval. If

1. there exists $M > 0$ such that for all $n \in \mathbf{N}$, $\|f_n\|_{BV(I, \mathbf{R})} < M$,
2. there exists $x_0 \in I$ such that $(f_n(x_0))_{n \in \mathbf{N}}$ is bounded.

Then $(f_n)_{n \in \mathbf{N}}$ has a pointwise convergent subsequence.

Radon measure For a real interval I , we consider the space $\mathcal{R}(I)$ of (signed) Radon measures on an real interval I . Recall that a positive Radon measure is a measure on borelians which is locally finite and inner regular. A signed Radon measure μ is (by means of Hahn-Jordan decomposition) the difference $\mu = \mu^+ - \mu^-$ of two positive Radon measures μ^+ and μ^- with disjoint support. We denote $|\mu| = \mu^+ + \mu^-$. The total variation of μ is $|\mu|(I)$.

We say that $(u_n)_{n \in \mathbf{N}} \in \mathcal{R}(I)$ converges to $u \in \mathcal{R}(I)$ if $\sup_n |u_n|(I) < +\infty$ (uniformly bounded total variations) and $u_n((0, t]) \rightarrow u((0, t])$ for every $t \in I$ as n tends to ∞ . Note that any cumulative function of a Radon measure is locally of bounded variation and the associated total variation (which do not depend on the choice of the cumulative function) coincides with the total variation of the Radon measure.

Every function in $L^1_{loc}(I)$ can be considered as Radon measure as the density of an absolutely continuous measure. When it does not create ambiguity we will identify the function with the associated Radon measure.

Lemma. Let $(u_n)_{n \in \mathbf{N}}$ be a sequence of integrable function over I converging in $L^1(I)$ to some u . Let $(\mu_n)_{n \in \mathbf{N}}$ defined by

$$\mu_n(J) = \int_J u_n d\lambda, \forall J \subset I \text{ borelian } \forall n \in \mathbf{N}.$$

Then $(\mu_n)_{n \in \mathbf{N}}$ converges to μ in $\mathcal{R}(I)$ where

$$\mu(J) = \int_J u d\lambda, \forall J \subset I \text{ borelian}.$$

We remark that for u in $L^1(I)$ the total variation of the associated Radon measure is the L^1 -norm of u .

The total variation is a norm on the subspace of $\mathcal{R}(I)$ formed by measures with finite total variation. Note that the notion of convergence we use is not the same as the topology induced by the norm of total variation, see also Remark 8.

Remark 2. For properties of bounded variation functions and signed Radon measures, we refer to Ambrosio, Fusco and Pallara [AFP00].

Other notations : For any real interval I , we define

$$\Delta_I := \{(s, t) \in I^2 \mid s \leq t\}.$$

In a metric space E , the notation $B_E(v_0, r)$ stands for the open ball of radius r and center v_0 in E . For a densely defined operator B on a Hilbert space, B^* stands for its adjoint. Recall that B^* is densely defined if and only if B is closable, in which case B^* is closed.

2 Well-posedness and continuity for BV controls

In this section, we present some global well-posedness results for a class of non autonomous perturbations of skew-adjoint linear Cauchy problem and as well as a continuity criteria for converging problem.

2.1 Abstract framework: definitions and notations

In the sequel we will consider a general framework, the classical definitions and tools we use can be found in [RS75, Section X.8], as well as the associated notes and problems. The only difference is that we consider the Hilbert framework and we have an opposite sign for the generators and thus following [Phi59] we use the word dissipative instead of accretive, see also [RS75, Notes of Section X.8]. We also restrict our analysis to the Hilbert space framework. In this case generators of contraction semi-groups and maximal dissipative operators coincide, see [Phi59, Theorem 1.1.3]. These framework is imposed for convenience as the existence of an hermitian product simplifies many parts of the analysis.

Let \mathcal{H} be a separable Hilbert space (possibly infinite dimensional) with scalar product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the corresponding norm, A, B be two (possibly unbounded) dissipative operators on \mathcal{H} . We consider the formal bilinear control system

$$\frac{d}{dt}\psi(t) = A\psi(t) + u(t)B\psi(t), \tag{2.1}$$

where the scalar control u is to be chosen in a set of real functions.

Given an initial data $\psi(0) = \psi_0 \in \mathcal{H}$, there is no reason for the system (2.1) to be well-posed. Indeed even the definition of $A + B$ is not obvious when A and B are unbounded. To this effect it is usually assumed that the operators A and B satisfy the following condition.

Definition 1. B is said relatively bounded with respect to A , or A -bounded, if $D(A) \subset D(B)$ and there exist $a, b > 0$ such that for every ψ in $D(A)$, $\|B\psi\| \leq a\|A\psi\| + b\|\psi\|$.

It is well-known that if A is skew-adjoint and B skew-symmetric, from Kato–Rellich theorem, (see for example [RS75, Theorem X.12]), if B is relatively bounded with respect to A , then for every real constant u such that $|u| < 1/a$, $A + uB$ is skew-adjoint with domain $D(A)$ and generates a group

of unitary operators. The system (2.1) is then well-posed for every initial condition¹. From [RS75, Corollary of theorem X.50], these can be extended to the dissipative case when A is maximal dissipative, B dissipative and u non-negative.

In most of the examples considered in Section 6, we consider the skew-adjoint case and a is arbitrary small, so that we can define the solutions of (2.1) for every piecewise constant control u with real values.

In the general cases, we will refer to the following assumptions.

Assumption 1. (A, B, K) is a triple with A a maximal dissipative operator on \mathcal{H} , B an operator on \mathcal{H} with $D(A) \subset D(B)$, and K a real interval containing 0 is such that for any $u \in K$, $A + uB$ is a maximal dissipative operator on \mathcal{H} with domain $D(A)$.

Assumption 1 implies that the operator B is a bounded from $D(A)$ to \mathcal{H} and allows to define

$$\|B\|_A := \inf_{\lambda > 0} \|B(\lambda - A)^{-1}\|. \quad (2.2)$$

The possibly zero number $\|B\|_A$ is the lower bound of all possible constant a in Definition 1. Notice that actually

$$\|B\|_A := \liminf_{\lambda \rightarrow +\infty} \|B(\lambda - A)^{-1}\|. \quad (2.3)$$

Indeed, it is clear that $\inf_{\lambda > 0} \|B(\lambda - A)^{-1}\| \leq \liminf_{\lambda \rightarrow +\infty} \|B(\lambda - A)^{-1}\|$. Conversely, for any $\epsilon > 0$ there exists $b_\epsilon > 0$ such that, for every ψ in $D(A)$,

$$\|B\psi\| \leq (\|B\|_A + \epsilon)\|A\psi\| + b_\epsilon\|\psi\|.$$

Hence, for every λ such that $A - \lambda$ is invertible, for every ϕ in \mathcal{H} ,

$$\|B(A - \lambda)^{-1}\phi\| \leq (\|B\|_A + \epsilon)\|A(A - \lambda)^{-1}\phi\| + b_\epsilon\|(A - \lambda)^{-1}\phi\|$$

and thus for $\lambda > 0$, since A is dissipative,

$$\|B(A - \lambda)^{-1}\psi\| \leq (\|B\|_A + \epsilon)\|\psi\| + \frac{b_\epsilon}{\lambda}\|\psi\|$$

so that

$$\lim_{\lambda \rightarrow +\infty} \|B(A - \lambda)^{-1}\| \leq (\|B\|_A + \epsilon).$$

The conclusion $\lim_{\lambda \rightarrow +\infty} \|B(A - \lambda)^{-1}\| \leq \|B\|_A = \inf_{\lambda > 0} \|B(\lambda - A)^{-1}\|$ follows as ϵ tends to zero.

In the next section, we will consider the following assumption.

Assumption 2. (A, B, K) , with A a maximal dissipative operator on \mathcal{H} and K a real interval containing 0, is such that

(2.1) there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on \mathcal{H} leaving $D(A)$ invariant,

(2.2) for every $u \in \mathcal{R}((0, T])$, with $u((0, t)) \in K$ for any $t \in [0, T]$,

$$t \in [0, T] \mapsto \mathcal{A}(t) := e^{u((0, t])B} A e^{-u((0, t])B}$$

is a family of maximal dissipative operators with common domain $D(A)$ such that :

¹In the skew-adjoint case the norm of the solutions of (2.1) is conserved.

- $\sup_{t \in [0, T]} \|(1 - \mathcal{A}(t))^{-1}\|_{L(\mathcal{H}, D(A))} < +\infty$,
- \mathcal{A} has bounded variation from $[0, T]$ to $L(D(A), \mathcal{H})$.

Notice that if $u \in \mathcal{R}((0, T])$, $u((0, 0]) = 0$ and $t \in [0, T] \mapsto u((0, t])$ is left continuous due to the inner regularity of Radon measures.

The triple (A, B, K) satisfies Assumption 2 for any interval K containing 0 if (A, B) satisfies the following one.

Assumption 3. (A, B) is such that

(3.1) A is a maximal dissipative operator on \mathcal{H} with domain $D(A)$,

(3.2) there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on \mathcal{H} leaving $D(A)$ invariant,

(3.3) the map $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$ is locally Lipschitz.

Assumption (3.3) is a strong assumption on the regularity of B with respect to the scale of A . Indeed it implies that B is the generator of a strongly continuous semigroup on $D(A)$. In Section 4, we consider higher regularity assumption.

2.2 Propagators

Definition 2 (Propagator on a Hilbert space). Let I be a real interval. A family $(s, t) \in \Delta_I \mapsto X(s, t)$ of linear contractions, that is Lipschitz maps with Lipschitz constant less than one, on a Hilbert space \mathcal{H} , strongly continuous in t and s and such that

- (i) for any $s < r < t$, $X(t, s) = X(t, r)X(r, s)$,
- (ii) $X(t, t) = I_{\mathcal{H}}$,

is called a contraction propagator on \mathcal{H} .

Assumption 4. Let I be a real interval and \mathcal{D} dense subset of \mathcal{H}

(4.1) $A(t)$ is a maximal dissipative operator on \mathcal{H} with domain \mathcal{D} ,

(4.2) $t \mapsto A(t)$ has bounded variation from I to $L(\mathcal{D}, \mathcal{H})$, where \mathcal{D} is endowed with the graph topology associated with $A(a)$ for $a = \inf I$,

(4.3) $M := \sup_{t \in I} \|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < \infty$,

Remark 3. In Assumption (4.2), the bounded variation of $t \mapsto A(t)$ ensures that any choice of $a \in I$ will be equivalent.

Remark 4. As $A(t)$ is a maximal dissipative operator, that is the generator of a contraction semi-group, its resolvent set contains the positive half line and from Hille-Yosida theorem, see Proposition 24, any generator of a contraction semi-group satisfies

$$\sup_{t \in I} \|(1 - A(t))^{-1}\|_{L(\mathcal{H})} < \infty.$$

Note that $\|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < +\infty$, the essence of Assumption (4.3) is that $\|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})}$ is bounded uniformly with respect to $t \in I$.

We do not assume $t \mapsto A(t)$ to be continuous. However as a consequence of Assumption (4.2) (see [Edw57, Theorem 3]) it admits right and left limit in $L(\mathcal{D}, \mathcal{H})$, $A(t - 0) = \lim_{\varepsilon \rightarrow 0^+} A(t - \varepsilon)$, $A(t + 0) = \lim_{\varepsilon \rightarrow 0^+} A(t + \varepsilon)$, for all $t \in I$, and $A(t - 0) = A(t + 0)$ for all $t \in I$ except an at most countable set.

The core of our analysis is the following result due to Kato (see [Kat53, Theorem 2 and Theorem 3]). It gives sufficient conditions for the well-posedness of the system (2.1).

Theorem 7. *If $t \in I \mapsto A(t)$ satisfies Assumption 4, then there exists a unique contraction propagator $X : \Delta_I \rightarrow L(\mathcal{H})$ such that if $\psi_0 \in \mathcal{D}$ then $X(t, s)\psi_0 \in \mathcal{D}$ and for $(t, s) \in \Delta_I$*

$$\|A(t)X(t, s)\psi_0\| \leq Me^{M\|A\|_{BV(I, L(\mathcal{D}, \mathcal{H}))}}\|A(s)\psi_0\|.$$

and in this case $X(t, s)\psi_0$ is strongly left differentiable in t and right differentiable in s with derivative (when $t = s$) $A(t+0)\psi_0$ and $-A(t-0)\psi_0$ respectively.

In the case in which $t \mapsto A(t)$ is continuous and skew-adjoint, if $\psi_0 \in \mathcal{D}$ then $t \in (s, +\infty) \mapsto X(t, s)\psi_0$ is strongly continuously differentiable in \mathcal{H} with derivative $A(t)X(t, s)\psi_0$.

Proof. The statement of this theorem is obtained by gathering statements of [Kat53]. The point which may not be stated clearly is the existence of $C > 0$ such that for $(t, s) \in \Delta_I$

$$\|A(t)X(t, s)\psi_0\| \leq C\|A(s)\psi_0\|,$$

for any $\psi_0 \in \mathcal{D}$. This is in [Kat53, §3.10] and $C = M \exp(MN)$ with

$$M = \sup_{t \in I} \|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} \quad \text{and} \quad N = \|A\|_{BV(I, L(\mathcal{D}, \mathcal{H}))}. \quad \square$$

We will consider $t \mapsto X(t, s)\phi_0$ as a “mild” solution in \mathcal{H} of

$$\begin{cases} \frac{d}{dt}\phi(t) = A(t)\phi(t) \\ \phi(s) = \phi_0 \end{cases} \quad (2.4)$$

even if it is not differentiable. However in the case in which $t \mapsto A(t)$ is continuous and skew-adjoint, if $\phi_0 \in \mathcal{D}$ then $t \mapsto X(t, s)\phi_0$ is a strong solution as it is continuously differentiable in \mathcal{H} and satisfies (2.4), see [Kat53].

We write that $X : \Delta_I \rightarrow L(\mathcal{H})$ is the propagator with generator $t \in I \mapsto A(t)$. The second argument s of $X(t, s)$ will be called the initial time.

Remark 5. If (A, B, K) satisfies Assumption 2, the operator $t \in [0, T] \mapsto \mathcal{A}(t) := e^{u((0, t)B}Ae^{-u((0, t)B}$ defined in Assumption (2.2) satisfies Assumption 4 for any Radon measure u on $(0, T)$ with $u((0, t]) \in K$ for any $t \in (0, T]$. If (A, B) satisfies Assumption 3 then (A, B, \mathbf{R}) satisfies Assumption 2)

The fact that Assumption 1 is stronger than Assumption 4 is the content of the following lemma.

Lemma 8. *If (A, B, K) satisfies Assumption 1 and $u : [0, T] \mapsto K$ has bounded variation then $A(t) := A + u(t)B$ satisfies Assumption 4 with $I = [0, T]$.*

Proof. The non obvious point to verify is Assumption (4.3). The map

$$u \in K \mapsto (1 - A)(1 - A - uB)^{-1}$$

is continuous. Indeed

$$\begin{aligned} & (1 - A)(1 - A - uB)^{-1} - (1 - A)(1 - A - vB)^{-1} \\ &= (1 - A) \left((1 - A - uB)^{-1} - (1 - A - vB)^{-1} \right) \\ &= (v - u)(1 - A) \left((1 - A - uB)^{-1}B(1 - A - vB)^{-1} \right) \\ &= (v - u)(1 - A) \left((1 - A - uB)^{-1}B(1 - A)^{-1}(1 - A)(1 - A - vB)^{-1} \right) \end{aligned}$$

so that

$$\begin{aligned} & (1-A)(1-A-uB)^{-1} - (1-A)(1-A-vB)^{-1} \\ & - (v-u)(1-A)(1-A-uB)^{-1}B(1-A)^{-1}((1-A)(1-A-vB)^{-1} - (1-A)(1-A-uB)^{-1}) \\ & = (v-u)(1-A)((1-A-uB)^{-1}B(1-A)^{-1}(1-A)(1-A-uB)^{-1}) \end{aligned}$$

Define

$$C(u) = \|(1-A)(1-A-uB)^{-1}\|_{L(\mathcal{H})} \quad \text{and} \quad b = \|B(1-A)^{-1}\|$$

so that

$$(1 - |v - u|bC(u))\|(1 - A)(1 - A - uB)^{-1} - (1 - A)(1 - A - vB)^{-1}\| \leq |v - u|C(u)^2b(1 - A)$$

which provides the desired continuity. Then as $|u(t) - u(0)| \leq \|u\|_{BV(I)}$ for any $t \in I$, $u(t)$ is in a bounded subset of K for all $t \in I$ thus the closure of its image is compact and by Heine theorem

$$t \in I \mapsto \|(1 - A - u(t)B)^{-1}\|_{L(\mathcal{H}, \mathcal{D})}$$

is bounded. □

2.3 Continuity

Our main concern on this analysis is the continuity of the solution in the control u . This is indeed a consequence of work [Kat53] by Kato as shown below.

Definition 3. A family $(A_n)_n$ of generators of contraction semi-groups tends to A , a generator of a contraction semi-group, in the strong resolvent sense if $((\lambda - A_n)^{-1}\phi)_n$ tends to $(\lambda - A)^{-1}\phi$ as n goes to infinity, for every ϕ in \mathcal{H} and some (and hence all, see [RS75]) λ in \mathbf{R}^+ .

Proposition 9. Let $(A_n)_{n \in \mathbf{N}}$ and A satisfy Assumption 4 on a bounded real interval I . Let $(\mathcal{D}_n)_{n \in \mathbf{N}}$ and \mathcal{D} be their respective domains (for any $t \in I$). Let X_n (respectively X) be the contraction propagator associated with A_n (respectively A).

Assume

- (i) $\sup_{n \in \mathbf{N}} \sup_{t \in I} \|(1 - A_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D}_n)} < +\infty$,
- (ii) $A_n(\tau)$ converges to $A(\tau)$ in the strong resolvent sense for almost every $\tau \in I$ as n tends to infinity,
- (iii) $\sup_{n \in \mathbf{N}} \|A_n\|_{BV(I, L(\mathcal{D}_n, \mathcal{H}))} < +\infty$,
- (iv) $\sup_{n \in \mathbf{N}} \|A_n(a)\|_{L(\mathcal{D}_n, \mathcal{H})} < +\infty$ for $a = \inf I$.

Then $X_n(t, s)$ tends strongly to $X(t, s)$ locally uniformly in $s, t \in I$.

Proof. From [Kat53, §3.8], we deduce that one can replace both X_n and X by any of their Riemann products that is the propagator associated with a piece-wise constant approximation, namely :

$$\|(X_n(t, s; \Delta) - X_n(t, s))\phi\| \leq Me^{MN}N|\Delta|\|A(a)\phi\|, \forall \phi \in \mathcal{E}$$

where $\Delta := \{s = t_0 < t_1 < \dots < t_n = t\}$ is a partition of the interval (t, s) , $X_n(\Delta)$ is the propagator

associated with $\sum_{j=1}^n A_{n,j-1}\chi_{[t_{j-1}, t_j]}$ where $A_{n,j-1} = A_n(t_{j-1})$,

$$M = \sup_{n \in \mathbf{N}} \sup_{t \in I} \{ \|(1 - A_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D}_n)}, \|(1 - A(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} \},$$

$$N = \max \{ \sup_{n \in \mathbf{N}} \|A_n\|_{BV(I, L(\mathcal{D}_n, \mathcal{H}))}, \|A\|_{BV(I, L(\mathcal{D}, \mathcal{H}))} \},$$

and $|\Delta| = \sup_{1 \leq j \leq n} |t_j - t_{j-1}|$. We define similarly $X(\Delta)$ and the following holds

$$\|X(t, s; \Delta) - X(t, s)\|_{L(\mathcal{D}, \mathcal{H})} \leq M e^{MN} N |\Delta|.$$

Following the proof of [RS75, Theorem X.47a (Hille-Yosida)] (see also Proposition 24), we have that

$$\left\| e^{tA_n(\tau)} \phi - e^{tA_n^\lambda(\tau)} \phi \right\| \leq t \left\| A_n(\tau) \phi - A_n^\lambda(\tau) \phi \right\|, \quad \forall \phi \in \mathcal{D}_n$$

with $A_n^\lambda(\tau) := \lambda(\lambda - A_n(\tau))^{-1} A_n(\tau)$, with $\lambda > 0$, and

$$\left\| e^{tA(\tau)} \phi - e^{tA^\lambda(\tau)} \phi \right\| \leq t \left\| A(\tau) \phi - A^\lambda(\tau) \phi \right\|, \quad \forall \phi \in \mathcal{D}$$

with $A^\lambda(\tau) := \lambda(\lambda - A(\tau))^{-1} A(\tau)$.

As A_n and A are generator of contraction semi-groups, then $\{\lambda(\lambda - A_n(\tau))^{-1}\}$ is uniformly bounded in n and τ . Notice that $\lambda(\lambda - A_n(\tau))^{-1} \phi$ tends to ϕ as λ goes to infinity for any $\phi \in \mathcal{D}_n$ uniformly in τ and n since from the assumptions we deduce $\sup_{n \in \mathbf{N}} \sup_{t \in I} \|A_n(t)\|_{L(\mathcal{D}_n, \mathcal{H})} < +\infty$. We deduce that $\lambda(\lambda - A_n(\tau))^{-1}$ tends strongly to the identity as λ goes to infinity uniformly in τ and n . Similarly $A^\lambda(\tau)$ tends strongly to $A(\tau)$ uniformly in τ and n as λ goes to infinity.

For $\phi \in \mathcal{H}$ and $\delta > 0$ there exists $\psi \in \mathcal{D}$ and $\psi^n \in \mathcal{D}_n$ such that

$$\|\phi - \psi\| \leq \delta \quad \text{and} \quad \|\phi - \psi^n\| \leq \delta$$

so that we can write

$$\begin{aligned} \left\| e^{tA(\tau)} \phi - e^{tA_n(\tau)} \phi \right\| &\leq 4\delta + \left\| e^{tA(\tau)} \psi - e^{tA^\lambda(\tau)} \psi \right\| + \left\| e^{tA^\lambda(\tau)} \phi - e^{tA_n^\lambda(\tau)} \phi \right\| + \left\| e^{tA_n^\lambda(\tau)} \psi^n - e^{tA_n(\tau)} \psi^n \right\| \\ &\leq 4\delta + t \left\| A(\tau) \psi - A^\lambda(\tau) \psi \right\| + t \left\| A_n(\tau) \psi^n - A_n^\lambda(\tau) \psi^n \right\| + \left\| e^{tA^\lambda(\tau)} \phi - e^{tA_n^\lambda(\tau)} \phi \right\|. \end{aligned}$$

As $e^{tA_n^\lambda(\tau)} = e^{-\lambda t} e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}$ and $e^{tA^\lambda(\tau)} = e^{-\lambda t} e^{t\lambda^2(\lambda - A(\tau))^{-1}}$ (see [RS75, Theorem X.47a (Hille-Yosida)]),

$$\begin{aligned} \left\| e^{tA^\lambda(\tau)} \phi - e^{tA_n^\lambda(\tau)} \phi \right\| &\leq \left\| e^{-\lambda t} e^{t\lambda^2(\lambda - A_n(\tau))^{-1}} \phi - e^{-\lambda t} e^{t\lambda^2(\lambda - A(\tau))^{-1}} \phi \right\| \\ &\leq e^{-\lambda t} \left\| e^{t\lambda^2(\lambda - A_n(\tau))^{-1}} \phi - e^{t\lambda^2(\lambda - A(\tau))^{-1}} \phi \right\| \end{aligned}$$

Recall that $\|(\lambda - A_n(\tau))^{-1}\| \leq \frac{1}{\lambda}$ (see Proposition 24 with $\omega = 0$) and hence $\|e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}\| \leq e^{\lambda t}$. Duhamel's identity writes, for $0 \leq t \leq T$,

$$\begin{aligned} &\left\| e^{t\lambda^2(\lambda - A_n(\tau))^{-1}} \phi - e^{t\lambda^2(\lambda - A(\tau))^{-1}} \phi \right\| \\ &= \left\| \int_0^t \lambda^2 e^{(t-s)\lambda^2(\lambda - A_n(\tau))^{-1}} \{(\lambda - A_n(\tau))^{-1} - (\lambda - A(\tau))^{-1}\} e^{s\lambda^2(\lambda - A(\tau))^{-1}} \phi \, ds \right\| \quad (2.5) \\ &\leq \lambda^2 e^{T\lambda} \int_0^T \left\| \{(\lambda - A_n(\tau))^{-1} - (\lambda - A(\tau))^{-1}\} e^{s\lambda^2(\lambda - A(\tau))^{-1}} \phi \right\| \, ds \end{aligned}$$

The result follows from Lebesgue dominated convergence theorem, using the convergence of $A_n(\tau)$ to $A(\tau)$ in the strong resolvent sense for almost every $\tau \in I$ as n tends to infinity. \square

Remark 6. Assuming that $\mathcal{D}_n = \mathcal{D}$, for all $n \in \mathbf{N}$, following [RS78, Theorem VIII.25] and using Banach-Steinhaus Theorem the assumptions of Proposition 9 are verified if

$$(i) \sup_{n \in \mathbf{N}} \sup_{t \in I} \|(1 - A_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < +\infty,$$

- (ii) $A_n(\tau)$ converges to $A(\tau)$ in the strong sense in \mathcal{D} for almost every $\tau \in I$ as n tends to infinity,
- (iii) $\sup_{n \in \mathbf{N}} \|A_n\|_{BV(I, L(\mathcal{D}, \mathcal{H}))} < +\infty$.

Recall that I is bounded, we have the following corollary.

Corollary 10. *Let (A, B, K) satisfy Assumption 2. Let $(v_n)_{n \in \mathbf{N}}$ be a uniformly bounded sequence in $\mathcal{R}(I)$ which is almost everywhere convergent to $v \in \mathcal{R}(I)$ with $v_n((0, t]) \in K$ and $v((0, t]) \in K$ for any $t \in (0, T]$ and $v_n(\{0\}) = v(\{0\}) = 0$. Set $\mathcal{A}_n(t) = e^{-v_n((0, t])B} A e^{v_n(0, t)B}$ and $\mathcal{A}(t) = e^{-v((0, t])B} A e^{v(0, t)B}$ and let X_n (respectively X) be the contraction propagators associated with \mathcal{A}_n (respectively \mathcal{A}).*

If $\sup_{n \in \mathbf{N}} \|\mathcal{A}_n\|_{BV(I, L(D(A), \mathcal{H}))} < +\infty$, then $X_n(t, s)$ tends to $X(t, s)$ locally uniformly $s, t \in I$ (and hence, uniformly in I^2).

Proof. Referring to the proof of Proposition 9, as $X_n(t, s; \Delta)$ is a product of contractions so is $X_n(t, s)$.

The proof consist of checking that the assumptions of Proposition 9 are fulfilled:

- (i) We have $\sup_{n \in \mathbf{N}} \sup_{t \in I} \|(1 - \mathcal{A}_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < \infty$. Indeed as

$$\begin{aligned}
& \|(1 - A)(1 - \mathcal{A}_n(t))^{-1}\|_{L(\mathcal{H})} \\
&= \|(1 - A)e^{v_n((0, t])B}(1 - A)^{-1}e^{-v_n((0, t])B}\|_{L(\mathcal{H})} \\
&\leq \|e^{v_n((0, t])B}\|_{L(\mathcal{H})} \|e^{-v_n((0, t])B}(1 - A)e^{v_n((0, t])B}(1 - A)^{-1}\|_{L(\mathcal{H})} \|e^{-v_n((0, t])B}\|_{L(\mathcal{H})} \\
&\leq \|e^{v_n((0, t])B}\|_{L(\mathcal{H})} \|(1 - \mathcal{A}_n(t))(1 - A)^{-1}\|_{L(\mathcal{H})} \|e^{-v_n((0, t])B}\|_{L(\mathcal{H})} \\
&\leq \|e^{v_n((0, t])B}\|_{L(\mathcal{H})} (\|\mathcal{A}_n(t) - \mathcal{A}_n(0)\|_{L(D(A), \mathcal{H})} + \|1 - A\|_{L(D(A), \mathcal{H})}) \|e^{-v_n((0, t])B}\|_{L(\mathcal{H})} \\
&\leq \|e^{v_n((0, t])B}\|_{L(\mathcal{H})} (\|\mathcal{A}_n\|_{BV(I, L(D(A), \mathcal{H}))} + 1) \|e^{-v_n((0, t])B}\|_{L(\mathcal{H})}.
\end{aligned}$$

We used $v_n(\{0\}) = 0$ to set $\mathcal{A}_n(0) = A$. Then from Assumption (2.1), there exists $\omega \in \mathbf{R}$ such that

$$\|e^{vB}\|_{L(\mathcal{H})} \leq e^{\omega|v|}, \forall v \in \mathbf{R} \quad (2.6)$$

which provides the boundedness.

- (ii) The sequence $\mathcal{A}_n(t)$ tends to $\mathcal{A}(t)$ in the strong resolvent sense. Indeed from

$$(1 - \mathcal{A}_n(t))^{-1} - (1 - \mathcal{A}(t))^{-1} = e^{-v_n((0, t])B}(1 - A)^{-1}e^{v_n((0, t])B} - e^{-v((0, t])B}(1 - A)^{-1}e^{v((0, t])B}$$

we have

$$\begin{aligned}
(1 - \mathcal{A}_n(t))^{-1} - (1 - \mathcal{A}(t))^{-1} &= (e^{-v_n((0, t])B} - e^{-v((0, t])B})(1 - A)^{-1}(e^{v_n((0, t])B} - e^{v((0, t])B}) \\
&\quad + (e^{-v_n((0, t])B} - e^{-v((0, t])B})(1 - A)^{-1}e^{v((0, t])B} \\
&\quad + e^{-v((0, t])B}(1 - A)^{-1}(e^{v_n((0, t])B} - e^{v((0, t])B})
\end{aligned}$$

then using (2.6) the boundedness of the sequence (v_n) and the strong continuity of $t \in \mathbf{R} \mapsto e^{tB}$, we conclude on the strong resolvent convergence.

- (iii) From the assumptions, we have $\sup_{n \in \mathbf{N}} \|\mathcal{A}_n\|_{BV(I, L(D(A), \mathcal{H}))} < +\infty$.

- (iv) As $\mathcal{A}_n(0) = A$, we have $\sup_{n \in \mathbf{N}} \|\mathcal{A}_n(0)\|_{L(D(A), \mathcal{H})} < +\infty$. □

Remark 7. To prove that $\sup_{n \in \mathbf{N}} \|\mathcal{A}_n\|_{BV(I, L(\mathcal{D}, \mathcal{H}))} < +\infty$ for $\mathcal{A}_n(t) = e^{-u_n((0, t])B} A e^{u_n(0, t)B}$, we use Assumption (3.3) as this provides $L_I(A, B)$ such that for every $s, t \in I$,

$$\|e^{-tB} A e^{tB} - e^{-sB} A e^{sB}\|_{L_I(\mathcal{D}, \mathcal{H})} \leq |t - s| L_I(A, B). \quad (2.7)$$

3 Interaction framework

In this section we consider the framework of Assumption 2 or 3. We show that they indeed lead to weak solutions of (2.1) and consider an extension of a negative result of Ball, Marsden and Slemrod [BMS82] on the attainable set.

In a subsequent paragraph we analyze the source problem in order to obtain a Duhamel formula and use it in the last paragraph to analyze perturbation by bounded control potentials.

3.1 Generalized propagators

In the following, we aim at explaining the link between Assumptions 1 and 2 and thus emphasize the fact that (2.1) has also solution for u being a Radon measure.

We will use approximation of Radon measures by piecewise constant functions.

Lemma 11. *For every $u \in \mathcal{R}((0, T])$ there exists a sequence $(u_n)_n$ of piece-wise constant functions such that $\int_0^t u_n$ tends to $u((0, t))$ and $\int_0^t |u_n|$ tends to $|u|((0, t))$ point-wise with respect to t in $[0, T]$ as n tends to infinity with $\int_0^T |u_n| \leq |u|((0, T))$.*

Moreover, if the cumulative function $t \mapsto u((0, t))$ is M -Lipschitz continuous absolutely continuous and bounded on $[0, T]$ then $(u_n)_n$ can be chosen such that $|u_n| \leq M$.

Proof. Using Hahn decomposition [Dos80] (see also [Fis12] for the Hahn-Jordan decomposition), any Radon measure u is the difference of two non negative Radon measures u^+ and u^- with disjoint supports. Hence its is equivalent to prove the statement for positive Radon measures.

Let us assume u is positive. Then $U : t \in (0, T] \mapsto u((0, t))$ is a decreasing function (with bounded variation). Except on a at most countable set, U is continuous. So U is the sum of an increasing step function, possibly with infinite steps, and an increasing continuous function. Both can be approximated by an increasing sequences of increasing continuous piecewise affine functions.

The last statement follows by considering approximation of Lipschitz continuous functions by continuous piecewise affine ones. \square

Remark 8. This lemma explains our choice of topology over the Radon measures while the total variation topology may seem more natural at first sight. Note that in this lemma, we consider the left-continuous cumulative function of u . Indeed below, we need convergence of the measures for intervals of the form $[a, b) \cap (0, T]$ thus which corresponds to characteristic functions right-continuous on $[0, T]$.

For instance, for a positive $u \in \mathcal{R}((0, T])$ the sequence $(u_n)_n$ of piece-wise constant functions such that $\int_t^T u_n$ tends to $u((0, t])$ point-wise is, in our construction, an increasing sequence.

As each $t \mapsto \int_0^t u_n$ is continuous, if $t \mapsto u((0, t))$ is not, we see that the same result for the total variation topology is excluded. In this case, we notice that $t \mapsto u((0, t)) - \int_0^t u_n$ cannot be, by any other construction, both positive and increasing in t as otherwise for any t $|u((0, t)) - \int_0^t u_n| \leq |u((0, T]) - \int_0^T u_n|$ which will implies the convergence for the total variation topology.

Definition 4. Let (A, B, K) satisfy Assumptions 1 and 2. Let $u \in \mathcal{R}([0, T])$. For any $v \in BV([0, T], K)$ with distributional derivative u let $t \mapsto Y_t^u$ be the contraction propagator with initial time $s = 0$ associated with $\mathcal{A}_v(t) := e^{-v(t)B} A e^{v(t)B}$. We define the generalized propagator associated with $A + u(t)B$ with initial time zero, to be $\Upsilon_{t,0}^{v,u} = e^{v(t)B} Y_t^u$ and t in $[0, T]$ and for every v in $BV([0, T], K)$ such that $v' = u$ in the distributional sense.

Remark 9. Let $u \in \mathcal{R}([0, T])$ and define $v_0(t) = u((0, t])$ the associated right-continuous cumulative function and $v \in BV([0, T], \mathbf{R})$ such that $v' = u$. Then $v - v_0$ is in $BV([0, T], \mathbf{R})$ and is almost

everywhere 0 as it is supported on the at most countable set where v is not right-continuous. A somehow pathological example could be $u = 0$ and v any characteristic function of a finite set.

The propagator Y_t^u will not depend on the choice of v being right-continuous, or not, at its discontinuities. Indeed the later is a negligible set and a Duhamel formula provides the equality of the propagators. In the other hand, the factor e^{vB} depends crucially on this choice. This explains the the notation $\Upsilon^{\partial v}$ instead of Υ^u .

We introduced this generalized propagator as imposing any extra requirement on the choice of v will lead to a loss in the compactness provided by Helly's selection theorem when we consider the bounded case, Section 3.4. Nonetheless, except on this particular case we will only consider right-continuous v_0 keeping in mind that the generalized propagator $\Upsilon^{\partial v}$ will be deduce multiplying with $e^{(v-v_0)B}$.

Compared to Definition 2, $X(t, s) = e^{v(t)B} Y_{t,s}^u e^{-v(s)B}$ defines a propagator when v is continuous, that is when u has no atoms. Otherwise, we no longer require any continuity keeping in mind that v_0 the right-continuous cumulative function of u will lead to a right-continuous propagator which seems more compatible with the initial condition requirement.

Proposition 12. *Let (A, B, K) satisfies Assumptions 1 and 2. Let $t \mapsto Y_t^u$ be the contraction propagator with initial time $s = 0$ associated with $\mathcal{A}(t) := e^{-u((0,t])B} A e^{u((0,t])B}$ for $u \in \mathcal{R}((0, T])$ and Υ_t^u the one associated with $A + u(t)B$ with initial time $s = 0$ for $u \in BV([0, T], \mathbf{R})$. Then for every $\psi_0 \in \mathcal{H}, t \in [0, T]$ the map $\Upsilon_t(\psi_0) : u \mapsto \Upsilon_t^u(\psi_0) \in \mathcal{H}$ admits a unique continuous extension on $\mathcal{R}((0, T])$ denoted $\Upsilon_t(\psi_0)$ which satisfies, for every u in $\mathcal{R}(I)$,*

$$\Upsilon_t^u(\psi_0) = e^{u((0,t])B} Y_t^u(\psi_0), \forall u \in \mathcal{R}((0, T]), \forall t \in [0, T]. \quad (3.1)$$

Proof. Let $u \in L^\infty([0, T], \mathbf{R})$ and consider the propagator Y_t^u with generator

$$t \in [0, T] \mapsto e^{-\int_0^t u(\tau) d\tau B} A e^{\int_0^t u(\tau) d\tau B}$$

and initial time $s = 0$ given by Theorem 7. If, in particular, u is left-continuous piecewise constant and $\psi_0 \in D(A)$ then the continuous function $\Psi : t \mapsto e^{-\int_0^t u(\tau) d\tau B} \Upsilon_t^u(\psi_0)$ is piecewise C^1 propagator with derivative $e^{-\int_0^t u(\tau) d\tau B} A e^{\int_0^t u(\tau) d\tau B} \Psi(t)$ and hence by uniqueness in Theorem 7, $Y_t^u(\psi_0) = e^{-\int_0^t u(\tau) d\tau B} \Upsilon_t^u(\psi_0)$.

For every $u \in \mathcal{R}((0, T])$ let $(u_n)_{n \in \mathbf{N}}$ be a sequence of (left-continuous) piecewise constant functions on $[0, T]$ that converges to u for the topology of $\mathcal{R}((0, T])$.

From Remark 7 and Corollary 10, for every $\psi_0 \in \mathcal{H}$, $Y_t^{u_n}(\psi_0)$ tends to $Y_t^u(\psi_0)$ as n tends to ∞ . We set $\Upsilon_t^u(\psi_0) = e^{u((0,t])B} Y_t^u(\psi_0)$. Then $\Upsilon_t^{u_n}(\psi_0)$ tends to $\Upsilon_t^u(\psi_0)$ as n tends to ∞ . \square

In the rest of this analysis, the definition of Υ_t^u will be given by (3.1). This means that in the case where $v = u((0, t])$, we write Υ^u instead of $\Upsilon^{\partial v}$.

Proposition 13. *Let $T \geq 0$. If (A, B) satisfies Assumption 3 and $D(A) \subset D(B)$. Then for every ψ_0 in $D(A)$, for every $u \in \mathcal{R}((0, T])$, the map $t \mapsto \Upsilon_t^u(\psi_0)$ is a weak solution of (2.1) in the sense that it satisfies*

$$\int_{[0, T]} \langle f'(t), \Upsilon_t^u(\psi_0) \rangle dt = \int_{[0, T]} \langle f(t), A \Upsilon_t^u(\psi_0) \rangle dt + \int_{(0, T]} \langle f(t), B \Upsilon_t^u(\psi_0) \rangle du(t),$$

for every $f \in C_0^1([0, T], \mathcal{H})$.

Proof. For every $u \in \mathcal{R}((0, T])$ let $(u_n)_{n \in \mathbf{N}}$ be a sequence of piecewise constant functions on $[0, T]$ that converges to u for the topology of $\mathcal{R}((0, T])$.

For every $f \in C_0^1([0, T], \mathcal{H})$,

$$-\int_{[0, T]} \langle f'(t), \Upsilon_t^{u_n}(\psi_0) \rangle dt = \int_{[0, T]} \langle f(t), A\Upsilon_t^{u_n}(\psi_0) \rangle dt + \int_{[0, T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0) \rangle du_n(t)$$

since from Theorem 7, $\Upsilon_t^{u_n}(\psi_0) \in D(A)$ for any $t \in [0, T]$.

The left hand side converges to

$$-\int_{[0, T]} \langle f'(t), \Upsilon_t^u(\psi_0) \rangle dt,$$

$$\int_{[0, T]} \langle f(t), A\Upsilon_t^{u_n}(\psi_0) \rangle dt \rightarrow \int_{[0, T]} \langle f(t), A\Upsilon_t^u(\psi_0) \rangle dt,$$

and

$$\int_{[0, T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0) \rangle du_n(t) \rightarrow \int_{(0, T]} \langle f(t), B\Upsilon_t^u(\psi_0) \rangle du(t)$$

as n tends to ∞ . Indeed, the integrand is a bounded pointwise converging sequence of right-continuous regulated functions which makes each of them the uniform limit of a sequence of piecewise constant functions which can be chosen to be right-continuous. Since

$$\int_{[0, T]} v(t) du_n(t) \rightarrow \int_{(0, T]} v(t) du(t)$$

for v a piecewise constant right-continuous function, see Lemma 11 and $(|\mu_n|((0, T]))_{n \in \mathbf{N}}$ is a bounded sequence, the desired convergence holds.

Eventually

$$\left| \int_{(0, T]} \langle Bf(t), \Upsilon_t^u(\psi_0) \rangle d(u - u_n)(t) \right| \leq \sup_{t \in [0, T]} \|Bf\| \|\psi_0\| |u - u_n|(\text{supp} f) \rightarrow_{n \rightarrow \infty} 0. \quad \square$$

Remark 10. The assumption $D(A) \subset D(B)$ is made for convenience to avoid the introduction of auxiliary spaces such as $D(B^*)^*$, the topological dual of the domain of the adjoint of B .

An interesting question would be to understand the relation between the assumptions associated to the two constructions of propagators we have considered in this section. For example, on what extent does Assumption 3 ensure that $A + uB$ has a maximal dissipative closure for $u \in \mathbf{R}$?

This seems to be a difficult question. However in the skew-adjoint case, the following considerations are in place. Let A and B be skew-adjoint (symmetric maximal dissipative) with $D(A) \cap D(B)$ dense. For any $v \in \mathcal{H}$, any $u \in D(A)$ the map

$$t \in K \mapsto \langle (1 - \varepsilon(B^2 + A^2))^{-1} v, e^{tB} A e^{-tB} (1 - \varepsilon(B^2 + A^2))^{-1} u \rangle$$

is Lipschitz, its distributional derivative is bounded uniformly in ε by the Banach-Steinhaus theorem. So that $[A, B] \in L(D(A) \cap D(B), (D(A) \cap D(B))^*)$ extends to an operator such that

$$[A, B] \in L(D(A), \mathcal{H}).$$

with a slight abuse of notation and hence for any $u \in \mathbf{R}$

$$[A, A + uB] \in L(D(A), \mathcal{H}).$$

The Nelson commutator theorem, see Appendix C, gives that $A + uB$ is essentially skew-adjoint for any $u \in \mathbf{R}$.

Proposition 12 extends the notion of solution to a very low regular case, namely Radon measure controls. To do so we made some regularity assumptions on the control potential B by using Assumption 3.

Corollary 14. *Let $T \geq 0$. Assume that (A, B) satisfies Assumption 3 and $D(A) \subset D(B)$. Then for every $\psi_0 \in D(A)$, for every $u \in \mathcal{R}((0, T])$, for every $t \in [0, T]$, $\Upsilon_t^u(\psi_0) \in D(A)$ and satisfies*

$$\Upsilon_t^u(\psi_0) = \psi_0 + \int_{[0,t]} A\Upsilon_s^u(\psi_0) ds + \int_{(0,t]} B\Upsilon_s^u(\psi_0) du(s). \quad (3.2)$$

Proof. From Theorem 7, $\Upsilon_t^u(\psi_0) \in D(A)$. From Proposition 13, the following holds

$$\int_{[0,T]} \langle f'(t), \Upsilon_t^u(\psi_0) \rangle dt = \int_{[0,T]} \langle Af(t), \Upsilon_t^u(\psi_0) \rangle dt + \int_{(0,T]} \langle Bf(t), \Upsilon_t^u(\psi_0) \rangle du(t),$$

for $f \in C_0^1([0, T], \mathcal{H})$. Then plugging

$$f(t) = \int_{[0,t]} f'(s) ds$$

and using Fubini's theorem for Bochner integral [DS88, Theorem III.11.9], we have

$$\int_{[0,T]} \langle f'(t), \Upsilon_t^u(\psi_0) \rangle dt = - \int_{[0,T]} \langle f'(s), \int_{[s,T]} A\Upsilon_t^u(\psi_0) dt \rangle ds - \int_{(0,T]} \langle f'(s), \int_{[s,T]} B\Upsilon_t^u(\psi_0) du(t) \rangle ds,$$

for $f' \in C_0([0, T], \mathcal{H})$ orthogonal to 1. Hence

$$t \in [0, T] \mapsto \Upsilon_t^u(\psi_0) + \int_{[s,T]} A\Upsilon_t^u(\psi_0) dt + \int_{[s,T]} B\Upsilon_t^u(\psi_0) du(t)$$

is almost everywhere equal to a fixed constant, so that (3.2) follows for almost every $t \in [0, T]$. Recalling that both sides of (3.2) are right-continuous we obtain the corollary. \square

3.2 The attainable set

Proposition 15. *Let $T > 0$. Let $\psi_0 \in \mathcal{H}$. Let (A, B) satisfy Assumption 3. Then, for every $L > 0$, the set $\{\alpha\Upsilon_t^u(\psi_0) : u \in \mathcal{R}((0, T]), |u|((0, T]) \leq L, t \in [0, T], |\alpha| \leq 1\}$ is relatively compact in \mathcal{H} .*

Proof. For every $u \in \mathcal{R}((0, T])$ consider $v(t) = u((0, t])$. Then $v \in BV([0, T])$ and $\|v\|_{BV([0, T])} \leq L$. Note that $\|v\|_{L^\infty(J)} = \sup_{t \in [0, T]} |v(t) - v(0)| \leq \|v\|_{BV([0, T])} \leq L$ since $v(0) = u(\emptyset) = 0$. Consider a sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{R}((0, T])$ such that $|u_n|((0, T]) \leq L$ for every n . The sequence $v_n : t \mapsto u_n((0, t])$, from Helly's selection theorem (see for example [Hel12, Nat55]) has a subsequence pointwise converging to some $v \in BV([0, T])$, $\|v\| \leq L$. Let u the Radon measure associated to v . Thus the sequence $(u_n)_{n \in \mathbb{N}}$ has a subsequence converging to $u \in \mathcal{R}((0, T])$. We relabel this convergent subsequence by $(u_n)_{n \in \mathbb{N}}$.

From (2.7) we have that $\mathcal{A}_n(t) = e^{-u_n((0,t])B} A e^{u_n((0,t])B}$ is uniformly bounded in $BV([0, T], L(\mathcal{D}, \mathcal{H}))$ by $L\|A, B\|_{L(\mathcal{D}, \mathcal{H})}$. By Corollary 10, $t \mapsto Y_t^{u_n}(\psi_0)$ converges uniformly for $t \in [0, T]$ to $t \mapsto Y_t^u(\psi_0)$. A special attention has to be paid to the sequence of factors $(t \mapsto e^{v_n((0,t])B})$, no uniform convergence can be ensured, and $t \mapsto \Upsilon_t^{u_n}(\psi_0)$ converges pointwise in $t \in [0, T]$ to $t \mapsto \Upsilon_t^{v,u}(\psi_0)$.

To conclude the lemma, we notice that for any sequence $(t_n)_n$, $(v_n((0, t_n]))_n$ is a bounded sequence and thus $(e^{v_n((0,t_n])B})$ is bounded and has a strongly convergent subsequence. \square

Remark 11. Note that the set $\{\Upsilon^u(\psi_0) : u \in L^1([0, T], \mathbf{R}), \|u\|_{L^1} \leq L\}$ is relatively compact in $L^\infty([0, T], \mathcal{H})$. However, despite the compactness of $[0, T]$, the set $\{\Upsilon^u(\psi_0) : u \in \mathcal{R}(J), |u|((0, T]) \leq L\}$ may be not relatively compact in $L^\infty([0, T], \mathcal{H})$. Indeed, if this set were relatively compact, the trajectories of (2.1) would be continuous, due to the density of continuous trajectories, which is not the case in general. The obstructions comes from the factor $e^{u((0,t])B}$ in (3.1).

From the above corollary the attainable set is contained in a countable union of precompact sets.

Corollary 16. *If \mathcal{H} is infinite dimensional and $\psi_0 \in \mathcal{H}$. If (A, B) satisfies Assumption 3 then*

$$\bigcup_{L, T, a > 0} \{\alpha \Upsilon_t^u(\psi_0), u \in \mathcal{R}((0, T]), |u|((0, T]) \leq L, t \in [0, T], |\alpha| \leq a\}$$

is a meagre set (in the sense of Baire).

Proof. The union can be actually restricted to L, T, a in a countable set, for instance \mathbf{N}^3 . Then Proposition 15 tells that each set of the union is relatively compact in \mathcal{H} and thus with empty interior. \square

Proof of Theorem 2. The well-posedness result for L^1 controls is a consequence of Proposition 13 proved for Radon controls. Emptiness of the attainable set for L^1 controls is a consequence of Corollary 16 proved for Radon controls. \square

Remark 12. This result represents an extension of [BMS82, Theorem 3.6], see also [Tur00], in the Hilbert case.

Similar results holds for bounded variation function in the framework of Theorem 7 with $A(t) = A + u(t)B$ where (A, B) satisfies Assumptions 1, see Section 4.

3.3 The source problem

In this section we want to consider the formal “source problem”

$$\frac{d}{dt}\psi(t) = A\psi(t) + u(t)B\psi(t) + v(t)F(t), \quad (3.3)$$

where $F : [0, T] \mapsto \mathcal{H}$ is integrable and $v \in \mathcal{R}((0, T])$.

Our aim is to establish a Duhamel formula in the Radon measure framework.

Corollary 17. *Let $T \geq 0$. Let $u \in \mathcal{R}((0, T])$. If (A, B) satisfy Assumption 3 and $D(A) \subset D(B)$. Let $X^u(t, s) = e^{u((0, t])B} Y^u(t, s) e^{u((0, s])B}$ where $Y^u(t, s)$ is the contraction propagator associated with $A(t) = e^{-u((0, t])B} A e^{u((0, t])B}$*

For every $\psi_0 \in D(A)$, for every $t \in [0, T]$ for every $u \in \mathcal{R}((0, T])$, $F \in L_{dv}^1([0, T], D(A))$ and $v \in \mathcal{R}((0, T])$ such that

$$\forall t \in (0, T], \int_{(0, t]} BF(r)v(\{r\})du(r) = 0$$

define

$$\psi(t) = X^u(t, 0)\psi_0 + \int_{(0, t]} X^u(t, s)F(s)dv(s).$$

Then

$$\psi(t) = \psi_0 + \int_{[0, t]} A\psi(s)ds + \int_{(0, t]} B\psi(s)du(s) + \int_{(0, t]} F(s)dv(s).$$

Proof. From Corollary 14, for $t \in [0, T]$ and $t_0 \in [0, t]$ after a translation by t_0 we deduce

$$X^u(t, t_0)\psi_0 = \psi_0 + \int_{[t_0, t]} AX^u(s, t_0)\psi_0 ds + \int_{(t_0, t]} BX^u(s, t_0)\psi_0 du(s).$$

Then

$$\begin{aligned}
\psi(t) &= \psi_0 + \int_{[0,t]} AX^u(s,0)\psi_0 ds + \int_{(0,t]} BX^u(s,0)\psi_0 du(s) \\
&+ \int_{(0,t]} F(s)dv(s) + \int_{(0,t]} \int_{[s,t]} AX^u(r,s)F(s)drdv(s) + \int_{(0,t]} \int_{(s,t]} BX^u(r,s)F(r)du(r)dv(s) \\
&= \psi_0 + \int_{(0,t]} F(s)dv(s) + \int_{[0,t]} AX^u(s,0)\psi_0 ds + \int_{(0,t]} \int_{[s,t]} AX^u(r,s)F(s)drdv(s) \\
&+ \int_{(0,t]} BX^u(s,0)\psi_0 du(s) + \int_{(0,t]} \int_{(s,t]} BX^u(r,s)F(s)du(r)dv(s).
\end{aligned}$$

So that we obtain

$$\begin{aligned}
\psi(t) &= \psi_0 + \int_{(0,t]} F(s)dv(s) + \int_{[0,t]} AX^u(r,0)\psi_0 dr + \int_{(0,t]} \int_{(0,r]} AX^u(r,s)F(s)dv(s)dr \\
&+ \int_{(0,t]} BX^u(r,0)\psi_0 du(r) + \int_{(0,t]} \int_{(0,r]} BX^u(r,s)F(s)dv(s)du(r) \\
&= \psi_0 + \int_{(0,t]} F(s)dv(s) + \int_{[0,t]} A \left(X^u(r,0)\psi_0 + \int_{(0,r]} X^u(r,s)F(s)dv(s) \right) dr \\
&+ \int_{(0,t]} B \left(X^u(r,0)\psi_0 + \int_{(0,r]} X^u(r,s)F(s)dv(s) \right) du(r)
\end{aligned}$$

which provides the conclusion as

$$\int_{(0,t]} BF(r)v(\{r\})du(r) = 0. \quad \square$$

Remark 13. We can now consider that for $\psi_0 \in \mathcal{H}$, for $u \in \mathcal{R}((0, T])$, $v \in \mathcal{R}((0, T])$ with no atoms and $F \in L_{dv}^1([0, T], \mathcal{H})$

$$t \in [0, T] \mapsto \psi(t) = X^u(t,0)\psi_0 + \int_{(0,t]} X^u(t,s)F(s)dv(s)$$

is a mild solution of (3.3).

3.4 Bounded control potentials

In this section only, we assume that A is the generator of a strongly continuous semigroup with domain $D(A)$ and B is bounded. For every u in \mathbf{R} , $A + uB$ is also a generator of a strongly continuous semigroup with domain $D(A)$.

Since A generates a strongly continuous semi-group there exist $C_A > 0$ and $\omega \in \mathbf{R}$ such that

$$\|e^{tA}\| \leq C_A e^{\omega t}, \quad \forall t > 0. \quad (3.4)$$

For the equivalent norm

$$N(\psi) = \sup_{t>0} \|e^{t(A-\omega)}\psi\|$$

$A - \omega$ is the generator of a contraction semigroup. Let B in $L(\mathcal{H})$ then B is also bounded for the norm N and if $\|B\|_N$ is its norm, $A - \omega + u(t)B - R\|B\|_N$ for any u with bounded variation and value in $B_{\mathbf{R}}(0, R)$ satisfies the assumptions of [Kat53] in the Banach space structure associated with

N . So that in this case the results of § 2.2 are still valid. Below we show that it is actually as simple to consider Radon measures controls.

It is classical (see [BMS82]) that the input-output mapping Υ admits a unique continuous continuation to $L^1(\mathbf{R}, \mathbf{R})$. We consider below the extension to Radon measures. Recall that C_A is defined in (3.4).

Theorem 18. *Let A be with domain $D(A)$ be the generator of a strongly continuous semigroup and let B be bounded. Then, the input-output mapping Υ admits a unique continuous extension to the set of Radon measures. Precisely, for any $u \in \mathcal{R}((0, T])$ there exists a unique family of operators $\Upsilon^u : \Delta_{[0, T]} \rightarrow L(\mathcal{H})$ strongly continuous in t and s such that*

$$(i) \text{ for any } s < r < t, \Upsilon_{t,s}^u = \Upsilon^u(t, r)\Upsilon^u(r, s),$$

$$(ii) \Upsilon^u(t, t) = I_{\mathcal{H}},$$

(iii) *if u has bounded variation on $[0, T]$, for any $\psi_0 \in \mathcal{H}$, $(s, t) \in \Delta_{[0, T]} \mapsto \Upsilon_{t,s}^u \psi_0$ is strongly left differentiable in t and right differentiable in s with derivative (when $t = s$) $(A + u(t+0)B)\psi_0$ and $-(A + u(t-0)B)\psi_0$ respectively,*

(iv) *the map $u \mapsto \Upsilon^u$ is continuous in $\mathcal{R}((0, T])$ in the sense that if $(u_l)_{l \in \mathbf{N}}$ is a sequence of Radon measures $(0, T]$ and u a Radon measure on $(0, T]$ such that $u_l((0, T])$ tends to $u((0, T])$ and for every t in $[0, T]$, $u_l((0, t))$ tends to $u((0, t))$ as l tends to infinity with $\sup_{l \in \mathbf{N}} |u_l|((0, T]) < +\infty$ then*

$$\Upsilon_{t,s}^{u_l} \psi_0 \rightarrow \Upsilon_{t,s}^u \psi_0$$

for any $\psi_0 \in \mathcal{H}$.

The family Υ^u is called propagator associated with $A + uB$. Moreover Υ^u satisfies the following estimate:

$$\|\Upsilon_{t,s}^u\|_{B(\mathcal{H})} \leq C_A e^{\omega|t-s| + |u|([s,t])C_A \|B\|}$$

for any Radon measure u .

Proof. Let $u : [0, T] \rightarrow \mathbf{R}$ be a piecewise constant function. For every ψ_0 in $D(A)$, the map $t \mapsto \Upsilon_{t,s}^u \psi_0$ is absolutely continuous and satisfies, for almost every t in $[s, T]$,

$$\frac{d}{dt} \Upsilon_{t,s}^u \psi_0 = (A + u(t)B) \Upsilon_{t,s}^u \psi_0.$$

Following [RS75, Theorem X.69], we write, thanks to the Duhamel's formula,

$$\Upsilon_{t,s}^u \psi_0 = e^{(t-s)A} \psi_0 + \int_{(s,t]} e^{(t-s_1)A} u(s_1) B \Upsilon_{s_1,s}^u \psi_0 ds_1$$

and replacing iteratively $\Upsilon_{t,s}^u \psi_0$ by its expression p times, we get

$$\begin{aligned} \Upsilon_{t,s}^u \psi_0 &= e^{(t-s)A} \psi_0 + \sum_{n=1}^{\infty} \int_{s < s_1 \leq s_2 \leq \dots \leq s_n \leq t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \circ \dots \\ &\quad \dots \circ B e^{(s_2-s_1)A} B e^{(s_1-s)A} \psi_0 u(s_1) u(s_2) \dots u(s_n) ds_1 \dots ds_n. \end{aligned}$$

This allows us to extend the propagator to Radon measures. Namely let $u \in \mathcal{R}((0, T])$, we define the linear operator

$$Y_{(n)}^u(t, s) \psi_0 := \int_{s < s_1 \leq s_2 \leq \dots \leq s_n \leq t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \dots B e^{(s_2-s_1)A} B e^{(s_1-s)A} \psi_0 du(s_1) du(s_2) \dots du(s_n).$$

As B is bounded,

$$\|Y_{(n)}^u(t, s)\psi_0\| \leq e^{\omega t} C_A^{n+1} \|B\|^n \|\psi_0\| \int_{s < s_1 \leq s_2 \leq \dots \leq s_n \leq t} d|u|(s_1) d|u|(s_2) \dots d|u|(s_n)$$

and since $[s, t]^n$ contains the disjoint union of $\{s < s_{\sigma(1)} < s_{\sigma(2)} < \dots < s_{\sigma(n)} \leq t\}$ over all permutations σ of $\{1, 2, \dots, n\}$

$$\|Y_{(n)}^u(t, s)\psi_0\| \leq e^{\omega t} C_A^{n+1} \|B\|^n \|\psi_0\| \frac{|u|([s, t])^n}{n!}$$

and the infinite expansion

$$\Upsilon_{t,s}^u = \sum_{n=0}^{\infty} Y_{(n)}^u(t, s) \quad (3.5)$$

converges in norm in the set $L(H)$ of the bounded operators of H .

Let $(u_l)_{l \in \mathbf{N}}$ be a sequence of piecewise constant functions on $(0, T]$ that converges to u a Radon measure on $(0, T]$ in the sense that: $(u_l)_{l \in \mathbf{N}}$ is a sequence of Radon measures $(0, T]$ and u a Radon measure on $(0, T]$ such that $u_l((0, T])$ tends to $u((0, T])$ and for every t in $[0, T]$, $u_l((0, t))$ tends to $u((0, t))$ as l tends to infinity with $\sup_{l \in \mathbf{N}} |u_l|((0, T]) < +\infty$.

Before proving the convergence of $(Y_{(n)}^{u_l}(t, s)\psi_0)_l$ in l , we remark that as in the proof of Proposition 13 for any right-continuous regulated function $F : [0, T] \rightarrow \mathcal{H}$

$$\int_{(s,t]} e^{(t-s_1)A} F(s_1) du_l(s_1) \rightarrow \int_{(s,t]} e^{(t-s_1)A} F(s_1) du(s_1).$$

For every (s, t) in $\Delta_{[0, T]}$, we have

$$\begin{aligned} Y_{(n)}^{u_l}(t, s)\psi_0 - Y_{(n)}^u(t, s)\psi_0 &= \sum_{k=1}^n \int_{s < s_1 \leq s_2 \leq \dots \leq s_n \leq t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \dots B e^{(s_2-s_1)A} \psi_0 \times \\ &\quad \times du_l(s_1) \dots du_l(s_k) du(s_{k+1}) \dots du(s_n) \\ &\quad - \int_{s < s_1 \leq s_2 \leq \dots \leq s_n \leq t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \dots B e^{(s_2-s_1)A} \psi_0 \times \\ &\quad \times du_l(s_1) \dots du_l(s_{k-1}) du(s_k) \dots du(s_n) \end{aligned}$$

so that for

$$F(t) = B \int_{s < s_1 \leq s_2 \leq \dots \leq s_{k-1} \leq t} e^{(t-s_{k-1})A} B e^{(s_{k-1}-s_{k-2})A} \dots B e^{(s_2-s_1)A} \psi_0 du(s_1) \dots du(s_{k-1}),$$

the s dependence being implicit, we write

$$\begin{aligned} &\|Y_{(n)}^{u_l}(t, s)\psi_0 - Y_{(n)}^u(t, s)\psi_0\| \\ &\leq \sum_{k=1}^n \left\| \int_{s < s_{k+1} \leq s_{k+2} \leq \dots \leq s_n \leq t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \dots B e^{(s_{k+2}-s_{k+1})A} \times \right. \\ &\quad \left(\int_{s < s_k \leq s_{k+1}} e^{(s_{k+1}-s_k)A} F(s_k) du_l(s_k) - \int_{s < s_k \leq s_{k+1}} e^{(s_{k+1}-s_k)A} F(s_k) du(s_k) \right) \times \\ &\quad \left. \times du_l(s_{k+1}) \dots du_l(s_n) \right\|. \end{aligned}$$

As F is regulated and right-continuous, we conclude on the convergence using the previous consideration and the Lebesgues theorem.

The differentiability properties in the bounded variation case are due to [Kat53, Theorem 1] as $A - \omega$ is the generator of a contraction semigroup and as B in $L(\mathcal{H})$ then B is also bounded for the norm N . So that $A - \omega + u(t)B - R\|B\|_N$ for any $R > |u|_\infty$ satisfies the assumptions of [Kat53, theorem 1] in the Banach space structure associated with N .

The properties

$$\Upsilon_{t,s}^u = \Upsilon_{t,r}^u \Upsilon_{r,s}^u \quad \text{for } s < r < t, \quad \Upsilon_{t,t}^u = I_{\mathcal{H}}$$

can be obtained by density and continuity from [Kat53, Theorem 1] for $A - \omega + u(t)B - R\|B\|_N$ as well. \square

Somehow $t \mapsto \Upsilon_{t,s}^u \phi_0$ can be considered as a ‘‘mild’’ solution in \mathcal{H} of (2.4) with $A(t) = A + u(t)B$.

Corollary 19. *Let A be with domain $D(A)$ be the generator of a strongly continuous semigroup and let B be bounded. Then for any $u \in BV([0, T], \mathbf{R})$ $\Upsilon_{T,0}^u D(A) \subset D(A)$ and there exists m (depending only on A, B and $\|u\|_{L^\infty([0, T])}$) such that*

$$\|\Upsilon_{T,0}^u(\psi_0)\|_2 \leq m e^{m\|u\|_{BV([0, T], \mathbf{R})}} e^{|\int_0^t u|} \|\psi_0\|_2$$

Proof. The proof is a consequence of estimates [Kat53, §3.8] applied to $A - \omega + u(t)B - R\|B\|_N$ which is the generator of a contraction semigroup for the norm N . \square

Proposition 20. *Let A be the generator of a strongly continuous semi-group and let B be bounded. Let $(u_l)_{l \in \mathbf{N}}$ be a sequence of Radon measures $(0, T]$ such that for every t in $[0, T]$, $\sup_{l \in \mathbf{N}} |u_l|((0, T]) < +\infty$. Let $(t_l)_{l \in \mathbf{N}} \in [0, T]$. Then $(\Upsilon_{t_l}^{u_l})_{l \in \mathbf{N}}$ has a strongly convergent subsequence.*

Proof. Let us first notice that $(t \mapsto |u_l|((0, t]))_l$ defines a sequence of increasing bounded functions. So that we can assume up to extracting a subsequence it is a pointwise convergent by Helly’s selection theorem. Using Hahn decomposition, we have that $(t \mapsto |u_l|((0, t]) - u_l((0, t]))_l$ is also a sequence of increasing functions so that one can extract a converging subsequence as well. For the rest the proof we assume that both sequences are pointwise convergent.

Let $\epsilon > 0$. We make several reductions to the formula (3.5). First for $S = \sup_{l \in \mathbf{N}} |u_l|((0, T])$, we have

$$\left\| \Upsilon_{t,s}^{u_l} - \sum_{n=0}^N Y_{(n)}^{u_l}(t, s) \right\| \leq C_A e^{\omega T} \left(\sum_{k>N} \frac{(C_A \|B\| S)^n}{n!} \right) \|\psi_0\|.$$

Therefore for some integer N_0 , this is smaller than ϵ whenever $N \geq N_0$.

Then recall that

$$Y_{(n)}^{u_l}(t, s)\psi_0 := \int_{s < s_1 \leq s_2 \leq \dots \leq s_n \leq t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \dots B e^{(s_2-s_1)A} B e^{(s_1-s)A} \psi_0 du_l(s_1) du_l(s_2) \dots du_l(s_n).$$

For $n < N_0$, the integrand is a continuous of all its variables in $[0, T]^{n+2}$, this can be replaced up to an error of size $\frac{1}{N_0} \epsilon T^{-N_0}$ by a piecewise constant function which is a finite linear combination of products of left-continuous characteristic functions. So the proof reduces to the convergence of

$$\int_{0 < s_l \leq s_1 \leq s_2 \leq \dots \leq s_n \leq t_l} \prod_{\ell=0}^{n+1} 1_{(a_\ell, b_\ell]}(s_\ell) du_l(s_1) du_l(s_2) \dots du_l(s_n).$$

where $s_{n+1} = t_l$ also depends on l . Since these N_0 sequences are all bounded we can extract convergent subsequences. \square

Corollary 21. *Let A be the generator of a strongly continuous semi-group and let B be bounded. Then for every ψ_0 in \mathcal{H} , the set*

$$\bigcup_{T \geq 0} \bigcup_{\alpha > 0} \bigcup_{u \in \mathcal{R}((0, T])} \{\alpha \Upsilon_{t,0}^u \psi_0 \mid t \in [0, T]\}$$

is a meager set in \mathcal{H} .

Proof. Due to Proposition 20, the proof is similar to the one of Corollary 16. \square

Proof of Proposition 3. As already mentioned, the well-posedness result is classical (see [BMS82] for instance), while the emptiness of the attainable set with L^1 controls follows from Corollary 21 proved for Radon controls. \square

Theorem 18 can be generalized to include a time dependent principal term as follows.

Definition 5. Let (A, B_1) be a couple of operators and $u_1 : [0, T] \rightarrow \mathbf{R}$ be piecewise constant such that $A + u_1(t)B_1$ defined on $D(A) \cap D(B_1)$ has maximal dissipative closure on \mathcal{H} for any $t \in [0, T]$. Let B_2 be a bounded operator and $u_2 : [0, T] \rightarrow \mathbf{R}$ be piecewise constant. Let $\Upsilon^{u_1, u_2} : [0, T] \rightarrow L(\mathcal{H}, \mathcal{H})$ be the operator that associates, with every t in $[0, T]$ and every u piecewise constant, the unitary linear transformation

$$\Upsilon_t^u = e^{(t-t_{j-1})(A+u_{1,j-1}B_1+u_{2,j-1}B_2)} e^{(t_j-1-t_{j-2})(A+u_{1,j-2}B_1+u_{2,j-2}B_2)} \dots e^{t_1(A+u_{1,1}B_1+u_{2,1}B_2)},$$

where $t_{j-1} \leq t < t_j$, $u_1 := \sum_{j \in \mathbf{N}} u_{1,j-1} \chi_{[t_{j-1}, t_j)}$ and $u_2 := \sum_{j \in \mathbf{N}} u_{2,j-1} \chi_{[t_{j-1}, t_j)}$.

We consider the map $\psi(t) = \Upsilon_t^{u_1, u_2}(\psi_0)$ as a “mild” solution of (2.4) with $A(t) = A + u_1(t)B_1 + u_2(t)B_2$ with initial condition $\psi_0 \in \mathcal{H}$.

Proposition 22. *Let $T > 0$. Let (A, B_1) satisfy Assumption 3 and B_2 be a bounded operator. Then for every $\psi_0 \in \mathcal{H}$, $t \in I$ the map $\Upsilon_t(\psi_0)$ admits a unique continuous extension on $\mathcal{R}((0, T]) \times \mathcal{R}((0, T])$.*

Moreover, if ψ_0 is in $D(A)$, for every $u_1 \in \mathcal{R}((0, T])$ and $u_2 \in \mathcal{R}((0, T])$, the map $t \mapsto \Upsilon_t^{u_1, u_2}(\psi_0)$ is a weak solution of (2.4) with $A(t) := A + u_1(t)B_1 + u_2(t)B_2$ in the sense that it satisfies

$$\begin{aligned} \int_I \langle f'(t), \Upsilon_t^{u_1, u_2}(\psi_0) \rangle dt &= \int_I \langle Af(t), \Upsilon_t^{u_1, u_2}(\psi_0) \rangle dt + \int_I \langle B_1 f(t), \Upsilon_t^{u_1, u_2}(\psi_0) \rangle du_1(t) \\ &\quad + \int_I \langle B_2 f(t), \Upsilon_t^{u_1, u_2}(\psi_0) \rangle du_2(t), \end{aligned}$$

for every $f \in C_c^1(I, \mathcal{H})$.

Proof. First notice that from Proposition 12 there is a map $\Upsilon_{t,s}^{u_1}$ such $t \mapsto \Upsilon^{u_1}(t, 0)\psi_0$ is a weak solution of (2.1) with B_1 replacing B and u_1 replacing u . Then the proof mimics the proof of Theorem 18 using Duhamel’s expansion, Corollary 17, with respect to $\Upsilon_t^{u_1}\psi_0$ and the fact that $\Upsilon_{t,s}^{u_1}$ is a contraction. \square

Remark 14. In the cases considered later u_1 and u_2 will be the same control u .

A comment on the case of bounded operators We take advantage of this part of our analysis devoted to bounded control potential to make some general remarks on propagators associated with bounded operators.

Proposition 23. *Let $t \in [0, T] \mapsto B \in L(\mathcal{H})$ be in $L^1_{s,loc}([0, T], L(\mathcal{H}))$, that is strongly measurable and strongly locally integrable. Then there exists a unique family of operators $\Upsilon_B : \Delta_{[0, T]} \rightarrow L(\mathcal{H})$ strongly continuous in t and s such that*

$$(i) \text{ for any } s < r < t, \Upsilon_B(t, s) = \Upsilon_B(t, r)\Upsilon_B(r, s),$$

$$(ii) \Upsilon_B(t, t) = I_{\mathcal{H}},$$

and for any $\psi_0 \in \mathcal{H}$, $(s, t) \in \Delta_{[0, T]} \mapsto \Upsilon_B(t, s)\psi_0$ is almost everywhere strongly left differentiable in t and right differentiable in s with derivative (when $t = s$) $B(t)\psi_0$ and $-B(t)\psi_0$ respectively.

The map

$$B \in L^1_{s,loc}([0, T], L(\mathcal{H})) \mapsto \Upsilon_B \in L^\infty(\Delta_{[0, T]}, L(\mathcal{H}))$$

is continuous.

The family Υ_B is called propagator associated with B .

Proof. The family Υ_B is built in the strong sense by generalizing the Dyson expansion as in Theorem 18 :

$$\Upsilon_B(t, s) = \sum_{n=0}^{\infty} Y_{(n)}^B(t, s)$$

with

$$Y_{(n)}^B(t, s)\psi_0 := \int_{s \leq s_1 < s_2 < \dots < s_n < t} B(s_n)B(s_{n-1}) \dots B(s_2)B(s_1)\psi_0 ds_1 ds_2 \dots ds_n.$$

The convergence follows exactly from similar arguments and provides the continuity. The differentiability properties are a consequence of Lebesgue's differentiation Theorem. Notice that the uniqueness of absolutely continuous function $t \in [0, T] \mapsto U(t) \in \mathcal{H}$ with Lebesgue derivative $t \mapsto B(t)U(t)$ and such that $U(0) = \psi_0$ follows from Lebesgue's differentiation Theorem and Gronwall's Lemma.

The fact that $\Upsilon_B(t, t)$ is the identity can be checked immediately while

$$\Upsilon_B(t, s)\psi_0 = \Upsilon_B(t, r)\Upsilon_B(r, s)\psi_0,$$

for any $s < r < t$, is a consequence of the uniqueness statement just mentioned. \square

4 Higher order norm estimates for weakly coupled systems

In the following we will restrict our analysis to the skew-adjoint case. This are the objects that appear in quantum mechanics. Moreover the restriction to self-adjoint operators makes the analysis simpler.

Our aim in this section is to analyze under which conditions the solution built previously are smoother in the scale of A . This is indeed the case if the assumptions of Assumptions 1, 2 or 3 are stated in $D(|A|^{k/2})$ instead of \mathcal{H} . Our aim is to provide a somewhat simpler criteria, we show that the extension of assumptions on B will be sufficient. In this respect, the A -boundedness of B as operators acting on $D(|A|^{k/2})$ is crucial and is stated in Lemma 28 which is the cornerstone of the analysis of this section. This is especially important if we want to obtain the regularity of propagators in the scale of A up to the order $k/2$. For lower orders, a simple interpolation provides the results. The criteria will be used in a perturbative framework (Kato-Rellich type argument) and we will not consider the whole of K for the values of u , unless we assume that the domain of powers of $A + uB$ are the same for any $u \in K$. We recall that in the dissipative framework in order to use Kato-Rellich

criterion u has to be non negative when B is dissipative, below we assume that both B and $-B$ have dissipativity properties (up to a shift by a constant) so that the sign of u does not have to be considered.

4.1 The weak coupling

Definition 6 (Weakly coupled). Let k be a non negative real. A couple of skew-adjoint operators (A, B) is k -weakly coupled if

- (i) A is invertible with bounded inverse from $D(A)$ to \mathcal{H} ,
- (ii) for any real t , $e^{tB}D(|A|^{k/2}) \subset D(|A|^{k/2})$,
- (iii) there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on $D(|A|^{k/2})$ for the norm $\psi \mapsto \| |A|^{k/2} \psi \|$.

We set, for every positive real k ,

$$\|\psi\|_{k/2} = \sqrt{\langle |A|^k \psi, \psi \rangle}.$$

The optimal exponential growth is defined by

$$c_k(A, B) := \sup_{t \in \mathbf{R}} \frac{\log \|e^{tB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))}}{|t|}.$$

Remark 15. As in Section 3.4, if

$$t \mapsto e^{tB}$$

is a strongly continuous semigroup on $D(|A|^{k/2})$ then there exists $\omega > 0$ and $C > 0$ such that

$$\|e^{tB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))} \leq C e^{\omega t}, \forall t > 0.$$

Up to a change of norm, namely consider the equivalent norm

$$\sup_{t > 0} \|e^{t(B-\omega)} \psi\|_{k/2}$$

then one can assume $C = 1$ but may lose the hilbertian structure and the skew adjoint character of A .

Remark 16. Notice that the content of Definition 6.(iii) is that the restriction to $D(|A|^{k/2})$ of the group generated by $B - c$ originally defined on \mathcal{H} defines a contraction semigroup. We thus have two generators and due to Definition 6.(i)-(ii) the domain of the second is included in the first. Hence again Definition 6.(i) gives that the smallest domain is dense in \mathcal{H} . The same comment can be made for $-B - c'$. Forgetting about c and c' and minus sign, we identify these three operators (which are closed in \mathcal{H} and $D(|A|^{k/2})$ respectively) and with an abuse of notation we denote them by the same B as they are restrictions of B . The domains of $B - c$ and $-B - c'$ acting on $D(|A|^{k/2})$ are actually both equal to $\{\phi \in D(|A|^{k/2}) \cap D(B), B\phi \in D(|A|^{k/2})\}$. They both contain this set as if ϕ is in this set then

$$e^{\pm tB} \phi - \phi \mp tB\phi = \int_0^t (e^{\pm sB} - 1) B\phi ds$$

is a $o(|t|)$ in \mathcal{H} or $D(|A|^{k/2})$. They are obviously contained in this set as if $t \mapsto e^{\pm tB} \phi$ is differentiable in $D(|A|^{k/2})$ it is also differentiable in \mathcal{H} .

The invertibility of A is needed to ensure that $\|\cdot\|_{k/2}$ is a norm equivalent to the graph norm of $D(|A|^{k/2})$. The use of the associated norm is due to the interpolation criterion we use below.

For the systems encountered in the physics literature, the operator A is skew-adjoint with at least two isolated eigenvalues. Hence the invertibility of A can be obtained by replacing A by $A - \lambda I$ for a suitable λ in \mathbf{R} . Notice that this translation on A only induces a global phase shift on the propagator that is physically irrelevant (i.e., undetectable by observations).

The following proposition gives another characterization of the weak-coupling using Hille-Yosida Theorem.

Proposition 24. *Let k be a non negative real. A couple of skew-adjoint operators (A, B) with A invertible is k -weakly coupled if and only if B is closed in $D(|A|^{k/2})$, and there exists ω such that for every real λ , $|\lambda| > \omega$ belongs to the resolvent set of B and for such λ*

$$\|(\lambda I - B)^{-1}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))} \leq \frac{1}{|\lambda| - \omega}.$$

The smallest ω satisfying the above property is $c_k(A, B)$.

Proof. If (A, B) be k -weakly coupled then $B - c_k(A, B)$ is the generator of a contraction semigroup in $D(|A|^{k/2})$. From Hille-Yosida Theorem, we deduce the stated equivalence. \square

The following proposition gives an equivalent definition which may be easier to check in practice.

Proposition 25. *Let k be a non negative real. A couple of skew-adjoint operators (A, B) with A invertible is k -weakly coupled if and only if for some $\omega > 0$,*

$$(\omega \mp B)^{-1}D(|A|^{k/2}) \subset D(|A|^{k/2})$$

and for any $\psi \in (\omega - B)^{-1}D(|A|^{k/2}) = (\omega + B)^{-1}D(|A|^{k/2})$

$$|\Re\langle |A|^k \psi, B\psi \rangle| \leq \omega \|\psi\|_{D(|A|^{k/2})}^2.$$

The smallest ω satisfying this property is $c_k(A, B)$.

Proof. We first notice that from the resolvent identity, the assumptions imply

$$(\omega - B)^{-1}D(|A|^{k/2}) \subset (\omega + B)^{-1}D(|A|^{k/2})$$

for any ω in the resolvent sets of B and $-B$.

Assume (A, B) is k -weakly coupled then as $B - c_k(A, B)$ and $-B - c_k(A, B)$ are generator of contraction semigroups on $D(|A|^{k/2})$, they are closed and maximal dissipative on $D(|A|^{k/2})$, their respective resolvent sets contains positive half lines (by means of Hille-Yosida theorem) and their domain is, by definition of the resolvent, $(\omega \pm B)^{-1}D(|A|^{k/2})$ for any $\omega > c_k(A, B)$. Since they are maximal dissipative, we have

$$\pm \Re\langle |A|^k \psi, B\psi \rangle \leq c_k(A, B) \|\psi\|_{D(|A|^{k/2})}^2.$$

for any $\psi \in (\omega - B)^{-1}D(|A|^{k/2}) = (\omega + B)^{-1}D(|A|^{k/2})$.

Reciprocally, $\pm B + \omega$ are closed as operators on \mathcal{H} so they are closed on $D(|A|^{k/2})$. As $\pm B + \omega$ are dissipative on $D(|A|^{k/2})$, they are generators of a contractions semigroups if they are surjective. As for any $f \in D(|A|^{k/2})$, $(\pm B + \omega)^{-1}f \in D(|A|^{k/2})$, they are indeed surjective. \square

The notion of weak coupling was introduced in [BCC13] but in a somewhat simpler form the relation between these two definitions can be obtained by means of the following lemma.

Lemma 26. *Let A be an invertible skew-adjoint operator and let B be a skew-symmetric operators such that for every u in \mathbf{R} , $A + uB$ is essentially skew-adjoint on the domain $D(A)$ of A and for some $k \geq 1$, for any real u , $D(|A + uB|^{k/2}) = D(|A|^{k/2})$ and there exists a constant C such that for every ψ in $D(|A|^k)$,*

$$|\Re\langle |A|^k \psi, B\psi \rangle| \leq C |\langle |A|^k \psi, \psi \rangle|.$$

Then (A, B) is k -weakly coupled .

Proof. Since for every u in \mathbf{R} , $A + uB$ is essentially skew-adjoint on the domain $D(A)$, this implicitly imply that $D(A) \subset D(B)$.

The assumption that there exist $k \geq 1$ and a constant C such that for every ψ in $D(|A|^k)$,

$$|\Re\langle |A|^k \psi, B\psi \rangle| \leq C |\langle |A|^k \psi, \psi \rangle|$$

and the Nelson commutator theorem, see Section C, imply that B with domain $D(|A|^{k/2})$ is essentially skew-adjoint so B is essentially skew-adjoint on $D(A)$. Then Trotter product formula, see [RS78, Theorem VIII.31], implies that :

$$\left(e^{\frac{t}{n}(A+uB)} e^{-\frac{t}{n}A} \right)^n$$

tends to e^{tuB} in the strong sense as n goes to infinity. As each of the term of the above sequence is bounded on $D(|A|^{k/2})$ with a bound $e^{C|t||u|}$, see [BCC13, proposition 2], we conclude that e^{tB} is bounded on $D(|A|^{k/2})$ with the same bound : $e^{C|t||u|}$. This provides (A, B) is k -weakly coupled . \square

Let us state state an interpolation result.

Lemma 27. *Let k be a positive real. If (A, B) is k -weakly coupled then (A, B) is s -weakly coupled for any $s \in [0, k]$ and*

$$c_s(A, B) \leq \frac{s}{k} c_k(A, B).$$

Proof. We will consider $s \in (0, k)$ as there is nothing to prove for $s = k$ and $s = 0$ is immediate since B is skew adjoint.

Notice that as B is skew-adjoint

$$\|e^{tB} u\|_{D(|A|^{k/2})} = \| |A|^{k/2} e^{tB} u \| = \| |\mathcal{A}(t)|^{k/2} u \|.$$

where $\mathcal{A}(t) = e^{-tB} A e^{tB}$.

As (A, B) is k -weakly coupled we deduce

$$\frac{1}{\|A^{-1}\|^k} \leq |\mathcal{A}(t)|^k \leq e^{2c|t|} |A|^k.$$

which from Proposition 58 yields

$$|\mathcal{A}(t)|^s \leq e^{2cs|t|/k} |A|^s$$

which concludes the proof. \square

A corollary of this interpolation result is the following result which is crucial in our analysis. It shows that if (A, B) is k -weakly coupled the A -boundedness of B extends naturally to $\mathcal{H} = D(|A|^{k/2})$.

Lemma 28. *Let k be a non-negative real. Let (A, B) be k -weakly coupled and such that B is A -bounded. Then*

$$\inf_{\lambda > 0} \|B(A - \lambda)^{-1}\|_{L(D(|A|^{\frac{k}{2}}), D(|A|^{\frac{k}{2}}))} \leq \|B\|_A$$

Proof. Let λ be a real. As (A, B) is k -weakly coupled, for any s in $[0, k]$ due to Lemma 27, and Proposition 25

$$|\langle \psi, [|A|^s, B]\psi \rangle| \leq \frac{s}{k} c_k(A, B) \langle \psi, |A|^s \psi \rangle$$

for any $\psi \in D(|A|^{s/2}) \cap D(B)$ with $B\psi \in D(|A|^{s/2})$ so that that

$$|A|^{-s/2} [|A|^s, B] |A|^{-s/2}$$

extends to a bounded operator in \mathcal{H} with bound lower than $\frac{s}{k} c_k(A, B)$. Then for $\phi \in D(|A|^{k/2}) \cap D(B)$ with $B\phi \in D(|A|^{k/2})$, we notice

$$\begin{aligned} \| |A|^{k/2} B\phi \|^2 &= \langle B\phi, |A|^k B\phi \rangle \\ &= \langle B\phi, [|A|^k, B]\phi \rangle + \langle B\phi, B|A|^k \phi \rangle \\ &= \langle |A|^{k/2} B\phi, |A|^{-k/2} [|A|^k, B] |A|^{-k/2} |A|^{k/2} \phi \rangle + \langle B\phi, B|A|^k \phi \rangle \\ &\leq c_k(A, B) \| |A|^{k/2} B\phi \| \| |A|^{k/2} \phi \| + |\langle B\phi, B|A|^k \phi \rangle| \end{aligned}$$

and for $s \geq 0$

$$\langle B\phi, B|A|^k \phi \rangle = \langle |A|^{k/2} B\phi, |A|^{s-k/2} B|A|^{k-s} \phi \rangle + \langle |A|^{s/2} B\phi, |A|^{-s/2} [|A|^s, B] |A|^{-s/2} |A|^{k-s/2} \phi \rangle$$

so that if we impose

$$0 \leq s - k/2 < k/2, \quad s/2 < k/2, \quad k - s/2 < k/2 + 1$$

or

$$k/2 \leq s < k, \quad k \leq s + 2$$

or if $k \in (s, \min\{s + 2, 2s\})$ for $s \geq 0$ this leads to

$$|\langle B\phi, B|A|^k \phi \rangle| \leq \| |A|^{k/2} B\phi \| \| |A|^{s-k/2} B|A|^{k-s} \phi \| + \frac{s}{k} c_k(A, B) \| |A|^{s/2} B\phi \| \| |A|^{k-s/2} \phi \|$$

and as $s < k$ to

$$\| |A|^{k/2} B\phi \| \leq c_k(A, B) \| |A|^{k/2} \phi \| + \| |A|^{s-k/2} B|A|^{k-s} \phi \| + \frac{s}{k} c_k(A, B) \| |A|^{\frac{s-k}{2}} \| \| |A|^{k-s/2} \phi \|.$$

Since $s - k/2 \in [0, k/2)$ a recurrence² emerges. First the value $s = k/2$ is allowed when $0 < k < k/2 + 2$ that is $k < 4$ and thus from the above inequality, we deduce, for every $\lambda > 0$,

$$\begin{aligned} &\| |A|^{k/2} B(A - \lambda)^{-1} \phi \| \\ &\leq c_k(A, B) \| |A|^{k/2} (A - \lambda)^{-1} \phi \| + \| B|A|^{k/2} (A - \lambda)^{-1} \phi \| + \frac{1}{2} c_k(A, B) \| |A|^{\frac{-k}{4}} \| \| |A|^{3k/4} (A - \lambda)^{-1} \phi \| \end{aligned}$$

and since $3/4k < k/2 + 1$ here, we deduce

$$\lim_{\lambda \rightarrow \infty} \| |A|^{k/2} B(A - \lambda)^{-1} \phi \| \leq \| B\|_A \| |A|^{k/2} \phi \|.$$

²This can be replaced by a connectedness argument on the set of $k' \in [0, k/2]$ for which the result holds

This initialize the recurrence. Then as $k - s/2 < k/2 + 1$

$$\lim_{\lambda \rightarrow \infty} \||A|^{k/2} B(A - \lambda)^{-1} \phi\| \leq \lim_{\lambda \rightarrow \infty} \||A|^{s-k/2} B(A - \lambda)^{-1} |A|^{-s+k/2} \phi\|$$

As the recurrence provides

$$\lim_{\lambda \rightarrow \infty} \||A|^{s-k/2} B(A - \lambda)^{-1} |A|^{-s+k/2} \phi\| \leq \|B\|_A$$

this concludes the proof due to the density of the set

$$\{\phi \in D(|A|^{k/2}) \cap D(B), B\phi \in D(|A|^{k/2})\}$$

in $D(|A|^{k/2})$. □

The following corollary shows that $D(|A+uB|^{k/2}) = D(|A|^{k/2})$. It seems a difficult issue otherwise. Recall that as $D(A) \subset D(B)$ and $A + uB$ denotes a self-adjoint extension of $A + uB$, we have $D(A) = D(A + uB)$ by Kato-Rellich theorem.

Lemma 29. *Let k be a positive real, (A, B) be k -weakly coupled and u be a real number such that $|u| < \|B\|_A$. Then $D(|A|^s) = D(|A + uB|^s)$ for every $s \in [0, k/2 + 1]$.*

Proof. We proceed by induction on j to prove $D(|A|^{k/2 - [k/2] + j}) = D(|A + uB|^{k/2 - [k/2] + j})$ for $j \leq [k/2]$. By Kato-Rellich theorem, $D(A) = D(A + uB)$ for every u in \mathbf{R} . By interpolation, see Lemma 62, $D(|A|^s) = D(|A + uB|^s)$ for $0 \leq s \leq 1$, in particular for $s = \frac{k}{2} - [\frac{k}{2}]$. This initializes the induction for $j = 0$.

Let us assume that $D(A^\ell) = D((A + uB)^\ell)$ for some $\ell \leq k/2$. By definition,

$$D(A^{\ell+1}) = \{f \in D(A^\ell) | Af \in D(A^\ell)\},$$

and, using the induction hypothesis,

$$D(|A + uB|^{\ell+1}) = \{f \in D(|A + uB|^\ell) | |A + uB|f \in D(|A + uB|^\ell)\} = \{f \in D(A^j) | |A + uB|f \in D(|A|^\ell)\}.$$

So that $D(|A + uB|^{\ell+1})$ is the domain of $A + uB$ as an operator acting on $D(|A|^\ell)$. The domain of A as an operator acting on $D(|A|^\ell)$ is $D(|A|^{\ell+1})$, as A is skew adjoint on $D(|A|^\ell)$ and $B - c$ or $-B - c'$ is dissipative in $D(|A|^\ell)$ due to Proposition 25, using Lemma 28 and Kato-Rellich theorem we conclude that $A + uB$ with domain $D(|A|^{\ell+1})$ is maximal dissipative in $D(|A|^\ell)$. □

Notice that if $D(|A + uB|^s) = D(|A|^s)$ for some positive real s then the associated norms are equivalent as the operators are closed.

4.2 Higher regularity

From Remark 8 and Proposition 9, we deduce the following statement.

Proposition 30. *Let k be a non negative real. Let (A, B) be k -weakly coupled and B be A -bounded.*

For any $u \in BV([0, T], \mathbf{R}) \cap B_{L^\infty([0, T])}(0, 1/\|B\|_A)$, there exists a family of contraction propagators in \mathcal{H} that extends uniquely as contraction propagators to $D(|A|^{k/2}) : \Upsilon^u : \Delta_{[0, T]} \rightarrow L(D(|A|^{k/2}))$ such that

(i) *for any $t \in [0, T]$, for any $\psi_0 \in D(|A|^{k/2})$*

$$\|\Upsilon_t(\psi_0)\|_{k/2} \leq e^{c_k(A, B) \int_0^t |u|} \|\psi_0\|_{k/2}$$

(ii) for any $t \in [0, T]$, for any $\psi_0 \in D(|A|^{1+k/2})$ for any $u \in BV([0, T], \mathbf{R}) \cap B_{L^\infty}([0, T])(0, 1/\|B\|_A)$, there exists m (depending only on A, B and $\|u\|_{L^\infty([0, T])}$)

$$\|\Upsilon_t(\psi_0)\|_{1+k/2} \leq m e^{m\|u\|_{BV([0, T], \mathbf{R})}} e^{c_k(A, B) \int_0^t |u|} \|\psi_0\|_{1+k/2}$$

Moreover, for every ψ_0 in $D(|A|^{k/2})$, the end-point mapping

$$\begin{aligned} \Upsilon(\psi_0) : BV([0, T], K) &\rightarrow D(|A|^{k/2}) \\ u &\mapsto \Upsilon^u(0, T)(\psi_0) \end{aligned}$$

is continuous.

Proof. The first part of the statement is a corollary of Theorem 7 after replacing B by $B - c_k(A, B)$ to ensure the dissipativity. Indeed consider $D(|A|^{\frac{k}{2}})$ in place of \mathcal{H} . Notice first that from Lemma 28, if $a > 1/\|B\|_A$ such that $a\|u\|_\infty < 1$ then there exists b_a such that for any $\psi \in D(|A|^{1+k/2})$

$$\|(1 - A - uB)\psi\|_{k/2} \geq \|(1 - A)\psi\|_{k/2} - |u|\|B\psi\|_{k/2} \geq (1 - a|u|)\|(1 - A)\psi\|_{k/2} - b_a\|\psi\|_{k/2}$$

or

$$\|(1 - A - uB)\psi\|_{k/2} + b_a\|\psi\| \geq (1 - a|u|)\|(1 - A)\psi\|_{k/2}$$

leading to

$$\|B\psi\|_{k/2} \leq a\|(1 - A)\psi\|_{k/2} \leq \frac{a}{1 - a|u|}\|(1 - A - uB)\psi\|_{k/2} + \frac{b_a}{1 - a|u|}\|\psi\|_{k/2}.$$

This gives

$$\begin{aligned} \|(1 - A(t))^{-1}\|_{L(D(|A|^{k/2}), D(|A|^{1+k/2}))} &= \|A(1 - A - u(t)B)^{-1}\|_{L(D(|A|^{k/2}))} \\ &\leq \|(A + u(t)B)(1 - A - u(t)B)^{-1}\|_{L(D(|A|^{k/2}))} \\ &\quad + |u(t)|\|B(1 - A - u(t)B)^{-1}\|_{L(D(|A|^{k/2}))} \\ &\leq 2 + |u(t)|\frac{a}{1 - a|u(t)|} + \frac{b_a}{1 - a|u(t)|}. \end{aligned}$$

Taking the supremum leads to

$$\sup_{t \in [0, T]} \|(1 - A(t))^{-1}\|_{L(D(|A|^{k/2}), D(|A|^{1+k/2}))} \leq 2 + \frac{1}{\|B\|_A} \frac{a}{1 - \frac{a}{\|B\|_A}} + \frac{b_a}{1 - \frac{a}{\|B\|_A}}.$$

Then Remark 8 gives the result using (A, B) to be k -weakly coupled using Lemma 28 and Kato-Rellich theorem for dissipative operators (see [RS78, Corollary of Theorem X.50]).

As moreover for $A_n(t) = A + u_n(t)B$ and $A(t) = A + u(t)B$ and a a sufficiently large real $\|A_n\|_{BV(I, L(D(A), \mathcal{H}))} \leq \|u_n\|_{BV(I, \mathbf{R})}\|B\|_{L(D(A), \mathcal{H})}$, $\|A_n(0)\|_{L(D(A), \mathcal{H})} \leq 1 + |u_n(0)|\|B\|_{L(D(A), \mathcal{H})}$. and

$$(A_n(t) - a)^{-1} - (A(t) - a)^{-1} = (u_n(t) - u(t))(A_n(t) - a)^{-1}B(A(t) - a)^{-1}$$

so that the strong resolvent convergence of A_n to A turns to be a consequence of the convergence of u_n to u in $BV(I, \mathbf{R})$. Proposition 9 applies as well. \square

We now state another version of Corollary 16.

Corollary 31. *Let k be a non negative real. Let (A, B) be k -weakly coupled B A -bounded and $\psi_0 \in D(|A|^{k/2})$. Then*

$$\bigcup_{L, T, a > 0} \{ \alpha \Upsilon^u(\psi_0), \|u\|_{BV([0, T], (-1/\|B\|_A, 1/\|B\|_A))} \leq L, t \in [0, T], |\alpha| \leq a \}$$

is a meagre set (in the sense of Baire) in $D(|A|^{k/2})$ as a union of relatively compact subsets.

Proof. This is similar to the one of Corollary 16. □

A comment on the exact controllability of time reversible system. Some comments on the exact controllability of time reversible system are in place.

Let (A, B, K) satisfy Assumption 1 (or Assumptions 2). Let \mathcal{C} be a bounded antilinear operator on \mathcal{H} such that $\mathcal{C}D(A) \subset D(A)$, $\mathcal{C}A = -A\mathcal{C}$, and $\mathcal{C}B = -B\mathcal{C}$. Let u be piecewise constant with value in K and let $\psi(t)$ be a strong solution of (2.1) associated with the control $t \mapsto u(t)$, then $\varphi(t) = \mathcal{C}\psi(T - t)$ satisfies

$$\begin{aligned} \frac{d}{dt}\varphi(t) &= -\mathcal{C}(\psi'(T - t)) \\ &= -\mathcal{C}(A + u(T - t)B)(\psi(T - t)) \\ &= A\varphi(t) + u(T - t)B\varphi(t). \end{aligned}$$

and therefore is a strong solution of (2.1) associated with the control $t \mapsto u(T - t)$.

This well-known fact for the Schrödinger equation (see for example [Ner09]) has been remarked for our abstract framework in [BCCS12, Section 6.1]. Using Proposition 12, it can be extended for the case in which u is a Radon measure as stated in the following proposition.

Proposition 32. *Let (A, B, K) satisfy Assumption 1 (or Assumption 2) with A skew-adjoint and B skew-symmetric. Let \mathcal{C} be a conjugation, that is a bounded antilinear operator on \mathcal{H} with $\mathcal{C}\mathcal{C} = \text{Id}_{\mathcal{H}}$, such that $\mathcal{C}D(A) \subset D(A)$, $\mathcal{C}A = -A\mathcal{C}$, and $\mathcal{C}B = -B\mathcal{C}$. Let u be a bounded variation function (or a Radon measure) on $(0, T)$ with value in K and let Υ_u be the associated contraction propagator. For any $(t, s) \in \Delta_{[0, T]}$, $\Upsilon_{t, s}^u$ is unitary and its inverse coincides with $\mathcal{C}\Upsilon_{u(T-\cdot)}(T - t, T - s)\mathcal{C}$ where $u(T - \cdot)$ denotes $t \in [0, T] \mapsto u(T - t)$ in the framework of Assumption 1 (or $t \in [0, T] \mapsto u((0, T]) - u((0, t]) = u([t, T])$ in the one of Assumption 2).*

Proof. The fact that the propagator is unitary is a consequence of the skew-adjointness. The invertibility follows and to identify the inverse it is enough to differentiate

$$s \mapsto \Upsilon_{t, s}^u \mathcal{C} \Upsilon_{T-t, T-s}^{u(T-\cdot)} \mathcal{C}$$

in s when u is continuous. The derivative is 0 while

$$\Upsilon_u(t, t) \mathcal{C} \Upsilon_{u(T-\cdot)}(T - t, T - t) \mathcal{C} = \text{Id}_{\mathcal{H}}.$$

Based on the continuity property, the identity

$$\Upsilon_{t, s}^u \mathcal{C} \Upsilon_{u(T-\cdot)}(T - t, T - s) \mathcal{C} = \text{Id}_{\mathcal{H}}$$

extends to all u in the framework of Assumption 1 (or Assumption 2). □

Remark 17. Proposition 30 shows that for time reversible systems, the input-output mapping does not change the regularity with respect to A . As eigenvectors belong to any $D(|A|^k)$ this shows that exact controllability clearly relies on the regularity of B in the scale of A .

4.3 Extension to Radon measures

The conclusion of Proposition 15 can be extended to $D(|A|^{k/2})$ if Assumption (3.3) is true in $D(|A|^{k/2})$ instead of \mathcal{H} . This is indeed the only missing assumption to apply Corollary 10 with $D(|A|^{k/2})$ instead of \mathcal{H} . Without this assumption the following result together with the interpolation result of Lemma 57 gives an interesting extension.

Proposition 33. *Let k be a positive real. Let (A, B, K) satisfy Assumption 2 such that (A, B) be k -weakly coupled. Then, Υ admits a unique continuous extension to the set of Radon measures and, for every $s \in [0, k]$, $\psi_0 \in D(|A|^{s/2})$, for every $T \geq 0$, one has $\Upsilon_T^u(\psi_0) \in D(|A|^{s/2})$ and*

$$\|\Upsilon_T^u(\psi_0)\|_{s/2} \leq e^{\frac{s}{k}c_k(A,B)|u|([0,T])}\|\psi_0\|_{s/2}$$

if $u \in \mathcal{R}((0, T])$ with $u((0, t]) \in K$ when $t \in [0, T]$.

Proof. We will make the proof for $s = k$, as from Lemma 27 the proof will be the same.

Consider a sequence v_n of piecewise constant functions converging to $t \in [0, T] \mapsto u((0, t])$ pointwise with $\|v_n\|_{BV((0,T])} \leq K$. Then v_n is the cumulative function of v'_n , a discrete sum of Dirac delta functions and, from (3.1), $\Upsilon_t^{v'_n}$ is a product of unitary operators of the form

$$e^{vB} e^{-vB} e^{tA} e^{vB} = e^{tA} e^{vB}.$$

So that

$$\|e^{vB} e^{-vB} e^{tA} e^{vB} \psi\|_{k/2} = \|e^{vB} \psi\|_{k/2} \leq M(v) \|\psi\|_{k/2}$$

where $M(v) := \|e^{vB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))}$. Noticing that Definition 6 and $M(v_1 + v_2) \leq M(v_1)M(v_2)$ for any pair (v_1, v_2) in $[0, \delta]^2$ imply

$$M(v) \leq e^{c_k(A,B)|v|}, \forall v \in \mathbf{R}.$$

Hence, for every n ,

$$\|\Upsilon_t^{v'_n}(\psi_0)\|_{k/2} < e^{c_k(A,B)K}.$$

For every f in $D(|A|^k)$,

$$|\langle |A|^k f, \Upsilon_t^{v'_n} \psi_0 \rangle| \leq \|f\| \|\psi_0\|_{k/2} e^{c_k(A,B)K}.$$

Because of the continuity result (Proposition 9), the left hand side tends to $|\langle |A|^k f, \Upsilon_t^u \psi_0 \rangle|$ as n tends to infinity. The right hand side is bounded by $\|u\| \|\psi_0\|_{k/2} e^{c_k(A,B)K}$. Hence, for every f in $D(|A|^k)$

$$|\langle |A|^k f, \Upsilon_t^u \psi_0 \rangle| \leq \|f\| \|\psi_0\|_{k/2} e^{c_k(A,B)K}.$$

As a consequence, $\Upsilon_t^u \psi_0$ belongs to $D((|A|^k)^*) = D(|A|^k)$ and

$$\||A|^k \Upsilon_t^u \psi_0\| \leq \|\psi_0\|_{k/2} e^{c_k(A,B)K}. \quad \square$$

Remark 18. Assumption (3.3) is a regularity assumption to make on top of (A, B) k -weakly coupled if one wants to extend the previous lemma to the limit case $s = k$. It is also interesting to notice that this Assumption (3.3) implies that (A, B) is 2-weakly coupled. Indeed, if (A, B) is a couple of skew-adjoint operators operators satisfying Assumption 3, then Assumption (3.3) implies for small $|t|$ that, for every ψ in $D(A)$,

$$\begin{aligned} \| |A| e^{-tB} \psi \| &= \| A e^{-tB} \psi \| = \| e^{tB} A e^{-tB} \psi \| \leq \| e^{tB} A e^{-tB} \psi - A \psi \| + \| A \psi \| \\ &\leq (1 + L|t|) \| A \psi \| \leq e^{L|t|} \| A \psi \| = e^{L|t|} \| |A| \psi \| \end{aligned}$$

as the map $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$ is locally Lipschitz with some constant L . Thus (A, B) is 2-weakly coupled.

As a consequence of Corollary 10 and Lemma 57 we have the following proposition.

Proposition 34. *Let k be a positive real. Let (A, B, K) satisfy Assumption 2 such that (A, B) is k -weakly coupled. Let*

$$\mathcal{R}((0, T], K) := \{u \in \mathcal{R}((0, T]), u((0, t]) \in K, \forall t \in [0, T]\}$$

Then for any $s \in [0, k)$, for every ψ_0 in $D(|A|^{s/2})$, and the end-point mapping

$$\begin{aligned} \Upsilon(\psi_0) : \mathcal{R}((0, T], K) &\rightarrow D(|A|^{s/2}) \\ u &\mapsto \Upsilon_T^u(\psi_0) \end{aligned}$$

is continuous.

Proof. Let (u_n) be a converging sequence in $\mathcal{R}((0, T])$ to some u . Then

$$\Upsilon_T^{u_n}(\psi_0) - \Upsilon_T^u(\psi_0)$$

is uniformly bounded in $D(|A|^{k/2})$ and converging to 0 in \mathcal{H} . By Lemma 57 it converges to 0 in $D(|A|^{s/2})$ for $s < k$. \square

Remark 19. One can notice that under the assumptions of Proposition 34 both Proposition 15 and Corollary 16 extend to $D(|A|^{s/2})$ for $s \in [0, k)$.

A remark on bounded control potentials If B is bounded in $D(|A|^{k/2})$, the latter can be considered in place of \mathcal{H} in all the analysis of § 3.4. These leads to a simpler version of the weak coupling theory.

5 Galerkin approximations

In this section, again A is skew-adjoint and B skew-symmetric. This assumption is introduced for convenience.

For every Hilbert basis $\Phi = (\phi_k)_{k \in \mathbf{N}}$ of \mathcal{H} , we define, for every N in \mathbf{N} ,

$$\begin{aligned} \pi_N^\Phi : \mathcal{H} &\rightarrow \mathcal{H} \\ \psi &\mapsto \sum_{j \leq N} \langle \phi_j, \psi \rangle \phi_j. \end{aligned}$$

Definition 7. Let (A, B) be a couple of (possibly unbounded) linear operators and $\Phi = (\phi_n)_{n \in \mathbf{N}}$ be an Hilbert basis of \mathcal{H} . Let $N \in \mathbf{N}$ and denote $\mathcal{L}_N^\Phi = \text{span}(\phi_1, \dots, \phi_N)$. The Galerkin approximation of order N of system (2.1), when it makes sense, is the system

$$\dot{x} = (A^{(\Phi, N)} + uB^{(\Phi, N)})x \tag{(\Sigma_N^\Phi)}$$

where $A^{(\Phi, N)}$ and $B^{(\Phi, N)}$, defined by

$$A^{(\Phi, N)} = \pi_N^\Phi A|_{\mathcal{L}_N} \quad \text{and} \quad B^{(\Phi, N)} = \pi_N^\Phi B|_{\mathcal{L}_N},$$

are the compressions of A and B (respectively) associated with \mathcal{L}_N .

Since \mathcal{L}_N is a finite dimensional space by definition, for every integer N , every A skew-adjoint and every Hilbert basis Φ made of eigenvectors of A then $(A^{(\Phi, N)}, B^{(\Phi, N)}, \mathbf{R})$ satisfies Assumptions 1. We can therefore define the contraction propagator $X_{(\Phi, N)}^u(t, 0)$ of (Σ_N^Φ) associated with bounded variation control u . We can also write that $(A^{(\Phi, N)}, B^{(\Phi, N)})$ is k -weakly coupled for any positive real k .

The weak coupling is actually invariant or at least does not deteriorate by compression with respect to a basis of eigenvectors of A .

Lemma 35. *Let k be a non-negative real. Let (A, B) is k -weakly coupled and let Φ be an Hilbert basis made of eigenvectors of A , (A, B) be k -weakly coupled. Then, for every N in \mathbf{N} , $(A^{(\Phi, N)}, B^{(\Phi, N)})$ is k -weakly coupled. Moreover,*

$$\|e^{vB^{(\Phi, N)}}\psi\|_{D(|A^{(\Phi, N)}|^{k/2})} \leq e^{c_k(A, B)|v|}\|\psi\|_{D(|A^{(\Phi, N)}|^{k/2})}$$

for every N in \mathbf{N} and ψ in \mathcal{L}_N^Φ .

Proof. From Proposition 25, one has

$$|\Re\langle |A|^k\psi, B\psi \rangle| \leq c_k(A, B)\|\psi\|_{D(|A|^{k/2})} \quad \forall \psi \in D(|A|^{k/2}).$$

Hence restricted to \mathcal{L}_N^Φ , we obtain $(A^{(\Phi, N)}, B^{(\Phi, N)})$ be k -weakly coupled. \square

In the sequel, we we will use some compactness properties for B in the scale of A . We make the following remark.

Lemma 36. *Let A be a skew-adjoint operator acting on \mathcal{H} such that there exists an Hilbert basis of \mathcal{H} made of eigenvectors of A . Let B be a bounded operator on \mathcal{H} . Then B is compact from $D(|A|^{k/2})$ to \mathcal{H} for some positive real k if and only it is compact for any positive real k .*

Proof. We only have to prove the first implication in the following form : B is compact from $D(|A|^{k/2})$ to \mathcal{H} for some positive real k if and only it is compact from $D(|A|^{s/2})$ to \mathcal{H} for any s in $(0, k)$.

Let Φ be an Hilbert basis made of eigenvectors of A . The associated orthogonal projector π_N^Φ tends strongly to the identity in \mathcal{H} and in $D(|A|^{k/2})$ as it commutes with $|A|^{k/2}$.

Hence if B is compact from $D(|A|^{k/2})$ to \mathcal{H} , $B\pi_N^\Phi$ tends to B in $L(D(|A|^{k/2}), \mathcal{H})$ as $(B - B\pi_N^\Phi)_N$ is bounded in $L(\mathcal{H})$, Lemma 57 shows that $(B - B\pi_N^\Phi)_N$ tends to 0 in $L(D(|A|^{s/2}), \mathcal{H})$ for any s in $(0, k)$. So that B is compact from $D(|A|^{s/2})$ to \mathcal{H} for any s in $(0, k)$. \square

Remark 20. If A has a discrete spectrum, which is equivalent to the compactness of the resolvent of A , if B is A -bounded, $B(|A| + 1)^{-1-\varepsilon}$ is compact for $\varepsilon > 0$. If moreover B is bounded, then $B(|A| + 1)^{-\varepsilon}$ is compact for $\varepsilon > 0$ that is B is A -compact.

5.1 For bounded variation controls

In this section is we consider the question of the Galerkin approximation in the framework of Assumption 1.

Proposition 37. *Let (A, B, K) satisfy Assumption 1 with $B(1-A)^{-1}$ compact. Let s be non-negative numbers with $0 \leq s < 1$. Then for every $\varepsilon > 0$, $L \geq 0$, $n \in \mathbf{N}$, and $(\psi_j)_{1 \leq j \leq n}$ in $D(A)^n$ there exists $N \in \mathbf{N}$ such that for any $u \in BV([0, T], K)$,*

$$\|u\|_{BV([0, T])} < L \Rightarrow \|\Upsilon_t^u(\psi_j) - X_{(N)}^u(t, 0)\pi_N\psi_j\|_{s/2} < \varepsilon,$$

for every $t \geq 0$ and $j = 1, \dots, n$.

Proof. As the propagators are contractions in \mathcal{H} , the proof for can be restricted to $(\psi_j)_{1 \leq j \leq n}$ in a the dense subset $D(A)$.

For every u in $BV([0, +\infty))$ such that $u(t) \in K$, for all $t \geq 0$, consider $y^u : t \mapsto \pi_N^\Phi \Upsilon_t^u(\psi)$ that is absolutely continuous and, for almost every $t \geq 0$ satisfies for almost every $t \geq 0$,

$$\dot{y}^u(t) = (A^{(\Phi, N)} + uB^{(\Phi, N)})y^u(t) + u(t)\pi_N^\Phi B(\text{Id}_{\mathcal{H}} - \pi_N^\Phi)\Upsilon_t^u(\psi).$$

Then, for every $t \geq 0$,

$$y^u(t) = X_{(\Phi, N)}^u(t, 0)\psi + \int_0^t X_{(\Phi, N)}^u(s, t)\pi_N^\Phi B(\text{Id}_{\mathcal{H}} - \pi_N^\Phi)\Upsilon_s^u(\psi)du(s)$$

Since $\text{Id}_{\mathcal{H}} - \pi_N^\Phi$ tends strongly to 0, as N goes to infinity, and $B(1 - A)^{-1}$ is compact, $B(1 - A)^{-1}(\text{Id}_{\mathcal{H}} - \pi_N^\Phi)$ tends to 0 in norm, as $X_{(\Phi, N)}^u(t, 0)$ is unitary and from Theorem 7,

$$\|\Upsilon_s^u(\psi)\|_{1/2} \leq M^2 e^{M\|B(1-A)^{-1}\| \|u\|_{BV([0, +\infty))}} \|\psi\|_2$$

with

$$M = \sup_{t \in [0, T]} \|(1 - A(t))^{-1}\|_{L(D(|A|^{k/2}), D(|A|^{1+k/2}))} \leq 2 + \frac{1}{\|B\|_A} \frac{a}{1 - \frac{a}{\|B\|_A}} + \frac{b_a}{1 - \frac{a}{\|B\|_A}}.$$

where $a > 1/\|B\|_A$ is such that $a\|u\|_\infty \leq aK < 1$ and b_a some real which only depend on a and thus K . We have for any $\varepsilon > 0$ the existence of N such that for $n \geq N$

$$\|y_j(t) - X_{(\Phi, N)}^u(t, 0)(\psi)\| \leq \varepsilon e^{c_k(A, B)M} \|u\|_{L^1([0, +\infty))} M^2 e^{M\|B(1-A)^{-1}\| \|u\|_{BV([0, +\infty))}} \|\psi\|_2.$$

This achieves the proof for $s = 0$.

Consider now the general case $s \in (0, 1)$. The sequence $\left(\pi_N^\Phi \Upsilon_t^u(\psi) - X_{(\Phi, N)}^u(t, 0)(\psi)\right)_{N \in \mathbf{N}}$ converges to zero in \mathcal{H} as N tends to infinity, uniformly with respect to $t \geq 0$ and u with total variation less than L (this is the case treated previously). The same sequence $\left(\pi_N^\Phi \Upsilon_t^u(\psi) - X_{(\Phi, N)}^u(t, 0)(\psi)\right)_{N \in \mathbf{N}}$ is bounded in $D(A)$ uniformly with respect to $t \geq 0$ and u with total variation less than L , since

$$\|\pi_N^\Phi \Upsilon_t^u(\psi) - X_{(\Phi, N)}^u(t, 0)(\psi)\|_1 \leq \|\pi_N^\Phi \Upsilon_t^u(\psi)\|_1 + \|X_{(\Phi, N)}^u(t, 0)(\psi)\|_1$$

and each of this term is bounded, uniformly with respect to N , u and t by Theorem 7 and since the projections π_N^Φ are bounded in $D(A)$ as they commute to A .

The conclusion follows from interpolation see Lemma 57. \square

Remark 21. The compactness of the operator $B(1 - A)^{-1}$ was crucial to control uniformly the norm of

$$\{ \|(\text{Id}_{\mathcal{H}} - \pi_N^\Phi)B\pi_N^\Phi\psi(s)\|; s \geq 0, |u|([0, +\infty)) \leq L \}.$$

One can notice that the compactness in Corollary 31 allows a weaker result where the N will depend on u or t .

One can also emphasize that this compactness was used only for the uniform smallness of the above set.

In case we have more regularity, we have the following result.

Proposition 38. *Let k be a positive real. Let (A, B) be k -weakly coupled with $B(1 - A)^{-1}$ is compact. Let s be non-negative numbers with $0 \leq s < k + 2$. Then for every $\varepsilon > 0$, $L \geq 0$, $n \in \mathbf{N}$, and $(\psi_j)_{1 \leq j \leq n}$ in $D(|A|^{1+k/2})^n$ there exists $N \in \mathbf{N}$ such that for any $u \in BV([0, T], \mathbf{R}) \cap B_{L^\infty}([0, T])(0, 1/\|B\|_A)$,*

$$\|u\|_{BV([0, T])} < L \Rightarrow \|\Upsilon_t^u(\psi_j) - X_{(N)}^u(t, 0)\pi_N\psi_j\|_{s/2} < \varepsilon,$$

for every $t \geq 0$ and $j = 1, \dots, n$.

Proof. The proof is exactly the same as this of Proposition 37 using the results of Section 4. Actually one only need to reconsider the step prior to the interpolation in the proof of Proposition 37, namely the boundedness of $\left(\pi_N^\Phi \Upsilon_t^u(\psi) - X_{(\Phi, N)}^u(t, 0)(\psi)\right)_{N \in \mathbf{N}}$ is bounded in $D(|A|^{1+k/2})$ which follows from Proposition 30. \square

5.2 For Radon measures

In the case of Radon measure, for the interacting framework we already exploited some regularity of B in the scale of A . In order to have a Galerkin approximation we need to assume further regularity.

Proposition 39 (Good Galerkin Approximation). *Let k be a positive real. Let (A, B) satisfy Assumptions 3 and be k -weakly coupled with B A -compact. Let Φ be an Hilbert basis made of eigenvectors of A .*

Then, for every $\varepsilon > 0$, $L \geq 0$, $l \in [0, k]$ there exists $N \in \mathbf{N}$ such that for every $u \in \mathcal{R}([0, +\infty))$ and ψ in \mathcal{L}_N^Φ , $t \geq 0$ and $n \geq N$

$$|u|([0, +\infty)) \leq L \implies \|\Upsilon_t^u(\psi) - X_{(\Phi, n)}^u(t, 0)\psi\|_{l/2} \leq 2^{\frac{l}{k}} L^{1-\frac{l}{k}} e^{c_k(A, B)M} \varepsilon \|\psi\|_{k/2}$$

Proof. First consider the case $l = 0$. For every u in $\mathcal{R}([0, +\infty))$ consider a sequence $(u_n)_n$ of piecewise constant functions that converges, as Radon measures, to u and the sequence $y^{u_n} : t \mapsto \pi_N^\Phi \Upsilon_t^{u_n}(\psi)$ of absolutely continuous mappings, that satisfies for almost every $t \geq 0$,

$$\dot{y}^{u_n}(t) = (A^{(\Phi, N)} + u_n(t)B^{(\Phi, N)})y^{u_n}(t) + u_n(t)\pi_N^\Phi B(\text{Id}_{\mathcal{H}} - \pi_N^\Phi)\Upsilon_t^{u_n}(\psi).$$

Then, for every $t \geq 0$, and $\alpha \in (0, k/2)$

$$y^{u_n}(t) = X_{(\Phi, N)}^{u_n}(t, 0)\psi + \int_0^t X_{(\Phi, N)}^{u_n}(s, t)\pi_N^\Phi B(\text{Id}_{\mathcal{H}} + |A|)^{-\alpha}(\text{Id}_{\mathcal{H}} - \pi_N^\Phi)(\text{Id}_{\mathcal{H}} + |A|)^\alpha \Upsilon_s^{u_n}(\psi) du(s)$$

Since $\text{Id}_{\mathcal{H}} - \pi_N^\Phi$ tends strongly to 0, as N goes to infinity, and $B(\text{Id}_{\mathcal{H}} + |A|)^{-\alpha}$ is compact, $B(\text{Id}_{\mathcal{H}} + |A|)^{-\alpha}(\text{Id}_{\mathcal{H}} - \pi_N^\Phi)$ tends to 0 in norm, as $X_{(\Phi, N)}^{u_n}(t, 0)$ is unitary it gives as $(\text{Id}_{\mathcal{H}} + |A|)^\alpha \Upsilon_s^{u_n}(\psi)$ is bounded in \mathcal{H} by $\|\psi\|_\alpha$

$$\|y_j(t) - X_{(\Phi, N)}^{u_n}(t, 0)(\psi)\| \leq \varepsilon e^{c_k(A, B)M} |u_n|([0, +\infty)) \|\psi\|_\alpha \leq L e^{c_k(A, B)M} \varepsilon \|\psi\|_\alpha.$$

Letting n tend to infinity achieves the proof for $l = 0$.

The rest of the proof is similar to the one of Proposition 37. \square

5.3 For bounded control potential

In the case of bounded potentials we can state the following proposition. As we need uniform in time estimates we cannot consider generators of continuous semigroup as in Theorem 18. Below we consider B be bounded in $D(|A|^{k/2})$ which implies that (A, B) is k -weakly coupled .

Proposition 40. *Let k be a positive real. Let A with domain $D(A)$ be the generator of a contraction semigroup and let B be bounded in \mathcal{H} and $D(|A|^{k/2})$ with $B(1 - A)^{-1}$ compact. Let s be non-negative numbers with $0 \leq s < k$. Then for every $\varepsilon > 0$, $L \geq 0$, $n \in \mathbf{N}$, and $(\psi_j)_{1 \leq j \leq n}$ in $D(|A|^{k/2})^n$ there exists $N \in \mathbf{N}$ such that for any $u \in \mathcal{R}([0, T])$,*

$$|u|([0, T]) < L \implies \|\Upsilon_t^u(\psi_j) - X_{(\Phi, N)}^u(t, 0)\pi_N^\Phi \psi_j\|_{s/2} < \varepsilon,$$

for every $t \geq 0$ and $j = 1, \dots, n$.

The proof is an adaptation of the previous ones and is omitted.

5.4 The general case of operator with continuous spectrum

A general extension of Propositions 37 or 39, with a bound of the error $\|\Upsilon_t^u \psi - X_{(\Phi, n)}^u(t, 0)\psi\|$ uniform in $t \geq 0$ is not possible in the case where there is no Hilbert basis of the ambient space made of eigenvectors of A . Indeed, the RAGE theorem (due to Ruelle [Rue69], Arnein and Georgescu [AG74] and Enss [Ens78]) states that if A is a skew-adjoint operator on the Hilbert space \mathcal{H} and \mathcal{H}^c is the continuity subspace of \mathcal{H} relatively to A , that is the space $\{v | v \text{ eigenvector of } A\}^\perp$, then for every ψ in \mathcal{H}^c , for every compact mapping $C : \mathcal{H} \rightarrow \mathcal{H}$ (such as the projection on a finite dimensional subspace), $\liminf_{t \rightarrow \infty} \|C e^{tA} \pi_C \psi\| = 0$. In other words: the part of the state which enters \mathcal{H}^c cannot be recovered by any finite dimensional approximation.

In order to consider the extension of our result to the case where there is no Hilbert basis made of eigenvectors for the free dynamics operator A , we recall a famous theorem by Weyl and Von Neumann, see [Kat66, Theorem X.2.1].

First we need the Schmidt class.

Definition 8. Let $\Phi = (\phi_n)_{n \in \mathbf{N}}$ be a complete orthonormal family of \mathcal{H} . Let $T \in L(\mathcal{H})$. If the series $\sum_{k \in \mathbf{N}} \|T \phi_k\|^2$ converges, then T is in the Schmidt class, denoted $L_2(\mathcal{H})$ and the Schmidt norm is given

$$\text{by } \|T\|_2 := \sqrt{\sum_{n \in \mathbf{N}} \|T \phi_n\|^2}.$$

The previous definitions are independent of the choice of the basis $\Phi = (\phi_n)_{n \in \mathbf{N}}$. The Schmidt norm endows the Schmidt class with an Hilbert structure. We have that

$$\|T\| \leq \|T\|_2$$

so that the Schmidt class is formed by compact operators.

Theorem 41 (Weyl-Von Neumann). *Let A be a skew-adjoint operator on \mathcal{H} . For any $\varepsilon > 0$, there exists $C_\varepsilon \in L_2(\mathcal{H})$ skew-adjoint with $\|C_\varepsilon\|_2 \leq \varepsilon$ and such that $A + C_\varepsilon$ has pure point spectrum.*

Proposition 42. *Let A be skew-adjoint, B be bounded with $B(1 - A)^{-1}$ compact and s be in $[0, 1)$. Then for every $\varepsilon > 0$, $L \geq 0$, $T > 0$, $n \in \mathbf{N}$, and $(\psi_j)_{1 \leq j \leq n}$ in $D(|A|^{k/2})^n$ there exists a Hilbert basis Φ of \mathcal{H} and $N \in \mathbf{N}$ such that for any $u \in BV(0, T]$,*

$$|u|([0, T]) < L \Rightarrow \|\Upsilon_t^u(\psi_j) - X_{(\Phi, N)}^u(t, 0)\pi_N^\Phi \psi_j\|_{s/2} < \varepsilon,$$

for every t in $[0, T]$ and $j = 1, \dots, n$.

Proof. We do the proof for $s = 0$, the rest of the proof is similar to the one of Proposition 37 using Corollary 19. By Weyl-Von Neumann theorem, there exists $C_\varepsilon \in L_2(\mathcal{H})$ skew-adjoint with $\|C_\varepsilon\|_2 \leq \varepsilon/T$ and such that $A_\varepsilon = A + C_\varepsilon$ has pure point spectrum. Let Φ be a Hilbert basis made of eigenvectors of A_ε . For u in $\mathcal{R}([0, T])$, we denote with $\Upsilon^{\varepsilon, u}$ the propagator associated with $\frac{d}{dt}x = A_\varepsilon x + u(t)Bx$.

By Proposition 40, there exists $N \in \mathbf{N}$ such that for any $u \in \mathcal{R}((0, T])$,

$$|u|([0, T]) < L \Rightarrow \|\Upsilon_t^{\varepsilon, u}(\psi_j) - X_{(\Phi, N)}^u(t, 0)\pi_N^\Phi \psi_j\| < \frac{\varepsilon}{2},$$

for every t in $[0, T]$ and $j = 1, \dots, n$. It is enough to notice that, for every $t \leq T$ and every u in $\mathcal{R}((0, T])$,

$$\|\Upsilon_t^{\varepsilon, u}(\psi_j) - \Upsilon_t^u(\psi_j)\| \leq t\varepsilon\|C_\varepsilon\|$$

to achieve the proof. \square

5.5 Application: the equivalence theorem

The notion of Good Galerkin Approximations has already been used in various contexts in order to obtain error bounds for numerical simulations ([BCC12a]) or quantitative estimates for approximate controllability (see an example with the controllability time in [BCC12b]). We mention in this paragraph another application based on a finite dimensional tracking result/.

Lemma 43. *Let k be a positive real. Let (A, B) satisfy Assumption 3 such that (A, B) is k -weakly coupled. Let Φ be an Hilbert basis made of eigenvectors of A , a, b be two real numbers such that $a < 0 < b$, u^* be a Radon measure on $(0, T)$ and ψ_0 be in \mathcal{H} . Then for every $\varepsilon > 0$ and every $l < k$, there exists a piecewise constant control $u_\varepsilon : [0, T_\varepsilon] \rightarrow \{a, 0, b\}$ such that*

$$\|\Upsilon_{T_\varepsilon}^{u_\varepsilon}(\psi_0) - \Upsilon_T^{u^*}(\psi_0)\|_l < \varepsilon \quad \text{and} \quad \|u_\varepsilon\|_{L^1([0, T_\varepsilon])} = |u^*|([0, T]).$$

If u^ is a positive Radon measure, then the conclusion holds true with piecewise constant controls u_ε taking value in $\{0, b\}$.*

Proof. For locally integrable controls u , this is [BCC14] [Lemma 12], based on the finite dimensional result [BCC14][Lemma 7] and an extension to infinite dimensional spaces thanks to the existence of Good Galerkin Approximations.

The result for general Radon measures controls follows by continuity. \square

Lemma 44. *Let k be a positive real. If (A, B) is k -weakly coupled with A have purely discrete spectrum and $B(1 - A)^{-1}$ is a compact operator, then for any a, b be two real numbers such that $a < b$, for any u^* be a Radon measure on $(0, T)$ for any ψ_0 in \mathcal{H} , for every $\varepsilon > 0$ and for every $l < k$, there exists a piecewise constant control $u_\varepsilon : [0, T_\varepsilon] \rightarrow \{a, b\}$ such that*

$$\|\Upsilon_{T_\varepsilon}^{u_\varepsilon}(\psi_0) - \Upsilon_T^{u^*}(\psi_0)\|_l < \varepsilon.$$

Proof. First recall that $D(|A + uB|^k) = D(|A|^k)$ for all $|u| < \|B\|_A$ from Lemma 29.

The proof goes in four steps:

1. First notice that it is enough to prove the result for $a < 0 < b$. In the case where $a > 0$ or $b < 0$, we translate A by $\frac{a+b}{2}B$ and write

$$A + uB = \left(A + \frac{a+b}{2}B\right) + \left(u - \frac{a+b}{2}\right)B.$$

The hypotheses of discrete spectrum on A and compactness of B from $D(A)$ to \mathcal{H} imply that $A + \frac{a+b}{2}B$ still have discrete spectrum and fulfills the other hypotheses of Lemma 44.

2. From Lemma 43, we can replace u by any control with values in $\{a, 0, b\}$. Again up to translating A by aB

$$A + uB = (A + aB) + (u - a)B$$

we can assume that u has value in $\{0, -a, b - a\}$.

3. From the positive case in Lemma 43, we can replace u by any control with values in $\{0, b - a\}$.
4. Now we translate the system back to restore the original operators and the desired values for the control. \square

Definition 9. Let \mathcal{K} be a set of Radon measures defined on \mathbf{R} , (A, B) satisfy Assumption 3 and ϕ be a vector of \mathcal{H} . For $T > 0$, we note \mathcal{K}_T the set of the restrictions of the measures of \mathcal{K} to $[0, T]$. The attainable set from ϕ by means of controls in \mathcal{K} is the set

$$\mathcal{Att}_{\mathcal{K}}\phi = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{K}_T} \{\Upsilon_{t,0}^u \phi | 0 \leq t \leq T\}$$

Proposition 45. Let k be a positive real. Assume that (A, B) satisfies Assumption 3, is k -weakly coupled and A has purely discrete spectrum, $B(1 - A)^{-1}$ is a compact operator and $D(|A + uB|^k) = D(|A|^k)$ for all $|u| < \|B\|_A$. Let ϕ be an eigenvector of A .

Then, for every $s < k$, for every $a < b$ in $(-1/\|B\|_A, 1/\|B\|_A)$, the closures in $D(|A|^s)$ of the three following sets are the same:

1. the attainable set X_1 from ϕ by means of controls piecewise constant taking value in $\{a, b\}$.
2. the attainable set X_2 from ϕ by means of controls piecewise constant taking value in (a, b) .
3. the attainable set X_3 from ϕ by means of Radon measures.

Proof. Clearly, $X_1 \subset X_2 \subset X_3$ hence the closures in $D(|A|^s)$ norm are the same. It is enough to show that X_3 is a subset of the closure of X_1 in $D(|A|^s)$ norm, which is exactly the content of Lemma 44. \square

Definition 10. Let $(X, \|\cdot\|_X)$ be a Banach subspace included in \mathcal{H} and let \mathcal{K} be a set of Radon measures. We say that system (2.1) is approximately controllable in norm $\|\cdot\|_X$ by means of control in \mathcal{K} if for every $\varepsilon > 0$ and for every pair ψ_0, ψ_1 in the intersection of the unit sphere of $L^2(\mathbf{R}, \mathbf{C})$ with X , there exists $T > 0$ and $u \in \mathcal{K}$ that steers the control system (2.1) from ψ_0 to an ε -neighborhood of ψ_1 for the norm $\|\cdot\|_X$.

Proposition 46. Let k be a positive real. If (A, B) satisfy Assumption 3 and is k -weakly coupled with A have purely discrete spectrum, $B(1 - A)^{-1}$ be a compact operator and $D(|A + uB|^k) = D(|A|^k)$ for all $|u| < \|B\|_A$

Then, for every $s < k$, the four following statements are equivalent:

1. For every $a < b$ in \mathbf{R} , the system (2.1) is approximately controllable in norm $|A|^s$ by means of piecewise constant functions taking value in (a, b) .
2. For every $a < b$ in \mathbf{R} , the system (2.1) is approximately controllable in norm $|A|^s$ by means of piecewise constant functions taking value in $\{a, b\}$.
3. There exist $a < b$ in \mathbf{R} such that the system (2.1) is approximately controllable in norm $|A|^s$ by means of piecewise constant functions taking value in $\{a, b\}$.
4. The system (2.1) is approximately controllable in norm $|A|^s$ by means of Radon measures.

Proof. Each of Assertion 1, 2 or 3 implies Assertion 4 as they consider a subclass.

The fact that Assertion 1 implies Assertion 2 is a consequence of the continuity of the end point mapping (Proposition 34) and density arguments.

Assertion 3 is obvious from Assertion 2 and Assertion 1 is obvious from Assertion 2.

At last, Assertion 4 implies Assertion 1 as a consequence of Lemma 44. \square

6 Examples

Most of the examples of bilinear control systems (1.2) encountered in the literature, also without any relation to quantum control, deal with bounded control operator B . Proposition 3 applies and allows, for instance, to complete the studies of the rod equation with clamped ends made in [BMS82, Section 6, Example 4] and [Bea08]. In the following, we concentrate on examples in relation with quantum control.

6.1 Quantum systems with smooth potentials on compact manifolds

This example motivated the present analysis because of its physical importance. We consider Ω a compact Riemannian manifold endowed with the associated Laplace-Beltrami operator Δ and the associated measure μ , $r \geq 2$ an integer, $V, W : \Omega \rightarrow \mathbf{R}$ two functions of class C^r , and the bilinear quantum system

$$i\frac{\partial\psi}{\partial t} = \Delta\psi + V\psi + u(t)W\psi. \quad (6.1)$$

With the notations of Section 2, $\mathcal{H} = L^2(\Omega, \mathbf{C})$ endowed with the Hilbert product $\langle f, g \rangle = \int_{\Omega} \bar{f}g d\mu$, $A = -i(\Delta + V)$ and $B = -iW$. For every $s \geq 0$, $D(|A|^s) = H^r(\Omega, \mathbf{C})$. There exists a Hilbert basis $(\phi_k)_{k \in \mathbf{N}}$ of H made of eigenvectors of A . Each eigenvalue of A has finite multiplicity. For every k , there exists λ_k in \mathbf{R} such that $A\phi_k = i\lambda_k\phi_k$. The sequence $(\lambda_k)_k$ tends to $+\infty$ and, up to a reordering, is non decreasing.

Since B is bounded from $D(|A|^{\frac{r}{2}})$ to $D(|A|^r)$, (A, B, \mathbf{R}) satisfies Assumption 1 and (A, B) is r -weakly coupled by Proposition 25.

We thus obtain Theorem 5 as a restatement of Proposition 46.

6.2 Potential well with dipolar interaction

In this example, $\Omega = (0, \pi)$ endowed with the standard Lebesgue measure, V is the constant zero function and $W : x \mapsto x$ is the identity function. This academic example is a simplification of the harmonic oscillator, presented in Section 6.3, in the sense that Ω is bounded. It has been thoroughly studied by K. Beauchard in a serie of analyses (see among others [Bea05, BL10]). These works give the first (and, at this time, almost the only one) satisfying description of the reachable set with L^2 controls from the first eigenvector for systems of the type of (1.1). Using Lyapounov techniques, V. Nersessyan gave practical algorithms for approximate controllability in [Ner10].

Equation (1.1) writes

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - u(t)x\psi \quad (6.2)$$

with boundary conditions $\psi(0) = \psi(\pi) = 0$.

The linear operators $A = \frac{i}{2}\Delta$ defined on $D(A) = (H^2 \cap H_0^1)((0, \pi), \mathbf{C})$ and $B : \psi \mapsto ix\psi$ are skew symmetric in the Hilbert space $H = L^2(\Omega, \mathbf{C})$ endowed with the hermitian product $L^2(\Omega, \mathbf{C})$,

$$\langle f, g \rangle = \int_0^\pi \overline{f(x)}g(x)dx.$$

Defining, for every k in \mathbf{N} ,

$$\phi_k : x \mapsto \sqrt{\frac{2}{\pi}}\sin(k\pi x)$$

the family $\Phi = (\phi_k)_{k \in \mathbf{N}}$ is an orthonormal basis of \mathcal{H} made of eigenvectors of A . (A, B, \mathbf{R}) satisfies Assumption 1.

For the sake of readability, we define for every (j, k) in \mathbf{N}^2 $b_{jk} := \langle \phi_j, B\phi_k \rangle$. Since B is skew-symmetric, $b_{jk} = -\bar{b}_{kj}$ for every j, k . Easy computations show that, for every j, k in \mathbf{N} , $b_{jk} = 0$ as soon as $j + k$ is even and, if $j + k$ is odd,

$$|b_{jk}| \leq \frac{2jk}{(j-k)^2(j+k)^2}.$$

We will use the following obvious technical remarks in the following.

Lemma 47. *For every j, k in \mathbf{N} such that $j \neq k$, we have $\frac{1}{|j-k|} \leq 2\frac{j}{k}$.*

Lemma 48. *Let a in $\left[0, \frac{9}{8}\right)$ and $b > \frac{1}{2}$. Then (the restriction of) B is bounded from $D(|A|^{a+b})$ to $D(|A|^a)$.*

Proof. Let ψ belong to $D(|A|^{a+b})$. Then $\sum_{k \in \mathbf{N}} k^{4(a+b)} |\langle \phi_k, \psi \rangle|^2 < +\infty$. By hypothesis, $8a - 1 < \min\{8, 8(a+b) - 5\}$. We chose $c \in (8a - 1, \min\{8(a+b) - 5, 8\})$. Notice that $4a - \frac{c}{2} - \frac{3}{2} < -1$ and $8(a+b) - 4 - c > 1$. Then

$$\begin{aligned} \| |A|^a B\psi \|^2 &= \sum_{k \in \mathbf{N}} k^{4a} |\langle \phi_k, B\psi \rangle|^2 \leq \sum_{k \in \mathbf{N}} k^{4a} \left| \sum_{j \in \mathbf{N}} b_{jk} \langle \phi_j, \psi \rangle \right|^2 \leq \sum_{k \in \mathbf{N}} k^{4a} \left(\sum_{j \in \mathbf{N}} |b_{jk} \langle \phi_j, \psi \rangle| \right)^2 \\ &\leq \sum_{k \in \mathbf{N}} k^{4a} \left[\left(\sum_{j \in \mathbf{N}} \frac{|b_{jk}|^2}{j^{4(a+b)}} \right) \left(\sum_{j \in \mathbf{N}} j^{4(a+b)} |\langle \phi_j, \psi \rangle|^2 \right) \right] \\ &\leq \left(\sum_{j \in \mathbf{N}} j^{4(a+b)} |\langle \phi_j, \psi \rangle|^2 \right) \sum_{k \in \mathbf{N}} k^{4a} \left[\left(\sum_{j \in \mathbf{N}, j \neq k} \frac{4j^2 k^2}{j^{4(a+b)} (k-j)^4 (k+j)^4} \right) \right] \\ &\leq 4 \left(\sum_{j \in \mathbf{N}} j^{4(a+b)} |\langle \phi_j, \psi \rangle|^2 \right) \sum_{k \in \mathbf{N}} k^{4a+2} \left[\left(\sum_{j \in \mathbf{N}, j \neq k} \frac{1}{j^{8(a+b)-4} (k-j)^8} \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbf{N}, j \neq k} \frac{1}{(k+j)^8} \right)^{\frac{1}{2}} \right] \\ &\leq 4 \left(\sum_{j \in \mathbf{N}} j^{4(a+b)} |\langle \phi_j, \psi \rangle|^2 \right) \sum_{k \in \mathbf{N}} k^{4a+2} \left[\left(2^c \sum_{j \in \mathbf{N}, j \neq k} \frac{j^c}{j^{8(a+b)-4} (k-j)^{4-c} k^c} \right)^{\frac{1}{2}} \left(\sum_{j > k} \frac{1}{j^8} \right)^{\frac{1}{2}} \right] \\ &\leq 2^{2+\frac{c}{2}} \left(\sum_{j \in \mathbf{N}} j^{4(a+b)} |\langle \phi_j, \psi \rangle|^2 \right) \sum_{k \in \mathbf{N}} k^{4a+2-\frac{c}{2}-\frac{7}{2}} \left[\left(\sum_{j \in \mathbf{N}} \frac{1}{j^{8(a+b)-4-c}} \right)^{\frac{1}{2}} \right]. \end{aligned}$$

In other words, the restriction of B is bounded from $D(|A|^{a+b})$ to $D(|A|^a)$ with bound less than $2^{\frac{1}{2}+\frac{c}{4}} \left(\sum_{j \in \mathbf{N}} \frac{1}{j^{8(a+b)-4-c}} \right)^{\frac{1}{4}} \left(\sum_{k \in \mathbf{N}} k^{4a-\frac{c}{2}-\frac{3}{2}} \right)^{\frac{1}{2}} \leq 2^{\frac{1}{2}+\frac{c}{4}} \left(1 + \frac{1}{8(a+b)-3-c} \right)^{\frac{1}{4}} \left(1 + \frac{2}{1+c-8a} \right)^{\frac{1}{2}}$. \square

We now present show the almost $5/2$ -weakly coupled character of the potential wall with dipolar interaction.

Lemma 49. *Let $1 < a < \frac{5}{4}$. Then B is bounded from $D(|A|^a)$ to $D(|A|^a)$, and (A, B) is $2a$ weakly coupled.*

Proof. For every j in \mathbf{N} , $B\phi_j = \sum_{k \in \mathbf{N}} b_{kj} \phi_k$. Noticing that $x \mapsto \frac{x^{2+4a}}{(j-x)^4(j+x)^4}$ is increasing on $[1, j-1]$ and decreasing on $[j+1, +\infty)$, and defining $y = x/j$, we get

$$\| |A|^a B\phi_j \|^2 = \sum_{k \in \mathbf{N}} k^{4a} |b_{jk}|^2 \quad (6.3)$$

$$\leq \sum_{k \in \mathbf{N}, k \neq j} \frac{j^2 k^{2+4a}}{(j-k)^4(j+k)^4} \quad (6.4)$$

$$\leq j^2 \left(\int_1^{j-1} \frac{x^{2+4a} dx}{(j-x)^4(j+x)^4} + \frac{1}{2^4} j^{4a-2} + \int_{j+1}^{+\infty} \frac{x^{2+4a} dx}{(j-x)^4(j+x)^4} \right) \quad (6.5)$$

$$\leq j^2 \left(j^{4a-5} \int_{\frac{1}{j}}^{1-\frac{1}{j}} \frac{y^{2+4a} dy}{(1-y)^4(1+y)^4} + \frac{1}{2^4} j^{4a-2} + j^{4a-5} \int_{1+\frac{1}{j}}^{+\infty} \frac{y^{2+4a} dy}{(1-y)^4(1+y)^4} \right) \quad (6.6)$$

Notice that the last integral in (6.6) converges in $+\infty$ if and only if $4a < 5$. The two integrals in (6.6) are both equivalent to $\frac{j^3}{3 \cdot 2^4}$ as j tends to infinity. Hence there exists a positive constant c such that $\| |A|^a B\phi_j \|^2 \leq c j^{4a} = c \| |A|^a \phi_j \|^2$.

We use Lemma 26 to prove that (A, B) is $2a$ weakly coupled. Since B is bounded from $D(A^a)$ to itself, as in Lemma 29 we obtain the equality of the domains of $|A|^s$ and $D(|A + uB|^s)$ for any real u and $s \in [0, 2]$.

For every ψ in $D(|A|^{2a})$, by Cauchy-Schwarz inequality,

$$|\langle |A|^{2a} \psi, B\psi \rangle| = |\langle |A|^a \psi, |A|^a B\psi \rangle| \quad (6.7)$$

$$\leq \| |A|^a \psi \| \| |A|^{\frac{a}{2}} B\psi \| \quad (6.8)$$

$$\leq \| |A|^a \psi \|^2 \| B \|_{L(D(|A|^a), D(|A|^a))} \quad (6.9)$$

$$\leq |\langle |A|^{2a} \psi, \psi \rangle| \| B \|_{L(D(|A|^a), D(|A|^a))} \quad (6.10)$$

Hence (A, B) is $2a$ weakly coupled with coupling constant less than $\| B \|_{L(D(|A|^a), D(|A|^a))}$. \square

Remark 22. Tedious but straightforward computations show that (A, B) is a -weakly coupled with

$$c_a(A, B) \leq \left(\sum_{k \in \mathbf{N}} k^{2(1-a)} \left(\sum_{j \in \mathbf{N}, j \neq k} \left| \frac{(k^{2a} - j^{2a})}{j^{a-1}(k+j)^2(k-j)^2} \right|^2 \right) \right)^{\frac{1}{2}}$$

for every $2 < a < \frac{5}{2}$.

The fact that the present system is not more than $5/2$ -weakly coupled is the purpose of the following lemma.

Lemma 50. For every a in $(0, +\infty)$, $e^B \phi_1 \in D(|A|^a) \Leftrightarrow a < \frac{5}{4}$.

Proof. Straightforward computations yield $e^B \phi_1 = \sum_{k \in \mathbf{N}} 2i \frac{(-1)^k - 1}{k^3 - 4k} \phi_k$ hence $e^B \phi_1 \in D(|A|^a)$ if and

only if $k^{4a} \left| \frac{(-1)^k - 1}{k^3 - 4k} \right|^2 \in \ell^1(\mathbf{C})$ that is $4a < 5$. \square

We sum up our results in the following

Proposition 51. Define $\text{Att}(\phi_1) = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0, T])} \{\Upsilon_{t,0}^u \phi_1 \mid 0 \leq t \leq T\}$. Then $\text{Att}(\phi_1) \subset \bigcap_{s < \frac{5}{4}} D(|A|^s)$

and $\text{Att}(\phi_1) \not\subset D(|A|^{\frac{5}{4}})$.

Remark 23. Notice that [BL10, Theorem 2] (exact controllability of (6.6) in $D(|A|^{\frac{5}{2}})$ with H_0^1 controls) is compatible with the result of Proposition 30 (no exact controllability of (6.6) in $D(|A|^{\frac{9}{4}})$ with BV controls).

From [BCCS12], we know that $\{(k, k+1) \mid k \in \mathbf{N}\}$ is a non-degenerate chain of connectedness for $(A + \eta B, B)$ for almost every real η . Hence Proposition 65 guarantees the approximate controllability of the system (6.2) from ϕ_1 for the norm of $D(|A + \eta B|^r) = D(|A|^r)$ for $\frac{3}{2} < r < \frac{5}{4} + 1$. For practical exact control in $D(|A|^{\frac{3}{2}})$, we can use the explicit controls given by Proposition 65 to reach a neighborhood of the target in $D(|A|^2)$ (see for instance [BCC12a]), and then use the L^2 control given by [BL10] for exact local controllability.

6.3 Quantum harmonic oscillator

The quantum harmonic oscillator with angular frequency ω describes the oscillations of a particle of mass m subject to the potential $V(x) = \frac{1}{2}m\omega x^2$. The corresponding uncontrolled Schrödinger equation is

$$i \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi(x, t) + \frac{1}{2} m \omega x^2 \psi(x, t).$$

With a suitable choice of units, it becomes

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi(x, t) + \frac{1}{2} x^2 \psi(x, t)$$

The operator $A = \frac{i}{2} \Delta - \frac{x^2}{2}$ is self-adjoint on $L^2(\mathbf{R}, \mathbf{C})$, A has a pure discrete spectrum. The k^{th} eigenvalue (corresponding to the k^{th} energy level) is equal to $\frac{2k+1}{2}i$ and is associated with the eigenstate

$$\phi_k : x \mapsto \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} \exp\left(-\frac{x^2}{2}\right) H_k(x)$$

where H_k is the k^{th} Hermite polynomial, namely $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} (e^{-x^2})$.

When considering the classical dipolar interaction, the control potential W takes the form $W(x) = x$ for every x in \mathbf{R} . It is well known (see [MR04] and references therein) that the resulting control system (1.1) is not controllable in any reasonable sense. Indeed the system splits in two uncoupled subsystems. The first one is a finite dimensional classical harmonic oscillator which is controllable. The second one is a free (that is, without control) quantum harmonic oscillator, whose evolution does not depend on the control and is therefore not controllable.

The proof given in [MR04, ILT06] (and especially the decomposition of the system in two decoupled systems) does not require more to the control than to be the derivative of a derivable function. Using the equivalence result (Proposition 46), it can be extended to Radon measures.

Proposition 52. The system (1.1) with $\Omega = \mathbf{R}$, $V : x \mapsto x^2$ and $W : x \mapsto x$ is not approximately controllable by means of Radon measures.

Because of its non-controllability properties, the system (A, B) does not admit any non resonant chain of connectedness. To circumvent the resonances on the spectrum of A , we apply the classical

techniques of analytical perturbations recalled in Appendix A. For every real number ν , we define the continuous real function $W_\nu : x \mapsto x + \nu \exp(-ax^2)$.

For every η in \mathbf{R} , define the real function $V_{\eta,\nu} = V + \eta W_\nu$. Our aim is to apply Proposition 65. We have, then, to check that for almost every $\nu \in \mathbf{R}$ that $\{(j, k) : |j - k| = 1\}$ is a nonresonant chain of connectedness for system 1.1.

The mapping $\eta \mapsto -\Delta + \eta V_{\eta,\nu}$ is analytic. Hence, there exists a family $(\lambda_k, \phi_k)_{k \in \mathbf{N}}$ of analytic functions $\lambda_k : \mathbf{R} \rightarrow \mathbf{C}$ and analytic mappings $\phi_k : \mathbf{R} \rightarrow L^2(\mathbf{R}, \mathbf{C})$ such that for every η , $-\Delta + V_{\eta,\nu}$ has pure point discrete spectrum $(\lambda_k(\nu))_{k \in \mathbf{N}}$ associated to the family of eigenvectors $(\phi_k(\nu))_{k \in \mathbf{N}}$.

We first compute, for every k in \mathbf{N} and every $b > 0$

$$\int_{\mathbf{R}} x^{2k} \exp(-bx^2) dx = \sqrt{\frac{\pi}{b}} \frac{(2k)!}{2^{2k} k! b^k}. \quad (6.11)$$

Recalling that the k^{th} Hermite polynomial H_k is a polynomial of degree exactly k with integer coefficients and with leading coefficient 2^k , one sees that, for every p in \mathbf{N} ,

$$\begin{aligned} F_p &:= \int_{\mathbf{R}} W_\nu(x) \phi_p(x)^2 dx \\ &= \frac{1}{2^p p! \sqrt{\pi}} \int_{\mathbf{R}} (x + \nu \exp(-(a+1)x^2)) H_p(x)^2 dx \\ &= \frac{\nu}{2^p p! \sqrt{\pi}} \int_{\mathbf{R}} \exp(-(a+1)x^2) H_p(x)^2 dx \end{aligned}$$

can be written as $\nu P_p \left(\frac{1}{\sqrt{a+1}} \right)$ where P_p is a polynomial with degree $2p+1$ with rational coefficients.

Moreover, we compute, for every k ,

$$\lambda'_k(0) = \int_{\mathbf{R}} \phi_k(0)^2 W_\nu = \nu F_k.$$

If a transcendental then the sequence $(F_p)_{p \in \mathbf{N}}$ is rationally independent, in particular, $F_{p'} - F_{q'} = F_p - F_q$ implies $\{p', q'\} = \{p, q\}$. Hence, for every $\nu \neq 0$, for almost every η in \mathbf{R} , there exists a nonresonant chain of connectedness for $(i(-\Delta + V_{\eta,\nu}), W_\nu)$.

Proposition 53. *For every $\eta, s > 0$, the perturbed harmonic oscillator $(i(-\Delta + V_{\eta,\nu}), W_\nu)$ is s -weakly coupled*

Proof. Notice first that $(i(-\Delta + V), W)$ is a tridiagonal system satisfying the conditions of [BCC13, Section IV] and is thus s -weakly-coupled for every $s > 0$. Since all the derivatives of the smooth function $x \mapsto \nu \exp(-ax^2)$ are bounded on \mathbf{R} , $(i(-\Delta + V, W, (W_\nu - W)))$ is s -weakly-coupled for every $s > 0$. \square

Proposition 65 can now be applied to $(i(-\Delta + V_{\eta,\nu}), W_\nu)$ and provides the following proposition.

Proposition 54. *For every $\nu \neq 0$, for every $s > 0$ and almost every $\eta \in \mathbf{R}$ the perturbed harmonic oscillator $(i(-\Delta + V_{\eta,\nu}), W_\nu)$ is simultaneously approximately controllable in $H^\eta(\mathbf{R})$.*

6.4 A controllable system without good Galerkin approximation

We present here a (non-physical) example which is simultaneously approximately controllable and cannot be suitably represented by its finite dimensional Galerkin approximations. It consists in Equation (2.1), in which the Hilbert space \mathcal{H} is equal to $L^2(\mathbf{R}, \mathbf{C})$ endowed with its standard scalar

product $\langle f, g \rangle = \int_{\mathbf{R}} \bar{f}g$, A is equal to $i((\Delta + x^2) + (\Delta + x^2)^{-1})$ and B is the multiplication by x^4 . The skew symmetric operator is skew adjoint, with spectrum $Sp(A) = i\{(2k-1)/2 + 2/(2k-1), k \in \mathbf{N}\}$. The k^{th} Hermite function ϕ_k is an eigenstate of A associated with the eigenvalue $i(2k+1)/2 + i2/(2k-1)$. The family $\Phi = (\phi_k)_{k \in \mathbf{N}}$ is an Hilbert basis of \mathcal{H} .

Notice that B is not relatively bounded with respect to A , while, for every (constant) u in $[0, +\infty)$, $A + uB$ is the generator of group of unitary transformations. We define the solutions of (2.1) only for controls u that are piecewise constant.

For parity reasons, B leaves invariant the sets $\Phi^e = (\phi_{2k})_{k \in \mathbf{N}}$ and $\Phi^o = (\phi_{2k+1})_{k \in \mathbf{N}}$. Hence, no global controllability in \mathcal{H} is to be expected.

Defining $\phi_k^o = \phi_{2k+1}$ for every k in \mathbf{N} and $\Phi^o = (\phi_k^o)_{k \in \mathbf{N}}$, the closure \mathcal{H}^o of $\text{span}\Phi^o$ is an Hilbert space stable by A and B . The restriction of A and B to \mathcal{H}^o are denoted by A^o and B^o respectively.

The set $\{(j, k) \in \mathbf{N}^2, |j - k| = 1\}$ is a non resonant chain of connectedness of (A^o, B^o, Φ^o) . By Proposition 64, the control system (A^o, B^o) is approximately controllable in \mathcal{H}^o . Moreover, for every j in \mathbf{N} , for every $\varepsilon > 0$, there exists a piecewise constant positive control $u_{j,\varepsilon}$ with L^1 norm less than $\sum_{l=1}^{j-1} \frac{\pi}{2l^2} < \frac{\pi^3}{12}$ that steers this system from the first eigenstate ϕ_1^o to an ε -neighborhood of ϕ_j^o . The $(A^o)^s$ norm of ϕ_j^o tends to $+\infty$ as j tends to infinity. Hence (A^o, B^o) does not satisfy the conclusions of Proposition 39.

Acknowledgments The financial support of the INRIA Color is acknowledge in this work. This long term analysis was possible due to the support of our respective institutions (the universities of Franche-Comté and Lorraine and the CNAM) as well as the facilities offered by the CNRS, the PIMS and the University of Victoria during the stay of the first author.

We are also grateful to many colleagues for the useful discussions that lead to many improvements since our first result, among them we especially thank Farid Ammar-Khodja, Marcelo Laca or Gilles Lancien.

A Analytical perturbations

To apply our sufficient condition for approximate controllability (Proposition 65), we need to find a non-resonant chain of connectedness, which may require some work on practical examples. A classical idea we already used in this study is to introduce a new control $\tilde{u} = u - \bar{u}$ and to consider the system $x' = (A + \bar{u}B) + (u - \bar{u})B$ for a suitably chosen constant \bar{u} .

We have the following results by Kato [Kat66, Section VII.2].

Definition 11. Let D_0 be a domain of the complex plane, a family $(T(z))_{z \in D_0}$ of closed operators from a Banach space X to a Banach space Y is said to be a holomorphic family of type (A) if

1. $D(T(z)) = D$ is independent of z ,
2. $T(z)u$ is holomorphic for z in D_0 for every u in D .

Theorem 55 ([Kat66, Theorem VII.3.9]). *Let $T(z)$ be a selfadjoint holomorphic family of type (A) defined for z in a neighborhood of an interval I_0 of the real axis such that $T(z)^* = T(\bar{z})$. Furthermore, let $T(z)$ have a compact resolvent. Then all eigenvalues of $T(z)$ can be represented by functions which are holomorphic in I_0 ³.*

More precisely, there is a sequence of scalar-valued functions $(z \mapsto \mu_n(z))_{n \in \mathbf{N}}$ and operator-valued functions $(z \mapsto \phi_n(z))_{n \in \mathbf{N}}$, all holomorphic on I_0 , such that for z in I_0 , the sequence $(\mu_n(z))_{n \in \mathbf{N}}$ represents all the repeated eigenvalues of $T(z)$ and $(\phi_n(z))_{n \in \mathbf{N}}$ forms a complete orthonormal family of the associated eigenvectors of $T(z)$.

Proposition 56. *If (A, B, K) satisfies Assumptions 1 then the family $i(A + zB)_{z \in \mathbf{C}, |z| < 1/\|B\|_A}$ is holomorphic of type (A).*

Proof. The question of domain is solved by the Kato-Rellich Theorem. The holomorphy is immediate as the family $i(A + zB)$ is affine in z . \square

B Interpolation

B.1 Convergence of sequences

Through the present analysis, the following simple interpolation lemma was useful.

Lemma 57. *Let A be a skew-adjoint operator, let S be a set and $(u_n)_{n \in \mathbf{N}}$ take value in the set of functions from S to $D(|A|^k)$, such that $(u_n)_{n \in \mathbf{N}}$ is uniformly bounded in S for the norm of $D(|A|^k)$, $k > 0$. If $(u_n)_{n \in \mathbf{N}}$ tends to zero in \mathcal{H} uniformly in S , then $(u_n)_{n \in \mathbf{N}}$ tends to zero in $D(|A|^l)$, uniformly in S for every $l < k$.*

Proof. The proof follows from the logarithmic convexity of $l \in [0, k] \mapsto \| |A|^l u \|$. Indeed

$$\| |A|^{\frac{l+j}{2}} u \| = \sqrt{\langle |A|^l u, |A|^j u \rangle} \leq \| |A|^l u \|^{1/2} \| |A|^j u \|^{1/2}.$$

If $l < k$ then

$$\| |A|^l u_n \| \leq \| u_n \|^{k-l} \| |A|^k u_n \|^{l/k}.$$

Let $C = \sup_{n \in \mathbf{N}} \| |A|^k u_n \|^2$ and $N > 0$ such that for any $n > N$, $\| u_n \|^2 \leq \varepsilon$ we obtain

$$n > N \implies \| |A|^l u_n \|^2 \leq \varepsilon^{\frac{k-l}{k}} C^{\frac{l}{k}},$$

which provides the lemma. \square

³Each of them is holomorphic in some neighborhood of I_0 but possibly different for each in such a way that their intersection is just I_0 .

B.2 Interpolation of fractional powers of operators

Let us now state a more sophisticated result. The following result can also be deduced from the content of [ABdMG96, Section 2.8].

Proposition 58. *Let A and B be two selfadjoint positive operators in \mathcal{H} such that there exists $c > 0$ with*

$$c \leq B \leq A$$

in the form sense. Then for any $\alpha \in (0, 1)$, the following is true

$$c^\alpha \leq B^\alpha \leq A^\alpha.$$

Proof. The proof follows from the following series of lemma.

For a selfadjoint operator \mathcal{H} and $z \in \mathbf{C} \setminus \mathbf{R}$, the functional calculus is the extension of the mapping:

$$\{x \in \mathbf{R} \mapsto (x - z)^{-1}\} \in B(\mathbf{R}) \rightarrow (A - z)^{-1} \in B(\mathcal{H})$$

as a strong continuous $*$ -algebra homomorphisme map on bounded borelian functions on the real line with the bounded pointwise topology : $B(\mathbf{R})$.

Let us recall the following functional calculus identity based on the Poisson formula.

Lemma 59. *Let A be a selfadjoint operator in \mathcal{H} . Let f be a bounded borelian function. Then*

$$f(A) := w - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2i\pi} \int_{\mathbf{R}} f(\lambda) \Im(A - \lambda - i\varepsilon)^{-1} d\lambda.$$

We also recall the formula for $\alpha \in (0, 1)$ and $x > 0$

$$x^{-\alpha} = \frac{\pi}{\sin(\pi\alpha)} \int_{\mathbf{R}_+} \frac{w^{-\alpha}}{x + w} dw.$$

and then the Fubini theorem with Lemma 59 we obtain the

Lemma 60. *Let A be a positive selfadjoint operator in \mathcal{H} . Then for $\alpha \in (0, 1)$*

$$A^\alpha = \frac{\pi}{\sin(\pi\alpha)} \int_{\mathbf{R}_+} \frac{w^{-1+\alpha} A}{A + w} dw.$$

With this identity and the following lemma we have the proof of the proposition.

Lemma 61. *Let A and B be two selfadjoint positive operators in \mathcal{H} such that there exists $c > 0$ with*

$$c \leq B \leq A.$$

Then

$$A^{-1} \leq B^{-1}.$$

Proof. First notice that both A and B are invertible from their domains to \mathcal{H} as well as their square roots. Then from

$$\sqrt{c}\|u\| \leq \|\sqrt{B}u\| \leq \|\sqrt{A}u\|,$$

we deduce that $\sqrt{B}\sqrt{A}^{-1}$ is a bounded operator with norm at most 1.

In the other hand the operator $\sqrt{A}^{-1}\sqrt{B}$ defined on $D(\sqrt{B})$ extends as the adjoint of $\sqrt{B}\sqrt{A}^{-1}$ to a closed operator on \mathcal{H} and hence is bounded with norm at most 1 and

$$\|\sqrt{A}^{-1}\sqrt{B}u\| \leq \|u\|, \forall u \in D(\sqrt{B})$$

and thus

$$\|\sqrt{A}^{-1}u\| \leq \|\sqrt{B}^{-1}u\|.$$

and the result follows. □

The proof of Proposition 58 then follows as

$$c \leq B \leq A$$

implies for any $w > 0$,

$$1 - w(B + w)^{-1} \leq 1 - w(A + w)^{-1}.$$

and thus

$$\frac{w^{-1+\alpha}B}{B+w} \leq \frac{w^{-1+\alpha}A}{A+w}$$

integrating on $w > 0$ gives the desired inequality. \square

The above result can be extended to the case $c = 0$ by replacing A and B by $A + \epsilon$ and $B + \epsilon$ as in [Ped72], we obtain

$$0 \leq B^\alpha \leq (B + \epsilon)^\alpha \leq (A + \epsilon)^\alpha.$$

The second inequality is immediate. We obtain

$$0 \leq (A + \epsilon)^{-\alpha/2} B^\alpha (A + \epsilon)^{-\alpha/2} \leq 1$$

so that taking ϵ to 0 giving

$$0 \leq B^\alpha \leq (A)^\alpha.$$

We immediately deduce the following corollary.

Corollary 62. *Let A and B be two positive self-adjoint operators sharing the same domains. For any $\alpha \in (0, 1)$, we have :*

$$D(A^\alpha) = D(B^\alpha)$$

Proof. Let us first assume that B has spectrum away from zero. As B is closed it is a bounded operator from $D(A)$ to \mathcal{H} . Thus

$$\exists c > 0, \forall \phi \in D(A), \|B\phi\| \leq c\|A\phi\|.$$

Hence

$$B^2 \leq c^2 A^2$$

from the above that $B^{2\alpha/2}$ is bounded from $D(A^{2\alpha/2})$ to \mathcal{H} .

The proof being now symmetric in A and B we can conclude \square

C An extension of the commutator theorem by Nelson

In this section only we will consider A as self-adjoint instead of skew-adjoint in order to follow the usual formulation of Nelson commutator theorem.

Theorem 63. *Let N be a selfadjoint operator on a Hilbert space H and A be a symmetric operator on H defined on $D(N)$. Let R be the resolvent of N at i and assume that*

$$\exists C > 0, \forall \psi \in D(A), \|[A, R]\psi\| \leq C\|R\psi\|$$

then A is essentially self-adjoint.

Proof. First we notice that $[A, R]$ extends uniquely to \mathcal{H} and thus

$$\exists C > 0, \forall \psi \in \mathcal{H}, \|[A, R]\psi\| \leq C\|R\psi\|.$$

Since the adjoint of R is the resolvent of N at $-i$, the resolvent identity provides

$$\exists C > 0, \forall \psi \in D(A), \|[A, R^*]\psi\| \leq C\|R^*\psi\|$$

or

$$\exists C > 0, \forall \psi \in \mathcal{H}, \|[A, R^*]\psi\| \leq C\|R^*\psi\|.$$

The adjoint of $[A, R^*]$ can be identified with an extension of $[A^*, R]$ defined on $D(A^*)$ (which contains $D(A)$), thus

$$\exists C > 0, \forall \psi \in \mathcal{H}, \|[A^*, R]\psi\| \leq C\|R\psi\|.$$

Let $c > C$ and $\psi \in D(A^*)$ such that $(A^* + ic)\psi = 0$. Then

$$R(A^* + ic)\psi = 0.$$

As $R\psi \in D(N) \subset D(A)$,

$$(A^* + ic)R\psi = [A^*, R]\psi$$

or

$$(A + ic)R\psi = [A^*, R]\psi.$$

We have

$$c\|R\psi\| \leq \|(A + ic)R\psi\| = \|[A^*, R]\psi\| \leq C\|R\psi\|$$

and

$$R\psi = 0$$

and thus $\psi = 0$. □

Notice that in the form sense $[A, R] = -R[A, N]R$ and the assumption is actually the boundedness of $R[A, N]$ or the boundedness of $[A, N]$ from $D(N)$ to H . This is stronger than the usual assumption on the Nelson Commutator Theorem.

Actually the operator N is not needed. The assumption can be replaced by the existence of a bounded operator R such that the range of R is in the domain of A (or R is bounded from $D(A^*)$ to $D(A)$).

D Some sufficient conditions for approximate controllability with bounded variation controls

The aim of this Section is to recall approximate controllability results obtained in other contexts and how this results may be adapted in our context.

We first recall the following definitions from [CMSB09].

Definition 12. Let (A, B, \mathbf{R}) satisfy Assumptions 1 such that A and B are skew-symmetric. Let $\Phi = (\phi_k)_k$ be a Hilbert basis of \mathcal{H} made of eigenvectors of A , $A\phi_k = i\lambda_k\phi_k$ for every k in \mathbf{N} . A pair (j, k) of integers is a non-degenerate transition of (A, B, Φ) if (i) $\langle \phi_j, B\phi_k \rangle \neq 0$ and (ii) for every (l, m) in \mathbf{N}^2 , $|\lambda_j - \lambda_k| = |\lambda_l - \lambda_m|$ implies $(j, k) = (l, m)$ or $\langle \phi_l, B\phi_m \rangle = 0$ or $\{j, k\} \cap \{l, m\} = \emptyset$.

Definition 13. Let (A, B, \mathbf{R}) satisfy Assumptions 1 such that A and B are skew-symmetric. Let $\Phi = (\phi_k)_k$ be a Hilbert basis of \mathcal{H} made of eigenvectors of A , $A\phi_k = i\lambda_k\phi_k$ for every k in \mathbf{N} . A subset S of \mathbf{N}^2 is a non-degenerate chain of connectedness of (A, B, Φ) if (i) for every (j, k) in S , (j, k) is a non-degenerate transition of (A, B) and (ii) for every r_a, r_b in \mathbf{N} , there exists a finite sequence $r_a = r_0, r_1, \dots, r_p = r_b$ in \mathbf{N} such that, for every $j \leq p - 1$, (r_j, r_{j+1}) belongs to S .

Proposition 64. Let (A, B, \mathbf{R}) satisfy Assumptions 1 such that A and B are skew-symmetric. Let $\Phi = (\phi_k)_k$ be a Hilbert basis of \mathcal{H} made of eigenvectors of A , $A\phi_k = i\lambda_k\phi_k$ for every k in \mathbf{N} . Let S be a non-degenerate chain of connectedness of (A, B) . Then, for every $\eta > 0$, (A, B) is simultaneously approximately controllable in $D(|A|^{1-\eta})$.

Proof. First of all, it is enough to prove the result for target propagators $\hat{\Upsilon}$ leaving invariant the space of co-dimension 2 spanned by (ϕ_j, ϕ_k) for (j, k) in S

$$\hat{\Upsilon} = e^{i\nu_l}(\cos(\theta)\phi_l^*\phi_l + \sin(\theta)\phi_l^*\phi_k) + e^{i\nu_k}(-\sin(\theta)\phi_k^*\phi_l + \cos(\theta)\phi_l^*\phi_k)$$

The result in \mathcal{H} -norm is a consequence of [Cha12, Theorem 1]: for every piecewise constant $u^* : \mathbf{R} \rightarrow \mathbf{R}$, $2\pi/|\lambda_j - \lambda_k|$ -periodic such that

$$\int_0^{\frac{2\pi}{|\lambda_j - \lambda_k|}} u^*(\tau) e^{i(\lambda_j - \lambda_k)\tau} d\tau \neq 0$$

and

$$\int_0^{\frac{2\pi}{|\lambda_j - \lambda_k|}} u^*(\tau) e^{i(\lambda_l - \lambda_m)\tau} d\tau = 0$$

for every l, m such that $(\lambda_l - \lambda_m) \in \mathbf{Z}(\lambda_j - \lambda_k)$ and $b_{l,m} \neq 0$, there exists T^* such that $\Upsilon^{u^*/n}(nT^*, 0)$ tends to $\hat{\Upsilon}$ as n tends to infinity.

The conclusion follows using Lemma 57 and the estimate in A -norm of Theorem 7. \square

Let us just mention the following result in case of higher regularity.

Proposition 65. Let k be a positive real. Let (A, B, \mathbf{R}) satisfy Assumptions 1 such that (A, B) is k -weakly coupled. Let $\Phi = (\phi_k)_k$ be a Hilbert basis of \mathcal{H} made of eigenvectors of A , $A\phi_k = i\lambda_k\phi_k$ for every k in \mathbf{N} . Let S be a non-degenerate chain of connectedness of (A, B) such that, for every (j, k) in S , the set $\{(l, m) \in \mathbf{N}^2 | (\lambda_l - \lambda_m) \in \mathbf{Z}(\lambda_j - \lambda_k) \text{ and } \langle \phi_l, B\phi_m \rangle \neq 0\}$ is finite. Then, for every $\eta > 0$, (A, B) is simultaneously approximately controllable in $D(|A|^{k/2+1-\eta})$.

Proof. The proof differs from the previous by the interpolation step by using Proposition 30. \square

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