

# Borel chromatic number of closed graphs

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**Abstract.** We construct, for each countable ordinal  $\xi$ , a closed graph with Borel chromatic number two and Baire class  $\xi$  chromatic number  $\aleph_0$ .

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# 1 Introduction

The study of the Borel chromatic number of analytic graphs on Polish spaces was initiated in [K-S-T]. In particular, the authors prove in this paper that the Borel chromatic number of the graph generated by a partial Borel function has to be in  $\{1, 2, 3, \aleph_0\}$ . They also provide a minimum graph  $\mathcal{G}_0$  of uncountable Borel chromatic number. This last result had a lot of developments. For example, B. Miller gave in [Mi] some other versions of it, which helped him to generalize a number of known dichotomy theorems in descriptive set theory. The first author generalized in [L2] the  $\mathcal{G}_0$ -dichotomy to any dimension making sense in classical descriptive set theory, and also used versions of  $\mathcal{G}_0$  to study the non-potentially closed subsets of a product of two Polish spaces (see [L1]).

A study of the  $\Delta_\xi^0$  chromatic number of analytic graphs on Polish spaces was initiated in [L-Z1] and was motivated by the  $\mathcal{G}_0$ -dichotomy. More precisely, let  $B$  be a Borel binary relation, on a Polish space  $X$ , having a Borel countable coloring (i.e., a Borel map  $c : X \rightarrow \omega$  such that  $c(x) \neq c(y)$  if  $(x, y) \in B$ ). Is there a relation between the Borel class of  $B$  and that of the coloring? In other words, is there a map  $k : \omega_1 \setminus \{0\} \rightarrow \omega_1 \setminus \{0\}$  such that any  $\Pi_\xi^0$  binary relation having a Borel countable coloring has in fact a  $\Delta_{k(\xi)}^0$ -measurable countable coloring, for each  $\xi \in \omega_1 \setminus \{0\}$ ?

In [L-Z2], the authors give a negative answer: for each countable ordinal  $\xi \geq 1$ , there is a partial injection with disjoint domain and range  $i : \omega^\omega \rightarrow \omega^\omega$ , whose graph

- is  $D_2(\Pi_1^0)$  (i.e., the difference of two closed sets),
- has Borel chromatic number two,
- has no  $\Delta_\xi^0$ -measurable countable coloring.

On the other hand, they note that an open binary relation having a finite coloring  $c$  has also a  $\Delta_2^0$ -measurable finite coloring (consider the differences of the  $c^{-1}(\{n\})$ 's, for  $n$  in the range of the coloring). Note that an irreflexive closed binary relation on a zero-dimensional space has a continuous countable coloring (this coloring is  $\Delta_2^0$ -measurable in non zero-dimensional spaces). So they wonder whether we can build, for each countable ordinal  $\xi \geq 1$ , a closed binary relation with a Borel finite coloring but no  $\Delta_\xi^0$ -measurable finite coloring. This is indeed the case:

**Theorem** *Let  $\xi \geq 1$  be a countable ordinal. Then there exists a partial injection with disjoint domain and range  $f : \omega^\omega \rightarrow \omega^\omega$  whose graph is closed (and thus has Borel chromatic number two), and has no  $\Delta_\xi^0$ -measurable finite coloring (and thus has  $\Delta_\xi^0$  chromatic number  $\aleph_0$ ).*

The previous discussion shows that this result is optimal. Its proof uses, among other things, the method used in [L-Z2] improving Theorem 4 in [M]. This method relates topological complexity and Baire category.

## 2 Mátrai sets

Before proving our main result, we recall some material from [L-Z2].

**Notation.** The symbol  $\tau$  denotes the usual product topology on the Baire space  $\omega^\omega$ .

**Definition 2.1** We say that a partial map  $f : \omega^\omega \rightarrow \omega^\omega$  is **nice** if its graph  $\text{Gr}(f)$  is a  $(\tau \times \tau)$ -closed subset of  $\omega^\omega \times \omega^\omega$ .

The construction of  $P_\xi$  and  $\tau_\xi$ , and the verification of the properties (1)-(3) from the next lemma (a corollary of Lemma 2.6 in [L-Z2]), can be found in [M], up to minor modifications.

**Lemma 2.2** Let  $1 \leq \xi < \omega_1$ . Then there are  $P_\xi \subseteq \omega^\omega$ , and a topology  $\tau_\xi$  on  $\omega^\omega$  such that

- (1)  $\tau_\xi$  is zero-dimensional perfect Polish and  $\tau \subseteq \tau_\xi \subseteq \Sigma_\xi^0(\tau)$ ,
- (2)  $P_\xi$  is a nonempty  $\tau_\xi$ -closed nowhere dense set,
- (3) if  $S \in \Sigma_\xi^0(\omega^\omega, \tau)$  is  $\tau_\xi$ -nonmeager in  $P_\xi$ , then  $S$  is  $\tau_\xi$ -nonmeager in  $\omega^\omega$ ,
- (4) if  $V, W$  are nonempty  $\tau_\xi$ -open subsets of  $\omega^\omega$ , then we can find a  $\tau_\xi$ -dense  $G_\delta$  subset  $H$  of  $V \setminus P_\xi$ , a  $\tau_\xi$ -dense  $G_\delta$  subset  $L$  of  $W \setminus P_\xi$ , and a nice  $(\tau_\xi, \tau_\xi)$ -homeomorphism from  $H$  onto  $L$ .

The following lemma (a corollary of Lemma 2.7 in [L-Z2]) is a consequence of the previous one. It provides, among other things, a topology  $T_\xi$  that we will use in the sequel.

**Lemma 2.3** Let  $1 \leq \xi < \omega_1$ . Then there is a disjoint countable family  $\mathcal{G}_\xi$  of subsets of  $\omega^\omega$  and a topology  $T_\xi$  on  $\omega^\omega$  such that

- (a)  $T_\xi$  is zero-dimensional perfect Polish and  $\tau \subseteq T_\xi \subseteq \Sigma_\xi^0(\tau)$ ,
- (b) for any nonempty  $T_\xi$ -open sets  $V, V'$ , there are disjoint  $G, G' \in \mathcal{G}_\xi$  with  $G \subseteq V$ ,  $G' \subseteq V'$ , and there is a nice  $(T_\xi, T_\xi)$ -homeomorphism from  $G$  onto  $G'$ ,  
and, for every  $G \in \mathcal{G}_\xi$ ,
- (c)  $G$  is nonempty,  $T_\xi$ -nowhere dense, and in  $\Pi_2^0(T_\xi)$ ,
- (d) if  $S \in \Sigma_\xi^0(\omega^\omega, \tau)$  is  $T_\xi$ -nonmeager in  $G$ , then  $S$  is  $T_\xi$ -nonmeager in  $\omega^\omega$ .

The construction of  $\mathcal{G}_\xi$  and  $T_\xi$  ensures that  $T_\xi$  is  $(\tau_\xi)^\omega$ , where  $\tau_\xi$  is as in Lemma 2.2. This topology is on  $(\omega^\omega)^\omega$ , identified with  $\omega^\omega$ . We will need the following consequence of the construction of  $\mathcal{G}_\xi$  and  $T_\xi$ .

**Lemma 2.4** Let  $1 \leq \xi < \omega_1$ , and  $V$  be a nonempty  $T_\xi$ -open set. Then  $\overline{V}^\tau$  is not  $\tau$ -compact.

**Proof.** The fact that  $T_\xi$  is  $(\tau_\xi)^\omega$  gives a finite sequence  $U_0, \dots, U_n$  of nonempty open subsets of  $(\omega^\omega, \tau_\xi)$  with  $U_0 \times \dots \times U_n \times (\omega^\omega)^\omega \subseteq V$ . Thus  $\overline{V}^\tau$  contains the  $\tau$ -closed set  $\overline{U_0}^\tau \times \dots \times \overline{U_n}^\tau \times (\omega^\omega)^\omega$ , and it is enough to see that this last set is not  $\tau$ -compact. This comes from the fact that the Baire space  $(\omega^\omega, \tau)$  is not compact.  $\square$

### 3 Proof of the main result

Before proving our main result, we give an example giving the flavour of the sequel. In [Za], the author gives a Hurewicz-like test to see when two disjoint subsets  $A, B$  of a product  $Y \times Z$  of Polish spaces can be separated by an open rectangle. We set  $\mathbb{A} := \{(n^\infty, n^\infty) \mid n \in \omega\}$ ,

$$\mathbb{B}_0 := \{(0^{m+1}(n+1)^\infty, (m+1)^{n+1}0^\infty) \mid m, n \in \omega\}$$

and  $\mathbb{B}_1 := \{((m+1)^{n+1}0^\infty, 0^{m+1}(n+1)^\infty) \mid m, n \in \omega\}$ . Then  $A$  is not separable from  $B$  by an open rectangle exactly when there are  $\varepsilon \in 2$  and continuous maps  $g : \omega^\omega \rightarrow Y$ ,  $h : \omega^\omega \rightarrow Z$  such that  $\mathbb{A} \subseteq (g \times h)^{-1}(A)$  and  $\mathbb{B}_\varepsilon \subseteq (g \times h)^{-1}(B)$ .

**Example.** Here we are looking for closed graphs with Borel chromatic number two and of arbitrarily high finite  $\Delta_\xi^0$  chromatic number  $n$ . There is an example with  $\xi = 1$  and  $n = 3$  where  $\mathbb{B}_0$  is involved. We set  $C := \{(2m)^\infty, (2m+1)^\infty \mid m \in \omega\} \cup \mathbb{B}_0$ ,

$$D := \{(2m)^\infty \mid m \in \omega\} \cup \{0^{m+1}(n+1)^\infty \mid m, n \in \omega\},$$

$$R := \{(2m+1)^\infty \mid m \in \omega\} \cup \{(m+1)^{n+1}0^\infty \mid m, n \in \omega\},$$

$$f((2m)^\infty) := (2m+1)^\infty \text{ and } f(0^{m+1}(n+1)^\infty) := (m+1)^{n+1}0^\infty.$$

This defines  $f : D \rightarrow R$  whose graph is  $C$ . The first part of  $C$  is discrete, and thus closed. Assume that  $(\alpha_k, \beta_k) := (0^{m_k+1}(n_k+1)^\infty, (m_k+1)^{n_k+1}0^\infty) \in \mathbb{B}_0$  and converges to  $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$  as  $k$  goes to infinity. We may assume that  $(m_k)$  is constant, and  $(n_k)$  too, so that  $(\alpha, \beta) \in \mathbb{B}_0$ , which is therefore closed. This shows that  $C$  is closed. Note that  $D, R$  are disjoint and Borel, so that  $C$  has Borel chromatic number two. Let  $\Delta$  be a clopen subset of  $\omega^\omega$ . Let us prove that  $C \cap \Delta^2$  or  $C \cap (\neg\Delta)^2$  is not empty. We argue by contradiction. Then  $\Delta$  or  $\neg\Delta$  has to contain  $0^\infty$ . Assume that it is  $\Delta$ , the other case being similar. Then  $0^{m+1}(n+1)^\infty \in \Delta$  if  $m$  is big enough. Thus  $(m+1)^{n+1}0^\infty \notin \Delta$  if  $m$  is big enough. Therefore  $(m+1)^\infty \notin \Delta$  if  $m$  is big enough. Thus  $((2m)^\infty, (2m+1)^\infty) \in C \cap (\neg\Delta)^2$  if  $m$  is big enough, which is absurd.

We now turn to the general case. Our main lemma is as follows. We equip  $\omega^m$  with the discrete topology  $\tau_d$ , for each  $m > 0$ .

**Lemma** *Let  $\xi \geq 1$  be a countable ordinal,  $n \geq 1$  be a natural number, and  $X := \omega \times \omega^\omega$ . Then we can find a partial injection  $f : X \rightarrow X$  and a disjoint countable family  $\mathcal{F}$  of subsets of  $X$  such that*

- (a)  $f$  has disjoint domain and range,
- (b)  $\text{Gr}(f)$  is  $((\tau_d \times \tau) \times (\tau_d \times \tau))$ -closed,
- (c) there is no sequence  $(\Delta_i)_{i < n}$  of  $\Delta_\xi^0$  subsets of  $(X, \tau_d \times \tau)$  such that
  - (i)  $\forall i < n \text{ Gr}(f) \cap \Delta_i^2 = \emptyset$ ,
  - (ii)  $\bigcup_{i < n} \Delta_i$  is  $(\tau_d \times T_\xi)$ -comeager in  $X$ ,
- (d)  $\mathcal{F}$  has the properties (b)-(d) in Lemma 2.3, where  $\mathcal{G}_\xi, \omega^\omega, T_\xi$  and  $\tau$  are respectively replaced with  $\mathcal{F}, X, \tau_d \times T_\xi$  and  $\tau_d \times \tau$ ,
- (e)  $(\bigcup \mathcal{F}) \cap (\text{Domain}(f) \cup \text{Range}(f)) = \emptyset$ .

**Proof.** We argue by induction on  $n$ .

**The basic case  $n = 1$**

Let  $\mathcal{G}_\xi$  be the family given by Lemma 2.3. We split  $\mathcal{G}_\xi$  into two disjoint subfamilies  $\mathcal{G}_\xi^0$  and  $\mathcal{G}_\xi^1$  having the property (b) in Lemma 2.3. This is possible since the elements of  $\mathcal{G}_\xi$  are  $T_\xi$ -nowhere dense. Let  $G_0, G_1 \in \mathcal{G}_\xi^0$  be disjoint, and  $\varphi$  be a nice  $(T_\xi, T_\xi)$ -homeomorphism from  $G_0$  onto  $G_1$ . We then set  $f(0, \alpha) := (0, \varphi(\alpha))$  if  $\alpha \in G_0$ , and  $\mathcal{F} := \{\{n\} \times G \mid n \in \omega \wedge G \in \mathcal{G}_\xi^1\}$ . It remains to check that the property (c) is satisfied. We argue by contradiction, which gives  $\Delta_0 \in \Delta_\xi^0$ . By property (d) in Lemma 2.3,  $\Delta_0 \cap (\{0\} \times G_\varepsilon)$  is  $(\tau_d \times T_\xi)$ -comeager in  $\{0\} \times G_\varepsilon$  for each  $\varepsilon \in 2$ . As  $f$  is a  $(\tau_d \times T_\xi, \tau_d \times T_\xi)$ -homeomorphism,  $\Delta_0 \cap (\{0\} \times G_0) \cap f^{-1}(\Delta_0 \cap (\{0\} \times G_1))$  is  $(\tau_d \times T_\xi)$ -comeager in  $\{0\} \times G_0$ , which contradicts the fact that  $\text{Gr}(f) \cap \Delta_0^2 = \emptyset$ .

## The induction step from $n$ to $n+1$

The induction assumption gives  $f$  and  $\mathcal{F}$ . Here again, we split  $\mathcal{F}$  into two disjoint subfamilies  $\mathcal{F}^0$  and  $\mathcal{F}^1$  having the property (b) in Lemma 2.3, where  $\mathcal{G}_\xi$ ,  $\omega^\omega$ ,  $T_\xi$  and  $\tau$  are respectively replaced with  $\mathcal{F}$ ,  $X$ ,  $\tau_d \times T_\xi$  and  $\tau_d \times \tau$ . Let  $(V_p)$  be a basis for the topology  $\tau_d \times T_\xi$  made of nonempty sets. Fix  $p \in \omega$ . By Lemma 2.4, there is a countable family  $(W_q^p)_{q \in \omega}$ , with  $(\tau_d \times \tau)$ -closed union, and made of pairwise disjoint  $(\tau_d \times \tau)$ -clopen subsets of  $X$  intersecting  $V_p$ .

• Let  $b : \omega \rightarrow \omega^2$  be a bijection. We construct, for  $\vec{v} = (p, q) \in \omega^2$  and  $\varepsilon \in 2$ , and by induction on  $b^{-1}(\vec{v})$ ,

- $G_\varepsilon^{\vec{v}} \in \mathcal{F}^0$ ,
- a nice  $(\tau_d \times T_\xi, \tau_d \times T_\xi)$ -homeomorphism  $\varphi^{\vec{v}} : G_0^{\vec{v}} \rightarrow G_1^{\vec{v}}$ .

We want these objects to satisfy the following:

- $G_0^{\vec{v}} \subseteq (V_p \cap W_q^p) \setminus (\bigcup_{m < b^{-1}(\vec{v})} \overline{G_0^{b(m)} \cup G_1^{b(m)}}^{\tau_d \times T_\xi})$ ,
- $G_1^{\vec{v}} \subseteq V_q \setminus (G_0^{\vec{v}} \cup \bigcup_{m < b^{-1}(\vec{v})} \overline{G_0^{b(m)} \cup G_1^{b(m)}}^{\tau_d \times T_\xi})$ .

• We now define the desired partial map  $\tilde{f} : \omega \times \omega \times \omega^\omega \rightarrow \omega \times \omega \times \omega^\omega$ , as well as  $\tilde{\mathcal{F}} \subseteq 2^{\omega \times \omega \times \omega^\omega}$ , as follows:

$$\tilde{f}(l, x) := \begin{cases} (p+1, \varphi^{p,q}(x)) & \text{if } l=0 \wedge x \in G_0^{p,q}, \\ (l, f(x)) & \text{if } l>0 \wedge x \in \text{Domain}(f). \end{cases}$$

and  $\tilde{\mathcal{F}} := \{\{l\} \times G \mid l \in \omega \wedge G \in \mathcal{F}^1\}$ . Note that  $\tilde{f}$  is well-defined and injective, by disjointness of the  $(G_0^{\vec{v}} \cup G_1^{\vec{v}})$ 's. Identifying  $X$  with  $\omega \times \omega \times \omega^\omega$ , we can consider  $\tilde{f}$  as a partial map from  $X$  into itself and  $\tilde{\mathcal{F}}$  as a family of subsets of  $X$  (this identification is based on the identification of  $\omega$  with  $\omega \times \omega$ ).

(a), (d) and (e) are clearly satisfied.

(b) Assume that  $((l_k, x_k), (m_k, y_k)) \in \text{Gr}(\tilde{f})$  tends to  $((l, x), (m, y)) \in (\omega \times X)^2$  as  $k$  goes to infinity. We may assume that  $(l_k)$  and  $(m_k)$  are constant.

If  $l=0$ , then there is  $p$  such that  $p+1=m$  and  $(x_k, y_k) \in G_0^{p,q_k} \times G_1^{p,q_k}$ . As  $G_0^{p,q_k} \subseteq W_{q_k}^p$ , we may also assume that  $(q_k)$  is also constant and equals  $q$ . As  $\varphi^{p,q}$  is nice,  $((l, x), (m, y)) \in \text{Gr}(f)$ .

If  $l>0$ , then  $(x_k, y_k) \in \text{Gr}(f)$ . As  $\text{Gr}(f)$  is  $((\tau_d \times \tau) \times (\tau_d \times \tau))$ -closed,  $((l, x), (m, y)) \in \text{Gr}(\tilde{f})$ .

(c) We argue by contradiction, which gives  $(\Delta_i)_{i \leq n}$ . We may assume, without loss of generality, that  $(\{0\} \times \omega \times \omega^\omega) \cap \Delta_n$  is not meager in  $(\{0\} \times \omega \times \omega^\omega, \tau_d \times T_\xi)$ . This gives  $p \in \omega$  such that  $(\{0\} \times V_p) \cap \Delta_n$  is  $(\tau_d \times T_\xi)$ -comeager in  $V_p' := \{0\} \times V_p$ . As  $V_p' \setminus \Delta_n \in \Sigma_\xi^0(\tau_d \times \tau)$ ,  $(\{0\} \times G_0^{p,q}) \cap \Delta_n$  is  $(\tau_d \times T_\xi)$ -comeager in  $\{0\} \times G_0^{p,q}$  for each  $q \in \omega$ .

As  $\text{Gr}(\tilde{f}) \cap \Delta_n^2 = \emptyset$  and the  $\varphi^{\vec{v}}$ 's are  $(\tau_d \times T_\xi, \tau_d \times T_\xi)$ -homeomorphisms,  $(\{p+1\} \times G_1^{p,q}) \cap \Delta_n$  is  $(\tau_d \times T_\xi)$ -meager in  $\{p+1\} \times G_1^{p,q}$ , for each  $q$ .

As  $(\omega \times \omega \times \omega^\omega) \setminus (\bigcup_{i \leq n} \Delta_i)$  is  $(\tau_d \times T_\xi)$ -meager in  $\omega \times \omega \times \omega^\omega$  and  $\Delta_\xi^0(\tau_d \times \tau)$ ,

$$(\{p+1\} \times G_1^{p,q}) \setminus (\bigcup_{i < n} \Delta_i)$$

is  $(\tau_d \times T_\xi)$ -meager in  $\{p+1\} \times G_1^{p,q}$ , for each  $q$ . Thus  $(\{p+1\} \times G_1^{p,q}) \cap (\bigcup_{i < n} \Delta_i)$  is  $(\tau_d \times T_\xi)$ -comeager in  $\{p+1\} \times G_1^{p,q}$ , for each  $q$ .

**Claim** *The set  $(\{p+1\} \times \omega \times \omega^\omega) \cap (\bigcup_{i < n} \Delta_i)$  is  $(\tau_d \times T_\xi)$ -comeager in  $\{p+1\} \times \omega \times \omega^\omega$ .*

Indeed, we argue by contradiction. This gives  $W \in (\tau_d \times T_\xi) \setminus \{\emptyset\}$  such that

$$(\{p+1\} \times W) \cap (\bigcup_{i < n} \Delta_i)$$

is  $(\tau_d \times T_\xi)$ -meager in  $W' := \{p+1\} \times W$ . Let  $q \in \omega$  be such that  $V_q \subseteq W$ . Then  $G_1^{p,q} \subseteq W$  and  $\{p+1\} \times G_1^{p,q} \subseteq W'$ . As  $W' \cap (\bigcup_{i < n} \Delta_i) \in \Sigma_\xi^0(\tau_d \times \tau)$  and  $(\{p+1\} \times G_1^{p,q}) \cap W' \cap (\bigcup_{i < n} \Delta_i)$  is  $(\tau_d \times T_\xi)$ -comeager in  $\{p+1\} \times G_1^{p,q}$ ,  $W' \cap (\bigcup_{i < n} \Delta_i)$  is not  $(\tau_d \times T_\xi)$ -meager in  $W'$ , which is absurd.  $\diamond$

Now we set  $\Delta'_i := (\{p+1\} \times \omega \times \omega^\omega) \cap \Delta_i$  if  $i < n$ . Note that  $\Delta'_i \in \Delta_\xi^0(\{p+1\} \times \omega \times \omega^\omega, \tau_d \times \tau)$ ,  $\text{Gr}(\tilde{f}) \cap (\Delta'_i)^2 = \emptyset$ , and  $\bigcup_{i < n} \Delta'_i$  is  $(\tau_d \times T_\xi)$ -comeager in  $\{p+1\} \times \omega \times \omega^\omega$ , which contradicts the induction assumption.  $\square$

In order to get our main result, it is enough to apply the main lemma to each  $n \geq 1$ . This gives  $f_n : \omega \times \omega^\omega \rightarrow \omega \times \omega^\omega$ . It remains to define  $f : \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega) \rightarrow \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega)$  by  $f(n, x) := f_n(x)$  (we identify  $(\omega \setminus \{0\}) \times \omega \times \omega^\omega$  with  $\omega^\omega$ ).

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