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Borel chromatic number of closed graphs

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Abstract. We construct, for each countable ordinal ξ , a closed graph with Borel chromatic number two and Baire class ξ chromatic number \aleph_0 .

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1 Introduction

The study of the Borel chromatic number of analytic graphs on Polish spaces was initiated in [K-S-T]. In particular, the authors prove in this paper that the Borel chromatic number of the graph generated by a partial Borel function has to be in $\{1, 2, 3, \aleph_0\}$. They also provide a minimum graph \mathcal{G}_0 of uncountable Borel chromatic number. This last result had a lot of developments. For example, B. Miller gave in [Mi] some other versions of it, which helped him to generalize a number of known dichotomy theorems in descriptive set theory. The first author generalized in [L2] the \mathcal{G}_0 -dichotomy to any dimension making sense in classical descriptive set theory, and also used versions of \mathcal{G}_0 to study the non-potentially closed subsets of a product of two Polish spaces (see [L1]).

A study of the Δ_ξ^0 chromatic number of analytic graphs on Polish spaces was initiated in [L-Z1], and was motivated by the \mathcal{G}_0 -dichotomy. More precisely, let B be a Borel binary relation, on a Polish space X , having a Borel countable coloring (i.e., a Borel map $c : X \rightarrow \omega$ such that $c(x) \neq c(y)$ if $(x, y) \in B$). Is there a relation between the Borel class of B and that of the coloring? In other words, is there a map $k : \omega_1 \setminus \{0\} \rightarrow \omega_1 \setminus \{0\}$ such that any Π_ξ^0 binary relation having a Borel countable coloring has in fact a $\Delta_{k(\xi)}^0$ -measurable countable coloring, for each $\xi \in \omega_1 \setminus \{0\}$?

In [L-Z2], the authors give a negative answer: for each countable ordinal $\xi \geq 1$, there is a partial injection with disjoint domain and range $i : \omega^\omega \rightarrow \omega^\omega$, whose graph

- is $D_2(\Pi_1^0)$ (i.e., the difference of two closed sets),
- has Borel chromatic number two,
- has no Δ_ξ^0 -measurable countable coloring.

On the other hand, they note that an open binary relation having a finite coloring c has also a Δ_2^0 -measurable finite coloring (consider the differences of the $c^{-1}(\{n\})$'s, for n in the range of the coloring). Note that an irreflexive closed binary relation on a zero-dimensional space has a continuous countable coloring (this coloring is Δ_2^0 -measurable in non zero-dimensional spaces). So they wonder whether we can build, for each countable ordinal $\xi \geq 1$, a closed binary relation with a Borel finite coloring but no Δ_ξ^0 -measurable finite coloring. This is indeed the case:

Theorem *Let $\xi \geq 1$ be a countable ordinal. Then there exists a partial function with disjoint domain and range $f : \omega^\omega \rightarrow \omega^\omega$ whose graph is closed (and thus has Borel chromatic number two), and has no Δ_ξ^0 -measurable finite coloring (and thus has Δ_ξ^0 chromatic number \aleph_0).*

The previous discussion shows that this result is optimal. Its proof uses, among other things, the method used in [L-Z2] improving Theorem 4 in [M]. This method relates topological complexity and Baire category.

2 Mátrai sets

Before proving our main result, we recall some material from [L-Z2].

Notation. The symbol τ denotes the usual product topology on the Baire space ω^ω .

Definition 2.1 We say that a partial map $f : \omega^\omega \rightarrow \omega^\omega$ is **nice** if its graph $\text{Gr}(f)$ is a $(\tau \times \tau)$ -closed subset of $\omega^\omega \times \omega^\omega$.

The construction of P_ξ and τ_ξ , and the verification of the properties (1)-(3) from the next lemma (a corollary of Lemma 2.6 in [L-Z2]), can be found in [M], up to minor modifications.

Lemma 2.2 Let $1 \leq \xi < \omega_1$. Then there are $P_\xi \subseteq \omega^\omega$, and a topology τ_ξ on ω^ω such that

- (1) τ_ξ is zero-dimensional perfect Polish and $\tau \subseteq \tau_\xi \subseteq \Sigma_\xi^0(\tau)$,
- (2) P_ξ is a nonempty τ_ξ -closed nowhere dense set,
- (3) if $S \in \Sigma_\xi^0(\omega^\omega, \tau)$ is τ_ξ -nonmeager in P_ξ , then S is τ_ξ -nonmeager in ω^ω ,
- (4) if V, W are nonempty τ_ξ -open subsets of ω^ω , then we can find a τ_ξ -dense G_δ subset H of $V \setminus P_\xi$, a τ_ξ -dense G_δ subset L of $W \setminus P_\xi$, and a nice (τ_ξ, τ_ξ) -homeomorphism from H onto L .

The following lemma (a corollary of Lemma 2.7 in [L-Z2]) is a consequence of the previous one. It provides, among other things, a topology T_ξ that we will use in the sequel.

Lemma 2.3 Let $1 \leq \xi < \omega_1$. Then there is a disjoint countable family \mathcal{G}_ξ of subsets of ω^ω and a topology T_ξ on ω^ω such that

- (a) T_ξ is zero-dimensional perfect Polish and $\tau \subseteq T_\xi \subseteq \Sigma_\xi^0(\tau)$,
- (b) for any nonempty T_ξ -open sets V, V' , there are disjoint $G, G' \in \mathcal{G}_\xi$ with $G \subseteq V$, $G' \subseteq V'$, and there is a nice (T_ξ, T_ξ) -homeomorphism from G onto G' ,
and, for every $G \in \mathcal{G}_\xi$,
- (c) G is nonempty, T_ξ -nowhere dense, and in $\Pi_2^0(T_\xi)$,
- (d) if $S \in \Sigma_\xi^0(\omega^\omega, \tau)$ is T_ξ -nonmeager in G , then S is T_ξ -nonmeager in ω^ω .

The construction of \mathcal{G}_ξ and T_ξ ensures the following facts:

- T_ξ is $(\tau_\xi)^\omega$, where τ_ξ is as in Lemma 2.2. This topology is on $(\omega^\omega)^\omega$, identified with ω^ω .
- $\bigcup \mathcal{G}_\xi$ is disjoint from $(\neg P_\xi)^\omega$.

We will need the following consequences of the construction of \mathcal{G}_ξ and T_ξ .

Lemma 2.4 Let $1 \leq \xi < \omega_1$, and V be a nonempty T_ξ -open set. Then \overline{V}^τ is not τ -compact.

Proof. The fact that T_ξ is $(\tau_\xi)^\omega$ gives a finite sequence U_0, \dots, U_n of nonempty open subsets of $(\omega^\omega, \tau_\xi)$ with $U_0 \times \dots \times U_n \times (\omega^\omega)^\omega \subseteq V$. Thus \overline{V}^τ contains the τ -closed set $\overline{U_0}^\tau \times \dots \times \overline{U_n}^\tau \times (\omega^\omega)^\omega$, and it is enough to see that this last set is not τ -compact. This comes from the fact that the Baire space (ω^ω, τ) is not compact. \square

Lemma 2.5 Let $1 \leq \xi < \omega_1$. Then we can find dense G_δ subsets C_0, C_1 of (ω^ω, T_ξ) , disjoint from $\bigcup \mathcal{G}_\xi$, and a nice (T_ξ, T_ξ) -homeomorphism from C_0 onto C_1 .

Proof. Lemma 2.2.(4) gives, for each $i \in \omega$, τ_ξ -dense G_δ subsets H_i, L_i of $\neg P_\xi$ and a nice (τ_ξ, τ_ξ) -homeomorphism ψ_i from H_i onto L_i . By Lemma 2.1 in [L-Z2], $\prod_{i \in \omega} \psi_i$ is a nice (T_ξ, T_ξ) -homeomorphism from $C_0 := \prod_{i \in \omega} H_i$ onto $C_1 := \prod_{i \in \omega} L_i$. It remains to note that C_0 and C_1 are dense G_δ subsets of (ω^ω, T_ξ) since $T_\xi = (\tau_\xi)^\omega$, and disjoint from $\bigcup \mathcal{G}_\xi$ since they are contained in $(\neg P_\xi)^\omega$. \square

3 Proof of the main result

Before proving our main result, we give an example giving the flavour of the sequel. In [Za], the author gives a Hurewicz-like test to see when two disjoint subsets A, B of a product $Y \times Z$ of Polish spaces can be separated by an open rectangle. We set $\mathbb{A} := \{(n^\infty, n^\infty) \mid n \in \omega\}$,

$$\mathbb{B}_0 := \{(0^{m+1}(n+1)^\infty, (m+1)^{n+1}0^\infty) \mid m, n \in \omega\}$$

and $\mathbb{B}_1 := \{((m+1)^{n+1}0^\infty, 0^{m+1}(n+1)^\infty) \mid m, n \in \omega\}$. Then A is not separable from B by an open rectangle exactly when there are $\varepsilon \in 2$ and continuous maps $g: \omega^\omega \rightarrow Y$, $h: \omega^\omega \rightarrow Z$ such that $\mathbb{A} \subseteq (g \times h)^{-1}(A)$ and $\mathbb{B}_\varepsilon \subseteq (g \times h)^{-1}(B)$.

Example. Here we are looking for closed graphs with Borel chromatic number two and of arbitrarily high finite Δ_ξ^0 chromatic number n . There is an example with $\xi = 1$ and $n = 3$ where \mathbb{B}_0 is involved. We set $C := \{(2m)^\infty, (2m+1)^\infty) \mid m \in \omega\} \cup \mathbb{B}_0$,

$$D := \{(2m)^\infty \mid m \in \omega\} \cup \{0^{m+1}(n+1)^\infty \mid m, n \in \omega\},$$

$R := \{(2m+1)^\infty \mid m \in \omega\} \cup \{(m+1)^{n+1}0^\infty \mid m, n \in \omega\}$, $f((2m)^\infty) := (2m+1)^\infty$ and

$$f(0^{m+1}(n+1)^\infty) := (m+1)^{n+1}0^\infty.$$

This defines $f: D \rightarrow R$ whose graph is C . The first part of C is discrete, and thus closed. Assume that $(\alpha_k, \beta_k) := (0^{m_k+1}(n_k+1)^\infty, (m_k+1)^{n_k+1}0^\infty) \in \mathbb{B}_0$ and converges to $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ as k goes to infinity. We may assume that (m_k) is constant, and (n_k) too, so that $(\alpha, \beta) \in \mathbb{B}_0$, which is therefore closed. This shows that C is closed. Note that D, R are disjoint and Borel, so that C has Borel chromatic number two. Let Δ be a clopen subset of ω^ω . Let us prove that $C \cap \Delta^2$ or $C \cap (\neg\Delta)^2$ is not empty. We argue by contradiction. Then Δ or $\neg\Delta$ has to contain 0^∞ . Assume that it is Δ , the other case being similar. Then $0^{m+1}(n+1)^\infty \in \Delta$ if m is big enough. Thus $(m+1)^{n+1}0^\infty \notin \Delta$ if m is big enough. Therefore $(m+1)^\infty \notin \Delta$ if m is big enough. Thus $((2m)^\infty, (2m+1)^\infty) \in C \cap (\neg\Delta)^2$ if m is big enough, which is absurd.

We now turn to the general case. Our main lemma is as follows. We equip ω^m with the discrete topology τ_d , for each $m > 0$.

Lemma *Let $\xi \geq 1$ be a countable ordinal, $n \geq 2$ be a natural number, and $X := \omega^{n-1} \times \omega^\omega$. Then there is a partial function $f: X \rightarrow X$ such that*

- (a) *f has disjoint domain and range,*
- (b) *$Gr(f)$ is $((\tau_d \times \tau) \times (\tau_d \times \tau))$ -closed,*
- (c) *there is no sequence $(\Delta_i)_{i < n}$ of Δ_ξ^0 subsets of $(X, \tau_d \times \tau)$ such that*
 - (i) $\forall i < n \quad Gr(f) \cap \Delta_i^2 = \emptyset$,
 - (ii) $\bigcup_{i < n} \Delta_i$ *is $(\tau_d \times T_\xi)$ -comeager in X .*

Proof. Let $b: \omega \rightarrow \Omega := \omega^4 \times (n-1)$ be a bijection.

• Let (V_m) be a basis for the topology T_ξ made of nonempty sets. Fix $m \in \omega$. By Lemma 2.4, there is a countable family $(W_p^m)_{p \in \omega}$, with τ -closed union, and made of pairwise disjoint τ -clopen subsets of X meeting V_m .

- We construct, for $\vec{v} = (p_0, q_0, p_1, q_1, \eta) \in \Omega$ and $\varepsilon \in 2$, and by induction on $b^{-1}(\vec{v})$,
 - $G_\varepsilon^{\vec{v}} \in \mathcal{G}_\xi$,
 - a nice (T_ξ, T_ξ) -homeomorphism $\varphi^{\vec{v}}: G_0^{\vec{v}} \rightarrow G_1^{\vec{v}}$.

We want these objects to satisfy the following: $G_\varepsilon^{\vec{v}} \subseteq (V_{p_\varepsilon} \cap W_{q_\varepsilon}^{p_\varepsilon}) \setminus (\bigcup_{m < b^{-1}(\vec{v})} \overline{G_0^{b(m)} \cup G_1^{b(m)}}^{T_\xi})$.

• Lemma 2.5 gives dense G_δ subsets C_0, C_1 of (ω^ω, T_ξ) , disjoint from $\bigcup \mathcal{G}_\xi$, and a nice (T_ξ, T_ξ) -homeomorphism φ from C_0 onto C_1 .

• We set

$$D := \bigcup_{(l_1, \dots, l_n, \varepsilon) \in \omega^{n+2}} \left((\{l_1+1\} \times \dots \times \{l_{n-4}+1\} \times \{l_{n-3}+1\} \times \{l_{n-2}+1\} \times \{2l_{n-1}+1\} \times C_0) \cup (\{l_1+1\} \times \dots \times \{l_{n-4}+1\} \times \{l_{n-3}+1\} \times \{l_{n-2}+1\} \times \{0\} \times G_0^{(l_{n-1}, 1+2l_n+\varepsilon, l_n, 0, 0)}) \cup (\{l_1+1\} \times \dots \times \{l_{n-4}+1\} \times \{l_{n-3}+1\} \times \{0\} \times \omega \times G_0^{(l_{n-1}, l_n, l_n, 0, 1)}) \cup (\{l_1+1\} \times \dots \times \{l_{n-4}+1\} \times \{0\} \times \omega \times \omega \times G_0^{(l_{n-1}, l_n, l_n, 0, 2)}) \cup \dots (\{0\} \times \omega \times \dots \times \omega \times G_0^{(l_{n-1}, l_n, l_n, 0, n-2)}) \right),$$

$$R := \bigcup_{(l_1, \dots, l_n, \varepsilon) \in \omega^{n+2}} \left((\{l_1+1\} \times \dots \times \{l_{n-4}+1\} \times \{l_{n-3}+1\} \times \{l_{n-2}+1\} \times \{2l_{n-1}+2\} \times C_1) \cup (\{l_1+1\} \times \dots \times \{l_{n-4}+1\} \times \{l_{n-3}+1\} \times \{l_{n-2}+1\} \times \{2l_{n-1}+1+\varepsilon\} \times G_1^{(l_{n-1}, 1+2l_n+\varepsilon, l_n, 0, 0)}) \cup (\{l_1+1\} \times \dots \times \{l_{n-4}+1\} \times \{l_{n-3}+1\} \times \{l_{n-1}+1\} \times \omega \times G_1^{(l_{n-1}, l_n, l_n, 0, 1)}) \cup (\{l_1+1\} \times \dots \times \{l_{n-4}+1\} \times \{l_{n-1}+1\} \times \omega \times \omega \times G_1^{(l_{n-1}, l_n, l_n, 0, 2)}) \cup \dots (\{l_{n-1}+1\} \times \omega \times \dots \times \omega \times G_1^{(l_{n-1}, l_n, l_n, 0, n-2)}) \right),$$

We define $f: D \rightarrow R$ as follows.

$$\begin{aligned} f(l_1+1, \dots, l_{n-2}+1, 2l_{n-1}+1, x) &:= (l_1+1, \dots, l_{n-2}+1, 2l_{n-1}+2, \varphi(x)) \text{ if } x \in C_0, \\ f(l_1+1, \dots, l_{n-2}+1, 0, x) &:= (l_1+1, \dots, l_{n-2}+1, 2l_{n-1}+1+\varepsilon, \varphi^{(l_{n-1}, 1+2l_n+\varepsilon, l_n, 0, 0)}(x)) \\ &\quad \text{if } x \in G_0^{(l_{n-1}, 1+2l_n+\varepsilon, l_n, 0, 0)}, \\ f(l_1+1, \dots, l_{n-3}+1, 0, l_{n-2}, x) &:= (l_1+1, \dots, l_{n-3}+1, l_{n-1}+1, l_{n-2}, \varphi^{(l_{n-1}, l_n, l_n, 0, 1)}(x)) \\ &\quad \text{if } x \in G_0^{(l_{n-1}, l_n, l_n, 0, 1)}, \\ f(l_1+1, \dots, l_{n-4}+1, 0, l_{n-3}, l_{n-2}, x) &:= \\ &\quad (l_1+1, \dots, l_{n-4}+1, l_{n-1}+1, l_{n-3}, l_{n-2}, \varphi^{(l_{n-1}, l_n, l_n, 0, 2)}(x)) \text{ if } x \in G_0^{(l_{n-1}, l_n, l_n, 0, 2)}, \\ f(0, l_1, \dots, l_{n-2}, x) &:= (l_{n-1}+1, l_1, \dots, l_{n-2}, \varphi^{(l_{n-1}, l_n, l_n, 0, n-2)}(x)) \\ &\quad \text{if } x \in G_0^{(l_{n-1}, l_n, l_n, 0, n-2)}. \end{aligned}$$

Note that f is well-defined, by disjointness of the $(G_0^{\vec{v}} \cup G_1^{\vec{v}})$'s.

(a) The sets D and R are clearly disjoint.

(b) Note first that if $U := \bigcup_{(l_1, \dots, l_{n-1}) \in \omega^n} (\{l_1+1\} \times \dots \times \{l_{n-2}+1\} \times \{2l_{n-1}+1\} \times C_0)$, then $\text{Gr}(f|_U)$ is $((\tau_d \times \tau) \times (\tau_d \times \tau))$ -closed. So assume that $((l_1^k, \dots, l_{n-1}^k, x_k), (m_1^k, \dots, m_{n-1}^k, y_k)) \in \text{Gr}(f)$ tends to $((l_1, \dots, l_{n-1}, x), (m_1, \dots, m_{n-1}, y)) \in X^2$ as k goes to infinity, and that $x_k \in G_0^{(l_{n+1}^k, l_{n+2}^k, l_n^k, 0, \eta_k)}$ for each k . Then we may assume that (η_k) is constant since n is finite, so that $l_{n-1-\eta}^k = 0$ and $l_m^k > 0$ if $m < n-1-\eta$. So we may assume that $(l_1^k), \dots, (l_{n-1}^k), (m_1^k), \dots, (m_{n-1}^k)$ are constant, $l_{n-1-\eta} = 0$ and $l_m > 0$ if $m < n-1-\eta$.

If $\eta = 0$, then $(x_k, y_k) \in G_0^{(l_{n+1}^k, 1+2l_n^k+\varepsilon_k, l_n^k, 0, 0)} \times G_1^{(l_{n+1}^k, 1+2l_n^k+\varepsilon_k, l_n^k, 0, 0)}$ and we may assume that $2l_{n+1}^k + 1 + \varepsilon_k = m_{n-1}$, and $(l_{n+1}^k), (\varepsilon_k)$ are constant with limits l_{n+1}, ε respectively. As $G_0^{(l_{n+1}^k, 1+2l_n^k+\varepsilon_k, l_n^k, 0, 0)} \subseteq W_{1+2l_n^k+\varepsilon}^{l_{n+1}}$, we may also assume that (l_n^k) is also constant, by the properties of $(W_p^m)_{p \in \omega}$. As $\varphi^{(l_{n+1}, 1+2l_n+\varepsilon, l_n, 0, 0)}$ is nice, $((l_1, \dots, l_{n-1}, x), (m_1, \dots, m_{n-1}, y)) \in \text{Gr}(f)$.

If $\eta > 0$, then $(x_k, y_k) \in G_0^{(l_{n+1}^k, l_n^k, l_n^k, 0, 1)} \times G_1^{(l_{n+1}^k, l_n^k, l_n^k, 0, 1)}$ and we may assume that

$$l_{n+1}^k + 1 = m_{n-1-\eta},$$

and (l_{n+1}^k) is constant. As $G_0^{(l_{n+1}^k, l_n^k, l_n^k, 0, 1)} \subseteq W_{l_n^k}^{l_{n+1}}$, we may also assume that (l_n^k) is also constant again. As $\varphi^{(l_{n+1}, l_n, l_n, 0, 1)}$ is nice, $((l_1, \dots, l_{n-1}, x), (m_1, \dots, m_{n-1}, y)) \in \text{Gr}(f)$.

(c) We argue by contradiction, which gives $(\Delta_i)_{i < n}$. Assume for example that

$$(\{0\} \times \omega^{n-2} \times \omega^\omega) \cap \Delta_{n-1}$$

is not meager in $(\{0\} \times \omega^{n-2} \times \omega^\omega, \tau_d \times T_\xi)$. This gives $p \in \omega$ such that $(\{0\} \times \omega^{n-2} \times V_p) \cap \Delta_{n-1}$ is $(\tau_d \times T_\xi)$ -comeager in $V_p' := \{0\} \times \omega^{n-2} \times V_p$.

Claim 1 $(\{0\} \times \omega^{n-2} \times G_0^{(p, q, q, 0, n-2)}) \cap \Delta_{n-1}$ is $(\tau_d \times T_\xi)$ -comeager in $\{0\} \times \omega^{n-2} \times G_0^{(p, q, q, 0, n-2)}$, for each $q \in \omega$.

Indeed, we argue by contradiction. This gives $q_0 \in \omega$ such that

$$((\{0\} \times \omega^{n-2} \times G_0^{(p, q_0, q_0, 0, n-2)}) \cap V_p') \setminus \Delta_{n-1}$$

is not $(\tau_d \times T_\xi)$ -meager in $\{0\} \times \omega^{n-2} \times G_0^{(p, q_0, q_0, 0, n-2)}$. As $V_p' \setminus \Delta_{n-1} \in \Sigma_\xi^0(\tau_d \times \tau)$, $V_p' \setminus \Delta_{n-1}$ is not $(\tau_d \times T_\xi)$ -meager in V_p' , which is absurd. \diamond

As $\text{Gr}(f) \cap \Delta_{n-1}^2 = \emptyset$ and the $\varphi^{\vec{v}}$'s are (T_ξ, T_ξ) -homeomorphisms,

$$(\{p+1\} \times \omega^{n-2} \times G_1^{(p, q, q, 0, n-2)}) \cap \Delta_{n-1}$$

is $(\tau_d \times T_\xi)$ -meager in $\{p+1\} \times \omega^{n-2} \times G_1^{(p, q, q, 0, n-2)}$, for each q .

As $X \setminus (\bigcup_{i < n} \Delta_i)$ is $(\tau_d \times T_\xi)$ -meager in X and $\Delta_\xi^0(\tau_d \times \tau)$,

$$(\{p+1\} \times \omega^{n-2} \times G_1^{(p,q,q,0,n-2)}) \setminus (\bigcup_{i < n} \Delta_i)$$

is $(\tau_d \times T_\xi)$ -meager in $\{p+1\} \times \omega^{n-2} \times G_1^{(p,q,q,0,n-2)}$, for each q . Thus

$$(\{p+1\} \times \omega^{n-2} \times G_1^{(p,q,q,0,n-2)}) \cap (\bigcup_{i < n-1} \Delta_i)$$

is $(\tau_d \times T_\xi)$ -comeager in $\{p+1\} \times \omega^{n-2} \times G_1^{(p,q,q,0,n-2)}$, for each q .

Claim 2 $(\{p+1\} \times \omega^{n-2} \times \omega^\omega) \cap (\bigcup_{i < n-1} \Delta_i)$ is $(\tau_d \times T_\xi)$ -comeager in $\{p+1\} \times \omega^{n-2} \times \omega^\omega$.

Indeed, we argue by contradiction. This gives $W \in T_\xi$ such that

$$(\{p+1\} \times \omega^{n-2} \times W) \cap (\bigcup_{i < n-1} \Delta_i)$$

is $(\tau_d \times T_\xi)$ -meager in $W' := \{p+1\} \times \omega^{n-2} \times W$. Let $q \in \omega$ be such that $V_q \subseteq W$. Then $G_1^{(p,q,q,0,n-2)} \subseteq W$ and $\{p+1\} \times \omega^{n-2} \times G_1^{(p,q,q,0,n-2)} \subseteq W'$. As $W' \cap (\bigcup_{i < n-1} \Delta_i) \in \Sigma_\xi^0(\tau_d \times \tau)$, $W' \cap (\bigcup_{i < n-1} \Delta_i)$ is not $(\tau_d \times T_\xi)$ -meager in W' , which is absurd. This proves the claim. \diamond

If we iterate this argument, we get $p_1, \dots, p_{n-2} \in \omega$ such that, if we set $\pi := \{p_1+1\} \times \dots \times \{p_{n-2}+1\}$, then for example $(\pi \times \omega \times \omega^\omega) \cap (\Delta_0 \cup \Delta_1)$ is $(\tau_d \times T_\xi)$ -comeager in $\pi \times \omega \times \omega^\omega$. Assume for example that $(\pi \times \{0\} \times \omega^\omega) \cap \Delta_1$ is not meager in $(\pi \times \{0\} \times \omega^\omega, \tau_d \times T_\xi)$. This gives $p \in \omega$ such that $(\pi \times \{0\} \times V_p) \cap \Delta_1$ is $(\tau_d \times T_\xi)$ -comeager in $V'_p := \pi \times \{0\} \times V_p$. As in Claim 1, we see that $(\pi \times \{0\} \times G_0^{(p,1+2q+\varepsilon,q,0,0)}) \cap \Delta_1$ is T_ξ -comeager in $\pi \times \{0\} \times G_0^{(p,1+2q+\varepsilon,q,0,0)}$, for each $(q, \varepsilon) \in \omega \times 2$. Here again, this implies that $(\pi \times \{0\} \times G_1^{(p,1+2q+\varepsilon,q,0,0)}) \cap \Delta_1$ is T_ξ -meager in $\pi \times \{2p+1+\varepsilon\} \times G_1^{(p,1+2q+\varepsilon,q,0,0)}$, for each $(q, \varepsilon) \in \omega \times 2$. As $(\pi \times \omega \times \omega^\omega) \setminus (\Delta_0 \cup \Delta_1)$ is $(\tau_d \times T_\xi)$ -meager in $\pi \times \omega \times \omega^\omega$ and $\Delta_\xi^0(\tau_d \times \tau)$,

$$(\pi \times \{2p+1+\varepsilon\} \times G_1^{(p,1+2q+\varepsilon,q,0,0)}) \setminus (\Delta_0 \cup \Delta_1)$$

is $(\tau_d \times T_\xi)$ -meager in $\pi \times \{2p+1+\varepsilon\} \times G_1^{(p,1+2q+\varepsilon,q,0,0)}$, for each $(q, \varepsilon) \in \omega \times 2$. Thus

$$(\pi \times \{2p+1+\varepsilon\} \times G_1^{(p,1+2q+\varepsilon,q,0,0)}) \cap \Delta_0$$

is $(\tau_d \times T_\xi)$ -comeager in $\pi \times \{2p+1+\varepsilon\} \times G_1^{(p,1+2q+\varepsilon,q,0,0)}$, for each $(q, \varepsilon) \in \omega \times 2$.

As in Claim 2, we see that $(\pi \times \{2p+1+\varepsilon\} \times \omega^\omega) \cap \Delta_0$ is $(\tau_d \times T_\xi)$ -comeager in

$$\pi \times \{2p+1+\varepsilon\} \times \omega^\omega,$$

for each $\varepsilon \in 2$. As φ is a (T_ξ, T_ξ) -homeomorphism and C_0, C_1 are T_ξ -comeager in ω^ω , $\Delta_0 \cap f^{-1}(\Delta_0)$ is $(\tau_d \times T_\xi)$ -comeager in $\pi \times \{2p+1+\varepsilon\} \times C_0$, which contradicts the fact that $\text{Gr}(f) \cap \Delta_0^2 = \emptyset$. \square

In order to get our main result, it is enough to apply the main lemma to each $n \geq 2$. This gives $f_n : \omega^{n-1} \times \omega^\omega \rightarrow \omega^{n-1} \times \omega^\omega$. It remains to define $f : \bigcup_{n \geq 2} (\{n\} \times \omega^{n-1} \times \omega^\omega) \rightarrow \bigcup_{n \geq 2} (\{n\} \times \omega^{n-1} \times \omega^\omega)$ by $f(n, x) := f_n(x)$.

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