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Borel chromatic number of closed graphs

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Abstract. We construct, for each countable ordinal ξ, a closed graph with Borel chromatic number two and Baire class ξ chromatic number ℵ0.
1 Introduction

The study of the Borel chromatic number of analytic graphs on Polish spaces was initiated in [K-S-T]. In particular, the authors prove in this paper that the Borel chromatic number of the graph generated by a partial Borel function has to be in \(\{1, 2, 3, \aleph_0\}\). They also provide a minimum graph \(G_0\) of uncountable Borel chromatic number. This last result had a lot of developments. For example, B. Miller gave in [Mi] some other versions of it, which helped him to generalize a number of known dichotomy theorems in descriptive set theory. The first author generalized in [L2] the \(G_0\)-dichotomy to any dimension making sense in classical descriptive set theory, and also used versions of \(G_0\) to study the non-potentially closed subsets of a product of two Polish spaces (see [L1]).

A study of the \(\Delta^0_\xi\) chromatic number of analytic graphs on Polish spaces was initiated in [L-Z1] and was motivated by the \(G_0\)-dichotomy. More precisely, let \(B\) be a Borel binary relation, on a Polish space \(X\), having a Borel countable coloring (i.e., a Borel map \(c: X \to \omega\) such that \(c(x) \neq c(y)\) if \((x, y) \in B\)). Is there a relation between the Borel class of \(B\) and that of the coloring? In other words, is there a map \(k: \omega_1 \setminus \{0\} \to \omega_1 \setminus \{0\}\) such that any \(\Pi^0_\xi\) binary relation having a Borel countable coloring has in fact a \(\Delta^0_\xi\) measurable countable coloring, for each \(\xi \in \omega_1 \setminus \{0\}\)?

In [L-Z2], the authors give a negative answer: for each countable ordinal \(\xi \geq 1\), there is a partial injection with disjoint domain and range \(i: \omega^\omega \to \omega^\omega\), whose graph
- is \(D_2(\Pi^0_\xi)\) (i.e., the difference of two closed sets),
- has Borel chromatic number two,
- has no \(\Delta^0_\xi\)-measurable countable coloring.

On the other hand, they note that an open binary relation having a finite coloring \(c\) has also a \(\Delta^0_\xi\)-measurable finite coloring (consider the differences of the \(c^{-1}(\{n\})\)'s, for \(n\) in the range of the coloring). Note that an irreflexive closed binary relation on a zero-dimensional space has a continuous countable coloring (this coloring is \(\Delta^0_\xi\)-measurable in non zero-dimensional spaces). So they wonder whether we can build, for each countable ordinal \(\xi \geq 1\), a closed binary relation with a Borel finite coloring but no \(\Delta^0_\xi\)-measurable finite coloring. This is indeed the case:

**Theorem** Let \(\xi \geq 1\) be a countable ordinal. Then there exists a partial injection with disjoint domain and range \(f: \omega^\omega \to \omega^\omega\) whose graph is closed (and thus has Borel chromatic number two), and has no \(\Delta^0_\xi\)-measurable finite coloring (and thus has \(\Delta^0_\xi\) chromatic number \(\aleph_0\)).

The previous discussion shows that this result is optimal. Its proof uses, among other things, the method used in [L-Z2] improving Theorem 4 in [M]. This method relates topological complexity and Baire category.

2 Mátrai sets

Before proving our main result, we recall some material from [L-Z2].

**Notation.** The symbol \(\tau\) denotes the usual product topology on the Baire space \(\omega^\omega\).
Definition 2.1 We say that a partial map \( f : \omega^\omega \to \omega^\omega \) is nice if its graph \( \text{Gr}(f) \) is a \((\tau \times \tau)\)-closed subset of \( \omega^\omega \times \omega^\omega \).

The construction of \( P_\xi \) and \( \tau_\xi \), and the verification of the properties (1)-(3) from the next lemma (a corollary of Lemma 2.6 in [L-Z2]), can be found in [M], up to minor modifications.

Lemma 2.2 Let \( 1 \leq \xi < \omega_1 \). Then there are \( P_\xi \subseteq \omega^\omega \), and a topology \( \tau_\xi \) on \( \omega^\omega \) such that

1. \( \tau_\xi \) is zero-dimensional perfect Polish and \( \tau \subseteq \tau_\xi \subseteq \Sigma^0_3(\tau) \).
2. \( P_\xi \) is a nonempty \( \tau_\xi \)-closed nowhere dense set,
3. if \( S \in \Sigma^0_3(\omega^\omega, \tau) \) is \( \tau_\xi \)-nonmeager in \( P_\xi \), then \( S \) is \( \tau_\xi \)-nonmeager in \( \omega^\omega \),
4. if \( V, W \) are nonempty \( \tau_\xi \)-open subsets of \( \omega^\omega \), then we can find a \( \tau_\xi \)-dense \( G_\delta \) subset \( H \) of \( V \setminus P_\xi \), a \( \tau_\xi \)-dense \( G_\delta \) subset \( L \) of \( W \setminus P_\xi \), and a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism from \( H \) onto \( L \).

The following lemma (a corollary of Lemma 2.7 in [L-Z2]) is a consequence of the previous one. It provides, among other things, a topology \( T_\xi \) that we will use in the sequel.

Lemma 2.3 Let \( 1 \leq \xi < \omega_1 \). Then there is a disjoint countable family \( \mathcal{G}_\xi \) of subsets of \( \omega^\omega \) and a topology \( T_\xi \) on \( \omega^\omega \) such that

1. \( T_\xi \) is zero-dimensional perfect Polish and \( \tau \subseteq T_\xi \subseteq \Sigma^0_3(\tau) \),
2. for any nonempty \( T_\xi \)-open sets \( V, V' \), there are disjoint \( G, G' \in \mathcal{G}_\xi \) with \( G \subseteq V \), \( G' \subseteq V' \), and there is a nice \((T_\xi, T_\xi)\)-homeomorphism from \( G \) onto \( G' \),
3. for every \( G \in \mathcal{G}_\xi \),
4. \( G \) is nonempty, \( T_\xi \)-nowhere dense, and in \( \Pi^0_3(T_\xi) \),
5. if \( S \in \Sigma^0_3(\omega^\omega, \tau) \) is \( T_\xi \)-nonmeager in \( G \), then \( S \) is \( T_\xi \)-nonmeager in \( \omega^\omega \).

The construction of \( \mathcal{G}_\xi \) and \( T_\xi \) ensures that \( T_\xi \) is \((\omega^\omega, \tau)\), where \( \tau_\xi \) is as in Lemma 2.2. This topology is on \((\omega^\omega, \tau)\), identified with \( \omega^\omega \). We will need the following consequence of the construction of \( \mathcal{G}_\xi \) and \( T_\xi \).

Lemma 2.4 Let \( 1 \leq \xi < \omega_1 \), and \( V \) be a nonempty \( T_\xi \)-open set. Then \( \overline{V}^\tau \) is not \( \tau \)-compact.

Proof. The fact that \( T_\xi \) is \((\omega^\omega, \tau)\) gives a finite sequence \( U_0, \ldots, U_n \) of nonempty open subsets of \((\omega^\omega, \tau_\xi)\) with \( U_0 \times \cdots \times U_n \subseteq V \). Thus \( \overline{V}^\tau \) contains the \( \tau \)-closed set \( \overline{U_0 \times \cdots \times U_n} \times (\omega^\omega, \tau) \), and it is enough to see that this last set is not \( \tau \)-compact. This comes from the fact that the Baire space \((\omega^\omega, \tau)\) is not compact. \( \square \)

3 Proof of the main result

Before proving our main result, we give an example giving the flavour of the sequel. In [Za], the author gives a Hurewicz-like test to see when two disjoint subsets \( A, B \) of a product \( Y \times Z \) of Polish spaces can be separated by an open rectangle. We set 

\[ A_0 := \{ (n^\infty, m^\infty) \mid n, m \in \omega \}, \]

and 

\[ B_1 := \{ (m^\infty, (n+1)^{m+1}0^\infty) \mid m, n \in \omega \}. \]

Then \( A \) is not separable from \( B \) by an open rectangle exactly when there are \( \varepsilon \in 2 \) and continuous maps \( g : \omega^\omega \to Y, h : \omega^\omega \to Z \) such that 

\[ \varepsilon \subseteq (g \times h)^{-1}(A) \text{ and } \overline{\varepsilon} \subseteq (g \times h)^{-1}(B). \]

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Example. Here we are looking for closed graphs with Borel chromatic number two and of arbitrarily high finite $\Delta_0^\xi$ chromatic number $n$. There is an example with $\xi = 1$ and $n = 3$ where $\Bbb 0$ is involved. We set $C := \{(2m)^\infty, (2m+1)^\infty \mid m \in \omega\} \cup \Bbb 0$, 

$$D := \{(2m)^\infty \mid m \in \omega\} \cup \{(n+1)^\infty \mid m, n \in \omega\},$$

and 

$$R := \{(2m+1)^\infty \mid m \in \omega\} \cup \{(m+1)^{n+1}0^\infty \mid m, n \in \omega\},$$

for each $\xi$ and $n$. We split $\{0\} \times G$ into two disjoint subfamilies $G_0^\xi$ and $G_1^\xi$ having the property (b) in Lemma 2.3. This is possible since the elements of $G_0$ are $T_\xi$-nowhere dense. Let $G_0, G_1 \in G_0^\xi$ be disjoint, and $\varphi$ be a nice $(T_\xi, T_\xi)$-homeomorphism from $G_0$ onto $G_1$. We then set $f(0, \alpha) := (0, \varphi(\alpha))$ if $\alpha \in G_0$, and $F := \{(n) \times G \mid n \in \omega \land G \in G_1^\xi\}$. It remains to check that the property (c) is satisfied. We argue by contradiction, which gives $\Delta_\omega^0 \in \Delta_\xi^0$. By property (d) in Lemma 2.3, $\Delta_\omega^0 \cap \{(0) \times G_0\}$ is $(\tau_\omega \times T_\xi)$-comeager in $\{0\} \times G_0$ for each $\varepsilon \in 2$. As $f$ is a $(\tau_\omega \times T_\xi, \tau_\omega \times T_\xi)$-homeomorphism, $\Delta_\omega^0 \cap \{(0) \times G_0\} \cap f^{-1}(\Delta_\omega^0 \cap \{(0) \times G_1\})$ is $(\tau_\omega \times T_\xi)$-comeager in $\{0\} \times G_0$, which contradicts the fact that $\text{Gr}(f) \cap \Delta_\omega^0 = \emptyset$. 

We now turn to the general case. Our main lemma is as follows. We equip $\omega^m$ with the discrete topology $\tau_d$, for each $m > 0$.

**Lemma** Let $\xi \geq 1$ be a countable ordinal, $n \geq 1$ be a natural number, and $X := \omega \times \omega^\xi$. Then we can find a partial injection $f : X \to X$ and a disjoint countable family $F$ of subsets of $X$ such that

(a) $f$ has disjoint domain and range,

(b) $\text{Gr}(f)$ is $((\tau_d \times \tau) \times (\tau_d \times \tau))$-closed,

(c) there is no sequence $(\Delta_i)_{i < n}$ of $\Delta_0^\xi$ subsets of $(\omega^m, \tau_d \times \tau)$ such that

(i) $\forall i < n \: \text{Gr}(f) \cap \Delta_i^2 = \emptyset$,

(ii) $\bigcup_{i < n} \Delta_i$ is $(\tau_d \times T_\xi)$-comeager in $X$,

(d) $F$ has the properties (b)-(d) in Lemma 2.3, where $G_\xi$, $\omega^\omega$, $T_\xi$ and $\tau$ are respectively replaced with $F$, $X$, $\tau_d \times T_\xi$ and $\tau_d \times \tau$.

(e) $(\bigcup F) \cap (\text{Domain}(f) \cup \text{Range}(f)) = \emptyset$.

**Proof.** We argue by induction on $n$.

**The basic case** $n = 1$

Let $G_\xi$ be the family given by Lemma 2.3. We split $G_\xi$ into two disjoint subfamilies $G_0^\xi$ and $G_1^\xi$ having the property (b) in Lemma 2.3. This is possible since the elements of $G_\xi$ are $T_\xi$-nowhere dense. Let $G_0, G_1 \in G_0^\xi$ be disjoint, and $\varphi$ be a nice $(T_\xi, T_\xi)$-homeomorphism from $G_0$ onto $G_1$. We then set $f(0, \alpha) := (0, \varphi(\alpha))$ if $\alpha \in G_0$, and $F := \{(n) \times G \mid n \in \omega \land G \in G_1^\xi\}$. It remains to check that the property (c) is satisfied. We argue by contradiction, which gives $\Delta_\omega^0 \in \Delta_\xi^0$. By property (d) in Lemma 2.3, $\Delta_\omega^0 \cap \{(0) \times G_0\}$ is $(\tau_\omega \times T_\xi)$-comeager in $\{0\} \times G_0$ for each $\varepsilon \in 2$. As $f$ is a $(\tau_\omega \times T_\xi, \tau_\omega \times T_\xi)$-homeomorphism, $\Delta_\omega^0 \cap \{(0) \times G_0\} \cap f^{-1}(\Delta_\omega^0 \cap \{(0) \times G_1\})$ is $(\tau_\omega \times T_\xi)$-comeager in $\{0\} \times G_0$, which contradicts the fact that $\text{Gr}(f) \cap \Delta_\omega^0 = \emptyset$. 

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We want these objects to satisfy the following:

- \( b \) \( \text{ Let } \{ \) pairwise disjoint \( \} \) and \( F \) \( \text{ We now define the desired partial map } \) \( p \) \( \text{ The induction step from } n \) \( \text{ Let } (V_p) \) \( \text{ be a basis for the topology } \tau_d \times T_\xi \text{ made of nonempty sets. Fix } p \in \omega. \) By Lemma 2.4, there is a countable family \( (W_q^p)_{q \in \omega}, \) with \( (\tau_d \times \tau) \)-closed union, and made of pairwise disjoint \( (\tau_d \times \tau) \)-clopen subsets of \( X \) intersecting \( V_p. \)

- \( b : \omega \to \omega^2 \) be a bijection. We construct, for \( \bar{v} = (p, q) \in \omega^2 \) and \( \varepsilon \in 2, \) and by induction on \( b^{-1}(\bar{v}), \)
  \( - G^p_\varepsilon \in F^0, \)
  \( - a \text{ nice } (\tau_d \times T_\xi, \tau_d \times T_\xi) \text{-homeomorphism } \varphi^p_\varepsilon : G^p_0 \to G^p_1. \)

We want these objects to satisfy the following:

- \( G^p_0 \subseteq (V_p \cap W_q^p) \setminus (\bigcup_{m < b^{-1}(\bar{v})} G^0_m \cup G^q_1 \tau_d \times T_\xi), \)
  \( - G^p_1 \subseteq V_q \setminus (G^p_0 \cup \bigcup_{m < b^{-1}(\bar{v})} G^0_m \cup G^q_1 \tau_d \times T_\xi). \)

- \( \text{ We now define the desired partial map } \tilde{f} : \omega \times \omega \to \omega \times \omega \omega, \) as well as \( \tilde{F} \subseteq 2^{\omega \times \omega \times \omega^\omega}, \) as follows:
  \[ \tilde{f}(l, x) := \begin{cases} (p + 1, \varphi^{p,q}(x)) & \text{if } l = 0 \land x \in G^{p,q}_0, \\ (l, f(x)) & \text{if } l > 0 \land x \in \text{Domain}(f). \end{cases} \]
  and \( \tilde{F} := \{ (l) \times G \mid l \in \omega \land G \in F^1 \}. \) Note that \( \tilde{f} \) is well-defined and injective, by disjointness of the \( (G^p_0 \cup G^q_1) \)'s. Identifying \( X \) with \( \omega \times \omega \times \omega^\omega, \) we can consider \( \tilde{f} \) as a partial map from \( X \) into itself and \( \tilde{F} \) as a family of subsets of \( X \) (this identification is based on the identification of \( \omega \) with \( \omega \times \omega \)).

(a), (d) and (e) are clearly satisfied.

(b) Assume that \( (l_k, x_k), (m_k, y_k) \in \text{Gr}(\tilde{f}) \) tends to \( (l, x), (m, y) \in (\omega \times \omega)^2 \) as \( k \) goes to infinity. We may assume that \( (l_k) \) and \( (m_k) \) are constant.

If \( l = 0, \) then there is \( p \) such that \( p + 1 = m \) and \( (x_k, y_k) \in G^{p,q}_0 \times G^{p,q}_1. \) As \( G^{p,q}_0 \subseteq W_q^p, \) we may also assume that \( (q_k) \) is also constant and equals \( q. \) As \( \varphi^{p,q} \) is nice, \( (l_k, x_k, y_k) \in \text{Gr}(\tilde{f}). \)

If \( l > 0, \) then \( (x_k, y_k) \in \text{Gr}(f). \) As \( \text{Gr}(f) \) is \( \{ (\tau_d \times \tau) \times (\tau_d \times \tau) \} \)-closed, \( (l, x), (m, y) \) \( \in \text{Gr}(\tilde{f}). \)

(c) We argue by contradiction, which gives \( (\Delta_n)_{i \leq n}. \) We may assume, without loss of generality, that \( \{0\} \cap \Delta_n \) is not meager in \( \{0\} \times \omega \times \omega^\omega, \) \( \tau_d \times T_\xi \). This gives \( p \in \omega \) such that \( \{0\} \cup V_p \cap \Delta_n \) is \( (\tau_d \times T_\xi) \)-comeager in \( V_p := \{0\} \times V_p. \) As \( V_p \setminus \Delta_n \in \Sigma^{\omega^\omega}_{\tau_d \times \tau}, \) \( \{0\} \times G^p_0 \cap \Delta_n \) is \( (\tau_d \times T_\xi) \)-comeager in \( \{0\} \times G^p_0 \) for each \( q \in \omega. \)
As $\text{Gr}(\bar{f}) \cap \Delta_n^2 = \emptyset$ and the $\varphi_i$'s are $(\tau_d \times T_\xi, \tau_d \times T_\xi)$-homeomorphisms, $\{(p + 1) \times G^{p,q}_1 \cap \Delta_n$ is $(\tau_d \times T_\xi)$-meager in $\{(p + 1) \times G^{p,q}_1$, for each $q$.

As $(\omega \times \omega \times \omega^\omega) \setminus (\bigcup_{i \leq n} \Delta_i)$ is $(\tau_d \times T_\xi)$-meager in $\omega \times \omega \times \omega^\omega$ and $\Delta^0_\xi(\tau_d \times \tau)$,
\[(\{p + 1\} \times G^{p,q}_1) \setminus (\bigcup_{i \leq n} \Delta_i)\]
is $(\tau_d \times T_\xi)$-meager in $\{p + 1\} \times G^{p,q}_1$, for each $q$. Thus $(\{p + 1\} \times G^{p,q}_1 \cap (\bigcup_{i < n} \Delta_i)$ is $(\tau_d \times T_\xi)$-comeager in $\{p + 1\} \times G^{p,q}_1$, for each $q$.

**Claim** The set $(\{p + 1\} \times \omega \times \omega^\omega) \cap (\bigcup_{i < n} \Delta_i)$ is $(\tau_d \times T_\xi)$-comeager in $\{p + 1\} \times \omega \times \omega^\omega$.

Indeed, we argue by contradiction. This gives $W \in (\tau_d \times T_\xi) \setminus \emptyset$ such that
\[(\{p + 1\} \times W) \cap (\bigcup_{i < n} \Delta_i)\]
is $(\tau_d \times T_\xi)$-meager in $W' := \{p + 1\} \times W$. Let $q \in \omega$ be such that $V_q \subseteq W$. Then $G^{p,q}_1 \subseteq W$ and $\{p + 1\} \times G^{p,q}_1 \subseteq W'$. As $W' \cap (\bigcup_{i < n} \Delta_i) \in \Sigma^0_\xi(\tau_d \times \tau)$ and $(\{p + 1\} \times G^{p,q}_1) \cap W' \cap (\bigcup_{i < n} \Delta_i)$ is $(\tau_d \times T_\xi)$-comeager in $\{p + 1\} \times G^{p,q}_1$, $W' \cap (\bigcup_{i < n} \Delta_i)$ is not $(\tau_d \times T_\xi)$-meager in $W'$, which is absurd. 

Now we set $\Delta'_i := (\{p + 1\} \times \omega \times \omega^\omega) \cap \Delta_i$ if $i < n$. Note that $\Delta'_i \in \Delta^0_\xi(\{p + 1\} \times \omega \times \omega^\omega, \tau_d \times \tau)$, $\text{Gr}(\bar{f}) \cap (\Delta'_i)^2 = \emptyset$, and $\bigcup_{i < n} \Delta'_i$ is $(\tau_d \times T_\xi)$-comeager in $\{p + 1\} \times \omega \times \omega^\omega$, which contradicts the induction assumption. 

In order to get our main result, it is enough to apply the main lemma to each $n \geq 1$. This gives $f_n : \omega \times \omega^\omega \to \omega \times \omega^\omega$. It remains to define $f : \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega) \to \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega)$ by $f(n, x) := f_n(x)$ (we identify $(\omega \setminus \{0\}) \times \omega \times \omega^\omega$ with $\omega^\omega$).

### 4 References


[Za] R. Zamora, Separation of analytic sets by rectangles of low complexity, *manuscript (see arXiv)*