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Borel chromatic number of closed graphs

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Abstract. We construct, for each countable ordinal $\xi$, a closed graph with Borel chromatic number two and Baire class $\xi$ chromatic number $\aleph_0$.

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1 Introduction

The study of the Borel chromatic number of analytic graphs on Polish spaces was initiated in [K-S-T]. In particular, the authors prove in this paper that the Borel chromatic number of the graph generated by a partial Borel function has to be in \( \{1, 2, 3, \aleph_0\} \). They also provide a minimum graph \( G_0 \) of uncountable Borel chromatic number. This last result had a lot of developments. For example, B. Miller gave in [Mi] some other versions of it, which helped him to generalize a number of known dichotomy theorems in descriptive set theory. The first author generalized in [L2] the \( G_0 \)-dichotomy to any dimension making sense in classical descriptive set theory, and also used versions of \( G_0 \) to study the non-potentially closed subsets of a product of two Polish spaces (see [L1]).

A study of the \( \Delta^0_\xi \) chromatic number of analytic graphs on Polish spaces was initiated in [L-Z1] and was motivated by the \( G_0 \)-dichotomy. More precisely, let \( B \) be a Borel binary relation, on a Polish space \( X \), having a Borel countable coloring (i.e., a Borel map \( c : X \to \omega \) such that \( c(x) \neq c(y) \) if \((x, y) \in B\)). Is there a relation between the Borel class of \( B \) and that of the coloring? In other words, is there a map \( k : \omega_1 \setminus \{0\} \to \omega_1 \setminus \{0\} \) such that any \( \Pi^0_\xi \) binary relation having a Borel countable coloring has in fact a \( \Delta^0_\xi \)-measurable countable coloring, for each \( \xi \in \omega_1 \setminus \{0\} \)?

In [L-Z2], the authors give a negative answer: for each countable ordinal \( \xi \geq 1 \), there is a partial injection with disjoint domain and range \( i : \omega^{\omega} \to \omega^{\omega} \), whose graph
- is \( D_2(\Pi^0_\xi) \) (i.e., the difference of two closed sets),
- has Borel chromatic number two,
- has no \( \Delta^0_\xi \)-measurable countable coloring.

On the other hand, they note that an open binary relation having a finite coloring \( c \) has also a \( \Delta^0_\xi \)-measurable finite coloring (consider the differences of the \( c^{-1}(\{n\}) \)'s, for \( n \) in the range of the coloring). Note that an irreflexive closed binary relation on a zero-dimensional space has a continuous countable coloring (this coloring is \( \Delta^0_2 \)-measurable in non zero-dimensional spaces). So they wonder whether we can build, for each countable ordinal \( \xi \geq 1 \), a closed binary relation with a Borel finite coloring but no \( \Delta^0_\xi \)-measurable finite coloring. This is indeed the case:

**Theorem** Let \( \xi \geq 1 \) be a countable ordinal. Then there exists a partial injection with disjoint domain and range \( f : \omega^{\omega} \to \omega^{\omega} \) whose graph is closed (and thus has Borel chromatic number two), and has no \( \Delta^0_\xi \)-measurable finite coloring (and thus has \( \Delta^0_\xi \) chromatic number \( \aleph_0 \)).

The previous discussion shows that this result is optimal. Its proof uses, among other things, the method used in [L-Z2] improving Theorem 4 in [M]. This method relates topological complexity and Baire category.

2 Mátrai sets

Before proving our main result, we recall some material from [L-Z2].

**Notation.** The symbol \( \tau \) denotes the usual product topology on the Baire space \( \omega^{\omega} \).
Definition 2.1 We say that a partial map \( f : \omega^\omega \to \omega^\omega \) is nice if its graph \( \text{Gr}(f) \) is a \((\tau \times \tau)\)-closed subset of \( \omega^\omega \times \omega^\omega \).

The construction of \( P_\xi \) and \( \tau_\xi \), and the verification of the properties (1)-(3) from the next lemma (a corollary of Lemma 2.6 in [L-Z2]), can be found in [M], up to minor modifications.

Lemma 2.2 Let \( 1 \leq \xi < \omega_1 \). Then there are \( P_\xi \subseteq \omega^\omega \), and a topology \( \tau_\xi \) on \( \omega^\omega \) such that

1. \( \tau_\xi \) is zero-dimensional perfect Polish and \( \tau \subseteq \tau_\xi \subseteq \Sigma^0_3(\tau) \).
2. \( P_\xi \) is a nonempty \( \tau_\xi \)-closed nowhere dense set,
3. if \( S \in \Sigma^0_3(\omega^\omega, \tau) \) is \( \tau_\xi \)-nonmeager in \( P_\xi \), then \( S \) is \( \tau_\xi \)-nonmeager in \( \omega^\omega \),
4. if \( V, W \) are nonempty \( \tau_\xi \)-open subsets of \( \omega^\omega \), then we can find a \( \tau_\xi \)-dense \( G_\delta \) subset \( H \) of \( V \setminus P_\xi \), a \( \tau_\xi \)-dense \( G_\delta \) subset \( L \) of \( W \setminus P_\xi \), and a nice \((\tau_\xi, \tau_\xi)\)-homeomorphism from \( H \) onto \( L \).

The following lemma (a corollary of Lemma 2.7 in [L-Z2]) is a consequence of the previous one. It provides, among other things, a topology \( T_\xi \) that we will use in the sequel.

Lemma 2.3 Let \( 1 \leq \xi < \omega_1 \). Then there is a disjoint countable family \( \mathcal{G}_\xi \) of subsets of \( \omega^\omega \) and a topology \( T_\xi \) on \( \omega^\omega \) such that

a. \( T_\xi \) is zero-dimensional perfect Polish and \( \tau \subseteq T_\xi \subseteq \Sigma^0_3(\tau) \).

b. for any nonempty \( T_\xi \)-open sets \( V, V' \), there are disjoint \( G, G' \in \mathcal{G}_\xi \) with \( G \subseteq V \) and \( G' \subseteq V' \), and there is a nice \((T_\xi, T_\xi)\)-homeomorphism from \( G \) onto \( G' \),

and, for every \( G \in \mathcal{G}_\xi \),

c. \( G \) is nonempty, \( T_\xi \)-nowhere dense, and in \( \Pi^0_3(T_\xi) \),

d. if \( S \in \Sigma^0_3(\omega^\omega, \tau) \) is \( T_\xi \)-nonmeager in \( G \), then \( S \) is \( T_\xi \)-nonmeager in \( \omega^\omega \).

The construction of \( \mathcal{G}_\xi \) and \( T_\xi \) ensures that \( T_\xi \) is \((\omega^\omega)^\omega \), where \( \tau_\xi \) is as in Lemma 2.2. This topology is on \((\omega^\omega)^\omega \), identified with \( \omega^\omega \). We will need the following consequence of the construction of \( \mathcal{G}_\xi \) and \( T_\xi \).

Lemma 2.4 Let \( 1 \leq \xi < \omega_1 \), and \( V \) be a nonempty \( T_\xi \)-open set. Then \( \overline{\tau \setminus V} \) is not \( \tau \)-compact.

Proof. The fact that \( T_\xi \) is \((\omega^\omega)^\omega \) gives a finite sequence \( U_0, \ldots, U_n \) of nonempty open subsets of \((\omega^\omega, \tau_\xi) \) with \( U_0 \times \cdots \times U_n \subseteq V \). Thus \( \overline{\tau \setminus V} \) contains the \( \tau \)-closed set \( \overline{U_0} \times \cdots \times \overline{U_n} \times (\omega^\omega)^\omega \), and it is enough to see that this last set is not \( \tau \)-compact. This comes from the fact that the Baire space \((\omega^\omega, \tau) \) is not compact. \( \square \)

3 Proof of the main result

Before proving our main result, we give an example giving the flavour of the sequel. In [Za], the author gives a Hurewicz-like test to see when two disjoint subsets \( A, B \) of a product \( Y \times Z \) of Polish spaces can be separated by an open rectangle. We set \( A := \{(n^\infty, n^\infty) \mid n \in \omega \} \), \( B_0 := \{(0^{m+1}(n+1)^\infty, (m+1)^{n+1}0^{\infty}) \mid m, n \in \omega \} \), and \( B_1 := \{(m+1)^{n+1}0^{\infty}, 0^{m+1}(n+1)^\infty} \mid m, n \in \omega \} \). Then \( A \) is not separable from \( B \) by an open rectangle exactly when there are \( \varepsilon \in 2 \) and continuous maps \( g : \omega^\omega \to Y \), \( h : \omega^\omega \to Z \) such that \( A \subseteq (g \times h)^{-1}(A) \) and \( B_\varepsilon \subseteq (g \times h)^{-1}(B) \).
Example. Here we are looking for closed graphs with Borel chromatic number two and of arbitrarily high finite $\Delta_0^0$ chromatic number $n$. There is an example with $\xi = 1$ and $n = 3$ where $\mathbb{B}_0$ is involved. We set $C := \{(2m)^\omega \mid m \in \omega\} \cup \mathbb{B}_0,$

$$D := \{(2m)^\omega \mid m \in \omega\} \cup \{0^{m+1}(n+1)^\omega \mid m, n \in \omega\},$$

$$R := \{(2m+1)^\omega \mid m \in \omega\} \cup \{(m+1)^{n+1}0^\omega \mid m, n \in \omega\},$$

$$f ((2m)^\omega) := (2m+1)^\omega \text{ and } f (0^{m+1}(n+1)^\omega) := (m+1)^{n+1}0^\omega.$$ This defines $f : D \to R$ whose graph is $C$. The first part of $C$ is discrete, and thus closed. Assume that $(\alpha_k, \beta_k) := (0^{m+k}(n_k+1)^\omega, (m_k+1)^{n_k+1}0^\omega) \in \mathbb{B}_0$ and converges to $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ as $k$ goes to infinity. We may assume that $(m_k)$ is constant, and $(n_k)$ too, so that $(\alpha, \beta) \in \mathbb{B}_0$, which is therefore closed. This shows that $C$ is closed. Note that $D, R$ are disjoint and Borel, so that $C$ has Borel chromatic number two. Let $\Delta$ be a clopen subset of $\omega^\omega$. Let us prove that $C \cap \Delta^2$ or $C \cap (-\Delta)^2$ is not empty. We argue by contradiction. Then $\Delta$ or $-\Delta$ has to contain $0^\omega$. Assume that it is $\Delta$, the other case being similar. Then $0^{m+1}(n+1)^\omega \in \Delta$ if $m$ is big enough. Thus $(m+1)^{n+1}0^\omega \notin \Delta$ if $m$ is big enough. Therefore $(m+1)^\omega \notin \Delta$ if $m$ is big enough. Thus $((2m)^\omega, (2m+1)^\omega) \in C \cap (-\Delta)^2$ if $m$ is big enough, which is absurd.

We now turn to the general case. Our main lemma is as follows. We equip $\omega^m$ with the discrete topology $\tau_d$, for each $m > 0$.

**Lemma** Let $\xi \geq 1$ be a countable ordinal, $n \geq 1$ be a natural number, and $X := \omega \times \omega^\omega$. Then we can find a partial injection $f : X \to X$ and a disjoint countable family $\mathcal{F}$ of subsets of $X$ such that

(a) $f$ has disjoint domain and range,

(b) $\text{Gr}(f)$ is $((\tau_d \times \tau) \times (\tau_d \times \tau))$-closed,

(c) there is no sequence $(\Delta_i)_{i < n}$ of $\Delta_0^0$ subsets of $(X, \tau_d \times \tau)$ such that

(i) $\forall i < n \text{ } \text{Gr}(f) \cap \Delta_i^2 = \emptyset$,

(ii) $\bigcup_{i < n} \Delta_i$ is $(\tau_d \times T_\xi)$-comeager in $X$,

(d) $\mathcal{F}$ has the properties (b)-(d) in Lemma 2.3, where $\mathcal{G}_\xi$, $\omega^\omega$, $T_\xi$ and $\tau$ are respectively replaced with $\mathcal{F}$, $X$, $\tau_d \times T_\xi$ and $\tau_d \times \tau$,

(e) $\big((\bigcup \mathcal{F}) \cap (\text{Domain}(f) \cup \text{Range}(f)) = \emptyset$.

**Proof.** We argue by induction on $n$.

**The basic case** $n = 1$

Let $\mathcal{G}_\xi$ be the family given by Lemma 2.3. We split $\mathcal{G}_\xi$ into two disjoint subfamilies $\mathcal{G}_{\xi}^0$ and $\mathcal{G}_{\xi}^1$ having the property (b) in Lemma 2.3. This is possible since the elements of $\mathcal{G}_\xi$ are $T_\xi$-nowhere dense. Let $G_0, G_1 \in \mathcal{G}_{\xi}^0$ be disjoint, and $\varphi$ be a nice $(T_\xi, T_\xi)$-homeomorphism from $G_0$ onto $G_1$. We then set $f (0, \alpha) := (0, \varphi(\alpha))$ if $\alpha \in G_0$, and $\mathcal{F} := \{\{n\} \times G \mid n \in \omega \land G \in \mathcal{G}_{\xi}^1\}$. It remains to check that the property (c) is satisfied. We argue by contradiction, which gives $\Delta_0 \in \Delta_0^0$. By property (d) in Lemma 2.3, $\Delta_0 \cap (\{0\} \times G_0)$ is $(\tau_d \times T_\xi)$-comeager in $\{0\} \times G_0$ for each $\varepsilon \in 2$. As $f$ is a $(\tau_d \times T_\xi, \tau_d \times T_\xi)$-homeomorphism, $\Delta_0 \cap (\{0\} \times G_0) \cap f^{-1}(\Delta_0 \cap (\{0\} \times G_1))$ is $(\tau_d \times T_\xi)$-comeager in $\{0\} \times G_0$, which contradicts the fact that $\text{Gr}(f) \cap \Delta_0^0 = \emptyset$. 4
We want these objects to satisfy the following:

- \(\varphi^\vartheta : G_0^\vartheta \to G_1^\vartheta\).

We may assume that \(\vartheta \) and \(\varphi^\vartheta\) are respectively replaced with \((\vartheta, \varphi^\vartheta)\)’s. Identifying \(\vartheta \) with \(\varphi^\vartheta\)’s, we can consider \(\varphi^\vartheta\) as a partial map from \(X\) into itself and \(\varphi\) as a family of subsets of \(X\) (this identification is based on the identification of \(\omega\) with \(\omega \times \omega\)).

(a), (d) and (e) are clearly satisfied.

(b) Assume that \([(l_k, x_k), (m_k, y_k)] \in \text{Gr}(\tilde{f})\) tends to \([(l, x), (m, y)] \in (\omega \times X)^2\) as \(k\) goes to infinity. We may assume that \((l_k)\) and \((m_k)\) are constant.

If \(l = 0\), then there is \(p\) such that \(p+1 = m\) and \((x_k, y_k) \in G_0^{\varphi^\vartheta} \times G_1^{\varphi^\vartheta}\). As \(G_0^{\varphi^\vartheta} \subseteq W_q^p\), we may also assume that \((q_k)\) is also constant and equals \(q\). As \(\varphi^\vartheta\) is nice, \([(l, x), (m, y)] \in \text{Gr}(\tilde{f})\).

If \(l > 0\), then \((x_k, y_k) \in \text{Gr}(\tilde{f})\). As \(\text{Gr}(\tilde{f})\) is \([(\tau_d \times \tau) \times (\tau_d \times \tau)]\)-closed, \([(l, x), (m, y)] \in \text{Gr}(\tilde{f})\).

(c) We argue by contradiction, which gives \((\Delta_n)_{i \leq n}\). We may assume, without loss of generality, that \([(\Delta_n)_{i \leq n}] \cap \Delta_n\) is not meager in \([(\Delta_n)_{i \leq n}]\cap(\tau_d \times \tau)\). This gives \(p \in \omega\) such that \([(\Delta_n)_{i \leq n}] \cap \Delta_n\) is \((\tau_d \times \tau)\)-comeager in \(V_0^p\) for each \(q \in \omega\).
As $\text{Gr}(\bar{f}) \cap \Delta^2_n = \emptyset$ and the $\varphi^i_s$ are $(\tau_d \times T_\xi, \tau_d \times T_\xi)$-homeomorphisms, \((p+1) \times G_1^{p,q}) \cap \Delta_n$$ is \((\tau_d \times T_\xi)\)-meager in \((p+1) \times G_1^{p,q}\), for each $q$.

As \((\omega \times \omega \times \omega^\omega) \setminus (\bigcup_{i \leq n} \Delta_i)$$ is \((\tau_d \times T_\xi)\)-meager in \(\omega \times \omega \times \omega^\omega$$ and \(\Delta^0_\xi(\tau_d \times \tau)$$, $(p+1) \times G_1^{p,q}) \setminus (\bigcup_{i \leq n} \Delta_i)$ is $(\tau_d \times T_\xi)$-meager in $(p+1) \times G_1^{p,q}$, for each $q$. Thus \((p+1) \times G_1^{p,q}) \cap (\bigcup_{i \leq n} \Delta_i)$$ is $(\tau_d \times T_\xi)$-comeager in $(p+1) \times G_1^{p,q}$, for each $q$.

Claim The set $(p+1) \times \omega \times \omega^\omega \setminus (\bigcup_{i \leq n} \Delta_i)$ is $(\tau_d \times T_\xi)$-comeager in $(p+1) \times \omega \times \omega^\omega$.

Indeed, we argue by contradiction. This gives $W \in (\tau_d \times T_\xi) \setminus \{\emptyset\}$ such that $(p+1) \times W \cap (\bigcup_{i \leq n} \Delta_i)$ is $(\tau_d \times T_\xi)$-meager in $(p+1) \times W$. Let $q \in \omega$ be such that $V_q \subseteq W$. Then $G_1^{p,q} \subseteq W$ and $(p+1) \times G_1^{p,q} \subseteq W'$. As $W' \cap (\bigcup_{i \leq n} \Delta_i)$ is not $(\tau_d \times T_\xi)$-meager in $W'$, which is absurd.

Now we set $\Delta'_i := \{(p+1) \times \omega \times \omega^\omega \setminus \{i\} \cup (p+1) \times \omega \times \omega^\omega, \tau_d \times \tau)$$, \text{Gr}(\bar{f}) \cap \Delta'_i^2 = \emptyset$, and $\bigcup_{i \leq n} \Delta'_i$ is $(\tau_d \times T_\xi)$-comeager in $(p+1) \times \omega \times \omega^\omega$, which contradicts the induction assumption.

In order to get our main result, it is enough to apply the main lemma to each $n \geq 1$. This gives $f_n : \omega \times \omega^\omega \to \omega \times \omega^\omega$. It remains to define $f : \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega) \to \bigcup_{n \geq 1} (\{n\} \times \omega \times \omega^\omega)$ by $f(n, x) := f_n(x)$ (we identify $\omega \\setminus \{0\} \times \omega \times \omega^\omega$ with $\omega^\omega$).

4 References


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