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Persistence criteria for populations with non-local dispersion

Henri Berestycki *, Jérôme Coville †, Hoang-Hung Vo ‡

July 20, 2014

Abstract

In this article, we analyse the non-local model :

$$\frac{\partial u}{\partial t} = J \star u - u + f(x,u) \quad \text{with} \quad x \in \mathbb{R}^N,$$

where J is a positive continuous dispersal kernel and $f(x,u)$ is a heterogeneous KPP type non-linearity describing the growth rate of the population. The ecological niche of the population is assumed to be bounded (i.e. outside a compact set, the environment is assumed to be lethal for the population). For compactly supported dispersal kernels J, we derive an optimal persistence criteria. We prove that a positive stationary solution exists if and only if the generalised principal eigenvalue λ_p of the linear problem

$$J \star \varphi(x) - \varphi(x) + \partial_s f(x,0)\varphi(x) + \lambda_p \varphi(x) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

is negative. λ_p is a spectral quantity that we defined in the spirit of the generalised first eigenvalue of an elliptic operator. In addition, for any continuous non-negative initial data that is bounded or integrable, we establish the long time behaviour of the solution $u(t,x)$. We also analyse the impact of the size of the support of the dispersal kernel on the persistence criteria. We exhibit situations where the dispersal strategy has "no impact" on the persistence of the species and other ones where the slowest dispersal strategy is not any more an "Ecological Stable Strategy". We also discuss persistence criteria for fat-tailed kernels.

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1 Introduction

In this article, we are interested in finding persistence criteria for a species that has a long range dispersal strategy. For such a model species, we can think of trees of which seeds and pollens are disseminated on a wide range. The possibility of a long range dispersal is well known in ecology, where numerous data now available support this assumptions [16, 23, 24, 60]. A commonly used model that integrates such long range dispersal is the following *nonlocal reaction diffusion equation* ([40, 43, 48, 52, 63]):

$$\frac{\partial u}{\partial t}(t,x) = J \star u(t,x) - u + f(x, u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N. \tag{1.1}$$

Here $u(t,x)$ is the density of the considered population, J is a dispersal kernel, $f(x,s)$ is a KPP type non-linearity describing the growth rate of the population.

In this setting the tail of the kernel can be thought of as the range of dispersion or as a measure of the frequency at which long dispersal events occur. A biological motivation for the use of (1.2) to describe the evolution of the population comes from the observation that the intrinsic variability in the capacity of the individuals to disperse generates, at the scale of a population, a long range dispersal of the population. The effect of such variability has been investigated in [46, 55] by means of correlated random walks. In such a framework, each individual moves according to classical random walks, however the diffusion coefficients is given by a probability law. It can be checked ([46, 55, 63]) that the density of population will then satisfy an integro-differential equation where the dispersal kernel J describes the probability to jump from one location to another.

Throughout this paper we will always make the following assumptions on the dispersal kernel J.

(H1) $J \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is nonnegative, symmetric and of unit mass (i.e. $\int_{\mathbb{R}^N} J(z)dz = 1$) .

(H2) $J(0) > 0$

In the present paper, we focus our analysis on species that have a bounded ecological niche. A simple way to model such a spatial repartition consists in considering that the environment is hostile to the species

outside a bounded set. For instance, biological populations that are sensitive to temperature thrive only in a limited latitude zone. Thus, if x is the latitude, we get such dependence. This fact is translated in our model by assuming that f satisfies:

(H3) $f \in C^{1,\alpha}(\mathbb{R}^{N+1})$ is of KPP type, that is :

$$\begin{cases} f(\cdot,0) \equiv 0, \\ \text{For all } x \in \mathbb{R}^N, f(x,s)/s \text{ is decreasing with respect to } s \text{ on } (0,+\infty). \\ \text{There exists } S(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ such that } f(x,S(x)) \leq 0 \text{ for all } x \in \mathbb{R}^N. \end{cases}$$

(H4) $\quad \limsup_{|x|\to\infty} \frac{f(x,s)}{s} < 0, \quad$ uniformly in $\quad s \geq 0.$

A typical example of such a nonlinearity is given by $f(x,s) := s(a(x) - b(x)s)$ with $b(x) > 0$ and $a(x)$ satisfies $\limsup_{|x|\to\infty} a(x) < 0$.

Our main purpose is to find conditions on J and f that characterise the persistence of the species modelled by (1.1). In this task, we focus our analysis on the description of the set of positive stationary solutions of (1.1), that is the positive solutions of the equation below

$$J \star u(x) - u(x) + f(x,u(x)) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

Existence of stationary solutions is naturally expected to provide the right persistence criterion. We will see that this is indeed the case.

In the literature, persistence criteria have been well studied for the local reaction diffusion version of (1.1)

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(x,t,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{1.3}$$

where Ω is a domain of $\mathbb{R}^N$, possibly $\mathbb{R}^N$ itself. Persistence criteria have been obtained for various media, ranging from periodic media to ergodic media [5, 6, 10, 17, 18, 54, 56, 61]. In the context of global warming, persistence criteria have been investigated in [10, 4, 11]. For such reaction diffusion equations the persistence criteria are often derived from the sign of the first eigenvalue of the linearised problem at the 0 solution. One is thus led to determine the sign of the first eigenvalue $\lambda_1(\Delta + \partial_s f(x,0), \Omega)$ of the spectral problem

$$\Delta\varphi(x) + \partial_s f(x,0)\phi(x) + \lambda_1\varphi(x) = 0 \quad \text{in} \quad \Omega \tag{1.4}$$

associated with the proper boundary conditions (if $\Omega \neq \mathbb{R}^N$).

In most situations, for KPP– like non-linearities, the existence of a positive stationary solution to (1.3) is indeed uniquely conditioned by the sign of λ_1. More precisely, there exists a unique positive stationary solution if and only if $\lambda_1 < 0$. If such type of criteria seems reasonable for problems defined on bounded set, it is less obvious for problems in unbounded domains. In particular, in unbounded domains, one of the main difficulty concerns the definition of λ_1. As shown in [9, 6, 12], the notion of first eigenvalue in unbounded domain can be quite delicate and several definitions of λ_1 exist rendering the question of sharp persistence criteria quite involved.

Much less is known for the non-local equation (1.2) and, to our knowledge persistence criteria have been essentially investigated in some specific situations such as periodic media : [33, 35, 62] or for a version of the problem (1.2) defined in a bounded domain Ω [2, 29, 31, 41, 51, 62] :

$$\frac{\partial u}{\partial t}(t,x) = \int_\Omega J(x-y)u(t,y)\,dy - u(t,x) + f(x,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega. \tag{1.5}$$

We also quote [8] for an analysis of a persistence criteria in periodic media for a non-local version of (1.3) involving a fractional diffusion and [59] for persistence criteria in time periodic versions of (1.5) . Similarly to the local diffusion case, for KPP like non-linearities, the existence of a positive solution of the non-local equation (1.5) can be characterised by the sign of a spectral quantity λ_p, called the generalised principal eigenvalue of

$$\int_\Omega J(x-y)\phi(y)\,dy - \phi(x) + \partial_s f(x,0)\phi(x) + \lambda\phi(x) = 0 \quad \text{in} \quad \Omega. \tag{1.6}$$

In the spirit of [7], this generalised principal eigenvalue λ_p is defined by :

$$\lambda_p := \sup\left\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\Omega), \varphi > 0, \text{ such that } \mathcal{L}_\Omega[\varphi](x) - \varphi(x) + \partial_s f(x,0)\varphi(x) + \lambda\varphi(x) \le 0 \quad \text{in} \quad \Omega.\right\},$$

where $\mathcal{L}_\Omega[\varphi](x)$ denotes

$$\mathcal{L}_\Omega[\varphi](x) := \int_\Omega J(x-y)\varphi(y)\,dy.$$

Unlike the elliptic PDE case, due to the lack of a regularising effect of the diffusion operator, the above spectral problem may not have a solution in spaces of functions like $L^p(\Omega), C(\Omega)$[33, 32, 51]. As a consequence, even in a bounded domain, simple sharp persistence criteria are quite delicate. Another difficulty inherent to the study of nonlocal equations (1.6) in unbounded domain concerns the lack of natural *a priori* estimates for the solution thus making standard approximations difficult to use in most cases.

1.1 Main Results:

Let us now state our main results. In the first one we establish a simple sharp persistence criteria assuming that the dispersal kernel J has compact support.

Theorem 1.1. *Assume that J, f satisfy (H1-H4) and assume further that J is compactly supported. Then, there exists a positive solution, $\tilde{u}$, of (1.2) if and only if $\lambda_p(\mathcal{M} + \partial_s f(x,0)) < 0$, where $\mathcal{M}$ denotes the continuous operator $\mathcal{M}[\varphi] = J \star \varphi(x) - \varphi(x)$ and*

$$\lambda_p(\mathcal{M} + \partial_s f(x,0)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\mathbb{R}^N), \phi > 0 \text{ so that } \mathcal{M}[\varphi] + \partial_s f(x,0)\varphi + \lambda\varphi \le 0\}.$$

When it exits, the solution is unique, that is, if v is another bounded solution, then $\tilde{u} = v$ almost everywhere. Moreover, for any non-negative initial data $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ we have the following asymptotic behaviour:

- *If $\lambda_p(\mathcal{M} + \partial_s f(x,0)) \ge 0$, then the solution satisfies $\|u(t)\|_\infty \to 0$ as $t \to \infty$,*
- *If $\lambda_p(\mathcal{M} + \partial_s f(x,0)) < 0$, then the solution satisfies $\|(u - \tilde{u})(t)\|_\infty \to 0$ as $t \to \infty$.*

In addition, if the initial data is such that $u_0 \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, then the convergence $u(t,x) \to \tilde{u}$ holds in $L^1(\mathbb{R}^N)$.

We observe that the stationary solution $\tilde{u}$ may not necessarily be continuous. For some homogeneous problems $f(x,u) = f(u)$, it is known that discontinuous solution may exists [34]. If f satisfies the stronger hypothesis that, for any x, $f(x,u)$ is concave with respect to u, then actually the solution $\tilde{u}$ is continuous.

To see this, it suffices to notice that $J \star \tilde{u} > 0$ in $\mathbb{R}^N$. The concavity of f with respect to u implies that for any x the map $u \mapsto u - f(x,u)$ is strictly increasing whenever $u - f(x,u) > 0$. Then from the continuity of $J \star \tilde{u}$ and (1.2), which can be rewritten as in the form $J \star \tilde{u} = \tilde{u} - f(x, \tilde{u})$, we deduce that $\tilde{u}$ is continuous.

Next, we aim at understanding the effect of the dispersal kernel on the persistence of the species, more precisely, of its range and scaling. To this end, we analyse the behaviour of the persistence criteria under some scaling of the dispersal operator. More precisely, let $J_\varepsilon := \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ and let $\mathcal{M}_\varepsilon$ denotes the operator $\mathcal{M}$ with the rescaled kernel, that is, $\mathcal{M}_\varepsilon[\varphi] := J_\varepsilon \star \varphi - \varphi$. We are interested in the behaviour of the solution to (1.2) as $\varepsilon \to 0$ or $\varepsilon \to +\infty$ where the dispersal operator $\mathcal{M}$ is replace by $\gamma(\varepsilon, m)\mathcal{M}_\varepsilon$, with $\gamma(\varepsilon, m) \sim \frac{\alpha_0}{\varepsilon^m}$.

These asymptotics represent two possible strategies that are observed in nature. The dispersal kernel $\gamma(\varepsilon, m) J_\varepsilon$ arises when the dispersal of the species is conditioned by *a dispersal budget* as defined in [48]. Roughly speaking, for a fixed cost, this budget is a way to measure the differences between different range strategies. For a given cost function of the order of $|y|^m$, the term $\gamma(\varepsilon, m)$ behaves like $\frac{\alpha_0}{\varepsilon^m}$ and in the analysis, the dispersal operator is then given by $\gamma(\varepsilon, m)\mathcal{M}_\varepsilon$. As explained in [48], the limit as $\varepsilon \to 0$ can be associated to a strategy of producing many offspring but with little capacity of movement. On the other extreme of the spectrum, the limit $\varepsilon \to +\infty$ corresponds to a strategy that aims at maximizing the exploration of the environment at the expense of the number of offspring produced. We will discuss more precisely this notion of *dispersal budget* in Section 2, where we also interpret our findings in this context. For simplicity, we introduce the notation $\mathcal{M}_{\varepsilon,m}$ to denote the operator $\gamma(\varepsilon, m)\mathcal{M}_\varepsilon$.

In the present paper, we analyse the cases $0 \le m \le 2$ and $\alpha_0 = 1$. The study of the case $m = 0$ corresponds to understanding the impact of the mean distance by itself on the persistence criteria. To simplify the presentation of these asymptotics, we restrict our discussion to nonlinearities $f(x,s)$ of the form

$$f(x,s) = s(a(x) - s).$$

However, most of the proofs apply to a more general nonlinearity $f(x,s)$, and with $a(x) = \partial_s f(x,0)$.

Our first result deals with the case $m = 0$.

Theorem 1.2. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $m = 0$. Then there exists $\varepsilon_0 \in (0, +\infty]$ so that for all $\varepsilon < \varepsilon_0$ there exists a positive solution u_ε to* (1.2)*. Moreover, at this value ε_0, we have*

$$\lim_{\varepsilon \to \varepsilon_0} u_\varepsilon(x) = (a(x) - 1)^+,$$

where s^+ denotes the positive part of s (i.e. $s^+ = sup\{0, s\}$). Assuming further that a is smooth, Lipschitz continuous, we have

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = v(x) \quad \textit{almot everywhere}$$

where v is a non-negative bounded solution of

$$v(x)(a(x) - v(x)) = 0 \quad \textit{in} \quad \mathbb{R}^N.$$

When $\varepsilon_0 < +\infty$ and, in addition, $a(x)$ is symmetric ($a(x) = a(-x)$ for all x) and the map $t \to a(tx)$ is non increasing for all $x, t > 0$, then ε_0 is sharp, in the sense that for all $\varepsilon \ge \varepsilon_0$ there is no positive solution of (1.2)*.*

The ecological interpretation of this result bears on the single range of expansion factor. It shows that a strategy for species to persist is to match the resource and not to move much. Note that it can happen that $\varepsilon_0 = +\infty$ and then there is no effect of the dispersal on the persistence criteria of the species. A natural sufficient condition for this to happen is

$$(a(x) - 1)^+ \neq 0.$$

In this context, the birth rates exceed all death rates and this guarantees the persistence of the population regardless the dispersal strategy. In particular, there exists a bounded positive solution to (1.2) for any positive kernel J. The uniqueness and the behaviour at infinity of the solution are still open questions for general kernels.

When $m > 0$, the characterisation of the existence of a positive solution changes and a new picture emerges. In particular, for large ε there is always a positive solution of (1.2), whereas for small ε, when $m = 2$, it may happen that no positive solution exists. Thus, when $m = 2$,the situation is, in a sense, opposite to the case when $m = 0$. Here is our precise results.

Theorem 1.3. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $0 < m < 2$. There exist $\varepsilon_0 \leq \varepsilon_1 \in (0, +\infty)$ such that for all $\varepsilon \leq \varepsilon_0$ and all $\varepsilon \geq \varepsilon_1$ there exists a positive solution u_ε of (1.2). Moreover, we have*

$$\lim_{\varepsilon \to +\infty} \|u_\varepsilon - a^+\|_\infty = 0, \qquad \lim_{\varepsilon \to +\infty} \|u_\varepsilon - a^+\|_{L^2(\mathbb{R}^N)} = 0.$$

In addition, assuming further that a is $C^2(\mathbb{R}^N)$, we have

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = v(x) \quad \text{almost everywhere},$$

where v is a non-negative bounded solution of

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

In the next Theorem, we require the following notation for the second moment of J:

$$D_2(J) := \int_{\mathbb{R}^N} J(z)|z|^2 \, dz.$$

Theorem 1.4. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $m = 2$. Then, there exists $\varepsilon_1 \in (0, \infty)$ so that for all $\varepsilon \geq \varepsilon_1$ there exists a positive solution u_ε to (1.2). Moreover,*

$$\lim_{\varepsilon \to +\infty} u_\varepsilon = a^+(x).$$

In addition, if J is radially symmetric, we have the following dichotomy

- *When $\lambda_1 \left(\frac{D_2(J)}{2N} \Delta + a(x) \right) < 0$, there exists $\varepsilon_0 \in (0, \infty)$ such that for all $\varepsilon \leq \varepsilon_0$ there exists a positive solution of (1.2) and*

$$u_\varepsilon \to v, \quad \text{in} \quad L^2_{loc}(\mathbb{R}^N),$$

  where v is the unique bounded non-trivial solution of

$$\frac{D_2(J)}{2N} \Delta v + v(a(x) - v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

- *When* $\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right) > 0$ *there exists* $\varepsilon_0 \in (0,\infty)$ *such that for all* $\varepsilon \le \varepsilon_0$ (1.2) *does not have any positive solution.*

This last result clearly highlights the dependence of the spreading strategy on the cost functions and the structure of ecological niche. Especially when $m = 2$, the smaller spreader strategy may not be an optimal strategy, in the sense that a population adopting such strategy can go extinct. This effect will be discussed in more detail in the next Section.

Lastly, we further establish existence/ non-existence criteria when we relax the compactly supported constraint on the dispersal kernel J. In this direction, we investigate a class of kernel J that can have a fat tail but still have some decay at infinity. More precisely, we assume that

(H5) $\int_{\mathbb{R}^N} J(z)|z|^{N+1} < +\infty.$

Theorem 1.5. *Assume that* J *and* f *satisfy (H1-H4) and assume further that* J *satisfies* $(H5)$*. Then*

(i) if $\lambda_p(\mathcal{M} + \partial_s f(x,0)) > 0$ *there is no bounded positive solution of* (1.2)*,*

(ii) if $\lim_{R\to\infty} \lambda_p(\mathcal{L}_R + \partial_s f(x,0)) < 0$ *then there exists a unique positive solution of* (1.2)*, where*

$$\mathcal{L}_R[\varphi] := \int_{B_R(0)} J(x-y)\varphi(y)\,dy - \varphi(x).$$

1.2 Remarks on the Principal eigenvalue

Before going into the proofs of these results, let us comment on the notion of generalised principal eigenvalue. Our results essentially hinge on the properties of the principal eigenvalue $\lambda_p(\mathcal{M} + a(x))$ and more precisely on the relations between the following spectral quantities:

$$\lambda_p(\mathcal{M}+a) := \sup\left\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\Omega), \varphi > 0, \text{ such that } \mathcal{M}[\varphi](x) + a(x)\varphi(x) + \lambda\varphi(x) \le 0 \quad \text{in} \quad \Omega\right\}.$$
$$\lambda_p'(\mathcal{M}+a) := \inf\left\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\Omega)\cap L^\infty(\Omega), \varphi > 0, \text{ such that } \mathcal{M}[\varphi](x) + a(x)\varphi(x) + \lambda\varphi(x) \ge 0 \quad \text{in} \quad \Omega\right\}.$$
$$\lambda_v(\mathcal{M}+a) := \inf_{\varphi\in L^2(\mathbb{R}^N),\varphi\not\equiv 0} \frac{\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J(x-y)[\varphi(x)-\varphi(y)]^2\,dxdy - \int_{\mathbb{R}^N} a(x)\varphi^2(x)\,dx}{\|\varphi\|_2^2}.$$

These quantities have been introduced in various contexts (see for example [29, 35, 31, 42, 50]). However until now, relations between them have not been fully investigated or only in some particular contexts such as when $a(x)$ is homogeneous or periodic. Some new results have been recently obtained in [3] now allowing us to have a clear description of the relation between λ_p, λ_p' and λ_v. Moreover, [3] provides a description of the asymptotic behaviour of these spectral quantities with respect to the scaling of the kernel. Since we strongly rely on these results, for the purpose of our analysis, we present a summary of these results in Section 3.

Finally, we also want to stress that although we have a clear description of the existence and the non-existence of a positive solution for small ε, the study of the convergence of u_ε as $\varepsilon \to 0$ is quite delicate. Indeed, it is to be expected that the limiting solution will satisfy the problem

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

But this problem has infinitely many bounded non negative solution in L^∞. E.g., for any set $Q \subset \mathbb{R}^N$ the function $a^+(x)\chi_Q$ is a solution. Therefore, owing to the lack of regularising effect of the dispersal operator,

we cannot rely on standard compactness results to obtain a smooth limit. If for the case $m = 2$ we could rely on the elliptic regularity and the new description of Sobolev Spaces developed in [13, 14, 57, 58] to get some compactness, this characterisation does not allow us to treat the case $m < 2$. We believe that a new characterisation of Fractional Sobolev space in the spirit of the work of Bourgain, Brezis and Mironescu [13, 14] will be helpful to resolve this issue.

The paper is organised as follows. In Section 2, we discuss the biological interpretations of our results. There, we describe notion such as *dispersal budget* and evolutionary stable strategies. In Section 3, we recall some known properties and describe our recent work on the principal eigenvalue $\lambda_p(\mathcal{L}_\Omega + a)$. We also describe the sharp persistence criteria for problem (1.5) defined in a bounded domain Ω that are derived in terms of principal eigenvalues. In Sections 4 and 5, we establish the sharp persistence criteria and prove the long time behaviour of the solution of (1.2) (Theorem 1.1). We analyse the dependence of the persistence criteria (Theorems 1.2 and 1.3) in Section 6. Finally, in the Section 7 we discuss the extension of the persistence criteria to kernels that are non longer assume to be compactly supported. In the concluding section, we emphasize some of our results and corresponding biological interpretations. We also indicate several open problems and directions that arise naturally from this work.

1.3 Notations

To simplify the presentation of the proofs, we introduce some notations and various linear operator that we will use throughout this paper:

- $B_R(x_0)$ denotes the standard ball of radius R centred at the point x_0
- χ_R will always refer to the characteristic function of $B_R(0)$.
- $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz space,[15]
- $C(\Omega)$ denotes the space of continuous function in Ω,
- $C_0(\Omega)$ denotes the Banach space of continuous function in Ω that vanishes at the boundary.
- For a positive integrable function $J \in \mathcal{S}(\mathbb{R}^N)$, the constant $\int_{\mathbb{R}^N} J(z)|z|^2\, dz$ will refer to

$$\int_{\mathbb{R}^N} J(z)|z|^2\, dz := \int_{\mathbb{R}^N} J(z) \left(\sum_{i=1}^N z_i^2 \right) dz$$

- We denote by $\mathcal{L}_\Omega$ the continuous linear operator

$$\begin{array}{rccl} \mathcal{L}_\Omega : & C(\bar{\Omega}) & \to & C(\bar{\Omega}) \\ & u & \mapsto & \int_\Omega J(x-y)u(y)\, dy, \end{array} \tag{1.7}$$

  where $\Omega \subset \mathbb{R}^N$.
- $\mathcal{L}_R$ corresponds to the continuous operator $\mathcal{L}_\Omega - I$ with $\Omega = B_R(0)$, (I denotes the identity)
- We will use $\mathcal{M}$ to denote the operators $\mathcal{L}_\Omega - I$ with $\Omega = \mathbb{R}^N$.
- Finally, $\mathcal{M}_\varepsilon$ will denote the operator $\mathcal{M}$ with a rescaled kernel $\frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ and $\mathcal{M}_{\varepsilon,m} := \frac{1}{\varepsilon^m}\mathcal{M}_\varepsilon$
- To simplify the presentation of the proofs, we will also use the notation $\beta(x) := \partial_s f(x, 0)$.

2 Ecological interpretation of our results

Here we define more precisely some concepts from ecology and discuss the ecological interpretation of our findings. We start with the notion of *dispersal budget* introduced in [48]. To this end, let us go back to the description of the dispersal of the population. The main ecological idea behind this "*dispersal budget*" is that we can consider that the amount of energy per individual that the organism can use to disperse is fixed (because of environmental or developmental constraints).

Let us denote by u, the density of the population and suppose, as an example, that it represents a population of trees that produces and disperses its seeds. Several dispersal strategies are then possible for this species: it can "choose" to disperse few seeds over long distances or produce many seed and disperse them over a short distance and, of course, there are the intermediate strategies. We can then assume that the costs involved in the dispersal are proportional to

- the number of individuals dispersed,
- a non decreasing and even function α of the distance moved.

For a population of trees, for instance, the function α can be somehow related, to the amount of energy used to produce seeds with sophisticated shapes and components that allow it to take advantage of wind or gravity or a transport by animals.

If we discretize uniformly the space $\mathbb{R}^N$ by small cubes of volume δx centered at points x_i and the time in time step δt, then we can compute the cost $C(x_i, x_j)$ associated to the transfer from a site x_j to a site x_i and we get

$$C(x_j, x_i) = J(x_j, x_i)\alpha(x_i - x_j)u(x_j, t)(\delta x)^2\delta t,$$

where the term $J(x_j, x_i)u(x_j, t)(\delta x)^2\delta t$ is the total number of individuals that is transferred through a dispersal kernel J, from a site x_j to a site x_i. The total cost in time δt for a typical site x_j is then:

$$u(x_j, t)\delta x\delta t \sum_i J(x_j, x_i)\alpha(x_i - x_j)\delta x.$$

Thus, if the amount of energy per individual is fixed, we get

$$\sum_i J(x_i, x_j)\alpha(x_j - x_i)\delta x = c_0$$

where $c_0 > 0$ is a constant. Letting the volume of the cube δx go down to 0, we obtain

$$\int_{\mathbb{R}^N} J(x, y)\alpha(y - x)\, dx = c_0.$$

For an dispersal that only depends on the distance moved, (i.e. $J(x, y) = J(x - y)$), we get

$$\int_{\mathbb{R}^N} J(z)\alpha(z)\, dz = c_0.$$

To investigate the effects of the dispersal range on the persistence of the population, it is then reasonable to fix the cost function α and to allow the dispersal kernel J to depend on a scaling factor, ε. Now,

for a fixed cost function $\alpha(z)$ proportional to $|z|^m$, by taking a rescaled dispersal kernel of the form $\gamma(\varepsilon,m)J_\varepsilon(z) := \frac{\gamma(\varepsilon,m)}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ we get

$$\int_{\mathbb{R}^N} \gamma(\varepsilon,m)J_\varepsilon(z)|z|^m\,dz = c_0.$$

Thus,

$$\gamma(\varepsilon,m) = \frac{1}{\varepsilon^m}\frac{c_0}{\int_{\mathbb{R}^N} J(z)|z|^m\,dz},$$

and the rescaled dispersal kernel associated with the cost function $\alpha(z)$ proportional to $|y|^m$ is

$$\frac{\alpha_0}{\varepsilon^m}J_\varepsilon(z), \qquad \text{with} \qquad \alpha_0 := \frac{c_0}{\int_{\mathbb{R}^N} J(z)|z|^m\,dz}.$$

Fixing the dispersal budget now means that the dispersal process involved is defined by the operator $\mathcal{M}_{\varepsilon,m} = \gamma(\varepsilon,m)\mathcal{M}_\varepsilon$ and that the species has the choice between large ε, which corresponds to a strategy that produces few offspring that are dispersed far away or small ε, which corresponds to the opposite strategy, that is, producing a large number of offspring dispersed on short range. The rescaled dispersal kernel, will then depend upon three parameters, α_0, ε and m. As explained in [48], in the above setting, the constant $\frac{\alpha_0}{\varepsilon^m}$ will refer to the *rate* of dispersal, whereas ε is a measure of the *range* of dispersal of the species. From the above formula, we clearly see how the cost function and the *range* factor affect the *rate* of dispersal.

The results of Theorems 1.2, 1.3 and 1.4 give some insight on the effects of the cost functions on the different strategies. When $2 > m > 0$, we see that strategies based on sufficiently large or sufficiently small *range* factors enable the population to persist. It is worth to mention that strategies based on sufficiently large *range* factors always enable the population to persist when $m > 0$. This is not necessarily true when $m = 0$.

To investigate further the effects of the *dispersal budget* on the different strategies, we can use the notion of Evolutionary Stable Strategy (ESS) introduced in Adaptive Dynamics, see [36, 37, 53, 64]. The concept of ESS comes from games theory and goes back to the work of Hamilton [45] on the evolution of sex-ratio. Roughly speaking, an Ecological Stable Strategy is a strategy such that if most of the members of a population adopt it, there is no "mutant" strategy that would yield a higher reproductive fitness. In this framework the strategies are compared using their relative pay-off. This concept has been recently used and adapted to investigate ecological stable strategies of dispersal in several contexts: unconditional dispersal [38, 47, 49], conditional dispersal [1, 19, 20, 21, 22, 28, 25, 44] and nonlocal dispersal [48, 51], see [26] for a review on this subject.

In these works, the general idea is to compare the dispersal strategies through the analysis of some invasion criteria. Here, following this idea, the strategies can be compared through the dynamics of a solution of a competitive system

$$\partial_t u(t,x) = \mathcal{M}_{\varepsilon_1,m}[u] + u(t,x)(a(x) - u(t,x) - v(t,x)) \quad \text{in} \quad \mathbb{R}^N \tag{2.1}$$

$$\partial_t v(t,x) = \mathcal{M}_{\varepsilon_2,m}[u] + v(t,x)(a(x) - u(t,x) - v(t,x)) \quad \text{in} \quad \mathbb{R}^N \tag{2.2}$$

where u is a population that has adopted the spreading strategy ε_1 and v the ε_2 strategy. The notion of ESS is then linked to some invasion condition which is related to the stability of the equilibria $(u^*, 0)$ where u^* is a positive solution of the following problem

$$\mathcal{M}_{\varepsilon_1,m}[u] + u(x)(a(x) - u(x)) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

The stability analysis of this equilibria, leads us to consider the sign of a principal eigenvalue of the operator $\mathcal{M}_{\varepsilon_2,m}+a-u^*$. When $\lambda_p(\mathcal{M}_{\varepsilon_2,m}+a-u^*)<0$ then the equilibria $(u^*,0)$ is unstable and a mutant may overtake the territory. Therefore, the strategy followed by u will not be an ESS. On the contrary, when $\lambda_p(\mathcal{M}_{\varepsilon_2,m}+a-u^*)>0$ the equilibria $(u^*,0)$ is stable and a mutant cannot invade its territory, making this strategy a possible candidate for an ESS.

In the context of local dispersion, this system has been introduced to discuss Ecological Stable Strategy of dispersal (see [1, 19, 21, 22, 28, 25, 38, 47, 44]). The only difference is that, in this case, $\mathcal{M}_{eps_i,m}$ is an elliptic operator possibly involving an advection (1st order) term.

In this framework, two classes of dispersal strategies are distinguished: the unconditional dispersal vs the conditional dispersal. Unconditional dispersal refers to dispersal without regard to the environment or the presence of other organisms. Pure diffusion and diffusion with physical advection (e.g. due to local climatic conditions) are examples of unconditional dispersal. Conditional dispersal refers to dispersal that is influenced by the environment or the presence of other organisms.

For particular conditional dispersal strategies, known as *ideal free*, it is known that such strategies are evolutionary stable [20, 21, 25, 26]. For long range dispersal, some nonlocal ideal free strategies have been recently exhibited [27].

For unconditional dispersal strategies, within the framework of reaction diffusion models, it is known that the smaller disperser is always favoured [21, 38, 47]. Such results are still valid for nonlocal dispersal strategies as soon as the cost function is a constant [48, 51]. In such cases, the *range* factor does not affect the *rate* of dispersal.

For cost functions proportional to $|y|^m$ with $m>0$, the *range* factor ε strongly affects the *rate* of dispersal, and the picture changes. From the asymptotics we have obtained (Theorems 1.3 and 1.4), we clearly see that the smaller *spreader* will not always be favoured. Indeed, consider two species that have the same ecological niche and suppose that this ecological niche is bounded.

Let ε_1 be the *range* factor associated to one of the species and let us denote by u^* the equilibrium reached by this population, i.e. $u^*>0$ (or $u^*=0$ if there is no positive equilibrium) is the solution of $\mathcal{M}_{\varepsilon_1,m}[u^*]+u^*(a(x)-u^*)=0$. Let ε_2 be the *range* factor associated with the other species. Let us look at the sign of $\lambda_p(\mathcal{M}_{\varepsilon_2,m}+a-u^*)$. From our results, for ε_2 large enough, we have $\lambda_p(\mathcal{M}_{\varepsilon_2,m}+a-u^*)\approx -\sup_{\mathbb{R}^N}(a-u^*)$. Since u^* satisfies $\mathcal{M}_{\varepsilon_1,m}[u^*]+u^*(a(x)-u^*)=0$, by the maximum principle, we infer that $\sup_{\mathbb{R}^N}(a-u^*)>0$. As a consequence, for ε_2 large enough, $\lambda_p(\mathcal{M}_{\varepsilon_2,m}+a-u^*)<0$ and the population having the *range* factor ε_1 will be wiped out. In other words, if a competing species disperses on a sufficiently long range, it will invade the territory occupied previously by the species of *range* factor ε_1. Conversely, when the cost functions is sub-quadradic, i.e. $m<2$, for ε_2 small enough, we have also $\lambda_p(\mathcal{M}_{\varepsilon_2,m}+a-u^*)\approx -\sup_{\mathbb{R}^N}(a-u^*)$. This means that a smaller spreading factor can also lead to invasion. For a cost function proportional to $|y|^m$ with $0<m<2$, the effect of the cost function is then twofold, both large and small *spreader* species can be favoured.

From our result, we also infer that within the framework of the space of a quadratic cost function ($m=2$), the ubiquity strategy ($\varepsilon=\infty$) seems to be an ESS. Indeed, for this case, we are led to consider the sign of $\lambda_p(\mathcal{M}_{\varepsilon_2,2}+a-a^+)$ which is positive for any $\varepsilon>0$. It is worth noticing that in this situation, the smallest spreader ($\varepsilon=0$) is never an ESS. For such singular strategy, what matter is the sign of $\lambda_p(\mathcal{M}_{\varepsilon_2,2}+a-u^*)$ where u^* is the solution of

$$\frac{D_2(J)}{2N}\Delta u^*+u^*(a(x)-u^*)=0.$$

For ε_2 large enough, we see that $\lambda_p(\mathcal{M}_{\varepsilon_2,2}+a-u^*)<0$. Thus the equilibrium $(u^*,0)$ is unstable, rendering

the singular strategy not evolutionary stable.

Such behaviour stands in contrast with known results on ESS strategy governed by the rate of dispersion [48, 51]. These properties show that nonlocal diffusion exhibits quite different behaviour with respect to what was known in the local diffusion case, where in such case the slowest possible rate is always the best strategy.

3 Preliminaries

In this section, we recall some known results on the principal eigenvalue of a linear non-local operator $\mathcal{L}_\Omega + a$ and on the nonlocal Fisher-KPP equation :

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}_\Omega[u] + f(x, u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{3.1}$$

considered in a bounded domain $\Omega \subset \mathbb{R}^N$.

3.1 Principal eigenvalue for non-local operators

In this subsection, we focus on the properties of the spectral problem

$$\mathcal{L}_\Omega[\varphi] + a\varphi + \lambda\varphi = 0 \quad \text{in} \quad \Omega. \tag{3.2}$$

In contrast with elliptic operators, when a is not a constant, neither $\mathcal{L}_\Omega + a + \lambda$ nor its inverse are compact operators and the description of the spectrum of $\mathcal{L}_\Omega + a$ using the Krein-Rutman Theory fails. However as shown in [29], some variational formula introduced in [7] to characterise the first eigenvalue of elliptic operators $\mathcal{E} := a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$,

$$\lambda_1(\mathcal{E}) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \varphi \in W^{2,n}(\Omega), \varphi > 0 \ \text{ so that } \ \mathcal{E}[\varphi] + \lambda\varphi \leq 0\}, \tag{3.3}$$

can be transposed to the operator $\mathcal{L}_\Omega + a$. Namely, we define the quantity

$$\lambda_p(\mathcal{L}_\Omega + a) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \varphi \in C(\Omega), \varphi > 0 \ \text{ so that } \ \mathcal{L}_\Omega[\varphi] + a\varphi + \lambda\varphi \leq 0\}. \tag{3.4}$$

$\lambda_p(\mathcal{L}_\Omega + a)$ is well defined and we call it the generalised principal eigenvalue.

As in [29], the quantity defined by (3.4) is not always an eigenvalue of $\mathcal{L}_\Omega + a$ in a reasonable Banach space. This means that there is not always a positive continuous eigenfunction associated with λ_p. This stands in contrast with elliptic PDE's. However, as proved in [29, 51, 62], when Ω is a bounded domain we can give some conditions on the coefficients that guarantee the existence of a positive continuous eigenfunction. For example, if the function a satisfies

$$\frac{1}{\sup_\Omega a - a} \notin L^1_{loc}(\bar{\Omega}),$$

then $\lambda_p(\mathcal{L}_\Omega + a)$ is an eigenvalue of $\mathcal{L}_\Omega + a$ in the Banach space $C(\bar{\Omega})$ that is, it is associated to a positive continuous eigenfunction, continuous up to the boundary.

Another useful criteria that guarantees the existence of a continuous principal eigenfunction is

Proposition 3.1. *Let Ω be a bounded domain and let $\mathcal{L}_\Omega$ be as in* (1.7) *then there exists a positive continuous eigenfunction associated to λ_p if and only if $\lambda_p(\mathcal{L}_\Omega + a(x)) < -\sup_\Omega a$.*

A proof of this proposition can be found for example in [35, 31]. To have a more complete description of the properties of λ_p in bounded domains see [32].

Next, we recall some properties of λ_p that we constantly use throughout this paper:

Proposition 3.2. *(i) Assume $\Omega_1 \subset \Omega_2$, then for the two operators $\mathcal{L}_{\Omega_1} + a$ and $\mathcal{L}_{\Omega_2} + a$ respectively defined on $C(\Omega_1)$ and $C(\Omega_2)$, we have :*

$$\lambda_p(\mathcal{L}_{\Omega_1} + a) \geq \lambda_p(\mathcal{L}_{\Omega_2} + a).$$

(ii) For a fixed Ω and assume that $a_1(x) \geq a_2(x)$, for all $x \in \Omega$. Then

$$\lambda_p(\mathcal{L}_\Omega + a_2) \geq \lambda_p(\mathcal{L}_\Omega + a_1).$$

(iii) $\lambda_p(\mathcal{L}_\Omega + a)$ is Lipschitz continuous with respect to a. More precisely,

$$|\lambda_p(\mathcal{L}_\Omega + a) - \lambda_p(\mathcal{L}_\Omega + b)| \leq \|a - b\|_\infty$$

(iv) The following estimate always holds

$$-\sup_\Omega \left(a(x) + \int_\Omega J(x-y)\, dy \right) \leq \lambda_p(\mathcal{L}_\Omega + a) \leq -\sup_\Omega a.$$

We refer to [29, 31] for the proofs of $(i) - (iv)$. Let us also recall the two following results proved in [3].

Lemma 3.3. *Assume that a achieves its maximum in Ω and let $\mathcal{L}_\Omega + a$ be defined as in (1.7) with J satisfying $(H1 - H2)$. Assume further that J is compactly supported. Let $(\Omega_n)_{n\in\mathbb{R}}$ be a sequence of subset of Ω such that $\lim_{n\to\infty} \Omega_n = \Omega$, $\Omega_n \subset \Omega_{n+1}$. Then, we have*

$$\lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a) = \lambda_p(\mathcal{L}_\Omega + a)$$

Lemma 3.4. *Assume that $a(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then for all $\varepsilon > 0$*

$$\lambda_p(\mathcal{M} + a) = \lambda_p(\mathcal{M}_\varepsilon + a_\varepsilon),$$

where $a_\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right)$ and $\mathcal{M}_\varepsilon[\varphi](x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right) \varphi(y)\, dy - \varphi(x)$.

Lastly, we recall some recent results obtained in [3] on the characterisation of the generalised principal eigenvalue $\lambda_p(\mathcal{M}_{\varepsilon,m} + a)$. Motivated by the works [9, 6, 10] on the generalised principal eigenvalue of an elliptic operators, let us introduce the two definitions :

Definition 3.5. Let $\mathcal{L}_\Omega + a$ be as in (1.7). We define the following quantities:

$$\lambda_p'(\mathcal{L}_\Omega + a) := \inf\{\lambda \in \mathbb{R} \,|\, \exists \varphi \geq 0, \varphi \in C(\Omega) \cap L^\infty(\Omega), \text{ such that } \mathcal{L}_\Omega[\varphi] + (a+\lambda)\varphi \geq 0 \text{ in } \Omega\}, \tag{3.5}$$

$$\lambda_v(\mathcal{L}_\Omega + a) := \inf_{\varphi \in L^2(\Omega), \varphi \not\equiv 0} -\frac{\langle \mathcal{L}_\Omega[\varphi] + a\varphi, \varphi\rangle}{\langle \varphi, \varphi\rangle}, \tag{3.6}$$

$$= \inf_{\varphi \in L^2(\Omega), \varphi \not\equiv 0} \frac{\int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2\, dxdy - \int_\Omega (a - 1 + k(x))\varphi^2(x)\, dx}{\|\varphi\|^2_{L^2(\Omega)}}. \tag{3.7}$$

where $k(x) := \int_\Omega J(y-x)\, dy$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $L^2(\Omega)$.

These definitions are natural extension of the definitions known for an elliptic operator. It is worth to mention that those definitions have already been used for the study of (1.2) in several papers [33, 35, 31, 42, 50], but the relation between λ_p, λ'_p and λ_v had not been clarified.

For elliptic operators, the analogues of these three quantities are equivalent on bounded domain [7]. This is not necessarily the case for unbounded domains, where examples can be constructed [9, 6, 12], for which $\lambda_1 > \lambda'_1$. Since the operator, $\mathcal{L}_\Omega + a$, shares many properties with elliptic operators, it is suspected that the three quantities, λ_p, λ'_p and λ_v, are not necessarily equal. However, for compactly supported kernel ,?, J, we have:

Theorem 3.6 ([3]). *Let J be compactly supported satisfying* (H1)–(H2). *Assume that $a \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then we have*

$$\lambda_p(\mathcal{M}_{\varepsilon,m} + a) = \lambda'_p(\mathcal{M}_{\varepsilon,m} + a) = \lambda_v(\mathcal{M}_{\varepsilon,m} + a).$$

Moreover, we have the following asymptotic behaviour as either $\varepsilon \to 0$ or $\varepsilon \to \infty$:

- *When $0 < m \leq 2$* $\quad \lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_{\varepsilon,m} + a) = -\sup_{x\in\mathbb{R}^N} a(x)$
- *When $m = 0$,* $\quad \lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_{\varepsilon} + a) = 1 - \sup_{x\in\mathbb{R}^N} a(x)$
- *When $0 \leq m < 2$,* $\quad \lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a) = -\sup_{x\in\mathbb{R}^N} a(x)$
- *When $m = 2$ and a is globally Lipschitz, then*

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,2} + a) = \lambda_1\left(\frac{D_2(J)}{2N}\Delta + a\right),$$

where we recall that

$$D_2(J) := \int_{\mathbb{R}^N} J(z)z^2\,dz.$$

and

$$\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a\right) := \inf_{\varphi\in H^1_0(\mathbb{R}^N),\varphi\not\equiv 0} \frac{D_2(J)}{2N}\frac{\int_{\mathbb{R}^N}|\nabla\varphi|^2(x)\,dx}{\|\varphi\|_2^2} - \frac{\int_{\mathbb{R}^N} a(x)\varphi^2(x)\,dx}{\|\varphi\|_2^2}.$$

A similar result also holds for the rescaled operator

$$\mathcal{L}_{R,\varepsilon,m} := \frac{1}{\varepsilon^m}\mathcal{L}_{R,\varepsilon}$$

with $\mathcal{L}_{R,\varepsilon}$ defined by :

$$\mathcal{L}_{R,\varepsilon}[\varphi](x) = \int_{B_R} J_\varepsilon(x-y)\varphi(y)\,dy - \varphi(x)$$

with the rescaled kernel J_ε. Namely,

Theorem 3.7 ([3]). *Assume J satisfies* (H1)–(H2) *and let $a(x) \in C(\bar{B}_R(0))$. Then we have*

$$\lambda_p(\mathcal{L}_{R,\varepsilon,m} + a) = \lambda'_p(\mathcal{L}_{R,\varepsilon,m} + a) = \lambda_v(\mathcal{L}_{R,\varepsilon,m} + a).$$

Moreover, we have the following asymptotic behaviour:

- *When* $0 < m \leq 2$ $\quad \lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{L}_{R,\varepsilon,m} + a) = -\sup_{B_R(0)} a(x)$
- *When* $m = 0$, $\quad \lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{L}_{R,\varepsilon} + a) = 1 - \sup_{B_R(0)} a(x)$
- *When* $0 \leq m < 2$, $\quad \lim_{\varepsilon\to 0} \lambda_p(\mathcal{L}_{R,\varepsilon,m} + a) = -\sup_{B_R(0)} a(x)$
- *When* $m = 2$ *and assuming that the function* a *is globally Lipschitz, we get*

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{L}_{R,\varepsilon,2} + a) = \lambda_1 \left(\frac{D_2(J)}{2N}\Delta + a, B_R(0) \right)$$

3.2 Existence criteria for the KPP-equation (3.1)

With this notion of generalised principal eigenvalue, it has been shown [2, 29] that on bounded domains, the existence of a positive stationary solution of (3.1) is conditioned by the sign of $\lambda_p(\mathcal{L}_\Omega + \partial_s f(x, 0))$. More precisely,

Theorem 3.8 ([2, 29])**.** *Let* Ω *be a bounded domain and* $\mathcal{L}_\Omega$ *defined as in* (1.7)*. Assume that* f *satisfies* $(H3)$*. Then there exists a unique positive continuous function,* $\bar{u}$*, stationary solution of* (3.1) *if and only if* $\lambda_p(\mathcal{L}_\Omega + \partial_s f(x, 0)) < 0$*. Moreover, if* $\lambda_p \geq 0$ *then* 0 *is the only non negative bounded stationary solution of* (3.1)*. In addition, for any positive continuous solutions of* (3.1) *we have the following dynamics :*

(i) When $\lambda_p \geq 0$*,*

$$\lim_{t\to\infty} u(t, x) \to 0 \quad \textit{uniformly in } \Omega,$$

(ii) When $\lambda_p < 0$*,*

$$\lim_{t\to\infty} u(t, x) \to \bar{u} \quad \textit{uniformly in } \Omega.$$

Remark 1. This existence criteria is similar to those known for the PDE reaction diffusion versions of (3.1) [5, 17, 18, 39].

4 Existence/non existence and uniqueness of a non-trivial solution

In this section we construct a non-trivial solution of (1.2) and prove the necessary and sufficient condition stated in Theorem 1.1. We treat successively the existence of a solution, its uniqueness and non-existence.

4.1 Existence of a non-trivial positive solution

The construction follows a basic approximation scheme previously used for example in [4]. We introduce the following approximated problem :

$$\mathcal{L}_R[u] + f(x, u) = 0 \quad \text{in} \quad \bar{B}(0, R) \tag{4.1}$$

where $B(0, R)$ denotes the ball of radius R centred at the origin. By Theorem 3.8, for any $R > 0$ the existence of a unique positive solution of (4.1) is conditioned by the sign of $\lambda_p(\mathcal{L}_R + \beta)$ where $\beta(x) := \partial_u f(x, 0)$. Since

$$\lim_{R\to+\infty} \lambda_p(\mathcal{L}_R + \beta) = \lambda_p(\mathcal{M} + \beta) < 0,$$

by Lemma 3.3 there exists $R_0 > 0$ such that

$$\forall R \geq R_0, \lambda_p(\mathcal{L}_R + \beta) < 0.$$

As a consequence, by Theorem 3.8, for all $R > R_0$ there exists a unique positive solution of (4.1) that we denote u_R. Moreover, since for all $R > 0, \sup_{B_R(0)} S(x)$ is a super-solution of (4.1), by a standard sweeping argument since the solution to (4.1) is unique, we get

$$\forall R > 0,\ u_R \leq \sup_{B_R(0)} S(x) \quad \text{in} \quad B(0,R).$$

On another hand, for any $R_1 > R_2$, the solution u_{R_1} is a super-solution for the problem

$$\mathcal{L}_{R_2}[u] + f(x,u) = 0 \quad \text{in} \quad B(0,R_2) \tag{4.2}$$

By the same sweeping argument we get

$$u_{R_2}(x) \leq u_{R_1}(x) \quad \text{in} \quad B(0,R_2).$$

Thus, the map $R \mapsto u_R$ is monotone increasing.

The idea is to obtain a positive solution of (1.2) as a limit of the positive solution of (4.1). To this end we construct a uniform super-solution of problem (1.2).

Lemma 4.1. *There exists $\bar{u} \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $\bar{u} > 0$ such that $\bar{u}$ is a super-solution of problem* (1.2).

Proof. Let us fix $\nu > 0$ and $R_0 > 1$ so that $\nu < -\limsup_{|x|\to\infty} \beta(x)$ and $\beta(x) \leq -\frac{\nu}{2}$ for all $|x| \geq R_0$. Consider the function

$$w(x) = Ce^{-\alpha|x|},$$

where C and α are to be chosen. For all $x \in \mathbb{R}^N \setminus B_{R_0}(0)$ we get:

$$\begin{aligned}\mathcal{M}[w](x) + \beta(x)w(x) &= Ce^{-\alpha|x|}\left(\int_{\mathbb{R}^N} J(x-y)e^{-\alpha(|y|-|x|)}\,dy - 1 + \beta(x)\right),\\ &\leq w(x)\left(\int_{\mathbb{R}^N} J(z)e^{\alpha(|z|)}\,dz - 1 - \frac{\nu}{2}\right).\end{aligned}$$

Therefore, w satisfies

$$\mathcal{M}[w](x) + \beta(x)w(x) \leq h(\alpha)w(x) \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_0}(0), \tag{4.3}$$

where $h(\alpha)$ is defined by

$$h(\alpha) = -1 - \frac{\nu}{2}.$$

Since J is compactly supported, thanks to Lebesgue's Theorems, we can check that $h(\cdot)$ is a smooth (C^2) convex increasing function of α. Moreover, we have

$$\lim_{\alpha\to 0} h(\alpha) = h(0) = -\frac{\nu}{2}.$$

Therefore, by continuity of h, we can choose α small enough such that $h(\alpha) < 0$. For such an α, we get

$$\mathcal{M}[w](x) + \beta(x)w(x) \leq h(\alpha)w(x) < 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_0}(0). \tag{4.4}$$

Let $M := \sup_{B_{2R_0}(0)} S(x)$ and let us fix $C = 2Me^{2\alpha R_0}$. We consider now the continuous function

$$\bar{u}(x) := \begin{cases} Ce^{-\alpha|x|} & \text{in} \quad \mathbb{R}^N \setminus B_{2R_0}(0), \\ 2M & \text{in} \quad B_{2R_0}(0). \end{cases}$$

By direct computation we can check that $\bar{u}$ is a super-solution of the problem (1.2). Indeed, for any $x \in B_{2R_0}(0)$, we have $\bar{u} = 2M > \sup_{B_{2R_0}(0)} S(x)$ which implies that $f(x, \bar{u}) = f(x, 2M) \leq 0$ and thus

$$\mathcal{M}[\bar{u}](x) + f(x, \bar{u}(x)) \leq 2M \int_{\mathbb{R}^N} J(x-y)\, dy - 2M + f(x, 2M) \leq f(x, 2M) \leq 0.$$

Then, for $x \in \mathbb{R}^N \setminus B_{2R_0}(0) \subset \mathbb{R}^N \setminus B_{R_0}(0)$ by (4.4) we have

$$\begin{aligned} \mathcal{M}[\bar{u}](x) + f(x, \bar{u}(x)) \leq \mathcal{M}[\bar{u}](x) + \beta(x)w(x) &\leq \mathcal{M}[w](x) + \beta(x)w(x), \\ &\leq h(\alpha)w(x) \leq 0. \end{aligned}$$

□

We are now in a position to construct a positive solution of (1.2). By Lemma 4.1, there exists $\bar{u}$ a positive continuous super-solution of problem (1.2). Therefore, for any $R > 0$, $\bar{u}$ is also a positive continuous super-solution of the problem (4.1). Therefore, by the standard sweeping argument, we can check that for all $R \geq R_0$ the unique positive continuous solution of (4.1) satisfies $u_R \leq \bar{u}$ in $B_R(0)$. By letting $R \to \infty$ and observing that $u_R \in C(B_R)$ is uniformly bounded and monotone with respect to R, we get $u_R \to \tilde{u} := \lim_{R\to\infty} u_R$. The function $\tilde{u}$ is a non-negative solution of (1.2) and it is non zero since $0 \leq \tilde{u} \leq \bar{u}$ and

$$0 < u_R \leq \tilde{u} \quad \text{in } B(0, R), \qquad \text{for all } R \geq R_0.$$

4.2 Uniqueness

Having constructed a positive solution of (1.2) in $L^1(\mathbb{R}^N)$, we now prove its uniqueness. Assume by contradiction that $v \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is another positive solution. Then v is a supersolution of problem (4.1) for any $R > 0$. Therefore $v \geq u_R$ for all $R \geq R_0$. Since u_R is monotone with respect to R, it follows that $v \geq \tilde{u} := \lim_R u_R(x)$. By assumption $v \not\equiv \tilde{u}$ almost everywhere. Recall that the functions v and $\tilde{u}$ satisfies:

$$\mathcal{M}[\tilde{u}] + f(x, \tilde{u}) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{4.5}$$

$$\mathcal{M}[v] + f(x, v) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{4.6}$$

Multiplying (4.5) by v and (4.6) by u we get after integration over $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\tilde{u}(y)v(x)\, dydx - \int_{\mathbb{R}^N} \tilde{u}(x)v(x)\, dx + \int_{\mathbb{R}^N} v(x)f(x, \tilde{u}(x))\, dx = 0, \tag{4.7}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\tilde{u}(x)v(y)\, dydx - \int_{\mathbb{R}^N} \tilde{u}(x)v(x)\, dx + \int_{\mathbb{R}^N} \tilde{u}(x)f(x, v(x))\, dx = 0. \tag{4.8}$$

Subtracting (4.8) from (4.7) yields

$$0 < \int_{\mathbb{R}^N} v(x)\tilde{u}\left[\frac{f(x,\tilde{u}(x))}{\tilde{u}(x)} - \frac{f(x,v(x))}{v(x)}\right] dx = 0,$$

which is a contradiction since $\tilde{u} \leq v$ and $f(x,s)/s$ is decreasing.

When v is just a L^∞ solution of (1.2), a similar argument holds using an adapted version of the maximum principle (Theorem 1.4 in [33]).

4.3 Non-existence of a solution

In this section, we deal with the non-existence of positive solution when $\lambda_p(\mathcal{M}+\beta) \geq 0$. To simplify the presentation of the proofs, we treat the two cases: $\lambda_p(\mathcal{M}+\beta > 0$ and $\lambda_p(\mathcal{M}+\beta) = 0$ separately, the proof in the second case being more involved.

Case $\lambda_p(\mathcal{M}+\beta) > 0$:

In this situation we argue as follows. Assume by contradiction that a positive bounded solution u exists. By assumption, u satisfies

$$\mathcal{M}[u](x) + \beta(x)u(x) \geq 0. \tag{4.9}$$

Therefore, u is a test function for $\lambda'_p(\mathcal{M}+\beta)$ and we get $\lambda'_p(\mathcal{M}+\beta) \leq 0$. Since by Theorem 3.6, $\lambda_p(\mathcal{M}+\beta) \leq \lambda'_p(\mathcal{M}+\beta)$ we get an obvious contradiction.

Case $\lambda_p(\mathcal{M}+\beta) = 0$:

We argue again by contradiction. Assume that a non-negative, non identically zero, bounded solution u exists. By a straightforward application of the maximum principle, since $u \not\equiv 0$ we have $u > 0$ in $\mathbb{R}^N$. By the above argument we have $\lambda_p(\mathcal{M}+\beta) = 0 = \lambda'_p(\mathcal{M}+\beta)$ and by (iv) of Proposition (3.2) we get the following estimate :

$$\sup_{\mathbb{R}^N}(\beta(x) - 1) \leq 0. \tag{4.10}$$

Let us denote $\gamma(x) := \frac{f(x,u(x))}{u(x)}$, then we obviously have

$$J \star u(x) - u(x) + \gamma(x)u(x) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{4.11}$$

By definition of λ'_p we have $\lambda'_p(\mathcal{M}+\gamma) \leq 0$. By construction, $\gamma(x) \leq \beta(x)$, so by combining (4.11) with Proposition 3.2, Theorem 3.6 and the definition of $\lambda_p(\mathcal{M}+\gamma)$ we can infer that

$$\lambda_p(\mathcal{M}+\gamma) \leq \lambda'_p(\mathcal{M}+\gamma) \leq 0 \leq \lambda_p(\mathcal{M}+\beta) \leq \lambda_p(\mathcal{M}+\gamma).$$

Hence $\lambda_p(\mathcal{M}+\gamma) = 0$. Let us denote $\eta \in C(\mathbb{R}^N)$ a smooth regularisation of $\chi_{B_1(0)}$ the characteristic function of the unit ball. Since $\gamma(x) < \beta(x)$ in $\mathbb{R}^N$, we can find $\varepsilon_0 > 0$ small enough so that for all $\varepsilon \leq \varepsilon_0$

$$\gamma(x) \leq \gamma(x) + \varepsilon\eta(x) < \beta(x) \quad \text{in} \quad \mathbb{R}^N.$$

By (i) of Proposition 3.2, we then have

$$0 = \lambda_p(\mathcal{M}+\beta) \leq \lambda_p(\mathcal{M}+\gamma+\varepsilon\eta) \leq \lambda_p(\mathcal{M}+\gamma) = 0.$$

Next we claim that

Lemma 4.2. *There exists $R_1 > 0$ and $\psi > 0$, $\psi \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ such that*

$$J \star \psi(x) - \psi(x) + (\gamma(x) + \varepsilon\eta(x))\psi(x) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Assume for a moment that the Lemma holds. Then by arguing as in subsection (4.2), since $\psi \in L^1$ we get the following contradiction

$$0 = -\varepsilon \int_{\mathbb{R}^N} u(x)\psi(x)\eta(x)\, dx < 0.$$

Proof of the Lemma. For convenience we denote $\tilde{\gamma} := \gamma + \varepsilon\eta$. By (4.10), since $\tilde{\gamma} < \beta$ we also have

$$0 < -\sup_{\mathbb{R}^N}(\tilde{\gamma} - 1). \tag{4.12}$$

From this inequality, by using Proposition 3.1 and Lemma 3.3 we see that there exists R_0 such that for all $R \geq R_0$ there exists a positive eigenfunction $\varphi_R \in C(\bar{B}(0,R))$ associated with the generalised principal eigenvalue $\lambda_p(\mathcal{L}_R + \tilde{\gamma})$ of the approximated problem

$$\mathcal{L}_R[\varphi] + (\tilde{\gamma} + \lambda)\varphi = 0 \quad \text{in} \quad B(0,R) \tag{4.13}$$

Consider now the increasing sequence $(R_n)_{n\in\mathbb{N}} := (R_0 + n)_{n\in\mathbb{N}}$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be the sequence of positive principal eigenfunction associated with $\lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma})$. Without loss of generality, we can assume that for all n, $\varphi_n(0) = 1$.

For all n, φ_n satisfies

$$\mathcal{L}_{R_n}[\varphi_n] + (\tilde{\gamma} + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}))\varphi_n = 0 \quad \text{in} \quad B_{R_n}. \tag{4.14}$$

Let us now define $b_n(x) := -\lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}) - \tilde{\gamma}(x)$. Then φ_n satisfies

$$\mathcal{L}_{R_n}[\varphi_n] = b_n\varphi_n \quad \text{in} \quad B_{R_n}.$$

By construction, for all $n \geq 0$ we have $b_n \geq -\lambda_p(\mathcal{L}_{R_{n_0}} + \tilde{\gamma}) - \sup_{\mathbb{R}^N}(\tilde{\gamma}(x) - 1) > 0$, therefore the Harnack inequality (Theorem 1.4 in [30]) applies to φ_n. Thus for $n \geq 0$ fixed and for all compact set $\omega \subset\subset B_{R_n}$ there exists a constant $C_n(\omega)$ such that

$$\varphi_n(x) \leq C_n(\omega)\varphi_n(y) \quad \forall\, x, y \in \omega.$$

Moreover, the constant $C_n(\omega)$ only depends on $\bigcup_{x\in\omega} B_{r_0}(x)$ and is monotone decreasing with respect to $\inf_{x\in B_{R_n}} b_n(x)$. For all $n \geq 0$, the function $b_n(x)$ being uniformly bounded from below by a constant independent of n, we can choose the constant C_n so that it is bounded from above independently of n by a constant $C(\omega)$. Thus,

$$\varphi_n(x) \leq C(\omega)\varphi_n(y) \quad \forall\, x, y \in \omega.$$

From the normalization $\varphi_n(0) = 1$, we infer that the sequence $(\varphi_n)_{n\geq 0}$ is locally uniformly bounded in $\mathbb{R}^N$. Moreover, from a standard diagonal extraction argument, there exists a subsequence still denoted $(\varphi_n)_{n\geq 0}$ such that $(\varphi_n)_{n\geq 0}$ converges locally uniformly to a continuous function φ. Furthermore, φ is a non-negative non trivial function and $\varphi(0) = 1$.

Since J is compactly supported, we can pass to the limit in equation (4.14) using the Lebesgue monotone convergence theorem and get

$$\mathcal{M}[\varphi] + (\tilde{\gamma} + \lambda_p(\mathcal{M} + \tilde{\gamma}))\varphi = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Hence, we have

$$\mathcal{M}[\varphi] + \tilde{\gamma}\varphi = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{4.15}$$

To conclude the proof of this Lemma, we characterise the behaviour of $\varphi(x)$ for $|x| >> 1$. Let us denote $0 < \nu < -\limsup_{|x|\to\infty} \beta(x)$ and let us fix R_1 so that $\beta(x) \leq -\frac{\nu}{2}$ for $|x| \geq R_1$.

Since by Lemma 3.3, $\lambda_p(\mathcal{L}_R + \tilde{\gamma}) \to \lambda_p(\mathcal{M} + \tilde{\gamma}) = 0$, we can take R_1 larger if necessary to achieve

$$\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_R + \tilde{\gamma}) \leq -\frac{\nu}{4} \quad \text{for} \quad |x| \geq R_1.$$

Let us now consider $\psi(x) := Ce^{-\alpha(|x|-R_1)}$ where C and α will be chosen later on. By a straightforward computation, we see that for all $R > R_1$

$$\begin{aligned}\mathcal{L}_R[\psi](x) + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_R + \tilde{\gamma}))\psi(x) &\leq \psi(x)\left(\int_{\mathbb{R}^N} J(z)e^{\alpha|z|}dz - 1 - \frac{\nu}{4}\right) \quad \text{for} \quad |x| \geq R_1,\\ &\leq h(\alpha)\psi(x) \quad \text{for} \quad |x| \geq R_1,\end{aligned}$$

with

$$h(\alpha) := \left(\int_{\mathbb{R}^N} J(z)e^{\alpha|z|}dz - 1 - \frac{\nu}{4}\right).$$

Since J is compactly supported, by the Lebesgue Theorem, the function h is continuous and $h(0) = -\frac{\nu}{4}$. By assumption $\nu > 0$, and by continuity of h there exists $\alpha_0 > 0$ such that $h(\alpha_0) < 0$. Thus, for $\alpha = \alpha_0$ we achieve

$$\mathcal{L}_R[\psi](x) + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_R + \tilde{\gamma}(x)))\psi(x) \leq 0 \quad \text{for} \quad |x| \geq R_1. \tag{4.16}$$

Recall that by construction, the function φ_n satisfies :

$$\mathcal{L}_{R_n}[\varphi_n](x) + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))\varphi_n(x) = 0 \quad \text{in} \quad B_{R_n}(0). \tag{4.17}$$

Since J is compactly supported and $J(0) > 0$ there exist positive constants $r_0 \geq r_1$ and $M \geq m$ so that

$$M\chi_{B_{r_0}(x)} \geq J(x-y) \geq m\chi_{B_{r_1}(x)} \quad \text{for all } \ x, y \in \mathbb{R}^N.$$

For n large enough, say $n \geq n_0$, we have $R_n > R_1 + r_0$ and by the Harnack inequality, for all $n \geq n_0$, we have

$$\varphi_n(x) \leq C(B_{R_1}, \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))\varphi_n(y) \quad \text{for all} \quad x, y \in B_{R_1}(0),$$

with $C(B_{R_1}, \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))$ a constant that only depends on $\bigcup_{x\in B_{R_1}} B_{r_0}(x)$ and is monotone decreasing with respect to $\inf_{x\in B_{R_n}}(\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}))$. For all $n \geq n_0$, the function $\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma})$ being uniformly bounded from below by a constant independent of n, the constant $C(B_{R_1}, \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}))$ is bounded from above independently of n by a constant $C(B_{R_1})$. Thus for all $n \geq n_0$;

$$\varphi_n(x) \leq C(B_{R_1})\varphi_n(y) \quad \forall\, x, y \in B_{R_1}.$$

In particular, for all $n \geq n_0$ we have :

$$\varphi_n(x) \leq C(B_{R_1})\varphi_n(0) = C(B_{R_1}) \quad \forall\, x \in B_{R_1}.$$

By choosing $C > C(B_{R_1})$, we get

$$\psi(x) \geq C > C(B_{R_1}) \geq \varphi_n(x) \quad \forall\, x \in B_{R_1}.$$

Set $w_n := \psi - \varphi_n$. From (4.16) and (4.17) we get

$$\mathcal{L}_{R_n}[w_n](x) + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))w_n(x) \leq 0 \quad \text{for} \quad R_1 \leq |x| < R_n, \tag{4.18}$$

$$w_n > 0 \quad \text{for} \quad |x| < R_1. \tag{4.19}$$

By a straightforward application of the Maximum principle, it follows that for all $n \geq n_0$ we have $\varphi_n(x) \leq \psi$. Indeed, since w_n is continuous, w_n achieves a minimum at some point $x_0 \in B_{R_n}$. Assume by contradiction that $w_n(x_0) < 0$. Then, thanks to (4.19), $x_0 \in B_{R_n} \setminus B_{R_1}$ and at this point, by (4.18) we have the following contradiction

$$\begin{aligned} 0 \geq \mathcal{L}_{R_n}[w_n](x_0) + (\tilde{\gamma}(x_0) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))w_n(x_0) &\geq \int_{B_{R_n}} J(x_0 - y)w_n(y)\,dy - w_n(x_0) + \frac{\nu}{4}|w_n(x_0)|, \\ &\geq \int_{B_{R_n}} J(x_0 - y)[w_n(y) - w_n(x_0)]\,dy + \frac{\nu}{4}|w_n(x_0)| > 0. \end{aligned}$$

Hence, for all $n \geq n_0$, we get $\varphi_n \leq \psi$ in B_{R_n} which, by sending $n \to \infty$, leads to $\varphi \leq \psi$ in $\mathbb{R}^N$. This concludes the proof of the Lemma. □

5 Long time Behaviour

In this section, we investigate the long-time behaviour of the positive solution $u(t,x)$ of

$$\frac{\partial u}{\partial t}(t,x) = J \star u(t,x) - u(t,x) + f(x, u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{5.1}$$

$$u(0,x) = u_0(x). \tag{5.2}$$

For any $u_0 \in C^k(\mathbb{R}^N) \cap L^\infty$ or in $C^k(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ the existence of a smooth solution $u(t,x) \in C^1((0,+\infty), C^{\min\{1,k\}}(\mathbb{R}^N))$ respectively $u(t,x) \in C^1((0,+\infty), C^{\min\{1,k\}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ is a straightforward consequence of the Cauchy-Lipschitz Theorem and of the KPP structure of the nonlinearity f. Before going to the proof of the asymptotic behaviour, let us recall some useful results

Lemma 5.1. *Assume that $u_0(x)$ is a sub-solution to* (5.1)*, then the solution $u(t,x)$ is increasing in time. Conversely, if $u_0(x)$ is a super-solution to* (5.1) *then $u(t,x)$ is decreasing in time.*

The proof of this Lemma follows from a straightforward application of the parabolic maximum principle and is left to reader. Let us now prove the asymptotic behaviour of the solution of (5.1) and finish the proof of Theorem 1.1.

Proof. Let $z(t,x)$ be the solution of

$$\frac{\partial z}{\partial t} = J \star z - z + f(x, z(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{5.3}$$

$$z(0,x) = C\|u_0\|_\infty. \tag{5.4}$$

Since $S(x) \in L^\infty$ by choosing C large enough, the constant $C\|u_0\|_\infty$ is a super-solution of (5.1). Therefore $z(t,x)$ is a decreasing function and by the parabolic maximum principle $u(t,x) \le z(t,x)$ for all $(t,x) \in [0,+\infty) \times \mathbb{R}^N$. Therefore,

$$\limsup_{t\to\infty} u(t,x) \le \limsup_{t\to\infty} z(t,x) \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{5.5}$$

Let us consider the approximated parabolic problem

$$\frac{\partial v_R}{\partial t}(t,x) = \int_{B_R(0)} J(x-y)v_R(t,y)\,dy - v_R(t,x) + f(x, v_R(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times B_R(0), \tag{5.6}$$

$$v_R(0,x) = \eta_R u_0(x), \tag{5.7}$$

where $\eta_R := \eta\left(\frac{|x|}{R}\right)$ with $\eta \in C(\mathbb{R}^+)$ a smooth cut-off function such that $\eta \ge 0$, $\eta \equiv 1$ in $[0,1]$ and $\eta \equiv 0$ in $\mathbb{R}^+ \setminus [0,2]$. By Theorem 3.8, for R large enough the solution v_R converges to u_R as $t \to \infty$, where u_R denotes the unique positive stationary solution of (5.6). By construction, since $u(t,x)$ is a super-solution of the problem (5.6), by the parabolic comparison principle for R large enough we have $v_R(t,x) \le u(t,x)$ for all $(t,x) \in [0,+\infty) \times B_R(0)$. Thus for R large enough, we get

$$\liminf_{t\to\infty} u(t,x) \ge u_R(x) \quad \text{for all} \quad x \in B_R(0). \tag{5.8}$$

By taking the limit in the above inequality as $R \to \infty$, we obtain

$$\liminf_{t\to\infty} u(t,x) \ge \lim_{R\to\infty} u_R(x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N \tag{5.9}$$

Note that we can reproduce the above arguments with $z(t,x)$, and thus we also get

$$v_R(t,x) \le z(t,x) \quad \text{for all} \quad (t,x) \in [0,+\infty) \times B_R(0), \tag{5.10}$$

$$\liminf_{t\to\infty} z(t,x) \ge \lim_{R\to\infty} u_R(x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{5.11}$$

By (5.10), $z(t,x)$ is locally uniformly bounded from below and, since $z(t,x)$ is a decreasing function of t, we get $\lim_{t\to\infty} z(t,x) = \bar{z}(x) > 0$, for all $x \in \mathbb{R}^N$. Moreover $\bar{z}$ is a bounded stationary solution of (5.1). By uniqueness of the positive stationary solution, we conclude that $\bar{z} = \tilde{u}$. It follows that

$$\lim_{t\to\infty} z(t,x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N \tag{5.12}$$

By collecting (5.5),(5.9) and (5.12) we get for all $x \in \mathbb{R}^N$

$$\tilde{u}(x) \le \liminf_{t\to\infty} u(t,x) \le \limsup_{t\to\infty} u(t,x) \le \limsup_{t\to\infty} z(t,x) = \lim_{t\to\infty} z(t,x) = \tilde{u}(x).$$

Now, to complete the proof it remains to show that $\|u - \tilde{u}\|_\infty \to 0$ as $t \to \infty$. To this end, we follow the argument in [10]. We argue by contradiction and assume that there exists $\varepsilon > 0$ and sequences $(t_n) \in \mathbb{R}^+$, $(x_n) \in \mathbb{R}^N$ such that

$$\lim_{n\to\infty} t_n = \infty, \qquad |u(t_n,x_n) - \tilde{u}(x_n)| > \varepsilon, \qquad \forall n \in \mathbb{N}. \tag{5.13}$$

By (5.12), we already know that $u \to \tilde{u}$ locally uniformly in $\mathbb{R}^N$, so, without loss of generality, we can assume that $|x_n| \to \infty$. From the construction of $\tilde{u}$, Subsection 4.1, we have $\lim_{|x|\to\infty} \tilde{u}(x) = 0$. Therefore, for some $R_0 > 0$, we have $\tilde{u}(x) \leq \frac{\varepsilon}{2}$ for all $|x| \geq R_0$. This, combined with (5.12) and (5.13) enforces

$$z(t_n, x_n) - \tilde{u}(x_n) \geq u(t_n, x_n) - \tilde{u}(x_n) > \varepsilon, \qquad \forall n \in \mathbb{N}. \tag{5.14}$$

Next we require the following limiting result

Lemma 5.2. *For all sequences $(t_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = +\infty$, we have $z(t_n, x_n) \to 0$.*

Assume for the moment that the Lemma holds. Then we obtain a straightforward contradiction since :

$$0 = \lim_{n\to\infty} z(t_n, x_n) - \tilde{u}(x_n) \geq \lim_{n\to\infty} u(t_n, x_n) - \tilde{u}(x_n) > \varepsilon.$$

We now prove the Lemma. Again, we argue by contradiction and assume that there exists $\varepsilon > 0$ and sequences $(t_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = \infty$ such that $z(t_n, x_n) > \varepsilon$ for all $n \in \mathbb{N}$. Let us define $z_n(t, x) := z(t, x + x_n)$. It satisfies

$$\begin{aligned}
&\frac{\partial z_n}{\partial t}(t, x) = \int_{\mathbb{R}} J(x - y) z_n(y)\, dy - z_n(t, x) + f(x + x_n, z_n(t, x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\
&z_n(0, x) = C\|u_0\|_\infty,
\end{aligned}$$

and $0 < z_n(t, x) < C\|u_0\|_\infty$ for $t > 0$. Since for all n, $z_n(0, x) \in C^\infty$, by the Cauchy Lipschitz Theorem, we see that $z_n \in C^1(\mathbb{R}^+, C^1(\mathbb{R}^N))$. Thus, there exists $C_0 > 0$ independent of n so that $\|z_n\|_{C^{1,1}(\mathbb{R}^+, C(\mathbb{R}^N))} < C_0$. From these estimates, the sequence $(z_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^{1,1}((0, T), C^{0,1}(\mathbb{R}^N))$ for any $T > 0$. By a diagonal extraction, there exists a subsequence of $(z_n)_{n\in\mathbb{N}}$ that converges locally uniformly to $\tilde{z}(t, x)$. Moreover, thanks to $\lim_{|x|\to\infty} \frac{f(x,s)}{s} < 0$, there exists $\kappa > 0$ so that $\tilde{z}(x, t)$ satisfies

$$\frac{\partial \tilde{z}}{\partial t}(t, x) \leq \int_{\mathbb{R}} J(x - y)\tilde{z}(t, y)\, dy - \tilde{z}(t, x) - \kappa\tilde{z}(t, x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{5.15}$$

$$\tilde{z}(0, x) = C\|u_0\|_\infty. \tag{5.16}$$

In addition, for all $t > 0$, $\tilde{z}(t, 0) = \lim_{n\to\infty} z_n(t, 0) \geq \varepsilon$. Since $\tilde{z}(0, x)$ is a super-solution of (5.15), by Lemma 5.1 the function $\tilde{z}(t, x)$ is monotone decreasing in time. By sending $t \to \infty$, since $\tilde{z} \geq 0$, $\tilde{z}$ converges locally uniformly to a non-negative function $\bar{z}$ that satisfies

$$\begin{aligned}
&\int_{\mathbb{R}} J(x - y)\bar{z}(y)\, dy - \bar{z}(x) - \kappa\bar{z}(x)) \geq 0 \quad \text{in} \quad \mathbb{R}^N, \\
&0 \leq \bar{z} \leq C\|u_0\|_\infty, \\
&\bar{z}(0) \geq \varepsilon.
\end{aligned}$$

Let us now consider the function $w(x) := \frac{\varepsilon}{2}e^{\alpha|x|} - \bar{z}$ with α to be chosen. It satisfies

$$\int_{\mathbb{R}} J(x - y)w(y)\, dy - w(x) - \kappa w(x) \leq \rho e^{\alpha|x|} \left(\int_{\mathbb{R}^N} J(z)e^{\alpha|z|}\, dy - 1 - \kappa \right) \quad \text{in} \quad \mathbb{R}^N.$$

The left hand side of the inequality is well defined and continuous with respect to α since J is compactly supported. Since $\int_{\mathbb{R}^N} J(z)dz = 1$, by choosing α small enough we achieve

$$\int_{\mathbb{R}} J(x-y)w(y)\,dy - w(x) - \kappa w(x) < 0 \quad \text{in} \quad \mathbb{R}^N.$$

By construction, since $\bar{z}$ is bounded, $\lim_{|x|\to\infty} w(x) = +\infty$ and w achieves a minimum in $\mathbb{R}^N$, say at x_0. Since $w(0) = \frac{\varepsilon}{2} - \bar{z}(0) \leq -\frac{\varepsilon}{2}$, we have $w(x_0) < 0$. At this point, we get the following contradiction

$$0 < \int_{\mathbb{R}} J(x_0-y)[w(y) - w(x_0)]\,dy - \kappa w(x_0) < 0 \quad \text{in} \quad \mathbb{R}^N.$$

□

Finally we establish the long time behaviour of the solution $u(t,x)$ starting from an integrable initial datum u_0,i.e $u_0 \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. To do so, we consider two auxiliary functions $h(t,x)$ and $v(t,x)$ that are respectively solutions of

$$\begin{cases} \frac{\partial h}{\partial t}(t,x) = J \star h(t,x) - h(t,x) + f(x,h(t,x)) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ h(0,x) = \sup\{\tilde{u}(x), u_0(x)\}, \end{cases} \tag{5.17}$$

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = J \star v(t,x) - v(t,x) + f(x,v(t,x)) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ v(0,x) = \inf\{\tilde{u}(x), u_0(x)\}. \end{cases} \tag{5.18}$$

By construction, from the comparison principle we see that $v(t,x) \leq u(t,x) \leq h(t,x)$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$. Therefore

$$\|u - \tilde{u}\|_{L^1(\mathbb{R}^N)} \leq \sup\{\|h - \tilde{u}\|_{L^1(\mathbb{R}^N)}, \|v - \tilde{u}\|_{L^1(\mathbb{R}^N)}\}.$$

Thus, to prove that $\|u - \tilde{u}\|_{L^1(\mathbb{R}^N)} \to 0$ it is enough to show that h and v converge to $\tilde{u}$ in $L^1(\mathbb{R}^N)$.

Let us show that v converges to $\tilde{u}$ in $L^1(\mathbb{R}^N)$. Since $\tilde{u}(x)$ is a super solution to (5.18) we deduce $v(t,x) \leq \tilde{u}(x)$ for all $x \in \mathbb{R}^N$. Let $\varepsilon > 0$ be fixed and choose R such that $\int_{\mathbb{R}^N \setminus B(0,R)} \tilde{u}(x)\,dx \leq \frac{\varepsilon}{4}$. We have

$$\begin{aligned} \|\tilde{u} - v\|_{L^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N \setminus B_R(0)} (\tilde{u}(x) - v(t,x))\,dx + \int_{B_R(0)} (\tilde{u}(x) - v(t,x))\,dx, \\ &\leq 2\int_{\mathbb{R}^N \setminus B_R(0)} \tilde{u}(x)\,dx + \int_{B_R(0)} (\tilde{u}(x) - v(t,x))\,dx, \\ &\leq \frac{\varepsilon}{2} + \int_{B_R(0)} (\tilde{u}(x) - v(t,x))\,dx. \end{aligned}$$

Recall that v converges pointwise to $\tilde{u}$ as t tends to infinity. Therefore, by Lebesgue's Theorem for some $t(\varepsilon)$ we get for $t \geq t(\varepsilon)$, $\int_{B_R(0)} (\tilde{u}(x) - v(t,x))\,dx \leq \frac{\varepsilon}{2}$ which yields

$$\|\tilde{u} - v\|_{L^1(\mathbb{R}^N)} \leq \varepsilon.$$

Since ε is arbitrary, we see that $\lim_{t\to\infty} \|\tilde{u} - v\|_{L^1(\mathbb{R}^N)} = 0$.

To obtain that $\|h-\tilde{u}\|_{L^1(\mathbb{R}^N)} \to 0$ we argue as follows. By construction, $\tilde{u}$ is a sub solution to (5.17), thus, $\tilde{u}(x) \le h(t,x)$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$. Let us denote $w(t,x) := h(t,x) - \tilde{u}(x)$. Then w satisfies for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$:

$$\begin{aligned}\frac{\partial w}{\partial t}(t,x) &= J \star w(t,x) - w(t,x) + \left(\frac{f(x,h(t,x))}{h(t,x)} - \frac{f(x,\tilde{u}(x))}{\tilde{u}}\right) h(t,x) + \frac{f(x,\tilde{u}(x))}{\tilde{u}} w(t,x),\\ &\le J \star w(t,x) - w(t,x) + \frac{f(x,\tilde{u}(x))}{\tilde{u}} w(t,x).\end{aligned}$$

Now since $\lim_{|x|\to\infty} \frac{f(x,s)}{s} < 0$, there exists $\kappa > 0$ and R_0 so that w satisfies

$$\frac{\partial w}{\partial t}(t,x) \le J \star w(t,x) - w(t,x) - \kappa w(t,x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \setminus B_{R_0}(0). \tag{5.19}$$

Fix now $\varepsilon > 0$. Recall that $h(t,x)$ converges pointwise to $\tilde{u}$. By Lebesgue's Theorem, there exists t_0 so that for $t \ge t_0$,

$$\int_{B_{R_0}(0)} w(t,x)\,dx \le \kappa\varepsilon.$$

Now let us estimate $\int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(x)\,dx$ for $t \ge t_0$. Integrating (5.19) over $\mathbb{R}^N \setminus B_{R_0}(0)$ yields

$$\frac{\partial \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx}{\partial t} \le \int_{\mathbb{R}^N \setminus B_{R_0}(0)} J \star w(t,x)\,dx - \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx - \kappa \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx.$$

By Fubini's Theorem, the uniform estimate on $\|w\|_\infty$ and the unit mass of the kernel, we can check that for $t \ge t_0$

$$\begin{aligned}\int_{\mathbb{R}^N \setminus B_{R_0}(0)} J \star w(t,x)\,dx &\le \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,y)\,dy + \int_{B_{R_0}(0)} w(t,y)\,dy,\\ &\le \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,y)\,dy + \kappa\varepsilon.\end{aligned}$$

Therefore for $t \ge t_0$, w satisfies

$$\frac{\partial \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx}{\partial t} \le \kappa\varepsilon - \kappa \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx.$$

From this differential inequality, there exists $t(\varepsilon) \ge t_0$ such that for all $t \ge t(\varepsilon)$ we have

$$\int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx \le 2\varepsilon.$$

Hence, for $t \ge t(\varepsilon)$ we have

$$\|w\|_{L^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx + \int_{B_{R_0}(0)} w(t,x)\,dx \le \left(2 + \frac{\kappa}{|B_{R_0}(0)|}\right)\varepsilon,$$

As above, ε being arbitrary, we see that $\lim_{t\to\infty} \|w\|_{L^1(\mathbb{R}^N)} = 0$.

6 Some asymptotics

In this section we analyse the qualitative behaviour of the solution of (1.2) with respect to the size of the support of J. For convenience we investigate the particular situation

$$\frac{1}{\varepsilon^m}(J_\varepsilon \star u - u) + u(a(x) - u) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{P_ε}$$

where $J_\varepsilon(z) = \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ with $\text{supp}(J) = \bar{B}(0,1)$ and $a \in C^1(\mathbb{R}^N)$ so that $a^+ \not\equiv 0$. This last condition on $a(\cdot)$ is necessary for the existence of a solution. Indeed, if $a^+ \equiv 0$ then for any positive constant c_0 we have

$$\mathcal{M}[c_0] + a(x)c_0 \leq 0.$$

Therefore, $\lambda_p(\mathcal{M}[c_0] + a(x)c_0) \geq 0$ and for all ε there is no solution of (P_ε) besides 0.

We analyse the behaviour of u_ε when $\varepsilon \to 0$ and $\to +\infty$ and seek to understand the influence of m on the resulting limits. Now we start by showing some *a priori* estimates for the solution u_ε.

Lemma 6.1. *There exists positive constants C_1, C_2, C_3 such that for any positive bounded solution u_ε of (P_ε) the following estimates hold*

(i) $\|u_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C_1, \quad \|u_\varepsilon\|_\infty < C_3,$

(ii) $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(x-y)(u_\varepsilon(x) - u_\varepsilon(y))^2\, dxdy \leq C_2\varepsilon^m$

(iii) $\sup_{supp(a^+)} u_\varepsilon \geq -\frac{\lambda_p(\mathcal{M}_{\varepsilon,m}+a(x))}{2}$.

(iv) $u_\varepsilon \geq (a(x) - \frac{1}{\varepsilon^m})^+$,

Proof. By construction the solution is unique and $u_\varepsilon \in L^1(\mathbb{R}^N) \cap L^\infty$. By (P_ε), $u_\varepsilon \leq M = \|a\|_\infty$. We derive (i) by integrating (P_ε) over $\mathbb{R}^N$. Indeed, we get

$$\int_{\mathbb{R}^N} u_\varepsilon^2(x)\, dx = \int_{\mathbb{R}^N} a(x)u_\varepsilon(x)\, dx \leq \int_{\mathbb{R}^N} a^+(x)u_\varepsilon(x)\, dx \leq M\int_{\mathbb{R}^N} a^+(x)dx =: C_1.$$

To obtain (ii), let us multiply (P_ε) by u_ε and integrate over $\mathbb{R}^N$ to get

$$\frac{1}{2\varepsilon^m}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u_\varepsilon(x) - u_\varepsilon(y))^2\, dxdy = \int_{\mathbb{R}^N} u_\varepsilon^2(x)(a(x) - u_\varepsilon(x))\, dx.$$

Since u_ε and $a(x)$ are uniformly bounded independently of ε, (ii) holds with $C_2 := 4C_1M$. Observe that $(a(x) - \frac{1}{\varepsilon^m})^+$ is always a sub-solution of (P_ε), so by the standard sweeping principle $u_\varepsilon \geq (a(x) - \frac{1}{\varepsilon^m})^+$ and (iv) holds.

Finally let us derive (iii). Since u_ε is a positive bounded solution of (P_ε) by Theorem 1.1 we know that $\lambda_p(\mathcal{M}_{\varepsilon,m} + a) < 0$. Consequently, and as J is compactly supported, by regularising a if need be, we can find $\varphi_\varepsilon \in C_c(\mathbb{R}^N)$ so that

$$\mathcal{M}_{\varepsilon,m}[\varphi_\varepsilon](x) + (a(x) + \frac{\lambda_p}{2})\varphi_\varepsilon(x) \geq 0 \quad \text{in} \quad \mathbb{R}^N,$$

see the proof of Lemma 3.1 in [3]. Moreover, we can normalised φ_ε by imposing $\|\varphi_\varepsilon\|_\infty = 1$. Plugging $\theta\varphi_\varepsilon$ in (P_ε), it follows that

$$\mathcal{M}_{\varepsilon,m}[\theta\varphi_\varepsilon] + \theta\varphi_\varepsilon(a(x) - \theta\varphi_\varepsilon) \geq \theta\varphi_\varepsilon\left(-\frac{\lambda_p}{2} - \theta\varphi_\varepsilon\right).$$

Therefore, for $0 < \theta \leq -\frac{\lambda_p}{2}$, the function $\theta\varphi_\varepsilon$ is a sub-solution to (P_ε). By the sweeping argument we have already mentioned, we get

$$-\frac{\lambda_p}{2}\varphi_\varepsilon \leq u_\varepsilon \quad \text{and} \quad \sup_{\mathbb{R}^N} u_\varepsilon \geq -\frac{\lambda_p}{2}.$$

Since $u_\varepsilon \in L^1(\mathbb{R}^N)$, u_ε achieves its maximum at some point, say x_0. From (P_ε), we infer

$$0 \geq \mathcal{M}_{\varepsilon,m}[u_\varepsilon](x_0) = -u_\varepsilon(x_0)(a(x_0) - u_\varepsilon(x_0)).$$

This implies that $x_0 \in \text{supp}(a^+)$ and thus $\|u_\varepsilon\|_\infty = \sup_{\text{supp}(a^+)} u_\varepsilon$ which proves (iii). □

Next we derive an upper bound for large ε.

Lemma 6.2. *There exists $\varepsilon_0 > 0$ such that for all $m \geq 0$ and $\varepsilon \geq \varepsilon_0$, any positive bounded solution u_ε of (P_ε) satisfies*

$$u_\varepsilon \leq a^+(x) + \frac{1}{\varepsilon^{\frac{N}{4}}}$$

Proof. Let $\delta \in (0, \frac{N}{2})$ and consider the function $\zeta_\varepsilon(x) := \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)$. We will show that ζ_ε is a super-solution to (P_ε) when ε is large enough.

Indeed, we have

$$\begin{aligned}\mathcal{M}_{\varepsilon,m}[\zeta_\varepsilon](x) + \zeta_\varepsilon(x)(a(x) - \zeta_\varepsilon(x)) &\leq \frac{\|J\|_\infty}{\varepsilon^{N+m}}\int_{\mathbb{R}^N} a^+(y)\,dy + \left(\frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)\right)\left[a(x) - \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} - a^+(x)\right] \\ &\leq \frac{\|J\|_\infty}{\varepsilon^{N+m}}\int_{\mathbb{R}^N} a^+(y)\,dy - \frac{1}{\varepsilon^{N-2\delta}}.\end{aligned}$$

where in the last inequality we use

$$\left(\frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)\right)\left[a(x) - \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} - a^+(x)\right] \leq -\frac{1}{\varepsilon^{N-2\delta}} \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Thus, for ε large enough, we get

$$\mathcal{M}[\zeta_\varepsilon](x) + \zeta_\varepsilon(x)(a(x) - \zeta_\varepsilon(x)) \leq \frac{\|J\|_\infty}{\varepsilon^{N+m}}\int_{\mathbb{R}^N} a^+(y)\,dy - \frac{1}{\varepsilon^{N-2\delta}} < 0.$$

Therefore for $\varepsilon >> 1$, we get $u_\varepsilon \leq \zeta_\varepsilon$. We end the proof by taking $\delta = \frac{N}{4}$.

□

Remark 2. When $m = 0$ and $(a(x) - 1)^+ \not\equiv 0$, the above computation holds as well with $\zeta_\varepsilon(x) := \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + (a(x) - 1)^+$. Thus in this case, for large ε, we have

$$u_\varepsilon(x) \leq \frac{1}{\varepsilon^{\frac{N}{4}}} + (a(x) - 1)^+.$$

Next, we prove the continuity of $\lambda_p(\mathcal{L}_{R,\varepsilon} + a(x))$ with respect to ε.

Lemma 6.3. *Let R, ε be fixed and positive then for all $\eta > 0$ there exists $\delta > 0$ so that*

$$|\lambda_p(\mathcal{L}_R + a_\varepsilon(x)) - \lambda_p(\mathcal{L}_R + a_{\varepsilon+\delta}(x))| \leq \eta,$$

where $a_\varepsilon(x) := a(\varepsilon x)$.

Proof. Let $\varepsilon > 0$ and $R > 0$ be fixed. We observe that for all $|\delta| < \varepsilon$ we have for all $x \in \mathbb{R}^N, a_{\varepsilon+\delta}(x) = a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)$ therefore

$$\|a_\varepsilon - a_{\varepsilon+\delta}\|_{\infty,R} = \sup_{B(0,R)} \left\| a_\varepsilon(x) - a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)\right\|.$$

Since a_ε is Lipschitz continuous in $\mathbb{R}^N$, we have

$$\left\| a_\varepsilon(x) - a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)\right\| \leq K(\varepsilon)\varepsilon\delta\|x\|,$$

where $K(\varepsilon)$ is the Lipschitz constant of a_ε. Thus

$$\|a_\varepsilon - a_{\varepsilon+\delta}\|_{\infty,R} \leq K(\varepsilon)R\varepsilon\delta.$$

Hence, by Proposition 3.2 (ii) we get

$$|\lambda_p(\mathcal{L}_R + a_\varepsilon(x)) - \lambda_p(\mathcal{L}_R + a_{\varepsilon+\delta}(x))| \leq K(\varepsilon)R\varepsilon\delta.$$

□

Lastly, we require the following identity.

Proposition 6.4. *Let $\rho \in C_c^\infty(\mathbb{R}^N)$ be a radial function, then for all $u \in L^2(\mathbb{R}^N), \varphi \in C_c^\infty(\mathbb{R}^N)$ we have*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)]\varphi(x)\,dzdx = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z)]\,dzdx.$$

Proof. Let us denote the left hand side of the equality by I, i.e.

$$I := \iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)]\varphi(x)\,dzdx.$$

By change of variables and thanks to the spherical symmetry of ρ, we get

$$\begin{aligned}
I &= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)]\varphi(x) + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(-z)[u(x-z) - u(x)]\varphi(x),\\
&= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)]\varphi(x) + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x) - u(x+z)]\varphi(x+z),\\
&= -\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)][\varphi(x+z) - \varphi(x)],\\
&= -\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x) - \varphi(x-z)] + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x+z) - \varphi(x)],\\
&= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z)].
\end{aligned}$$

□

From this Proposition, we get the following identity for all $u \in L^2(\mathbb{R}^N), \varphi \in C_c^\infty(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} \mathcal{M}_{\varepsilon,m}[u](x)\varphi(x)\,dx = \frac{\varepsilon^{2-m}D_2(J)}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_\varepsilon(z)}{|z|^2} u_\varepsilon(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z)]\,dxdz \tag{6.1}$$

where $\rho_\varepsilon(z) = \frac{1}{\varepsilon^N D_2(J)} J\left(\frac{z}{\varepsilon}\right) \frac{|z|^2}{\varepsilon^2}$.

With these various *apriori* estimates, we can now analyse the asymptotic behaviour of u_ε.

6.1 The case $m = 0$

In this situation, from Theorem 3.6 we know that

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{6.2}$$

$$\lim_{\varepsilon\to +\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = 1 - \sup_{\mathbb{R}^N} a(x) \tag{6.3}$$

As a consequence, for ε small enough, $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \le -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$ and, by Theorem (1.1), there exists a positive solution of (P_ε). Moreover the following quantity is well defined

$$\varepsilon^* := \sup\{\varepsilon > 0 \,|\, \text{ for all } \varepsilon' < \varepsilon, \text{ there exists a positive solution to } (P_{\varepsilon'})\}.$$

In view of (6.3) $\varepsilon^* \in (0, +\infty]$ and $\varepsilon^* < +\infty$ if and only if $(a(x) - 1)^+ \not\equiv 0$.

Let us now determine the limit of u_ε as $\varepsilon \to 0$ and $\varepsilon \to +\infty$. We start by proving that

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = v(x) \quad \text{a.e.} \tag{6.4}$$

where v is a non negative bounded solution of

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{6.5}$$

Let $w_\varepsilon := a(x) - u_\varepsilon$, then from (P_ε), w_ε satisfy

$$-J_\varepsilon \star w_\varepsilon + w_\varepsilon + u_\varepsilon(x)w_\varepsilon(x) = a(x) - J_\varepsilon \star a(x). \tag{6.6}$$

Multiplying this equation by w_ε^+ and integrating over $\mathbb{R}^N$, it follows that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy + \int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2 = \int_{\mathbb{R}^N} w_\varepsilon^+ g_\varepsilon,$$

with $g_\varepsilon(x) := a(x) - J_\varepsilon \star a(x)$.

Let us now estimate the above integrals. First, we observe that the double integral is positive. Indeed, since $w(y) = w^+(y) - w^-(y)$ we get

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy = \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon^+(y)w_\varepsilon^+(x))\,dxdy$$
$$+ \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\,dxdy.$$

Thus,

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x)-w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)(x)-w_\varepsilon^+(y))^2\,dxdy + \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\,dxdy. \tag{6.7}$$

Let us denote $Q := \mathrm{supp}(a^+)$. Since u_ε is positive and uniformly bounded, we have $\mathrm{supp}(w^+)\subset Q$ and

$$\left|\int_{\mathbb{R}^N} w_\varepsilon^+ g_\varepsilon\right| \le C\int_Q |g_\varepsilon|.$$

Since a is Lipschitz continuous, a Taylor expansion leads to $|g_\varepsilon(x)|\le \varepsilon D_2(J)\|\nabla a\|_\infty$. Therefore,

$$\left|\int_{\mathbb{R}^N} w_\varepsilon^+ g_\varepsilon\right| \le C|Q|\varepsilon. \tag{6.8}$$

Collecting (6.7) and (6.8), we get

$$\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)(x)-w_\varepsilon^+(y))^2\,dxdy + \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\,dxdy + \int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2 \le C\varepsilon.$$

Thus,

$$\int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2 \le C\varepsilon$$

and $u_\varepsilon w_\varepsilon^+\to 0$ almost everywhere in Q.

Recalling that

$$\int_{\mathbb{R}^N} u_\varepsilon w_\varepsilon = 0,$$

from the above estimates we conclude that

$$\int_{\mathbb{R}^N\setminus Q} u_\varepsilon(a(x)-u_\varepsilon) = \int_Q u_\varepsilon(a(x)-u_\varepsilon)\to 0 \quad \text{when} \quad \varepsilon\to 0.$$

Since $u_\varepsilon(a(x)-u_\varepsilon)\le 0$ in $\mathbb{R}^N\setminus Q$, it follows that $u_\varepsilon(x)w_\varepsilon\to 0$ almost everywhere in $\mathbb{R}^N\setminus Q$. Since $u_\varepsilon>0$ and $w_\varepsilon=(a(x)-u_\varepsilon(x))\le 0$, it follows that $u_\varepsilon\to 0$ almost everywhere in $\mathbb{R}^N\setminus Q$. Hence, u_ε converges pointwise almost everywhere to a bounded non-neqative solution of (6.5).

Remark 3. Note that the above proof can easily be adapted to $\mathcal{M}_{\varepsilon,m}$ for $m<2$ as soon as the function a is smooth enough. Indeed, for $a\in C^2(\mathbb{R}^N)$, following the above arguments, we get by using the Taylor expansion up to order 2 of a

$$\int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2(x)\,dx \le C\varepsilon^{2-m},$$

with a constant C only depending on $\|\nabla^2 u\|_\infty$. When a is only Lipschitz, this argument is valid for $\mathcal{M}_{\varepsilon,m}$ only when $m<1$.

Finally, to complete our analysis, we need to check that

$$\lim_{\varepsilon\to\varepsilon^*} u_\varepsilon = (a(x)-1)^+. \tag{6.9}$$

We treat separately the following two cases : (i) $\varepsilon^* < +\infty$, (ii) $\varepsilon^* = \infty$. The latter case arises when $\sup_{\mathbb{R}^N}(a(x)-1) > 0$. In this situation, there exists $R_0 > 0$ such that the continuous function $\varphi = (a(x)-1)^+ \not\equiv 0$ in $B_R(0)$ for $R \geq R_0$ and we can check that φ is a sub-solution for the approximated problem:

$$\int_{B_R(0)} J_\varepsilon(x-y)u(y)\,dy - u(x) + u(x)(a(x)-u) = 0 \quad \text{in} \quad B_R(0). \tag{6.10}$$

Since large constants are super-solutions of (6.10) for any $\varepsilon \geq 0, R > R_0$, there exists a unique solution $u_{\varepsilon,R}$ with $\varphi \leq u_{\varepsilon,R} \leq M$. By sending $R \to \infty$ and by the uniqueness of the solution of (P_ε) we have $\varphi \leq u_\varepsilon \leq M$ in $\mathbb{R}^N$.

Case $\varepsilon^* = +\infty$:

Owing to Lemma 6.1 and by Remark 2, for all $x \in \mathbb{R}^N$ for large ε we get

$$(a(x)-1)^+ \leq u_\varepsilon(x) \leq (a(x)-1)^+ + \frac{1}{\varepsilon^{\frac{N}{4}}}.$$

Hence, u_ε converge uniformly to $(a(x)-1)^+$.

Case $\varepsilon^* < +\infty$:

In this situation, the function $(a(x)-1)^+ \equiv 0$ in $\mathbb{R}^N$ and the problem is reduced to prove that

$$\lim_{\varepsilon\to\varepsilon^*} u_\varepsilon(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Note that by definition of ε^* we must have $\lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) \geq 0$. Indeed, if not, then $\lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) < 0$ and by Lemma 3.4, $\lambda_p(\mathcal{M} + a_{\varepsilon^*}(x)) < 0$. This implies that for some R, $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*}(x)) < 0$. By continuity of $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*}(x))$ with respect to ε, (Lemma 6.3) we get for some $\delta_0 > 0$, $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*+\delta}(x)) < 0$ for any $\delta \leq \delta_0$. Hence, $\lambda_p(\mathcal{M}_{\varepsilon^*+\delta} + a(x)) = \lambda_p(\mathcal{M} + a_{\varepsilon^*+\delta}(x)) < 0$ for any $\delta \leq \delta_0$ and by Theorem 1.1 there exists a positive solution of (P_ε) for all $\varepsilon \leq \varepsilon^* + \delta_0$ thus contradicting the definition of ε^*.

Note also that since $\varepsilon^* < +\infty$, the construction of the supersolution in Section 4 holds for any $\varepsilon \in [\frac{\varepsilon^*}{2}, \varepsilon^*]$, thus u_ε is uniformly bounded in $L^1(\mathbb{R}^N)$.

Let $g(x,s) := s(a(x)-1-s)$ then for all ε we have

$$J_\varepsilon \star u_\varepsilon = -g(x, u_\varepsilon(x)) \quad \text{in} \quad \mathbb{R}^N.$$

Since J is C^1 and $\varepsilon^* > 0$, for $\varepsilon \in [\frac{1}{2}\varepsilon^*, \varepsilon^*)$, we have

$$\begin{aligned}
|g(x,u_\varepsilon(x)) - g(z,u_\varepsilon(z))| &= \left|\int_{\mathbb{R}^N} [J_\varepsilon(x-y) - J(z-y)]u_\varepsilon(y)\,dy\right|, \\
&\leq |x-z| \int_{\mathbb{R}^N} \frac{|J_\varepsilon(x-y) - J(z-y)|}{|x-z|} u_\varepsilon(y)\,dy, \\
&\leq C(\varepsilon^*)|x-z|.
\end{aligned}$$

This leads to :

$$\begin{aligned} C(\varepsilon^*)|x-z| &\geq |[1-a(x)+u_\varepsilon(x)+u_\varepsilon(z)][u_\varepsilon(x)-u_\varepsilon(z)]+[a(z)-a(x)]u_\varepsilon(x)|, \\ &\geq |[1-a(x)+u_\varepsilon(x)+u_\varepsilon(z)]||u_\varepsilon(x)-u_\varepsilon(z)|-|x-z|\frac{|a(z)-a(x)|}{|x-z|}M. \end{aligned}$$

From the last inequality, it follows that u_ε is uniformly Lipschitz in $Q := \{y \in \mathbb{R}^N \,|\, a(y) < 1\}$ u_ε with a Lipschitz constant independent of ε. Thus, $(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)}$ is uniformly bounded in $C^{0,\frac{1}{2}}_{loc}(Q)$. If $Q^c = \emptyset$, then $(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)}$ is uniformly bounded in $C^{0,\frac{1}{2}}_{loc}(\mathbb{R}^N)$. Otherwise, $Q^c \neq \emptyset$ and on Q^c we have $a(x) \equiv 1$. Therefore, on Q^c, $u^2_\varepsilon(x) = J_\varepsilon \star u_\varepsilon$ and the $C^{0,\frac{1}{2}}(\overset{\circ}{Q^c})$ norm of u_ε is bounded independently of ε. Hence,

$$(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)} \quad \text{is uniformly bounded in} \quad C^{0,\frac{1}{2}}_{loc}(Q)\cap C^{0,\frac{1}{2}}(\overset{\circ}{Q^c}). \tag{6.11}$$

In both case, since $a(x) < 0$ for $|x| >> 1$, Q^c is a compact set and $|\bar{Q}\cap Q^c| = 0$. From (6.11), for all sequence $\varepsilon_n \to \varepsilon^*$ by a diagonal extraction procedure there exists a subsequence still denoted $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ that converges locally uniformly in $\mathbb{R}^N \setminus (\bar{Q}\cap Q^c)$ to some non-negative function v. By passing to the limit in (P_ε) we can see that v is a bounded non negative solution of

$$J_{\varepsilon^*} \star v(x) - v(x) + v(x)(a(x)-v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N \setminus (\bar{Q}\cap Q^c).$$

Since $\bar{Q}\cap Q^c$ is of zero measure v is a solution to

$$\mathcal{L}_{\varepsilon^*,\mathbb{R}^N\setminus(\bar{Q}\cap Q^c)}[v] + v(x)(a(x)-v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N \setminus (\bar{Q}\cap Q^c).$$

Since $0 \leq \lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) \leq \lambda_p(\mathcal{L}_{\varepsilon^*,\mathbb{R}^N\setminus(\bar{Q}\cap Q^c)} + a(x))$, we deduce that $v \equiv 0$ which concludes the proof of the limit.

Remark 4. When $a(x)$ is a radially symmetric non-increasing function we remark that ε^* is a sharp threshold. That is for all $\varepsilon \geq \varepsilon^*$ then (P_ε) does not have any positive solutions. Indeed in this situation, the function $a_\varepsilon(x)$ is monotone non increasing with respect to ε. Thus, by (i) of Proposition 3.2, for all $\varepsilon \geq \varepsilon^*$ we have

$$0 = \lambda_p(\mathcal{M} + a_{\varepsilon^*}(x)) \leq \lambda_p(\mathcal{M} + a_\varepsilon(x)).$$

Hence, by Theorem 1.1, 0 is the unique non negative solution to (P_ε) for $\varepsilon \geq \varepsilon^*$.

6.2 The case $0 < m < 2$

In this situation, from Theorem 3.6 we know that

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{6.12}$$

$$\lim_{\varepsilon\to +\infty} \lambda_p(\mathcal{M}_{\varepsilon} + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{6.13}$$

As a consequence, for ε small enough and for large ε we have $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \leq -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$. Therefore, by Theorem (1.1) there exists a solution of (P_ε) for both small and large ε.

The limit of u_ε when $\varepsilon \to \infty$ is an obvious consequence of (iv) of Lemma 6.1 and Lemma 6.2 since for ε large enough

$$(a(x) - \frac{1}{\varepsilon^m})^+ \leq u_\varepsilon \leq a^+(x) + \frac{1}{\varepsilon^{\frac{N}{4}}}.$$

To obtain the limits in L^2, we just observe that since by Lemma u_ε is uniformly bounded in L^2 and converges pointwise to a^+, we get $u_\varepsilon \rightharpoonup a^+$ in L^2. Moreover by Fatou's Lemma, we infer that

$$\int_{\mathbb{R}^N} (a^+)^2(x)\, dx \leq \liminf_{\varepsilon\to\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\, dx.$$

On the other hand, by integrating (P_ε) over $\mathbb{R}^N$ we get for all ε

$$\int_{\mathbb{R}^N} u_\varepsilon^2(x)\, dx = \int_{\mathbb{R}^N} a(x)u_\varepsilon(x)\, dx \leq \int_{\mathbb{R}^N} a^+(x)u_\varepsilon(x)\, dx.$$

By the Cauchy-Schwartz inequality, for all ε

$$\left(\int_{\mathbb{R}^N} u_\varepsilon^2(x)\, dx\right)^{1/2} \leq \left(\int_{\mathbb{R}^N} (a^+)^2(x)\, dx\right)^{1/2}$$

and we get

$$\int_{\mathbb{R}^N} (a^+)^2(x)\, dx \leq \liminf_{\varepsilon\to\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\, dx \leq \limsup_{\varepsilon\to+\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\, dx \leq \int_{\mathbb{R}^N} (a^+)^2(x)\, dx.$$

Hence, $\|u_\varepsilon\|_2 \to \|a^+\|_2$ and by the parallelogram identity $u_\varepsilon \to a^+$ in $L^2(\mathbb{R}^N)$ since u_ε converges weakly to a^+ in L^2.

As already mentioned in Remark 3, it can be seen that u_ε has a limit when $\varepsilon \to 0$ as soon as a is smooth enough. Thus,

$$\lim_{\varepsilon\to 0} u_\varepsilon = v(x)$$

where v is a non-negative bounded solution of (6.5).

6.3 The case $m = 2$

In this situation, from Theorem 3.6 we have

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = \lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right) \tag{6.14}$$

$$\lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{6.15}$$

As a consequence, for large ε we have $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \leq -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$ and by Theorem (1.1) there exists a solution of (P_ε) for large ε. For ε small, the existence of a positive solution is conditioned by the sign of $\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right)$. When $\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right) > 0$, for ε small there exists no positive solution of (P_ε). The limit of u_ε when $\varepsilon \to +\infty$ is obtained in the same way as in the case $2 > m > 0$. Hence, we only focus here on the limit when $\varepsilon \to 0$.

Assume for the moment that $\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right) < 0$. We now show that $u_\varepsilon \to v$ where v is the positive solution of

$$\frac{D_2(J)}{2N}\Delta v + v(a(x) - v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive reals converging to 0. We write u_n instead of u_{ε_n}. By Lemma 6.1, $\|u_n\|_2$ is bounded uniformly and a simple algebraic computation yields :

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho_n(z)\frac{(u_n(x+z) - u_n(x))^2}{|z|^2}\,dxdz < C,$$

with C independent of n. Therefore, for any $R > 0$, we see that

$$\iint_{B_R\times B_R} \rho_\varepsilon(z)\frac{(u_n(x+z) - u_n(x))^2}{|z|^2}\,dxdz < C.$$

For $R > 0$ fixed, since $\|u_n\|_2$ is uniformly bounded in L^2, there exists a subsequence $u_n \rightharpoonup v$ in $L^2(B_R)$ and from the characterisation of Sobolev Space [58, 57], we have $u_n \to v$ in $L^2(B_R)$.

By a standard diagonal extraction argument, from the sequence $(u_n)_{n\in\mathbb{N}}$ we can then extract a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ which converges to some v in $L^2_{loc}(\mathbb{R}^N)$. Moreover, by Lemma 6.1, u_n is uniformly bounded and there exists $\delta(\lambda_1) > 0$ independent of ε such that $\max_{supp(a^+)}(u_n) > \delta$.

Multiplying (P_ε) by $\varphi \in C_c^\infty(\mathbb{R}^N)$ and integrating yields :

$$\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z)]\,dxdz + \int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x) - u_n(x))\,dx = 0,$$

where we use (6.1) to compute $\int_{\mathbb{R}^N}\mathcal{M}_{\varepsilon,2}[u_n](x)\varphi(x)\,dx$. This leads us to

$$\begin{aligned}\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)\,{}^t z\nabla^2\varphi(x)z\,dxdz + \int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x) - u_n(x))\,dx \\ = -\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z) - {}^t z\nabla^2\varphi(x)z]\,dxdz,\end{aligned}$$

where $\nabla^2\varphi(x) := (\partial_{ij}\varphi(x))_{i,j}$. Since $\rho_n(z)$ is radially symmetric, we can see that

$$\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)\,{}^t z\nabla^2\varphi(x)z\,dxdz = \frac{D_2(J)K_{2,N}}{2}\int_{\mathbb{R}^N}u_n(x)\Delta\varphi(x)\,dx$$

with

$$K_{2,N} := \fint_{S^{N-1}}(\sigma\cdot e_1)^2 d\sigma = \frac{1}{N}.$$

Thus, we get

$$\begin{aligned}\frac{D_2(J)}{2N}\int_{\mathbb{R}^N}u_n(x)\Delta\varphi(x)\,dx + \int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x) - u_n(x))\,dx \\ = -\frac{D_2(J)}{2}\iint_{\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z) - {}^t z\nabla^2\varphi(x)z]\,dxdz, \quad (6.16)\end{aligned}$$

Note that since u_n converges to v in $L^2_{loc}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \varphi(x)u_n(x)(a(x)-u_n(x))\,dx \to \int_{\mathbb{R}^N} \varphi(x)v(x)(a(x)-v(x))\,dx \tag{6.17}$$

Recall that $\varphi \in C_c^\infty(\mathbb{R}^N)$, so there exists $C(\varphi)$ and $R(\varphi)$ so that

$$|\varphi(x+z)-2\varphi(x)+\varphi(x-z)-{}^t z\nabla^2\varphi(x)z| < C(\varphi)|z|^3\chi_{B_{R(\varphi)}}(x).$$

Because u_n is bounded uniformly we obtain

$$\frac{D_2(J)}{2}\iint_{\mathbb{R}^N} \frac{\rho_\varepsilon(z)}{|z|^2}u_n(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)-{}^t z\nabla^2\varphi(x)z]\,dxdz \le CC(\varphi)\int_{\mathbb{R}^N}\rho_n(z)|z| \to 0. \tag{6.18}$$

Passing to the limit $\varepsilon \to 0$ in (6.16), using (6.17) and (6.18), we get

$$\frac{D_2(J)}{2N}\int_{\mathbb{R}^N} v(x)\Delta\varphi(x)\,dx + \int_{\mathbb{R}^N}\varphi(x)v(x)(a(x)-v(x))\,dx = 0. \tag{6.19}$$

(6.19) being true for any $\varphi \in C_c^\infty$ this implies that v satisfies

$$\frac{D_2(J)}{2N}\Delta v + v(a(x)-v) = 0 \quad \text{a.e. in} \quad \mathbb{R}^N.$$

Since v is bounded, by elliptic regularity v is smooth. To conclude we need to prove that v is non zero. To this end, we claim that

Lemma 6.5. *There exists R_0, τ and ε_0 positive constants so that for all $\varepsilon \le \varepsilon_0$ we have $u_\varepsilon \ge \tau$ almost everywhere in $B_{R_0}(0)$.*

From the above claim, we deduce that $v \ge \tau > 0$ *a.e.* and therefore $v \equiv u$, the unique smooth non-trivial solution of

$$\frac{D_2(J)}{2N}\Delta u + u(a(x)-u) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

The sequence $(\varepsilon_n)_n$ being arbitrary, it follows that $u_\varepsilon \to u$ in $L^2_{loc}(\mathbb{R}^N)$.

Similarly, if we assume now that $\lambda_1(\frac{D_2(J)}{2N}\Delta + a(x)) = 0$ and there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}, \varepsilon_n \to 0$ of non trivial solution of (P_ε). The above argument is valid and we get $u_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$ with v a smooth solution to

$$\frac{D_2(J)}{2N}\Delta v + v(a(x)-v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Since $\lambda_1(\frac{D_2(J)}{2N}\Delta + a(x)) = 0$, $v \equiv 0$ is the only solution and we get $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$.

Let us complete our proof and establish the Lemma.

Proof. Let us denote, $\mathcal{L}$ the operator

$$\mathcal{L}[\varphi] := \frac{1}{\varepsilon^2}\left[\int_{B_R(0)} J_\varepsilon(x-y)\varphi(y)\,dy - \varphi(x)\right].$$

Since $\sup_{\mathbb{R}^N} a(x)$ is achieved on $\mathbb{R}^N$ we regularise a by a_σ independently of ε, so that for all ε and $R \ge R_1$ the principal eigenvalue $\lambda_p(\mathcal{L}_+ a_\sigma(x))$ is associated with a continuous principal eigenfunction $\varphi_{p,\varepsilon}$ and

$$|\lambda_p(\mathcal{L}_{R,\varepsilon,2} + a_\sigma(x)) - \lambda_p(\mathcal{L}_{R,\varepsilon,2} + a(x))| \le \|a_\sigma(x) - a(x)\|_\infty \le \kappa\sigma,$$

where κ is the Lipschitz constant of a.

By the Lipschitz continuity of $\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right)$ with respect to a, we can choose σ small enough so that

$$\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a_\sigma(x)\right) \leq \frac{1}{2}\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right) < 0.$$

Recall that

$$\lim_{R\to\infty} \lambda_1\left(\frac{D_2(J)}{2N}\Delta + a_\sigma(x), B_R\right) = \lambda_1\left(\frac{D_2(J)}{2N}\Delta + a_\sigma(x)\right),$$

So we can choose R_0 large so that

$$\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a_\sigma(x), B_{R_0}\right) \leq \frac{1}{4}\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right).$$

Thanks to Theorem 3.7, we have $\lim_{\varepsilon\to 0}\lambda_p(\mathcal{L}_{R,\varepsilon,2}+a_\sigma(x)) = \lambda_1\left(\frac{D_2(J)}{2N}\Delta + a_\sigma(x), B_R\right)$ so for ε small,say $\varepsilon \leq \varepsilon_0$ by choosing σ smaller if necessary, we achieve

$$\lambda_p(\mathcal{L}_{R,\varepsilon,2} + a_\sigma(x)) \leq \frac{1}{8}\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right) \quad \text{for all} \quad \varepsilon \leq \varepsilon_0.$$

Let $\varphi_{p,\varepsilon}$ be the principal eigenfunction associated with $\mathcal{L}_{R_0,\varepsilon,2} + a_\sigma(x)$, then we have

$$\mathcal{L}_{R_0,\varepsilon,2}[\varphi_{p,\varepsilon}](x) + a(x)\varphi_{p,\varepsilon}(x) \geq \left[-\frac{1}{8}\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right) - \kappa\sigma\right]\varphi_{p,\varepsilon}(x) \quad \text{for all} \quad \varepsilon \leq \varepsilon_0.$$

By choosing σ smaller if necessary,

$$\left[-\frac{1}{8}\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right) - \kappa\sigma\right] \geq -\frac{1}{16}\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right)$$

and we achieve

$$\mathcal{L}_{R_0,\varepsilon,2}[\varphi_{p,\varepsilon}](x) + a(x)\varphi_{p,\varepsilon}(x) \geq -\frac{1}{16}\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right)\varphi_{p,\varepsilon}(x) \quad \text{for all} \quad \varepsilon \leq \varepsilon_0. \tag{6.20}$$

To conclude our proof, it is then enough to show that for some well chosen normalisation of $\varphi_{p,\varepsilon}$ we have

$$\varphi_{p,\varepsilon}(x) \to \varphi_1(x), \quad \text{a.e. in} \quad B_{R_0} \tag{6.21}$$

φ_1 is a positive principal eigenfunction associated with $\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a_\sigma(x), B_{R_0}\right)$. Indeed, assume for the moment that (6.21) holds true. Then there exists $\alpha > 0$ so that

$$\alpha\varphi_{p,\varepsilon}(x) \to \alpha\varphi_1(x) < \frac{1}{2} \quad \text{a.e. in} \quad B_{R_0}.$$

Now thanks to (6.20), we can now adapt the proof the proof of (iii) of Lemma 6.1 to get for ε small, says $\varepsilon \leq \varepsilon_1$,

$$u_\varepsilon(x) \geq -\frac{\alpha}{32}\lambda_1\left(\frac{D_2(J)}{2N}\Delta + a(x)\right)\varphi_{p,\varepsilon}(x) \quad \text{a.e. in} \quad B_{R_0}, \tag{6.22}$$

which combined with (6.21) enforces

$$u_\varepsilon(x) \geq \gamma\varphi_1(x) \quad \text{a.e. in} \quad B_{R_0}, \quad \text{for all} \quad \varepsilon \leq \varepsilon_2,$$

for some $\gamma, \varepsilon_2 > 0$. Since $\varphi_1 > 0$ in B_{R_0}, the claim holds true in any smaller ball B_R.

To prove (6.21), let us normalise $\varphi_{p,\varepsilon}$ by $\|\varphi_{p,\varepsilon}\|_{L^2(B_{R_0})} = 1$. Let k_ε be the function defined by

$$k_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus B_{R_0}} J_\varepsilon(x-y)\,dy.$$

Multiplying by $\varphi_{p,\varepsilon}$ the equation satisfied by $\varphi_{p,\varepsilon}$ and integrating over B_{R_0} yields

$$\begin{aligned}\frac{D_2(J)}{2}\iint_{B_{R_0}\times B_{R_0}} \rho_\varepsilon(x-y)\frac{|\varphi_{p,\varepsilon}(y)-\varphi_{p,\varepsilon}(x)|^2}{|x-y|^2}\,dxdy &= \int_{B_{R_0}}(a_\sigma(x)+\lambda_{p,\varepsilon})\varphi_{p,\varepsilon}^2(x)\,dx - \int_{B_{R_0}} k_\varepsilon(x)\varphi_{p,\varepsilon}^2(x)\,dx \\ &\leq C.\end{aligned}$$

Therefore by the characterisation of Sobolev space [58, 57], along a sequence we have $\varphi_{p,\varepsilon} \to \psi$ in $L^2(B_{R_0})$ with $\|\psi\|_{L^2(B_{R_0})} = 1$. Moreover by extending $\varphi_{p,\varepsilon}$ and φ by 0 outside B_{R_0} and by arguing as above for any $\varphi \in C_c^2(B_{R_0})$ we have

$$\begin{aligned}\frac{D_2(J)}{2}\iint_{B_{R_0}\times\mathbb{R}^N} \frac{\rho_\varepsilon(z)}{|z|^2}\varphi_{p,\varepsilon}(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)]\,dxdz &= -\int_{B_{R_0}}\varphi(x)\varphi_{p,\varepsilon}(a(x)+\lambda_{p,\varepsilon})\,dx \\ &\quad + \int_{B_{R_0}} k_\varepsilon(x)\varphi_{p,\varepsilon}(x)\varphi(x)\,dx.\end{aligned}$$

Since $\varphi \in C_c^2(B_{R_0})$ we get for ε small enough $supp(k_\varepsilon) \cap supp(\varphi) = \emptyset$. Thus passing to the limit along a sequence in the above equation yields

$$\frac{D_2(J)K_{2,N}}{2}\int_{B_{R_0}}\psi(x)\Delta\varphi(x)\,dx + \int_{B_{R_0}}\varphi(x)\psi(x)(a(x)+\lambda_1)\,dx = 0. \tag{6.23}$$

The relation (6.23) being true for any φ, it follows that ψ is the smooth positive eigenfunction associated to λ_1 normalised by $\|\psi\|_{L^2(B_{R_0})} = 1$. ψ being uniquely defined, we get $\varphi_{p,\varepsilon} \to \psi$ in $L^2(B_{R_0})$ when $\varepsilon \to 0$. Thus along any sequence $\varphi_{p,\varepsilon}(x) \to \varphi_1(x)$ almost everywhere in B_{R_0}.

□

7 Extension to non-compactly supported kernels

In this section, we discuss the extension of our persistence criteria to more general dispersal kernel J and prove Theorem 1.5. Observe that the construction of positive solution only required that $\lambda_p(\mathcal{L}_R + \beta(x)) < 0$ for some R, regardless of what the dispersal kernel J is. Therefore as soon as $\lim_{R\to\infty}\lambda_p(\mathcal{L}_R + \beta(x)) < 0$

there exists a positive solution to (1.2) with no restriction on the decay of the kernel. Similarly, when $\lambda_p(\mathcal{M}+\beta(x)) > 0$ the proof of the non-existence of positive bounded solution essentially relies on the inequality between $\lambda_p(\mathcal{M}+\beta(x))$ and $\lambda'_p(\mathcal{M}+\beta(x))$ which holds for quite general kernels including those satisfying the assumption $H5$ as proved in [3].Concerning the proof of the uniqueness of the positive solution, it relies on the construction of an integrable uniform super-solution of (1.2) which guarantes the existence of a positive L^1 solution to (1.2). Such super-solution still exists for kernels J that satisfies the decay assumption $H5$. Indeed, we can show

Lemma 7.1. *Assume that J satisfies $H5$ and there exists a periodic function $\mu(x) : \mathbb{R}^N \to \mathbb{R}$ such that*

$$\limsup_{|x|\to\infty}(\beta(x)-\mu(x)) \le 0 \qquad \text{and} \qquad \lambda_p(\mathcal{M}+\mu(x)) > 0.$$

Then there exists $\bar{u} \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $\bar{u} > 0$ so that $\bar{u}$ is a super-solution to (1.2).

Observe that the construction of the super-solution thus covers a class of nonlinearities $f(x,u)$ larger than those that satisfy $H4$. As an immediate consequence, the persistence criteria obtained in Theorem 1.1 still holds for any nonlinearity that satisfies:

(H7) There exists $\mu(x) \in C_p(\mathbb{R}^N)$ such that : $\begin{cases} \lambda_p(\mathcal{M}+\mu(x)) > 0, \\ \limsup_{|x|\to\infty}\left(\frac{f(x,s)}{s} - \mu(x)\right) \le 0 \quad \text{uniformly in } s. \end{cases}$

From an ecological point of view, such a nonlinearity allows one to consider a more complex niche structure for the species. Thus, we can consider ecological niches that are the superposition of a compact niche structure with a periodic structure. Assume that in the (unbounded) periodic structure, there is extinction. Then, this framework allows us to discuss perturbation with compact support from the periodic structure and derive conditions for persistence. The perspective offered by this approach are quite promising and we believe that it may also be applied to investigate a climate change version of (1.2).

Proof. The construction of the super-solution in this situation follows the same general scheme as for a compactly supported kernel. By assumption since $\limsup_{|x|\to\infty}(\beta(x)-\mu(x)) \le 0$, for any $\delta > 0$ there exists $R_\delta > 1$ such that

$$\beta(x) \le \mu(x) + \delta \qquad \text{for all } x, \quad |x| \ge R_\delta.$$

Fix $\delta < \lambda_p(\mathcal{M}+\mu(x))$ and observe that by the definition of $\lambda_p(\mathcal{M}+\mu(x))$ there exists a constant $\delta < \lambda < \lambda_p(\mathcal{M}+\mu(x))$ and a positive periodic function φ such that

$$\mathcal{M}[\varphi](x) + (\mu(x)+\lambda)\varphi(x) \le 0 \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{7.1}$$

Let $w = C\frac{\varphi(x)}{1+\tau|x|^{N+1}}$ with C, τ to be chosen.

$$\begin{aligned}
\mathcal{M}[w] + (\mu(x)+\delta)w(x) &= C(1+\tau|x|^{N+1})^{-1}\left(\int_{\mathbb{R}^N} J(x-y)\frac{(1+\tau|x|^{N+1})}{(1+\tau|y|^{N+1})}\varphi(y)\,dy - \varphi(x) + (\mu(x)+\delta)\varphi(x)\right), \\
&\le C(1+\tau|x|^{N+1})^{-1}\left(\int_{\mathbb{R}^N} J(z)\left[\frac{(1+\tau|x|^{N+1})}{(1+\tau|x+z|^{N+1})} - 1\right]\varphi(x+z)\,dz + (\delta-\lambda)\varphi(x)\right), \\
&\le C(1+\tau|x|^{N+1})^{-1}\left(\tau\int_{\mathbb{R}^N} J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]\varphi(x+z)\,dz + (\delta-\lambda)\varphi(x)\right), \\
&\le w(x)\left(\tau\int_{\mathbb{R}^N} J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]\frac{\varphi(x+z)}{\varphi(x)}\,dz + \delta - \lambda\right),
\end{aligned}$$

where we use (7.1) and $\inf_{\mathbb{R}^N} \varphi > 0$.

Set

$$h(\tau, x) := \tau \int_{\mathbb{R}^N} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] \frac{\varphi(x+z)}{\varphi(x)} \, dz + \delta - \lambda.$$

Since $\varphi \in L^\infty(\mathbb{R}^N)$ and $\inf_{\mathbb{R}^N} \varphi > 0$, there exists a positive constant C_0 such that

$$\frac{\varphi(x+z)}{\varphi(x)} \leq C_0 \quad \text{for all} \quad x, z \in \mathbb{R}^N.$$

For all $x \in \mathbb{R}^N$, we have

$$h(\tau, x) \leq C_0 \tau \int_{\mathbb{R}^N} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz + \delta - \lambda. \tag{7.2}$$

Let

$$I := C_0 \tau \int_{\mathbb{R}^N} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz,$$

then we have

$$I = C_0 \tau \int_{\{|x| \leq 2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz + C_0 \tau \int_{\{|x| > 2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz.$$

Let us estimate the first integral. Since $|x| \leq 2|z|$ we have

$$C_0 \tau \int_{\{|x| \leq 2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz \leq C_0 \tau 2^{N+1} \int_{\mathbb{R}^N} J(z) |z|^{N+1} \, dz. \tag{7.3}$$

Let us now estimate the second term. Since $|x+z|^{N+1} \geq (|x| - |z|)^{N+1}$, we have

$$\begin{aligned} C_0 \tau \int_{\{|x| > 2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz &\leq C_0 \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\{|x| > 2|z|\}} J(z) (-1)^{i+1} |z|^i \left[\frac{\tau |x|^{N+1-i}}{(1+\tau|x+z|^{N+1})} \right] dz, \\ &\leq C_0 \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\{|x| > 2|z|\}} J(z) |z|^i \left[\frac{\tau |x|^{N+1-i}}{(1+\tau|x+z|^{N+1})} \right] dz. \end{aligned}$$

Since $|x| > 2|z|$, we have

$$\frac{1}{1+\tau|x+z|^{N+1}} \leq \frac{2^{N+1}}{2^{N+1} + \tau|x|^{N+1}}$$

and for $|x| \geq R_0 > 1$

$$\begin{aligned} C_0 \tau \int_{\{|x| > 2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz &\leq C_0 2^{N+1} \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\{|x| > 2|z|\}} J(z) \frac{|z|^i}{|x|^i} \left[\frac{\tau |x|^{N+1}}{(2^{N+1} + \tau|x|^{N+1})} \right] dz \\ &\leq \frac{C_0 2^{N+1}}{R_0} \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\mathbb{R}^N} J(z) |z|^i \left[\frac{\tau |x|^{N+1}}{2^{N+1} + \tau|x|^{N+1}} \right] dz. \end{aligned}$$

Since for all $|x|$,

$$\left[\frac{\tau|x|^{N+1}}{2^{N+1}+\tau|x|^{N+1}}\right] < 1,$$

we get for $|x| \geq R_0$

$$C_0\tau \int_{\{|x|>2|z|\}} J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right] dz \leq \frac{C_0 2^{N+1}}{R_0}\sum_{i=1}^{N+1}\binom{N+1}{i}\int_{\mathbb{R}^N} J(z)|z|^i\, dz. \tag{7.4}$$

Combining (7.3), (7.4) and (7.2), we get for $|x| > R_0$

$$h(x,\tau) \leq \frac{C_0 2^{N+1}}{R_0}\sum_{i=1}^{N+1}\binom{N+1}{i}\int_{\mathbb{R}^N} J(z)|z|^i\, dz + C_0\tau 2^{N+1}\int_{\mathbb{R}^N} J(z)|z|^{N+1}\, dz + \delta - \lambda.$$

Thanks to (H5), for τ small enough, says $\tau \leq \tau_1$ and R_0 large enough we derive $h(x,\tau) \leq \frac{\delta-\lambda}{2} < 0$,

Hence, for all $\tau \leq \tau_1$, we have

$$\mathcal{M}[w] + (\mu(x)+\delta)w(x) \leq w(x)h(x,\tau) \leq w(x)\frac{\delta-\lambda}{2} < 0 \quad \text{for all} \quad x \in \mathbb{R}^N \setminus B_{R_0}. \tag{7.5}$$

Fix now $\tau \leq \tau_1$ and fix $R_0 > R_\delta$ so that $h(x,\tau) < 0$ in $\mathbb{R}^N \setminus B_{R_0}(0)$. Let $\kappa_0 := \sup_{\mathbb{R}^N\setminus B_{R_0}(0)} \frac{\varphi(x)}{1+\tau|x|^{N+\alpha}}$ Let $0 < \kappa < \kappa_0$ and consider the set

$$\Omega_\kappa := \left\{x \in \mathbb{R}^N \mid \frac{\varphi(x)}{1+\tau|x|^{N+\alpha}} \leq \kappa\right\}.$$

By construction, since $\varphi > 0$ in $\mathbb{R}^N$, we can choose κ small so that

$$\Omega_\kappa \subset \mathbb{R}^N \setminus B_{R_0}(0).$$

Moreover, $\mathbb{R}^N \setminus \Omega_\kappa$ is a bounded domain and $M := \sup_{\mathbb{R}^N\setminus\Omega_\kappa} S(x)$ is well defined. Choose now C such that $C = \frac{2M}{\kappa}$ and consider the continuous function

$$\bar{u}(x) := \begin{cases} C\frac{\varphi(x)}{1+\tau|x|^{N+\alpha}} & \text{in} \quad \Omega_\kappa, \\ C\kappa \quad \text{in} \quad \mathbb{R}^N \setminus \Omega_\kappa. \end{cases}$$

We can check that $\bar{u}$ is a super-solution to (1.2). Indeed, for any $x \in \mathbb{R}^N \setminus \Omega_\kappa$, we have $\bar{u} = C\kappa = 2M > \sup_{\mathbb{R}^N\setminus\Omega_\kappa} S(x)$ which implies that $f(x, C\kappa) \leq 0$ and

$$\mathcal{M}[\bar{u}](x) + f(x,\bar{u}(x)) \leq \int_{\mathbb{R}^N} J(x-y)\bar{u}(y)\, dy - C\kappa + f(x,C\kappa) \leq f(x,C\kappa) \leq 0.$$

On the other hand, for $x \in \Omega_\kappa$ we directly have

$$\begin{aligned}\mathcal{M}[\bar{u}](x) + f(x,\bar{u}(x)) \leq \mathcal{M}[\bar{u}](x) + \beta(x)w(x) &\leq \mathcal{M}[w] + (\mu(x)+\delta)w(x), \\ &\leq h(x,\tau)w(x) \leq 0.\end{aligned}$$

This completes the proof. □

8 Conclusion

In this paper, we obtain an optimal persistence criteria for a population that has a long range dispersal. The dynamics of the population is described by a reaction dispersion equation with a convolution for dispersal and a Fisher-KPP type nonlinearity that describes the reproduction and mortality of the population. The model reads :

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{M}[u] + f(x, u(t,x)),$$

with $\mathcal{M}[u] = J \star u - u$.

We consider here the case of a bounded ecological niche, that is when the environment is lethal to the population outside a bounded region. In our model, this fact is translated by an assumption on the Fisher-KPP nonlinearity, th This fact is translated in our model by assuming that the Fisher -KPP nonlinearity f satisfies:

$$\limsup_{|x|\to\infty} \frac{f(x,s)}{s} < 0, \qquad \text{uniformly in} \qquad s \geq 0.$$

When the dispersal kernel is compactly supported, we prove that the existence of a positive solution of the above equation is characterised by the sign of the generalised principal eigenvalue $\lambda_p(\mathcal{M} + \partial_s f(x,0))$ defined by

$$\lambda_p(\mathcal{M} + \partial_s f(x,0)) := \sup\{\lambda \in \mathbb{R} \,|\exists\, \varphi \in C(\mathbb{R}^N), \varphi > 0, \text{ such that } \mathcal{M}[\varphi] + \partial_s f(x,0)\varphi + \lambda\varphi \leq 0\}.$$

Moreover, when such a positive stationary solution exists, it is unique. In addition, we describe completely the long time behaviour of positive solution of the above nonlocal Fisher-KPP equation. We also obtain persistence criteria for fat-tailed kernel, in terms of the sign of $\lim_{R\to\infty} \lambda_p(\mathcal{L}_R + \partial_s f(x,0))$. However, due to the lack of Harnack type *a priori* estimate, the optimality of this criteria is still an open problem. A better understanding of the properties of the generalised principal eigenvalue λ_p in such context would allow one to resolve this issue. In particular, the optimality of this criteria would follow from proving that

$$\lim_{R\to\infty} \lambda_p(\mathcal{L}_R + \partial_s f(x,0)) = \lambda_p(\mathcal{M} + \partial_s f(x,0)).$$

In the context of compactly supported kernel, we also analyse the effect of the range of dispersal on the persistence of a species for some rescaled dispersal kernels e. g. $\frac{1}{\varepsilon^m} J_\varepsilon(z)$. These rescaled kernels arise, when the dispersal is conditioned by a *dispersal budget* in which the cost functions are of the form $|z|^m$. For $0 \leq m < 2$, we prove that small spreaders, i.e. ε small, always survive. This is not necessarily true when $m = 2$. In that case, it may happen that having a small dispersal leads to extinction. Conversely, when $m > 0$, we prove that large spreaders, i.e ε large, always survive. We also provide the asymptotics of the solution of the associated nonlocal Fisher-KPP equation. These asymptotics provide valuable informations, when we try to compare the different dispersal strategies and for the search of a Evolutionary Stable Strategy.

Many new open problems and new directions come up naturally as the continuation of the present study. For instance, to clarify the effect of the dispersal budget on the dispersal strategies, we would need a deeper understanding of the generalised principal eigenvalue. We suspect that for a quadratic cost function, at least in some situation, the effect of the dispersal budget should be the opposite of the one usually observed for unconditional dispersal strategies. That is, the larger *spreader* should be always favoured.

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