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Nonlocal heterogeneous KPP equations in $\mathbb{R}^N$

Henri Berestycki [*], Jérôme Coville [†], Hoang-Hung Vo [‡]

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Abstract

In this article we analyse the non-local niche model

$$\frac{\partial u}{\partial t} = J \star u - u + f(x,u) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N,$$

where J is a positive continuous dispersal kernel and $f(x,u)$ is a heterogeneous KPP type non-linearity describing the growth rate of the population. The ecological niche of the population is assumed to be bounded (i.e. outside a compact set the environment is assumed to be lethal for the population). For compactly supported dispersal kernels J, we derive an optimal survival criteria. We prove that the existence of a positive stationary solution exists if and only if the principal eigenvalue λ_p of the linear problem

$$J \star \varphi - \varphi + \partial_s f(x,0) + \lambda_p \varphi = 0 \quad \text{in} \quad \mathbb{R}^N,$$

is negative. In addition, for any continuous non-negative initial data that is bounded or integrable, we establish the long time behaviour of the solution $u(t,x)$. We also analyse the impact of the size of the support of the dispersal kernel on the persistence criteria. We exhibit situations where the dispersal strategy has "no impact" on the survival of the species and situations where the slowest dispersal strategy is not any more an Ecological Stable Strategy. Some generalisations of the survival criteria are also discussed for fat-tailed kernels.

Contents

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1 Introduction

In this article, we are interested in finding survival criteria for a species that has a long range dispersal strategy. As a model species, we can think of trees whose seed and pollens are disseminated on a long range. In ecology a commonly used model that integrate such long range dispersal [32, 36, 39, 42, 44, 55] is the following:

$$\frac{\partial u}{\partial t}(t,x) = J \star u(t,x) - u + f(x,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.1}$$

where $u(t,x)$ is the density of the considered population, J is a dispersal kernel, $f(x,s)$ is a KPP type non-linearity describing the growth rate of the population. The possibility of a long range dispersal is well known in ecology, where numerous data now available support this assumptions [14, 19, 20, 27, 52, 57]. In this setting the tail of the kernel can be thought of as a measure of the frequency at which long dispersal events occur. A biological motivation for the use of (1.2) to describe the evolution of the population comes from the observation that the intrinsic variability in the capacity of the individuals to disperse generates, at the scale of a population, a long range dispersal of the population. The effect of such variability has been investigated in [38, 47] by means of the study of correlated random walks. In such a framework, all individuals follow a simple random walk where the diffusion coefficient follows a probability law. It can be checked that then the probability of the density of population will follow an integro-differential equation [38, 47, 55] where a dispersal kernel J describes the probability to jump from one location to another.

Throughout this paper we will always make the following assumptions on the dispersal kernel J:

(H1) $J \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is nonnegative, radially symmetric and of unit mass (i.e. $\int_{\mathbb{R}^N} J(z)dz = 1$) .

(H2) $J(0) > 0$

In the present paper, we focus our analysis on species that have a bounded ecological niche. A simple way to model such a spatial repartition consists in considering that the environment is hostile to the species outside a bounded set. This fact is translated in our model by assuming that f satisfies:

(H3) $f \in C^{1,\alpha}(\mathbb{R}^{N+1})$ is of KPP type, that is :

$$\begin{cases} f(\cdot,0) \equiv 0, \\ \text{For all } x \in \mathbb{R}^N, f(x,s)/s \text{ is decreasing with respect to } s \text{ on } (0,+\infty). \\ \text{There exists } S(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ such that } f(x,S(x)) \le 0 \text{ for all } x \in \mathbb{R}^N. \end{cases}$$

(H4) $\limsup_{|x|\to\infty} \frac{f(x,s)}{s} < 0, \quad$ uniformly in $\quad s \ge 0.$

A typical example of such a nonlinearity is given by $f(x,s) := s(a(x) - b(x)s)$ with $b(x) > 0$ and $a(x)$ satisfies $\limsup_{|x|\to\infty} a(x) < 0$.

Our main purpose is to seek conditions on J and f that characterise the persistence of the species modelled by (1.1). In this task, we focus our analysis on the description of the set of positive stationary solution of (1.1). That is the set of positive solution of the equation below

$$J \star u(x) - u(x) + f(x,u(x)) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

This description is expected to provide useful persistence criteria.

In the literature, persistence criteria have been well studied for the reaction diffusion version of (1.1)

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(x,t,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{1.3}$$

where Ω is a domain of $\mathbb{R}^N$, possibly $\mathbb{R}^N$ itself. Survival criteria have been obtained for various media, ranging from periodic media to ergodic media [4, 5, 9, 15, 16, 17, 33, 45, 46, 48, 53]. In the context of global warming, survival criteria have been investigated in [9, 3, 10]. For such reaction diffusion equations the survival criteria are often obtained by looking at the sign of the first eigenvalue of the linear problem obtained by linearising (1.3) around the 0 solution. That is the sign of the first eigenvalue $\lambda_1(\Delta + \partial_s f(x,0), \Omega)$ of the spectral problem

$$\Delta\varphi(x) + \partial_s f(x,0)\phi(x) + \lambda_1\varphi(x) = 0 \quad \Omega \tag{1.4}$$

associated with the proper boundary conditions (if $\Omega \neq \mathbb{R}^N$).

In most situations, for KPP– like non-linearities, the existence of a positive stationary solution to (1.3) is uniquely conditioned by the sign of λ_1. More precisely, there exists a unique positive stationary solution if and only if $\lambda_1 < 0$. If such type of criteria seems reasonable in problems defined on bounded set, it is less obvious for problems in unbounded domains. In particular, in unbounded domains, one of the main difficulty concerns the definition of λ_1. As shown in [8, 5], the notion of first eigenvalue in unbounded domain can be ambiguous and several definition of λ_1 exists rendering the question of a sharp survival criteria quite involved.

For the non-local equation (1.2) less is known and to our knowledge survival criteria have been essentially investigated in some specific situations, periodic media [25, 26, 54] or for a version of the problem (1.2) defined in a bounded domain Ω,

$$\frac{\partial u}{\partial t}(t,x) = \int_\Omega J(x-y)u(t,y)\,dy - u(t,x) + f(x,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{1.5}$$

[1, 21, 23, 34, 41, 54]. We also quote [7] for an analysis of a persistence criteria in periodic media for a non-local version of (1.3) involving a fractional diffusion and [51] for survival criteria in time periodic versions of (1.5) . Similarly to the local diffusion case, for KPP like non-linearities, the existence of a positive solution of the non-local equation (1.5) can be characterised by the sign of a spectral quantity λ_p, called the generalised principal eigenvalue of

$$\int_\Omega J(x-y)\phi(y)\,dy - \phi(x) + \partial_s f(x,0)\phi(x) + \lambda\phi(x) = 0 \quad \text{in} \quad \Omega. \tag{1.6}$$

In fact, λ_p is defined by

$$\lambda_p := \sup\left\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\Omega), \varphi > 0, \text{ so that } \int_\Omega J(x-y)\varphi(y)\,dy - \varphi(x) + \partial_s f(x,0)\varphi(x) + \lambda\varphi(x) \le 0 \quad \text{in} \quad \Omega.\right\}.$$

Unlike the elliptic PDE case, due to the lack of a regularising effect of the diffusion operator, the above spectral problem may not have a solution in any reasonable space of functions i.e $(L^p(\Omega), C(\Omega))$[25, 24, 41]. As a consequence, even in bounded domain, simple sharp survival criteria are already quite involved to obtain. Another difficulty inherent to the study of nonlocal equations (1.6) in unbounded domain concerns the lack of "reasonable" *a priori* estimates for the solution thus making standard approximations difficult to use in most cases.

1.1 Main Results:

Let us now state our main results. We first prove a simple sharp survival criteria assuming that the dispersal kernel J satisfy an extra assumption.

Theorem 1.1. *Assume that J, f satisfy (H1-H4) and assume further that J is compactly supported. Then, there exists a unique positive solution to* (1.2) *if and only if $\lambda_p(\mathcal{M} + \partial_s f(x,0)) < 0$, where*

$$\lambda_p(\mathcal{M} + \partial_u f(x,0)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\mathbb{R}^N), \phi > 0 \ \text{ so that } \ \mathcal{M}[\varphi] + \partial_s f(x,0)\varphi + \lambda\varphi \le 0\},$$

where $\mathcal{M}$ denotes the continuous operator $\mathcal{M}[\varphi] = J \star \varphi(x) - \varphi(x)$. Moreover, for any non-negative initial data $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ we have the following asymptotic behaviour:

- *If $\lambda_p(\mathcal{M} + \partial_s f(x,0)) \ge 0$, then the solution satisfies $\|u(t)\|_\infty \to 0$ as $t \to \infty$,*

- *If $\lambda_p(\mathcal{M}+\partial_s f(x,0))<0$, then the solution $\|u-\tilde{u}\|_\infty(t)\to 0$ as $t\to\infty$, where $\tilde{u}=\tilde{u}(x)$ denotes the unique positive solution to* (1.2)

In addition, if the initial data $u_0\in C(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, then the convergence $u(t,x)\to\tilde{u}$ holds in $L^1(\mathbb{R}^N)$.

Next, we aim at understanding the effect of the dispersal kernel on the survival of the species. To this end, we analyse the behaviour of the survival criteria under some scaling of the dispersal operator. More precisely, let $J_\varepsilon := \frac{1}{\varepsilon^N}J\left(\frac{z}{\varepsilon}\right)$ and let $\mathcal{M}_\varepsilon$ denotes the operator $\mathcal{M}$ with the rescaled kernel, then we look at the behaviour of the solution to (1.2) as $\varepsilon\to 0$ or $\varepsilon\to+\infty$ where the dispersal operator $\mathcal{M}$ is replace by $\alpha(\varepsilon)\mathcal{M}_\varepsilon$, with $\alpha(\varepsilon)\sim\frac{\alpha_0}{\varepsilon^m}$. These asymptotics represent two possible strategies that are observed in nature. The terms $\alpha(\varepsilon)$ refers to a dispersal budget of the species as defined in [39]. Roughly speaking, for a fixed cost, this budget is a way to measure the differences between different strategies. For a given dispersal cost function of the order of $|y|^m$, the term $\alpha(\varepsilon)$ behaves like $\frac{\alpha_0}{\varepsilon^m}$ and in the analysis, the dispersal operator is then given by $\alpha(\varepsilon)\mathcal{M}_\varepsilon$. As explained in [39], the limit as $\varepsilon\to 0$ can be associated to a strategy of producing a lot of offspring but with little capacity of movement. Whereas the limit $\varepsilon\to+\infty$ corresponds to a strategy that aims at maximizing the possibility to explore the environment at the expense of the number of offspring.

Here, we analyse the cases $0\le m\le 2$ and $\alpha_0=1$, the case $m=0$ corresponding to understand the impact of the mean distance on the survival criteria. To simplify the presentation of these asymptotics, we restrict to nonlinearities $f(x,s)$ of the form $f(x,s)=s(a(x)-s)$. However, the proofs apply more generally.

In this situation, we first obtain

Theorem 1.2. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $m=0$. Then there exists $\varepsilon_0\in(0,+\infty]$ so that for all $\varepsilon\le\varepsilon_0$ there exists a positive solution u_ε to* (1.2). *Moreover, we have*

$$\lim_{\varepsilon\to\varepsilon_0} u_\varepsilon(x)=(a(x)-1)^+,$$

where s^+ denotes the positive part of s (i.e. $s^+=sup\{0,s\}$). Assuming further that a is smooth, at least $C^{0,1}(\mathbb{R}^N)$, we have

$$\lim_{\varepsilon\to 0} u_\varepsilon(x)=v(x)\quad \textit{almot everywhere}$$

where v is a non-negative bounded solution of

$$v(x)(a(x)-v(x))=0\quad \textit{in}\quad \mathbb{R}^N.$$

In addition, when $a(x)$ is radially symmetric non increasing and $\varepsilon_0<+\infty$ then ε_0 is sharp, in the sense that for all $\varepsilon\ge\varepsilon_0$ there is no positive solution to (1.2).

The ecologically interpretation of this result is that one way of persistence for a species is to match the resource and not to move much. In some situation, $\varepsilon_0=+\infty$ and there is no effect of the dispersal on the survival criteria of the species. A natural condition that ensure that $\varepsilon_0=+\infty$ is

$$(a(x)-1)^+\not\equiv 0.$$

In this context, the birth rates exceed all death rates and guarantee the persistence of the population no matter what the dispersal strategy is. In particular, there exists a bounded positive solution to (1.2) for any positive kernel J. The uniqueness and the behaviour at infinity of the solution are still open questions for general kernels.

When $m>0$, then the characterisation of the existence of a positive solution changes and a new picture emerges. In particular, for large ε there is always of solution to (1.2) whereas for small ε it may happen that no positive solution exists. Thus, the situation is, in a sense, opposite to the case when $m=0$. Non existence for small value of ε appears only when $m\ge 2$. More precisely we prove

Theorem 1.3. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $0<m<2$. Then there exists $\varepsilon_0\le\varepsilon_1\in(0,+\infty)$ so that for all $\varepsilon\le\varepsilon_0$ and all $\varepsilon\ge\varepsilon_1$ there exists a positive solution u_ε to* (1.2). *Moreover, we have*

$$\lim_{\varepsilon\to+\infty}\|u_\varepsilon - a^+\|_\infty = 0, \qquad \lim_{\varepsilon\to+\infty}\|u_\varepsilon - a^+\|_{L^2(\mathbb{R}^N)} = 0.$$

In addition, assuming further that a is smooth, at least $C^2(\mathbb{R}^N)$, we have

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = v(x) \quad \textit{almot everywhere},$$

where v is a non-negative bounded solution of

$$v(x)(a(x) - v(x)) = 0 \quad \textit{in} \quad \mathbb{R}^N.$$

Theorem 1.4. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $m = 2$. Then there exists $\varepsilon_1 \in (0,\infty)$ so that for all $\varepsilon \geq \varepsilon^*$ there exists a positive solution u_ε to (1.2). Moreover,*

$$\lim_{\varepsilon\to+\infty} u_\varepsilon = a^+(x).$$

In addition, we have the following dichotomy

- *When $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) < 0$, there exists $\varepsilon_0 \in (0,\infty)$ so that for all $\varepsilon \leq \varepsilon_0$ there exists a positive solution to (1.2) and*

  $$u_\varepsilon \to v, \quad \textit{in} \quad L^2_{loc}(\mathbb{R}^N),$$

  where v is the unique bounded non-trivial solution to

  $$\frac{K_{2,N}D_2(J)}{2}\Delta v + v(a(x) - v) = 0 \quad \textit{in} \quad \mathbb{R}^N.$$

- *When $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) > 0$, then there $\varepsilon_0 \in (0,\infty)$ so that for all $\varepsilon \leq \varepsilon_0$ there exists no positive solution to (1.2).*

This last result clearly highlights the dependence of the spreading strategy on the cost functions and the structure of ecological niche. Especially when $m = 2$, the smaller spreader strategy may not be an optimal strategy, in the sense that a population adopting such strategy can go extinct. From the view point of Adaptive dynamics [28, 29, 43, 56], the smaller spreader strategy (SSS) will not be a Ecological Stable Strategy (ESS). The concept of ESS comes from game's theory and goes back to the work of Hamilton [37] on the evolution of sex-ratio. Roughly speaking, within this framework, an Ecological Stable Strategy, is a strategy such that if most of the members of a population adopt it,there is no "mutant" strategy that would give higher reproductive fitness. In such viewpoint the strategies are compared using their relative pay-off. Here, following [18, 30, 39, 41] the strategies can be compared through the faith of a solution of a competitive system

$$\partial_t u(t,x) = \mathcal{M}_{m,\varepsilon_1}[u] + u(t,x)(a(x) - u(t,x) - v(t,x)) \quad \text{in} \quad \mathbb{R}^N \tag{1.7}$$

$$\partial_t v(t,x) = \mathcal{M}_{m,\varepsilon_2}[u] + v(t,x)(a(x) - u(t,x) - v(t,x)) \quad \text{in} \quad \mathbb{R}^N \tag{1.8}$$

where u is a population that has adopted the spreading strategy ε_1 and v an another one. The notion of ESS is then linked to some invasion condition which is related to the stability of the equilibria $(u^*, 0)$ where u^* is a positive solution of the following problem

$$\mathcal{M}_{m,\varepsilon_1}[u] + u(x)(a(x) - u(x)) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

The stability analysis of this equilibria, leads to consider the sign of a principal eigenvalue of the operator $\mathcal{M}_{m,\varepsilon_2} + a(x) - u^*(x)$. When $\lambda_p(\mathcal{M}_{m,\varepsilon_2} + a(x) - u^*(x))$ is negative then the equilibria $(u^*, 0)$ is unstable and a mutant can invade. Therefore the strategy followed by u will not be an ESS. On the contrary, when $\lambda_p(\mathcal{M}_{2,\varepsilon_2} + a(x) - u^*(x))$

is positive the equilibria $(u^*, 0)$ is stable and a mutant cannot invade, making this strategy a potential candidate for an ESS.

From our result, we can observe that within the strategy with a quadratic cost function ($m = 2$), the ubiquity strategy ($\varepsilon = \infty$) is an ESS. Indeed, for such case, from the above Theorem, we are lead to consider the sign of $\lambda_p(\mathcal{M}_{2,\varepsilon_2} + a(x) - a^+(x))$ which is positive for any $\varepsilon > 0$. Whereas, the smallest spreader ($\varepsilon = 0$) is never an ESS. Such behaviour stands in contrast with known results on ESS strategy governed by the rate of dispersion [39, 41], where in such case the slowest rate possible is the best strategy.

Finally, we obtained existence/ non-existence criteria when we relax the compactly supported constraint on the dispersal kernel J. In this direction, we investigate a class of kernel J that can have a fat tail but still have some decay at infinity. More precisely, we assume that

(H5) $\int_{\mathbb{R}^N} J(z)|z|^{N+1} < +\infty.$

Theorem 1.5. *Assume that J, f satisfy (H1-H4) and assume further that J satisfies $(H5)$ then we have*

(i) ***if*** $\lambda_p(\mathcal{M} + \partial_s f(x, 0)) > 0$ *there is no bounded positive solution to* (1.2).

(ii) ***if*** $\lim_{R\to\infty} \lambda_p(\mathcal{L}_R + \partial_s f(x, 0)) < 0$ *where*

$$\mathcal{L}_R[\varphi] := \int_{B_R(0)} J(x - y)\varphi(y)\, dy - \varphi(x),$$

then there exists a unique positive solution to (1.2).

1.2 Comments

Before going into the proofs of these results, we would like to make some further comments. Our proofs essentially rely on the properties of the principal eigenvalue $\lambda_p(\mathcal{M} + a(x))$ and more precisely on the relations between the following spectral quantities:

$$\lambda_p(\mathcal{M} + a(x)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\varphi \in C(\Omega), \varphi > 0, \ \text{ so that } \ \mathcal{M}[\varphi](x) + a(x)\varphi(x) + \lambda\varphi(x) \leq 0 \quad \text{in} \quad \Omega\}.$$
$$\lambda_p'(\mathcal{M} + a(x)) := \inf\{\lambda \in \mathbb{R} \,|\, \exists\varphi \in C(\Omega) \cap L^\infty(\Omega), \varphi > 0, \ \text{ so that } \ \mathcal{M}[\varphi](x) + a(x)\varphi(x) + \lambda\varphi(x) \geq 0 \quad \text{in} \quad \Omega\}.$$
$$\lambda_v(\mathcal{M} + a(x)) := \inf_{\varphi\in L^2(\mathbb{R}^N),\varphi\not\equiv 0} \frac{\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J(x - y)[\varphi(x) - \varphi(y)]^2\, dxdy - \int_{\mathbb{R}^N} a(x)\varphi^2(x)\, dx}{\|\varphi\|_2^2}.$$

Although these quantities have been introduced in various context see for example [21, 26, 23, 35, 40], the relation between them have not been fully investigated or only in some particular contexts such as when $a(x)$ is homogeneous or periodic. Some new results have been recently obtained in [2] allowing now to have a clear description of the relation between λ_p, λ_p' and λ_v. Moreover, [2] provides a description of the asymptotic behaviour of these spectral quantities with respect to the scaling of the kernel. For the purpose of our analysis, we present a summary of these results in Section2.

Finally, we also want to stress that although we have a clear description of the existence/non-existence of a positive solution for small ε, the study of the convergence of u_ε as $\varepsilon \to 0$ is quite delicate. Indeed, in $L^\infty(\mathbb{R}^N)$, the problem

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

has infinitely many bounded non negative solution (e.g. for any set $Q \subset \mathbb{R}^N$, the function $a^+(x)\chi_Q$ is a solution) and owing to the lack of regularising effect of the dispersal operator, we cannot rely on standard compactness result in the usual manner to obtain a smooth limit. If for the case $m = 2$ we could rely on the elliptic regularity and the new description of Sobolev Spaces developed in [12, 13, 49, 50] to get some compactness, this characterisation does not allow us to treat the case $m < 2$. We believe that a new characterisation of Fractional Sobolev space in the spirit of the work of Bourgain, Brezis and Mironescu [12, 13] may be helpful to resolve this issue.

The paper is organised as follows. In Section 2, we recall some known results and properties of the principal eigenvalue $\lambda_p(\mathcal{L}_\Omega + a(x))$. We also describe the sharp persistence criteria for problem (1.5) defined in a bounded domain Ω that are derived in terms of principal eigenvalues. In Sections 3 and 4, we establish the sharp survival criteria and prove the long time behaviour of the solution of (1.2) (Theorem 1.1). We analyse the dependence of the persistence criteria (Theorems 1.2 and 1.3) in Section 5. Finally, in the last Section we discuss the extension of the persistence criteria to non compactly supported kernel.

1.3 Notations

To simplify the presentation of the proofs, we introduce some notations and various linear operator that we will use along this paper:

- $B_R(x_0)$ will denotes the standard ball of radius R centred at the point x_0
- χ_R will always refer to the characteristic function of $B_R(0)$.
- $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz space.
- For a positive integrable function $J \in \mathcal{S}(\mathbb{R}^N)$, the constant $\int_{\mathbb{R}^N} J(z)|z|^2\,dz$ will refer to

$$\int_{\mathbb{R}^N} J(z)|z|^2\,dz := \int_{\mathbb{R}^N} J(z)\left(\sum_{i=0}^{N} z_i^2\right)dz$$

- We denote $\mathcal{L}_\Omega$ the continuous linear operator

$$\begin{array}{rccl} \mathcal{L}_\Omega : & C(\bar{\Omega}) & \to & C(\bar{\Omega}) \\ & u & \mapsto & \int_\Omega J(x-y)u(y)\,dy, \end{array} \tag{1.9}$$

  where $\Omega \subset \mathbb{R}^N$.
- $\mathcal{L}_R$ correspond to the continuous operator $\mathcal{L}_\Omega - Id$ with $\Omega = B_R(0)$.
- We will use $\mathcal{M}$ to denote the operators $\mathcal{L}_\Omega - Id$ with $\Omega = \mathbb{R}^N$.
- Finally, $\mathcal{M}_\varepsilon$ will denote the operator $\mathcal{M}$ with a rescaled kernel $\frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ and $\mathcal{M}_{\varepsilon,m} := \frac{1}{\varepsilon^m}\mathcal{M}_\varepsilon$
- To simplify the presentation of the proofs, we will also use the notation $\beta(x) := \partial_s f(x,0)$.

2 Preliminaries

In this section, we recall some known results on the principal eigenvalue of a linear non-local operator $\mathcal{L}_\Omega + a(x)$ and on the KPP equation below

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}_\Omega[u] + f(x,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{2.1}$$

defined in a bounded domain $\Omega \subset \mathbb{R}^N$. For simplicity, we divide this section into two subsections, one devoted to the principal eigenvalue and the other dedicated to known survival criteria for the problem (2.1).

2.1 Principal eigenvalue for non-local operators

In this subsection, we focus on the properties of the spectral problem

$$\mathcal{L}_{\Omega}[\varphi] + a(x)\varphi + \lambda\varphi = 0 \quad \text{in} \quad \Omega. \tag{2.2}$$

In contrast with elliptic operators, when $a(x) \not\equiv Cste$, neither $\mathcal{L}_{\Omega} + a(x) + \lambda$ nor its inverse are compact operators and the description of the spectrum of $\mathcal{L}_{\Omega} + a$ using the Krein-Rutman Theory fails. However as shown in [21], some variational formula introduced in [6] to characterise the first eigenvalue of elliptic operators $\mathcal{E} := a_{ij}\partial_{ij} + b_i(x)\partial_i + c(x)$,

$$\lambda_1(\mathcal{E}) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \varphi \in W^{2,n}(\Omega), \varphi > 0 \text{ so that } \mathcal{E}[\varphi] + \lambda\varphi \leq 0\}, \tag{2.3}$$

can be transposed to the operator $\mathcal{L}_{\Omega} + a(x)$. Namely, the quantity

$$\lambda_p(\mathcal{L}_{\Omega} + a(x)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \varphi \in C(\Omega), \varphi > 0 \text{ so that } \mathcal{L}_{\Omega}[\varphi] + a(x)\varphi + \lambda\varphi \leq 0\}. \tag{2.4}$$

is well defined

As noted also in [21], the quantity defined by (2.4) is not always an eigenvalue of $\mathcal{L}_{\Omega} + a(x)$ in a reasonable Banach space. This means that there is not always a positive continuous eigenfunction associated with λ_p. However, as proved in [21, 41, 54], when Ω is a bounded domain we can find some conditions on the coefficients that guarantee the existence of a positive continuous eigenfunction. For example , if we assume that the function $a(x)$ satisfies

$$\frac{1}{\sup_{\Omega} a - a(x)} \notin L^1_{loc}(\bar{\Omega}),$$

then $\lambda_p(\mathcal{L}_{\Omega} + a(x))$ is an eigenvalue of $\mathcal{L}_{\Omega} + a(x)$ in the Banach space $C(\bar{\Omega})$ and is associated to a positive continuous eigenfunction.

Another useful criteria that guarantees the existence of a continuous principal eigenfunction is

Proposition 2.1. *Let Ω be a bounded domain and let $\mathcal{L}_{\Omega}$ be as in* (1.9) *then there exists a positive continuous eigenfunction associated to λ_p* ***if and only if*** $\lambda_p(\mathcal{L}_{\Omega} + a(x)) < -\sup_{\Omega} a$.

A proof of this proposition can be found for example in [26, 23]. To have a more complete description of the properties of λ_p in bounded domains see [24].

Next, we recall some properties of λ_p that we constantly use along this paper:

Proposition 2.2. *(i) Assume $\Omega_1 \subset \Omega_2$, then for the two operators*

$$\mathcal{L}_{\Omega_1}[u] + a(x)u := \int_{\Omega_1} J(x-y)u(y)\,dy + a(x)u$$
$$\mathcal{L}_{\Omega_2}[u] + a(x)u := \int_{\Omega_2} J(x-y)u(y)\,dy + a(x)u$$

respectively defined on $C(\Omega_1)$ and $C(\Omega_2)$ we have

$$\lambda_p(\mathcal{L}_{\Omega_1} + a(x)) \geq \lambda_p(\mathcal{L}_{\Omega_2} + a(x)).$$

(ii) Fix Ω and assume that $a_1(x) \geq a_2(x)$, then

$$\lambda_p(\mathcal{L}_{\Omega} + a_2(x)) \geq \lambda_p(\mathcal{L}_{\Omega} + a_1(x)).$$

(iii) $\lambda_p(\mathcal{L}_{\Omega} + a(x))$ is Lipschitz continuous in $a(x)$. More precisely,

$$|\lambda_p(\mathcal{L}_{\Omega} + a(x)) - \lambda_p(\mathcal{L}_{\Omega} + b(x))| \leq \|a(x) - b(x)\|_{\infty}$$

(iv) We always have the following estimate

$$-\sup_{\Omega}\left(a(x)+\int_{\Omega}J(x-y)\,dy\right)\leq\lambda_p(\mathcal{L}_{\Omega}+a)\leq-\sup_{\Omega}a.$$

We refer to [21, 23] for the proofs of $(i)-(iv)$. Let us also recall the two following results proved in [2].

Lemma 2.3. *Assume that a achieves its maximum in Ω and let $\mathcal{L}_{\Omega}+a(x)$ be defined as in* (1.9) *with J satisfying $(H1-H2)$. Assume further that J is compactly supported. Let $(\Omega_n)_{n\in\mathbb{R}}$ be a sequence of subset of Ω so that $\lim_{n\to\infty}\Omega_n=\Omega$, $\Omega_n\subset\Omega_{n+1}$. Then we have*

$$\lim_{n\to\infty}\lambda_p(\mathcal{L}_{\Omega_n}+a(x))=\lambda_p(\mathcal{L}_{\Omega}+a(x))$$

Lemma 2.4. *Assume that $a(x)\in C(\mathbb{R}^N)\cap L^{\infty}(\mathbb{R}^N)$. Then for all $\varepsilon>0$ one has*

$$\lambda_p(\mathcal{M}+a(x))=\lambda_p(\mathcal{M}_{\varepsilon}+a_{\varepsilon}(x)).$$

where $a_{\varepsilon}(x):=a\left(\frac{x}{\varepsilon}\right)$ and $\mathcal{M}_{\varepsilon}[\varphi]:=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}J\left(\frac{x-y}{\varepsilon}\right)\varphi(y)\,dy-\varphi(x)$.

Finally, we recall some recent results obtained in [2] on the characterisation of the principal eigenvalue $\lambda_p(\mathcal{M}_{\varepsilon,m}+a(x))$. To this end, we recall some variational quantities of interest. Motivated by the works [8, 5, 9] on the generalised first eigenvalue of an elliptic operators, let us introduce the two definitions :

Definition 2.5. Let $\mathcal{L}_{\Omega}+a(x)$ be as in (1.9). We define the following quantities:

$$\lambda_p'(\mathcal{L}_{\Omega}+a(x)) :=\inf\{\lambda\in\mathbb{R}\,|\,\exists\varphi\geq0,\varphi\in C(\Omega)\cap L^{\infty}(\Omega),\text{ s.t }\mathcal{L}_{\Omega}[\varphi]+\lambda\varphi\geq0\text{ in }\Omega\},\tag{2.5}$$

$$\lambda_v(\mathcal{L}_{\Omega}+a(x)) :=\inf_{\varphi\in L^2(\Omega),\varphi\not\equiv0}-\frac{\langle\mathcal{L}_{\Omega}[\varphi]+a(x)\varphi,\varphi\rangle}{\langle\varphi,\varphi\rangle},\tag{2.6}$$

$$=\inf_{\varphi\in L^2(\Omega),\varphi\not\equiv0}\frac{\int_{\Omega}\int_{\Omega}J(x-y)(\varphi(x)-\varphi(y))^2\,dxdy-\int_{\Omega}(a(x)-1+k(x))\varphi^2(x)\,dx}{\|\varphi\|^2_{L^2(\Omega)}}.\tag{2.7}$$

where $k(x):=\int_{\Omega}J(y-x)\,dy$ and $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $L^2(\Omega)$.

In the context of the study of nonlocal operators, these definitions are natural extension of the definitions known for an elliptic operator. It is worth to mention that those definitions have already been used in the context of the study of (1.2) in several papers [25, 26, 23, 35, 40], but the relation between λ_p,λ_p' and λ_v has never been clarified.

For elliptic operators, it is known that the analogous of these three quantities are equivalent on bounded domain [6]. This is not necessarily the case for unbounded domains, where examples can be constructed [8, 5, 11], showing that $\lambda_1>\lambda_1'$. Since the operator, $\mathcal{L}_{\Omega}+a(x)$, shares many properties with elliptic operators, it is suspected that the three quantities, λ_p,λ_p' and λ_v, are not necessarily equal. However, for particular kernel J, we have:

Theorem 2.6 ([2])**.** *Let J be compactly supported satisfying* (H1)–(H2). *Assume that $a(x)\in C(\mathbb{R}^N)\cap L^{\infty}(\mathbb{R}^N)$. Then we have*

$$\lambda_p(\mathcal{M}_{\varepsilon,m}+a(x))=\lambda_p'(\mathcal{M}_{\varepsilon,m}+a(x))=\lambda_v(\mathcal{M}_{\varepsilon,m}+a(x)).$$

Moreover, we have the following asymptotic behaviour:

- *When $0<m\leq2$* $\quad\lim_{\varepsilon\to+\infty}\lambda_p(\mathcal{M}_{\varepsilon,m}+a(x))=-\sup_{\mathbb{R}^N}a(x)$
- *When $m=0$,* $\quad\lim_{\varepsilon\to+\infty}\lambda_p(\mathcal{M}_{\varepsilon}+a(x))=1-\sup_{\mathbb{R}^N}a(x)$
- *When $0\leq m<2$,* $\quad\lim_{\varepsilon\to0}\lambda_p(\mathcal{M}_{\varepsilon,m}+a(x))=-\sup_{\mathbb{R}^N}a(x)$

- *When $m=2$ and $a(x)$ is globally Lipschitz, then*

$$\lim_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x))=\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right),$$

where

$$D_2(J):=\int_{\mathbb{R}^N}J(z)z^2\,dz,\quad K_{2,N}:=\frac{1}{|S^{N-1}|}\int_{S^{N-1}}(\sigma.e_1)^2\,d\sigma=\frac{1}{N},$$

and

$$\lambda_1\left(K_{2,N}D_2(J)\Delta+a(x)\right):=\inf_{\varphi\in H_0^1(\mathbb{R}^N),\varphi\not\equiv 0}\frac{K_{2,N}D_2(J)}{2}\frac{\int_{\mathbb{R}^N}|\nabla\varphi|^2(x)\,dx}{\|\varphi\|_2^2}-\frac{\int_{\mathbb{R}^N}a(x)\varphi^2(x)\,dx}{\|\varphi\|_2^2}.$$

A similar result also holds for the rescaled operator $\mathcal{L}_{R,\varepsilon,m}:=\frac{1}{\varepsilon^m}\mathcal{L}_{R,\varepsilon}$ with $\mathcal{L}_{R,\varepsilon}$ denotes the operator $\mathcal{L}_R$ taken with the rescaled kernel $J_\varepsilon(z)$. Namely,

Theorem 2.7 ([2])**.** *Assume J satisfies* (H1)–(H2) *and let $a(x)\in C(\bar{B}_R(0))$. Then we have*

$$\lambda_p(\mathcal{L}_{R,\varepsilon,m}+a(x))=\lambda_p'(\mathcal{L}_{R,\varepsilon,m}+a(x))=\lambda_v(\mathcal{L}_{R,\varepsilon,m}+a(x)).$$

Moreover, we have the following asymptotic behaviour:

- *When $0<m\le 2$* $\quad\lim_{\varepsilon\to+\infty}\lambda_p(\mathcal{L}_{R,\varepsilon,m}+a(x))=-\sup_{B_R(0)}a(x)$
- *When $m=0$,* $\quad\lim_{\varepsilon\to+\infty}\lambda_p(\mathcal{L}_{R,\varepsilon}+a(x))=1-\sup_{B_R(0)}a(x)$
- *When $0\le m<2$,* $\quad\lim_{\varepsilon\to 0}\lambda_p(\mathcal{L}_{R,\varepsilon,m}+a(x))=-\sup_{B_R(0)}a(x)$
- *When $m=2$ and $a(x)$ is globally Lipschitz, then*

$$\lim_{\varepsilon\to 0}\lambda_p(\mathcal{L}_{R,\varepsilon,2}+a(x))=\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x),B_R(0)\right)$$

2.2 Existence criteria for the KPP-equation (2.1)

Equipped with this notion of principal eigenvalue, it has been shown [1, 21] that on bounded domains, the existence of a positive stationary solution of (2.1) is conditioned by the sign of $\lambda_p(\mathcal{L}_\Omega+\partial_s f(x,0))$. That is to say

Theorem 2.8 ([1, 21])**.** *Let Ω be a bounded domain and $\mathcal{L}_\Omega$ defined as in* (1.9)*. Assume that f satisfies $(H3)$. Then there exists a unique positive continuous function, $\bar{u}$, stationary solution of* (2.1) ***if and only if*** $\lambda_p(\mathcal{L}_\Omega+\partial_s f(x,0))<0$*. Moreover, if $\lambda_p\ge 0$ then 0 is the only non negative bounded stationary solution of* (2.1)*. In addition, for any positive continuous solutions of* (2.1) *we have the following dynamics :*

(i) When $\lambda_p\ge 0$,

$$\lim_{t\to\infty}u(t,x)\to 0\qquad\textit{uniformly in }\Omega,$$

(ii) When $\lambda_p<0$,

$$\lim_{t\to\infty}u(t,x)\to\bar{u}\qquad\textit{uniformly in }\Omega.$$

Remark 1. This existence criteria is similar to those known for the reaction diffusion versions of (2.1) [4, 15, 16, 17, 31, 33].

3 Existence/non existence and uniqueness of a non-trivial solution

In this section we construct a non-trivial solution of (1.2) and prove the necessary and sufficient condition stated in Theorem 1.1. For convenience the section is split up into three subsections, each of them respectively devoted to the proofs of existence of a solution, the uniqueness and non-existence.

3.1 Existence of a non-trivial positive solution

The construction follows a basic approximation scheme previously used for example in [3]. To this end we introduce the following approximated problem :

$$\mathcal{L}_R[u] + f(x,u) = 0 \quad \text{in} \quad \bar{B}(0,R) \tag{3.1}$$

where $B(0,R)$ denotes the ball of radius R centred at the origin. By Theorem 2.8, for any $R>0$ the existence of a unique positive solution of (3.1) is conditioned to the sign of $\lambda_p(\mathcal{L}_R + \beta(x))$ where $\beta(x) := \partial_u f(x,0)$. Since

$$\lim_{R\to+\infty} \lambda_p(\mathcal{L}_R + \beta(x)) = \lambda_p(\mathcal{M} + \beta(x)) < 0,$$

by Lemma 2.3 there exists $R_0 > 0$ so that

$$\forall R \geq R_0, \lambda_p(\mathcal{L}_R + \beta(x)) < 0.$$

As a consequence, by Theorem 2.8, for all $R > R_0$ there exists a unique positive solution of (3.1) that we denote u_R. Moreover, since for all $R > 0, \sup_{B_R(0)} S(x)$ is a super-solution of (3.1), by a standard sweeping argument since the solution to (3.1) is unique, we get

$$\forall R > 0,\ u_R \leq \sup_{B_R(0)} S(x) \quad \text{in} \quad B(0,R).$$

On another hand, for any $R_1 > R_2$, the solution u_{R_1} is a super-solution to the problem

$$\mathcal{L}_{R_2}[u] + f(x,u) = 0 \quad \text{in} \quad \bar{B}(0,R_2) \tag{3.2}$$

So as above by a standard sweeping argument we get

$$u_{R_2} \leq u_{R_1}(x) \quad \text{in} \quad B(0,R_2).$$

Thus the map $R \mapsto u_R$ is monotone increasing.

The idea is to obtain a positive solution to (1.2) as a limit of the positive solution of (3.1). To this end we construct a uniform super-solution of the problem (1.2).

Lemma 3.1. *There exists $\bar{u} \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $\bar{u} > 0$ so that $\bar{u}$ is a super-solution of the problem* (1.2).

Proof. Let us fix $\nu > 0$ and $R_0 > 1$ so that $\nu < -\limsup_{|x|\to\infty} \beta(x)$ and $\beta(x) \leq -\frac{\nu}{2}$ for all $|x| \geq R_0$. Consider now

$$w(x) = Ce^{-\alpha|x|},$$

where C and α are to be chosen. By direct computations, for all $x \in \mathbb{R}^N \setminus B_{R_0}(0)$ we get:

$$\begin{aligned}\mathcal{M}[w](x) + \beta(x)w(x) &= Ce^{-\alpha|x|}\left(\int_{\mathbb{R}^N} J(x-y)e^{-\alpha(|y|-|x|)}\,dy - 1 + \beta(x)\right),\\ &\leq w(x)\left(\int_{\mathbb{R}^N} J(z)e^{\alpha(|z|)}\,dz - 1 - \frac{\nu}{2}\right).\end{aligned}$$

Therefore w satisfies

$$\mathcal{M}[w](x) + \beta(x)w(x) \leq h(\alpha)w(x) \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_0}(0), \tag{3.3}$$

where $h(\alpha)$ is defined by

$$h(\alpha) = -1 - \frac{\nu}{2}.$$

Since J is compactly supported, thanks to the Lebesgue's Theorems, we can check that $h(\cdot)$ is a smooth (C^2) convex increasing function of α. Moreover, we have

$$\lim_{\alpha \to 0} h(\alpha) = h(0) = -\frac{\nu}{2}.$$

Therefore, by continuity of h, we can choose α small so that $h(\alpha) < 0$. For such α, we get

$$\mathcal{M}[w](x) + \beta(x)w(x) \leq h(\alpha)w(x) < 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_0}(0). \tag{3.4}$$

Let $M := \sup_{B_{2R_0}(0)} S(x)$ and let us fix $C = 2Me^{2\alpha R_0}$. We consider now the continuous function

$$\bar{u}(x) := \begin{cases} Ce^{-\alpha|x|} & \text{in} \quad \mathbb{R}^N \setminus B_{2R_0}(0), \\ 2M & \text{in} \quad B_{2R_0}(0). \end{cases}$$

By direct computation we can check that $\bar{u}$ is a super-solution of the problem (1.2). Indeed, for any $x \in B_{2R_0}(0)$, we have $\bar{u} = 2M > \sup_{B_{2R_0}(0)} S(x)$ which implies that $f(x, \bar{u}) = f(x, 2M) \leq 0$ and

$$\mathcal{M}[\bar{u}](x) + f(x, \bar{u}(x)) \leq 2M \int_{\mathbb{R}^N} J(x-y)\, dy - 2M + f(x, 2M) \leq f(x, 2M) \leq 0.$$

Whereas, for $x \in \mathbb{R}^N \setminus B_{2R_0}(0) \subset \mathbb{R}^N \setminus B_{R_0}(0)$ we have by (3.4)

$$\begin{aligned} \mathcal{M}[\bar{u}](x) + f(x, \bar{u}(x)) \leq \mathcal{M}[\bar{u}](x) + \beta(x)w(x) &\leq \mathcal{M}[w](x) + \beta(x)w(x), \\ &\leq h(\alpha)w(x) \leq 0. \end{aligned}$$

□

We are now in position to construct a positive solution of (1.2). By Lemma 3.1, there exists $\bar{u}$ a positive continuous super-solution of the problem (1.2). Therefore for any $R > 0$, $\bar{u}$ is also a positive continuous super-solution of the problem (3.1). Therefore by using a standard sweeping argument, we can check that for all $R \geq R_0$ the unique positive continuous solution of (3.1) satisfies $u_R \leq \bar{u}$ in $B_R(0)$. By sending now $R \to \infty$ and observing that $u_R \in C(\mathbb{R}^N)$ is locally uniformly bounded and monotone with respect to R, we get $u_R \to \tilde{u} := \lim_{R\to\infty} u_R$ a non-negative solution of (1.2). $\tilde{u}$ is non trivial since $0 \leq \tilde{u} \leq \bar{u}$ and

$$u_R \leq \tilde{u} \quad \text{in } B(0,R), \quad \text{for all } R \geq R_0.$$

Moreover, by adapting the proof in [1] we can show that $\tilde{u} \in C_0(\mathbb{R}^N)$.

3.2 Uniqueness

Having constructed a $L^1(\mathbb{R}^N)$ positive solution to (1.2), the uniqueness of the solutions of (1.2) is then obtained by the following argument. Assume by contradiction that $v \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is another positive solution. Then v is a supersolution of the problem (3.1) for any $R > 0$. Therefore $v \geq u_R$ for all $R \geq R_0$. Since u_R is monotone with respect to R, it follows that $v \geq \tilde{u} := \lim_R u_R(x)$. By assumption $v \not\equiv \tilde{u}$. Recall that the functions v and $\tilde{u}$ are verifying:

$$\mathcal{M}[\tilde{u}] + f(x, \tilde{u}) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{3.5}$$

$$\mathcal{M}[v] + f(x, v) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{3.6}$$

So, multiplying (3.5) by v and (3.6) by u we get after integration over $\mathbb{R}^N$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\tilde{u}(y)v(x)\,dydx - \int_{\mathbb{R}^N} \tilde{u}(x)v(x)\,dx + \int_{\mathbb{R}^N} v(x)f(x,\tilde{u}(x))\,dx = 0, \tag{3.7}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\tilde{u}(x)v(y)\,dydx - \int_{\mathbb{R}^N} \tilde{u}(x)v(x)\,dx + \int_{\mathbb{R}^N} \tilde{u}(x)f(x,v(x))\,dx = 0. \tag{3.8}$$

Therefore by subtracting the two above equality, we get the contradiction

$$0 < \int_{\mathbb{R}^N} v(x)\tilde{u}\left[\frac{f(x,\tilde{u}(x))}{\tilde{u}(x)} - \frac{f(x,v(x))}{v(x)}\right] dx = 0,$$

since $\tilde{u} \le v$ and $f(x,s)/s$ is decreasing.

3.3 Non-existence of a solution

In this section, we deal with the non-existence of positive solution when $\lambda_p(\mathcal{M} + \beta(x)) \ge 0$. To simplify the presentation of the proofs, we treat the two cases: $\lambda_p(\mathcal{M} + \beta(x)) > 0$ and $\lambda_p(\mathcal{M} + \beta(x)) = 0$ separately. The proof in the second case being more involved, we start by showing the non existence results when $\lambda_p(\mathcal{M} + \beta(x)) > 0$.

Case $\lambda_p(\mathcal{M} + \beta(x)) > 0$:

In this situation we argue as follows. Assume by contradiction that a positive bounded solution u exists. By assumption, u satisfies

$$\mathcal{M}[u](x) + \beta(x)u(x) \ge 0. \tag{3.9}$$

Therefore u is a test function for $\lambda_p'(\mathcal{M}+\beta(x))$ and we get $\lambda_p'(\mathcal{M}+\beta(x)) \le 0$. Since by Theorem 2.6 $\lambda_p(\mathcal{M}+\beta(x)) \le \lambda_p'(\mathcal{M} + \beta(x))$ we get a straightforward contradiction.

Case $\lambda_p(\mathcal{M} + \beta(x)) = 0$:

In this situation, as above we argue by contradiction. Assume by contradiction that a non-negative, non identically zero, bounded solution u exists. By a straightforward application of the maximum principle, since $u \not\equiv 0$ we have $u > 0$ in $\mathbb{R}^N$. Now let us observe that in this situation, by the above argumentation we have $\lambda_p(\mathcal{M} + \beta(x)) = 0 = \lambda_p'(\mathcal{M} + \beta(x))$ and by (iv) of Proposition (2.2) we get the following estimate

$$\sup_{\mathbb{R}^N}(\beta(x) - 1) \le 0. \tag{3.10}$$

Let us denote $\gamma(x) := \frac{f(x,u(x))}{u(x)}$, then we obviously have

$$J \star u(x) - u(x) + \gamma(x)u(x) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{3.11}$$

Therefore by definition of λ_p' we have $\lambda_p'(\mathcal{M} + \gamma(x)) \le 0$. By construction $\gamma(x) \le \beta(x)$, so by combining (3.11) with the Proposition 2.2, the Theorem 2.6 and the definition of $\lambda_p(\mathcal{M} + \gamma(x))$ we can infer that

$$\lambda_p(\mathcal{M} + \gamma(x)) \le \lambda_p'(\mathcal{M} + \gamma(x)) \le 0 \le \lambda_p(\mathcal{M} + \beta(x)) \le \lambda_p(\mathcal{M} + \gamma(x)).$$

Therefore $\lambda_p(\mathcal{M} + \gamma(x)) = 0$. Let us denote $\eta \in C(\mathbb{R}^N)$ a smooth regularisation of $\chi_{B_1(0)}$ the characteristic function of the unit ball. Since $\gamma(x) < \beta(x)$ in $\mathbb{R}^N$, we can find $\varepsilon_0 > 0$ small so that for all $\varepsilon \le \varepsilon_0$

$$\gamma(x) < \gamma(x) + \varepsilon\eta(x) < \beta(x) \quad \text{in} \quad \mathbb{R}^N.$$

By (i) of Proposition 2.2, we then have

$$0 = \lambda_p(\mathcal{M} + \beta) \le \lambda_p(\mathcal{M} + \gamma + \varepsilon\eta) \le \lambda_p(\mathcal{M} + \gamma) = 0.$$

Now we claim that

Claim 3.2. There exists $R_1 > 0$ and $\psi > 0$, $\psi \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ so that

$$J \star \psi(x) - \psi(x) + (\gamma(x) + \varepsilon\eta(x))\psi(x) = 0 \quad \text{in} \quad \mathbb{R}^N$$

Assume for a moment that the claim holds then by arguing as in the subsection (3.2), since $\psi \in L^1$ we get the following contradiction

$$0 = -\varepsilon \int_{\mathbb{R}^N} u(x)\psi(x)\eta(x)\,dx < 0.$$

Proof of the Claim. For convenience we denote $\tilde{\gamma}(x) := \gamma + \varepsilon\eta$. By (3.10), since $\tilde{\gamma} < \beta$ we also have

$$0 < -\sup_{\mathbb{R}^N}(\tilde{\gamma} - 1). \tag{3.12}$$

From the latter inequality, by using Proposition 2.1 and Lemma 2.3 we see that there exists R_0 so that for all $R \geq R_0$ there exists a positive eigenfunction $\varphi_R \in C(\bar{B}(0,R))$ associated to the principal eigenvalue $\lambda_p(\mathcal{L}_R + \tilde{\gamma}(x))$ of the approximated problem

$$\mathcal{L}_R[\varphi] + (\tilde{\gamma}(x) + \lambda)\varphi = 0 \quad \text{in} \quad B(0,R) \tag{3.13}$$

Take now the increasing sequence $(R_n)_{n\in\mathbb{N}} := (R_0 + n)_{n\in\mathbb{N}}$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be the sequence of function where φ_n is the positive principal eigenfunction associated to $\lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x))$. Without loss of generality, we can assume that for all n, $\varphi_n(0) = 1$.

Recall that for all n, φ_n satisfies

$$\mathcal{L}_{R_n}[\varphi_n] + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))\varphi_n = 0 \quad \text{in} \quad B_{R_n}. \tag{3.14}$$

Let us now define $b_n(x) := -\lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)) - \tilde{\gamma}(x)$. Then φ_n satisfies

$$\mathcal{L}_{R_n}[\varphi_n] = b_n(x)\varphi_n \quad \text{in} \quad B_{R_n}.$$

By construction for all $n \geq 0$ we have $b_n(x) \geq -\lambda_p(\mathcal{L}_{R_{n_0}} + \tilde{\gamma}(x)) - \sup_{\mathbb{R}^N}(\tilde{\gamma}(x) - 1) > 0$, therefore the Harnack inequality (Theorem 1.4 in [22]) applies to φ_n. Thus for $n \geq 0$ fixed and for all compact set $\omega \subset\subset B_{R_n}$ there exists a constant $C_n(\omega)$ such that

$$\varphi_n(x) \leq C_n(\omega)\varphi_n(y) \quad \forall \quad x, y \in \omega.$$

Moreover the constant $C_n(\omega)$ only depends on $\bigcup_{x\in\omega} B_{r_0}(x)$ and is monotone decreasing with respect to $\inf_{x\in B_{R_n}} b_n(x)$. For all $n \geq 0$, the function $b_n(x)$ being uniformly bounded from below by a constant independent of n, the constant C_n is bounded from above independently of n by a constant $C(\omega)$. Thus we have

$$\varphi_n(x) \leq C(\omega)\varphi_n(y) \quad \forall \quad x, y \in \omega.$$

From the normalization $\varphi_n(0) = 1$, we deduce that the sequence $(\varphi_n)_{n\geq 0}$ is locally uniformly bounded in $\mathbb{R}^N$. Moreover, from a standard diagonal extraction argument, there exists a subsequence still denoted $(\varphi_n)_{n\geq 0}$ such that $(\varphi_n)_{n\geq 0}$ converges locally uniformly to a continuous function φ. Furthermore, φ is a non-negative non trivial function and $\varphi(0) = 1$.

Since J is compactly supported, we can pass to the limit in the equation (3.14) using the Lebesgue monotone convergence theorem and get

$$\mathcal{M}[\varphi] + (\tilde{\gamma}(x) + \lambda_p(\mathcal{M} + \tilde{\gamma}(x)))\varphi(x) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Hence, we have

$$\mathcal{M}[\varphi] + \tilde{\gamma}(x)\varphi = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{3.15}$$

To conclude the proof of this claim, we characterise the behaviour of $\varphi(x)$ for $|x| >> 1$.
Let us denote $0 < \nu < -\limsup_{|x|\to\infty} \beta(x)$ and let us fix R_1 so that $\beta(x) \leq -\frac{\nu}{2}$ for $|x| \geq R_1$.

Since by Lemma 2.3 $\lambda_p(\mathcal{L}_R + \tilde{\gamma}(x)) \to \lambda_p(\mathcal{M} + \tilde{\gamma}(x)) = 0$, we can take R_1 larger if necessary to achieve

$$\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_R + \tilde{\gamma}(x)) \leq -\frac{\nu}{4} \quad \text{for} \quad |x| \geq R_1.$$

Now let us consider $\psi(x) := Ce^{-\alpha(|x|-R_1)}$ where C and α will be chosen later on. By a straightforward computation, we can see that for all $R > R_1$

$$\begin{aligned}\mathcal{L}_R[\psi](x) + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_R + \tilde{\gamma}(x)))\psi(x) &\leq \psi(x)\left(\int_{\mathbb{R}^N} J(z)e^{\alpha|z|}dz - 1 - \frac{\nu}{4}\right) \quad \text{for} \quad |x| \geq R_1,\\ &\leq h(\alpha)\psi(x) \quad \text{for} \quad |x| \geq R_1,\end{aligned}$$

with

$$h(\alpha) := \left(\int_{\mathbb{R}^N} J(z)e^{\alpha|z|}dz - 1 - \frac{\nu}{4}\right).$$

Since J is compactly supported, by the Lebesgue Theorem, the function h is continuous and $h(0) = -\frac{\nu}{4}$. By assumption $\nu > 0$, so by continuity of h there exists $\alpha_0 > 0$ so that $h(\alpha_0) < 0$. Thus we achieve for $\alpha = \alpha_0$

$$\mathcal{L}_R[\psi](x) + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_R + \tilde{\gamma}(x)))\psi(x) \leq 0 \quad \text{for} \quad |x| \geq R_1. \tag{3.16}$$

Recall that by construction, the function φ_n satisfies

$$\mathcal{L}_{R_n}[\varphi_n](x) + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))\varphi_n(x) = 0 \quad \text{in} \quad B_{R_n}(0). \tag{3.17}$$

Since J is compactly supported and $J(0) > 0$ there exists positive constants $r_0 \geq r_1; M \geq m$ so that

$$M\chi_{B_{r_0}(x)} \geq J(x-y) \geq m\chi_{B_{r_1}(x)} \quad \text{for all} \;\; x, y \in \mathbb{R}^N.$$

Therefore for n large enough say $n \geq n_0$, we have $R_n > R_1 + r_0$ and by the Harnack inequality, for all $n \geq n_0$ we have

$$\varphi_n(x) \leq C(B_{R_1}, \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))\varphi_n(y) \quad \text{for all} \quad x, y \in B_{R_1}(0).$$

with $C(B_{R_1}, \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))$ a constant that only depends on $\bigcup_{x\in B_{R_1}} B_{r_0}(x)$ and is monotone decreasing with respect to $\inf_{x\in B_{R_n}}(\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))$. For all $n \geq n_0$, the function $\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x))$ being uniformly bounded from below by a constant independent of n, the constant $C(B_{R_1}, \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))$ is bounded from above independently of n by a constant $C(B_{R_1})$. Thus we have for all $n \geq n_0$

$$\varphi_n(x) \leq C(B_{R_1})\varphi_n(y) \quad \forall \quad x, y \in B_{R_1}.$$

In particular, we have for all $n \geq n_0$,

$$\varphi_n(x) \leq C(B_{R_1})\varphi_n(0) = C(B_{R_1}) \quad \forall \quad x \in B_{R_1}.$$

By choosing $C > C(B_{R_1})$, we achieve

$$\psi(x) \geq C > C(B_{R_1}) \geq \varphi_n(x) \quad \forall \quad x \in B_{R_1}.$$

Set now $w_n := \psi - \varphi_n$, from (3.16) and (3.17) we get

$$\mathcal{L}_{R_n}[w_n](x) + (\tilde{\gamma}(x) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))w_n(x) \leq 0 \quad \text{for} \quad R_1 \leq |x| < R_n, \tag{3.18}$$

$$w_n > 0 \quad \text{for} \quad |x| < R_1. \tag{3.19}$$

By a straightforward application of the Maximum principle, it follows that for all $n \geq n_0$ we have $\varphi_n(x) \leq \psi$. Indeed, since w_n is continuous, w_n achieves a minimum at some point $x_0 \in B_{R_n}$. Assume by contradiction that $w_n(x_0) < 0$. Then, thanks to (3.19) $x_0 \in\in B_{R_n} \setminus B_{R_1}$ and at this point by (3.18) we have the following contradiction

$$0 \geq \mathcal{L}_{R_n}[w_n](x_0) + (\tilde{\gamma}(x_0) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))w_n(x_0) \geq \int_{B_{R_n}} J(x_0 - y)w_n(y)\, dy - w_n(x_0) + +\frac{\nu}{4}|w_n(x_0)|,$$

$$\geq \int_{B_{R_n}} J(x_0 - y)[w_n(y) - w_n(x_0)]\, dy + \frac{\nu}{4}|w_n(x_0)| > 0.$$

Hence, for all $n \geq n_0$ $\varphi_n \leq \psi$ in B_{R_n} which by sending $n \to \infty$ leads to $\varphi \leq \psi$ in $\mathbb{R}^N$ which conclude the proof of the Claim. □

4 Long time Behaviour

In this section, we investigate the long-time behaviour of the positive solution $u(t,x)$ of

$$\frac{\partial u}{\partial t}(t,x) = J \star u(t,x) - u(t,x) + f(x, u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{4.1}$$

$$u(0,x) = u_0(x). \tag{4.2}$$

For any $u_0 \in C^k(\mathbb{R}^N) \cap L^\infty$ or in $C^k(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ the existence of a solution $u(t,x) \in C^1((0,+\infty), C^{\min\{1,k\}}(\mathbb{R}^N))$ respectively $u(t,x) \in C^1((0,+\infty), C^{\min\{1,k\}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ is a straightforward consequence of the Cauchy-Lipschitz Theorem and of the KPP structure of the nonlinearity f. Before going to the proof of the asymptotic behaviour, let us recall some useful results

Lemma 4.1. *Assume that $u_0(x)$ is a sub-solution to* (4.1)*, then the solution $u(t,x)$ is increasing in time. Conversely, if $u_0(x)$ is a super-solution to* (4.1) *then $u(t,x)$ is decreasing in time.*

The proof of this Lemma follows from a straightforward used of the parabolic maximum principle and is let to reader. Let us now prove the asymptotic behaviour of the solution of (4.1) and end the proof of Theorem 1.1.

Proof. Let $z(t,x)$ be the solution to

$$\frac{\partial z}{\partial t} = J \star z - z + f(x, z(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \tag{4.3}$$

$$z(0,x) = C\|u_0\|_\infty \tag{4.4}$$

Since $S(x) \in L^\infty$ by choosing C large enough, the constant $C\|u_0\|_\infty$ is a super-solution to (4.1) therefore $z(t,x)$ is a decreasing function and by the parabolic maximum principle we have $u(t,x) \leq z(t,x)$ for all $(t,x) \in [0,+\infty) \times \mathbb{R}^N$, leading to

$$\limsup_{t\to\infty} u(t,x) \leq \limsup_{t\to\infty} z(t,x) \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{4.5}$$

Now let us consider the approximated parabolic problem

$$\frac{\partial v_R}{\partial t}(t,x) = \int_{B_R(0)} J(x-y)v_R(t,y)\, dy - v_R(t,x) + f(x, v_R(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times B_R(0), \tag{4.6}$$

$$v_R(0,x) = \eta_R u_0(x), \tag{4.7}$$

where $\eta_R := \eta\left(\frac{|x|}{R}\right)$ with $\eta \in C(\mathbb{R}^+)$ a smooth cut-off function so that $\eta \geq 0$, $\eta \equiv 1$ in $[0,1]$ and $\eta \equiv 0$ in $\mathbb{R}^+ \setminus [0,2]$. By Theorem 2.8, for R large enough the solution $v_R(t,x)$ converges to $u_R(x)$ the unique positive stationary solution of (4.6). By construction since $u(t,x)$ is a super-solution of the problem (4.6), by the parabolic comparison principle we have for all R large enough $v_R(t,x) \leq u(t,x)$ for all $(t,x) \in [0,+\infty) \times B_R(0)$. Therefore we have for all R large enough

$$\liminf_{t\to\infty} u(t,x) \geq u_R(x) \quad \text{for all} \quad x \in B_R(0) \tag{4.8}$$

By taking the limit as $R \to \infty$, in the above inequality we get

$$\liminf_{t\to\infty} u(t,x) \geq \lim_{R\to\infty} u_R(x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N \tag{4.9}$$

Note that we can reproduce the above arguments with $z(t,x)$, thus we also get

$$v_R(t,x) \leq z(t,x) \quad \text{for all} \quad (t,x) \in [0,+\infty) \times B_R(0) \tag{4.10}$$

$$\liminf_{t\to\infty} z(t,x) \geq \lim_{R\to\infty} u_R(x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N \tag{4.11}$$

By (4.10) $z(t,x)$ is locally uniformly bounded from below and since $z(t,x)$ is a decreasing function of t we get $\lim_{t\to\infty} z(t,x) = \bar{z}(x) > 0$ for all $x \in \mathbb{R}^N$. Moreover $\bar{z}$ is a bounded stationary solution to (4.1). By uniqueness of the positive stationary solution, we conclude that $\bar{z} = \tilde{u}$. Thus we have

$$\lim_{t\to\infty} z(t,x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N \tag{4.12}$$

Hence by collecting (4.5),(4.9), (4.12) we get for all $x \in \mathbb{R}^N$

$$\tilde{u}(x) \leq \liminf_{t\to\infty} u(t,x) \leq \limsup_{t\to\infty} u(t,x) \leq \limsup_{t\to\infty} z(t,x) = \lim_{t\to\infty} z(t,x) = \tilde{u}(x).$$

Now, to complete the proof we are left to show that $\|u - \tilde{u}\|_\infty \to 0$ as $t \to \infty$. To this end we follow the argument in [9]. We argue by contradiction and assume there exists $\varepsilon > 0$ and the sequences $(t_n) \in \mathbb{R}^+$, $(x_n) \in \mathbb{R}^N$ such that

$$\lim_{n\to\infty} t_n = \infty, \qquad |u(t_n,x_n) - \tilde{u}(x_n)| > \varepsilon, \qquad \forall n \in \mathbb{N}. \tag{4.13}$$

By (4.12), we already know that $u \to \tilde{u}$ locally uniformly in $\mathbb{R}^N$, so without loss of generality, we can assume that $|x_n| \to \infty$. From the construction of $\tilde{u}$, Subsection 3.1, we have $\lim_{|x|\to\infty} \tilde{u}(x) = 0$. Therefore for some $R_0 > 0$, we have $\tilde{u}(x) \leq \frac{\varepsilon}{2}$ for all $|x| \geq R_0$. The latter combined with (4.12) and (4.13) enforces

$$z(t_n,x_n) - \tilde{u}(x_n) \geq u(t_n,x_n) - \tilde{u}(x_n) > \varepsilon, \qquad \forall n \in \mathbb{N}. \tag{4.14}$$

We claim that

Claim 4.2. For all sequences $(t_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ so that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = +\infty$, then $z(t_n,x_n) \to 0$.

Assume for the moment the claim holds true. Then we obtain a straightforward contradiction

$$0 = \lim_{n\to\infty} z(t_n,x_n) - \tilde{u}(x_n) \geq \lim_{n\to\infty} u(t_n,x_n) - \tilde{u}(x_n) > \varepsilon.$$

Let us prove the Claim. Again we argue by contradiction and assume there exists $\varepsilon > 0$ and sequences $(t_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = \infty$ so that $z(t_n,x_n) > \varepsilon$ for all $n \in \mathbb{N}$. Let us define $z_n(t,x) := z(t,x+x_n)$ then by definition z_n satisfies

$$\begin{aligned}
&\frac{\partial z_n}{\partial t}(t,x) = \int_{\mathbb{R}} J(x-y) z_n(y)\,dy - z_n(t,x) + f(x+x_n, z_n(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N,\\
&z_n(0,x) = C\|u_0\|_\infty,
\end{aligned}$$

and $0 < z_n(t,x) < C\|u_0\|_\infty$. Since for all n, $z_n(0,x) \in C^\infty$ by the Cauchy Lipschitz Theorem we see that $z_n \in C^1(\mathbb{R}^+, C^1(\mathbb{R}^N))$. Thus, there exists $C_0 > 0$ independent of n so that $\|z_n\|_{C^{1,1}(\mathbb{R}^+, C(\mathbb{R}^N))} < C_0$. From these estimates, the sequence $(z_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^{1,1}((0,T), C^{0,1}(\mathbb{R}^N))$ for any $T > 0$. By a diagonal extraction, there exists a subsequence of $(z_n)_{n\in\mathbb{N}}$ that converges locally uniformly to $\tilde z(t,x)$. Moreover, thanks to $\lim_{|x|\to\infty}\frac{f(x,s)}{s}<0$, there exists $\kappa>0$ so that $\tilde z(x,t)$ satisfies

$$\frac{\partial \tilde z}{\partial t}(t,x)\le \int_{\mathbb{R}} J(x-y)\tilde z(t,y)\,dy-\tilde z(t,x)-\kappa\tilde z(t,x)) \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^N, \tag{4.15}$$

$$\tilde z(0,x)=C\|u_0\|_\infty. \tag{4.16}$$

In addition, for all $t>0$, $\tilde z(t,0)=\lim_{n\to\infty} z_n(t,0)\ge\varepsilon$. Since $\tilde z(0,x)$ is a super-solution of (4.15), by Lemma 4.1 the function $\tilde z(t,x)$ is monotone decreasing in time. By sending $t\to\infty$, since $\tilde z\ge 0$, $\tilde z$ converges locally uniformly to a non-negative function $\bar z$ that satisfies

$$\begin{aligned}
&\int_{\mathbb{R}} J(x-y)\bar z(y)\,dy-\bar z(x)-\kappa\bar z(x))\ge 0 \quad \text{in} \quad \mathbb{R}^N,\\
&0\le \bar z\le C\|u_0\|_\infty,\\
&\bar z(0)\ge\varepsilon.
\end{aligned}$$

Now let consider the function $w(x):=\frac{\varepsilon}{2}e^{\alpha|x|}-\bar z$ with α to be chosen, then w satisfies

$$\int_{\mathbb{R}} J(x-y)w(y)\,dy-w(x)-\kappa w(x)\le \rho e^{\alpha|x|}\left(\int_{\mathbb{R}^N} J(z)e^{\alpha|z|}\,dy-1-\kappa\right) \quad \text{in} \quad \mathbb{R}^N.$$

The left hand side of the inequality is well defined and continuous with respect to α since J is compactly supported. Thanks to $\int_{\mathbb{R}^N} J(z)dz=1$, by choosing α small enough, we achieve

$$\int_{\mathbb{R}} J(x-y)w(y)\,dy-w(x)-\kappa w(x)<0 \quad \text{in} \quad \mathbb{R}^N.$$

By construction, since $\bar z$ is bounded $\lim_{|x|\to\infty} w(x)=+\infty$ and w achieves a minimum in $\mathbb{R}^N$ says at x_0. Since $w(0)=\frac{\varepsilon}{2}-\bar z(0)\le-\frac{\varepsilon}{2}$, we have $w(x_0)<0$. Now at this point, we get the following contradiction

$$0<\int_{\mathbb{R}} J(x_0-y)[w(y)-w(x_0)]\,dy-\kappa w(x_0)<0 \quad \text{in} \quad \mathbb{R}^N.$$

$\square$

Finally we establish the long time behaviour of the solution $u(t,x)$ starting from an integrable initial datum u_0,i.e $u_0\in L^1(\mathbb{R}^N)\cap C(\mathbb{R}^N)$. To do so, we define two auxiliary functions $h(t,x)$ and $v(t,x)$ that are respectively solution to

$$\begin{cases}\frac{\partial h}{\partial t}(t,x)=J\star h(t,x)-h(t,x)+f(x,h(t,x)) \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^N,\\ h(0,x)=\sup\{\tilde u(x),u_0(x)\},\end{cases} \tag{4.17}$$

$$\begin{cases}\frac{\partial v}{\partial t}(t,x)=J\star v(t,x)-v(t,x)+f(x,v(t,x)) \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^N,\\ v(0,x)=\inf\{\tilde u(x),u_0(x)\}.\end{cases} \tag{4.18}$$

By construction, from the comparison principle we deduce that $v(t,x)\le u(t,x)\le h(t,x)$ for all $(t,x)\in\mathbb{R}^+\times\mathbb{R}^N$. Therefore

$$\|u-\tilde u\|_{L^1(\mathbb{R}^N)}\le\sup\{\|h-\tilde u\|_{L^1(\mathbb{R}^N)},\|v-\tilde u\|_{L^1(\mathbb{R}^N)}\}.$$

Thus to prove that $\|u-\tilde u\|_{L^1(\mathbb{R}^N)}\to 0$ it is enough to show that h and v converge to $\tilde u$ in $L^1(\mathbb{R}^N)$.

Let us show that v converges to $\tilde u$ in $L^1(\mathbb{R}^N)$. Since $\tilde u(x)$ is a super solution to (4.18) we deduce $v(t,x)\le\tilde u(x)$ for all $x\in\mathbb{R}^N$. Let $\varepsilon>0$ be fixed and choose R so that $\int_{\mathbb{R}^N\setminus B(0,R)}\tilde u(x)\,dx\le\frac{\varepsilon}{4}$ then we have

$$
\begin{aligned}
\|\tilde u - v\|_{L^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N\setminus B_R(0)} (\tilde u(x) - v(t,x))\,dx + \int_{B_R(0)} (\tilde u(x) - v(t,x))\,dx,\\
&\le 2\int_{\mathbb{R}^N\setminus B_R(0)} \tilde u(x)\,dx + \int_{B_R(0)} (\tilde u(x) - v(t,x))\,dx,\\
&\le \frac{\varepsilon}{2} + \int_{B_R(0)} (\tilde u(x) - v(t,x))\,dx.
\end{aligned}
$$

Recall that v converges pointwise to $\tilde u$ as t tends to infinity. Therefore, by Lebesgue Theorem for some $t(\varepsilon)$ we get for all $t \ge t(\varepsilon)$, $\int_{B_R(0)}(\tilde u(x) - v(t,x))\,dx \le \frac{\varepsilon}{2}$ which enforces

$$
\|\tilde u - v\|_{L^1(\mathbb{R}^N)} \le \varepsilon.
$$

ε being chosen arbitrary, the latter inequality shows that $\lim_{t\to\infty} \|\tilde u - v\|_{L^1(\mathbb{R}^N)} = 0$ which proves that v converges to $\tilde u$ in $L^1(\mathbb{R}^N)$.

To obtain that $\|h - \tilde u\|_{L^1(\mathbb{R}^N)} \to 0$ we argue as follow. By construction $\tilde u$ is a sub solution to (4.17), thus $\tilde u(x) \le h(t,x)$ for all $(t,x) \in \mathbb{R}^+\times\mathbb{R}^N$. Let us denote $w(t,x) := h(t,x) - \tilde u(x)$. Then w satisfies for all $(t,x) \in \mathbb{R}^+\times\mathbb{R}^N$:

$$
\begin{aligned}
\frac{\partial w}{\partial t}(t,x) &= J\star w(t,x) - w(t,x) + \left(\frac{f(x,h(t,x))}{h(t,x)} - \frac{f(x,\tilde u(x))}{\tilde u}\right) h(t,x) + \frac{f(x,\tilde u(x))}{\tilde u} w(t,x),\\
&\le J\star w(t,x) - w(t,x) + \frac{f(x,\tilde u(x))}{\tilde u} w(t,x).
\end{aligned}
$$

Now thanks to $\lim_{|x|\to\infty} \frac{f(x,s)}{s} < 0$, there exists $\kappa > 0$ and R_0 so that w satisfies

$$
\frac{\partial w}{\partial t}(t,x) \le J\star w(t,x) - w(t,x) - \kappa w(t,x) \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^N\setminus B_{R_0}(0). \tag{4.19}
$$

Fix now $\varepsilon > 0$. Recall that $h(t,x)$ converges pointwise to $\tilde u$, then by Lebesgue Theorem there exists t_0 so that for all $t \ge t_0$,

$$
\int_{B_{R_0}(0)} w(t,x)\,dx \le \kappa\varepsilon.
$$

Now let us estimate $\int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(x)\,dx$ for $t \ge t_0$. By integrating (4.19) over $\mathbb{R}^N\setminus B_{R_0}(0)$ it yields

$$
\frac{\partial \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx}{\partial t} \le \int_{\mathbb{R}^N\setminus B_{R_0}(0)} J\star w(t,x)\,dx - \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx - \kappa\int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx.
$$

By using Fubini's Theorem, the uniform estimate on $\|w\|_\infty$ and the unit mass of the kernel, we can check that for $t \ge t_0$

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus B_{R_0}(0)} J\star w(t,x)\,dx &= \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,y)\left(\int_{\mathbb{R}^N\setminus B_{R_0}(0)} J(x-y)\,dx\right) dy + \int_{B_{R_0}(0)} w(t,y)\left(\int_{\mathbb{R}^N\setminus B_{R_0}(0)} J(x-y)\,dx\right) dy,\\
&\le \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,y)\,dy + \int_{B_{R_0}(0)} w(t,y)\,dy,\\
&\le \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,y)\,dy + \kappa\varepsilon.
\end{aligned}
$$

Therefore for $t \ge t_0$, w satisfies

$$
\frac{\partial \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx}{\partial t} \le \kappa\varepsilon - \kappa\int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx.
$$

From the later differential inequality, there exists $t(\varepsilon) \geq t_0$ so that for all $t \geq t(\varepsilon)$ we have

$$\int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx \leq 2\varepsilon.$$

Hence, we have for all $t \geq t(\varepsilon)$

$$\|w\|_{L^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N \setminus B_{R_0}(0)} w(t,x)\,dx + \int_{B_{R_0}(0)} w(t,x)\,dx \leq \left(2 + \frac{\kappa}{|B_{R_0}(0)|}\right)\varepsilon,$$

As above, ε being chosen arbitrary, the latter inequality shows that $\lim_{t\to\infty} \|w\|_{L^1(\mathbb{R}^N)} = 0$ which proves that h converges to $\tilde{u}$ in $L^1(\mathbb{R}^N)$.

5 Some asymptotics

In this section we analyse the qualitative behaviour of the solution of (1.2) with respect to the size of the support of J. For convenience we investigate the particular situation

$$\frac{1}{\varepsilon^m}(J_\varepsilon \star u - u) + u(a(x) - u) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{P_ε}$$

where $J_\varepsilon(z) = \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ with $supp(J) = B(0,1)$ and $a \in C^1(\mathbb{R}^N)$ so that $a^+ \not\equiv 0$.

The latter condition on $a(\cdot)$ is necessary to observe the possible existence of a solution. Indeed, if $a^+ \equiv 0$ then for any positive constant c_0 we have

$$\mathcal{M}[c_0] + a(x)c_0 \leq 0$$

therefore $\lambda_p(\mathcal{M}[c_0] + a(x)c_0) \geq 0$ and for all ε there is no solution to (P_ε) besides 0.

We analyse the behaviour of u_ε when $\varepsilon \to 0$ and $\to +\infty$ and try to understand the influence of m on the resulting limits.

We start by showing some *a priori* estimate for the solution u_ε.

Lemma 5.1. *There exist positive constants C_1, C_2, C_3 so that we have for any positive bounded solution u_ε of (P_ε)*

(i) $\|u_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C_1, \quad \|u_\varepsilon\|_\infty < C_3,$

(ii) $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(x-y)(u_\varepsilon(x) - u_\varepsilon(y))^2\,dxdy \leq C_2\varepsilon^m$

(iii) For all $x \in supp(a^+)$, there exists ρ, $\sup_{supp(a^+)} u_\varepsilon \geq -\frac{\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x))}{2}$.

(iv) $u_\varepsilon \geq (a(x) - \frac{1}{\varepsilon^m})^+$,

Proof. Since by construction the solution is unique and $u_\varepsilon \in L^1(\mathbb{R}^N) \cap L^\infty$. Moreover by (P_ε) $u_\varepsilon \leq M = \|a\|_\infty$. We obtain (i) by integrating (P_ε) over $\mathbb{R}^N$. Indeed, we get

$$\int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx = \int_{\mathbb{R}^N} a(x)u_\varepsilon(x)\,dx \leq \int_{\mathbb{R}^N} a^+(x)u_\varepsilon(x)\,dx \leq M\int_{\mathbb{R}^N} a^+(x)dx =: C_1.$$

To obtain (ii), let us multiply (P_ε) by u_ε and integrate over $\mathbb{R}^N$, then we get

$$\frac{1}{2\varepsilon^m}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u_\varepsilon(x) - u_\varepsilon(y))^2\,dxdy = \int_{\mathbb{R}^N} u_\varepsilon^2(x)(a(x) - u_\varepsilon(x))\,dx$$

Since u_ε and $a(x)$ are uniformly bounded independently, (ii) holds true with $C_2 := 4C_1M$. Observe that $(a(x) - \frac{1}{\varepsilon^m})^+$ is always a sub-solution to (P_ε), so by a standard sweeping principle $u_\varepsilon \geq (a(x) - \frac{1}{\varepsilon^m})^+$ and (iv) holds true. Finally to obtain (iii) we argue as follows.

Since u_ε is a positive bounded solution of (P_ε) by Theorem 1.1 we have $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) < 0$. Now since $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) < 0$ and J is compactly supported, by regularising a if necessary, we can find (see the proof of Lemma ?? in [2]) $\varphi_\varepsilon \in C_c(\mathbb{R}^N)$ so that

$$\mathcal{M}_{\varepsilon,m}[\varphi_\varepsilon] + a(x) + \frac{\lambda_p}{2}\varphi_\varepsilon \geq 0 \quad \text{in} \quad \mathbb{R}^N.$$

Moreover, we can normalised φ_ε so that $\|\varphi_\varepsilon\|_\infty = 1$. Plugging $\theta\varphi_\varepsilon$ with in (P_ε) it follows that

$$\mathcal{M}_{\varepsilon,m}[\theta\varphi_\varepsilon] + \theta\varphi_\varepsilon(a(x) - \theta\varphi_\varepsilon) \geq \theta\varphi_\varepsilon(-\frac{\lambda_p}{2} - \theta\varphi_\varepsilon).$$

Therefore for $0 < \theta \leq -\frac{\lambda_p}{2}$, the function $\theta\varphi_\varepsilon$ is a sub-solution to (P_ε). By a standard sweeping argument, we get

$$-\frac{\lambda_p}{2}\varphi_\varepsilon \leq u_\varepsilon \quad \text{and} \quad \sup_{\mathbb{R}^N} u_\varepsilon \geq -\frac{\lambda_p}{2}.$$

Since $u_\varepsilon \in L^1(\mathbb{R}^N)$, u_ε achieves its maximum at some point, says x_0. At this point from (P_ε) we have

$$0 \geq \mathcal{M}_{\varepsilon,m}[u_\varepsilon](x_0) = -u_\varepsilon(x_0)(a(x_0) - u_\varepsilon(x_0)).$$

Thus $x_0 \in supp(a^+)$ and $\|u_\varepsilon\|_\infty = \sup_{supp(a^+)} u_\varepsilon$ which proves (iii). $\square$

Next we obtain derive some useful super-solution for large ε.

Lemma 5.2. *There exists $\varepsilon_0 > 0$ so that for all $m \geq 0$ and $\varepsilon \geq \varepsilon_0$ any positive bounded solution u_ε of (P_ε) satisfies*

$$u_\varepsilon \leq a^+(x) + \frac{1}{\varepsilon^{\frac{N}{4}}}$$

Proof. Let $\delta \in (0, \frac{N}{2})$ and consider the function $\zeta_\varepsilon(x) := \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)$. We will show that ζ_ε is a super-solution to (P_ε) when $\varepsilon >> 1$.

Indeed, we have

$$\begin{aligned}\mathcal{M}_{\varepsilon,m}[\zeta_\varepsilon](x) + \zeta_\varepsilon(x)(a(x) - \zeta_\varepsilon(x)) &\leq \frac{\|J\|_\infty}{\varepsilon^{N+m}}\int_{\mathbb{R}^N} a^+(y)\,dy + \left(\frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)\right)\left[a(x) - \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} - a^+(x)\right] \\ &\leq \frac{\|J\|_\infty}{\varepsilon^{N+m}}\int_{\mathbb{R}^N} a^+(y)\,dy - \frac{1}{\varepsilon^{N-2\delta}}.\end{aligned}$$

where we use in the last inequality that

$$\left(\frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)\right)\left[a(x) - \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} - a^+(x)\right] \leq -\frac{1}{\varepsilon^{N-2\delta}} \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Thus for $\varepsilon >> 1$, we achieve

$$\mathcal{M}[\zeta_\varepsilon](x) + \zeta_\varepsilon(x)(a(x) - \zeta_\varepsilon(x)) \leq \frac{\|J\|_\infty}{\varepsilon^{N+m}}\int_{\mathbb{R}^N} a^+(y)\,dy - \frac{1}{\varepsilon^{N-2\delta}} < 0.$$

Therefore for $\varepsilon >> 1$, by a sweeping argument we get $u_\varepsilon \leq \zeta_\varepsilon$. We end the proof by taking $\delta = \frac{N}{4}$. $\square$

Remark 2. When $m = 0$ and $(a(x)-1)^+ \not\equiv 0$, the above computation holds as well with $\zeta_\varepsilon(x) := \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + (a(x)-1)^+$. Thus in this case we have for large ε

$$u_\varepsilon(x) \leq \frac{1}{\varepsilon^{\frac{N}{4}}} + (a(x)-1)^+.$$

Next, we prove some continuity of $\lambda_p(\mathcal{L}_{R,\varepsilon}+a(x))$ with respect to ε.

Lemma 5.3. *Let R,ε be fixed and positive then for all $\eta>0$ there exists $\delta>0$ so that*

$$|\lambda_p(\mathcal{L}_R+a_\varepsilon(x))-\lambda_p(\mathcal{L}_R+a_{\varepsilon+\delta}(x))|\le\eta,$$

where $a_\varepsilon(x):=a(\varepsilon x)$.

Let us prove now the continuity of $\lambda_p(\mathcal{L}_R+a_\varepsilon(x))$.

Proof of the Claim. Let $\varepsilon>0$ and $R>0$ be fixed. We observe that for all $|\delta|<\varepsilon$ we have for all $x\in\mathbb{R}^N$, $a_{\varepsilon+\delta}(x)=a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)$ therefore

$$\|a_\varepsilon-a_{\varepsilon+\delta}\|_{\infty,R}=\sup_{B(0,R)}\left\|a_\varepsilon(x)-a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)\right\|.$$

Since a_ε is a Lipschitz in $\mathbb{R}^N$, we have

$$\left\|a_\varepsilon(x)-a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)\right\|\le K(\varepsilon)\varepsilon\delta\|x\|,$$

where $K(\varepsilon)$ is the Lipschitz constant of a_ε. Thus

$$\|a_\varepsilon-a_{\varepsilon+\delta}\|_{\infty,R}\le K(\varepsilon)R\varepsilon\delta.$$

Hence, by (ii) of Proposition 2.2 we get

$$|\lambda_p(\mathcal{L}_R+a_\varepsilon(x))-\lambda_p(\mathcal{L}_R+a_{\varepsilon+\delta}(x))|\le K(\varepsilon)R\varepsilon\delta.$$

□

Finally, we establish a useful identity.

Proposition 5.4. *Let $\rho\in C_c^\infty(\mathbb{R}^N)$ be a radial function, then for all $u\in L^2(\mathbb{R}^N),\varphi\in C_c^\infty(\mathbb{R}^N)$ we have*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)[u(x+z)-u(x)]\varphi(x)\,dzdx=\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)u(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)]\,dzdx.$$

Proof. Thanks to the symmetry of ρ, using standard changes variables we have

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)[u(x+z)-u(x)]\varphi(x)&=\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)[u(x+z)-u(x)]\varphi(x)+\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(-z)[u(x-z)-u(x)]\varphi(x),\\&=\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)[u(x+z)-u(x)]\varphi(x)+\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)[u(x)-u(x+z)]\varphi(x+z),\\&=-\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)[u(x+z)-u(x)][\varphi(x+z)-\varphi(x)],\\&=-\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)u(x)[\varphi(x)-\varphi(x-z)]+\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)u(x)[\varphi(x+z)-\varphi(x)],\\&=\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho(z)u(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)].\end{aligned}$$

□

From the Proposition, for all $u\in L^2(\mathbb{R}^N),\varphi\in C_c^\infty(\mathbb{R}^N)$ we straightforwardly get the following identity

$$\int_{\mathbb{R}^N}\mathcal{M}_{\varepsilon,m}[u](x)\varphi(x)\,dx=\frac{\varepsilon^{2-m}D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\rho_\varepsilon(z)}{|z|^2}u_\varepsilon(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)]\,dxdz \tag{5.1}$$

where $\rho_\varepsilon(z)=\frac{1}{\varepsilon^N D_2(J)}J\left(\frac{z}{\varepsilon}\right)\frac{|z|^2}{\varepsilon^2}$.

Equipped with all these *apriori* estimates, we can now analyse the asymptotic behaviour of u_ε.

5.1 The case $m = 0$

In this situation, from Theorem 2.6 we know that

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{5.2}$$

$$\lim_{\varepsilon\to +\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = 1 - \sup_{\mathbb{R}^N} a(x) \tag{5.3}$$

As a consequence for ε small enough we have $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \le -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$ and by Theorem (1.1) there exists a solution to (P_ε). Moreover the following quantity is well defined

$$\varepsilon^* := \sup\{\varepsilon > 0 \,|\, \text{ for all } \ \varepsilon' < \varepsilon, \ \text{ there exists a positive solution to } (P_{\varepsilon'})\}.$$

In view of (5.3) $\varepsilon^* \in (0, +\infty]$ and $\varepsilon^* < +\infty$ **if and only if** $(a(x) - 1)^+ \not\equiv 0$.

Let us now deal with the limit of u_ε as $\varepsilon \to 0$ and $\varepsilon \to +\infty$ and let us start by proving that

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = v(x) \quad \text{a.e.} \tag{5.4}$$

where v is a non negative bounded solution of

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{5.5}$$

Let $w_\varepsilon := a(x) - u_\varepsilon$, then from (P_ε), w_ε satisfy

$$-J_\varepsilon \star w_\varepsilon + w_\varepsilon + u_\varepsilon(x) w_\varepsilon(x) = a(x) - J_\varepsilon \star a(x). \tag{5.6}$$

Multiplying the above equation by w_ε^+ and integrating over $\mathbb{R}^N$, it follows that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy + \int_{\mathbb{R}^N} u_\varepsilon(x)(w_\varepsilon^+)^2(x) = \int_{\mathbb{R}^N} w_\varepsilon^+(x) g_\varepsilon(x)\,dx$$

with $g_\varepsilon(x) := a(x) - J_\varepsilon \star a(x)$.

Let us now estimate the above integrals. First we observe that the double integral is positive. Indeed, since $w(y) = w^+(y) - w^-(x)$ we get

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon^+(y)w_\varepsilon^+(x))\,dxdy \\ &+ \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\,dxdy\end{aligned}$$

Thus

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy &= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)(x) - w_\varepsilon^+(y))^2\,dxdy \\ &+ \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\,dxdy\end{aligned} \tag{5.7}$$

Let us denote $Q := supp(a^+)$. Since u_ε is positive and uniformly bounded, we have $supp(w^+) \subset Q$ and

$$\left|\int_{\mathbb{R}^N} w_\varepsilon^+(x) g_\varepsilon(x)\,dx\right| \le C\int_Q |g_\varepsilon|(x)\,dx.$$

Since a is Lipschitz, by using a Taylor expansion, we can see that $|g_\varepsilon(x)| \le \varepsilon D_2(J)\|\nabla a\|_\infty$. Therefore

$$\left| \int_{\mathbb{R}^N} w_\varepsilon^+(x) g_\varepsilon(x)\, dx \right| \le C|Q|\varepsilon. \tag{5.8}$$

Collecting (5.7),(5.8), we get

$$\frac{1}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)(x) - w_\varepsilon^+(y))^2\, dxdy + \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\, dxdy + \int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2(x)\, dx \le C\varepsilon.$$

Thus

$$\int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2(x)\, dx \le C\varepsilon$$

and $u_\varepsilon w_\varepsilon^+(x) \to 0$ almost everywhere in Q.

Recall that

$$\int_{\mathbb{R}^N} u_\varepsilon w_\varepsilon(x)\, dx = 0$$

then from above estimates we conclude that

$$\int_{\mathbb{R}^N\setminus Q} u_\varepsilon(a(x)-u_\varepsilon) = \int_Q u_\varepsilon(a(x)-u_\varepsilon) \to 0 \quad \text{when} \quad \varepsilon\to 0.$$

Since $u_\varepsilon(a(x)-u_\varepsilon) \le 0$ in $\mathbb{R}^N\setminus Q$, from the above inequality it follows that $u_\varepsilon(x)w_\varepsilon(x) \to 0$ almost everywhere in $\mathbb{R}^N\setminus Q$. Since $u_\varepsilon > 0$ and $w_\varepsilon = (a(x)-u_\varepsilon(x)) \le 0$, it follows that $u_\varepsilon(x)\to 0$ almost everywhere in $\mathbb{R}^N\setminus Q$. Thus u_ε converges pointwise almost everywhere to a bounded non-neqative solution of (5.5).

Remark 3. Note that the above proof can be easily adapted to $\mathcal{M}_{\varepsilon,m}$ for $m<2$ as soon as the function a is smooth enough. Indeed, for $a\in C^2(\mathbb{R}^N)$, following the above arguments, we get by using the Taylor expansion up to order 2 of a

$$\int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2(x)\, dx \le C\varepsilon^{2-m},$$

with C a constant depending on $\|\nabla^2 u\|_\infty$. When a is only Lipschitz, the above argument holds only for $\mathcal{M}_{\varepsilon,m}$ with $m<1$.

Finally, to complete our analysis, we need to check that

$$\lim_{\varepsilon\to\varepsilon^*} u_\varepsilon = (a(x)-1)^+. \tag{5.9}$$

We treat separately the following two cases : (i) $\varepsilon^* < +\infty$, (ii) $\varepsilon^*=\infty$. The latter arise when $\sup_{\mathbb{R}^N}(a(x)-1)>0$. In this situation, there exists $R_0>0$ so that the continuous function $\varphi = (a(x)-1)^+ \not\equiv 0$ in $B_R(0)$ for $R\ge R_0$ and we can check that φ is a sub-solution to the approximated problem:

$$\int_{B_R(0)} J_\varepsilon(x-y)u(y)\, dy - u(x) + u(x)(a(x)-u) = 0 \quad \text{in} \quad B_R(0). \tag{5.10}$$

Since large constants are super-solutions of (5.10) for any $\varepsilon\ge 0, R>R_0$ there exists a unique solution $\varphi \le u_{\varepsilon,R} \le M$. By sending $R\to\infty$ and by the uniqueness of the solution to (P_ε) we have $\varphi\le u_\varepsilon \le M$ in $\mathbb{R}^N$.

Case $\varepsilon^* = +\infty$:

Thanks to Lemma 5.1 and Remark 2, for all $x\in\mathbb{R}^N$ for large ε we have

$$(a(x)-1)^+ \le u_\varepsilon(x) \le (a(x)-1)^+ + \frac{1}{\varepsilon^{\frac{N}{4}}}.$$

Hence, u_ε converge uniformly to $(a(x)-1)^+$.

Case $\varepsilon^* < +\infty$:

In this situation, the function $(a(x)-1)^+ \equiv 0$ in $\mathbb{R}^N$ and we are reduce to prove that

$$\lim_{\varepsilon\to\varepsilon^*} u_\varepsilon(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Note that by definition of ε^* we must have $\lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) \geq 0$. Indeed, if not then $\lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) < 0$ and by Lemma 2.4 $\lambda_p(\mathcal{M} + a_{\varepsilon^*}(x)) < 0$. Therefore for some R we have $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*}(x)) < 0$. By continuity of $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*}(x))$ with respect to ε, (Lemma 5.3) we get for some $\delta_0 > 0$, $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*+\delta}(x)) < 0$ for any $\delta \leq \delta_0$. Hence, $\lambda_p(\mathcal{M}_{\varepsilon^*+\delta} + a(x)) = \lambda_p(\mathcal{M} + a_{\varepsilon^*+\delta}(x)) < 0$ for any $\delta \leq \delta_0$ and by Theorem 1.1 there exists a positive solution to (P_ε) for all $\varepsilon \leq \varepsilon^* + \delta_0$ contradicting the definition of ε^*.

Note also that since $\varepsilon^* < +\infty$, the construction of the supersolution in Section 3 holds for any $\varepsilon \in [\frac{\varepsilon^*}{2}, \varepsilon^*]$, thus u_ε is uniformly bounded in $L^1(\mathbb{R}^N)$.

Let $g(x,s) := s(a(x) - 1 - s)$ then for all ε we have

$$J_\varepsilon \star u_\varepsilon = -g(x, u_\varepsilon(x)) \quad \text{in} \quad \mathbb{R}^N.$$

Now since J is C^1 and ε^* then for $\varepsilon \in [\frac{1}{2}\varepsilon^*, \varepsilon^*)$, we have

$$\begin{aligned}
|g(x,u_\varepsilon(x)) - g(z,u_\varepsilon(z))| &= \left|\int_{\mathbb{R}^N} [J_\varepsilon(x-y) - J(z-y)]u_\varepsilon(y)\,dy\right|, \\
&\leq |x-z| \int_{\mathbb{R}^N} \frac{|J_\varepsilon(x-y) - J(z-y)|}{|x-z|} u_\varepsilon(y)\,dy, \\
&\leq C(\varepsilon^*)|x-z|.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
C(\varepsilon^*)|x-z| &\geq |[1 - a(x) + u_\varepsilon(x) + u_\varepsilon(z)][u_\varepsilon(x) - u_\varepsilon(z)] + [a(z) - a(x)]u_\varepsilon(x)|, \\
&\geq |[1 - a(x) + u_\varepsilon(x) + u_\varepsilon(z)]||u_\varepsilon(x) - u_\varepsilon(z)| - |x-z|\frac{|a(z) - a(x)|}{|x-z|} M,
\end{aligned}$$

and for any $x \in Q := \{y \in \mathbb{R}^N \mid a(y) < 1\}$ u_ε is uniformly Lipschitz in x with a constant independent of ε. Thus $(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)}$ is uniformly bounded in $C^{0,\frac{1}{2}}_{loc}(Q)$. If $Q^c = \emptyset$, then $(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)}$ is uniformly bounded in $C^{0,\frac{1}{2}}_{loc}(\mathbb{R}^N)$. Otherwise, $Q^c \neq \emptyset$ and on Q^c $a(x) \equiv 1$. Therefore $u_\varepsilon^2(x) = J_\varepsilon \star u_\varepsilon$ and the $C^{0,\frac{1}{2}}(Q^c)$ norm of u_ε is bounded independently of ε. Hence,

$$(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)} \quad \text{is uniformly bounded in} \quad C^{0,\frac{1}{2}}_{loc}(Q) \cap C^{0,\frac{1}{2}}(Q^c). \tag{5.11}$$

In both case, since $a(x) < 0$ for $|x| >> 1$, Q^c is a compact set and $|\bar{Q} \cap Q^c| = 0$. From (5.11), for all sequence $\varepsilon_n \to \varepsilon^*$ by a diagonal extraction procedure there exists a subsequence still denoted $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ that converges locally uniformly in $\mathbb{R}^N \setminus (\bar{Q} \cap Q^c)$ to some non-negative function v. By passing to the limit in (P_ε) we can see that v is a bounded non negative solution of

$$J_{\varepsilon^*} \star v(x) - v(x) + v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N \setminus (\bar{Q} \cap Q^c).$$

Since $\bar{Q} \cap Q^c$ is of zero measure v is a solution to

$$\mathcal{L}_{\varepsilon^*,\mathbb{R}^N\setminus(\bar{Q}\cap Q^c)}[v] + v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N \setminus (\bar{Q} \cap Q^c).$$

Since $0 \leq \lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) \leq \lambda_p(\mathcal{L}_{\varepsilon^*,\mathbb{R}^N\setminus(\bar{Q}\cap Q^c)} + a(x))$, we deduce that $v \equiv 0$ which proves the limit.

Remark 4. When $a(x)$ is a radially symmetric non-increasing function we remark that ε^* is a sharp threshold. That is for all $\varepsilon \geq \varepsilon^*$ then (P_ε) does not have any positive solutions. Indeed in this situation the function $a_\varepsilon(x)$ is monotone non increasing with respect to ε. Thus by (i) of Proposition 2.2, for all $\varepsilon \geq \varepsilon^*$ we have

$$0 = \lambda_p(\mathcal{M} + a_{\varepsilon^*}(x)) \leq \lambda_p(\mathcal{M} + a_\varepsilon(x)).$$

Hence, by Theorem 1.1, 0 is the unique non negative solution to (P_ε) for $\varepsilon \geq \varepsilon^*$.

5.2 The case $0 < m < 2$

In this situation, from Theorem 2.6 we know that

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{5.12}$$

$$\lim_{\varepsilon\to +\infty} \lambda_p(\mathcal{M}_{\varepsilon} + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{5.13}$$

As a consequence for ε small enough and for large ε we have $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \le -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$. Therefore by Theorem (1.1) there exists a solution to (P_ε) for small and large ε.

The limit of u_ε when $\varepsilon \to \infty$ is easy to obtain. Indeed it is a straightforward consequence of (iv) of Lemma 5.1 and Lemma 5.2 since we have for ε large

$$(a(x) - \frac{1}{\varepsilon^m})^+ \le u_\varepsilon \le a^+(x) + \frac{1}{\varepsilon^{\frac{N}{4}}}.$$

To obtain the limits in L^2, we just observe that since by Lemma u_ε is uniformly bounded in L^2 and converges pointwise to a^+, we get $u_\varepsilon \rightharpoonup a^+$ in L^2. Moreover by Fatou's Lemma, we have

$$\int_{\mathbb{R}^N} (a^+)^2(x)\,dx \le \liminf_{\varepsilon\to\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx.$$

On another hand, by integrating (P_ε) over $\mathbb{R}^N$ we get for all ε

$$\int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx = \int_{\mathbb{R}^N} a(x)u_\varepsilon(x)\,dx \le \int_{\mathbb{R}^N} a^+(x)u_\varepsilon(x)\,dx.$$

Thus by the Cauchy-Schwartz inequality, for all ε

$$\left(\int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx\right)^{1/2} \le \left(\int_{\mathbb{R}^N} (a^+)^2(x)\,dx\right)^{1/2}$$

and we have

$$\int_{\mathbb{R}^N} (a^+)^2(x)\,dx \le \liminf_{\varepsilon\to\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx \le \limsup_{\varepsilon\to+\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx \le \int_{\mathbb{R}^N} (a^+)^2(x)\,dx$$

Hence, $\|u_\varepsilon\|_2 \to \|a^+\|_2$ and by the parallelogram identity $u_\varepsilon \to a^+$ in $L^2(\mathbb{R}^N)$ since u_ε converges weakly to a^+ in L^2.

As already mentioned in Remark 3, the limit of u_ε when $\varepsilon \to 0$ can be obtained using a similar arguments as soon as a is smooth enough. So

$$\lim_{\varepsilon\to 0} u_\varepsilon = v(x)$$

with v is a non-negative bounded solution to (5.5).

5.3 The case $m = 2$

In this situation, from Theorem 2.6 we have

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) \tag{5.14}$$

$$\lim_{\varepsilon\to +\infty} \lambda_p(\mathcal{M}_{\varepsilon} + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{5.15}$$

As a consequence for large ε we have $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \le -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$ and by Theorem (1.1) there exists a solution to (P_ε) for large ε. Whereas the existence of a positive solution for ε small is conditioned to the sign of

$\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x))$. When $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)) > 0$ then for ε small there exists no positive solution to (P_ε). The limit of u_ε when $\varepsilon \to +\infty$ can be obtain as in the case $2 > m > 0$ so we focus only on the limit when $\varepsilon \to 0$.

Assume for the moment that $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)) < 0$, we will show that $u_\varepsilon \to v$ where v is the positive solution to

$$\frac{K_{2,N}D_2(J)}{2}\Delta v + v(a(x) - v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive number converging to 0 and let u_n denote u_{ε_n}. By Lemma 5.1, $\|u_n\|_2$ is bounded uniformly and after simple algebraic computation

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho_n(z)\frac{(u_n(x+z) - u_n(x))^2}{|z|^2}\,dxdz < C,$$

with C independent of ε. Therefore for any $R > 0$, we have

$$\iint_{B_R\times B_R} \rho_\varepsilon(z)\frac{(u_n(x+z) - u_n(x))^2}{|z|^2}\,dxdz < C.$$

For $R > 0$ fixed, since $\|u_n\|_2$ is uniformly there exists a subsequence $u_n \rightharpoonup v$ in $L^2(B_R)$ and from the characterisation of Sobolev Space [50, 49], we have $u_n \to v$ in $L^2(B_R)$.

By a standard diagonal extraction argument, from the sequence $(u_n)_{n\in\mathbb{N}}$ we can then extract a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ which converges to some v in $L^2_{loc}(\mathbb{R}^N)$. Moreover by Lemma 5.1 u_n is uniformly bounded and there exists $\delta(\lambda_1) > 0$ independent of ε so that $\max_{supp(a^+)}(u_n) > \delta$.

Multiplying (P_ε) by $\varphi \in C_c^\infty(\mathbb{R}^N)$ and integrating we get

$$\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z)]\,dxdz + \int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x) - u_n(x))\,dx = 0,$$

where we use (5.1) to compute $\int_{\mathbb{R}^N}\mathcal{M}_{\varepsilon,2}[u_n](x)\varphi(x)\,dx$. Thus we get

$$\begin{aligned}\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)^t z\nabla^2\varphi(x)z\,dxdz + \int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x) - u_n(x))\,dx\\ = -\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z) - {}^t z\nabla^2\varphi(x)z]\,dxdz,\end{aligned}$$

where $\nabla^2\varphi(x) := (\partial_{ij}\varphi(x))_{i,j}$. Since $\rho_n(z)$ is radial, we can see that

$$\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)^t z\nabla^2\varphi(x)z\,dxdz = \frac{D_2(J)K_{2,N}}{2}\int_{\mathbb{R}^N}u_n(x)\Delta\varphi(x)\,dx$$

and we get

$$\begin{aligned}\frac{D_2(J)K_{2,N}}{2}\int_{\mathbb{R}^N}u_n(x)\Delta\varphi(x)\,dx + \int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x) - u_n(x))\,dx\\ = -\frac{D_2(J)}{2}\iint_{\mathbb{R}^N} \frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z) - {}^t z\nabla^2\varphi(x)z]\,dxdz, \quad (5.16)\end{aligned}$$

Note that since u_n converges to v in $L^2_{loc}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x) - u_n(x))\,dx \to \int_{\mathbb{R}^N}\varphi(x)v(x)(a(x) - v(x))\,dx \quad (5.17)$$

Recall now that $\varphi \in C_c^\infty(\mathbb{R}^N)$, so there exists $C(\varphi)$ and $R(\varphi)$ so that

$$|\varphi(x+z) - 2\varphi(x) + \varphi(x-z) - {}^t z \nabla^2\varphi(x) z| < C(\varphi)|z|^3 \chi_{B_{R(\varphi)}}(x).$$

Therefore since u_n is bounded uniformly,

$$\frac{D_2(J)}{2} \iint_{\mathbb{R}^N} \frac{\rho_\varepsilon(z)}{|z|^2} u_n(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z) - {}^t z\nabla^2\varphi(x)z]\,dxdz \le CC(\varphi)\int_{\mathbb{R}^N} \rho_n(z)|z| \to 0. \tag{5.18}$$

Passing to the limit $\varepsilon \to 0$ in (5.16), thanks to (5.17) and (5.18) we get

$$\frac{D_2(J)K_{2,N}}{2} \int_{\mathbb{R}^N} v(x)\Delta\varphi(x)\,dx + \int_{\mathbb{R}^N} \varphi(x)v(x)(a(x) - v(x))\,dx = 0. \tag{5.19}$$

(5.19) being true for any $\varphi \in C_c^\infty$ this implies that v satisfies

$$\frac{K_{2,N}D_2(J)}{2}\Delta v + v(a(x) - v) = 0 \quad \text{a.e. in} \quad \mathbb{R}^N.$$

Since v is bounded, by elliptic regularity v is smooth. To conclude we need to prove that v is non trivial. To do so we claim that

Claim 5.5. There exists R_0, τ and ε_0 positive constants so that for all $\varepsilon \le \varepsilon_0$ we have $u_\varepsilon \ge \tau$ almost everywhere in $B_{R_0}(0)$.

From the above claim, we deduce that $v \ge \tau > 0$ *a.e.*
and therefore $v \equiv u$, the unique smooth non-trivial solution of

$$\frac{K_{2,N}D_2(J)}{2}\Delta u + u(a(x) - u) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

The sequence $(\varepsilon_n)_n$ being arbitrary, it follows that $u_\varepsilon \to u$ in $L^2_{loc}(\mathbb{R}^N)$.

Similarly, if we assume now that $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)) = 0$ and there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}, \varepsilon_n \to 0$ of non trivial solution of (P_ε). The above argumentation then holds true and we get $u_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$ with v a smooth solution to

$$\frac{K_{2,N}D_2(J)}{2}\Delta v + v(a(x) - v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Since $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)) = 0$, $v \equiv 0$ is the only solution and we get $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$.

Let us complete our proof and establish the claim.

Proof. Let us denote $\mathcal{L}_{R,\varepsilon,2}$ the operator

$$\mathcal{L}_{R,\varepsilon,2}[\varphi] := \frac{1}{\varepsilon^2}\left[\int_{B_R(0)} J_\varepsilon(x-y)\varphi(y)\,dy - \varphi(x)\right].$$

Since $sup_{\mathbb{R}^N} a(x)$ is achieve in $\mathbb{R}^N$ we regularise a by a_σ independently of ε, so that for all ε and $R \ge R_1$ the principal eigenvalue $\lambda_p(\mathcal{L}_{R,\varepsilon,2} + a_\sigma(x))$ is associated to a continuous principal eigenfunction $\varphi_{p,\varepsilon}$ and

$$|\lambda_p(\mathcal{L}_{R,\varepsilon,2} + a_\sigma(x)) - \lambda_p(\mathcal{L}_{R,\varepsilon,2} + a(x))| \le \|a_\sigma(x) - a(x)\|_\infty \le \kappa\sigma,$$

with κ the Lipschitz constant of a.

By the Lipschitz continuity of $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right)$ with respect to a, we can choose σ small enough so that

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x)\right) \le \frac{1}{2}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) < 0.$$

Recall that

$$\lim_{R\to\infty} \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x), B_R\right) = \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x)\right),$$

So we can choose R_0 large so that

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x), B_{R_0}\right) \leq \frac{1}{4}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right).$$

Thanks to Theorem 2.7, we have $\lim_{\varepsilon\to 0}\lambda_p(\mathcal{L}_{R,\varepsilon,2} + a_\sigma(x)) = \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x), B_R\right)$ so for ε small,say $\varepsilon \leq \varepsilon_0$ by choosing σ smaller if necessary, we achieve

$$\lambda_p(\mathcal{L}_{R,\varepsilon,2} + a_\sigma(x)) \leq \frac{1}{8}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) \quad \text{for all} \quad \varepsilon \leq \varepsilon_0.$$

Let $\varphi_{p,\varepsilon}$ be the principal eigenfunction associated with $\mathcal{L}_{R_0,\varepsilon,2} + a_\sigma(x)$, then we have

$$\mathcal{L}_{R_0,\varepsilon,2}[\varphi_{p,\varepsilon}](x) + a(x)\varphi_{p,\varepsilon}(x) \geq \left[-\frac{1}{8}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) - \kappa\sigma\right]\varphi_{p,\varepsilon}(x) \quad \text{for all} \quad \varepsilon \leq \varepsilon_0.$$

By choosing σ smaller if necessary,

$$\left[-\frac{1}{8}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) - \kappa\sigma\right] \geq -\frac{1}{16}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right)$$

and we achieve

$$\mathcal{L}_{R_0,\varepsilon,2}[\varphi_{p,\varepsilon}](x) + a(x)\varphi_{p,\varepsilon}(x) \geq -\frac{1}{16}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right)\varphi_{p,\varepsilon}(x) \quad \text{for all} \quad \varepsilon \leq \varepsilon_0. \tag{5.20}$$

To conclude our proof, it is then enough to show that for some well chosen normalisation of $\varphi_{p,\varepsilon}$ we have

$$\varphi_{p,\varepsilon}(x) \to \varphi_1(x), \quad \text{a.e. in} \quad B_{R_0} \tag{5.21}$$

φ_1 is a positive principal eigenfunction associated with $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x), B_{R_0}\right)$. Indeed, assume for the moment that (5.21) holds true. Then there exists $\alpha > 0$ so that

$$\alpha\varphi_{p,\varepsilon}(x) \to \alpha\varphi_1(x) < \frac{1}{2} \quad \text{a.e. in} \quad B_{R_0}.$$

Now thanks to (5.20), we can now adapt the proof the proof of (iii) of Lemma 5.1 to get for ε small, says $\varepsilon \leq \varepsilon_1$,

$$u_\varepsilon(x) \geq -\frac{\alpha}{32}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right)\varphi_{p,\varepsilon}(x) \quad \text{a.e. in} \quad B_{R_0}, \tag{5.22}$$

which combined with (5.21) enforces

$$u_\varepsilon(x) \geq \gamma\varphi_1(x) \quad \text{a.e. in} \quad B_{R_0}, \quad \text{for all} \quad \varepsilon \leq \varepsilon_2,$$

for some $\gamma, \varepsilon_2 > 0$.

Since $\varphi_1 > 0$ in B_{R_0}, the claim holds true in any smaller ball B_R.

To prove (5.21), let us normalise $\varphi_{p,\varepsilon}$ by $\|\varphi_{p,\varepsilon}\|_{L^2(B_{R_0})} = 1$. Let k_ε be the function

$$k_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus B_{R_0}} J_\varepsilon(x-y)\,dy,$$

then by multiplying by $\varphi_{p,\varepsilon}$ the equation satisfied by $\varphi_{p,\varepsilon}$ and integrating it over B_{R_0}, we get

$$\frac{D_2(J)}{2}\iint_{B_{R_0}\times B_{R_0}}\rho_\varepsilon(x-y)\frac{|\varphi_{p,\varepsilon}(y)-\varphi_{p,\varepsilon}(x)|^2}{|x-y|^2}\,dxdy = \int_{B_{R_0}}(a_\sigma(x)+\lambda_{p,\varepsilon})\varphi_{p,\varepsilon}^2(x)\,dx - \int_{B_{R_0}}k_\varepsilon(x)\varphi_{p,\varepsilon}^2(x)\,dx, \\ \leq C.$$

Therefore by the characterisation of Sobolev space [50, 49], along a sequence we have $\varphi_{p,\varepsilon}\to\psi$ in $L^2(B_{R_0})$ with $\|\psi\|_{L^2(B_{R_0})}=1$. Moreover by extending $\varphi_{p,\varepsilon}$ and φ by 0 outside B_{R_0} and by arguing as above for any $\varphi\in C_c^2(B_{R_0})$ we have

$$\frac{D_2(J)}{2}\iint_{B_{R_0}\times\mathbb{R}^N}\frac{\rho_\varepsilon(z)}{|z|^2}\varphi_{p,\varepsilon}(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)]\,dxdz = -\int_{B_{R_0}}\varphi(x)\varphi_{p,\varepsilon}(a(x)+\lambda_{p,\varepsilon})\,dx \\ +\int_{B_{R_0}}k_\varepsilon(x)\varphi_{p,\varepsilon}(x)\varphi(x)\,dx$$

Since $\varphi\in C_c^2(B_{R_0})$ we get for ε small enough $supp(k_\varepsilon)\cap supp(\varphi)=\emptyset$. Thus passing to the limit along a sequence in the above equation yields

$$\frac{D_2(J)K_{2,N}}{2}\int_{B_{R_0}}\psi(x)\Delta\varphi(x)\,dx+\int_{B_{R_0}}\varphi(x)\psi(x)(a(x)+\lambda_1)\,dx=0. \tag{5.23}$$

(5.23) being true for any φ, it follows that ψ is the smooth positive eigenfunction associated to λ_1 normalised by $\|\psi\|_{L^2(B_{R_0})}=1$. ψ being uniquely defined, we get $\varphi_{p,\varepsilon}\to\psi$ in $L^2(B_{R_0})$ when $\varepsilon\to 0$. Thus along any sequence $\varphi_{p,\varepsilon}(x)\to\varphi_1(x)$ almost everywhere in B_{R_0}.

□

6 Extension to non-compactly supported kernels

In this section, we discuss the extension of our survival criteria to more general dispersal kernel J and prove Theorem 1.5. Observe that the construction of positive solution only required that $\lambda_p(\mathcal{L}_R+\beta(x))<0$ for some R, no matter the dispersal kernel J is. Therefore as soon as $\lim_{R\to\infty}\lambda_p(\mathcal{L}_R+\beta(x))<0$ there exists a positive solution to (1.2) with no restriction on the decay of the kernel. Similarly, when $\lambda_p(\mathcal{M}+\beta(x))>0$ the proof of the non-existence of positive bounded solution essentially relies on the inequality between $\lambda_p(\mathcal{M}+\beta(x))$ and $\lambda_p'(\mathcal{M}+\beta(x))$ which holds for quite general kernels including those satisfying the assumption $H5$ as proved in [2].Concerning the proof of the uniqueness of the positive solution, it relies on the construction of a integrable uniform super-solution of (1.2) which guarantes the existence of a positive L^1 solution to (1.2). Such super-solution still exists for kernels J that satisfies the decay assumption $H5$. Indeed, we can show

Lemma 6.1. *Assume that J satisfies $H5$ and there exists a periodic function $\mu(x):\mathbb{R}^N\to\mathbb{R}$ such that*

$$\limsup_{|x|\to\infty}(\beta(x)-\mu(x))\leq 0 \qquad and \qquad \lambda_p(\mathcal{M}+\mu(x))>0.$$

Then there exists $\bar{u}\in C_0(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, $\bar{u}>0$ so that $\bar{u}$ is a super-solution to (1.2).

Observe that the construction of the super-solution covers a larger class of nonlinearity $f(x,u)$ than those that satisfy $H4$. As a immediate consequence, the survival criteria obtained in Theorem 1.1 is still true for nonlinearity that satisfies:

(H7) There exists $\mu(x)\in C_p(\mathbb{R}^N)$ so that : $\begin{cases}\lambda_p(\mathcal{M}+\mu(x))>0,\\ \limsup_{|x|\to\infty}\left(\frac{f(x,s)}{s}-\mu(x)\right)\leq 0 \quad \text{uniformly in } s.\end{cases}$

From the ecological point of view, such nonlinearity allows to consider a more complex niche structure for the species. In particular, we can consider ecological niches that are the superposition of a compact niche structure with a periodic structure. The condition being that on the periodic structure alone, the species could not survive. The perspective offers by this approach are quite promising and we believe that it may be applied to investigate a climate change version of (1.2).

Proof. The construction of the super-solution in this situation follows the same scheme as for a compactly supported kernel. By assumption since $\limsup_{|x|\to\infty}(\beta(x)-\mu(x))\le 0$, for any $\delta>0$ there exists $R_\delta>1$ such that

$$\beta(x)\le\mu(x)+\delta \qquad |x|\ge R_\delta.$$

Fix $\delta<\lambda_p(\mathcal{M}+\mu(x))$ and observe that by definition of $\lambda_p(\mathcal{M}+\mu(x))$ there exists a constant $\delta<\lambda<\lambda_p(\mathcal{M}+\mu(x))$ and a positive periodic function φ so that

$$\mathcal{M}[\varphi](x)+(\mu(x)+\lambda)\varphi(x)\le 0 \quad \text{ for all } \quad x\in\mathbb{R}^N. \tag{6.1}$$

Let $w=C\frac{\varphi(x)}{1+\tau|x|^{N+1}}$ with C,τ to be chosen.

$$\begin{aligned}
\mathcal{M}[w]+(\mu(x)+\delta)w(x) &= C(1+\tau|x|^{N+1})^{-1}\left(\int_{\mathbb{R}^N}J(x-y)\frac{(1+\tau|x|^{N+1})}{(1+\tau|y|^{N+1})}\varphi(y)\,dy-\varphi(x)+(\mu(x)+\delta)\varphi(x)\right),\\
&\le C(1+\tau|x|^{N+1})^{-1}\left(\int_{\mathbb{R}^N}J(z)\left[\frac{(1+\tau|x|^{N+1})}{(1+\tau|x+z|^{N+1})}-1\right]\varphi(x+z)\,dz+(\delta-\lambda)\varphi(x)\right),\\
&\le C(1+\tau|x|^{N+1})^{-1}\left(\tau\int_{\mathbb{R}^N}J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]\varphi(x+z)\,dz+(\delta-\lambda)\varphi(x)\right),\\
&\le w(x)\left(\tau\int_{\mathbb{R}^N}J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]\frac{\varphi(x+z)}{\varphi(x)}\,dz+\delta-\lambda\right),
\end{aligned}$$

where we use (6.1) and $inf_{\mathbb{R}^N}\varphi>0$.

Set

$$h(\tau,x)=\tau\int_{\mathbb{R}^N}J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]\frac{\varphi(x+z)}{\varphi(x)}\,dz+\delta-\lambda.$$

Thanks to $\varphi\in L^\infty(\mathbb{R}^N),\inf_{\mathbb{R}^N}\varphi>0$ there exists a positive constant C_0 so that

$$\frac{\varphi(x+z)}{\varphi(x)}\le C_0 \quad \text{ for all } \quad x,z\in\mathbb{R}^N.$$

Thus for all $x\in\mathbb{R}^N$, we have

$$h(\tau,x)\le C_0\tau\int_{\mathbb{R}^N}J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]dz+\delta-\lambda. \tag{6.2}$$

Let

$$I:=C_0\tau\int_{\mathbb{R}^N}J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]dz,$$

then we have

$$I=C_0\tau\int_{\{|x|\le 2|z|\}}J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]dz+C_0\tau\int_{\{|x|>2|z|\}}J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]dz.$$

Let us estimate the first integral. Since $|x|\le 2|z|$ we have

$$C_0\tau\int_{\{|x|\le 2|z|\}}J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]dz\le C_0\tau 2^{N+1}\int_{\mathbb{R}^N}J(z)|z|^{N+1}\,dz. \tag{6.3}$$

Let us now estimate second term. Since $|x+z|^{N+1} \geq (|x|-|z|)^{N+1}$, we have

$$C_0\tau\int_{\{|x|>2|z|\}} J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]dz \leq C_0\sum_{i=1}^{N+1}\binom{N+1}{i}\int_{\{|x|>2|z|\}} J(z)(-1)^{i+1}|z|^i\left[\frac{\tau|x|^{N+1-i}}{(1+\tau|x+z|^{N+1})}\right]dz,$$
$$\leq C_0\sum_{i=1}^{N+1}\binom{N+1}{i}\int_{\{|x|>2|z|\}} J(z)|z|^i\left[\frac{\tau|x|^{N+1-i}}{(1+\tau|x+z|^{N+1})}\right]dz.$$

Since $|x|>2|z|$, we have

$$\frac{1}{1+\tau|x+z|^{N+1}} \leq \frac{2^{N+1}}{2^{N+1}+\tau|x|^{N+1}}$$

and for $|x|\geq R_0>1$

$$C_0\tau\int_{\{|x|>2|z|\}} J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]dz \leq C_0 2^{N+1}\sum_{i=1}^{N+1}\binom{N+1}{i}\int_{\{|x|>2|z|\}} J(z)\frac{|z|^i}{|x|^i}\left[\frac{\tau|x|^{N+1}}{(2^{N+1}+\tau|x|^{N+1})}\right]dz$$
$$\leq \frac{C_0 2^{N+1}}{R_0}\sum_{i=1}^{N+1}\binom{N+1}{i}\int_{\mathbb{R}^N} J(z)|z|^i\left[\frac{\tau|x|^{N+1}}{2^{N+1}+\tau|x|^{N+1}}\right]dz.$$

Since for all $|x|$,

$$\left[\frac{\tau|x|^{N+1}}{2^{N+1}+\tau|x|^{N+1}}\right]<1,$$

we achieve for $|x|\geq R_0$

$$C_0\tau\int_{\{|x|>2|z|\}} J(z)\left[\frac{|x|^{N+1}-|x+z|^{N+1}}{(1+\tau|x+z|^{N+1})}\right]dz \leq \frac{C_0 2^{N+1}}{R_0}\sum_{i=1}^{N+1}\binom{N+1}{i}\int_{\mathbb{R}^N} J(z)|z|^i\,dz. \tag{6.4}$$

Combining (6.3), (6.4) and (6.2), we get for $|x|>R_0$

$$h(x,\tau) \leq \frac{C_0 2^{N+1}}{R_0}\sum_{i=1}^{N+1}\binom{N+1}{i}\int_{\mathbb{R}^N} J(z)|z|^i\,dz + C_0\tau 2^{N+1}\int_{\mathbb{R}^N} J(z)|z|^{N+1}\,dz+\delta-\lambda.$$

Thanks to (H5), for τ small enough, says $\tau\leq\tau_1$ and R_0 large enough we achieve $h(x,\tau)\leq\frac{\delta-\lambda}{2}<0$,
Hence, for all $\tau\leq\tau_1$, we have

$$\mathcal{M}[w]+(\mu(x)+\delta)w(x)\leq w(x)h(x,\tau)\leq w(x)\frac{\delta-\lambda}{2}<0 \quad \text{for all} \quad x\in\mathbb{R}^N\setminus B_{R_0}. \tag{6.5}$$

Fix now $\tau\leq\tau_1$ and fix $R_0>R_\delta$ so that $h(x,\tau)<0$ in $\mathbb{R}^N\setminus B_{R_0}(0)$.

Let $\kappa_0:=\sup_{\mathbb{R}^N\setminus B_{R_0}(0)}\frac{\varphi(x)}{1+\tau|x|^{N+\alpha}}$. Let $0<\kappa<\kappa_0$ and consider the set

$$\Omega_\kappa:=\left\{x\in\mathbb{R}^N\mid\frac{\varphi(x)}{1+\tau|x|^{N+\alpha}}\leq\kappa\right\}.$$

By construction since $\varphi>0$ in $\mathbb{R}^N$ we can choose κ small so that

$$\Omega_\kappa\subset\mathbb{R}^N\setminus B_{R_0}(0).$$

Moreover, $\mathbb{R}^N\setminus\Omega_\kappa$ is a bounded domain and $M:=\sup_{\mathbb{R}^N\setminus\Omega_\kappa}S(x)$ is well defined. Choose now C so that $C=\frac{2M}{\kappa}$ and consider the continuous function

$$\bar{u}(x):=\begin{cases}C\frac{\varphi(x)}{1+\tau|x|^{N+\alpha}} & \text{in} \quad \Omega_\kappa,\\ C\kappa & \text{in} \quad \mathbb{R}^N\setminus\Omega_\kappa.\end{cases}$$

By direct computation we can check that $\bar{u}$ is a super-solution to (1.2). Indeed, for any $x \in \mathbb{R}^N \setminus \Omega_\kappa$, we have $\bar{u} = C\kappa = 2M > \sup_{\mathbb{R}^N \setminus \Omega_\kappa} S(x)$ which implies that $f(x, C\kappa) \leq 0$ and

$$\mathcal{M}[\bar{u}](x) + f(x, \bar{u}(x)) \leq \int_{\mathbb{R}^N} J(x-y)\bar{u}(y)\,dy - C\kappa + f(x, C\kappa) \leq f(x, C\kappa) \leq 0.$$

Whereas, for $x \in \Omega_\kappa$ we have

$$\begin{aligned}\mathcal{M}[\bar{u}](x) + f(x, \bar{u}(x)) \leq \mathcal{M}[\bar{u}](x) + \beta(x)w(x) &\leq \mathcal{M}[w] + (\mu(x) + \delta)w(x), \\ &\leq h(x, \tau)w(x) \leq 0.\end{aligned}$$

□

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