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# APPROXIMATE SUBGROUPS

JEAN-CYRILLE MASSICOT AND FRANK O. WAGNER

ABSTRACT. Given a definably amenable approximate subgroup  $A$  of a (local) group in some first-order structure, there is a type-definable subgroup  $H$  normalised by  $A$  and contained in  $A^4$  such that every definable superset of  $H$  has positive measure.

## INTRODUCTION

Let  $G$  be a group, or more generally a local group, and  $K > 0$  an integer. A symmetric subset  $A \subseteq G$  is a  $K$ -approximate subgroup if there is a finite subset  $E \subseteq G$  with  $|E| \leq K$  such that  $A \cdot A \subseteq E \cdot A$ .

Following work of Hrushovski [6] and many others, Breuillard, Green and Tao [1] have classified finite approximate subgroups of local groups (see [9] for an excellent survey). In particular, they show that there is an approximate subgroup  $A^* \subseteq A^4$  and an actual  $A^*$ -invariant subgroup  $H^* \subseteq A^*$  such that

- finitely many left translates of  $A^*$  cover  $A$ , and
- $\langle A^* \rangle / H^*$  is nilpotent.

The result and its proof are inspired not only by Gleason's and Yamabe's solution of Hilbert's 5th problem [3, 10] and its extension to the local context by Goldbring [4], but also by Gromov's Theorem on groups with polynomial growth, and is indeed a way to generalize this theorem. The three articles [6, 1, 9] provide some applications to geometric group theory.

The proof proceeds by considering a non-principal ultraproduct of a sequence of finite counterexamples  $(A_n : n < \omega)$  with  $|A_n| \rightarrow \infty$ , giving rise to a pseudo-finite counterexample  $A$ . Then an  $A$ -invariant subgroup  $H \subset A^4$  is constructed such that  $\langle A \rangle / H$  is locally compact. From Yamabe's theorem on the approximation of locally compact groups by Lie groups, it follows that there are suitable  $A^*$  and  $H^*$  such that  $\langle A^* \rangle / H^*$  is a real Lie group; using pseudofiniteness, the final result is obtained.

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The construction of the locally compact quotient  $\langle A \rangle / H$  and the Lie model  $\langle A^* \rangle / H^*$  was first shown by Hrushovski [6] by model-theoretic means inspired by and reminiscent of stability theory. Breuillard, Green and Tao use instead a (subsequent) theorem of Sanders [8] from finite combinatorics, constructing successively the traces of the definable supersets of  $H$  on the various finite approximate groups  $A_n$ . Using the ultraproduct construction, the profinite counting measure and the translation-invariant ideal of measure zero sets, Hrushovski's theorem allows to recover Sanders' result at least qualitatively.

The topology of  $\langle A \rangle / H$  constructed analytically in [1] has a natural definition as the *logic topology* induced on a bounded quotient of a local group by a type-definable normal subgroup: A proper subset  $X \subset \langle A \rangle / H$  is closed if its preimage in  $\langle A \rangle$  is type-definable. While the group  $H$  constructed by Hrushovski is naturally type-definable in the structure given, it only becomes so in Sanders' Theorem (either in the ultraproduct or in a suitable version using a bi-invariant measure instead of cardinality) after expanding the language. Hrushovski, on the other hand, assumes the existence of a bi-invariant S1 ideal (which should be thought of as the ideal of sets of measure zero) which in addition is automorphism invariant; in order for an ideal to become automorphism invariant, one would generally also have to expand the language. Such an expansion does not matter much if the structure to start with is arbitrary, but becomes inconvenient if more is assumed at the outset. For example, Eleftheriou and Peterzil [2] construct  $H$  type-definably without expanding the language in the case when  $A$  is definable in an  $\mathcal{o}$ -minimal extension of an ordered group, provided that  $\langle A \rangle$  is abelian. Pillay [7], generalizing an argument in [5], generalises this result if  $A$  is definable in a NIP theory and definably amenable. In fact, in this setting there is a unique minimal choice for  $H$ , namely the type-definable connected component  $\langle A \rangle^{00}$ . Let us also remark that from a model-theoretic point of view, a definable approximate subgroup  $A$  is just a symmetric generic set in the locally definable group  $\langle A \rangle$ , i.e. a definable symmetric subset of  $\langle A \rangle$  such that every definable subset of  $\langle A \rangle$  is covered by finitely many left translates of  $A$ .

In this paper we shall show that in any local group, a definably amenable approximate subgroup  $A$  (see the beginning of the next section for the definitions) gives rise to a type-definable subgroup  $H \subseteq A^4$ , such that finitely many left translates of any definable superset of  $H$  cover  $A$ . Hence such an approximate subgroup  $A$  allows a real Lie model, and definably amenable approximate subgroups are closely related to approximate subgroups of real Lie groups. The classification of approximate subgroups of real Lie groups is still an open problem. Since in a real Lie group any compact neighbourhood of the identity is an approximate subgroup, in particular no nilpotency (or even solubility) result can hold in general. We conjecture that even without the definable amenability assumption a suitable Lie model exists.

**Remark 1.** Definability is with respect to the ambient structure  $M$ ; a subset of  $M^n$  is *definable* if it is the set of realizations of a first-order formula in the language of the structure (with quantifiers!), and *type-definable* if it is the set of realizations of some (usually infinite but fixed) collection of formulæ. Thus expanding the language by new predicates for previously undefinable sets clearly enriches the collection of definable sets, but generally destroys desirable model-theoretic properties the initial structure  $M$  might have had, such as  $\omega$ -minimality or the lack of the independence property (NIP).

We shall assume that all structures under consideration are  $\omega$ -saturated, which means that any countable intersection of definable sets is non-empty as soon as all finite subintersections are. The compactness theorem of model theory implies that we can replace any structure  $M$  by a superstructure  $M^*$  satisfying the same first-order sentences with parameters in  $M$  (an elementary extension) which in addition is  $\omega$ -saturated.

### 1. A TYPE-DEFINABLE VERSION OF SANDERS' THEOREM

**Definition 2.** A subset  $A$  of a (local) group  $G$  is *symmetric* if  $1 \in A$ , and  $a^{-1} \in A$  for all  $a \in A$ .

If  $K < \omega$ , a symmetric subset  $A$  of  $G$  is a  *$K$ -approximate subgroup* if  $A^2 = \{aa' : a, a' \in A\}$  is contained in  $K$  left cosets of  $A$ . An *approximate subgroup* is a symmetric subset which is a  $K$ -approximate subgroup for some  $K < \omega$ .

**Definition 3.** A definable approximate subgroup  $A$  is *definably amenable* if there is a (left) translation-invariant finitely additive measure  $\mu$  on the definable subsets of  $\langle A \rangle$  with  $\mu(A) = 1$ .

Note that by  $\omega$ -saturation, for any definable subset  $X$  of  $\langle A \rangle$  there is  $n < \omega$  with  $X \subseteq A^n$ .

**Remark 4.** If  $\lim_{n \rightarrow \infty} \mu(A^n) < \infty$ , then there is  $n < \omega$  with  $A^n = \langle A \rangle$ .

*Proof:* Suppose not. Then for every  $n < \omega$  there is  $a_n \in A^{n+1} \setminus A^n$ . But then  $(a_{3k}A : k \leq n)$  is a sequence of disjoint left translates of  $A$  inside  $A^{3n+2}$ , whence  $\mu(A^{3n+2}) \geq (n+1)\mu(A) = n+1$ , a contradiction.  $\square$

For the remainder of the paper we fix a  $K$ -approximate subgroup  $A$  of a (local) group  $G$ , and consider the structure whose domain is  $G$ , with a predicate for  $A$ , and with group multiplication (which is a partial map in case  $G$  is only local). We assume that  $A^n \subseteq G$  for all  $n < \omega$  (in fact  $n \leq 100$  would be enough). Definability and type-definability will be with respect to this structure.

We assume that  $A$  is definably amenable with  $\lim_{n \rightarrow \infty} \mu(A^n) = \infty$ . We also fix a set  $E$  of size  $K$  with  $A^2 \subseteq EA$ .

**Fact 5** (Ruzsa's covering lemma). *Let  $X, Y \subset G$  be definable such that  $\mu(XY) \leq K\mu(Y)$ . Then  $X \subset ZYY^{-1}$  for some finite  $Z \subset X$  with  $|Z| \leq K$ .*

*Proof:* If  $X = \emptyset$  there is nothing to show. Otherwise, consider a finite subset  $Z \subseteq X$  such that  $zY \cap z'Y = \emptyset$  for all  $z \neq z'$  in  $Z$ . By left invariance,

$$|Z|\mu(Y) = \mu(ZY) \leq \mu(XY) \leq K\mu(Y),$$

so  $|Z| \leq K$ , and there is a maximal such  $Z$ . But then for any  $x \in X$  there is  $z \in Z$  with  $zY \cap xY \neq \emptyset$  by maximality, whence  $x \in zYY^{-1}$ .  $\square$

**Definition 6.** A definable subset  $B \subset \langle A \rangle$  is *wide in  $A$*  if  $A$  is covered by finitely many translates of  $B$ .

Two approximate subgroups are said to be *equivalent* if each one is contained in finitely many translates of the other.

We will sometimes make explicit the finite constants and say that  $B$  is  $L$ -wide in  $A$ , or that  $A$  and  $A^*$  are  $L$ -equivalent.

**Lemma 7.** *Let  $B \subset \langle A \rangle$  be definable and symmetric.*

- (1) *If  $\mu(B) > 0$ , then  $B^2$  is wide in  $A$ .*
- (2) *If  $B$  is wide in  $A$ , then  $B$  is also an approximate subgroup equivalent to  $A$ .*

*Proof:*

- (1) As  $\mu(B) > 0$  and  $\mu(A^2) \leq K\mu(A) < \infty$ , there are at most  $K\mu(A)/\mu(B)$  disjoint translates  $h_i B$  of  $B$  by elements of  $A$ . Then  $\bigcup_i h_i B^2$  covers  $A$ .
- (2) There is  $n < \omega$  such that  $B^2 \subseteq A^{2n} \subseteq E^{2n-1}A$ . Suppose  $Y$  is finite with  $A \subseteq YB$ . Then

$$B^2 \subseteq E^{2n-1}A \subseteq E^{2n-1}YB.$$

Thus  $B$  is an approximate subgroup; being wide in  $A$ , it must be equivalent to  $A$ .  $\square$

**Definition 8.** A type-definable subgroup  $H$  of a (local) group  $G$  has *bounded index* if there is some cardinal  $\kappa$  such that in any elementary extension the index  $|G : H|$  is bounded by  $\kappa$ .

**Remark 9.** By  $\omega$ -saturation  $H$  has bounded index in  $G$  if and only if for every definable subset  $X$  of  $G$  and every definable superset  $Y$  of  $H$ , finitely many left translates of  $Y$  cover  $X$ .

**Lemma 10.** *If  $A$  and  $A^*$  are equivalent, there exists an approximate subgroup in which both are wide, and another one which is wide in both. In particular,  $\langle A \rangle \cap \langle A^* \rangle$  will have bounded index in both  $\langle A \rangle$  and  $\langle A^* \rangle$ .*

*Proof:* Suppose  $A^{*2} \subseteq E^*A^*$ , and put  $B = AA^*A$ , a symmetric set containing  $A$  and  $A^*$ . If  $A \subseteq XA^*$  and  $A^* \subseteq X^*A$ , then

$$B = AA^*A \subseteq XA^*A^*A \subseteq XE^*A^*A \subseteq XE^*X^*AA \subseteq XE^*X^*EA \subseteq XE^*X^*EXA^*,$$

so  $A$  and  $A^*$  are wide in  $B$ . Moreover

$$B^2 \subseteq XE^*X^*EAB = XE^*X^*EAAA^*A \subseteq XE^*X^*E^2AA^*A = XE^*X^*E^2B,$$

so  $B$  is also an approximate subgroup. As  $\langle A \rangle$  and  $\langle A^* \rangle$  have bounded index in  $\langle B \rangle$ , the intersection  $\langle A \rangle \cap \langle A^* \rangle$  has bounded index in  $\langle B \rangle$ , and thus in  $\langle A \rangle$  and in  $\langle A^* \rangle$ .

Now note that  $\langle A \rangle \cap \langle A^* \rangle = \bigcup_{n < \omega} (A^n \cap A^{*n})$ . By  $\omega$ -saturation there is  $n < \omega$  such that finitely many translates of  $A^n \cap A^{*n}$  cover  $B$ . So  $A^n \cap A^{*n}$  is wide in  $B$ , whence in  $A$  and in  $A^*$ .  $\square$

We now turn to the main result. We shall need the following Lemma due to Sanders.

**Lemma 11.** *Let  $f : ]0, 1] \rightarrow [1, K]$  and  $\epsilon > 0$ . Then there exists  $n < \omega$  depending only on  $K, \epsilon$  and  $t > 1/(2K)^{2^n - 1}$  such that*

$$f\left(\frac{t^2}{2K}\right) \geq (1 - \epsilon)f(t).$$

*Proof:* Define a sequence  $(t_n)$  by  $t_0 = 1$  and  $t_{k+1} = \frac{t_k^2}{2K}$ , so  $t_n = 1/(2K)^{2^n - 1}$ .

For  $n < \omega$  suppose that for all  $i < n$  we have  $f(t_{i+1}) < (1 - \epsilon)f(t_i)$ . Then

$$f(t_n) < (1 - \epsilon)^n f(t_0) \leq (1 - \epsilon)^n K.$$

But  $f(t_n) \geq 1$ , so if  $n < \omega$  is such that  $(1 - \epsilon)^n K < 1$ , there must be some  $i < n$  with  $f(t_{i+1}) \geq (1 - \epsilon)f(t_i)$ .  $\square$

**Theorem 12.** *Let  $A$  be a  $K$ -approximate subgroup. For any  $m < \omega$  there is a definable  $L$ -wide approximate subgroup  $S$  with  $S^m \subseteq A^4$ , where  $L$  depends only on  $K$  and  $m$ .*

*Proof:* Let us show first that if  $B \subseteq A$  is definable with  $\mu(B) \geq t\mu(A)$  for some  $0 < t \leq 1$  and  $s = \frac{t}{2K}$ , then  $A$  is covered by  $N = \lfloor \frac{1}{s} \rfloor$  translates of

$$X = \{g \in A^2 : \mu(gB \cap B) \geq st\mu(A)\}$$

by elements of  $A$ . So suppose not. Then inductively we find a sequence  $(g_i : i \leq N)$  of elements of  $A$  such that  $\mu(g_i B \cap g_j B) < st\mu(A)$  for all  $i < j \leq N$ , since for  $j \leq N$

the set  $\bigcup_{i < j} g_i X$  cannot cover  $A$ . But then

$$\begin{aligned}
K\mu(A) &\geq \mu(A^2) \geq \mu\left(\bigcup_{i \leq N} g_i B\right) \\
&\geq (N+1)\mu(B) - \sum_{i < j \leq N} \mu(g_i B \cap g_j B) \\
&> (N+1)t\mu(A) - \frac{N(N+1)}{2}st\mu(A) = (1 - N\frac{s}{2})(N+1)t\mu(A) \\
&\geq (1 - \frac{1}{s}\frac{s}{2})\frac{1}{s}t\mu(A) = \frac{1}{2}\frac{2K}{t}t\mu(A) = K\mu(A),
\end{aligned}$$

a contradiction.

However, as  $\mu$  is not supposed to be definable,  $X$  need not be definable either. We shall hence look for definable sets with similar properties. To this end, consider the following conditions  $P_n^t(X)$  on definable subsets of  $A$ , for  $n < \omega$  and  $0 < t \leq 1$ :

- $P_0^t(B)$  if  $B \neq \emptyset$ .
- $P_{n+1}^t(B)$  if  $P_n^t(B)$ , and  $A$  is covered by  $\lfloor \frac{2K}{t} \rfloor$  translates of

$$X_{n+1}^t(B) = \{g \in A^2 : P_n^{t^2/2K}(gB \cap B) \text{ and } P_n^{t^2/2K}(g^{-1}B \cap B)\}.$$

Clearly, if  $(B_x)_x$  is a family of uniformly definable subsets of  $A$ , then  $P_n^t(B_x)$  is definable by a formula  $\theta_n^t(x)$  for all  $n < \omega$  and  $0 < t \leq 1$ . As  $X_{n+1}^t(B) \subseteq A^2$ , the translating elements for the covering of  $A$  must come from  $A^3$ , so the  $P_n^t$  are definable even in a local group where we can only quantify over finite powers of  $A$ .

For  $0 < t \leq 1$  we consider the family  $\mathfrak{B}_t$  of definable subsets  $B$  of  $A$  with  $P_n^t(B)$  for all  $n < \omega$ . The first paragraph implies inductively that for definable  $B \subseteq A$ , if  $\mu(B) \geq t\mu(A)$  then  $P_n^t(B)$  holds, whence  $B \in \mathfrak{B}_t$ . In particular,  $A \in \mathfrak{B}_t$  so  $\mathfrak{B}_t$  is non-empty. Note that  $P_n^t$  implies  $P_n^{t'}$  for  $t \geq t'$ , so  $\mathfrak{B}_t \subseteq \mathfrak{B}_{t'}$ .

Define a function  $f : ]0, 1] \rightarrow \mathbb{R}$  by

$$f(t) = \inf\left\{\frac{\mu(BA)}{\mu(A)} : B \in \mathfrak{B}_t\right\}.$$

Fix  $\epsilon > 0$ . Since  $1 \leq f(t) \leq K$  for all  $0 < t \leq 1$ , by Lemma 11 there is  $t > 0$  depending only on  $K$  and  $\epsilon$  such that

$$f\left(\frac{t^2}{2K}\right) \geq (1 - \epsilon)f(t).$$

Choose  $B \in \mathfrak{B}_t$  with  $\frac{\mu(BA)}{\mu(A)} \leq (1 + \epsilon)f(t)$ . Put

$$X_n = X_n^t(B) = \{g \in A^2 : P_n^{t^2/2K}(gB \cap B) \text{ and } P_n^{t^2/2K}(g^{-1}B \cap B)\}$$

and  $X = \bigcap_{n < \omega} X_n$ . Then  $X_n$  is symmetric,  $X_{n+1} \subseteq X_n$  and  $\lfloor \frac{2K}{t} \rfloor$  translates of  $X_n$  cover  $A$ , for all  $n < \omega$ . By  $\omega$ -saturation,  $\lfloor \frac{2K}{t} \rfloor$  translates of  $X$  cover  $A$ , so  $X$  is nonempty. Moreover, for  $g \in X$  we have  $gB \cap B \in \mathfrak{B}_{t^2/2K}$ , whence

$$\begin{aligned} \mu(gBA \cap BA) &\geq \mu((gB \cap B)A) \geq f\left(\frac{t^2}{2K}\right)\mu(A) \\ &\geq (1 - \epsilon)f(t)\mu(A) \geq \frac{1 - \epsilon}{1 + \epsilon}\mu(BA). \end{aligned}$$

Hence for  $g \in X$ ,

$$\mu(gBA \triangle BA) \leq \frac{4\epsilon}{1 + \epsilon}\mu(BA) < 4\epsilon\mu(BA).$$

It follows that for  $g_1, \dots, g_m \in X$ ,

$$\begin{aligned} \mu(g_1 \cdots g_m BA \triangle BA) &\leq \mu((BA \triangle g_1 BA) \cup g_1(BA \triangle g_2 BA) \cup \cdots \cup g_1 \cdots g_{m-1}(BA \triangle g_m BA)) \\ &\leq \mu(BA \triangle g_1 BA) + \mu(BA \triangle g_2 BA) + \cdots + \mu(BA \triangle g_m BA) \\ &< 4m\epsilon\mu(BA). \end{aligned}$$

In particular, if  $\epsilon \leq \frac{1}{4m}$ , then  $g_1 \cdots g_m BA \cap BA \neq \emptyset$ , whence  $X^m \subseteq A^4$ . By  $\omega$ -saturation there is  $n < \omega$  such that  $X_n^m \subseteq A^4$ . Note that  $S := X_n$  is  $\lfloor \frac{2K}{t} \rfloor$ -wide in  $A$ , and thus an approximate subgroup equivalent to  $A$  by Lemma 7.  $\square$

**Corollary 13.** *There is a type-definable subgroup  $H \subseteq A^4$  such that every definable superset of  $H$  contained in  $\langle A \rangle$  is wide in  $A$ .*

*Proof:* Put  $S_0 = A$  and apply inductively Theorem 12 for  $m = 8$  with  $S_i$  instead of  $A$ , in order to obtain a sequence of approximate subgroups  $(S_i : i < \omega)$  with  $S_{i+1}$  wide in  $S_i$  (whence in  $A$ ) and  $S_{i+1}^8 \subseteq S_i^4$ . Then  $H = \bigcap_{i < \omega} S_i^4$  is a type-definable subgroup of  $A^4$ . Any definable superset of  $H$  must contain some  $S_i^4$  by  $\omega$ -saturation, and hence be wide in  $A$ .  $\square$

## 2. NORMALITY

Since we want to consider the quotient  $\langle A \rangle / H$ , we shall look for a stronger version of Theorem 12 where  $H$  will be normal.

**Lemma 14.** *Let  $X_1, \dots, X_n$  be definable subsets of  $A$  with  $N_i \mu(X_i) \geq \mu(A)$  for some  $N_i < \omega$ . Then there is a definable  $D \subseteq A$  such that*

$$D^{-1}D \subset (X_1 X_1^{-1})^2 \cap \dots \cap (X_n X_n^{-1})^2 \quad \text{and} \quad K^{n-1} N_1 \dots N_n \mu(D) \geq \mu(A).$$



*Proof:* Let us find  $D_0$  with

$$KN_1N_2\mu(D_0) \geq \mu(A) \quad \text{and} \quad D_0^{-1}D_0 \subseteq X_1^{-1}X_1 \cap (X_2X_2^{-1})^2.$$

Then we can replace  $X_1$  by  $D_0$  and  $X_2$  by  $X_3$  and iterate the construction. Inductively we obtain  $D$  with  $K^{n-1}N_1\dots N_n\mu(D) \geq \mu(A)$  such that

$$D^{-1}D \subseteq X_1^{-1}X_1 \cap (X_2X_2^{-1})^2 \cap \dots \cap (X_nX_n^{-1})^2.$$

Notice that  $D^{-1}D$  is  $K^{n-1}N_1\dots N_n$ -wide in  $A$  by Fact 5.

Since  $\mu(AX_2) \leq K\mu(A) \leq KN_2\mu(X_2)$ , by Fact 5 there are  $g_1, \dots, g_{KN_2}$  such that

$$A \subseteq \bigcup_{i=1}^{KN_2} g_i X_2 X_2^{-1}.$$

Then there is an  $i$  such that

$$KN_1N_2\mu(X_1 \cap g_i X_2 X_2^{-1}) \geq \mu(A).$$

We set  $D_0 = X_1 \cap g_i X_2 X_2^{-1}$  and note that  $D_0^{-1}D_0 \subseteq X_1^{-1}X_1 \cap (X_2X_2^{-1})^2$ .  $\square$

**Theorem 15.** *Let  $A$  be a  $K$ -approximative subgroup, and  $R$  an  $N$ -wide symmetric subset with  $R^4 \subseteq A^4$ . Then there exists an  $L$ -wide symmetric subset  $S$  with  $(S^8)^A \subseteq R^4$ , where  $L$  depends only on  $K$  and  $N$ .*

*Proof:* If  $A \subseteq XR$ , then

$$R^2 \subseteq A^8 \subseteq E^7 A \subseteq E^7 XR,$$

so  $R$  is a  $K^7N$ -approximative subgroup. Theorem 12 yields the existence of some  $T \subseteq R^4$  equivalent to  $R$  with  $T^{48} \subset R^4$ . Then  $T$  is wide in  $A$  and there exists  $n < \omega$  depending only on  $K$  and  $N$  and some elements  $a_i$  of  $A$  such that

$$A \subset \bigcup_{i=1}^n a_i T.$$

Consider the measure  $\bar{\mu}$  on definable subsets of  $\langle A \rangle$  defined by

$$\bar{\mu}(X) := \frac{1}{n} \sum_{i=1}^n \mu(Xa_i).$$

Clearly  $\bar{\mu}$  is left translation invariant,

$$\bar{\mu}(A) = \frac{1}{n} \sum_{i=1}^n \mu(Aa_i) \leq \frac{n}{n} \mu(A^2) \leq K\mu(A),$$

and

$$\bar{\mu}(a_i T a_i^{-1}) \geq \frac{1}{n} \mu(T) \geq \frac{1}{n^2} \mu(A) \geq \frac{1}{Kn^2} \bar{\mu}(A).$$

Since all the  $a_i T a_i^{-1}$  are subsets of  $A^6$  and  $K^6 n^2 \cdot \bar{\mu}(a_i T a_i^{-1}) \geq \bar{\mu}(A^6)$ , Lemma 14 applied to the  $K^6$ -approximate subgroup  $A^6$  yields a subset  $D \subseteq A^6$  with

$$(K^6)^{n-1} (K^6 n^2)^n \bar{\mu}(D) \geq \bar{\mu}(A^6)$$

such that for  $i = 1, 2, \dots, n$  we have

$$S := D^{-1} D \subseteq a_i T^4 a_i^{-1}.$$

Then  $S$  is symmetric, wide in  $A$  and  $S^{a_i} \subset T^4$  for  $i = 1, \dots, n$ . Since  $A \subset \bigcup a_i T$ , this means that  $S^A \subseteq T^6$ , so  $(S^8)^A \subseteq T^{48} \subseteq R^4$ .  $\square$

**Corollary 16.** *There is a type-definable normal subgroup  $H$  of  $\langle A \rangle$  contained in  $A^4$  such that every definable superset of  $H$  contained in  $\langle A \rangle$  is wide in  $A$ .*

*Proof:* As Corollary 13, using Theorem 15 instead of Theorem 12.  $\square$

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