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# A Minimal Model Program for $\mathbb{Q}$ -Gorenstein varieties

Boris Pasquier\*

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## Abstract

The Minimal Model Program is constructed for projective varieties with at most  $\mathbb{Q}$ -factorial terminal singularities. Here, we adapt the definitions of divisorial contractions and flips to construct a Minimal Model Program for projective varieties with at most  $\mathbb{Q}$ -Gorenstein terminal singularities. This new construction can be naturally extended to klt pairs. In the family of  $\mathbb{Q}$ -Gorenstein spherical varieties, we answer positively to the questions of existence of flips and of finiteness of sequences of flips.

**Mathematics Subject Classification.** 14E30 14M25 14M17

**Keywords.** Minimal Model Program.

## 1 Introduction

We only consider normal algebraic varieties over  $\mathbb{C}$ .

The Minimal Model Program (MMP) is now well-known for projective varieties with at most  $\mathbb{Q}$ -factorial terminal singularities. Moreover, the first two main results of the theory, the Contraction Theorem and the Cone Theorem, are given for projective varieties with at most  $\mathbb{Q}$ -Gorenstein terminal singularities [KMM87]. It is then natural to ask if the MMP also works for  $\mathbb{Q}$ -Gorenstein projective varieties. With the study of toric varieties (for example), we can answer negatively to this question: there exist divisorial contractions of extremal rays from a  $\mathbb{Q}$ -Gorenstein variety to a non- $\mathbb{Q}$ -Gorenstein variety and flips do not always exist. Nevertheless, still for toric varieties (and more generally for horospherical varieties), we observed in [Pas13] that a program very similar to the MMP works for  $\mathbb{Q}$ -Gorenstein varieties. In this latter paper, where the MMP is reduced to the study of a family of moment polytopes, the theory for  $\mathbb{Q}$ -Gorenstein projective varieties even seems to be more natural than the classical theory for  $\mathbb{Q}$ -factorial projective varieties.

In this paper, we define a MMP for  $\mathbb{Q}$ -Gorenstein projective varieties with terminal singularities, and we give the first results of the theory.

Before to describe explicitly the results, we recall some basic definitions and notations.

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\*Boris PASQUIER, E-mail: boris.pasquier@univ-montp2.fr

**Definition 1.** • A normal variety  $X$  is  $\mathbb{Q}$ -Gorenstein if its canonical divisor  $K_X$  is  $\mathbb{Q}$ -Cartier (ie if there exists a multiple of  $K_X$  that is Cartier).

- A normal  $\mathbb{Q}$ -Gorenstein variety  $X$  has terminal singularities if there exists a desingularization  $\sigma : V \rightarrow X$  of  $X$  such that

$$K_V = \sigma^* K_X + \sum_{E \text{ prime divisor of } V} a_E E,$$

with  $a_E > 0$  for any exceptional prime divisor  $E$  of  $\sigma$  (and  $a_E = 0$  otherwise).

- A contraction (ie a projective morphism  $\phi : X \rightarrow Y$  such that  $\phi_* \mathcal{O}_X = \mathcal{O}_Y$ ) is a small contraction if its exceptional locus is at least of codimension 2.

Now, fix a normal  $\mathbb{Q}$ -Gorenstein projective variety  $X$  with terminal singularities. We denote by  $NE(X)$  the nef cone of curves of  $X$  and by  $NE(X)_{K_X < 0}$  (resp.  $NE(X)_{K_X > 0}$ ) the intersection of the nef cone with the open half-space of curves negative (resp. positive) along the divisor  $K_X$ . Then, by the Cone Theorem,  $NE(X) = NE(X)_{K_X > 0} + \sum_{i \in I} R_i$ , where the set  $\{R_i \mid i \in I\}$  is a discrete set of (extremal) rays of  $NE(X)_{K_X < 0}$ . Fix a ray  $R$  of  $NE(X)_{K_X < 0}$ . Then, by the Contraction Theorem, there exists a unique contraction  $\phi : X \rightarrow Y$ , such that, for any curve  $C$  of  $X$ ,  $\phi(C)$  is a point if and only if the class of  $C$  is in  $R$ .

We also need to introduce a new notion of flips.

**Definition 2.** Let  $X$  be a normal  $\mathbb{Q}$ -Gorenstein projective variety, and  $\phi : X \rightarrow Y$  be a birational contraction of a ray of  $NE(X)_{K_X < 0}$ , such that  $Y$  is not  $\mathbb{Q}$ -Gorenstein.

A  $\mathbb{Q}$ -Gorenstein flip of  $\phi$  is a small contraction  $\phi : X^+ \rightarrow Y$ , where

- $X^+$  is a normal  $\mathbb{Q}$ -Gorenstein projective variety;
- and for any contracted curve  $C$  of  $X^+$ ,  $K_{X^+} \cdot C > 0$ .

In this paper, we prove the following result.

**Theorem 1.** *Let  $X$  be a normal  $\mathbb{Q}$ -Gorenstein projective variety with terminal singularities. Let  $R$  be a ray of  $NE(X)_{K_X < 0}$  and denote by  $\phi : X \rightarrow Y$  the contraction of  $R$ . Suppose that  $\phi$  is birational.*

- *If  $Y$  is  $\mathbb{Q}$ -Gorenstein, then  $Y$  has terminal singularities and  $\phi$  contracts a  $\mathbb{Q}$ -Cartier divisor.*
- *If  $Y$  is not  $\mathbb{Q}$ -Gorenstein, then there exists a  $\mathbb{Q}$ -Gorenstein flip of  $\phi$  if and only if  $\bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lmK_X)$  is a finitely generated sheaf of  $\mathcal{O}_Y$ -algebras. Moreover, if it exists, it is unique and  $X^+$  has terminal singularities.*

The proof of Theorem 1 is inspired by the proofs of the same results in the original MMP [KMM87], which are also detailed in [Mat02].

The paper is organized as follows.

In Section 2, we list several probably well-known and general results, which are very useful in the rest of the paper.

Sections 3 and 4 are devoted to the proof of Theorem 1.

In Section 5, we illustrate the MMP for  $\mathbb{Q}$ -Gorenstein projective varieties in an example of a 3-dimensional toric variety. In particular, we observe that a  $\mathbb{Q}$ -Gorenstein flip is not necessarily a contraction of an extremal ray.

In Section 6, we explain how to run a  $\mathbb{Q}$ -Gorenstein log MMP.

And we conclude in Section 7 with open questions.

## 2 General lemmas

We begin by a very classical result.

**Lemma 2.** *Let  $X$  be a normal variety, and let  $D$  be a Cartier divisor of  $X$ . Then the following assertions are equivalent.*

1.  $D$  is effective.
2.  $\mathcal{O}_X \subset \mathcal{O}_X(D)$ .
3.  $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ .

From Lemma 2, we deduce the following, probably well-known, result.

**Lemma 3.** *Let  $\phi : X \rightarrow Y$  be a morphism between two normal varieties  $X$  and  $Y$ . Let  $D$  be an effective  $\mathbb{Q}$ -Cartier divisor of  $Y$ .*

*Then  $\phi^*D$  is an effective  $\mathbb{Q}$ -Cartier divisor of  $X$ .*

*Proof.* Let  $m$  be a positive integer such that  $mD$  is Cartier. Since  $mD$  is effective, we have  $\mathcal{O}_Y \subset \mathcal{O}_Y(mD)$ . Then the image of  $\phi^*(\mathcal{O}_Y)$  in  $\phi^*(\mathcal{O}_Y(mD))$ , is the pull-back by  $\phi$  of the image of  $\mathcal{O}_Y$  in  $\mathcal{O}_Y(mD)$ . But all these invertible sheaves are subsheaves of the constant sheaf  $\mathcal{K} = \mathbb{C}(X)$ . Thus,  $\phi^*(\mathcal{O}_Y) = \mathcal{O}_X$  is contained in  $\phi^*(\mathcal{O}_Y(mD))$ . Then  $\phi^*(mD)$  is effective and also is  $\phi^*(D) := \frac{1}{m}\phi^*(mD)$ .  $\square$

We now prove the key lemma of the paper.

**Lemma 4.** *Let  $\phi : X \rightarrow Y$  be a surjective birational morphism between two normal varieties  $X$  and  $Y$ . Let  $D$  be a Cartier divisor of  $X$  and denote by  $D_Y$  the Weil divisor  $\phi_*D$ .*

*Suppose that the morphism  $\phi^*\phi_*\mathcal{O}_X(-D) \rightarrow \mathcal{O}_X(-D)$  is surjective.*

*Then the image of  $\phi^*\mathcal{O}_Y(D_Y) \rightarrow \mathcal{K}$  is contained in  $\mathcal{O}_X(D)$ .*

*In particular,  $\phi_*\mathcal{O}_X(D) = \mathcal{O}_Y(D_Y)$ .*

Note that  $D_Y$  is not necessarily Cartier, and  $\mathcal{O}_Y(D_Y)$  is the (not necessarily invertible) subsheaf of the constant sheaf  $\mathcal{K} = \mathbb{C}(X) = \mathbb{C}(Y)$  defined by, for any open set  $\mathcal{U}$  of  $Y$ ,

$$\mathcal{O}_Y(D_Y)(\mathcal{U}) = \{f \in \mathbb{C}(Y) \mid \operatorname{div} f|_{\mathcal{U}} + D_Y|_{\mathcal{U}} \geq 0\}.$$

*Proof.* Consider the inclusion of sheaves of  $\mathcal{O}_Y$ -modules  $\phi_*\mathcal{O}_X(-D) \subset \mathcal{O}_Y(-D_Y)$ . Tensoring by  $\mathcal{O}_Y(D_Y)$ , and composing by the natural morphism  $\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-D_Y) \rightarrow \mathcal{O}_Y$ , we get the natural morphism of sheaves of  $\mathcal{O}_Y$ -modules:

$$\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_Y} \phi_*\mathcal{O}_X(-D) \rightarrow \mathcal{O}_Y.$$

The pull-back of this morphism

$$\phi^*\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_X} \phi^*\phi_*\mathcal{O}_X(-D) \rightarrow \mathcal{O}_X$$

factors through the following surjective morphism

$$\phi^*\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_X} \phi^*\phi_*\mathcal{O}_X(-D) \rightarrow \phi^*\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D).$$

Then the image of  $\phi^*\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$  in  $\mathcal{K}$  is contained in  $\mathcal{O}_X$ . We tensor by the invertible sheaf  $\mathcal{O}_X(D)$ , and we get that  $\phi^*\mathcal{O}_Y(D_Y)$  maps to  $\mathcal{O}_X(D)$ .

To prove the last statement we use that, since  $\phi$  is surjective and  $\phi_*\mathcal{O}_X = \mathcal{O}_Y$ , for any sheaf of  $\mathcal{O}_Y$ -modules  $\mathcal{F}$ , we have  $\phi_*\phi^*\mathcal{F} = \mathcal{F}$ . Hence,  $\mathcal{O}_Y(D_Y) = \phi_*\phi^*\mathcal{O}_Y(D_Y)$  maps to  $\phi_*\mathcal{O}_X(D)$ . Both are subsheaves of  $\mathcal{K}$ , so we deduce that this map is an inclusion. The other inclusion is obvious, so that  $\phi_*\mathcal{O}_X(D) = \mathcal{O}_Y(D_Y)$ .  $\square$

From these lemmas, we get a useful corollary.

**Corollary 5.** *Let  $\phi : X \rightarrow Y$  be a surjective birational morphism between two normal varieties  $X$  and  $Y$ . Suppose that there exists a positive integer  $m$  such that  $mK_X$  and  $mK_Y$  are Cartier and such that the morphism  $\phi^*\phi_*\mathcal{O}_X(-mK_X) \rightarrow \mathcal{O}_X(-mK_X)$  is surjective.*

*Then  $K_X - \phi^*K_Y$  is effective.*

*Proof.* We apply Lemma 4 to  $D = mK_X$  (and  $D_Y = mK_Y$ ). Since  $mK_Y$  is Cartier, then  $\phi^*\mathcal{O}_Y(mK_Y)$  is a subsheaf of  $\mathcal{K}$  and the map  $\phi^*\mathcal{O}_Y(mK_Y) \rightarrow \mathcal{O}_X(mK_X)$  is an inclusion. We deduce that  $\mathcal{O}_X \subset \mathcal{O}_X(mK_X) \otimes_{\mathcal{O}_X} \phi^*\mathcal{O}_Y(-mK_Y)$ . And we apply Lemma 2 to conclude.  $\square$

### 3 $\mathbb{Q}$ -Gorenstein divisorial contractions

In this section we study the contractions that play the role of divisorial contractions.

**Definition 3.** The contraction  $\phi : X \rightarrow Y$  is called a  $\mathbb{Q}$ -Gorenstein divisorial contraction if it is birational and  $Y$  is  $\mathbb{Q}$ -Gorenstein.

The aim of this section is to prove that  $Y$  has the same singularities as  $X$ , and that  $\phi$  contracts a (not necessarily irreducible) Cartier divisor.

**Theorem 6.** *Let  $X$  be a normal  $\mathbb{Q}$ -Gorenstein projective variety with terminal singularities. Let  $R$  be a ray of  $NE(X)_{K_X < 0}$  and denote by  $\phi : X \rightarrow Y$  the contraction of  $R$ .*

*If  $\phi$  is a  $\mathbb{Q}$ -Gorenstein divisorial contraction, then  $Y$  is  $\mathbb{Q}$ -Gorenstein with terminal singularities and  $\phi$  contracts the (not zero) Cartier divisor  $E := K_X - \phi^*K_Y$ .*

*Proof.* By hypothesis,  $E$  is a  $\mathbb{Q}$ -Cartier divisor of  $X$  such that  $E \cdot C < 0$  for any curve  $C$  of class in  $R$ . In particular  $E$  is not zero. Write  $E = \sum_{i \in I} a_i E_i$  where for any  $i \in I$ , where  $\{E_i \mid i \in I\}$  is the set of irreducible exceptional divisors of  $\phi$ .

Let  $\sigma : V \rightarrow X$  be a desingularization of  $X$ . Denote by  $F_j$ , with  $j \in J$  the irreducible exceptional divisors of  $\sigma$ , and for any  $i \in I$ , denote by  $G_i$  the strict transform of  $E_i$  by  $\sigma$ . Write  $K_V - \sigma^* K_X = \sum_{j \in J} b_j F_j$  and  $\sigma^*(K_X - \phi^* K_Y) = \sum_{j \in J} c_j F_j + \sum_{i \in I} a_i G_i$ . since  $X$  has terminal singularities, for any  $j \in J$ , the rational numbers  $b_j$  are positive. And, by Lemma 3 and Corollary 5, for any  $i \in I$  and any  $j \in J$ , the rational numbers  $a_i$  and  $c_j$  are non-negative.

Hence, it is now enough to prove that the rational numbers  $a_i$  are positive (or non-zero). Let  $i_0 \in I$ . There exists a curve  $C$  in  $V$  that is contracted by  $\phi \circ \sigma$ , contained in  $G_{i_0}$  but not in  $\text{Exc}(\sigma) \cup \bigcup_{i \in I, i \neq i_0} G_i$ . In particular,  $\sigma(C)$  is a curve of  $X$  that is contracted by  $\phi$ , for any  $j \in J$  we have  $F_j \cdot C \geq 0$ , and for any  $i \in I, i \neq i_0$  we have  $G_i \cdot C \geq 0$ . Then, on the one hand  $(K_X - \phi^* K_Y) \cdot \sigma(C) = K_X \cdot \sigma(C)$  so that  $\sigma^*(K_X - \phi^* K_Y) \cdot C < 0$ , and on the other hand

$$\sigma^*(K_X - \phi^* K_Y) \cdot C = \sum_{i \in I} a_i (G_i \cdot C) + \sum_{j \in J} c_j (F_j \cdot C) \geq a_{i_0} (G_{i_0} \cdot C).$$

We deduce that  $a_{i_0}$  cannot be zero (we necessarily have  $a_{i_0} > 0$  and  $G_{i_0} \cdot C < 0$ ).  $\square$

**Remark 1.** In the proof of Theorem 6, we actually prove that  $E := K_X - \phi^* K_Y$  is an exceptional and effective  $\mathbb{Q}$ -Cartier divisor of  $X$  such that  $E \cdot C < 0$  for any curve  $C$  of  $X$  contracted by  $\phi$ . Also,  $E = \sum_{i \in I} a_i E_i$  where for any  $i \in I$ ,  $a_i$  is a positive rational number and where  $\{E_i \mid i \in I\}$  is the set of exceptional divisors of  $\phi$ .

## 4 $\mathbb{Q}$ -Gorenstein flips

In this section we interest at the existence of  $\mathbb{Q}$ -Gorenstein flips (see Definition 2).

Note that, if  $\phi : X \rightarrow Y$  is a birational contraction of a ray of  $NE(X)_{K_X < 0}$  and  $m$  is a positive integer such that  $mK_X$  is Cartier, then by Lemma 4 applied to  $D = mK_X$  (and  $D_Y = mK_Y$ ), we have

$$\bigoplus_{l \geq 0} \mathcal{O}_Y(lmK_Y) = \bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lmK_X).$$

We denote by  $\mathcal{A}$  this sheaf of  $\mathcal{O}_Y$ -algebras.

Then, we get an analogue result as in the classical MMP.

**Theorem 7.** *There exists a  $\mathbb{Q}$ -Gorenstein flip of  $\phi$  if and only if  $\mathcal{A}$  is finitely generated as sheaf of  $\mathcal{O}_Y$ -algebras. In that case, the  $\mathbb{Q}$ -Gorenstein flip is unique, given by  $\phi^+ : X^+ := \text{Proj}(\mathcal{A}) \rightarrow Y$ .*

*Moreover, if  $X$  has terminal singularities, then  $X^+$  has also terminal singularities.*

*Proof.* Suppose that there exists a  $\mathbb{Q}$ -Gorenstein flip  $\phi^+ : X^+ \rightarrow Y$  of  $\phi$ . Since  $\phi^+$  is a small contraction, for any  $l \geq 0$  we have  $\phi_*^+(\mathcal{O}_{X^+}(lK_{X^+})) = \mathcal{O}_Y(lK_Y)$ . In particular,  $\mathcal{A} = \bigoplus_{l \geq 0} \phi_* \mathcal{O}_{X^+}(lmK_{X^+})$ . But  $mK_{X^+}$  is  $\phi^+$ -ample (because for any contracted curve  $C$  of  $X^+$ ,  $K_{X^+} \cdot C > 0$ ), hence  $\text{Proj}(\mathcal{A})$  is a finitely generated sheaf of  $\mathcal{O}_Y$ -algebra and  $X^+ = \text{Proj}(\mathcal{A})$ .

Suppose now that  $\mathcal{A}$  is finitely generated as sheaf of  $\mathcal{O}_Y$ -algebras, and define  $X^+ := \text{Proj}(\mathcal{A})$ . Denote by  $\phi^+$  the corresponding morphism  $\text{Proj}(\mathcal{A}) \rightarrow Y$ . Then  $X^+$  is clearly a normal variety.

For the rest of the proof, we choose  $m$  sufficiently large such that  $\mathcal{A}$  is generated by  $\phi_*\mathcal{O}_X(mK_X)$ , which equals  $\mathcal{O}_Y(mK_Y)$ ; it does not change  $\text{Proj}(\mathcal{A})$ . Then we denote by  $\mathcal{O}_{X^+}(1)$  the  $\phi^+$ -very ample invertible sheaf on  $X^+$  such that  $\phi_*^+\mathcal{O}_{X^+}(1) = \phi_*\mathcal{O}_X(mK_X)$ .

We know prove, by contradiction, that  $\phi^+$  does not contract a divisor. Let  $E$  be an irreducible exceptional divisor of  $\phi^+$ . We get the following exact sequence

$$0 \rightarrow \mathcal{O}_{X^+} \rightarrow \mathcal{O}_{X^+}(E) \rightarrow \text{Coker} \rightarrow 0$$

where Coker cannot be zero. For a positive integer  $l$ , we apply the functor  $\phi_*^+(- \otimes \mathcal{O}_{X^+}(l))$  to this sequence, to obtain

$$0 \rightarrow \phi_*^+\mathcal{O}_{X^+}(l) \rightarrow \phi_*^+(\mathcal{O}_{X^+}(E) \otimes \mathcal{O}_{X^+}(l)) \rightarrow \phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l)) \rightarrow R^1\phi_*^+\mathcal{O}_{X^+}(l) \rightarrow \dots$$

Choose  $l$  sufficiently large such that  $R^1\phi_*^+\mathcal{O}_{X^+}(l) = 0$  and such that the map  $(\phi^+)^*\phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l)) \rightarrow \text{Coker} \otimes \mathcal{O}_{X^+}(l)$  is surjective (it is possible because  $\mathcal{O}_{X^+}(1)$  is  $\phi^+$ -ample). Then

$$0 \rightarrow \phi_*^+\mathcal{O}_{X^+}(l) \rightarrow \phi_*^+(\mathcal{O}_{X^+}(E) \otimes \mathcal{O}_{X^+}(l)) \rightarrow \phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l)) \rightarrow 0.$$

We claim that the first map of the above sequence is surjective. Indeed, since  $E$  is exceptional, for any open set  $\mathcal{U}$  of  $Y$ , we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_X(lmK_Y)(\mathcal{U}) & \xlongequal{\quad\quad\quad} & \mathcal{O}_X(lmK_Y)(\mathcal{U} \setminus \phi^+(E)) \\ \parallel & & \parallel \\ \phi_*^+\mathcal{O}_{X^+}(l)(\mathcal{U}) & \hookrightarrow & \phi_*^+\mathcal{O}_{X^+}(l)(\mathcal{U} \setminus \phi^+(E)) \\ \downarrow & & \parallel \\ \phi_*^+(\mathcal{O}_{X^+}(E) \otimes \mathcal{O}_{X^+}(l))(\mathcal{U}) & \hookrightarrow & \phi_*^+(\mathcal{O}_{X^+}(E) \otimes \mathcal{O}_{X^+}(l))(\mathcal{U} \setminus \phi^+(E)), \end{array}$$

where all inclusions have to be equalities.

Hence,  $\phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l)) = 0$ . But  $\phi^{+*}\phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l))$  surjects to  $\text{Coker} \otimes \mathcal{O}_{X^+}(l)$ , so  $\text{Coker} \otimes \mathcal{O}_{X^+}(l) = 0$  and then  $\text{Coker} = 0$ . We get a contraction.

Now, since  $\phi^+$  is a small contraction, we get that  $\mathcal{O}(mK_{X^+})$  is isomorphic to  $\mathcal{O}_{X^+}(1)$  so that  $X^+$  is clearly  $\mathbb{Q}$ -Gorenstein and  $K_{X^+}$  is  $\phi^+$ -ample (ie for any contracted curve  $C$  of  $X^+$ ,  $K_{X^+} \cdot C > 0$ ).

Suppose now that  $X$  has terminal singularities. We consider a common desingularization of  $X$  and  $X^+$ :

$$\begin{array}{ccc} & V & \\ \sigma \swarrow & & \searrow \sigma^+ \\ X & & X^+ \\ \phi \searrow & & \swarrow \phi^+ \\ & Y & \end{array}$$

Since  $mK_{X^+}$  is  $\phi^+$ -ample, we have

$$\begin{aligned}\sigma^{+*}\mathcal{O}_{X^+}(mK_{X^+}) &= \sigma^{+*}(\mathrm{Im}(\phi^{+*}\phi_*^+\mathcal{O}_{X^+}(mK_{X^+}) \longrightarrow \mathcal{O}_{X^+}(mK_{X^+}))) \\ &= \mathrm{Im}(\sigma^{+*}\phi^{+*}\phi_*^+\mathcal{O}_{X^+}(mK_{X^+}) \longrightarrow \sigma^{+*}\mathcal{O}_{X^+}(mK_{X^+})).\end{aligned}$$

But  $\sigma^{+*}\phi^{+*}\phi_*^+\mathcal{O}_{X^+}(mK_{X^+}) = (\phi^+ \circ \sigma^+)^*\mathcal{O}_Y(mK_Y) = (\phi \circ \sigma)^*\phi_*\mathcal{O}_X(mK_X) = (\phi \circ \sigma)^*(\phi \circ \sigma)_*\sigma^*\mathcal{O}_X(mK_X)$ . Thus,  $\sigma^{+*}\mathcal{O}_{X^+}(mK_{X^+})$  is contained in  $\sigma^*\mathcal{O}_X(mK_X)$ . In particular, by Lemma 2, the divisor  $\sigma^*K_X - \sigma^{+*}K_{X^+}$  is effective.

Hence, the divisor  $K_V - \sigma^{+*}K_{X^+} = (K_V - \sigma^*K_X) + (\sigma^*K_X - \sigma^{+*}K_{X^+})$  is effective, and moreover, it has positive coefficient in the irreducible divisors of  $V$  contracted by  $\sigma$  (because  $X$  has terminal singularities). It remains to prove that it has positive coefficient in the irreducible divisors of  $V$  contracted by  $\sigma^+$ .

Let  $E$  be an irreducible divisor of  $V$  contracted by  $\sigma^+$ , but not contracted by  $\sigma$  (if it exists). The irreducible divisor  $\sigma(E)$  of  $X$  is contracted by  $\phi$ . Then there exists a curve  $C$  in  $E$  that is contracted by  $\sigma^+$  but not by  $\sigma$ . In particular,  $\sigma(C)$  is a curve of  $X$  contracted by  $\phi$ . On the one hand  $\sigma^{+*}K_{X^+} \cdot C = 0$ , and on the other hand  $K_X \cdot \sigma(C) < 0$  so that  $\sigma^*K_X \cdot C < 0$ . Hence,  $(\sigma^*K_X - \sigma^{+*}K_{X^+}) \cdot C < 0$ . We can choose the curve  $C$  such that it is contained in no other exceptional divisor of  $\sigma$  and  $\sigma^+$ . Then we conclude as in the proof of Theorem 6, that the coefficient in  $E$  of  $\sigma^*K_X - \sigma^{+*}K_{X^+}$  is not zero.  $\square$

**Corollary 8** (Corollary of the proof of Theorem 7). *Let  $\phi^+ : X^+ \rightarrow Y$  be a  $\mathbb{Q}$ -Gorenstein flip of an extremal contraction  $\phi : X \rightarrow Y$ . Let  $V$  be a common desingularization of  $X$  and  $X^+$ . Denote  $\sigma : V \rightarrow X$  and  $\sigma^+ : V \rightarrow X^+$ . Then  $K_V = \sigma^*K_X + \sum_{i \in I} a_i E_i = \sigma^{+*}K_{X^+} + \sum_{i \in I} a_i^+ E_i$ , such that, for any  $i \in I$ ,  $a_i^+ \geq a_i$ . Moreover, there exists  $i_0 \in I$  with  $a_{i_0}^+ > a_{i_0}$ .*

Note that the set  $\{E_i \mid i \in I\}$  is the union of the sets of irreducible exceptional divisors of  $\sigma$  and  $\sigma^+$ , the hypothesis of Corollary 8 implies that for any  $i \in I$ ,  $a_i^+ > 0$ ,  $a_i \geq 0$  and  $a_i = 0$  if and only if  $E_i$  is not an exceptional divisor of  $\sigma$ .

*Proof.* It is enough to prove that, the effective divisor  $\sigma^*K_X - \sigma^{+*}K_{X^+}$  is not zero. Let  $C$  be a curve of  $V$  contracted by  $\phi \circ \sigma = \phi^+ \circ \sigma^+$  but not by  $\sigma$ . Then, by hypothesis on  $\phi$ ,  $\sigma^*K_X \cdot C < 0$ . If  $C$  is contracted by  $\phi^+$ , we have  $\sigma^{+*}K_{X^+} \cdot C = 0$ , and if not, we have  $\sigma^{+*}K_{X^+} \cdot C > 0$ . In any cases,  $\sigma^{+*}K_{X^+} \cdot C \geq 0$  so that  $(\sigma^*K_X - \sigma^{+*}K_{X^+}) \cdot C > 0$ . In particular  $\sigma^*K_X - \sigma^{+*}K_{X^+}$  is not zero.  $\square$

This corollary will be useful to prove the finiteness of sequences of  $\mathbb{Q}$ -Gorenstein flips in the family of  $\mathbb{Q}$ -Gorenstein spherical varieties.

## 5 An example

Here, we give an example of the  $\mathbb{Q}$ -Gorenstein MMP for a 3-dimensional toric variety. For the basics of theory of toric varieties, the reader can see [Ful93] or [Oda88].

- In  $\mathbb{Z}^3 \subset \mathbb{Q}^3$ , we consider the six following vectors:

$$\begin{aligned}e_1 &= (-1, -1, 1), & e_2 &= (1, -1, 1), & e_3 &= (1, 1, 2), \\ e_4 &= (-1, 1, 2), & e_5 &= (0, 1, 1), & e_6 &= (0, 0, -1).\end{aligned}$$

We denote by  $\mathcal{C}(e_{i_1}, \dots, e_{i_k})$  the cone in  $\mathbb{Q}^3$  generated by  $e_{i_1}, \dots, e_{i_k}$ . Then

$$\mathbb{F} := \{\mathcal{C}(e_1, e_2, e_3, e_4), \mathcal{C}(e_3, e_4, e_5), \mathcal{C}(e_1, e_2, e_6), \mathcal{C}(e_1, e_4, e_6), \mathcal{C}(e_2, e_3, e_6), \mathcal{C}(e_3, e_5, e_6), \mathcal{C}(e_4, e_5, e_6)\}$$

is a complete fan of  $\mathbb{Z}^3$ . We denote by  $X$  the toric variety of fan  $\mathbb{F}$ .

Note that since  $\mathbb{F}$  contains a non-simplicial cone, the variety  $X$  is not  $\mathbb{Q}$ -factorial.

Denote by  $X_1, X_2, X_3, X_4, X_5$  and  $X_6$  the  $(\mathbb{C}^*)^3$ -stable irreducible divisors respectively associated to the rays of  $\mathbb{F}$  generated by  $e_1, e_2, e_3, e_4, e_5$  and  $e_6$ . We can compute that

$$\text{Pic}(X)_{\mathbb{Q}} := \left\{ \sum_{i=1}^6 a_i X_i \mid \begin{array}{l} a_1 - a_2 + a_3 - a_4 = 0 \\ a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Q} \end{array} \right\} \Big/ \left\langle \begin{array}{l} X_1 - X_2 - X_3 + X_4, \\ X_1 + X_2 - X_3 - X_4 - X_5, \\ X_1 + X_2 + 2X_3 + 2X_4 + X_5 - X_6 \end{array} \right\rangle.$$

In particular,  $-K_X = \sum_{i=1}^6 X_i$  is  $\mathbb{Q}$ -Cartier.

We know that  $NE(X)$  is generated by the classes of  $(\mathbb{C}^*)^3$ -stable (and rational) curves  $C_{12}, C_{14}, C_{16}, C_{23}, C_{26}, C_{34}, C_{35}, C_{36}, C_{45}, C_{46}, C_{56}$ , where  $C_{ij}$  denotes the  $(\mathbb{C}^*)^3$ -stable curve associated to the 2-codimensional cone of  $\mathbb{F}$  generated by  $e_i$  and  $e_j$ .

We choose the basis  $(X_1 + X_2, X_1 - X_3)$  of  $\text{Pic}(X)_{\mathbb{Q}}$  and we compute the classes of these curves in the corresponding dual basis:

$$\begin{array}{llll} [C_{12}] & = & (\frac{1}{3}, 0), & [C_{14}] & = & (\frac{1}{6}, 0), & [C_{16}] & = & (\frac{1}{2}, 0), & [C_{23}] & = & (\frac{1}{6}, 0), \\ [C_{26}] & = & (\frac{1}{2}, 0), & [C_{34}] & = & (\frac{1}{3}, \frac{1}{2}), & [C_{35}] & = & (0, -\frac{1}{2}), & [C_{36}] & = & (\frac{1}{2}, \frac{1}{2}), \\ [C_{45}] & = & (0, -\frac{1}{2}), & [C_{46}] & = & (\frac{1}{2}, \frac{1}{2}), & [C_{56}] & = & (0, -1). \end{array}$$

In particular, the cone  $NE(X)$  is generated by  $(2, 3)$  and  $(0, -1)$ , ie by  $[C_{34}]$  and  $[C_{35}]$ .

Note also that  $-K_X$  is linearly equivalent to  $5(X_1 + X_2) - 2(X_1 - X_3)$  so that it is positive on all effective curves. Hence, in order to run the  $(\mathbb{Q}$ -Gorenstein) MMP, we need to choose one of the two extremal rays in  $NE(X)_{K_X < 0}$ .

- First, consider the contraction  $\phi : X \rightarrow Y$  of the extremal ray generated by  $[C_{35}]$ . We remark that this ray contains the classes of  $C_{35}, C_{45}$ , and  $C_{56}$ . Then  $Y$  is the 3-dimensional toric variety whose fan is

$$\mathbb{F}_Y := \{\mathcal{C}(e_1, e_2, e_3, e_4), \mathcal{C}(e_3, e_4, e_6), \mathcal{C}(e_1, e_2, e_6), \mathcal{C}(e_1, e_4, e_6), \mathcal{C}(e_2, e_3, e_6)\}.$$

We still denote by  $X_1, X_2, X_3, X_4$  and  $X_6$  the  $(\mathbb{C}^*)^3$ -stable irreducible divisors of  $Y$ . And we compute that

$$\text{Pic}(Y)_{\mathbb{Q}} := \left\{ \sum_{i=1}^4 a_i X_i + a_6 X_6 \mid \begin{array}{l} a_1 - a_2 + a_3 - a_4 = 0 \\ a_1, a_2, a_3, a_4, a_6 \in \mathbb{Q} \end{array} \right\} \Big/ \left\langle \begin{array}{l} X_1 - X_2 - X_3 + X_4, \\ X_1 + X_2 - X_3 - X_4, \\ X_1 + X_2 + 2X_3 + 2X_4 - X_6 \end{array} \right\rangle.$$

In particular,  $-K_Y = \sum_{i=1}^4 X_i + X_6$  is  $\mathbb{Q}$ -Cartier (and generates  $\text{Pic}(Y)_{\mathbb{Q}}$ ). The contraction  $\phi$  is a  $\mathbb{Q}$ -Gorenstein divisorial contraction.

• Now, consider the contraction  $\phi : X \rightarrow Y$  of the extremal ray generated by  $[C_{34}]$ . Then  $Y$  is the 3-dimensional toric variety whose fan is

$$\mathbb{F}_Y := \{\mathcal{C}(e_1, e_2, e_3, e_4, e_5), \mathcal{C}(e_1, e_2, e_6), \mathcal{C}(e_1, e_4, e_6), \mathcal{C}(e_2, e_3, e_6), \mathcal{C}(e_3, e_5, e_6), \mathcal{C}(e_4, e_5, e_6)\}.$$

We still denote by  $X_1, X_2, X_3, X_4, X_5$  and  $X_6$  the  $(\mathbb{C}^*)^3$ -stable irreducible divisors of  $Y$ . And we compute that

$$\text{Pic}(Y)_{\mathbb{Q}} := \left\{ \sum_{i=1}^6 a_i X_i \mid \begin{array}{l} a_1 - a_2 + a_3 - a_4 = 0 \\ a_1 - 3a_2 + 4a_3 - 6a_5 = 0 \\ a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Q} \end{array} \right\} / \left\langle \begin{array}{l} X_1 - X_2 - X_3 + X_4, \\ X_1 + X_2 - X_3 - X_4 - X_5, \\ X_1 + X_2 + 2X_3 + 2X_4 + X_5 - X_6 \end{array} \right\rangle.$$

In particular,  $-K_Y = \sum_{i=1}^6 X_i$  is not  $\mathbb{Q}$ -Cartier.

The  $\mathbb{Q}$ -Gorenstein flip of  $\phi$  is given by the  $\mathbb{Q}$ -factorial 3-dimensional toric variety  $X^+$  whose fan is

$$\mathbb{F}_{X^+} := \{\mathcal{C}(e_1, e_2, e_3), \mathcal{C}(e_1, e_3, e_5), \mathcal{C}(e_1, e_4, e_5), \\ \mathcal{C}(e_1, e_2, e_6), \mathcal{C}(e_1, e_4, e_6), \mathcal{C}(e_2, e_3, e_6), \mathcal{C}(e_3, e_5, e_6), \mathcal{C}(e_4, e_5, e_6)\},$$

and the  $(\mathbb{C}^*)^3$ -equivariant map  $\phi^+ : X^+ \rightarrow Y$ . We still denote by  $X_1, X_2, X_3, X_4, X_5$  and  $X_6$  the  $(\mathbb{C}^*)^3$ -stable irreducible divisors of  $X^+$ . In the basis dual to  $(X_1, X_2, X_3)$ , the classes of the  $(\mathbb{C}^*)^3$ -stable curves of  $X^+$  are

$$\begin{array}{llll} [C_{12}] & = & (\frac{1}{3}, 0, \frac{1}{3}), & [C_{13}] & = & (-\frac{1}{2}, \frac{1}{2}, -\frac{2}{3}), & [C_{14}] & = & (\frac{1}{2}, 0, 0), & [C_{15}] & = & (-\frac{1}{3}, 0, \frac{1}{3}), \\ [C_{16}] & = & (0, \frac{1}{2}, 0), & [C_{23}] & = & (\frac{1}{6}, 0, \frac{1}{6}), & [C_{26}] & = & (\frac{1}{2}, 0, \frac{1}{2}), & [C_{35}] & = & (\frac{1}{3}, 0, \frac{1}{3}), \\ [C_{36}] & = & (0, \frac{1}{2}, -\frac{1}{2}), & [C_{45}] & = & (1, 0, 0), & [C_{46}] & = & (\frac{1}{2}, 0, 0), & [C_{56}] & = & (0, 0, 1). \end{array}$$

We deduce that the cone  $NE(X^+)$  is generated by  $(-3, 3, -4)$ ,  $(1, 0, 0)$  and  $(-1, 0, 1)$ . Moreover, since  $-K_{X^+}$  is linearly equivalent to  $3X_1 + 5X_2 + 2X_3$ , there are two extremal rays in  $NE(X^+)_{K_{X^+} > 0}$ , respectively generated by  $[C_{13}]$  and  $[C_{15}]$ . The map  $\phi^+$  is the contraction of the 2-dimensional face of  $NE(X)$  generated by  $[C_{13}]$  and  $[C_{15}]$ .

For more examples, we refer to [Pas13] where a  $\mathbb{Q}$ -Gorenstein flip of a contraction that contracts a divisor is given.

## 6 Log $\mathbb{Q}$ -Gorenstein MMP

As for  $\mathbb{Q}$ -factorial varieties, we can run a  $\mathbb{Q}$ -Gorenstein MMP for klt pairs  $(X, D)$ .

Let  $X$  be a normal variety and let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $K_X + D$  is  $\mathbb{Q}$ -Cartier.

**Definition 4.** The pair  $(X, D)$  is said to be klt (Kawamata log terminal) if there exists a desingularization  $\sigma : V \rightarrow X$  of  $X$  such that  $K_V = \sigma^*(K_X + D) + \sum_{i \in I} a_i E_i$  where the  $E_i$ 's are irreducible divisors of  $V$  and for any  $i \in I$ ,  $a_i > -1$ .

**Remark 2.** 1. If a pair  $(X, D)$  is klt, then the above property is true for every desingularization of  $X$ .

2. The condition "for any  $i \in I$ ,  $a_i > -1$ " can be replaced by:  $[D] = 0$  and for any  $i \in I$  such that  $E_i$  is exceptional for  $\sigma$ ,  $a_i > -1$ .

Suppose now that  $(X, D)$  is klt.

By [KMM87], the Contraction Theorem and the Cone Theorem are still valid. In particular, for any ray  $R$  of  $NE(X)_{K_X + D < 0}$ , there exists a unique contraction  $\phi : X \rightarrow Y$ , such that, for any curve  $C$  of  $X$ ,  $\phi(C)$  is a point if and only if the class of  $C$  is in  $R$ . Moreover, we have equivalent results of Theorems 6 and 7.

**Theorem 9.** *We denote by  $D_Y$  the  $\mathbb{Q}$ -divisor  $\phi_*D$  of  $Y$ . And we fix  $m \geq 1$  such that  $m(K_X + D)$  is Cartier.*

1. *If  $K_Y + D_Y$  is  $\mathbb{Q}$ -Cartier, then the pair  $(Y, D_Y)$  is klt and  $E := K_X + D - \phi^*(K_Y + D_Y)$  is an exceptional and effective  $\mathbb{Q}$ -Cartier divisor such that  $E \cdot C < 0$  for any curve  $C$  of  $X$  contracted by  $\phi$ .*
2. *The sheaf of  $\mathcal{O}_Y$ -algebras  $\mathcal{A} := \bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lm(K_X + D))$  equals  $\bigoplus_{l \geq 0} \mathcal{O}_Y(lm(K_Y + D_Y))$ .*
3. *If  $K_Y + D_Y$  is not  $\mathbb{Q}$ -Cartier,  $\mathcal{A}$  is finitely generated if and only if there exists a small contraction  $\phi^+ : X^+ \rightarrow Y$  such that the pair  $(X^+, (\phi^+)^{-1}D_Y)$  is klt and for any curve  $C^+$  of  $X^+$  contracted by  $\phi^+$ ,  $(K_{X^+} + (\phi^+)^{-1}D_Y) \cdot C^+ > 0$ . In that case,  $X^+$  is  $\text{Proj}(\mathcal{A})$  over  $Y$ .*

The proof is very similar to the proofs of Theorems 6 and 7. The key of the proof of (1) is that, as in Corollary 5, the divisor  $K_X + D - \phi^*(K_Y + D_Y)$  is effective. Lemma 4, applied to  $m(K_X + D)$  (and  $m(K_Y + D_Y)$ ) directly gives (2). And to prove (3), we do the same proof as in Theorem 7, by replacing  $K_X$  by  $K_X + D$ ,  $K_Y$  by  $K_Y + D_Y$  and  $K_{X^+}$  by  $K_{X^+} + (\phi^+)^{-1}D_Y$  (excepting the last paragraph, which is not necessary).

## 7 Open questions

The same questions as in the classical MMP can be done.

**Question 1.** Do  $\mathbb{Q}$ -Gorenstein flips always exist? Or equivalently, in the case where  $Y$  is not  $\mathbb{Q}$ -Gorenstein, is  $\mathcal{A} := \bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lmK_X) = \bigoplus_{l \geq 0} \mathcal{O}_Y(lmK_Y)$  finitely generated as sheaf of  $\mathcal{O}_Y$ -algebras?

**Question 2.** Are sequences of  $\mathbb{Q}$ -Gorenstein flips always finite?

We can answer positively these two questions in the case of spherical varieties.

Let  $G$  be a connected reductive algebraic group (over  $\mathbb{C}$ ). A normal  $G$ -variety is spherical if there exists an open orbit in  $X$  under the action of a Borel subgroup of  $G$ .

Let  $H$  be a spherical subgroup of  $G$  (ie such that there exists a Borel subgroup of  $G$  satisfying that  $BH$  is open in  $G$ ). A  $G/H$ -embedding is a normal  $G$ -variety containing an open  $G$ -orbit isomorphic to  $G/H$ . (A  $G/H$ -embedding is a spherical  $G$ -variety, and inversely a spherical  $G$ -variety is a  $G/H$ -embedding for some spherical subgroup  $H$  of  $G$ .)

**Proposition 10.** 1. ([Bri93, Lemme 4.3]) *Let  $X$  and  $Y$  be two spherical  $G$ -varieties and let  $\phi$  be a proper  $G$ -equivariant morphism. Then for any Cartier divisor  $D$  of  $X$ , the  $\mathcal{O}_Y$ -algebra  $\bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lD)$  is finitely generated.*

2. *Let  $X$  be a  $G/H$ -embeddings (ie a normal  $G$ -variety containing an open  $G$ -orbit isomorphic to  $G/H$ ). There are only finitely many varieties  $Z$  that can be obtained from  $X$  by  $\mathbb{Q}$ -Gorenstein flips (and these varieties are still  $G/H$ -embeddings). Moreover, there exists a common desingularization for all these varieties, ie there exist a smooth  $G/H$ -embedding  $V$  and birational proper  $G$ -equivariant morphisms  $\sigma_Z : V \rightarrow Z$ , for any  $Z$ .*

The second part of the proposition is a consequence of the classification of  $G/H$ -embeddings in terms of colored fans and the fact that a  $\mathbb{Q}$ -Gorenstein flip adds no divisor.

Hence, in the family of spherical varieties, we immediately get the existence of  $\mathbb{Q}$ -Gorenstein flips, and the finiteness of sequences of  $\mathbb{Q}$ -Gorenstein flips is a consequence of 2 of Proposition 10 and Corollary 8.

**Question 3.** Let  $\phi : X \rightarrow Y$  be a contraction of an extremal ray of  $NE(X)_{K_X < 0}$  such that  $\dim(Y) < \dim(X)$ . If  $X$  is  $\mathbb{Q}$ -Gorenstein,  $Y$  is unfortunately not necessarily  $\mathbb{Q}$ -Gorenstein. But can we give some properties about the fibers of these contractions?

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