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A Minimal Model Program for \mathbb{Q} -Gorenstein varieties

Boris Pasquier*

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Abstract

The Minimal Model Program is constructed for projective varieties with at most \mathbb{Q} -factorial terminal singularities. Here, we adapt the definitions of divisorial contractions and flips to construct a Minimal Model Program for projective varieties with at most \mathbb{Q} -Gorenstein terminal singularities. This new construction can be naturally extended to klt pairs. In the family of \mathbb{Q} -Gorenstein spherical varieties, we answer positively to the questions of existence of flips and of finiteness of sequences of flips.

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1 Introduction

We only consider normal algebraic varieties over \mathbb{C} .

The Minimal Model Program (MMP) is now well-known for projective varieties with at most \mathbb{Q} -factorial terminal singularities. Moreover, the first two main results of the theory, the Contraction Theorem and the Cone Theorem, are given for projective varieties with at most \mathbb{Q} -Gorenstein terminal singularities [KMM87]. It is then natural to ask if the MMP also works for \mathbb{Q} -Gorenstein projective varieties. With the study of toric varieties (for example), we can answer negatively to this question: there exist divisorial contractions of extremal rays from a \mathbb{Q} -Gorenstein variety to a non- \mathbb{Q} -Gorenstein variety and flips do not always exist. Nevertheless, still for toric varieties (and more generally for horospherical varieties), we observed in [Pas13] that a program very similar to the MMP works for \mathbb{Q} -Gorenstein varieties. In this latter paper, where the MMP is reduced to the study of a family of moment polytopes, the theory for \mathbb{Q} -Gorenstein projective varieties even seems to be more natural than the classical theory for \mathbb{Q} -factorial projective varieties.

In this paper, we define a MMP for \mathbb{Q} -Gorenstein projective varieties with terminal singularities, and we give the first results of the theory.

Before to describe explicitly the results, we recall some basic definitions and notations.

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Definition 1. • A normal variety X is \mathbb{Q} -Gorenstein if its canonical divisor K_X is \mathbb{Q} -Cartier (ie if there exists a multiple of K_X that is Cartier).

- A normal \mathbb{Q} -Gorenstein variety X has terminal singularities if there exists a desingularization $\sigma : V \rightarrow X$ of X such that

$$K_V = \sigma^* K_X + \sum_{E \text{ prime divisor of } V} a_E E,$$

with $a_E > 0$ for any exceptional prime divisor E of σ (and $a_E = 0$ otherwise).

- A contraction (ie a projective morphism $\phi : X \rightarrow Y$ such that $\phi_* \mathcal{O}_X = \mathcal{O}_Y$) is a small contraction if its exceptional locus is at least of codimension 2.

Now, fix a normal \mathbb{Q} -Gorenstein projective variety X with terminal singularities. We denote by $NE(X)$ the nef cone of curves of X and by $NE(X)_{K_X < 0}$ (resp. $NE(X)_{K_X > 0}$) the intersection of the nef cone with the open half-space of curves negative (resp. positive) along the divisor K_X . Then, by the Cone Theorem, $NE(X) = NE(X)_{K_X > 0} + \sum_{i \in I} R_i$, where the set $\{R_i \mid i \in I\}$ is a discrete set of (extremal) rays of $NE(X)_{K_X < 0}$. Fix a ray R of $NE(X)_{K_X < 0}$. Then, by the Contraction Theorem, there exists a unique contraction $\phi : X \rightarrow Y$, such that, for any curve C of X , $\phi(C)$ is a point if and only if the class of C is in R .

We also need to introduce a new notion of flips.

Definition 2. Let X be a normal \mathbb{Q} -Gorenstein projective variety, and $\phi : X \rightarrow Y$ be a birational contraction of a ray of $NE(X)_{K_X < 0}$, such that Y is not \mathbb{Q} -Gorenstein.

A \mathbb{Q} -Gorenstein flip of ϕ is a small contraction $\phi : X^+ \rightarrow Y$, where

- X^+ is a normal \mathbb{Q} -Gorenstein projective variety;
- and for any contracted curve C of X^+ , $K_{X^+} \cdot C > 0$.

In this paper, we prove the following result.

Theorem 1. *Let X be a normal \mathbb{Q} -Gorenstein projective variety with terminal singularities. Let R be a ray of $NE(X)_{K_X < 0}$ and denote by $\phi : X \rightarrow Y$ the contraction of R . Suppose that ϕ is birational.*

- *If Y is \mathbb{Q} -Gorenstein, then Y has terminal singularities and ϕ contracts a \mathbb{Q} -Cartier divisor.*
- *If Y is not \mathbb{Q} -Gorenstein, then there exists a \mathbb{Q} -Gorenstein flip of ϕ if and only if $\bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lmK_X)$ is a finitely generated sheaf of \mathcal{O}_Y -algebras. Moreover, if it exists, it is unique and X^+ has terminal singularities.*

The proof of Theorem 1 is inspired by the proofs of the same results in the original MMP [KMM87], which are also detailed in [Mat02].

The paper is organized as follows.

In Section 2, we list several probably well-known and general results, which are very useful in the rest of the paper.

Sections 3 and 4 are devoted to the proof of Theorem 1.

In Section 5, we illustrate the MMP for \mathbb{Q} -Gorenstein projective varieties in an example of a 3-dimensional toric variety. In particular, we observe that a \mathbb{Q} -Gorenstein flip is not necessarily a contraction of an extremal ray.

In Section 6, we explain how to run a \mathbb{Q} -Gorenstein log MMP.

And we conclude in Section 7 with open questions.

2 General lemmas

We begin by a very classical result.

Lemma 2. *Let X be a normal variety, and let D be a Cartier divisor of X . Then the following assertions are equivalent.*

1. D is effective.
2. $\mathcal{O}_X \subset \mathcal{O}_X(D)$.
3. $\mathcal{O}_X(-D) \subset \mathcal{O}_X$.

From Lemma 2, we deduce the following, probably well-known, result.

Lemma 3. *Let $\phi : X \rightarrow Y$ be a morphism between two normal varieties X and Y . Let D be an effective \mathbb{Q} -Cartier divisor of Y .*

*Then ϕ^*D is an effective \mathbb{Q} -Cartier divisor of X .*

Proof. Let m be a positive integer such that mD is Cartier. Since mD is effective, we have $\mathcal{O}_Y \subset \mathcal{O}_Y(mD)$. Then the image of $\phi^*(\mathcal{O}_Y)$ in $\phi^*(\mathcal{O}_Y(mD))$, is the pull-back by ϕ of the image of \mathcal{O}_Y in $\mathcal{O}_Y(mD)$. But all these invertible sheaves are subsheaves of the constant sheaf $\mathcal{K} = \mathbb{C}(X)$. Thus, $\phi^*(\mathcal{O}_Y) = \mathcal{O}_X$ is contained in $\phi^*(\mathcal{O}_Y(mD))$. Then $\phi^*(mD)$ is effective and also is $\phi^*(D) := \frac{1}{m}\phi^*(mD)$. \square

We now prove the key lemma of the paper.

Lemma 4. *Let $\phi : X \rightarrow Y$ be a surjective birational morphism between two normal varieties X and Y . Let D be a Cartier divisor of X and denote by D_Y the Weil divisor ϕ_*D .*

Suppose that the morphism $\phi^\phi_*\mathcal{O}_X(-D) \rightarrow \mathcal{O}_X(-D)$ is surjective.*

Then the image of $\phi^\mathcal{O}_Y(D_Y) \rightarrow \mathcal{K}$ is contained in $\mathcal{O}_X(D)$.*

In particular, $\phi_\mathcal{O}_X(D) = \mathcal{O}_Y(D_Y)$.*

Note that D_Y is not necessarily Cartier, and $\mathcal{O}_Y(D_Y)$ is the (not necessarily invertible) subsheaf of the constant sheaf $\mathcal{K} = \mathbb{C}(X) = \mathbb{C}(Y)$ defined by, for any open set \mathcal{U} of Y ,

$$\mathcal{O}_Y(D_Y)(\mathcal{U}) = \{f \in \mathbb{C}(Y) \mid \operatorname{div} f|_{\mathcal{U}} + D_Y|_{\mathcal{U}} \geq 0\}.$$

Proof. Consider the inclusion of sheaves of \mathcal{O}_Y -modules $\phi_*\mathcal{O}_X(-D) \subset \mathcal{O}_Y(-D_Y)$. Tensoring by $\mathcal{O}_Y(D_Y)$, and composing by the natural morphism $\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-D_Y) \rightarrow \mathcal{O}_Y$, we get the natural morphism of sheaves of \mathcal{O}_Y -modules:

$$\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_Y} \phi_*\mathcal{O}_X(-D) \rightarrow \mathcal{O}_Y.$$

The pull-back of this morphism

$$\phi^*\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_X} \phi^*\phi_*\mathcal{O}_X(-D) \rightarrow \mathcal{O}_X$$

factors through the following surjective morphism

$$\phi^*\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_X} \phi^*\phi_*\mathcal{O}_X(-D) \rightarrow \phi^*\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D).$$

Then the image of $\phi^*\mathcal{O}_Y(D_Y) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ in \mathcal{K} is contained in \mathcal{O}_X . We tensor by the invertible sheaf $\mathcal{O}_X(D)$, and we get that $\phi^*\mathcal{O}_Y(D_Y)$ maps to $\mathcal{O}_X(D)$.

To prove the last statement we use that, since ϕ is surjective and $\phi_*\mathcal{O}_X = \mathcal{O}_Y$, for any sheaf of \mathcal{O}_Y -modules \mathcal{F} , we have $\phi_*\phi^*\mathcal{F} = \mathcal{F}$. Hence, $\mathcal{O}_Y(D_Y) = \phi_*\phi^*\mathcal{O}_Y(D_Y)$ maps to $\phi_*\mathcal{O}_X(D)$. Both are subsheaves of \mathcal{K} , so we deduce that this map is an inclusion. The other inclusion is obvious, so that $\phi_*\mathcal{O}_X(D) = \mathcal{O}_Y(D_Y)$. \square

From these lemmas, we get a useful corollary.

Corollary 5. *Let $\phi : X \rightarrow Y$ be a surjective birational morphism between two normal varieties X and Y . Suppose that there exists a positive integer m such that mK_X and mK_Y are Cartier and such that the morphism $\phi^*\phi_*\mathcal{O}_X(-mK_X) \rightarrow \mathcal{O}_X(-mK_X)$ is surjective.*

*Then $K_X - \phi^*K_Y$ is effective.*

Proof. We apply Lemma 4 to $D = mK_X$ (and $D_Y = mK_Y$). Since mK_Y is Cartier, then $\phi^*\mathcal{O}_Y(mK_Y)$ is a subsheaf of \mathcal{K} and the map $\phi^*\mathcal{O}_Y(mK_Y) \rightarrow \mathcal{O}_X(mK_X)$ is an inclusion. We deduce that $\mathcal{O}_X \subset \mathcal{O}_X(mK_X) \otimes_{\mathcal{O}_X} \phi^*\mathcal{O}_Y(-mK_Y)$. And we apply Lemma 2 to conclude. \square

3 \mathbb{Q} -Gorenstein divisorial contractions

In this section we study the contractions that play the role of divisorial contractions.

Definition 3. The contraction $\phi : X \rightarrow Y$ is called a \mathbb{Q} -Gorenstein divisorial contraction if it is birational and Y is \mathbb{Q} -Gorenstein.

The aim of this section is to prove that Y has the same singularities as X , and that ϕ contracts a (not necessarily irreducible) Cartier divisor.

Theorem 6. *Let X be a normal \mathbb{Q} -Gorenstein projective variety with terminal singularities. Let R be a ray of $NE(X)_{K_X < 0}$ and denote by $\phi : X \rightarrow Y$ the contraction of R .*

*If ϕ is a \mathbb{Q} -Gorenstein divisorial contraction, then Y is \mathbb{Q} -Gorenstein with terminal singularities and ϕ contracts the (not zero) Cartier divisor $E := K_X - \phi^*K_Y$.*

Proof. By hypothesis, E is a \mathbb{Q} -Cartier divisor of X such that $E \cdot C < 0$ for any curve C of class in R . In particular E is not zero. Write $E = \sum_{i \in I} a_i E_i$ where for any $i \in I$, where $\{E_i \mid i \in I\}$ is the set of irreducible exceptional divisors of ϕ .

Let $\sigma : V \rightarrow X$ be a desingularization of X . Denote by F_j , with $j \in J$ the irreducible exceptional divisors of σ , and for any $i \in I$, denote by G_i the strict transform of E_i by σ . Write $K_V - \sigma^* K_X = \sum_{j \in J} b_j F_j$ and $\sigma^*(K_X - \phi^* K_Y) = \sum_{j \in J} c_j F_j + \sum_{i \in I} a_i G_i$. since X has terminal singularities, for any $j \in J$, the rational numbers b_j are positive. And, by Lemma 3 and Corollary 5, for any $i \in I$ and any $j \in J$, the rational numbers a_i and c_j are non-negative.

Hence, it is now enough to prove that the rational numbers a_i are positive (or non-zero). Let $i_0 \in I$. There exists a curve C in V that is contracted by $\phi \circ \sigma$, contained in G_{i_0} but not in $\text{Exc}(\sigma) \cup \bigcup_{i \in I, i \neq i_0} G_i$. In particular, $\sigma(C)$ is a curve of X that is contracted by ϕ , for any $j \in J$ we have $F_j \cdot C \geq 0$, and for any $i \in I, i \neq i_0$ we have $G_i \cdot C \geq 0$. Then, on the one hand $(K_X - \phi^* K_Y) \cdot \sigma(C) = K_X \cdot \sigma(C)$ so that $\sigma^*(K_X - \phi^* K_Y) \cdot C < 0$, and on the other hand

$$\sigma^*(K_X - \phi^* K_Y) \cdot C = \sum_{i \in I} a_i (G_i \cdot C) + \sum_{j \in J} c_j (F_j \cdot C) \geq a_{i_0} (G_{i_0} \cdot C).$$

We deduce that a_{i_0} cannot be zero (we necessarily have $a_{i_0} > 0$ and $G_{i_0} \cdot C < 0$). \square

Remark 1. In the proof of Theorem 6, we actually prove that $E := K_X - \phi^* K_Y$ is an exceptional and effective \mathbb{Q} -Cartier divisor of X such that $E \cdot C < 0$ for any curve C of X contracted by ϕ . Also, $E = \sum_{i \in I} a_i E_i$ where for any $i \in I$, a_i is a positive rational number and where $\{E_i \mid i \in I\}$ is the set of exceptional divisors of ϕ .

4 \mathbb{Q} -Gorenstein flips

In this section we interest at the existence of \mathbb{Q} -Gorenstein flips (see Definition 2).

Note that, if $\phi : X \rightarrow Y$ is a birational contraction of a ray of $NE(X)_{K_X < 0}$ and m is a positive integer such that mK_X is Cartier, then by Lemma 4 applied to $D = mK_X$ (and $D_Y = mK_Y$), we have

$$\bigoplus_{l \geq 0} \mathcal{O}_Y(lmK_Y) = \bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lmK_X).$$

We denote by \mathcal{A} this sheaf of \mathcal{O}_Y -algebras.

Then, we get an analogue result as in the classical MMP.

Theorem 7. *There exists a \mathbb{Q} -Gorenstein flip of ϕ if and only if \mathcal{A} is finitely generated as sheaf of \mathcal{O}_Y -algebras. In that case, the \mathbb{Q} -Gorenstein flip is unique, given by $\phi^+ : X^+ := \text{Proj}(\mathcal{A}) \rightarrow Y$.*

Moreover, if X has terminal singularities, then X^+ has also terminal singularities.

Proof. Suppose that there exists a \mathbb{Q} -Gorenstein flip $\phi^+ : X^+ \rightarrow Y$ of ϕ . Since ϕ^+ is a small contraction, for any $l \geq 0$ we have $\phi_*^+(\mathcal{O}_{X^+}(lK_{X^+})) = \mathcal{O}_Y(lK_Y)$. In particular, $\mathcal{A} = \bigoplus_{l \geq 0} \phi_* \mathcal{O}_{X^+}(lmK_{X^+})$. But mK_{X^+} is ϕ^+ -ample (because for any contracted curve C of X^+ , $K_{X^+} \cdot C > 0$), hence $\text{Proj}(\mathcal{A})$ is a finitely generated sheaf of \mathcal{O}_Y -algebra and $X^+ = \text{Proj}(\mathcal{A})$.

Suppose now that \mathcal{A} is finitely generated as sheaf of \mathcal{O}_Y -algebras, and define $X^+ := \text{Proj}(\mathcal{A})$. Denote by ϕ^+ the corresponding morphism $\text{Proj}(\mathcal{A}) \rightarrow Y$. Then X^+ is clearly a normal variety.

For the rest of the proof, we choose m sufficiently large such that \mathcal{A} is generated by $\phi_*\mathcal{O}_X(mK_X)$, which equals $\mathcal{O}_Y(mK_Y)$; it does not change $\text{Proj}(\mathcal{A})$. Then we denote by $\mathcal{O}_{X^+}(1)$ the ϕ^+ -very ample invertible sheaf on X^+ such that $\phi_*^+\mathcal{O}_{X^+}(1) = \phi_*\mathcal{O}_X(mK_X)$.

We know prove, by contradiction, that ϕ^+ does not contract a divisor. Let E be an irreducible exceptional divisor of ϕ^+ . We get the following exact sequence

$$0 \rightarrow \mathcal{O}_{X^+} \rightarrow \mathcal{O}_{X^+}(E) \rightarrow \text{Coker} \rightarrow 0$$

where Coker cannot be zero. For a positive integer l , we apply the functor $\phi_*^+(- \otimes \mathcal{O}_{X^+}(l))$ to this sequence, to obtain

$$0 \rightarrow \phi_*^+\mathcal{O}_{X^+}(l) \rightarrow \phi_*^+(\mathcal{O}_{X^+}(E) \otimes \mathcal{O}_{X^+}(l)) \rightarrow \phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l)) \rightarrow R^1\phi_*^+\mathcal{O}_{X^+}(l) \rightarrow \dots$$

Choose l sufficiently large such that $R^1\phi_*^+\mathcal{O}_{X^+}(l) = 0$ and such that the map $(\phi^+)^*\phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l)) \rightarrow \text{Coker} \otimes \mathcal{O}_{X^+}(l)$ is surjective (it is possible because $\mathcal{O}_{X^+}(1)$ is ϕ^+ -ample). Then

$$0 \rightarrow \phi_*^+\mathcal{O}_{X^+}(l) \rightarrow \phi_*^+(\mathcal{O}_{X^+}(E) \otimes \mathcal{O}_{X^+}(l)) \rightarrow \phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l)) \rightarrow 0.$$

We claim that the first map of the above sequence is surjective. Indeed, since E is exceptional, for any open set \mathcal{U} of Y , we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_X(lmK_Y)(\mathcal{U}) & \xlongequal{\quad\quad\quad} & \mathcal{O}_X(lmK_Y)(\mathcal{U} \setminus \phi^+(E)) \\ \parallel & & \parallel \\ \phi_*^+\mathcal{O}_{X^+}(l)(\mathcal{U}) & \hookrightarrow & \phi_*^+\mathcal{O}_{X^+}(l)(\mathcal{U} \setminus \phi^+(E)) \\ \downarrow & & \downarrow \\ \phi_*^+(\mathcal{O}_{X^+}(E) \otimes \mathcal{O}_{X^+}(l))(\mathcal{U}) & \hookrightarrow & \phi_*^+(\mathcal{O}_{X^+}(E) \otimes \mathcal{O}_{X^+}(l))(\mathcal{U} \setminus \phi^+(E)), \end{array}$$

where all inclusions have to be equalities.

Hence, $\phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l)) = 0$. But $\phi^{+*}\phi_*^+(\text{Coker} \otimes \mathcal{O}_{X^+}(l))$ surjects to $\text{Coker} \otimes \mathcal{O}_{X^+}(l)$, so $\text{Coker} \otimes \mathcal{O}_{X^+}(l) = 0$ and then $\text{Coker} = 0$. We get a contraction.

Now, since ϕ^+ is a small contraction, we get that $\mathcal{O}(mK_{X^+})$ is isomorphic to $\mathcal{O}_{X^+}(1)$ so that X^+ is clearly \mathbb{Q} -Gorenstein and K_{X^+} is ϕ^+ -ample (ie for any contracted curve C of X^+ , $K_{X^+} \cdot C > 0$).

Suppose now that X has terminal singularities. We consider a common desingularization of X and X^+ :

$$\begin{array}{ccc} & V & \\ \sigma \swarrow & & \searrow \sigma^+ \\ X & & X^+ \\ \phi \searrow & & \swarrow \phi^+ \\ & Y & \end{array}$$

Since mK_{X^+} is ϕ^+ -ample, we have

$$\begin{aligned}\sigma^{+*}\mathcal{O}_{X^+}(mK_{X^+}) &= \sigma^{+*}(\mathrm{Im}(\phi^{+*}\phi_*^+\mathcal{O}_{X^+}(mK_{X^+}) \longrightarrow \mathcal{O}_{X^+}(mK_{X^+}))) \\ &= \mathrm{Im}(\sigma^{+*}\phi^{+*}\phi_*^+\mathcal{O}_{X^+}(mK_{X^+}) \longrightarrow \sigma^{+*}\mathcal{O}_{X^+}(mK_{X^+})).\end{aligned}$$

But $\sigma^{+*}\phi^{+*}\phi_*^+\mathcal{O}_{X^+}(mK_{X^+}) = (\phi^+ \circ \sigma^+)^*\mathcal{O}_Y(mK_Y) = (\phi \circ \sigma)^*\phi_*\mathcal{O}_X(mK_X) = (\phi \circ \sigma)^*(\phi \circ \sigma)_*\sigma^*\mathcal{O}_X(mK_X)$. Thus, $\sigma^{+*}\mathcal{O}_{X^+}(mK_{X^+})$ is contained in $\sigma^*\mathcal{O}_X(mK_X)$. In particular, by Lemma 2, the divisor $\sigma^*K_X - \sigma^{+*}K_{X^+}$ is effective.

Hence, the divisor $K_V - \sigma^{+*}K_{X^+} = (K_V - \sigma^*K_X) + (\sigma^*K_X - \sigma^{+*}K_{X^+})$ is effective, and moreover, it has positive coefficient in the irreducible divisors of V contracted by σ (because X has terminal singularities). It remains to prove that it has positive coefficient in the irreducible divisors of V contracted by σ^+ .

Let E be an irreducible divisor of V contracted by σ^+ , but not contracted by σ (if it exists). The irreducible divisor $\sigma(E)$ of X is contracted by ϕ . Then there exists a curve C in E that is contracted by σ^+ but not by σ . In particular, $\sigma(C)$ is a curve of X contracted by ϕ . On the one hand $\sigma^{+*}K_{X^+} \cdot C = 0$, and on the other hand $K_X \cdot \sigma(C) < 0$ so that $\sigma^*K_X \cdot C < 0$. Hence, $(\sigma^*K_X - \sigma^{+*}K_{X^+}) \cdot C < 0$. We can choose the curve C such that it is contained in no other exceptional divisor of σ and σ^+ . Then we conclude as in the proof of Theorem 6, that the coefficient in E of $\sigma^*K_X - \sigma^{+*}K_{X^+}$ is not zero. \square

Corollary 8 (Corollary of the proof of Theorem 7). *Let $\phi^+ : X^+ \rightarrow Y$ be a \mathbb{Q} -Gorenstein flip of an extremal contraction $\phi : X \rightarrow Y$. Let V be a common desingularization of X and X^+ . Denote $\sigma : V \rightarrow X$ and $\sigma^+ : V \rightarrow X^+$. Then $K_V = \sigma^*K_X + \sum_{i \in I} a_i E_i = \sigma^{+*}K_{X^+} + \sum_{i \in I} a_i^+ E_i$, such that, for any $i \in I$, $a_i^+ \geq a_i$. Moreover, there exists $i_0 \in I$ with $a_{i_0}^+ > a_{i_0}$.*

Note that the set $\{E_i \mid i \in I\}$ is the union of the sets of irreducible exceptional divisors of σ and σ^+ , the hypothesis of Corollary 8 implies that for any $i \in I$, $a_i^+ > 0$, $a_i \geq 0$ and $a_i = 0$ if and only if E_i is not an exceptional divisor of σ .

Proof. It is enough to prove that, the effective divisor $\sigma^*K_X - \sigma^{+*}K_{X^+}$ is not zero. Let C be a curve of V contracted by $\phi \circ \sigma = \phi^+ \circ \sigma^+$ but not by σ . Then, by hypothesis on ϕ , $\sigma^*K_X \cdot C < 0$. If C is contracted by ϕ^+ , we have $\sigma^{+*}K_{X^+} \cdot C = 0$, and if not, we have $\sigma^{+*}K_{X^+} \cdot C > 0$. In any cases, $\sigma^{+*}K_{X^+} \cdot C \geq 0$ so that $(\sigma^*K_X - \sigma^{+*}K_{X^+}) \cdot C > 0$. In particular $\sigma^*K_X - \sigma^{+*}K_{X^+}$ is not zero. \square

This corollary will be useful to prove the finiteness of sequences of \mathbb{Q} -Gorenstein flips in the family of \mathbb{Q} -Gorenstein spherical varieties.

5 An example

Here, we give an example of the \mathbb{Q} -Gorenstein MMP for a 3-dimensional toric variety. For the basics of theory of toric varieties, the reader can see [Ful93] or [Oda88].

- In $\mathbb{Z}^3 \subset \mathbb{Q}^3$, we consider the six following vectors:

$$\begin{aligned}e_1 &= (-1, -1, 1), & e_2 &= (1, -1, 1), & e_3 &= (1, 1, 2), \\ e_4 &= (-1, 1, 2), & e_5 &= (0, 1, 1), & e_6 &= (0, 0, -1).\end{aligned}$$

We denote by $\mathcal{C}(e_{i_1}, \dots, e_{i_k})$ the cone in \mathbb{Q}^3 generated by e_{i_1}, \dots, e_{i_k} . Then

$$\mathbb{F} := \{\mathcal{C}(e_1, e_2, e_3, e_4), \mathcal{C}(e_3, e_4, e_5), \mathcal{C}(e_1, e_2, e_6), \mathcal{C}(e_1, e_4, e_6), \mathcal{C}(e_2, e_3, e_6), \mathcal{C}(e_3, e_5, e_6), \mathcal{C}(e_4, e_5, e_6)\}$$

is a complete fan of \mathbb{Z}^3 . We denote by X the toric variety of fan \mathbb{F} .

Note that since \mathbb{F} contains a non-simplicial cone, the variety X is not \mathbb{Q} -factorial.

Denote by X_1, X_2, X_3, X_4, X_5 and X_6 the $(\mathbb{C}^*)^3$ -stable irreducible divisors respectively associated to the rays of \mathbb{F} generated by e_1, e_2, e_3, e_4, e_5 and e_6 . We can compute that

$$\text{Pic}(X)_{\mathbb{Q}} := \left\{ \sum_{i=1}^6 a_i X_i \mid \begin{array}{l} a_1 - a_2 + a_3 - a_4 = 0 \\ a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Q} \end{array} \right\} \Big/ \left\langle \begin{array}{l} X_1 - X_2 - X_3 + X_4, \\ X_1 + X_2 - X_3 - X_4 - X_5, \\ X_1 + X_2 + 2X_3 + 2X_4 + X_5 - X_6 \end{array} \right\rangle.$$

In particular, $-K_X = \sum_{i=1}^6 X_i$ is \mathbb{Q} -Cartier.

We know that $NE(X)$ is generated by the classes of $(\mathbb{C}^*)^3$ -stable (and rational) curves $C_{12}, C_{14}, C_{16}, C_{23}, C_{26}, C_{34}, C_{35}, C_{36}, C_{45}, C_{46}, C_{56}$, where C_{ij} denotes the $(\mathbb{C}^*)^3$ -stable curve associated to the 2-codimensional cone of \mathbb{F} generated by e_i and e_j .

We choose the basis $(X_1 + X_2, X_1 - X_3)$ of $\text{Pic}(X)_{\mathbb{Q}}$ and we compute the classes of these curves in the corresponding dual basis:

$$\begin{array}{llll} [C_{12}] & = & (\frac{1}{3}, 0), & [C_{14}] & = & (\frac{1}{6}, 0), & [C_{16}] & = & (\frac{1}{2}, 0), & [C_{23}] & = & (\frac{1}{6}, 0), \\ [C_{26}] & = & (\frac{1}{2}, 0), & [C_{34}] & = & (\frac{1}{3}, \frac{1}{2}), & [C_{35}] & = & (0, -\frac{1}{2}), & [C_{36}] & = & (\frac{1}{2}, \frac{1}{2}), \\ [C_{45}] & = & (0, -\frac{1}{2}), & [C_{46}] & = & (\frac{1}{2}, \frac{1}{2}), & [C_{56}] & = & (0, -1). \end{array}$$

In particular, the cone $NE(X)$ is generated by $(2, 3)$ and $(0, -1)$, ie by $[C_{34}]$ and $[C_{35}]$.

Note also that $-K_X$ is linearly equivalent to $5(X_1 + X_2) - 2(X_1 - X_3)$ so that it is positive on all effective curves. Hence, in order to run the $(\mathbb{Q}$ -Gorenstein) MMP, we need to choose one of the two extremal rays in $NE(X)_{K_X < 0}$.

- First, consider the contraction $\phi : X \rightarrow Y$ of the extremal ray generated by $[C_{35}]$. We remark that this ray contains the classes of C_{35}, C_{45} , and C_{56} . Then Y is the 3-dimensional toric variety whose fan is

$$\mathbb{F}_Y := \{\mathcal{C}(e_1, e_2, e_3, e_4), \mathcal{C}(e_3, e_4, e_6), \mathcal{C}(e_1, e_2, e_6), \mathcal{C}(e_1, e_4, e_6), \mathcal{C}(e_2, e_3, e_6)\}.$$

We still denote by X_1, X_2, X_3, X_4 and X_6 the $(\mathbb{C}^*)^3$ -stable irreducible divisors of Y . And we compute that

$$\text{Pic}(Y)_{\mathbb{Q}} := \left\{ \sum_{i=1}^4 a_i X_i + a_6 X_6 \mid \begin{array}{l} a_1 - a_2 + a_3 - a_4 = 0 \\ a_1, a_2, a_3, a_4, a_6 \in \mathbb{Q} \end{array} \right\} \Big/ \left\langle \begin{array}{l} X_1 - X_2 - X_3 + X_4, \\ X_1 + X_2 - X_3 - X_4, \\ X_1 + X_2 + 2X_3 + 2X_4 - X_6 \end{array} \right\rangle.$$

In particular, $-K_Y = \sum_{i=1}^4 X_i + X_6$ is \mathbb{Q} -Cartier (and generates $\text{Pic}(Y)_{\mathbb{Q}}$). The contraction ϕ is a \mathbb{Q} -Gorenstein divisorial contraction.

• Now, consider the contraction $\phi : X \rightarrow Y$ of the extremal ray generated by $[C_{34}]$. Then Y is the 3-dimensional toric variety whose fan is

$$\mathbb{F}_Y := \{\mathcal{C}(e_1, e_2, e_3, e_4, e_5), \mathcal{C}(e_1, e_2, e_6), \mathcal{C}(e_1, e_4, e_6), \mathcal{C}(e_2, e_3, e_6), \mathcal{C}(e_3, e_5, e_6), \mathcal{C}(e_4, e_5, e_6)\}.$$

We still denote by X_1, X_2, X_3, X_4, X_5 and X_6 the $(\mathbb{C}^*)^3$ -stable irreducible divisors of Y . And we compute that

$$\text{Pic}(Y)_{\mathbb{Q}} := \left\{ \sum_{i=1}^6 a_i X_i \mid \begin{array}{l} a_1 - a_2 + a_3 - a_4 = 0 \\ a_1 - 3a_2 + 4a_3 - 6a_5 = 0 \\ a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Q} \end{array} \right\} / \left\langle \begin{array}{l} X_1 - X_2 - X_3 + X_4, \\ X_1 + X_2 - X_3 - X_4 - X_5, \\ X_1 + X_2 + 2X_3 + 2X_4 + X_5 - X_6 \end{array} \right\rangle.$$

In particular, $-K_Y = \sum_{i=1}^6 X_i$ is not \mathbb{Q} -Cartier.

The \mathbb{Q} -Gorenstein flip of ϕ is given by the \mathbb{Q} -factorial 3-dimensional toric variety X^+ whose fan is

$$\mathbb{F}_{X^+} := \{\mathcal{C}(e_1, e_2, e_3), \mathcal{C}(e_1, e_3, e_5), \mathcal{C}(e_1, e_4, e_5), \\ \mathcal{C}(e_1, e_2, e_6), \mathcal{C}(e_1, e_4, e_6), \mathcal{C}(e_2, e_3, e_6), \mathcal{C}(e_3, e_5, e_6), \mathcal{C}(e_4, e_5, e_6)\},$$

and the $(\mathbb{C}^*)^3$ -equivariant map $\phi^+ : X^+ \rightarrow Y$. We still denote by X_1, X_2, X_3, X_4, X_5 and X_6 the $(\mathbb{C}^*)^3$ -stable irreducible divisors of X^+ . In the basis dual to (X_1, X_2, X_3) , the classes of the $(\mathbb{C}^*)^3$ -stable curves of X^+ are

$$\begin{array}{llll} [C_{12}] & = & (\frac{1}{3}, 0, \frac{1}{3}), & [C_{13}] & = & (-\frac{1}{2}, \frac{1}{2}, -\frac{2}{3}), & [C_{14}] & = & (\frac{1}{2}, 0, 0), & [C_{15}] & = & (-\frac{1}{3}, 0, \frac{1}{3}), \\ [C_{16}] & = & (0, \frac{1}{2}, 0), & [C_{23}] & = & (\frac{1}{6}, 0, \frac{1}{6}), & [C_{26}] & = & (\frac{1}{2}, 0, \frac{1}{2}), & [C_{35}] & = & (\frac{1}{3}, 0, \frac{1}{3}), \\ [C_{36}] & = & (0, \frac{1}{2}, -\frac{1}{2}), & [C_{45}] & = & (1, 0, 0), & [C_{46}] & = & (\frac{1}{2}, 0, 0), & [C_{56}] & = & (0, 0, 1). \end{array}$$

We deduce that the cone $NE(X^+)$ is generated by $(-3, 3, -4)$, $(1, 0, 0)$ and $(-1, 0, 1)$. Moreover, since $-K_{X^+}$ is linearly equivalent to $3X_1 + 5X_2 + 2X_3$, there are two extremal rays in $NE(X^+)_{K_{X^+} > 0}$, respectively generated by $[C_{13}]$ and $[C_{15}]$. The map ϕ^+ is the contraction of the 2-dimensional face of $NE(X)$ generated by $[C_{13}]$ and $[C_{15}]$.

For more examples, we refer to [Pas13] where a \mathbb{Q} -Gorenstein flip of a contraction that contracts a divisor is given.

6 Log \mathbb{Q} -Gorenstein MMP

As for \mathbb{Q} -factorial varieties, we can run a \mathbb{Q} -Gorenstein MMP for klt pairs (X, D) .

Let X be a normal variety and let D be an effective \mathbb{Q} -divisor such that $K_X + D$ is \mathbb{Q} -Cartier.

Definition 4. The pair (X, D) is said to be klt (Kawamata log terminal) if there exists a desingularization $\sigma : V \rightarrow X$ of X such that $K_V = \sigma^*(K_X + D) + \sum_{i \in I} a_i E_i$ where the E_i 's are irreducible divisors of V and for any $i \in I$, $a_i > -1$.

Remark 2. 1. If a pair (X, D) is klt, then the above property is true for every desingularization of X .

2. The condition "for any $i \in I$, $a_i > -1$ " can be replaced by: $[D] = 0$ and for any $i \in I$ such that E_i is exceptional for σ , $a_i > -1$.

Suppose now that (X, D) is klt.

By [KMM87], the Contraction Theorem and the Cone Theorem are still valid. In particular, for any ray R of $NE(X)_{K_X + D < 0}$, there exists a unique contraction $\phi : X \rightarrow Y$, such that, for any curve C of X , $\phi(C)$ is a point if and only if the class of C is in R . Moreover, we have equivalent results of Theorems 6 and 7.

Theorem 9. *We denote by D_Y the \mathbb{Q} -divisor ϕ_*D of Y . And we fix $m \geq 1$ such that $m(K_X + D)$ is Cartier.*

1. *If $K_Y + D_Y$ is \mathbb{Q} -Cartier, then the pair (Y, D_Y) is klt and $E := K_X + D - \phi^*(K_Y + D_Y)$ is an exceptional and effective \mathbb{Q} -Cartier divisor such that $E \cdot C < 0$ for any curve C of X contracted by ϕ .*
2. *The sheaf of \mathcal{O}_Y -algebras $\mathcal{A} := \bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lm(K_X + D))$ equals $\bigoplus_{l \geq 0} \mathcal{O}_Y(lm(K_Y + D_Y))$.*
3. *If $K_Y + D_Y$ is not \mathbb{Q} -Cartier, \mathcal{A} is finitely generated if and only if there exists a small contraction $\phi^+ : X^+ \rightarrow Y$ such that the pair $(X^+, (\phi^+)^{-1}D_Y)$ is klt and for any curve C^+ of X^+ contracted by ϕ^+ , $(K_{X^+} + (\phi^+)^{-1}D_Y) \cdot C^+ > 0$. In that case, X^+ is $\text{Proj}(\mathcal{A})$ over Y .*

The proof is very similar to the proofs of Theorems 6 and 7. The key of the proof of (1) is that, as in Corollary 5, the divisor $K_X + D - \phi^*(K_Y + D_Y)$ is effective. Lemma 4, applied to $m(K_X + D)$ (and $m(K_Y + D_Y)$) directly gives (2). And to prove (3), we do the same proof as in Theorem 7, by replacing K_X by $K_X + D$, K_Y by $K_Y + D_Y$ and K_{X^+} by $K_{X^+} + (\phi^+)^{-1}D_Y$ (excepting the last paragraph, which is not necessary).

7 Open questions

The same questions as in the classical MMP can be done.

Question 1. Do \mathbb{Q} -Gorenstein flips always exist? Or equivalently, in the case where Y is not \mathbb{Q} -Gorenstein, is $\mathcal{A} := \bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lmK_X) = \bigoplus_{l \geq 0} \mathcal{O}_Y(lmK_Y)$ finitely generated as sheaf of \mathcal{O}_Y -algebras?

Question 2. Are sequences of \mathbb{Q} -Gorenstein flips always finite?

We can answer positively these two questions in the case of spherical varieties.

Let G be a connected reductive algebraic group (over \mathbb{C}). A normal G -variety is spherical if there exists an open orbit in X under the action of a Borel subgroup of G .

Let H be a spherical subgroup of G (ie such that there exists a Borel subgroup of G satisfying that BH is open in G). A G/H -embedding is a normal G -variety containing an open G -orbit isomorphic to G/H . (A G/H -embedding is a spherical G -variety, and inversely a spherical G -variety is a G/H -embedding for some spherical subgroup H of G .)

Proposition 10. 1. ([Bri93, Lemme 4.3]) *Let X and Y be two spherical G -varieties and let ϕ be a proper G -equivariant morphism. Then for any Cartier divisor D of X , the \mathcal{O}_Y -algebra $\bigoplus_{l \geq 0} \phi_* \mathcal{O}_X(lD)$ is finitely generated.*

2. *Let X be a G/H -embeddings (ie a normal G -variety containing an open G -orbit isomorphic to G/H). There are only finitely many varieties Z that can be obtained from X by \mathbb{Q} -Gorenstein flips (and these varieties are still G/H -embeddings). Moreover, there exists a common desingularization for all these varieties, ie there exist a smooth G/H -embedding V and birational proper G -equivariant morphisms $\sigma_Z : V \rightarrow Z$, for any Z .*

The second part of the proposition is a consequence of the classification of G/H -embeddings in terms of colored fans and the fact that a \mathbb{Q} -Gorenstein flip adds no divisor.

Hence, in the family of spherical varieties, we immediately get the existence of \mathbb{Q} -Gorenstein flips, and the finiteness of sequences of \mathbb{Q} -Gorenstein flips is a consequence of 2 of Proposition 10 and Corollary 8.

Question 3. Let $\phi : X \rightarrow Y$ be a contraction of an extremal ray of $NE(X)_{K_X < 0}$ such that $\dim(Y) < \dim(X)$. If X is \mathbb{Q} -Gorenstein, Y is unfortunately not necessarily \mathbb{Q} -Gorenstein. But can we give some properties about the fibers of these contractions?

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