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MINIMAL MASS BLOW UP SOLUTIONS FOR A DOUBLE POWER NONLINEAR SCHRÖDINGER EQUATION

STEFAN LE COZ, YVAN MARTEL, AND PIERRE RAPHAËL

Abstract. We consider a nonlinear Schrödinger equation with double power nonlinearity

$$i\partial_t u + \Delta u + |u|^{\frac{4}{d}}u + \epsilon|u|^{p-1}u = 0, \quad \epsilon \in \{-1, 0, 1\}, \quad 1 < p < \frac{4}{d}$$

in $\mathbb{R}^d$ ($d = 1, 2, 3$). Classical variational arguments ensure that $H^1(\mathbb{R}^d)$ data with $\|u_0\|_2 < \|Q\|_2$ lead to global in time solutions, where Q is the ground state of the mass critical problem ($\epsilon = 0$). We are interested by the threshold dynamic $\|u_0\|_2 = \|Q\|_2$ and in particular by the existence of finite time blow up minimal solutions. For $\epsilon = 0$, such an object exists thanks to the explicit conformal symmetry, and is in fact unique from the seminal work [22]. For $\epsilon = -1$, simple variational arguments ensure that minimal mass data lead to global in time solutions. We investigate in this paper the case $\epsilon = 1$, exhibiting a new class of minimal blow up solutions with blow up rates deeply affected by the double power nonlinearity. The analysis adapts the recent approach [31] for the construction of minimal blow up elements.

1. Introduction

We consider the following double power nonlinear Schrödinger equation in $\mathbb{R}^d$

$$\text{(NLS)} \quad \begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}}u + \epsilon|u|^{p-1}u = 0, \\ u_{|t=0} = u_0, \end{cases} \qquad 1 < p < 1 + \frac{4}{d}, \quad \epsilon \in \{-1, 0, 1\}. \tag{1}$$

This model corresponds to a subcritical perturbation of the classical mass critical problem $\epsilon = 0$ which rules out the scaling symmetry of the problem. It is well-known (see e.g [6] and the references therein) that for any $u_0 \in H^1(\mathbb{R}^d)$, there exists a unique maximal solution $u \in \mathcal{C}((-T_\star, T^\star), H^1(\mathbb{R}^d)) \cap \mathcal{C}^1((-T_\star, T^\star), H^{-1}(\mathbb{R}^d))$ of (1). Moreover, the mass (i.e. L^2 norm) and energy E of the solution are conserved by the flow where:

$$E(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2 + \frac{4}{d}}\|u\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} - \epsilon\frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Moreover, there holds the blow up criterion:

$$T^\star < +\infty \ \text{ implies } \ \lim_{t \uparrow T^\star} \|\nabla u(t)\|_2 = +\infty. \tag{2}$$

In this paper, we are interested in the derivation of a sharp global existence criterion for (1) in connection with the existence of *minimal mass blow up solutions* of (1).

1.1. The mass critical problem.

Let us briefly recall the structure of the mass critical problem $\epsilon = 0$. In this case, the scaling symmetry

$$u_\lambda(t, x) = \lambda^{\frac{d}{2}}u(\lambda^2 t, \lambda x)$$

acts on the set of solutions and leaves the mass invariant

$$\|u_\lambda(t, \cdot)\|_2 = \|u(\lambda^2 t, \cdot)\|_2.$$

From variational argument [32], the unique ([3, 14]) up to symmetry ground state solution to

$$-\Delta Q + Q - |Q|^{\frac{4}{d}}Q = 0, \quad Q \in H^1(\mathbb{R}^d), \quad Q > 0, \quad Q \text{ radial}$$

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attains the best constant in the Gagliardo-Nirenberg inequality

$$\|u\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} \leq C\|u\|_2^{\frac{4}{d}}\|\nabla u\|_2^2,$$

so that

$$\forall u \in H^1(\mathbb{R}^d), \quad E_{\rm crit} = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2+\frac{4}{d}}\|u\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} \geq \frac{1}{2}\|\nabla u\|_2^2\left[1-\left(\frac{\|u\|_2}{\|Q\|_2}\right)^{\frac{4}{d}}\right]. \tag{3}$$

Together with the conservation of mass and energy and the blow up criterion (2), this implies the global existence of all solutions with data $\|u_0\|_2 < \|Q\|_2$. In fact, there holds scattering, see [10] and references therein.

At the threshold $\|u_0\|_2 = \|Q\|_2$, the pseudo-conformal symmetry

$$\frac{1}{|t|^{\frac{d}{2}}}u\left(\frac{1}{t},\frac{x}{t}\right)e^{i\frac{|x|^2}{4t}} \tag{4}$$

applied to the solitary wave solution $u(t,x) = Q(x)e^{it}$ yields the existence of the following explicit minimal blow up solution

$$S(t,x) = \frac{1}{|t|^{\frac{d}{2}}}Q\left(\frac{x}{|t|}\right)e^{-i\frac{|x|^2}{4|t|}}e^{\frac{i}{|t|}}, \quad \|S(t)\|_2 = \|Q\|_2, \quad \|\nabla S(t)\|_2 \underset{t\sim 0^-}{\sim} \frac{1}{|t|}. \tag{5}$$

From [22], minimal blow up elements are *classified* in $H^1(\mathbb{R}^d)$ in the following sense

$$\|u(t)\|_2 = \|Q\|_2 \ \text{ and } \ T^* < +\infty \ \text{ imply } \ u \equiv S$$

up to the symmetries of the flow. Note that the minimal blow up dynamic (5) can be extended to the super critical mass case $\|u_0\|_2 > \|Q\|_2$ (see [5]) and that it corresponds to an unstable threshold dynamics between global in time scattering solutions and finite time blow up solutions in the *stable* blow up regime

$$\|\nabla u(t)\|_2 \underset{t\sim T^*}{\sim} \sqrt{\frac{\log|\log|T^*-t||}{T^*-t}}. \tag{6}$$

We refer to [26] and references therein for an overview of the existing literature for the L^2 critical blow up problem.

1.2. **The case $\epsilon = -1$.** Let us now consider the case of a defocusing perturbation. First, there are no solitary waves with subcritical mass $\|u_0\|_2 < \|Q\|_2$ from a standard Pohozaev integration by parts argument. At the threshold, we claim:

Lemma 1 (Global existence at threshold for $\epsilon = -1$)**.** *Let $\epsilon = -1$. Let $u_0 \in H^1(\mathbb{R}^d)$ with $\|u_0\|_2 = \|Q\|_2$, then the solution of* (1) *is global and bounded in $H^1(\mathbb{R}^d)$.*

The proof follows from standard concentration compactness argument, see Appendix A. The global existence criterion of Lemma 1 is sharp in the sense that for all $\alpha^* > 0$, we can build an $H^1(\mathbb{R}^d)$ finite time blow up solution to (7) with $\|u_0\|_2 = \|Q\|_2 + \alpha^*$ and blow up speed given by the log-log law (6). This is a consequence of the strong structural stability of the log log regime and the proof would follow the lines of [28, 29, 30].

1.3. **The case $\epsilon = 1$.** We now turn to the case $\epsilon = 1$ for the rest of the paper, i.e. we consider the model

$$i\partial_t u + \Delta u + |u|^{\frac{4}{d}}u + |u|^{p-1}u = 0 \quad \text{where} \quad 1 < p < 1 + \frac{4}{d}. \tag{7}$$

First, from mass and energy conservation, using (3) and (72), $H^1(\mathbb{R}^d)$ solutions with $\|u_0\|_2 < \|Q\|_2$ are global and bounded in $H^1(\mathbb{R}^d)$. However, large time scattering is not true in general, even for small L^2 solutions, since there exist arbitrarily small solitary waves.

Lemma 2 (Small solitary waves)**.** *For all $M \in (0, \|Q\|_2)$, there exists $\omega(M) > 0$ and a Schwartz radially symmetric solution of*

$$\Delta Q_M - \omega(M)Q_M + Q_M^{1+\frac{4}{d}} + Q_M^p = 0, \quad \|Q_M\|_2 = M.$$

The proof follows from classical variational methods, see Appendix B.

The main result of this paper is the existence of a minimal mass blow up solution for (7), in contrast with the defocusing case $\epsilon = -1$.

Theorem 1 (Existence of a minimal blow up element)**.** *Let $d = 1, 2, 3$ and $1 < p < 1 + \frac{4}{d}$. Then for all energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric Cauchy data $u(t_0) \in H^1(\mathbb{R}^d)$ with*

$$\|u(t_0)\|_2 = \|Q\|_2, \quad E(u(t_0)) = E_0,$$

such that the corresponding solution $u(t)$ of (7) blows up at time $T^* = 0$ with speed:

$$\|\nabla u(t)\|_2 = \frac{C(p) + o_{t\uparrow 0}(1)}{|t|^\sigma} \tag{8}$$

for some universal constants

$$\sigma = \frac{4}{4 + d(p-1)} \in \left(\tfrac{1}{2}, 1\right), \quad C(p) > 0.$$

Comments on the result.

1. On the existence of minimal elements. Since the pioneering work [22], it has long been believed that the existence of a minimal blow up bubble was related to the exceptional pseudo conformal symmetry (4), or at least to the existence of a sufficiently sharp approximation of it, see [2, 16]. However, a new methodology to construct minimal mass elements for a inhomogeneous (NLS) problem, *non perturbative* of critical (NLS), was developed in [31], and later successfully applied to problems without any sort of pseudo conformal symmetry, [4, 12, 19]. More generally, the heart of the matter is to be able to compute the trajectory of the solution on the soliton manifold, see [13, 18] for related problems for two solitary waves motion. The present paper adapts this approach which relies on the *direct* computation of the blow up speed and the control of non dispersive bubbles as in [15].

Observe that the blow up speed (8) is quite surprising since it approaches the self simiar blow up speed $|t|^{-\frac{1}{2}}$ as $p \to \left(1 + \frac{4}{d}\right)^-$.

2. Uniqueness. A delicate question investigated in [4, 19, 31] is the uniqueness of the minimal blow up element. Such a uniqueness statement should involve Galilean drifts since the Galilean symmetry applied to (7) is an L^2 isometry and automatically induces minimal elements with non trivial momentum. Uniqueness issues lie within the general question of classifying the compact elements of the flow in the Kenig-Merle road map [11]. A more limited question is to determine the global behavior of the minimal element for negative time, which is poorly understood in general. Here, at least in the case $E_0 \geq 0$, one can see from Virial type estimates that the solution is global in negative time.

3. Detailed structure of the singular bubble. The analysis provides the following detailed structure of the blow up bubble

$$u(t,x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} Q\left(\frac{x}{\lambda(t)}\right) e^{-i\sigma \frac{|x|^2}{4t}} e^{i\gamma(t)} + v\,(t,x) \tag{9}$$

where Q is the mass critical ground state, and

$$\lim_{t\to 0} \|v(t)\|_2 = 0, \quad \lambda(t) \sim C_p |t|^\sigma \quad \text{as } t \to 0^-,$$

for some constant $C_p > 0$. Note also that the dimension restriction $d \in \{1, 2, 3\}$ is for the sake of simplicity but not essential.

The construction of the minimal blow up element for (1) can be viewed as part of a larger program of understanding what kind of blow up speeds are possible for (NLS) type models. Let us repeat that log-log type solutions with super critical mass can be constructed for (1), but then the question becomes: do these examples illustrate all possible blow up types, at least near the ground state profile? The recent series of works [19, 20, 21] for the mass critical gKdV equation indicate that this is a delicate problem, and that the role played by the topology used to measure the perturbation is essential. More generally, symmetry breaking perturbations are very common in nonlinear analysis, and while they are expected to be lower order for generic stable blow up dynamics, our analysis shows that they can dramatically influence the structure of unstable threshold dynamics such as in our case minimal blow up bubbles.

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1.4. **Notation.** Let us collect the main notation used throughout the paper. For the sake of simplicity, we work in the radial setting only. The L^2 scalar product and L^q norm ($q \geq 1$) are denoted by

$$(u,v)_2 = \mathrm{Re}\left(\int_{\mathbb{R}^d} u(x)\bar{v}(x)dx\right), \quad \|u\|_q = \left(\int_{\mathbb{R}^d} |u|^q\right)^{\frac{1}{q}}.$$

We fix the notation:

$$f(z) = |z|^{\frac{4}{d}}z; \quad g(z) = |z|^{p-1}z; \quad F(z) = \frac{1}{\frac{4}{d}+2}|z|^{\frac{4}{d}+2}; \quad G(z) = \frac{1}{p+1}|z|^{p+1}.$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we denote the differential of these functions by df, dg, dF and dG. Let Λ be the generator of L^2-scaling i.e.

$$\Lambda = \frac{d}{2} + y\cdot\nabla.$$

The linearized operator close to Q comes as a matrix

$$L_+ := -\Delta + 1 - \left(1+\frac{4}{d}\right)Q^{\frac{4}{d}}, \qquad L_- := -\Delta + 1 - Q^{\frac{4}{d}}.$$

and the generalized kernel of

$$\begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}$$

is non-degenerate and spanned by the symmetries of the problem (see [14, 33] for the original results and [8] for a short proof). It is completely described in $H^1_{\mathrm{rad}}(\mathbb{R}^d)$ by the relations (we define ρ as the unique radial solution to $L_+\rho = |y|^2Q$)

$$L_-Q = 0, \quad L_+\Lambda Q = -2Q, \quad L_-|y|^2Q = -4\Lambda Q, \quad L_+\rho = |y|^2Q. \tag{10}$$

Denote by $\mathcal{Y}$ the set of radially symmetric functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\forall \alpha \in \mathbb{N}^d, \quad \exists C_\alpha,\ \kappa_\alpha > 0,\ \forall x \in \mathbb{R}^d, \quad |\partial^\alpha f(x)| \leq C_\alpha(1+|x|)^{\kappa_\alpha}Q(x).$$

It follows from the kernel properties of L_+ and L_-, and from well-known properties of the Helmholtz kernel (see [1] for the properties of Helmholtz kernel (i.e. Bessel and Hankel functions) and [9, Appendix A] or proof of Lemma 3.2 in [27] for related arguments) that

$$\forall g \in \mathcal{Y},\ \exists f_+ \in \mathcal{Y},\ L_+f_+ = g, \tag{11}$$

$$\forall g \in \mathcal{Y},\ (g,Q)_2 = 0,\ \exists f_- \in \mathcal{Y},\ L_-f_- = g. \tag{12}$$

It is also well known (see e.g. [23, 24, 31, 34]) that L_+ and L_- verify the following coercivity property: *there exists $\mu > 0$ such that for all $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1_{\mathrm{rad}}(\mathbb{R}^d)$,*

$$\langle L_+\varepsilon_1, \varepsilon_1\rangle + \langle L_-\varepsilon_2, \varepsilon_2\rangle \geq \mu\|\varepsilon\|^2_{H^1} - \frac{1}{\mu}\left((\varepsilon_1, Q)_2^2 + \left(\varepsilon_1, |y|^2Q\right)_2^2 + (\varepsilon_2, \rho)_2^2\right). \tag{13}$$

Throughout the paper, C denotes various positive constants whose exact values may vary from line to line but are of no importance in the analysis. When an inequality is true up to such a constant, we also use the notation $\lesssim$, $\gtrsim$ or $\approx$.

2. Construction of the blow-up profile

In this section, we define the blow-up profile which is relevant to construct the minimal mass solution – see Proposition 3 below.

2.1. **Blow up profile.** Let us start with some heuristic arguments justifying the construction. As usual in blow up contexts, we look for a solution of the following form, with rescaled variables (s,y):

$$u(t,x) = \frac{1}{\lambda^{\frac{d}{2}}(s)}w(s,y)e^{i\gamma(s) - i\frac{b(s)|y|^2}{4}}, \qquad \frac{ds}{dt} = \frac{1}{\lambda^2}, \qquad y = \frac{x}{\lambda(s)},$$

where the function w, and the time dependent parameters $\lambda > 0$, b and γ are to be determined satisfying the following equation

$$iw_s + \Delta w - w + f(w) + \lambda^\alpha g(w) \\ - i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda w + (1-\gamma_s)w + (b_s + b^2)\frac{|y|^2}{4}w - b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{|y|^2}{2}w = 0, \quad (14)$$

where

$$\alpha = 2 - \frac{d(p-1)}{2} \in (0,2).$$

Since we look for blow up solutions, the parameter $\lambda(s)$ should converge to zero as $s \to \infty$. Therefore,

$$w(s,y) = Q(y), \qquad b + \frac{\lambda_s}{\lambda} = b_s + b^2 = 1 - \gamma_s = 0 \quad (15)$$

is a solution of (14) at the first order, i.e. when neglecting $\lambda^\alpha |w|^{p-1}w$. However, the first order error term $\lambda^\alpha Q^p$ cannot be neglected in the minimal mass blow up analysis (while it could be neglected easily in the log-log regime where $\lambda \sim e^{-e^{\frac{c}{b}}}$). Therefore, starting from Q, we need to look for a refined blow up ansatz. Actually, to close the analysis for any $\alpha \in (0,2)$, we need to remove error terms at any order of λ^α and b in the equation of w. It is important to note that in the process of constructing the approximate solution, we cannot exactly solve (14) since we need to introduce new terms in the equation (due to degrees of freedom necessary to construct the ansatz) that will modify the modulation equations in (15). These terms (gathered in the time dependent function $\theta(s)$ below) are responsible for the specific blow up law obtained in Theorem 1.

Fix $K \in \mathbb{N}$, $K \gg 1$ ($K > 20/\alpha$ is sufficient in the proof of Theorem 1), and

$$\Sigma_K = \{(j,k) \in \mathbb{N}^2 \mid j + k \le K\}.$$

Proposition 3. *Let $\lambda(s) > 0$ and $b(s) \in \mathbb{R}$ be $\mathcal{C}^1$ functions of s such that $\lambda(s) + |b(s)| \ll 1$.*
(i) Existence of a blow up profile. *For any $(j,k) \in \Sigma_K$, there exist real-valued functions $P^+_{j,k} \in \mathcal{Y}$, $P^-_{j,k} \in \mathcal{Y}$ and $\beta_{j,k} \in \mathbb{R}$ such that $P(s,y) = \tilde{P}_K(y; b(s), \lambda(s))$, where $\tilde{P}_K$ is defined by*

$$\tilde{P}_K(y;b,\lambda) := Q(y) + \sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{(k+1)\alpha}P^+_{j,k}(y) + i\sum_{(j,k)\in\Sigma_K} b^{2j+1}\lambda^{(k+1)\alpha}P^-_{j,k}(y) \quad (16)$$

satisfies

$$i\partial_s P + \Delta P - P + f(P) + \lambda^\alpha g(P) + \theta\frac{|y|^2}{4}P = \Psi_K$$

where $\theta(s) = \tilde{\theta}(b(s), \lambda(s))$,

$$\tilde{\theta}(b,\lambda) = \sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{(k+1)\alpha}\beta_{j,k}$$

and

$$\sup_{y\in\mathbb{R}^d}\left(e^{\frac{|y|}{2}}\left(|\Psi_K(y)| + |\nabla\Psi_K(y)|\right)\right) \lesssim \lambda^\alpha\left(\left|b + \frac{\lambda_s}{\lambda}\right| + \left|b_s + b^2 - \theta\right|\right) + (|b|^2 + \lambda^\alpha)^{K+2}. \quad (17)$$

(ii) Rescaled blow up profile. *Let*

$$P_b(s,y) := P(s,y)e^{-i\frac{b(s)|y|^2}{4}}. \quad (18)$$

Then

$$i\partial_s P_b + \Delta P_b - P_b + f(P_b) + \lambda^\alpha g(P_b) - i\frac{\lambda_s}{\lambda}\Lambda P_b \\ = -i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda P_b + (b_s + b^2 - \theta)\frac{|y|^2}{4}P_b + \Psi_K e^{-i\frac{b|y|^2}{4}}. \quad (19)$$

(iii) Mass and energy properties of the blow up profile. *Let*

$$P_{b,\lambda,\gamma}(s,y) = \frac{1}{\lambda^{\frac{d}{2}}}P_b\left(s, \frac{x}{\lambda}\right)e^{i\gamma}.$$

Then,

$$\left|\frac{d}{ds}\int |P_{b,\lambda,\gamma}|^2\right| \lesssim \lambda^\alpha \left(\left|b+\frac{\lambda_s}{\lambda}\right| + \left|b_s+b^2-\theta\right|\right) + (|b|^2+\lambda^\alpha)^{K+2}, \tag{20}$$

$$\left|\frac{d}{ds}E(P_{b,\lambda,\gamma})\right| \lesssim \frac{1}{\lambda^2}\left(\left|b+\frac{\lambda_s}{\lambda}\right| + \left|b_s+b^2-\theta\right| + (|b|^2+\lambda^\alpha)^{K+2}\right). \tag{21}$$

Moreover, for any $(j,k)\in\Sigma_K$, there exist $\eta_{j,k}\in\mathbb{R}$ such that

$$\left|E(P_{b,\lambda,\gamma}) - \frac{\int |y|^2Q^2}{8}\mathcal{E}(b,\lambda)\right| \lesssim \frac{(b^2+\lambda^\alpha)^{K+2}}{\lambda^2}, \tag{22}$$

where

$$\mathcal{E}(b,\lambda) = \frac{b^2}{\lambda^2} - \frac{2\beta}{2-\alpha}\lambda^{\alpha-2} + \lambda^{\alpha-2}\sum_{(j,k)\in\Sigma_K, j+k\geq 1} b^{2j}\lambda^{k\alpha}\eta_{j,k}. \tag{23}$$

See a similar construction of a blow up profile at any order of b in [27]. One sees in (19) the impact of the subcritical nonlinearity $g(u)$ on the blow up law $b_s+b^2-\theta=0$, which differs from the unperturbed equation $b_s+b^2=0$, and leads to leading order to $\lambda^\alpha\approx b^2$, see (32).

Proof of Proposition 3. Proof of (i). For time dependent functions $\lambda(s)>0$, $b(s)$, we set

$$P = Q+\lambda^\alpha Z \quad \text{where} \quad Z = \sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{k\alpha}P^+_{j,k} + i\sum_{(j,k)\in\Sigma_K} b^{2j+1}\lambda^{k\alpha}P^-_{j,k},$$

$$\theta(s) = \sum_{(j,k)\in\Sigma_K} b^{2j}(s)\lambda^{(k+1)\alpha}(s)\beta_{j,k},$$

where $P^+_{j,k}\in\mathcal{Y}$, $P^-_{j,k}\in\mathcal{Y}$ and $\beta_{j,k}$ are to be determined. Set

$$\Psi_K = i\partial_s P + \Delta P - P + |P|^{\frac{4}{d}}P + \lambda^\alpha|P|^{p-1}P + \theta\frac{|y|^2}{4}P.$$

The objective is to choose the unknown functions and parameters so that the error term Ψ_K is controlled as in (17). First,

$$\begin{aligned}
iP_s &= i\frac{\lambda_s}{\lambda}\sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2j}\lambda^{(k+1)\alpha}P^+_{j,k} + ib_s\sum_{(j,k)\in\Sigma_K}2jb^{2j-1}\lambda^{(k+1)\alpha}P^+_{j,k}\\
&\quad - \frac{\lambda_s}{\lambda}\sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2j+1}\lambda^{(k+1)\alpha}P^-_{j,k} - b_s\sum_{(j,k)\in\Sigma_K}(2j+1)b^{2j}\lambda^{(k+1)\alpha}P^-_{j,k}\\
&= -i\sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2j+1}\lambda^{(k+1)\alpha}P^+_{j,k}\\
&\quad - i\left(b^2 - \sum_{(j',k')\in\Sigma_K} b^{2j'}\lambda^{(k'+1)\alpha}\beta_{j',k'}\right)\sum_{(j,k)\in\Sigma_K}2jb^{2j-1}\lambda^{(k+1)\alpha}P^+_{j,k}\\
&\quad + \sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2(j+1)}\lambda^{(k+1)\alpha}P^-_{j,k}\\
&\quad + \left(b^2 - \sum_{(j',k')\in\Sigma_K} b^{2j'}\lambda^{(k'+1)\alpha}\beta_{j',k'}\right)\sum_{(j,k)\in\Sigma_K}(2j+1)b^{2j}\lambda^{(k+1)\alpha}P^-_{j,k} + \Psi^{P_s}
\end{aligned}$$

where

$$\begin{aligned}
\Psi^{P_s} &= \left(\frac{\lambda_s}{\lambda}+b\right)\sum_{(j,k)\in\Sigma_K}(k+1)\alpha b^{2j}\lambda^{(k+1)\alpha}\left(iP^+_{j,k} - bP^-_{j,k}\right)\\
&\quad + \left(b_s+b^2-\theta\right)\sum_{(j,k)\in\Sigma_K} b^{2j-1}\lambda^{(k+1)\alpha}\left(2jiP^+_{j,k} - (2j+1)bP^-_{j,k}\right).
\end{aligned} \tag{24}$$

We rewrite

$$iP_s = -i \sum_{(j,k)\in\Sigma_K} ((k+1)\alpha + 2j)\, b^{2j+1}\lambda^{(k+1)\alpha} P^+_{j,k} + i\sum_{j,k\geq 0} b^{2j+1}\lambda^{(k+1)\alpha} F^{P_s,-}_{j,k} + \sum_{j,k\geq 0} b^{2j}\lambda^{(k+1)\alpha} F^{P_s,+}_{j,k} + \Psi^{P_s},$$

where for $j,k\geq 0$, $F^{P_s,\pm}_{j,k}$ depends on various functions $P^\pm_{j',k'}$ and parameters $\beta_{j',k'}$ for $(j',k')\in\Sigma_K$ such that either $k'\leq k-1$ and $j'\leq j+1$ or $k'\leq k$ and $j'\leq j-1$. Only a finite number of these functions are nonzero.

Next, using $\Delta Q - Q + Q^{\frac{4}{d}+1} = 0$, we get

$$\Delta P - P + |P|^{\frac{4}{d}}P = -\sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{(k+1)\alpha} L_+ P^+_{j,k} - i\sum_{(j,k)\in\Sigma_K} b^{2j+1}\lambda^{(k+1)\alpha} L_- P^-_{j,k} + f(Q+\lambda^\alpha Z) - f(Q) - \lambda^\alpha df(Q)Z.$$

Let

$$\begin{aligned}\Psi^f &= f(Q+\lambda^\alpha Z) - \sum_{k=0}^K d^k f(Q)(\lambda^\alpha Z,\dots,\lambda^\alpha Z)\\ &= |Q+\lambda^\alpha Z|^{\frac{4}{d}}(Q+\lambda^\alpha Z) - Q^{\frac{4}{d}+1}\left(1+\sum_{n=1}^{K+1}\frac{\left(\frac{2}{d}+1\right)\left(\frac{2}{d}\right)\dots\left(\frac{2}{d}-n+2\right)}{n!}\left(\frac{\lambda^\alpha Z}{Q}\right)^n\right)\\ &\qquad\times\left(1+\sum_{n'=1}^{K+1}\frac{\left(\frac{2}{d}\right)\left(\frac{2}{d}-1\right)\dots\left(\frac{2}{d}-n'+1\right)}{n'!}\left(\frac{\lambda^\alpha\overline{Z}}{Q}\right)^{n'}\right). \qquad (25)\end{aligned}$$

Then

$$\Delta P - P + |P|^{\frac{4}{d}}P = -\sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{(k+1)\alpha} L_+ P^+_{j,k} - i\sum_{(j,k)\in\Sigma_K} b^{2j+1}\lambda^{(k+1)\alpha} L_- P^-_{j,k} + i\sum_{j\geq 0,k\geq 1} b^{2j+1}\lambda^{(k+1)\alpha} F^{f,-}_{j,k} + \sum_{j,k\geq 0} b^{2j}\lambda^{(k+1)\alpha} F^{f,+}_{j,k} + \Psi^f.$$

where for $j,k\geq 0$, $F^{f,\pm}_{j,k}$ depends on Q and on various functions $P^\pm_{j',k'}$ for $(j',k')\in\Sigma_K$ such that $k'\leq k-1$ and $j'\leq j$.

Using a similar argument for $\lambda^\alpha|P|^{p-1}P$, we obtain

$$\lambda^\alpha|P|^{p-1}P = i\sum_{j\geq 0,k\geq 1} b^{2j+1}\lambda^{(k+1)\alpha} F^{g,-}_{j,k} + \sum_{j\geq 0,k\geq 1} b^{2j}\lambda^{(k+1)\alpha} F^{g,+}_{j,k} + \Psi^g,$$

where

$$\begin{aligned}\Psi^g = \lambda^\alpha\Bigg(&|Q+\lambda^\alpha Z|^{p-1}(Q+\lambda^\alpha Z) - Q^{\frac{p}{2}+1}\left(1+\sum_{n=1}^{K+1}\frac{\left(\frac{p}{2}\right)\left(\frac{p}{2}-1\right)\dots\left(\frac{p}{2}-n+1\right)}{n!}\left(\frac{\lambda^\alpha Z}{Q}\right)^n\right)\\ &\times\left(1+\sum_{n=1}^{K+1}\frac{\left(\frac{p}{2}-1\right)\left(\frac{p}{2}-2\right)\dots\left(\frac{p}{2}-n\right)}{n!}\left(\frac{\lambda^\alpha\overline{Z}}{Q}\right)^n\right)\Bigg),\end{aligned}$$

and where for $j,k\geq 0$, $F^{g,\pm}_{j,k}$ depends on Q and on various functions $P^\pm_{j',k'}$ for $(j',k')\in\Sigma_K$ such that $k'\leq k-1$ and $j'\leq j$.

Finally,

$$\theta\frac{|y|^2}{4}P = \left(\sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{(k+1)\alpha}\beta_{j,k}\right)\frac{|y|^2}{4}Q + i\sum_{j,k\geq 0} b^{2j+1}\lambda^{(k+1)\alpha} F^{\theta,-}_{j,k} + \sum_{j,k\geq 0} b^{2j}\lambda^{(k+1)\alpha} F^{\theta,+}_{j,k},$$

where $F^{\theta,\pm}_{j,k}$ depends on Q and on various functions $P^{\pm}_{j',k'}$ or parameters $\beta_{j',k'}$ for $(j',k') \in \Sigma_K$ such that $k' \le k-1$ and $j' \le j$.
Combining these computations, we obtain

$$\begin{aligned}\Psi_K = &- \sum_{(j,k)\in\Sigma_K} b^{2j}\lambda^{(k+1)\alpha}\left(L_+P^+_{j,k} - F^+_{j,k} - \beta_{j,k}|y|^2Q\right)\\ &- i\sum_{(j,k)\in\Sigma_K} b^{2j+1}\lambda^{(k+1)\alpha}\left(L_-P^-_{j,k} - F^-_{j,k} + ((k+1)\alpha+2j)P^+_{j,k}\right)\\ &+ \Psi^{>K} + \Psi^{P_s} + \Psi^f + \Psi^g,\end{aligned}$$

where

$$F^{\pm}_{j,k} = F^{P_s,\pm}_{j,k} + F^{f,\pm}_{j,k} + F^{g,\pm}_{j,k} + F^{\theta,\pm}_{j,k},$$

and

$$\Psi^{>K} = \sum_{j,k>0,\ (j,k)\notin\Sigma_K} b^{2j}\lambda^{(k+1)\alpha}F^+_{j,k} + i\sum_{j,k>0,\ (j,k)\notin\Sigma_K} b^{2j+1}\lambda^{(k+1)\alpha}F^-_{j,k}.$$

(Note that the series in the expression of $\Psi^{>K}$ contains only a finite number of terms.) Now, for any $(j,k)\in\Sigma_K$, we want to choose recursively $P^{\pm}_{j,k}\in\mathcal{Y}$ and $\beta_{j,k}$ to solve the system

$$(S_{j,k}) \qquad \begin{cases} L_+P^+_{j,k} - F^+_{j,k} - \beta_{j,k}|y|^2Q = 0\\ L_-P^-_{j,k} - F^-_{j,k} + ((k+1)\alpha+2j)P^+_{j,k} = 0,\end{cases}$$

where $F^{\pm}_{j,k}$ are source terms depending of previously determined $P^{\pm}_{j',k'}$ and $\beta_{j',k'}$. We argue by a suitable induction argument on the two parameters j and k. For $(j,k)=(0,0)$, we see that the system writes

$$\begin{aligned}&L_+P^+_{0,0} - Q^p - \beta_{0,0}|y|^2Q = 0\\ &L_-P^-_{0,0} + \alpha P^+_{0,0} = 0,\end{aligned}$$

(the term Q^p in the first line is coming from Ψ^g). By (12), for any $\beta_{0,0}\in\mathbb{R}$, there exists a unique $P^+_{0,0}\in\mathcal{Y}$ so that $L_+P^+_{0,0} - Q^p - \beta_{0,0}|y|^2Q = 0$. We choose $\beta_{0,0}\in\mathbb{R}$ so that

$$\left(P^+_{0,0}, Q\right)_2 = -\frac{1}{2}\left(L_+P^+_{0,0}, \Lambda Q\right)_2 = -\frac{1}{2}\left(Q^p + \beta_{0,0}\frac{|y|^2}{4}Q, \Lambda Q\right)_2 = 0$$

(recall from (10) that $L_+\Lambda Q = -2Q$), which gives

$$\beta := \beta_{0,0} = -\frac{4\left(Q^p,\Lambda Q\right)_2}{\left(|y|^2Q,\Lambda Q\right)_2} = \frac{2d(p-1)}{p+1}\frac{\|Q\|^{p+1}_{p+1}}{\|yQ\|^2_2} > 0. \tag{26}$$

By (12), there exists $P^-_{0,0}\in\mathcal{Y}$ (unique up to the addition of cQ) such that $L_-P^-_{0,0} + \alpha P^+_{0,0} = 0$. Now, we assume that for some $(j_0,k_0)\in\Sigma_K$, the following assertion is true:

$H(j_0,k_0)$: for all $(j,k)\in\Sigma_K$ such that either $k<k_0$, or $k=k_0$ and $j<j_0$, the system $(S_{j,k})$ has a solution $(P^+_{j,k}, P^-_{j,k}, \beta_{j,k})$, $P^{\pm}_{j,k}\in\mathcal{Y}$.

In view of the definition of $F^{\pm}_{j_0,k_0}$, $H(j_0,k_0)$ implies in particular that $F^{\pm}_{j_0,k_0}\in\mathcal{Y}$. We now solve the system (S_{j_0,k_0}) as before. By (12), for any $\beta_{j_0,k_0}\in\mathbb{R}$, there exists a unique $P^+_{j_0,k_0}\in\mathcal{Y}$ so that $L_+P^+_{j_0,k_0} - F^+_{j_0,k_0} - \beta_{j_0,k_0}|y|^2Q = 0$. We uniquely choose $\beta_{j_0,k_0}\in\mathbb{R}$ so that

$$\left(-F^-_{j_0,k_0} + ((k_0+1)\alpha+2j_0)P^+_{j_0,k_0}, Q\right)_2 = 0.$$

By (12), there exists $P^-_{j_0,k_0}\in\mathcal{Y}$ (unique up to the addition of cQ) such that $L_-P^-_{j_0,k_0} - F^-_{j_0,k_0} + ((k_0+1)\alpha+2j_0)P^+_{j_0,k_0} = 0$. In particular, we have proved that if $j_0<K$, then $H(j_0,k_0)$ implies $H(j_0+1,k_0)$, and $H(K,k_0)$ implies $H(1,k_0+1)$. This is enough to complete an induction argument on the two parameters (j,k). Therefore, system $(S_{j,k})$ is solved for all $(j,k)\in\Sigma_K$.
It remains to estimate Ψ_K and $\nabla\Psi_K$. It is straightforward to check that

$$\sup_{y\in\mathbb{R}^d}\left(e^{\frac{|y|}{2}}\left(|\Psi^{P_s}(y)| + |\nabla\Psi^{P_s}(y)|\right)\right) \lesssim \lambda^{\alpha}\left(\left|\frac{\lambda_s}{\lambda} + b\right| + \left|b_s + b^2 - \theta\right|\right).$$

Next, we claim

$$|\Psi^f| \lesssim \left(\lambda^{(K+2)\alpha} + \lambda^\alpha b^{2K+2}\right) Q. \tag{27}$$

Indeed, first, if y is such that $\left|\lambda^\alpha \frac{Z(y)}{Q(y)}\right| < \frac{1}{2}$ then the result follows from (25) and a order Taylor expansion of order $K+1$ of $(1+\frac{\lambda^\alpha Z}{Q})^{\frac{2}{d}+1}$ and $(1+\frac{\lambda^\alpha Z}{Q})^{\frac{2}{d}}$. Second, if on the contrary, $\left|\lambda^\alpha \frac{Z(y)}{Q(y)}\right| \geq \frac{1}{2}$, then, since $Z \in \mathcal{Y}$, we have, for such y,

$$Q(y) \leq 2\lambda^\alpha |Z(y)| \lesssim \lambda^\alpha (1+|y|^\kappa) Q(y) \quad \text{and so} \quad Q(y) + |Z(y)| \lesssim e^{-\frac{1}{2}\lambda^{\alpha/\kappa}},$$

which completes the proof of (27). The proofs of estimates for $\nabla\Psi_f$, Ψ_g and $\nabla\Psi_g$ are similar. Finally the following estimates for $\Psi^{>K}$ and $\nabla\Psi^{>K}$ are clear:

$$|\Psi^{>K}| + |\nabla\Psi^{>K}| \lesssim \left(\lambda^{(K+2)\alpha} + \lambda^\alpha |b|^{2K+2}\right) Q^{\frac{1}{2}}.$$

The result follows from $K \geq \frac{20}{\alpha}$.

Proof of (ii). This is a straightforward computation which is left to the reader.
Proof of (iii). To prove (20), we hit (19) with iP_b and compute using the critical relation $(P, \Lambda P)_2 = 0$:

$$\frac{1}{2}\frac{d}{ds}\|P_b\|_2^2 = (i\partial_s P_b, iP_b)_2 = (\Psi_K e^{-i\frac{b|y|^2}{4}}, iP_b)$$

and (20) follows from (17). For (21), we have from scaling:

$$E(P_{b,\lambda,\gamma}) = \frac{1}{\lambda^2}\left(\frac{1}{2}\int |\nabla P_b|^2 - \int F(P_b) - \lambda^\alpha \int G(P_b)\right) =: \frac{1}{\lambda^2}\tilde{E}(\lambda, P_b)$$

Therefore,

$$\frac{d}{ds}E(P_{b,\lambda,\gamma}) = \frac{1}{\lambda^2}\left(-2\frac{\lambda_s}{\lambda}\tilde{E}(\lambda, P_b) + \left\langle \tilde{E}'(\lambda, P_b), \partial_s P_b\right\rangle - \alpha\lambda^\alpha \frac{\lambda_s}{\lambda}\int G(P_b)\right). \tag{28}$$

Using the equation (19) of P_b, we compute:

$$\begin{aligned}\left\langle \tilde{E}'(\lambda, P_b), \partial_s P_b\right\rangle = \frac{\lambda_s}{\lambda}\left\langle \tilde{E}'(\lambda, P_b), \Lambda P_b\right\rangle - \left(\frac{\lambda_s}{\lambda} + b\right)\left\langle \tilde{E}'(\lambda, P_b), \Lambda P_b\right\rangle \\ + (b_s + b^2 - \theta)\left\langle i\tilde{E}'(\lambda, P_b), \frac{|y|^2}{4}P_b\right\rangle + \left\langle i\tilde{E}'(\lambda, P_b), \Psi_K e^{-i\frac{b|y|^2}{4}}\right\rangle.\end{aligned} \tag{29}$$

We now integrate by parts to estimate

$$\left\langle \tilde{E}'(\lambda, P_b), \Lambda P_b\right\rangle = \int |\nabla P_b|^2 - 2\int F(P_b) - \frac{d(p-1)}{2}\int G(P_b) = 2\tilde{E}(\lambda, P_b) + \alpha\lambda^\alpha \int G(P_b), \tag{30}$$

where we have used $\alpha = 2 - \frac{d(p-1)}{2}$, from which:

$$\begin{aligned}\frac{d}{ds}E(P_{b,\lambda,\gamma}) &= \frac{1}{\lambda^2}\left[-2\frac{\lambda_s}{\lambda}\tilde{E}(\lambda, P_b) - \alpha\lambda^\alpha\frac{\lambda_s}{\lambda}\int G(P_b) + \frac{\lambda_s}{\lambda}\left[2\tilde{E}(\lambda, P_b) + \alpha\lambda^\alpha\int G(P_b)\right]\right] \\ &+ \frac{1}{\lambda^2}O\left(\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s + b^2 - \theta| + (b^2 + \lambda^\alpha)^{K+2}\right).\end{aligned}$$

The estimate (21) on the time-derivative of the energy then follows from (28), (29), (30), and (17). Next,

$$\begin{aligned}\lambda^2 E(P_{b,\lambda,\gamma}) &= \frac{1}{2}\int |\nabla P_b|^2 - \int F(P_b) - \lambda^\alpha \int G(P_b) \\ &= \frac{1}{2}\int |\nabla P|^2 + \frac{b^2}{8}\int |y|^2 |P|^2 - \int F(P) - \lambda^\alpha \int G(P).\end{aligned}$$

Thus, replacing $P = Q + \lambda^\alpha Z$,

$$\begin{aligned}\lambda^2 E(P_{b,\lambda,\gamma}) &= \frac{1}{2}\int |\nabla Q|^2 - \int F(Q) + \frac{b^2}{8}\int |y|^2 Q^2 - \lambda^\alpha \int G(Q)\\ &+ \lambda^\alpha \int (-\Delta Q - f(Q))\mathrm{Re}Z - \lambda^{2\alpha}\int g(Q)\mathrm{Re}Z + \frac{b^2}{4}\lambda^\alpha \int |y|^2 Q\mathrm{Re}Z\\ &+ \frac{\lambda^{2\alpha}}{2}\int |\nabla Z|^2 + \frac{b^2\lambda^{2\alpha}}{8}\int |y|^2|Z|^2 - \int \{F(Q+\lambda^\alpha Z) - F(Q) - \lambda^\alpha f(Q)\mathrm{Re}Z\}\\ &- \lambda^\alpha \int \{G(Q+\lambda^\alpha Z) - G(Q) - \lambda^\alpha g(Q)\mathrm{Re}Z\}.\end{aligned}$$

On the one hand, we recall that from Pohozaev identity,

$$\frac{1}{2}\int |\nabla Q|^2 - \int F(Q) = 0,$$

and from the definition (26) of $\beta_{0,0}$,

$$\int G(Q) = \frac{\beta}{2d(p-1)}\int |y|^2 Q^2 = \frac{\beta}{4(2-\alpha)}\int |y|^2 Q^2$$

and moreover

$$\Delta Q + f(Q) = Q.$$

On the other hand, we observe, since $\int P_{0,0}^+ Q = 0$,

$$\lambda^\alpha \int ZQ = \lambda^\alpha \sum_{(j,k)\in\Sigma_K, j+k\geq 1} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{I}}_{j,k},$$

for some $\eta^{\mathrm{I}}_{j,k} \in \mathbb{R}$;

$$\lambda^{2\alpha}\int Zg(Q) = \lambda^\alpha \sum_{(j,k)\in\Sigma_K, k\geq 1} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{II}}_{j,k},$$

for some $\eta^{\mathrm{II}}_{j,k} \in \mathbb{R}$;

$$\lambda^\alpha b^2 \int |y|^2 Q\mathrm{Re}Z = \lambda^\alpha \sum_{(j,k)\in\Sigma_K, j\geq 1} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{III}}_{j,k},$$

for some $\eta^{\mathrm{III}}_{j,k} \in \mathbb{R}$;

$$\lambda^{2\alpha}\int |\nabla Z|^2 + \frac{b^2\lambda^{2\alpha}}{8}\int |y|^2 Z^2 = \lambda^\alpha \sum_{(j,k)\in\Sigma_K, j\geq 1, k\geq 0} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{IV}}_{j,k},$$

for some $\eta^{\mathrm{IV}}_{j,k} \in \mathbb{R}$. Moreover, by Taylor expansion as before, for some $\eta^{\mathrm{V}}_{j,k}, \eta^{\mathrm{IV}}_{j,k} \in \mathbb{R}$

$$\left|\int \left\{F(Q+\lambda^\alpha Z) - F(Q) - \lambda^\alpha f(Q)\mathrm{Re}Z - \lambda^\alpha \sum_{(j,k)\in\Sigma_K, k\geq 1} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{V}}_{j,k}\right\}\right| \lesssim \lambda^{(K+2)\alpha},$$

$$\left|\lambda^\alpha \int \left\{G(Q+\lambda^\alpha Z) - G(Q) - \lambda^\alpha g(Q)\mathrm{Re}Z - \lambda^\alpha \sum_{(j,k)\in\Sigma_K, k\geq 2} b^{2j}\lambda^{k\alpha}\eta^{\mathrm{VI}}_{j,k}\right\}\right| \lesssim \lambda^{(K+2)\alpha}.$$

Gathering these computations, we obtain (22). □

2.2. **Approximate blow up law.** For simplicity of notation, we set

$$\beta = \beta_{0,0} = \frac{2d(p-1)}{p+1}\frac{\|Q\|_{p+1}^{p+1}}{\|yQ\|_2^2}.$$

First, we find a relevant solution to the following approximate system

$$b_s + b^2 - \beta\lambda^\alpha = 0, \quad b + \frac{\lambda_s}{\lambda} = 0. \tag{31}$$

Indeed, for $|b| + \lambda \ll 1$, $\beta\lambda^\alpha$ is the main term in θ, and the only term in θ that will modify at the main order the blow up rate.

Lemma 4. *Let*

$$\lambda_{\rm app}(s)=\left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{2}{\alpha}}s^{-\frac{2}{\alpha}},\quad b_{\rm app}(s)=\frac{2}{\alpha s}. \tag{32}$$

Then $(\lambda_{\rm app}(s),b_{\rm app}(s))$ *solves* (31) *for* $s>0$.

Proof. We compute:

$$\left(\frac{b^2}{\lambda^2}\right)_s=2\frac{b}{\lambda}\frac{b_s+b^2}{\lambda}=-2\beta\frac{\lambda_s}{\lambda}\lambda^{\alpha-2},$$

and so

$$\frac{b^2}{\lambda^2}-\frac{2\beta}{2-\alpha}\lambda^{\alpha-2}=c_0. \tag{33}$$

Taking the constant $c_0=0$, and using $b=-\frac{\lambda_s}{\lambda}>0$, we find

$$\frac{\lambda_s}{\lambda^{1+\frac{\alpha}{2}}}=\sqrt{\frac{2\beta}{2-\alpha}}.$$

Therefore,

$$\lambda(s)=\left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{2}{\alpha}}s^{-\frac{2}{\alpha}},\quad b(s)=-\frac{\lambda_s}{\lambda}(s)=\frac{2}{\alpha}\frac{1}{s}$$

is solution of (31). □

Remark 1. We now express this solution in the time variable $t_{\rm app}$ related to $\lambda_{\rm app}$. Let

$$dt_{\rm app}=\lambda_{\rm app}^2ds=\left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{4}{\alpha}}s^{-\frac{4}{\alpha}}ds.$$

Therefore (with the convention that $t_{\rm app}\to 0^-$ as $s\to+\infty$)

$$t_{\rm app}=-C_s s^{-\frac{4-\alpha}{\alpha}}\quad\text{where}\quad C_s=\frac{\alpha}{4-\alpha}\left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{4}{\alpha}}. \tag{34}$$

As a consequence, we obtain for $t_{\rm app}<0$,

$$\lambda_{\rm app}(t_{\rm app})=C_\lambda|t_{\rm app}|^{\frac{2}{4-\alpha}}\quad\text{where}\quad C_\lambda=\left(\frac{4-\alpha}{\alpha}C_s^{-\frac{\alpha}{4-\alpha}}\right)^{\frac12}, \tag{35}$$

$$b_{\rm app}(t_{\rm app})=C_b|t_{\rm app}|^{\frac{\alpha}{4-\alpha}},\quad\text{where}\quad C_b=\frac{2}{\alpha}C_s^{-\frac{\alpha}{4-\alpha}}. \tag{36}$$

Now, we choose suitable initial conditions b_1 and λ_1 for $b(s)$ and $\lambda(s)$ at some large time s_1, first to adjust the value of the energy of $P_{b,\lambda,\gamma}$ (up to the small error term in (22)) and second to be able to close the perturbed dynamical system of (λ,b) at the end of the proof (see proof of Lemma 16 below). Let $E_0\in\mathbb{R}$ and

$$C_0=\frac{8E_0}{\int|y|^2Q^2}.$$

Fix $0<\lambda_0\ll 1$ such that $\frac{2\beta}{2-\alpha}+C_0\lambda_0^{2-\alpha}>0$. For $\lambda\in(0,\lambda_0]$, let

$$\mathcal{F}(\lambda)=\int_\lambda^{\lambda_0}\frac{d\mu}{\mu^{\frac{\alpha}{2}+1}\sqrt{\frac{2\beta}{2-\alpha}+C_0\mu^{2-\alpha}}}. \tag{37}$$

Note that the function $\mathcal{F}$ is related to the resolution of the system (33) for $c_0=C_0$, see proof of Lemma 16.

Lemma 5. *Let* $s_1\gg 1$. *There exist* b_1 *and* λ_1 *such that*

$$\left|\frac{\lambda_1^{\frac{\alpha}{2}}}{\lambda_{\rm app}^{\frac{\alpha}{2}}(s_1)}-1\right|+\left|\frac{b_1}{b_{\rm app}(s_1)}-1\right|\lesssim s_1^{-\frac12}+s_1^{2-\frac{4}{\alpha}}, \tag{38}$$

$$\mathcal{F}(\lambda_1)=s_1,\quad \mathcal{E}(b_1,\lambda_1)=C_0. \tag{39}$$

Proof. First, we choose λ_1. Note that $\mathcal{F}$ is a decreasing function of λ satisfying $\mathcal{F}(\lambda_0) = 0$ and $\lim_{\lambda\downarrow 0} \mathcal{F}(\lambda) = +\infty$. Thus, there exists a unique $\lambda_1 \in (0, \lambda_0)$ such that $\mathcal{F}(\lambda_1) = s_1$.

For $\lambda \in (0, \lambda_0]$,

$$\left| \mathcal{F}(\lambda) - \frac{2}{\alpha\sqrt{\frac{2\beta}{2-\alpha}}\lambda^{\frac{\alpha}{2}}} \right| \lesssim 1 + \left| \int_\lambda^{\lambda_0} \frac{d\mu}{\mu^{\frac{\alpha}{2}+1}} \left[\frac{1}{\sqrt{\frac{2\beta}{2-\alpha} + C_0\mu^{2-\alpha}}} - \frac{1}{\sqrt{\frac{2\beta}{2-\alpha}}} \right] \right|$$
$$\lesssim 1 + \int_\lambda^{\lambda_0} \frac{d\mu}{\mu^{1+\frac{\alpha}{2}-(2-\alpha)}}.$$

Thus,

$$\left| \mathcal{F}(\lambda) - \frac{2}{\alpha\sqrt{\frac{2\beta}{2-\alpha}}\lambda^{\frac{\alpha}{2}}} \right| \lesssim \begin{cases} 1 & \text{for } \alpha \in (0, \frac{4}{3}), \\ |\log\lambda| & \text{for } \alpha = \frac{4}{3}, \\ \lambda^{2-\frac{3\alpha}{2}} & \text{for } \alpha \in (\frac{4}{3}, 2). \end{cases}$$

To simplify, we will use the non sharp but sufficient estimate

$$\left| \mathcal{F}(\lambda) - \frac{2}{\alpha\sqrt{\frac{2\beta}{2-\alpha}}\lambda^{\frac{\alpha}{2}}} \right| \lesssim \lambda^{-\frac{\alpha}{4}} + \lambda^{2-\frac{3\alpha}{2}}. \tag{40}$$

Applied to λ_1, it gives

$$\left| s_1 - \frac{2}{\alpha\sqrt{\frac{2\beta}{2-\alpha}}\lambda_1^{\frac{\alpha}{2}}} \right| \lesssim \lambda_1^{-\frac{\alpha}{4}} + \lambda_1^{2-\frac{3\alpha}{2}} \quad \text{and thus} \quad \left| \frac{\lambda_1^{\frac{\alpha}{2}}}{\lambda_{\rm app}^{\frac{\alpha}{2}}(s_1)} - 1 \right| \lesssim s_1^{-\frac{1}{2}} + s_1^{2-\frac{4}{\alpha}}.$$

Second, we choose b_1. From the definition of $\mathcal{E}$, we have

$$h(b) := \lambda_1^2\mathcal{E}(b, \lambda_1) = b^2 - \left(\frac{2}{\alpha s_1}\right)^2 - \frac{2\beta}{2-\alpha}\left(\lambda_1^\alpha - \lambda_{\rm app}^\alpha(s_1)\right) + \lambda_1^\alpha \sum_{(j,k)\in\Sigma_K,\ j+k\geq 1} b^{2j}\lambda_1^{-k\alpha}\eta_{j,k}$$
$$= b^2 - \left(\frac{2}{\alpha s_1}\right)^2 + O(s_1^{-\frac{5}{2}}) + O(s_1^{-\frac{4}{\alpha}}).$$

Observe that

$$|h(b_{\rm app}(s_1))| \lesssim s_1^{-\frac{4}{\alpha}}, \quad |h'(b_{\rm app}(s_1))| \geq 2b_{\rm app}(s_1) + O(s_1^{-3}) \geq s_1^{-1}.$$

Since $\lambda_1^2 \approx s_1^{-\frac{4}{\alpha}}$, it follows that there exists a unique b_1 such that

$$|b_1 - b_{\rm app}(s_1)| \lesssim s_1^{-\frac{3}{2}} + s_1^{1-\frac{4}{\alpha}}, \quad h(b_1) = C_0\lambda_1^2,$$

and so $\mathcal{E}(b_1, \lambda_1) = C_0$. □

3. Existence proof assuming uniform estimates

This section is devoted to the proof of Theorem 1 by a compactness argument, assuming uniform estimates on specific solutions of (7). These estimates are given in Proposition 7.

3.1. Uniform estimates in rescaled time variable.

The rescaled time depending on a suitable modulation of the solution $u(t)$, we first recall without proof the following standard result (see e.g. [24]).

Lemma 6 (Modulation)**.** *Let $u(t) \in \mathcal{C}(I, H^1(\mathbb{R}^d))$ for some interval I, be such that*

$$\sup_{t\in I} \inf_{\lambda_0>0,\gamma_0} \left\| \lambda_0^{\frac{d}{2}} u(t, \lambda_0 y)e^{i\gamma_0} - Q(y) \right\|_{H^1} \leq \delta, \tag{41}$$

for $\delta > 0$ small enough. Then, there exist $\mathcal{C}^1$ functions $\lambda \in (0, +\infty)$, $b \in \mathbb{R}$, $\gamma \in \mathbb{R}$ on I such that u admits a unique decomposition of the form

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} \left(P_{b(t)} + \varepsilon(t, y)\right) e^{i\gamma(t)}, \qquad y = \frac{x}{\lambda(t)} \tag{42}$$

where ε satisfies the following orthogonality conditions on I ($\rho_b(t, y) = \rho(y)e^{-i\frac{b(t)|y|^2}{4}}$)

$$(\varepsilon, i\Lambda P_b)_2 = \left(\varepsilon, |y|^2 P_b\right)_2 = (\varepsilon, i\rho_b)_2 = 0. \tag{43}$$

See (18) for the definition of P_b.

Let $E_0 \in \mathbb{R}$. Given $t_1 < 0$ close to 0, following Remark 1, we define the initial rescaled time s_1 as

$$s_1 := \left|C_s^{-1} t_1\right|^{-\frac{\alpha}{4-\alpha}}.$$

Let λ_1 and b_1 be given by Lemma 5 for this value of s_1. Let $u(t)$ be the solution of (7) for $t \le t_1$, with data

$$u(t_1, x) = \frac{1}{\lambda_1^{\frac{d}{2}}} P_{b_1}\left(\frac{x}{\lambda_1}\right). \tag{44}$$

As long as the solution $u(t)$ satisfies (41), we consider its decomposition $(\lambda, b, \gamma, \varepsilon)$ from Lemma 6 and we define the rescaled time s by

$$s = s_1 - \int_t^{t_1} \frac{1}{\lambda^2(\tau)} d\tau. \tag{45}$$

The heart of the proof of Theorem 1 is the following result, giving uniform backwards estimates on the decomposition of $u(s)$ on $[s_0, s_1]$ for some s_0 independent of s_1.

Proposition 7 (Uniform estimates in rescaled time)**.** *There exists $s_0 > 0$ independent of s_1 such that the solution u of (7) defined by (44) exists and satisfies (41) on $[s_0, s_1]$. Moreover, its decomposition*

$$u(s, x) = \frac{1}{\lambda^{\frac{d}{2}}(s)} \left(P_b + \varepsilon\right)(s, y)\, e^{i\gamma(s)}, \qquad y = \frac{x}{\lambda(s)},$$

satisfies the following uniform estimates on $[s_0, s_1]$,

$$\|\varepsilon(s)\|_{H^1} \lesssim s^{-(K+1)}, \quad \left|\frac{\lambda^{\frac{\alpha}{2}}(s)}{\lambda_{\rm app}^{\frac{\alpha}{2}}(s)} - 1\right| + \left|\frac{b(s)}{b_{\rm app}(s)} - 1\right| \lesssim s^{-\frac{1}{2}} + s^{2-\frac{4}{\alpha}}. \tag{46}$$

In addition,

$$|E(P_{b,\lambda,\gamma}(s)) - E_0| \le O(s^{-6}).$$

Let us insist again that the key point in Proposition 7 is that s_0 and the constants in the estimates are independent of $s_1 \to +\infty$.

3.2. **Proof of Theorem 1 assuming Proposition 7.** First, we convert the estimates of Proposition 7 in the original time variable t. We claim:

Lemma 8 (Estimates in the t variable)**.** *There exists $t_0 < 0$ such that under the assumptions of Proposition 7, for all $t \in [t_0, t_1]$,*

$$b(t) = C_b |t|^{\frac{\alpha}{4-\alpha}} (1 + o_{t\uparrow 0}(1)), \quad \lambda(t) = C_\lambda |t|^{\frac{2}{4-\alpha}} (1 + o_{t\uparrow 0}(1)) \tag{47}$$

$$\|\varepsilon(t)\|_{H^1} \lesssim |t|^{\frac{(K+1)\alpha}{4-\alpha}} \tag{48}$$

$$|E(P_{b,\lambda,\gamma}(t)) - E_0| = o_{t\uparrow 0}(1) \tag{49}$$

Proof of Lemma 8. Using (46), (45), for all large $s < s_1$,

$$t_1 - t(s) = \int_s^{s_1} \lambda^2(\sigma) d\sigma = \int_s^{s_1} \lambda_{\rm app}^2(\sigma) \left[1 + O(\sigma^{-\frac{1}{2}}) + O(\sigma^{2-\frac{4}{\alpha}})\right] d\sigma.$$

Recall that $t_{\rm app}$ given by (34) corresponds to the normalization

$$t_{\rm app}(s) = -\int_s^{+\infty} \lambda_{\rm app}^2(\sigma), \quad t_{\rm app}(s_1) = t_1,$$

from which we obtain

$$t(s) = t_{\rm app}(s)(1 + o(1)) = -C_s s^{-\frac{4-\alpha}{\alpha}} \left[1 + o(1)\right].$$

The estimates of Lemma 8 now follow directly follow from (32) and Proposition 7 (see the definition of C_λ and C_b in (35) and (36)). □

Now, we finish the proof of Theorem 1 assuming Proposition 7.

Proof of Theorem 1. Let $(t_n) \subset (t_0, 0)$ be an increasing sequence such that $\lim_{n\to\infty} t_n = 0$. For each n, let u_n be the solution of (7) on $[t_0, t_n]$ with final data at t_n

$$u_n(t_n, x) = \frac{1}{\lambda^{\frac{d}{2}}(t_n)} P_{b(t_n)}\left(\frac{x}{\lambda(t_n)}\right), \tag{50}$$

where $\lambda(t_n) = \lambda_1$ and $b(t_n) = b_1$ are given by Lemma 5 for $s_1 = |C_s^{-1} t_n|^{-\frac{\alpha}{4-\alpha}}$, so that $u_n(t)$ satisfies the conclusions of Proposition 7 and of Lemma 8 on the interval $[t_0, t_n]$. The minimal mass blow up solution for (7) is now obtained as the limit of a subsequence of (u_n). In a first step, we prove that a subsequence of $(u_n(t_0))$ converges to a suitable initial data. Indeed, from Lemma 8, we infer that $(u_n(t_0))$ is bounded in $H^1(\mathbb{R}^d)$. Hence there exists a subsequence of $(u_n(t_0))$ (still denoted by $(u_n(t_0))$ and $u_\infty(t_0) \in H^1(\mathbb{R}^d)$ such that

$$u_n(t_0) \rightharpoonup u_\infty(t_0) \quad \text{weakly in } H^1(\mathbb{R}^d) \text{ as } n \to +\infty.$$

Now, we obtain strong convergence in H^s (for some $0 < s < 1$) by direct arguments. Let $\chi : [0, +\infty) \to [0, 1]$ be a smooth cut-off function such that $\chi \equiv 0$ on $[0, 1]$ and $\chi \equiv 1$ on $[2, +\infty)$. For $R > 0$, define $\chi_R : \mathbb{R}^d \to [0, 1]$ by $\chi_R(x) = \chi(|x|/R)$. Take any $\delta > 0$. By the expression of $u_n(t_n)$ in (50), we can choose R large enough (independent of n) so that

$$\int_{\mathbb{R}^d} |u_n(t_n)|^2 \chi_R dx \le \delta. \tag{51}$$

It follows from elementary computations that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u_n|^2 \chi_R dx = 2\,\mathrm{Im}\int_{\mathbb{R}^d} \nabla\chi_R \cdot \nabla u_n\, \bar{u}_n dx.$$

Hence from the geometrical decomposition

$$u_n(t, x) = \frac{1}{\lambda_n^{\frac{d}{2}}(t)} \left(P_{b_n(t)} + \varepsilon_n\right)(t, y)\big)\, e^{i\gamma_n(t)}, \qquad y = \frac{x}{\lambda_n(t)},$$

and the smallness (47)-(48) of ε_n and λ_n we infer

$$\left|\frac{d}{dt}\int_{\mathbb{R}^d} |u_n(t)|^2 \chi_R dx\right| \le \frac{C}{\lambda_n(t) R}\left(e^{-\frac{R}{2\lambda_n(t)}} + \|\varepsilon_n(t)\|_{H^1}^2\right) \le \frac{C}{R}|t|^{\left(-\frac{2}{\alpha}+K+1\right)\frac{\alpha}{4-\alpha}}.$$

Integrating between t_0 and t_n, we obtain

$$\int_{\mathbb{R}^d} |u_n(t_0)|^2 \chi_R dx \le \frac{C}{R}|t_0|^{\left(-\frac{2}{\alpha}+K+1\right)\frac{\alpha}{4-\alpha}+1} + \int_{\mathbb{R}^d} |u_n(t_n)|^2 \chi_R dx.$$

Combined with (51), for a possibly larger R, this implies

$$\int_{\mathbb{R}^d} |u_n(t_0)|^2 \chi_R dx \le 2\delta.$$

We conclude from the local compactness of Sobolev embeddings that for $0 \le s < 1$:

$$u_n(t_0) \to u_\infty(t_0) \quad \text{strongly in } H^s(\mathbb{R}^d), \quad \text{as } n \to +\infty.$$

Let $u_\infty(t)$ be the solution of (7) with $u_\infty(t_0)$ as initial data at $t = t_0$. From [6, 7] there exists $0 < s_0 < 1$ such that the Cauchy problem for (7) is locally well-posed in $H^{s_0}(\mathbb{R}^d)$. This implies that u_∞ exists on $[t_0, 0)$ and for any $t \in [t_0, 0)$,

$$u_n(t) \to u_\infty(t) \quad \text{strongly in } H^{s_0}(\mathbb{R}^d), \ \text{ weakly in } H^1(\mathbb{R}^d), \ \text{ as } n \to +\infty.$$

Moreover, since $\lim_{n\to\infty} \int u_n^2(t_n) = \int Q^2$, we have $\int u_\infty^2 = \int Q^2$. By weak convergence in $H^1(\mathbb{R}^d)$ and the estimates from Lemma 8 applied to u_n, $u_\infty(t)$ satisfies (41), and denoting $(\varepsilon_\infty, \lambda_\infty, b_\infty, \gamma_\infty)$ its decomposition, we have by standard arguments (see e.g. [24]), for any $t \in [t_0, 0)$,

$$\lambda_n(t) \to \lambda_\infty(t), \quad b_n(t) \to b_\infty(t), \quad \gamma_n(t) \to \gamma_\infty(t), \quad \varepsilon_n(t) \rightharpoonup \varepsilon_\infty(t) \quad H^1(\mathbb{R}^d) \text{ weak, as } n \to \infty.$$

The uniform estimates on u_n from Lemma 8 give, on $[t_0, 0)$,

$$b_\infty(t) = C_b|t|^{\frac{\alpha}{4-\alpha}}\left(1 + o_{t\uparrow 0}(1)\right), \quad \lambda_\infty(t) = C_\lambda|t|^{\frac{2}{4-\alpha}}\left(1 + o_{t\uparrow 0}(1)\right), \quad \|\varepsilon_\infty(t)\|_{H^1} \lesssim |t|^{\frac{(K+1)\alpha}{4-\alpha}}, \tag{52}$$

$$\frac{b_\infty(t)}{\lambda_\infty^2(t)} = \frac{C_b}{C_\lambda^2}|t|^{\frac{\alpha}{4-\alpha}-\frac{4}{4-\alpha}}\left(1 + o_{t\uparrow 0}(1)\right) = \frac{2}{4-\alpha}\frac{1}{|t|}\left(1 + o_{t\uparrow 0}(1)\right) = \frac{\sigma}{|t|}\left(1 + o_{t\uparrow 0}(1)\right), \tag{53}$$

which justifies the form (9) and the blow up rate (8). Finally, we prove that $E(u_\infty) = E_0$. Let $t_0 < t < 0$. We have by (49) and (22),

$$\mathcal{E}(b_n(t), \lambda_n(t)) - \frac{8E_0}{\int |y|^2 Q^2} = o_{t\uparrow 0}(1)$$

where the $o_{t\uparrow 0}(1)$ is independent of n, and thus

$$\mathcal{E}(b_\infty(t), \lambda_\infty(t)) - \frac{8E_0}{\int |y|^2 Q^2} = o_{t\uparrow 0}(1)$$

Using (22), we deduce

$$E(P_{b_\infty, \lambda_\infty, \gamma_\infty}(t)) - E_0 = o_{t\uparrow 0}(1)$$

and thus, by (52),

$$E(u_\infty(t)) - E_0 = o_{t\uparrow 0}(1).$$

Thus, by conservation of energy and passing to the limit $t \uparrow 0$, we obtain $E(u_\infty(t)) = E_0$. □

3.3. **Bootstrap estimates.** The rest of the paper is devoted to the proof of Proposition 7. We use a bootstrap argument involving the following estimates:

$$\|\varepsilon(s)\|_{H^1} < s^{-K}, \quad \left| \frac{\lambda^{\frac{\alpha}{2}}(s)}{\lambda^{\frac{\alpha}{2}}_{\rm app}(s)} - 1 \right| + \left| \frac{b(s)}{b_{\rm app}(s)} - 1 \right| < s^{-\delta(\alpha)} \tag{54}$$

for some small enough universal constant $\delta(\alpha) > 0$. The following value is suitable in this paper

$$\delta(\alpha) = \min\left(\frac{1}{4}, \frac{2}{\alpha} - 1\right) > 0. \tag{55}$$

For $s_0 > 0$ to be chosen large enough (independently of s_1), we define

$$s_* = \inf\{\tau \in [s_0, s_1]; (54) \text{ holds on } [\tau, s_1]\}. \tag{56}$$

Observe from (38) that

$$\left| \frac{\lambda_1^{\frac{\alpha}{2}}}{\lambda^{\frac{\alpha}{2}}_{\rm app}(s_1)} - 1 \right| + \left| \frac{b_1}{b_{\rm app}(s_1)} - 1 \right| \lesssim s_1^{-\frac{1}{2}} + s_1^{2-\frac{4}{\alpha}} \ll s_1^{-\delta(\alpha)},$$

for s_1 large, and hence by the definition (44) of $u(s_1)$, s_* is well-defined and $s_* < s_1$. In §5, §6 and §7, we prove that (46) holds on $[s_*, s_1]$. By a standard continuity argument, provided that s_0 is large enough, we obtain $s_* = s_0$ which implies Proposition 7. The main lines of the proof are as follows: first, we derive modulation equations from the construction of P_b, second we control the remaining error using a mixed Energy/Morawetz functional first derived in [31].

4. Modulation equations

In this section, we work with the solution $u(t)$ of Proposition 7 on the time interval $[s_*, s_1]$ (see (54)-(56)). We justify that the dynamical system satisfied by the modulation parameters λ, b is at the main order given by (31). Define

$$\mathrm{Mod}(s) = \begin{pmatrix} b + \frac{\lambda_s}{\lambda} \\ b_s + b^2 - \theta \\ 1 - \gamma_s \end{pmatrix}.$$

Lemma 9 (Modulation equations and additional orthogonality)**.** *For all $s \in [s_*, s_1]$,*

$$|\,\mathrm{Mod}(s)| \lesssim \frac{1}{s^{K+2}}, \tag{57}$$

$$|\,(\varepsilon(s), Q)_2\,| \lesssim \frac{1}{s^{K+1}}. \tag{58}$$

Proof of Lemma 9. The proofs of the two estimates are combined. Since $\varepsilon(s_1) \equiv 0$, we may define

$$s_{**} = \inf\{s \in [s_*, s_1];\ |\,(\varepsilon(\tau), P_b)_2\,| < \tau^{-(K+2)} \text{ holds on } [s, s_1]\}.$$

We work on the interval $[s_{**}, s_1]$.

Since P_b verifies equation (19), we obtain the following equation for ε:

$$\begin{aligned} i\varepsilon_s + \Delta\varepsilon - \varepsilon + ib\Lambda\varepsilon + (f(P_b+\varepsilon) - f(P_b)) + \lambda^\alpha(g(P_b+\varepsilon) - g(P_b)) \\ - i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda(P_b+\varepsilon) + (1-\gamma_s)(P_b+\varepsilon) + (b_s + b^2 - \theta)\frac{|y|^2}{4}P_b \\ = -\Psi e^{-i\frac{b|y|^2}{4}}. \quad (59) \end{aligned}$$

where $\Psi := \Psi_K$. Recall that equation (59) combined with the orthogonality conditions chosen on ε – see (43) – contains the equations of the modulation parameters. Technically, one differentiates in time the orthogonality conditions for ε, then uses the equation (59) on ε and the estimate (17) on the error term Ψ. Here, as in [31], the orthogonality conditions are chosen to obtain quadratic control in ε. Since it is a standard argument (see e.g. [25, 28, 31]), we only sketch relevant computations.

Consider for example the orthogonality condition $(\varepsilon, i\Lambda P_b)_2 = 0$. Differentiating in s, we obtain $\langle \varepsilon_s, i\Lambda P_b\rangle + \langle \varepsilon, i\partial_s(\Lambda P_b)\rangle = 0$. Since

$$\frac{d}{ds}(\Lambda P_b) = \left((\Lambda P)_s - i\frac{b_s}{4}|y|^2\Lambda P\right)e^{-i\frac{b}{4}|y|^2},$$

and

$$\begin{aligned} (\Lambda P)_s = \lambda^\alpha\Bigg(\alpha\frac{\lambda_s}{\lambda}\Bigg(Z + \sum_{(j,k)\in\Sigma_K} kb^{2j}\lambda^{k\alpha-1}(P^+_{j,k} + bP^-_{j,k})\Bigg) \\ + b_s\Bigg(\sum_{(j,k)\in\Sigma_K} 2jb^{2j-1}\lambda^{k\alpha}P^+_{j,k} + \sum_{(j,k)\in\Sigma_K}(2j+1)b^{2j}\lambda^{k\alpha}P^-_{j,k}\Bigg)\Bigg), \end{aligned}$$

proceeding as in the proof of Proposition 3, and using the properties of the functions $P^\pm_{j,k}$, we note that

$$\sup_{y\in\mathbb{R}}\left(e^{\frac{y}{2}}\left|\frac{d}{ds}(\Lambda P_b)(y)\right|\right) \lesssim |\operatorname{Mod}(s)| + b^2(s) + \lambda^\alpha(s).$$

Thus, by (54),

$$|\,(\varepsilon, i\partial_s(\Lambda P_b))_2\,| \lesssim \|\varepsilon(s)\|_2\left(|\operatorname{Mod}(s)| + b^2(s) + \lambda^\alpha(s)\right) \lesssim s^{-2}|\operatorname{Mod}(s)| + s^{-(K+2)}.$$

Next, we write $\langle \varepsilon_s, i\Lambda P_b\rangle = -\langle i\varepsilon_s, \Lambda P_b\rangle$ and we use the equation of ε. We start by the contribution of the first line of (59). Remark that by (54),

$$\begin{aligned} f(P_b+\varepsilon) - f(P_b) &= e^{-ib\frac{|y|^2}{4}}\left(f\left(P + e^{ib\frac{|y|^2}{4}}\varepsilon\right) - f(P)\right) = e^{-ib\frac{|y|^2}{4}}df(P)\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right) + O(|\varepsilon|^2) \\ &= e^{-ib\frac{|y|^2}{4}}df(P)\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right) + O(s^{-2}|\varepsilon|), \end{aligned}$$

$$\lambda^\alpha\left(g(P_b+\varepsilon) - g(P_b)\right) = O(\lambda^\alpha|\varepsilon|) = O(s^{-2}|\varepsilon|),$$

and

$$\Delta\varepsilon + ib\Lambda\varepsilon = e^{-ib\frac{|y|^2}{4}}\Delta\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right) + b^2\frac{|y|^2}{4}\varepsilon, \qquad \Lambda P_b = e^{-ib\frac{|y|^2}{4}}\left(\Lambda P - ib\frac{|y|^2}{2}P\right).$$

Therefore, using (54) and $P = Q + O_{H^1}(s^{-2})$ (see the definition of P in (16)), we have

$$\begin{aligned}
&\langle -\Delta\varepsilon + \varepsilon - ib\Lambda\varepsilon - (f(P_b+\varepsilon) - f(P_b)) + \lambda^\alpha \left(g(P_b+\varepsilon) - g(P_b)\right), \Lambda P_b\rangle \\
&\quad = \left\langle -\Delta\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right) + e^{ib\frac{|y|^2}{4}}\varepsilon - pQ^{p-1}\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right), \Lambda Q - ib\frac{|y|^2}{2}Q\right\rangle + O(s^{-2}\|\varepsilon\|_2) \\
&\quad = \left\langle L_+\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right), \Lambda Q\right\rangle - \frac{b}{2}\left\langle L_-\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right), i|y|^2 Q\right\rangle + O(s^{-2}\|\varepsilon\|_2) \\
&\quad = \left\langle e^{ib\frac{|y|^2}{4}}\varepsilon, L_+(\Lambda Q)\right\rangle - \frac{b}{2}\left\langle e^{ib\frac{|y|^2}{4}}\varepsilon, iL_-(|y|^2 Q)\right\rangle + O(s^{-2}\|\varepsilon\|_2) \\
&\quad = -2\left(\varepsilon, e^{-ib\frac{|y|^2}{4}}Q\right)_2 + 2b\left(\varepsilon, ie^{-ib\frac{|y|^2}{4}}\Lambda Q\right)_2 + O(s^{-2}\|\varepsilon\|_2) \\
&\quad = -2\left(\varepsilon, P_b\right)_2 + 2b\left(\varepsilon, i\Lambda P_b\right)_2 + O(s^{-2}\|\varepsilon\|_2) = O(s^{-(K+2)}).
\end{aligned}$$

Note that we have used algebraic relations from (10), then (54), $(\varepsilon, i\Lambda P_b)_2 = 0$ and the definition of s_{**}.

The part corresponding to the second line of (59) gives

$$\begin{aligned}
&\left(-i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda(P_b+\varepsilon) + (1-\gamma_s)(P_b+\varepsilon) + (b_s + b^2 - \theta)\frac{|y|^2}{4}P_b, \Lambda P_b\right)_2 \\
&\qquad = -(b_s + b^2 - \theta)\|yP_b\|_2^2 + O(|\operatorname{Mod}(s)|\|\varepsilon\|_2) \\
&\qquad = -(b_s + b^2 - \theta)(\|yQ\|_2^2 + O\left(s^{-2}\right)) + O(s^{-2}|\operatorname{Mod}(s)|).
\end{aligned}$$

Finally, from the estimate (17) on Ψ, we have

$$\left|\left(\Psi, \Lambda P - ib\frac{|y|^2}{2}P\right)_2\right| \lesssim s^{-2}|\operatorname{Mod}(s)| + s^{-2(K+2)}.$$

Combining the previous estimates, we find

$$|b_s + b^2 - \theta| \lesssim s^{-2}|\operatorname{Mod}(s)| + s^{-(K+2)}.$$

Using the other orthogonality conditions in (43) in a similar way, together with (10), we find

$$|\operatorname{Mod}(s)| \lesssim s^{-2}|\operatorname{Mod}(s)| + s^{-(K+2)}.$$

We deduce that for all $s \in [s_{**}, s_1]$,

$$|\operatorname{Mod}(s)| \lesssim s^{-(K+2)}. \tag{60}$$

By conservation of the L^2 norm and (44), we have

$$\|u(s)\|_2^2 = \|u(s_1)\|_2^2 = \|P_b(s_1)\|_2^2.$$

Thus, by (42),

$$(\varepsilon(s), P_b)_2 = \frac{1}{2}\left(\|u(s)\|_2^2 - \|P_b(s)\|_2^2 - \|\varepsilon(s)\|_2^2\right) = -\frac{1}{2}\|\varepsilon(s)\|_2^2 + \frac{1}{2}\left(\|P_b(s_1)\|_2^2 - \|P_b(s)\|_2^2\right).$$

Moreover, by (20), (54) and (60),

$$\frac{d}{ds}\int |P_b|^2 \lesssim s^{-(K+4)}.$$

Integrating and combining the previous estimates with (54), we obtain, for all $s \in [s_{**}, s_1]$,

$$|\left(\varepsilon(s), P_b\right)_2| \lesssim s^{-(K+3)}. \tag{61}$$

Therefore, $s_{**} = s_*$ and the estimates (60) and (61) are proved on $[s_*, s_1]$. Since $|P_b - Q| \lesssim Q^{\frac{1}{2}}s^{-1}$, we obtain (58). □

5. The mixed energy Morawetz monotonicity formula

In this section, following [31], we introduce a mixed Energy/Morawetz functional to control the remaining part of the solution in $H^1(\mathbb{R}^d)$. First, define the energy of ε

$$H(s,\varepsilon) := \frac{1}{2}\|\nabla\varepsilon\|_2^2 + \frac{1}{2}\|\varepsilon\|_2^2 - \int_{\mathbb{R}^d}(F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon)dy \\ - \lambda^\alpha \int_{\mathbb{R}^d}(G(P_b+\varepsilon) - G(P_b) - dG(P_b)\varepsilon)dy.$$

Note that as in [31], the time derivative of the linearized energy H for ε cannot be controlled alone, and one has to add a virial type functional such as $\frac{b}{2}\mathrm{Im}\int_{\mathbb{R}^d}\nabla\left(\frac{|y|^2}{2}\right)\nabla\varepsilon\bar{\varepsilon}dy$. In practice, due to the lack of control on $\|y\varepsilon\|_2$, we replace $\frac{1}{2}|y|^2$ by a function whose gradient is bounded, which we introduce now.

Let $\phi : \mathbb{R}\to\mathbb{R}$ be a smooth even and convex function, nondecreasing on $\mathbb{R}^+$, such that

$$\phi(r) = \begin{cases} \frac{1}{2}r^2 & \text{for } r<1, \\ 3r + e^{-r} & \text{for } r>2, \end{cases}$$

and set $\phi(x) = \phi(|x|)$. Let $A \gg 1$ to be fixed. Define ϕ_A by $\phi_A(y) = A^2\phi\left(\frac{y}{A}\right)$ and

$$J(\varepsilon) = \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^d}\nabla\phi_A\cdot\nabla\varepsilon\bar{\varepsilon}dy.$$

Finally, set

$$S(s,\varepsilon) = \frac{1}{\lambda^4(s)}(H(s,\varepsilon) + b(s)J(\varepsilon(s))).$$

The relevance of the functional S lies on the following two properties.

Proposition 10 (Coercivity of S)**.** *For any $s\in[s_*,s_1]$,*

$$S(s,\varepsilon(s)) \gtrsim \frac{1}{\lambda^4(s)}\left(\|\varepsilon(s)\|_{H^1}^2 + O(s^{-2(K+1)}\right).$$

Proposition 11. *For any $s\in[s_*,s_1]$,*

$$\frac{d}{ds}\left[S(s,\varepsilon(s))\right] \gtrsim \frac{b}{\lambda^4(s)}\left(\|\varepsilon(s)\|_{H^1}^2 + O(s^{-2(K+1)}\right).$$

The rest of this section is organized as follows. We first prove Proposition 10 in §5.1. In §5.2 we compute the time derivative of H and in §5.3, the time derivative of J. We finish the proof of Proposition 11 in §5.4.

5.1. Coercivity of S.

We prove Proposition 10. We first claim a coercivity property for H, consequence of the properties of L_+ and L_- (see (13)) and of the orthogonality conditions of ε (see (43)).

Lemma 12 (Coercivity of H)**.** *For all $s\in[s_*,s_1]$,*

$$H(s,\varepsilon) \gtrsim \|\varepsilon\|_{H^1}^2 + O(s^{-2(K+1)}).$$

Proof. From the orthogonality conditions (43), (58), and estimates (54), the following holds:

$$\begin{aligned} \left(\varepsilon, |y|^2Q\right)_2 &= \left(\varepsilon,|y|^2P_b\right)_2 + O(|b|\|\varepsilon\|_2) + O(\lambda^\alpha\|\varepsilon\|_2) = O(s^{-1}\|\varepsilon\|_{H^1}), \\ (\varepsilon, i\rho)_2 &= (\varepsilon, i\rho_b)_2 + O(|b|\|\varepsilon\|_2) = O(s^{-1}\|\varepsilon\|_{H^1}), \\ (\varepsilon, Q)_2 &= O(s^{-(K+1)}). \end{aligned}$$

From (54), we have

$$\lambda^\alpha\int_{\mathbb{R}^d}(G(P_b+\varepsilon) - G(P_b) - dG(P_b)\varepsilon)dx = O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

Next, (denoting $\varepsilon = \varepsilon_1 + i\varepsilon_2$),

$$\left|F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon - \left(1+\frac{4}{d}\right)Q^{\frac{4}{d}}\varepsilon_1^2 - \frac{1}{2}Q^{\frac{4}{d}}\varepsilon_2^2\right| \lesssim e^{-\frac{1}{2}|y|}|\varepsilon|^3 + |\varepsilon|^{2+\frac{4}{d}} + |\varepsilon|^2(|b|+\lambda^\alpha).$$

Thus, from (54),

$$\left| \int F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon - \left(1+\frac{4}{d}\right) Q^{\frac{4}{d}}\varepsilon_1^2 - \frac{1}{2}Q^{\frac{4}{d}}\varepsilon_2^2 \right| \lesssim O(s^{-1}\|\varepsilon\|_{H^1}^2),$$

and

$$\left| H(s,\varepsilon) - \frac{1}{2}\langle L_+\varepsilon_1,\varepsilon_1\rangle - \frac{1}{2}\langle L_-\varepsilon_2,\varepsilon_2\rangle \right| \lesssim O(s^{-1}\|\varepsilon\|_{H^1}^2).$$

Combining these estimates with the coercivity properties of L_+, L_- (see (13)), we obtain the result. □

Since

$$|bJ(\varepsilon)| \le |b|\|\nabla\phi_A\|_\infty \|\varepsilon\|_{H^1}^2 \lesssim O(s^{-1}\|\varepsilon\|_{H^1}^2)$$

(from (54)), Lemma 12 implies Proposition 10.

For future reference, we also claim the following localized coercivity property (see similar statement in [17] and [31]).

Lemma 13. *There exists $A_0 > 1$ such that for any $A > A_0$,*

$$\frac{1}{2}\int_{\mathbb{R}^d} \nabla\varepsilon^T\nabla^2\phi_A\nabla\bar{\varepsilon}dy + \frac{1}{2}\|\varepsilon\|_2^2 - \int_{\mathbb{R}^d}(F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon)dx \gtrsim \|\varepsilon\|_2^2 + O(s^{-2(K+1)}).$$

For now on, we consider $A > A_0$.

5.2. Time variation of the energy of ε.

Lemma 14. *For all $s \in [s_*, s_1]$,*

$$\frac{d}{ds}[H(s,\varepsilon(s))] = \frac{\lambda_s}{\lambda}\left(\|\nabla\varepsilon\|_2^2 - \langle f(P_b+\varepsilon) - f(P_b), \Lambda\varepsilon\rangle\right) + O(s^{-(2K+3)}) + O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

Proof of Lemma 14. The time derivative for H separates into two parts:

$$\frac{d}{ds}[H(s,\varepsilon(s))] = D_sH(s,\varepsilon) + \langle D_\varepsilon H(s,\varepsilon), \varepsilon_s\rangle,$$

where D_s (respectively, D_ε) denotes differentiation of the functional with respect to s (respectively, ε). In particular,

$$\begin{aligned} D_sH(s,\varepsilon) = &-\int (P_b)_s\left(f(P_b+\varepsilon) - f(P_b) - df(P_b)\varepsilon\right) - \lambda^\alpha\int (P_b)_s\left(g(P_b+\varepsilon) - g(P_b) - dg(P_b)\varepsilon\right) \\ &- \alpha\frac{\lambda_s}{\lambda}\lambda^\alpha\int \left(G(P_b+\varepsilon) - G(P_b) - dG(P_b)\varepsilon\right). \end{aligned}$$

Note that

$$e^{i\frac{b|y|^2}{4}}(P_b)_s = P_s - ib_s\frac{|y|^2}{4}P = P_s - i\left(b_s + b^2 - \beta\lambda^\alpha\right)\frac{|y|^2}{4}P + i\left(b^2 - \beta\lambda^\alpha\right)\frac{|y|^2}{4}P.$$

By (24), (54) and Lemma 9, we obtain

$$|(P_b)_s| \lesssim s^{-2}e^{-\frac{|y|}{2}} \quad \text{and} \quad \left|\frac{\lambda_s}{\lambda}\right|\lambda^\alpha \lesssim s^{-3}.$$

Thus,

$$|D_sH(s,\varepsilon)| \lesssim s^{-2}\|\varepsilon\|_{H^1}^2.$$

Now, we compute $\langle D_\varepsilon H(s,\varepsilon), \varepsilon_s\rangle$. Note that (59) rewrites

$$i\varepsilon_s - D_\varepsilon H(s,\varepsilon) + \mathrm{Mod}_{\mathrm{op}}(s)P_b - i\frac{\lambda_s}{\lambda}\Lambda\varepsilon + (1-\gamma_s)\varepsilon + e^{-ib\frac{|y|^2}{4}}\Psi = 0, \tag{62}$$

where

$$\mathrm{Mod}_{\mathrm{op}}(s)P_b := -i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda P_b + (1-\gamma_s)P_b + (b_s + b^2 - \theta)\frac{|y|^2}{4}P_b.$$

Using (62), since $\langle iD_\varepsilon H(s,\varepsilon), D_\varepsilon H(s,\varepsilon)\rangle = 0$, we have

$$\langle D_\varepsilon H(s,\varepsilon), \varepsilon_s\rangle = \langle iD_\varepsilon H(s,\varepsilon), i\varepsilon_s\rangle = -\langle iD_\varepsilon H(s,\varepsilon), \mathrm{Mod}_{\mathrm{op}}(s)P_b\rangle + \frac{\lambda_s}{\lambda}\langle iD_\varepsilon H(s,\varepsilon), i\Lambda\varepsilon\rangle$$
$$- (1-\gamma_s)\langle iD_\varepsilon H(s,\varepsilon),\varepsilon\rangle - \left\langle iD_\varepsilon H(s,\varepsilon), e^{-ib\frac{|y|^2}{4}}\Psi\right\rangle. \quad (63)$$

From the proof of Lemma 9

$$D_\varepsilon H(s,\varepsilon) = -\Delta\varepsilon + \varepsilon - (f(P_b+\varepsilon) - f(P_b)) - \lambda^\alpha(g(P_b+\varepsilon) - g(P_b))$$
$$= e^{-ib\frac{|y|^2}{4}}\left(L_+\mathrm{Re}\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right) + iL_-\mathrm{Im}\left(e^{ib\frac{|y|^2}{4}}\varepsilon\right)\right) + ib\Lambda\varepsilon + b^2\frac{|y|^2}{4}\varepsilon + O(s^{-2}|\varepsilon|).$$

Therefore, using the orthogonality conditions (43), (58) and estimates (54), we have (see also proof of Lemma 9),

$$\langle D_\varepsilon H(s,\varepsilon), \Lambda P_b\rangle = -2(\varepsilon, P_b)_2 + b(\varepsilon, i\Lambda P_b)_2 + O(s^{-2}\|\varepsilon\|_2) = O(s^{-(K+1)}).$$

Thus, from Lemma 9,

$$\left|\frac{\lambda_s}{\lambda} + b\right| |\langle D_\varepsilon H(s,\varepsilon), \Lambda P_b\rangle| \lesssim O(s^{-(2K+3)}).$$

Using similar arguments we get

$$\langle D_\varepsilon H(s,\varepsilon), iP_b\rangle = -4(\varepsilon, \Lambda P_b)_2 + O(s^{-1}\|\varepsilon\|_2) = O(s^{-1}\|\varepsilon\|_2) = O(s^{-(K+1)})$$

and

$$\left\langle D_\varepsilon H(s,\varepsilon), i\frac{|y|^2}{4}P_b\right\rangle = (\varepsilon,\rho_b)_2 + O(s^{-1}\|\varepsilon\|_2) = O(s^{-1}\|\varepsilon\|_2) = O(s^{-(K+1)}).$$

Using Lemma 9, we obtain in conclusion for this term

$$\langle iD_\varepsilon H(s,\varepsilon), \mathrm{Mod}_{\mathrm{op}}(s)P_b\rangle = O(s^{-(2K+3)}).$$

Next, we have

$$\langle iD_\varepsilon H(s,\varepsilon), i\Lambda\varepsilon\rangle = \langle D_\varepsilon H(s,\varepsilon), \Lambda\varepsilon\rangle =$$
$$\langle -\Delta\varepsilon + \varepsilon - (f(P_b+\varepsilon) - f(P_b)) - \lambda^\alpha(g(P_b+\varepsilon) - g(P_b)), \Lambda\varepsilon\rangle.$$

Note that (by direct computations)

$$\langle -\Delta\varepsilon, \Lambda\varepsilon\rangle = \|\nabla\varepsilon\|_2^2, \quad \langle\varepsilon, \Lambda\varepsilon\rangle = 0,$$

and by (54),

$$|\langle\lambda^\alpha(g(P_b+\varepsilon) - g(P_b)), \Lambda\varepsilon\rangle| \lesssim O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

Thus,

$$\frac{\lambda_s}{\lambda}\langle iD_\varepsilon H(s,\varepsilon), i\Lambda\varepsilon\rangle = \frac{\lambda_s}{\lambda}\left(\|\nabla\varepsilon\|_2^2 - \langle f(P_b+\varepsilon) - f(P_b), \Lambda\varepsilon\rangle\right) + O(s^{-3}\|\varepsilon\|_{H^1}^2).$$

For the third term in the right-hand side of (63), we claim

$$|(1-\gamma_s)\langle iD_\varepsilon H(s,\varepsilon),\varepsilon\rangle| = \left|(1-\gamma_s)\langle(f(P_b+\varepsilon) - f(P_b)) + \lambda^\alpha(g(P_b+\varepsilon) - g(P_b)),\varepsilon\rangle\right|$$
$$\lesssim |\mathrm{Mod}(s)|\left(\|\varepsilon\|_2^2 + \|\varepsilon\|_{H^1}^{2+\frac{4}{d}}\right) = O(s^{-4}\|\varepsilon\|_{H^1}^2).$$

Finally, the fourth term in the right-hand side of (63) is estimated by (17) combined with Lemma 9 and (54)

$$|\langle iD_\varepsilon H(s,\varepsilon), \Psi\rangle| \le O(s^{-(K+4)}\|\varepsilon\|_{H^1}) \le O(s^{-(2K+3)}) + O(s^{-5}\|\varepsilon\|_{H^1}^2).$$

Gathering these estimates, we have proved the lemma. □

5.3. **The time derivative of the Morawetz part.**

Lemma 15. *For all $s \in [s_*, s_1]$,*

$$\frac{d}{ds}[J(\varepsilon(s))] = \int_{\mathbb{R}^d} \nabla\varepsilon^T \nabla^2\phi_A \nabla\bar{\varepsilon} dy - \frac{1}{4}\int_{\mathbb{R}^d} |\varepsilon|^2 \Delta^2\phi_A dy \\ - \left\langle f(P_b+\varepsilon) - f(P_b), \frac{1}{2}\Delta\phi_A\varepsilon + \nabla\phi_A\nabla\varepsilon \right\rangle + O(s^{-(2K+2)}) + O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

Proof. From the definition of $J(\varepsilon)$, we have

$$\frac{d}{ds}[J(\varepsilon(s))] = \mathrm{Re}\int_{\mathbb{R}^d} i\varepsilon_s \left(\frac{1}{2}\Delta\phi_A\bar{\varepsilon} + \nabla\phi_A\nabla\bar{\varepsilon}\right) dy.$$

We replace $i\varepsilon_s$ using (59). First, from standard computations

$$\mathrm{Re}\int_{\mathbb{R}^d} -\Delta\varepsilon \left(\frac{1}{2}\Delta\phi_A\bar{\varepsilon} + \nabla\phi_A\nabla\bar{\varepsilon}\right) dy = \int_{\mathbb{R}^d} \nabla\varepsilon^T\nabla^2\phi_A\nabla\bar{\varepsilon}dy - \frac{1}{4}\int_{\mathbb{R}^d}|\varepsilon|^2\Delta^2\phi_A dy,$$

$$\mathrm{Re}\int_{\mathbb{R}^d} \varepsilon \left(\frac{1}{2}\Delta\phi_A\bar{\varepsilon} + \nabla\phi_A\nabla\bar{\varepsilon}\right) dy = 0,$$

$$\frac{\lambda_s}{\lambda}\mathrm{Re}\int_{\mathbb{R}^d} i\Lambda\varepsilon \left(\frac{1}{2}\Delta\phi_A\bar{\varepsilon} + \nabla\phi_A\nabla\bar{\varepsilon}\right) dy = 0.$$

Next,

$$\lambda^\alpha \mathrm{Re}\int_{\mathbb{R}^d} (g(P_b+\varepsilon) - g(P_b)) \left(\frac{1}{2}\Delta\phi_A\bar{\varepsilon} + \nabla\phi_A\nabla\bar{\varepsilon}\right) dy = O(\lambda^\alpha\|\varepsilon\|_{H^1}^2) = O(s^{-2}\|\varepsilon\|_{H^1}^2).$$

The term corresponding to the second line of (59) is estimated as follows.

$$\left|\left\langle -i(b+\frac{\lambda_s}{\lambda})\Lambda(P_b+\varepsilon) + (1-\gamma_s)(P_b+\varepsilon) - (b_s+b^2-\theta)\frac{|y|^2}{4}P_b, \frac{1}{2}\Delta\phi_A\varepsilon + \nabla\phi_A\nabla\varepsilon\right\rangle\right| \\ \lesssim |\,\mathrm{Mod}(s)|\|\varepsilon\|_{H^1} \lesssim O(s^{-(2K+2)}).$$

Finally, by (17) and Lemma 9,

$$\left|\left\langle \Psi e^{-i\frac{b|y|^2}{4}}, \frac{1}{2}\Delta\phi_A\bar{\varepsilon} + \nabla\phi_A\nabla\bar{\varepsilon}\right\rangle\right| \le O(s^{-(K+4)}\|\varepsilon\|_{H^1}) \le O(s^{-(2K+4)}).$$

The result follows. □

5.4. **The Lyapunov property.**

Proof of Proposition 11. By definition of S, we have

$$\frac{d}{ds}[S(s,\varepsilon(s))] = \frac{1}{\lambda^4}\left(-4\frac{\lambda_s}{\lambda}(H(s,\varepsilon)+bJ(\varepsilon)) + \frac{d}{ds}[H(s,\varepsilon(s))] + b\frac{d}{ds}[J(\varepsilon(s))] + b_sJ(\varepsilon)\right)$$

First, we claim the following estimate

$$\frac{d}{ds}[H(s,\varepsilon(s))] + b\frac{d}{ds}[J(\varepsilon(s))] = b\int_{\mathbb{R}^d}\nabla\varepsilon^T\nabla^2\phi_A\nabla\bar{\varepsilon}dy - b\|\nabla\varepsilon\|_2^2 + \frac{b}{A}O(\|\varepsilon\|_{H^1}^2) + O(s^{-(2K+3)}). \quad (64)$$

Proof of (64). It is essential to see from Lemmas 14 and 15 that the main nonlinear terms are cancelling. Indeed, by integration by parts,

$$-\mathrm{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\Lambda\bar{\varepsilon}dy \\ = -\frac{d}{2}\mathrm{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\bar{\varepsilon}dy - \mathrm{Re}\int_{\mathbb{R}^d} y\nabla(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)dy \\ + \mathrm{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)y\nabla\bar{P}_b dy \\ = -\frac{d}{2}\mathrm{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b))\bar{\varepsilon}dy + d\,\mathrm{Re}\int_{\mathbb{R}^d}(F(P_b+\varepsilon)-F(P_b)-dF(P_b)\varepsilon)dy \\ + \mathrm{Re}\int_{\mathbb{R}^d}(f(P_b+\varepsilon)-f(P_b)-df(P_b)\varepsilon)y\nabla\bar{P}_b dy,$$

$$
\begin{aligned}
&- \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b)) \left(\frac{1}{2} \Delta \phi_A \bar{\varepsilon} + \nabla \phi_A \nabla \bar{\varepsilon} \right) dy \\
&\quad = -\frac{1}{2} \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b)) \Delta \phi_A \bar{\varepsilon} dy + \operatorname{Re} \int_{\mathbb{R}^d} \Delta \phi_A (F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon) dy \\
&\qquad\qquad + \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b) - df(P_b)\varepsilon) \nabla \phi_A \nabla \bar{P}_b dy.
\end{aligned}
$$

Writing these two terms as above, it becomes clear that when y or $\nabla \phi_A$ appear, they are multiplied by ∇P_b, which is exponentially decaying in space (see Proposition 3). Therefore, such terms are controlled by expressions involving only $\|\varepsilon\|_{H^1}$.

Therefore, combining Lemma 14 and Lemma 15, we have

$$
\begin{aligned}
&\frac{d}{ds}[H(s,\varepsilon(s))] + b\frac{d}{ds}[J(\varepsilon(s))] = b \int_{\mathbb{R}^d} \nabla \varepsilon^T \nabla^2 \phi_A \nabla \bar{\varepsilon} dy - b\|\nabla \varepsilon\|_2^2 \\
&+ \left(b + \frac{\lambda_s}{\lambda} \right) \left(\|\nabla \varepsilon\|_2^2 - \frac{d}{2} \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b)) \bar{\varepsilon} dy + d \operatorname{Re} \int_{\mathbb{R}^d} (F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon) dy \right. \\
&\qquad\qquad \left. + \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b) - df(P_b)\varepsilon) y \nabla \bar{P}_b dy \right) \\
&+ b \left(-\frac{1}{2} \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b))(\Delta \phi_A - d) \bar{\varepsilon} dy + \operatorname{Re} \int_{\mathbb{R}^d} (F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon)(\Delta \phi_A - d) dy \right. \\
&\qquad\qquad \left. + \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b) - df(P_b)\varepsilon)(\nabla \phi_A - y) \nabla \bar{P}_b dy \right) \\
&\qquad\qquad - b\frac{1}{4} \int_{\mathbb{R}^d} |\varepsilon|^2 \Delta^2 \phi_A dy + O(s^{-(2K+3)}) + O(s^{-2}\|\varepsilon\|_{H^1}^2).
\end{aligned}
$$

By $\left| b + \frac{\lambda_s}{\lambda} \right| \lesssim O(s^{-4})$, we have

$$
\begin{aligned}
&\left| \left(b + \frac{\lambda_s}{\lambda} \right) \left(\|\nabla \varepsilon\|_2^2 - \frac{d}{2} \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b)) \bar{\varepsilon} dy + d \operatorname{Re} \int_{\mathbb{R}^d} (F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon) dy \right. \right. \\
&\qquad\qquad \left. \left. + \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b) - df(P_b)\varepsilon) y \nabla \bar{P}_b dy \right) \right| \lesssim s^{-4} \|\varepsilon\|_{H^1}^2.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&|b| \left| -\frac{1}{2} \operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b)) \Delta(\phi_A - d) \bar{\varepsilon} dy \right| \\
&\qquad\qquad \lesssim \frac{1}{s} \int_{\mathbb{R}^d} \left| |P|^{\frac{4}{d}} |\varepsilon|^2 |\Delta \phi_A - d| + |\varepsilon|^{2+\frac{4}{d}} \right| dy \lesssim \frac{e^{-\frac{A}{2}}}{s} \|\varepsilon\|_2^2 + O\left(s^{-1} \|\varepsilon\|_{H^1}^{2+\frac{4}{d}} \right),
\end{aligned}
$$

and similarly for $b\operatorname{Re} \int_{\mathbb{R}^d} (F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon)(\Delta \phi_A - d) dy$ and $b\operatorname{Re} \int_{\mathbb{R}^d} (f(P_b+\varepsilon) - f(P_b) - df(P_b)\varepsilon)(\nabla \phi_A - y) \nabla \bar{P}_b dy$. Next,

$$
\left| -b \int_{\mathbb{R}^d} |\varepsilon|^2 \Delta^2 \phi_A dy \right| \lesssim \frac{b}{A^2} \|\varepsilon\|_2^2.
$$

In conclusion for this term, we have obtained (64)

Using $-\frac{\lambda_s}{\lambda} = b + O(s^{-2})$ and the expression of H we have

$$\begin{aligned}
&-4\frac{\lambda_s}{\lambda}H(s,\varepsilon) + \frac{d}{ds}[H(s,\varepsilon(s))] + b\frac{d}{ds}[J(\varepsilon(s))] \\
&\gtrsim 4bH(s,\varepsilon) + b\int_{\mathbb{R}^d} \nabla\varepsilon^T\nabla^2\phi_A\nabla\bar{\varepsilon}dy - b\|\nabla\varepsilon\|_2^2 + O(s^{-2}\|\varepsilon\|_{H^1}^2) + \frac{b}{A}O(\|\varepsilon\|_{H^1}^2) + O(s^{-(2K+3)}) \\
&\gtrsim b\left(\int_{\mathbb{R}^d} \nabla\varepsilon^T\nabla^2\phi_A\nabla\bar{\varepsilon}dy + \|\varepsilon\|_2^2 - 2\int_{\mathbb{R}^d}(F(P_b+\varepsilon) - F(P_b) - dF(P_b)\varepsilon)dx\right) \\
&\quad + 2bH(s,\varepsilon) + \frac{b}{A}O(\|\varepsilon\|_{H^1}^2) + O(s^{-(2K+3)})
\end{aligned}$$

Thus, and the coercivity properties Lemma 12 and Lemma 13, we obtain (for A large enough)

$$-4\frac{\lambda_s}{\lambda}H(s,\varepsilon) + \frac{d}{ds}[H(s,\varepsilon(s))] + b\frac{d}{ds}[J(\varepsilon(s))] \gtrsim b\|\varepsilon\|_{H^1}^2 + O(s^{-(2K+3)}).$$

Since $b = O(s^{-1})$, $b_s = O(s^{-2})$ and $J(\varepsilon) = O(\|\varepsilon\|_{H^1}^2)$, we have

$$\left(\left|\frac{\lambda_s}{\lambda}\right| b + |b_s|\right)|J(\varepsilon)| \lesssim s^{-2}O(\|\varepsilon\|_{H^1}^2)$$

and thus

$$\frac{d}{ds}[S(s,\varepsilon(s))] \gtrsim \frac{b}{\lambda^4}\left(\|\varepsilon\|_{H^1}^2 + O(s^{-(2K+2)})\right).$$

This finishes the proof. □

6. End of the proof of Proposition 7

In this section, we finish the proof of Proposition 7. Recall from §3.3 that our objective is to prove $s_* = s_0$ by improving estimates (54) into (46). Therefore, it is sufficient to prove the following lemma which closes the bounds (54) provided $\delta(\alpha) > 0$ has been chosen small enough (e.g. as in (55)).

Lemma 16 (Refined estimates)**.** *For all $s \in [s_*, s_1]$,*

$$\|\varepsilon(s)\|_{H^1} \lesssim s^{-(K+1)}, \tag{65}$$

$$\left|\frac{\lambda^{\frac{\alpha}{2}}(s)}{\lambda_{\mathrm{app}}^{\frac{\alpha}{2}}(s)} - 1\right| + \left|\frac{b(s)}{b_{\mathrm{app}}(s)} - 1\right| \lesssim s^{-\frac{1}{2}} + s^{2-\frac{4}{\alpha}}. \tag{66}$$

Proof. First, we prove (65). From Proposition 10, and the expression of S, there exists a universal constant $\kappa > 1$ such that for any $s \in [s_*, s_1]$,

$$\frac{1}{\kappa}\frac{1}{\lambda^4}\left(\|\varepsilon\|_{H^1}^2 - \kappa^2 s^{-2(K+1)}\right) \le S(s,\varepsilon) \le \frac{\kappa}{\lambda^4}\|\varepsilon\|_{H^1}^2. \tag{67}$$

From Proposition 11, possibly taking a larger κ,

$$\frac{d}{ds}[S(s,\varepsilon(s))] \ge \frac{1}{\kappa}\frac{b}{\lambda^4}\left(\|\varepsilon\|_{H^1}^2 - \kappa^2 s^{-2(K+1)}\right). \tag{68}$$

Define

$$s_\dagger := \inf\{s \in [s_*, s_1], \quad \|\varepsilon(\tau)\|_{H^1} \le 2\kappa^2\tau^{-(K+1)} \quad \text{for all } \tau \in [s, s_1]\}.$$

Since $\varepsilon(s_1) = 0$, by continuity $s_\dagger$ is well-defined and $s_\dagger < s_1$. For the sake of contradiction, assume that $s_\dagger > s_*$. In particular, $\|\varepsilon(s_\dagger)\|_{H^1} = 2\kappa^2 s_\dagger^{-(K+1)}$. Define

$$s_\ddagger := \sup\{s \in [s_\dagger, s_1], \quad \|\varepsilon(\tau)\|_{H^1} \ge \kappa\tau^{-(K+1)} \quad \text{for all } \tau \in [s_\dagger, s]\}.$$

In particular, $s_\dagger < s_\ddagger < s_1$ and $\|\varepsilon(s_\ddagger)\|_{H^1} = \kappa s_\ddagger^{-(K+1)}$, and from (68), S is nondecreasing on $[s_\dagger, s_\ddagger]$. From equations (67)-(68) and the estimates on λ (see (46)), we obtain

$$\begin{aligned}
\|\varepsilon(s_\dagger)\|_{H^1}^2 - \kappa^2 s_\dagger^{-2(K+1)} &\le \kappa\lambda^4(s_\dagger)S(s_\dagger,\varepsilon(s_\dagger)) \le \kappa\lambda^4(s_\dagger)S(s_\ddagger,\varepsilon(s_\ddagger)) \\
&\le \kappa^2\frac{\lambda^4(s_\dagger)}{\lambda^4(s_\ddagger)}\|\varepsilon(s_\ddagger)\|_{H^1}^2 \le \kappa^4\frac{\lambda^4(s_\dagger)}{\lambda^4(s_\ddagger)}s_\ddagger^{-2(K+1)} \le 2\kappa^4\left(\frac{s_\ddagger}{s_\dagger}\right)^{\frac{8}{\alpha}} s_\ddagger^{-2(K+1)} \le 2\kappa^4 s_\dagger^{-2(K+1)},
\end{aligned}$$

since $K > 4/\alpha$. Therefore $\|\varepsilon(s_\dagger)\|_{H^1}^2 \leq 3\kappa^4 s_\dagger^{-2(K+1)}$, which is a contradiction. Hence $s_\dagger = s_*$ and (65) is proved.

Now, we prove (66). The main idea is to use a conservation law on (b,λ) which can be found from the differential system satisfied by (b,λ), but that we rather derive from energy properties of the blow up profile. Recall that $\lambda(s_1) = \lambda_1$ and $b(s_1) = b_1$ are chosen in Lemma 5 so that $\mathcal{F}(\lambda(s_1)) = s_1$ and $\mathcal{E}(b(s_1),\lambda(s_1)) = \frac{8E_0}{\int |y|^2 Q^2}$. In particular, we deduce from (22) that $|E(P_{b_1,\lambda_1,\gamma_1}) - E_0| \lesssim s_1^{-6}$. Using (21) and (54), (57), for all $s \in [s_*, s_1]$,

$$\left|\frac{d}{ds} E(P_{b,\lambda,\gamma})\right| \lesssim s^{-(K+2)+\frac{4}{\alpha}}.$$

In particular, by integration, we find, for all $s \in [s_*, s_1]$, $|E(P_{b,\lambda,\gamma}(s)) - E_0| \lesssim s^{-6}$ (recall $K > 20/\alpha$) and using (22) at s,

$$\left|\mathcal{E}(b(s),\lambda(s)) - \frac{8E_0}{\int |y|^2 Q^2}\right| \lesssim s^{-6}.$$

We obtain from the expression (23) of $\mathcal{E}$ with $C_0 = \frac{8E_0}{\int |y|^2 Q^2}$:

$$\left|b^2 - \frac{2\beta}{2-\alpha}\lambda^\alpha - C_0\lambda^2\right| \lesssim \frac{\lambda^\alpha}{s^2}$$

where the error term $O(\frac{\lambda^\alpha}{s^2})$ comes from θ and cannot be improved. In this estimate, since $\lambda^2 \approx s^{-\frac{4}{\alpha}}$ and $\frac{\lambda^\alpha}{s^2} \approx s^{-4}$, whether or not $C_0\lambda^2$ is controled by the error term depends on the value of α. We address both cases at once in what follows. Since $b \approx \lambda^{\frac{\alpha}{2}}$,

$$\left|b - \sqrt{\frac{2\beta}{2-\alpha}\lambda^\alpha + C_0\lambda^2}\right| \lesssim \frac{\lambda^{\frac{\alpha}{2}}}{s^2}, \tag{69}$$

and with $\left|\frac{\lambda_s}{\lambda} + b\right| \lesssim s^{-(K+1)}$, we obtain (see (37) for the definition of $\mathcal{F}$)

$$\left|\frac{\lambda_s}{\lambda^{\frac{\alpha}{2}+1}\sqrt{\frac{2\beta}{2-\alpha} + C_0\lambda^{2-\alpha}}} + 1\right| = |\mathcal{F}'(s) - 1| \lesssim s^{-2}. \tag{70}$$

Integrating (70) on $[s, s_1]$, we obtain

$$|\mathcal{F}(\lambda(s_1)) - \mathcal{F}(\lambda(s)) - (s_1 - s)| \lesssim s^{-1}$$

and thus, by the choice $\mathcal{F}(\lambda(s_1)) = s_1$, we obtain

$$\mathcal{F}(\lambda(s)) = s + O(s^{-1}).$$

Therefore, using (40) and the definition of $\lambda_{\rm app}(s)$ in (32),

$$\left|\frac{\lambda_{\rm app}^{\frac{\alpha}{2}}(s)}{\lambda^{\frac{\alpha}{2}}(s)} - 1\right| \lesssim s^{-\frac{1}{2}} + s^{2-\frac{4}{\alpha}}.$$

We reinject this estimate into (69) and use the definition of $b_{\rm app}$ to conclude:

$$b(s) = b_{\rm app}(s) + O(s^{-\frac{3}{2}} + s^{-\frac{4-\alpha}{\alpha}}).$$

This finishes the proof. □

Appendix A. Proof of Lemma 1

By contradiction, assume that there exists a blow up solution $u(t)$ of (1) with $\epsilon = -1$ and $\|u(t)\|_2 = \|Q\|_2$. Let a sequence $t_n \to T^* \in (0, +\infty]$ with $\|\nabla u(t_n)\|_2 \to +\infty$ and consider the renormalized sequence

$$v_n(x) = \lambda(t_n)^{\frac{d}{2}} u(t_n, \lambda(t_n)x), \quad \lambda(t_n) = \frac{\|\nabla Q\|_2}{\|\nabla u(t_n)\|_2}.$$

Then, by conservation of mass,

$$\|v_n\|_2 = \|Q\|_2$$

and conservation of energy and $\epsilon < 0$,

$$E_0 = E(u_n) \geq E_{\rm crit}(u_n) = \frac{E_{\rm crit}(v_n)}{\lambda^2(t_n)}.$$

Therefore, the sequence v_n satisfies:

$$\|v_n\|_2 = \|Q\|_2, \quad \|\nabla v_n\|_2 = \|\nabla Q\|_2, \quad \limsup_{n\to+\infty} E_{\rm crit}(v_n) \leq 0.$$

From standard concentration compactness argument, see [24, 32], there holds, up to a subsequence, for some $x_n \in \mathbb{R}^d, \gamma_n \in \mathbb{R}$,

$$v_n(.-x_n)e^{i\gamma_n} \underset{n\to+\infty}{\to} Q \ \text{ in } \ H^1(\mathbb{R}^d).$$

In particular,

$$\|u(t_n)\|_{p+1} = \frac{\|v_n\|_{p+1}}{\lambda^{\frac{d(p-1)}{2(p+1)}}(t_n)} \to +\infty \ \text{ as } \ n \to \infty,$$

which contradicts the a priori bound from the energy conservation law and (3):

$$E_0 = E(u) \geq E_{\rm crit}(u) + \frac{1}{p+1}\int |u|^{p+1} \geq \frac{1}{p+1}\int |u|^{p+1}.$$

Appendix B. Proof of Lemma 2

For the sake of simplicity, we give the proof only for $d \geq 2$. The case $d = 1$ would require an additionnal (standard) concentration compactness argument (see [32]). For $M < \|Q\|_2$, set

$$A_M = \{u \in H^1_{\rm rad}(\mathbb{R}^d) \ \text{ with } \ \|u\|_2 = M\}$$

and consider the minimization problem

$$I_M = \inf_{u\in A_M} E(u).$$

First, we claim

$$-\infty < I_M < 0. \tag{71}$$

Indeed, from (3) and

$$\int |u|^{p+1} \leq C_{\rm GN}(p)\|\nabla u\|_2^{\frac{d(p-1)}{2}}\|u\|_2^{p+1-\frac{d(p-1)}{2}}, \tag{72}$$

with $1 < p < 1 + \frac{4}{d}$, we note that $I_M > -\infty$ and that any minimizing sequence is bounded in $H^1(\mathbb{R}^d)$. Let $u \in A_M$ and $v_\lambda(x) = \lambda^{\frac{d}{2}}u(\lambda x)$, then $v_\lambda \in A_M$ and

$$E(v_\lambda) = \lambda^2\left[E_{\rm crit}(u) - \frac{1}{\lambda^{2-\frac{d(p-1)}{2}}}\frac{1}{p+1}\int |u|^{p+1}\right].$$

In particular, for $0 < \lambda \ll 1$ and $u \not\equiv 0$, $E(v_\lambda) < 0$ and (71) follows.

Second, let $u_\lambda = \lambda^{\frac{2}{p-1}}u(\lambda x)$, so that

$$E(u_\lambda) = \lambda^{\frac{4}{p-1}+2-d}\left[\frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1}\right] - \frac{\lambda^{\frac{2}{p-1}(2+\frac{4}{d})-d}}{2+\frac{4}{d}}\int |u|^{2+\frac{4}{d}}.$$

We observe that

$$\begin{aligned}\frac{d}{d\lambda}E(u_\lambda)_{|\lambda=1} &= \left(\frac{4}{p-1}+2-d\right)\left[\frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1}\right] - \frac{\frac{2}{p-1}(2+\frac{4}{d})-d}{2+\frac{4}{d}}\int |u|^{2+\frac{4}{d}} \\ &= \left(\frac{4}{p-1}+2-d\right)E(u) - \frac{\frac{4}{d}}{2+\frac{4}{d}}\left(\frac{2}{p-1}-\frac{d}{2}\right)\int |u|^{2+\frac{4}{d}}.\end{aligned}$$

Together with $\|u_\lambda\|_2 = \lambda^{\frac{2}{p-1}-\frac{d}{2}}\|u\|_2$, which implies $\frac{d}{d\lambda}\|u_\lambda\|_{2|\lambda=1} > 0$, this proves that

$$I(M) \text{ is decreasing in } M. \tag{73}$$

To finish, let (u_n) be a minimizing sequence. Up to a subsequence and from the standard radial compactness of Sobolev embeddings (see [3])

$$u_n \rightharpoonup u \ \text{ in } \ H^1(\mathbb{R}^d), \quad u_n \to u \ \text{ in } \ L^q, \quad 2 < q \leq 2 + \frac{4}{d}.$$

Hence

$$E(u) \leq I_M \quad \text{and} \quad \|u\|_2 \leq M.$$

From (73) and the definition of I_M, we deduce $\|u\|_2 = M$ and $E(u) = I_M$. From a standard Lagrange multiplier argument, u satisfies

$$\Delta u + |u|^{1+\frac{4}{d}} u + |u|^{p-1} u = \omega u$$

for a constant $\omega \in \mathbb{R}$. The sign $\omega > 0$ now follows from a standard Pohozaev type argument.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964.

[2] V. Banica, R. Carles, and T. Duyckaerts. Minimal blow-up solutions to the mass-critical inhomogeneous NLS equation. *Comm. Partial Differential Equations*, 36(3):487–531, 2011.

[3] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations I. *Arch. Ration. Mech. Anal.*, 82:313–346, 1983.

[4] T. Boulenger. *Blow-up solutions for the 2-dimensional critical Schrödinger equation on a riemannian manifold*. PhD thesis, Université Paris Sud - Orsay, 2012.

[5] J. Bourgain and W. Wang. Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(1-2):197–215 (1998), 1997. Dedicated to Ennio De Giorgi.

[6] T. Cazenave. *Semilinear Schrödinger equations*. New York University – Courant Institute, New York, 2003.

[7] T. Cazenave and F. B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal.*, 14(10):807–836, 1990.

[8] S.-M. Chang, S. Gustafson, K. Nakanishi, and T.-P. Tsai. Spectra of linearized operators for NLS solitary waves. *SIAM J. Math. Anal.*, 39(4):1070–1111, 2007/08.

[9] R. Côte and S. Le Coz. High-speed excited multi-solitons in nonlinear Schrödinger equations. *J. Math. Pures Appl. (9)*, 96(2):135–166, 2011.

[10] B. Dodson. Global well-posedness and scattering for the mass critical nonlinear Schr{\"o}dinger equation with mass below the mass of the ground state. *ArXiv:1104.1114*, 2011.

[11] C. E. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.*, 166(3):645–675, 2006.

[12] J. Krieger, E. Lenzmann, and P. Raphaël. Nondispersive solutions to the L^2-critical half-wave equation. *Arch. Ration. Mech. Anal.*, 209(1):61–129, 2013.

[13] J. Krieger, Y. Martel, and P. Raphaël. Two-soliton solutions to the three-dimensional gravitational Hartree equation. *Comm. Pure Appl. Math.*, 62(11):1501–1550, 2009.

[14] M. K. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$. *Arch. Rational Mech. Anal.*, 105(3):243–266, 1989.

[15] Y. Martel. Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *Amer. J. Math.*, 127(5):1103–1140, 2005.

[16] Y. Martel and F. Merle. Nonexistence of blow-up solution with minimal L^2-mass for the critical gKdV equation. *Duke Math. J.*, 115(2):385–408, 2002.

[17] Y. Martel and F. Merle. Stability of blow-up profile and lower bounds for blow-up rate for the critical generalized KdV equation. *Ann. of Math. (2)*, 155(1):235–280, 2002.

[18] Y. Martel and F. Merle. Description of two soliton collision for the quartic gKdV equation. *Ann. of Math. (2)*, 174(2):757–857, 2011.

[19] Y. Martel, F. Merle, and P. Raphael. Blow up for the critical gKdV equation II: minimal mass dynamics. *ArXiv:1204.4624*, 2012.

[20] Y. Martel, F. Merle, and P. Raphael. Blow up for the critical gKdV equation III: exotic regimes. *ArXiv:1209.2510*, 2012.

[21] Y. Martel, F. Merle, and P. Raphaël. Blow up for the critical generalized Korteweg–de Vries equation. I: Dynamics near the soliton. *Acta Math.*, 212(1):59–140, 2014.

[22] F. Merle. Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. *Duke Math. J.*, 69(2):427–454, 1993.

[23] F. Merle and P. Raphael. On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation. *Invent. Math.*, 156(3):565–672, 2004.

[24] F. Merle and P. Raphael. The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Ann. of Math. (2)*, 161(1):157–222, 2005.

[25] F. Merle and P. Raphael. On a sharp lower bound on the blow-up rate for the L^2 critical nonlinear Schrödinger equation. *J. Amer. Math. Soc.*, 19(1):37–90 (electronic), 2006.

[26] F. Merle, P. Raphaël, and J. Szeftel. The instability of Bourgain-Wang solutions for the L^2 critical NLS. *Amer. J. Math.*, 135(4):967–1017, 2013.

[27] F. Merle, P. Raphaël, and J. Szeftel. On collapsing ring blow-up solutions to the mass supercritical nonlinear Schrödinger equation. *Duke Math. J.*, 163(2):369–431, 2014.

[28] F. Planchon and P. Raphaël. Existence and stability of the log-log blow-up dynamics for the L^2-critical nonlinear Schrödinger equation in a domain. *Ann. Henri Poincaré*, 8(6):1177–1219, 2007.

[29] P. Raphael. Stability of the log-log bound for blow up solutions to the critical non linear Schrödinger equation. *Math. Ann.*, 331(3):577–609, 2005.
[30] P. Raphaël. Existence and stability of a solution blowing up on a sphere for an L^2-supercritical nonlinear Schrödinger equation. *Duke Math. J.*, 134(2):199–258, 2006.
[31] P. Raphaël and J. Szeftel. Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical NLS. *J. Amer. Math. Soc.*, 24(2):471–546, 2011.
[32] M. I. Weinstein. Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.*, 87(4):567–576, 1982/83.
[33] M. I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16:472–491, 1985.
[34] M. I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39(1):51–67, 1986.

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