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STABILITY IN THE DETERMINATION OF A TIME-DEPENDENT COEFFICIENT FOR WAVE EQUATIONS FROM PARTIAL DATA

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Abstract. Let Ω be a C^2 bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, and fix $Q = (0,T) \times \Omega$ with $T > 0$. We consider the stability in the inverse problem of determining a time-dependent coefficient of order zero q, appearing in a Dirichlet initial-boundary value problem for a wave equation $\partial_t^2 u - \Delta_x u + q(t,x)u = 0$ in Q, from partial observations on ∂Q. The observation is given by a boundary operator associated to the wave equation. Using suitable geometric optics solutions and Carleman estimates, we prove a stability estimate in the determination of q from the boundary operator.

Keywords: Inverse problems, wave equation, scalar time-dependent potential, Carleman estimates, stability inequality.

Mathematics subject classification 2010 : 35R30, 35L05.

1. Introduction

1.1. **Statement of the problem.** In the present paper we consider a $\mathcal{C}^2$ bounded domain Ω of $\mathbb{R}^n$, $n \geqslant 2$. We set $\Sigma = (0,T) \times \partial\Omega$ and $Q = (0,T) \times \Omega$ with $0 < T < \infty$. We introduce the following initial-boundary value problem (IBVP in short) for the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u + q(t,x)u = 0, & \text{in } Q, \\ u(0,\cdot) = v_0, \quad \partial_t u(0,\cdot) = v_1, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma, \end{cases} \tag{1.1}$$

where the potential $q \in L^\infty(Q)$ is assumed to be real valued. We study the inverse problem of determining q from observations of the solutions of (1.1) on ∂Q. For $v_0 = v_1 = 0$, we associate to (1.1) the hyperbolic Dirichlet-Neumann (DN in short) $\Lambda_q : g \mapsto \partial_\nu u$ with u the solution of (1.1) and ν the outward unit normal vector to Ω. It is well known that for $T > \mathrm{Diam}(\Omega)$ the DN map Λ_q determines uniquely a time-independent potential q (e.g. [34]). In contrast to time-independent potentials, due to domain of dependence arguments, there is no hope to recover the restriction of a general time dependent potential q on the set

$$D = \{(t,x) \in Q : \ 0 < t < \mathrm{Diam}(\Omega)/2, \ \mathrm{dist}(x, \partial\Omega) > t\}$$

from the DN map Λ_q (see [27, Subsection 1.1]). In light of this obstruction to uniqueness, it seems that the minimal data that allows to recover globally a general time-dependent potential q so far (at finite time) is given by [27, Theorem 1] where uniqueness is stated. The main goal of the present paper is to prove stability in the recovery of a general time-dependent potential q from similar data.

Practically, our inverse problem is to determine physical properties such as the time evolving density of an inhomogeneous medium by probing it with disturbances generated on the boundary and at initial time. The data is the response of the medium to these disturbances, measured on the boundary and at the end of the experiment, and the purpose is to recover the function which measures the property of the medium. Note that time-dependent potential can also be considered for models where the evolution in time of the perturbation can not be avoided.

We also remark that, according to [22], the determination of time-dependent potentials can be an important tool for the determination of a semilinear term appearing in a semilinear hyperbolic equation from boundary measurements.

1.2. **Existing papers.** In recent years the problem of recovering coefficients for hyperbolic equations from boundary measurements has attracted many attention. Many authors have considered this problem with an observation given by the DN map Λ_q. In [34], the authors proved that the DN map determines uniquely a time-independent potential and in [21] Isakov considered the determination of a coefficient of order zero and a damping coefficient. These results are concerned with measurements on the whole boundary. The uniqueness by local DN map has been considered in [12]. The stability estimate in the case where the DN map is considered on the whole lateral boundary was treated by Stefanov and Uhlmann [38, 39]. The uniqueness and Hölder stability estimate in a subdomain were established by Isakov and Sun [23] and, assuming that the coefficients are known in a neighborhood of the boundary, Bellassoued, Choulli and Yamamoto [3] proved a log-type stability estimate in the case where the Neumann data are observed in an arbitrary subdomain of the boundary. We mention also [31], where the stability issue have been considered for large class of coefficients. In a recent work [26] extended the results of [34] to determine a large class of time-independent coefficients of order zero in an unbounded cylindrical domain. It has been proved that only measurements on a bounded subset are required for the determination of some class of coefficients including periodic coefficients and compactly supported coefficients.

Let us also mention that the method using Carleman inequalities was first considered by Bukhgeim and Klibanov [6]. For the application of Carleman estimates to the problem of recovering time-independent coefficients for hyperbolic equations we refer to [2, 19, 25].

All the above mentioned results are concerned only with time-independent coefficients. Several authors considered the problem of determining time-dependent coefficients for hyperbolic equations. In [37], Stefanov proved unique determination of a time-dependent potential for the wave equation from the knowledge of scattering data. The result of [37] is equivalent to the consideration of the problem with boundary measurements. In [35], Ramm and Sjöstrand considered the problem of determining a time-dependent coefficient q from the DN map Λ_q associated to (1.1). For this purpose, they considered the problem on the infinite time-space cylindrical domain $\mathbb{R}_t \times \Omega$ instead of Q ($t \in \mathbb{R}$ instead of $0 < t < T < \infty$) and their DN map was associated to solutions vanishing for large negative time. Then, under suitable additional assumptions, [35] proved a result of uniqueness. The result of [35] has been extended to more general coefficients by [36] where stability estimate is also stated for compactly supported coefficients provided T is sufficiently large. In [33], Rakesh and Ramm considered the same problem at finite time on Q, with $T > \mathrm{Diam}(\Omega)$, and they proved a uniqueness result for the determination of q restricted to the subset S of Q, made of lines with angle 45° with the t-axis and which meet the planes $t = 0$ and $t = T$ outside $\overline{Q}$, from the DN map Λ_q. In [20, Theorem 4.2], Isakov established a result of uniqueness for a time-dependent potential on the whole domain Q from observations of the solution on ∂Q. Applying a result of unique continuation borrowed from [41], Eskin [13] proved that the DN map uniquely determines time-dependent coefficients that are analytic with respect to the time variable t. In some recent work, [43] proved stability in the recovery of X-ray transforms of time-dependent potentials on a Riemannian manifold. We also mention that [5], proved log-type stability in the recovery of time-dependent potentials from the data considered by [33] and [20]. Finally in [27], the author proved determination of general time dependent potentials from, roughly speaking, half of the data considered by [20].

Let us also remark that [8, 9, 10, 14] consider the problem of determining a time-dependent coefficient for parabolic and Schrödinger equations and derive stability estimate for these problems.

1.3. **Main result.** In order to state our main result, we first introduce some intermediate tools and notations. For all $\omega \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n : \ |y| = 1\}$ we introduce the ω-illuminated and the ω-shadowed faces

$$\partial\Omega_{-,\omega} = \{x \in \partial\Omega : \ \nu(x) \cdot \omega \leqslant 0\}, \quad \partial\Omega_{+,\omega} = \{x \in \partial\Omega : \ \nu(x) \cdot \omega \geqslant 0\}$$

of $\partial\Omega$. We associate to $\partial\Omega_{\pm,\omega}$ the part of the lateral boundary Σ given by $\Sigma_{\pm,\omega} = (0,T) \times \partial\Omega_{\pm,\omega}$. From now on we fix $\omega_0 \in \mathbb{S}^{n-1}$ and we consider $F = (0,T) \times F'$ (resp $G = (0,T) \times G'$) with F' (resp G') an open neighborhood of $\partial\Omega_{+,\omega_0}$ (resp $\partial\Omega_{-,\omega_0}$) in $\partial\Omega$.

We denote by $\Box$ the differential operator $\partial_t^2 - \Delta_x$. According to [27, Proposition 4], we can extend the trace maps

$$\tau_{0,1}v = v_{|\Sigma}, \ \tau_{0,2}v = v_{|t=0}, \ \tau_{0,3}v = \partial_t v_{|t=0}, \quad v \in \mathcal{C}^\infty(\overline{Q})$$

on $H_\Box(Q) = \{u \in L^2(Q) : \ \Box u \in L^2(Q)\}$. Then we define

$$\mathcal{H}_F(\partial Q) := \{(\tau_{0,1}u, \tau_{0,3}u) : \ u \in H_\Box(Q), \ \tau_{0,2}u = 0, \ \mathrm{supp}\tau_{0,1}u \subset F\}.$$

We refer to [27, Section 2] (see also Section 2) for more details about these spaces and the definition of $\|.\|_{\mathcal{H}_F(\partial Q)}$. In view of [27, Section 2], we can associate to (1.1) with $v_0 = 0$ the boundary operator

$$B_q : \mathcal{H}_F(\partial Q) \ni (g, v_1) \mapsto (\partial_\nu u_{|G}, u_{|t=T}) \tag{1.2}$$

where u solves (1.1) with $v_0 = 0$. We refer to [27, Proposition 2] (see also Section 2) for a more rigorous definition of this operator. In Section 2, we prove that for every $q_1, \ q_2 \in L^\infty(Q)$ the operator

$$B_{q_1} - B_{q_2} : \mathcal{H}_F(\partial Q) \to L^2(G) \times H^1(\Omega)$$

is bounded. Then our main result can be stated as follows.

Theorem 1. *Let $p > n+1$ and $q_1, \ q_2 \in W^{1,p}(Q)$. Assume that the conditions*

$$q_1(t,x) = q_2(t,x), \quad (t,x) \in \Sigma \tag{1.3}$$

$$\|q_1\|_{W^{1,p}(Q)} + \|q_2\|_{W^{1,p}(Q)} \leqslant M$$

are fulfilled. Then, there exist C and γ_*, depending on n, p, M, T, Ω, F', G', such that

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leqslant Ch\left(\|B_{q_1} - B_{q_2}\|\right) \tag{1.4}$$

with

$$h(\gamma) = \begin{cases} \gamma, & \gamma \geqslant \gamma^*, \\ \ln(|\ln \gamma|)^{-1}, & 0 < \gamma < \gamma^*, \\ 0, & \gamma = 0. \end{cases}$$

Here $\|B_{q_1} - B_{q_2}\|$ stands for the norm of $B_{q_1} - B_{q_2}$ as an element of $\mathcal{B}(\mathcal{H}_F(\partial Q); L^2(G) \times H^1(\Omega))$.

Let us observe that this stability estimate is the first that is stated with the data considered in [27], where uniqueness is proved with conditions that seems to be one of the weakest so far. Moreover, it appears that with the paper of [5], this paper is the first where stability is stated for global determination of general time dependent potentials appearing in a wave equation from boundary measurements.

The main tools in our analysis are suitable geometric optics (GO in short) solutions, Carleman estimates and results of stability in analytic continuation. More precisely, following the approach of [27] combined with arguments used by [4, 11, 16] (see also [7, 24, 32] for the original aproach in the case of elliptic equations), we consider suitable geometric optics solutions for our problem associated to Carleman estimate with linear weight. In contrast to [27], we recover the time dependent potential not from its Fourier transform but from its light-ray transform (see the proof of Theorem 1). This approach make it possible to derive stability even in the case $n = 2$. Note also that contrary to [27], for the stability issue it is necessary to consider GO lying in $H^2(Q)$ (and not only in $H^1(Q)$).

1.4. **Outline.** This paper is organized as follows. In Section 2 we treat the direct problem. We recall some properties of solutions of (1.1) and we give a result of smoothing for the difference of boundary operators $B_{q_1}-B_{q_2}$ associated to this problem. In Section 3, using some results of [8, 17, 18], we build GO solutions, similar to [4, 27], associated to (1.1) and lying in $H^2(Q)$. In Section 4, we recall some results of [27] about Carleman estimates for the wave equation with linear weight and GO solutions vanishing on parts of the boundary. Then combining these tools with the GO solutions of Section 3 we prove Theorem 1.

2. Functional spaces

In this section following [27] we recall some properties of the IBVP (1.1). According to [27, Proposition 1], for any $(g,v_1)\in\mathcal H_F(\partial Q)$ the IBVP (1.1) with $q=v_0=0$ admits a unique solution $\mathcal P_0(g,v_1)$ and we can define $\|\cdot\|_{\mathcal H_F(\partial Q)}$ by

$$\|(g,v_1)\|_{\mathcal H_F(\partial Q)}=\|\mathcal P_0(g,v_1)\|_{L^2(Q)}\,.$$

Applying [27, Proposition 4], we can extend the map

$$\tau_{1,1}v=\partial_\nu v_{|\Sigma},\quad \tau_{1,2}v=v_{|t=T},\quad v\in\mathcal C^\infty(\overline{Q})$$

on $H_\Box(Q)$. Then, in light of [27, Proposition 2], we can define the boundary operator

$$B_q:\mathcal H_F(\partial Q)\ni(g,v_1)\mapsto(\tau_{1,1}u_{|G},\tau_{1,2}u)$$

with $u\in L^2(Q)$ the unique weak solution of the IBVP (1.1) with $v_0=0$. Moreover, in view of [27, Proposition 2], B_q is bounded from $\mathcal H_F(\partial Q)$ to $H^{-3}(0,T;H^{-\frac12}(G'))\times H^{-2}(\Omega)$.

Now consider the operator $B_{q_1}-B_{q_2}$ for $q_1,\ q_2\in L^\infty(Q)$. We have the following smoothing result.

Proposition 1. *Let $q_1,\ q_2\in L^\infty(Q)$. Then the operator $B_{q_1}-B_{q_2}$ is a bounded operator from $\mathcal H_F(\partial Q)$ to $L^2(G)\times H^1(\Omega)$.*

Proof. For $j=1,2$, let u_j be the unique solution of the IBVP (1.1) for $q=q_j$, $v_0=0$ and supp$g\subset F$. Then, $u=u_1-u_2$ solves

$$\left\{\begin{array}{rcl}\partial_t^2u-\Delta_xu+q_1u&=&(q_2-q_1)u_2,\quad (t,x)\in Q,\\ u_{|t=0}=\partial_tu_{|t=0}&=&0,\\ u_{|\Sigma}&=&0.\end{array}\right.$$

Since $(q_2-q_1)u_2\in L^2(Q)$, in view of [3, Theorem A.2] (see also [28, Theorem 2.1] for $q=0$), $u\in\mathcal C^1([0,T];L^2(\Omega))\cap\mathcal C([0,T];H^1_0(\Omega))$ with $\partial_\nu u\in L^2(\Sigma)$. Moreover, we have the following energy estimate

$$\|u\|_{\mathcal C^1([0,T];L^2(\Omega))}+\|u\|_{\mathcal C([0,T];H^1_0(\Omega))}+\|\partial_\nu u\|_{L^2(\Sigma)}\leqslant C\,\|q_1-q_2\|_{L^\infty(Q)}\,\|u_2\|_{L^2(Q)}\,.$$

It follows $\tau_{1,1}u_{|G}\in L^2(G)$, $\tau_{1,2}u\in H^1(\Omega)$ and [27, Proposition 2] implies

$$\|\tau_{1,1}u\|_{L^2(G)}+\|\tau_{1,2}u\|_{H^1(\Omega)}\leqslant C\,\|(g,v_1)\|_{\mathcal H_F(\partial Q)}\,,$$

where C depends on Ω, T and $M\geqslant\|q_1\|_{L^\infty(Q)}+\|q_2\|_{L^\infty(Q)}$. Finally, we complete the proof by recalling that

$$(\tau_{1,1}u_{|G},\tau_{1,2}u)=(\tau_{1,1}u_{1|G},\tau_{1,2}u_1)-(\tau_{1,1}u_{2|G},\tau_{1,2}u_2)=(B_{q_1}-B_{q_2})(g,v_1).$$

$\square$

3. Smooth geometric optics solutions without boundary conditions

The goal of this section is to build GO solutions $u\in H^2(Q)$ associated to the equation

$$\partial_t^2u-\Delta_xu+q(t,x)u=0\quad\text{in }Q.\tag{3.1}$$

More precisely, for $\lambda>0$, $\omega\in\mathbb S^{n-1}=\{y\in\mathbb R^n:\ |y|=1\}$, $\varphi\in\mathcal C^\infty(\mathbb R^n)$ we consider solutions of this equation of the form

$$u=e^{-\lambda(t+x\cdot\omega)}(\chi(t,x)+w(t,x))\tag{3.2}$$

with $u \in H^2(Q)$ and $\chi(t,x) = \varphi(x + t\omega)$. Here w is the remainder term in the asymptotic expansion of u with respect to λ and we have

$$\|w\|_{H^1(Q)} \leqslant \frac{C}{\lambda}$$

with $C > 0$ independent of λ. In order to build such GO solutions, we first introduce some well known results of Hörmander about solutions of PDEs with constant coefficients of the form $P(D)u = f$ in Q with P a polynomial of $n+1$ variables with complex valued coefficients and $D = -i(\partial_t, \partial_x)$.

3.1. **Solutions of PDEs with constant coefficients.** We start this subsection by recalling some properties of solutions of PDEs of the form $P(D)u = f$ with constant coefficients. For P a polynomial of $n+1$ variable, let $\tilde{P}$ be defined by

$$\tilde{P}(\mu,\xi) = \left(\sum_{k\in\mathbb{N}}\sum_{\alpha\in\mathbb{N}^n}\left|\partial_\mu^k\partial_\xi^\alpha P(\mu,\xi)\right|^2\right)^{\frac{1}{2}}, \quad \mu \in \mathbb{R},\ \xi \in \mathbb{R}^n.$$

Theorem 2. (Theorem 7.3.10, [17]) *For every $P \neq 0$ polynomial of $n+1$ variables one can find a distribution of finite order $E_P \in D'(\mathbb{R}^{1+n})$ such that $P(D)E_P = \delta$.*

Such distributions E_P are called fundamental solutions of P. Note that

$$E_P * (P(D)u) = u, \quad u \in \mathcal{E}'(\mathbb{R}^{1+n}),$$

$$P(D)(E_P * f) = f, \quad f \in \mathcal{E}'(\mathbb{R}^{1+n}),$$

where $\mathcal{E}'(\mathbb{R}^{1+n})$ is the set of distributions with compact support. Thus, for all $f \in \mathcal{E}'(\mathbb{R}^{1+n})$, $u = E_P * f$ is a solution of $P(D)u = f$. Let us give some information about the regularity of such a solution. For this purpose we need the following definitions introduced in [18].

Definition 1. *A positive function κ defined in $\mathbb{R}^{1+n}$ will be called a temperate weight function if there exist positive constants C and N such that*

$$\kappa(\zeta + \eta) \leqslant C(1 + |\zeta|)^N\kappa(\eta), \quad \zeta, \eta \in \mathbb{R}^{1+n}.$$

The set of all such functions κ will be denoted by $\mathcal{K}$.

Notice that, for all polynomial of $n+1$ variables P, $\tilde{P} \in \mathcal{K}$.

Definition 2. *If $\kappa \in \mathcal{K}$ and $1 \leqslant p \leqslant \infty$, we denote by $B_{p,\kappa}$ the set of all temperate distribution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ such that its Fourier transform $\hat{u}$ is a function and*

$$\|u\|_{p,\kappa} = \left(\frac{1}{(2\pi)^{1+n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}}|\kappa(\mu,\xi)\hat{u}(\mu,\xi)|^p\, d\mu d\xi\right)^{\frac{1}{p}} < \infty.$$

When $p = \infty$ we shall interpret $\|u\|_{p,\kappa}$ as ess. $\sup|\kappa(\mu,\xi)\hat{u}(\mu,\xi)|$. We denote by $B_{p,\kappa}^{loc}$ the set of $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ such that for all $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n})$ we have $\chi u \in B_{p,\kappa}$.

Remark 1. *Let*

$$\kappa_1(\mu,\xi) = (1 + |(\mu,\xi)|^2)^{\frac{1}{2}}, \quad \mu \in \mathbb{R},\ \xi \in \mathbb{R}^n.$$

Then, in view of [18, Example 10.1.2]*, one can easily show that $\kappa_1 \in \mathcal{K}$ and $B_{2,\kappa_1} = H^1(\mathbb{R}^{1+n})$.*

Remark 2. *In view of* [18, Theorem 10.1.12]*, for $\kappa_1', \kappa_2' \in \mathcal{K}$, $\kappa = \kappa_1' \cdot \kappa_2'$, $u_1 \in B_{p,\kappa_1'} \cap \mathcal{E}'(\mathbb{R}^{1+n})$ and $u_2 \in B_{\infty,\kappa_2'}$, we have $u_1 * u_2 \in B_{p,\kappa}$ and*

$$\|u_1 * u_2\|_{B_{p,\kappa}} \leqslant \|u_1\|_{B_{p,\kappa_1'}}\|u_2\|_{B_{\infty,\kappa_2'}}. \tag{3.3}$$

Theorem 3. (Theorem 10.21, [18]) *Every polynomial of $n+1$ variables $P \neq 0$, has a fundamental solution $E_P \in B^{loc}_{\infty,\tilde{P}}$ such that $\frac{E_P}{\cosh(|(t,x)|)} \in B_{\infty,\tilde{P}}$ and*

$$\left\|\frac{E_P}{\cosh(|(t,x)|)}\right\|_{B_{\infty,\tilde{P}}} \leqslant C \tag{3.4}$$

with $C > 0$ a constant depending only on the degree of P.

Such a fundamental solution will be denoted by regular fundamental solution. Let us remark that in our construction of geometric optics solutions we need to consider an operator E such that $P(D)E = \mathrm{Id}$ for some polynomial of $n+1$ variables $P \neq 0$. Using the properties of regular fundamental solutions, Hörmander proved in [18, Theorem 10.3.7] that such a operator exists and it is a bounded operator of $L^2(X)$ for X a bounded open set of $\mathbb{R}^{1+n}$. In contrast to elliptic equations and parabolic equations (see [8, Subsection 2.1 and 3.6]), for hyperbolic equations we can not build GO lying in $H^2(Q)$ by applying the result of [18]. What we can actually build from this result is GO lying in $H^1(Q)$ (e.g. [26, Proposition 3]) . Therefore, we need to consider the following.

Proposition 2. *Let $P \neq 0$ be a polynomial of $n+1$ variables. Then there exists an operator*

$$E: \ H^1(Q) \to H^1(Q)$$

such that:

(1) $P(D)Ef = f, \quad f \in H^1(Q)$,
(2) *for all polynomial of $n+1$ variables S such that $\frac{\tilde{S}}{\tilde{P}}$ is bounded, we have $S(D)E \in \mathcal{B}(H^1(Q))$, and*

$$\|S(D)E\|_{\mathcal{B}(H^1(Q))} \leqslant C \sup_{(\mu,\xi)\in\mathbb{R}\times\mathbb{R}^n} \frac{|S(\mu,\xi)|}{\tilde{P}(\mu,\xi)}, \quad k = 0,1, \tag{3.5}$$

where $C > 0$ depends only on the degree of P, Ω and T.

Proof. Let $f \in H^1(Q)$. In view of [40, Theorem 5, p 181], there exists an extension operator $\mathcal{E} \in \mathcal{B}\left(H^1(Q), H^1(\mathbb{R}^{1+n})\right)$ such that $\mathcal{E}f_{|Q} = f$. Set $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^{1+n})$ and $R > 0$ such that $\chi = 1$ on a neighborhood of $\overline{Q}$ and $\mathrm{supp}\chi \subset B_R$ with B_R the ball of radius R and of center 0 of $\mathbb{R}^{1+n}$. Let E_P be a regular fundamental solution of P. Now consider the operator

$$E : f \longmapsto (E_P * (\chi\mathcal{E}f))_{|Q} .$$

Clearly we have

$$P(D)E_P * (\chi\mathcal{E}f) = \chi\mathcal{E}f$$

and it follows that

$$P(D)Ef = (\chi\mathcal{E}f)_{|Q} = f$$

which proves (1). Now let us show (2). For this purpose, let $\psi \in \mathcal{C}^\infty_0(\mathbb{R}^{1+n})$ be such that $\psi = 1$ on the closure of $B_R - B_R = \{x - y : \ x, y \in B_R\}$ and notice that

$$(E_P * (\chi\mathcal{E}f))_{|Q} = ((\psi E_P) * (\chi\mathcal{E}f))_{|Q} . \tag{3.6}$$

Moreover, the Fourier transform $\mathcal{F}(S(D)\psi E_P)$ of $S(D)\psi E_P$ satisfies

$$|\mathcal{F}(S(D)\psi E_P)(\mu,\xi)| \leqslant \frac{|S(\mu,\xi)|}{\tilde{P}(\mu,\xi)}\tilde{P}(\mu,\xi)\left|\mathcal{F}\left(\psi\cosh(|(t,x)|)\frac{E_P}{\cosh(|(t,x)|)}\right)(\mu,\xi)\right|, \ \mu\in\mathbb{R}, \ \xi\in\mathbb{R}^n.$$

Here $\mathcal{F}$ denotes the Fourier transform in the sense of $\mathcal{S}'(\mathbb{R}^{1+n})$. Then, since $\psi\cosh(|(t,x)|) \in \mathcal{C}^\infty_0(\mathbb{R}^{1+n})$, from [8, Lemma 2.1] we deduce that

$$\psi\cosh(|(t,x)|)\frac{E_P}{\cosh(|(t,x)|)} \in B_{\infty,\tilde{P}}$$

and

$$\left\|\psi\cosh(|(t,x)|)\frac{E_P}{\cosh(|(t,x)|)}\right\|_{B_{\infty,\tilde{P}}}\leqslant C_1\left\|\frac{E_P}{\cosh(|(t,x)|)}\right\|_{B_{\infty,\tilde{P}}}\leqslant C'$$

with $C'>0$ a constant depending only on the degree of P and χ. It follows that $S(D)\psi E_P\in B_{\infty,1}$ and

$$\|S(D)\psi E_P\|_{B_{\infty,1}}\leqslant C'\sup_{(\mu,\xi)\in\mathbb{R}\times\mathbb{R}^n}\frac{|S(\mu,\xi)|}{\tilde{P}(\mu,\xi)}.$$

In view of Remark 2, since $\chi\mathcal{E}f\in H^1(\mathbb{R}^{1+n})=B_{2,\kappa_1}$ with κ_1 introduced in Remark 1, we have $S(D)(\psi E_P)*(\chi\mathcal{E}f)=(S(D)\psi E_P)*(\chi\mathcal{E}f)\in B_{2,\kappa_1}$ and

$$\begin{aligned}\|S(D)(\psi E_P)*(\chi\mathcal{E}f)\|_{H^1(\mathbb{R}^{1+n})}&=\|S(D)(\psi E_P)*(\chi\mathcal{E}f)\|_{B_{2,\kappa_1}}\\&\leqslant\|S(D)\psi E_P\|_{B_{\infty,1}}\|\chi\mathcal{E}f\|_{H^1(\mathbb{R}^{1+n}))}\\&\leqslant C\sup_{(\mu,\xi)\in\mathbb{R}\times\mathbb{R}^n}\frac{|S(\mu,\xi)|}{\tilde{P}(\mu,\xi)}\|f\|_{H^1(Q)}\end{aligned}$$

with $C>0$ a constant depending only on the degree of P, χ, Ω and T. Thus, in view of (3.6), we have $S(D)Ef\in H^1(Q)$ and

$$\|S(D)Ef\|_{H^1(Q)}\leqslant\|(\psi E_P)*(\chi\mathcal{E}f)\|_{H^1(Q)}\leqslant C\sup_{(\mu,\xi)\in\mathbb{R}\times\mathbb{R}^n}\frac{|S(\mu,\xi)|}{\tilde{P}(\mu,\xi)}\|f\|_{H^1(Q)}.$$

□

Armed with this result, we are now in position to build GO of the form (3.2) lying in $H^2(Q)$.

3.2. **Construction of geometric optics solutions.** The goal of this subsection is to apply the results of the previous subsection in order to build geometric optics of the form (3.2). For this purpose, for all $s\in\mathbb{R}$ and all $\omega\in\mathbb{S}^{n-1}$, we consider the operators $P_{s,\omega}$ defined by $P_{s,\omega}=e^{-s(t+x\cdot\omega)}\Box e^{s(t+x\cdot\omega)}$. One can check that

$$P_{s,\omega}=p_{s,\omega}(D_t,D_x)=\Box+2s(\partial_t-\omega\cdot\nabla_x)$$

with $D_t=-i\partial_t$, $D_x=-i\nabla_x$ and $p_{s,\omega}(\mu,\xi)=-\mu^2+|\xi|^2+2is(\mu-\omega\cdot\xi)$, $\mu\in\mathbb{R}$, $\xi\in\mathbb{R}^n$. Applying Proposition 2 to $P_{-\lambda,\omega}$ we obtain the following intermediate result.

Lemma 1. *For every $\lambda>1$ and $\omega\in\mathbb{S}^{n-1}$ there exists a bounded operator $E_{\lambda,\omega}:\ H^1(Q)\to H^1(Q)$ such that:*

$$P_{-\lambda,\omega}E_{\lambda,\omega}f=f,\quad f\in H^1(Q),\tag{3.7}$$

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(H^1(Q))}\leqslant C\lambda^{-1},\tag{3.8}$$

$$E_{\lambda,\omega}\in\mathcal{B}(H^1(Q);H^2(Q))\quad\textit{and}\quad\|E_{\lambda,\omega}\|_{\mathcal{B}(H^1(Q);H^2(Q))}\leqslant C\tag{3.9}$$

with $C>$ depending only on T and Ω.

Proof. In light of Proposition 2, there exists a bounded operator $E_{\lambda,\omega}:\ H^1(Q)\to H^1(Q)$, defined from a fundamental solution associated to $P_{-\lambda,\omega}$, such that (3.7) is fulfilled. In addition, for all differential operator $Q(D_t,D_x)$ such that $\frac{Q(\mu,\xi)}{\tilde{p}_{-\lambda,\omega}(\mu,\xi)}$ is bounded, we have $Q(D_t,D_x)E_{\lambda,\omega}\in\mathcal{B}(H^1(Q))$ and

$$\|Q(D_t,D_x)E_{\lambda,\omega}\|_{\mathcal{B}(H^1(Q))}\leqslant C\sup_{(\mu,\xi)\in\mathbb{R}^{1+n}}\frac{|Q(\mu,\xi)|}{\tilde{p}_{-\lambda,\omega}(\mu,\xi)},\tag{3.10}$$

where $\tilde{p}_{-\lambda,\omega}$ is given by

$$\tilde{p}_{-\lambda,\omega}(\mu,\xi)=\left(\sum_{k\in\mathbb{N}}\sum_{\alpha\in\mathbb{N}^n}|\partial_\mu^k\partial_\xi^\alpha p_{-\lambda,\omega}(\mu,\xi)|^2\right)^{\frac{1}{2}}$$

and $C>0$ depends only on Ω, T. Note that $\tilde{p}_{-\lambda,\omega}(\mu,\xi)\geqslant|\Im\partial_\mu p_{-\lambda,\omega}(\mu,\xi)|=2\lambda$. Therefore, (3.10) implies

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(H^1(Q))}\leqslant C\sup_{(\mu,\xi)\in\mathbb{R}^{1+n}}\frac{1}{\tilde{p}_{-\lambda,\omega}(\mu,\xi)}\leqslant C\lambda^{-1}$$

and (3.8) is fulfilled. In a same way, we have $\tilde{p}_{-\lambda,\omega}(\mu,\xi)\geqslant|\Re\partial_\mu p_{-\lambda,\omega}(\mu,\xi)|=2|\mu|$ and $\tilde{p}_{-\lambda,\omega}(\mu,\xi)\geqslant|\Re\partial_{\xi_i}p_{-\lambda,\omega}(\mu,\xi)|=2|\xi_i|$, $i=1,\ldots,n$ and $\xi=(\xi_1,\ldots,\xi_n)$. Therefore, in view of condition (2) of Proposition 2, for all $h\in H^1(Q)$, we have $\partial_t E_{\lambda,\omega}h$, $\partial_{x_1}E_{\lambda,\omega}h,\ldots,\partial_{x_n}E_{\lambda,\omega}h\in H^1(Q)$ with

$$\|\partial_t E_{\lambda,\omega}h\|_{H^1(Q)}+\sum_{j=1}^n\left\|\partial_{x_j}E_{\lambda,\omega}h\right\|_{H^1(Q)}\leqslant C\left(\sup_{(\mu,\xi)\in\mathbb{R}^{1+n}}\frac{|\mu|+|\xi_1|+\ldots+|\xi_n|}{\tilde{p}_{-\lambda,\omega}(\mu,\xi)}\right)\|h\|_{H^1(Q)}\leqslant C(n+1)\|h\|_{H^1(Q)}.$$

Thus, we get $E_{\lambda,\omega}\in\mathcal{B}(H^1(Q);H^2(Q))$ with

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(H^1(Q);H^2(Q))}\leqslant C\sup_{(\mu,\xi)\in\mathbb{R}^{1+n}}\frac{|\mu|+|\xi_1|+\ldots+|\xi_n|}{\tilde{p}_{-\lambda,\omega}(\mu,\xi)}\leqslant C(n+1)$$

and (3.9) is proved. □

In light of this result, we are now in position to build geometric optics solutions of the form (3.2) lying in $H^2(Q)$.

Theorem 4. *Let $q\in W^{1,p}(Q)$, with $p>n+1$, be such that $\|q\|_{W^{1,p}(Q)}\leqslant M$, $\omega\in\mathbb{S}^{n-1}$, $\lambda>1$. Then, there exists $\lambda_0>1$ such that for $\lambda\geqslant\lambda_0\|\chi\|_{H^3(Q)}$ the equation* (3.1) *admits a solution $u\in H^2(Q)$ of the form* (3.2) *with*

$$\|w\|_{H^k(Q)}\leqslant C\lambda^{k-2}\|\chi\|_{H^3(Q)},\quad k=1,2,\tag{3.11}$$

where C and λ_0 depend on Ω, T, M, n, p.

Proof. We start by recalling that

$$\Box e^{-\lambda(t+x\cdot\omega)}\chi(t,x)=e^{-\lambda(t+x\cdot\omega)}\Box\chi(t,x),\quad(t,x)\in Q.$$

Thus, w should be a solution of

$$\partial_t^2w-\Delta_xw-2\lambda(\partial_t-\omega\cdot\nabla_x)w=-\left((\Box+q)\chi(t,x)+qw\right).\tag{3.12}$$

Note that since $q\in W^{1,p}(Q)$ with $p>n+1$, using the Sobolev embedding theorem (e.g. [15, Theorem 1.4.4.1]) and Hölder inequality, one can check for all $w\in H^1(Q)$, $qw\in H^1(Q)$ with

$$\|qw\|_{H^1(Q)}\leqslant CM\|w\|_{H^1(Q)}\tag{3.13}$$

with C depending only on T, Ω, n, p. Therefore, according to Lemma 1, we can define w as a solution of the equation

$$w=-E_{\lambda,\omega}\left((\Box+q)\chi(t,x)+qw\right),\quad w\in H^1(Q)$$

with $E_{\lambda,\omega}\in\mathcal{B}(H^1(Q))$ given by Lemma 1. For this purpose, we will use a standard fixed point argument associated to the map

$$\begin{array}{rccl}\mathcal{G}: & H^1(Q) & \to & H^1(Q),\\ & F & \mapsto & -E_{\lambda,\omega}\left[(\Box+q)\chi(t,x)+qF\right].\end{array}$$

Indeed, in view of (3.8), fixing $M_1>0$, there exists $\lambda_0>1$ such that for $\lambda\geqslant\lambda_0\|\chi\|_{H^3(Q)}$ the map $\mathcal{G}$ admits a unique fixed point w in $\{u\in H^1(Q):\|u\|_{H^1(Q)}\leqslant M_1\}$. In addition, condition (3.8)-(3.9) imply that $w\in H^2(Q)$ fulfills (3.11). This completes the proof. □

4. Stability estimate

This section is devoted to the proof of Theorem 1. We start by collecting some tools of [27] that will play an important role in the proof of Theorem 1.

4.1. **Carleman estimate and geometric optics solutions vanishing on parts of the boundary.** The goal of this section is to recall some useful tools for the proof of Theorem 1. We first consider the following Carleman estimate.

Theorem 5. (Theorem 2, [27]) *Let $q \in L^\infty(Q)$, $\omega \in \mathbb{S}^{n-1}$ and $u \in \mathcal{C}^2(\overline{Q})$. If u satisfies the condition*

$$u_{|\Sigma} = 0, \quad u_{|t=0} = \partial_t u_{|t=0} = 0 \tag{4.1}$$

then there exists $\lambda_1 > 1$ depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$ such that the estimate

$$\begin{aligned}&\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|\partial_t u_{|t=T}\right|^2 dx + \lambda \int_{\Sigma_{+,\omega}} e^{-2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt + \lambda^2 \int_Q e^{-2\lambda(t+\omega\cdot x)} |u|^2 dxdt \\ &\leqslant C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta_x + q)u\right|^2 dxdt + \lambda^3 \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|u_{|t=T}\right|^2 dx + \lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} \left|\nabla_x u_{|t=T}\right|^2 dx\right) \\ &\quad + C\lambda \int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)} \left|\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt\end{aligned} \tag{4.2}$$

holds true for $\lambda \geqslant \lambda_1$ with C and λ_1 depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$.

We precise that this Carleman estimate has been proved in [27] following some arguments of [4].

From now on, for all $y \in \mathbb{S}^{n-1}$ and all $r > 0$, we set

$$\partial\Omega_{+,r,y} = \{x \in \partial\Omega :\ \nu(x)\cdot y > r\}, \quad \partial\Omega_{-,r,y} = \{x \in \partial\Omega :\ \nu(x)\cdot y \leqslant r\}$$

and $\Sigma_{\pm,r,y} = (0,T)\times\partial\Omega_{\pm,r,y}$. Here and in the remaining of this text we always assume, without mentioning it, that y and r are chosen in such way that $\partial\Omega_{\pm,r,\pm y}$ contain a non-empty relatively open subset of $\partial\Omega$. Without lost of generality we can assume that there exists $0 < \varepsilon < 1$ such that for all $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$ we have $\partial\Omega_{-,\varepsilon,-\omega} \subset F'$. We consider $u \in H_\Box(Q)$ satisfying

$$\left\{\begin{array}{l} (\partial_t^2 - \Delta_x + q(t,x))u = 0 \ \text{ in } Q, \\ u_{|t=0} = 0, \\ u = 0, \quad \text{ on } \Sigma_{+,\varepsilon/2,-\omega}, \end{array}\right. \tag{4.3}$$

of the form

$$u(t,x) = e^{\lambda(t+\omega\cdot x)}\left(\chi(t,x) + z(t,x)\right), \quad (t,x) \in Q. \tag{4.4}$$

Here $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$, $\chi(t,x) = \varphi(x + t\omega)$, $z \in e^{-\lambda(t+\omega\cdot x)}H_\Box(Q)$ fulfills: $z(0,x) = -\chi(0,x)$, $x \in \Omega$, $z = -\chi(t,x)$ on $\Sigma_{+,\varepsilon/2,-\omega}$ and

$$\|z\|_{L^2(Q)} \leqslant C\lambda^{-\frac{1}{2}} \|\chi\|_{H^2(Q)} \tag{4.5}$$

with C depending on F', Ω, T, p, n and M. Since $\Sigma \setminus F \subset \Sigma_{+,\varepsilon,-\omega}$ and since $\Sigma_{+,\varepsilon/2,-\omega}$ is a neighborhood of $\Sigma_{+,\varepsilon,-\omega}$ in Σ, it is clear that condition (4.3) implies $(\tau_{0,1}u, \tau_{0,3}u) \in \mathcal{H}_F(\partial Q)$ (recall that for $v \in \mathcal{C}^\infty(\overline{Q})$, $\tau_{0,1}v = v_{|\Sigma}$, $\tau_{0,3}v = \partial_t v_{|t=0}$). Repeating some arguments of [27, Theorem 3], we prove the following.

Theorem 6. *Let $q \in L^\infty(Q)$. For all $\lambda \geqslant \lambda_1$, with λ_1 the constant of Theorem 5, there exists a solution $u \in H_\Box(Q)$ of (4.3) of the form (4.4) with z satisfying (4.5).*

Proof. Note first that z must satisfy

$$\left\{\begin{array}{l} z \in L^2(Q) \\ (\partial_t^2 - \Delta_x + q)(e^{\lambda(t+\omega\cdot x)}z) = -e^{\lambda(t+\omega\cdot x)}(\Box + q)\chi(t,x) \ \text{ in } Q \\ z(0,x) = -\chi(0,x), \quad x \in \Omega, \\ z = -\chi(t,x) \quad \text{on } \Sigma_{+,\varepsilon/2,-\omega}. \end{array}\right. \tag{4.6}$$

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\mathrm{supp}\psi \cap \partial\Omega \subset \{x \in \partial\Omega :\ \omega\cdot\nu(x) < -\varepsilon/3\}$ and $\psi = 1$ on $\{x \in \partial\Omega : \omega\cdot\nu(x) < -\varepsilon/2\} = \partial\Omega_{+,\varepsilon/2,-\omega}$. Choose $v_-(t,x) = -e^{\lambda(t+\omega\cdot x)}\psi(x)\chi(t,x)$, $v(t,x) = -e^{\lambda(t+\omega\cdot x)}(\Box + q)\chi(t,x)$ and $v_0(x) = -e^{\lambda\omega\cdot x}\chi(0,x)$. Then, in view of [27, Lemma 3], there exists $w \in H_\Box(Q)$ such that

$$\left\{\begin{array}{ll} (\partial_t^2 - \Delta_x + q)w = v(t,x) = -e^{\lambda(t+\omega\cdot x)}(\Box + q)\chi(t,x) & \text{in } Q, \\ w(0,x) = v_0(x) = -e^{\lambda\omega\cdot x}\chi(0,x), & x \in \Omega, \\ w(t,x) = v_-(t,x) = -e^{\lambda(t+\omega\cdot x)}\psi(x)\chi(t,x), & (t,x) \in \Sigma_{-,\omega}. \end{array}\right.$$

For $z=e^{-\lambda(t+\omega\cdot x)}w$ condition (4.6) will be fulfilled. Moreover, in light of [27, Lemma 3], we have (4.5). $\square$

Armed with these results we are now in position to complete the proof of Theorem 1.

4.2. **Proof of Theorem 1.** In this subsection we complete the proof of Theorem 1. We start with two intermediate results. From now on we set $q=q_2-q_1$ on Q and assume that $q=0$ on $\mathbb{R}^{1+n}\setminus Q$. Without lost of generality we assume that $0\in\Omega$ and that for all $\omega\in\{y\in\mathbb{S}^{n-1}:|y-\omega_0|\leqslant\varepsilon\}$ we have $\partial\Omega_{-,\varepsilon,\omega}\subset G'$ with ε introduced in the previous subsection (see 2 lines before (4.3)). Let us consider the light-ray transform of q (see [35] and [37]) given by

$$\mathcal{R}q(x,\omega)=\int_{\mathbb{R}}q(t,x+t\omega)dt,\quad x\in\mathbb{R}^n,\ \omega\in\mathbb{S}^{n-1}.$$

Using the Carleman estimate introduced in (4.2) and the geometric optics solutions of Theorem 4 and Theorem 6, we obtain the following estimate of $\mathcal{R}q$.

Lemma 2. *Assume that the conditions of Theorem 1 are fulfilled. Then, there exists $\lambda_2>1$, such that for all $\lambda>\lambda_2$, $\omega\in\{y\in\mathbb{S}^{n-1}:|y-\omega_0|\leqslant\varepsilon\}$, we have*

$$\|\mathcal{R}q(\cdot,\omega)\|_{L^1(\mathbb{R}^n)}\leqslant C\left(\lambda^{-\frac{\alpha}{4+2\alpha}}+e^{d\lambda}\|B_{q_1}-B_{q_2}\|\right)\tag{4.7}$$

with $\alpha=1-\frac{n+1}{p}$ and d,C depending only on Ω, M, T, F', G', p, n.

Proof. Let $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $0\leqslant\varphi\leqslant1$, supp$\varphi\subset\{x\in\mathbb{R}^n:\ |x|\leqslant1\}$, $\|\varphi\|_{L^2(\mathbb{R}^n)}=1$. For $0<\delta<1$, we set

$$\chi_\delta(t,x,y)=\delta^{-n/2}\varphi\left(\delta^{-1}(y-x-t\omega)\right),\quad t\in\mathbb{R},\ x\in\mathbb{R}^n,\ y\in\mathbb{R}^n.$$

Note that

$$\|\chi_\delta(.,.,y)\|_{H^k(Q)}\leqslant C\delta^{-k},\quad y\in\mathbb{R}^n\tag{4.8}$$

with C independent of δ and y. We fix $\lambda_2=\max(C\lambda_0+1,\lambda_1)^{\frac{\alpha+3}{\alpha}}$ with λ_0 the constant introduced in Theorem 4, λ_1 the constant introduced in Theorem 5 and C the constant of the previous estimate. Let $\omega\in\{y\in\mathbb{S}^{n-1}:|y-\omega_0|\leqslant\varepsilon\}$ and let $\lambda>\lambda_2$ with $\lambda\geqslant\delta^{-\alpha-3}$, $\omega\in\{y\in\mathbb{S}^{n-1}:|y-\omega_0|\leqslant\varepsilon\}$. Then, we have

$$\lambda>\lambda_2^{\frac{\alpha}{\alpha+3}}\lambda^{\frac{3}{\alpha+3}}\geqslant\lambda_0C\delta^{-3}\geqslant\lambda_0\|\chi_\delta(.,.,y)\|_{H^3(Q)},\quad y\in\mathbb{R}^n$$

and, in view of Theorem 4, we can introduce

$$u_1(t,x)=e^{-\lambda(t+x\cdot\omega)}\left(\chi_\delta(t,x,y)+w(t,x)\right),\ (t,x)\in Q,\ y\in\mathbb{R}^n,$$

where $u_1\in H^2(Q)$ satisfies $\partial_t^2u_1-\Delta_xu_1+q_1u_1=0$ and w satisfies (3.11). Moreover, in view of Theorem 6, we consider $u_2\in H_\Box(Q)$ a solution of (4.3) with $q=q_2$ of the form

$$u_2(t,x)=e^{\lambda(t+x\cdot\omega)}\left(\chi_\delta(t,x,y)+z(t,x)\right),\quad (t,x)\in Q,\ y\in\mathbb{R}^n$$

with z satisfying (4.5), such that supp$\tau_{0,1}u_2\subset F$ and $\tau_{0,2}u_2=0$. Let w_1 be the solution of

$$\begin{cases}\partial_t^2w_1-\Delta_xw_1+q_1w_1=0 & \text{in } Q,\\ w_{1|t=0}=0,\ \partial_tw_{1|t=0}=\tau_{0,3}u_2 & \text{on } \Omega,\\ w_{1|\Sigma}=\tau_{0,1}u_2.\end{cases}\tag{4.9}$$

Then, $u=w_1-u_2$ solves

$$\begin{cases}\partial_t^2u-\Delta_xu+q_1u=(q_2-q_1)u_2 & \text{in } Q,\\ u(0,x)=\partial_tu(0,x)=0 & \text{on } \Omega,\\ u=0 & \text{on } \Sigma\end{cases}\tag{4.10}$$

and since $(q_2-q_1)u_2\in L^2(Q)$, in view of [3, Theorem A.2], we deduce that $u\in\mathcal{C}^1([0,T];L^2(\Omega))\cap\mathcal{C}([0,T];H_0^1(\Omega))$ with $\partial_\nu u\in L^2(\Sigma)$. Moreover we have

$$\|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))}+\|u\|_{\mathcal{C}([0,T];H_0^1(\Omega))}+\|\partial_\nu u\|_{L^2(\Sigma)}\leqslant 2CM\|u_2\|_{L^2(Q)}.$$

Applying the Green formula with respect to $x \in \Omega$ and integration by parts with respect to $t \in (0,T)$, we find

$$\begin{aligned}\int_Q qu_2u_1dxdt &= \int_Q (\partial_t^2 - \Delta_x + q_1)uu_1dxdt \\ &= -\int_G \partial_\nu uu_1d\sigma(x)dt - \int_{\Sigma\setminus G} \partial_\nu uu_1d\sigma(x)dt \\ &\quad + \int_\Omega \partial_t u(T,x)u_1(T,x)dx - \int_\Omega u(T,x)\partial_t u_1(T,x)dx.\end{aligned} \tag{4.11}$$

Combining (4.8) and (3.11), we find

$$\|u_1\|_{H^2(Q)} \leqslant C\delta^{-3}e^{c\lambda} \tag{4.12}$$

with $c = T + \mathrm{Diam}(\Omega) + 1$, C depending on M, T, Ω, n, p. Moreover, in view of estimate (3.11), we have

$$\|w\|_{L^2(\Sigma)} \leqslant C\|w\|_{L^2(0,T;H^1(\Omega))} \leqslant C\|w\|_{H^1(Q)} \leqslant C,$$

where C depends on Ω, T, n, p and M. Applying this estimate, (4.8), (4.12) and the Cauchy Schwarz inequality, we obtain

$$\begin{aligned}\left|\int_{\Sigma\setminus G} \partial_\nu uu_1d\sigma(x)dt\right| &\leqslant \int_{\Sigma_{+,\varepsilon,\omega}} \left|\partial_\nu ue^{-\lambda(t+x\cdot\omega)}(\chi_\delta + w)\right| dtd\sigma(x) \\ &\leqslant C\delta^{-1}\left(\int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-\lambda(t+x\cdot\omega)}\partial_\nu u\right|^2 d\sigma(x)dt\right)^{\frac{1}{2}},\end{aligned}$$

$$\left|\int_G \partial_\nu uu_1d\sigma(x)dt\right| \leqslant C\delta^{-1}e^{c\lambda}\|\partial_\nu u\|_{L^2(G)}$$

for some C depending only on Ω, T, n, p and M. Here we use the fact that $\Sigma\setminus G \subset \Sigma_{+,\varepsilon,\omega}$ In the same way, (3.11) imply

$$\left\|w_{|t=T}\right\|_{L^2(\Omega)} \leqslant C\|w\|_{H^1(0,T;L^2(\Omega))} \leqslant C\|w\|_{H^1(Q)} \leqslant C$$

and

$$\left\|\partial_t w_{|t=T}\right\|_{L^2(\Omega)} \leqslant C\|w\|_{H^2(0,T;L^2(\Omega))} \leqslant C\|w\|_{H^2(Q)} \leqslant C\delta^{-3}$$

with C a generic constant depending on Ω, T, M, n, p. Thus, we obtain

$$\left|\int_\Omega \partial_t u(T,x)u_1(T,x)dx\right| \leqslant C\left(\int_\Omega \left|e^{-\lambda(T+x\cdot\omega)}\partial_t u(T,x)\right|^2 dx\right)^{\frac{1}{2}},$$

$$\left|\int_\Omega u(T,x)\partial_t u_1(T,x)dx\right| \leqslant C\delta^{-3}e^{c\lambda}\left(\int_\Omega |u(T,x)|^2dx\right)^{\frac{1}{2}}.$$

In view of these estimates and (4.11), we have

$$\begin{gathered}\left|\int_Q qu_2u_1dxdt\right|^2 \\ \leqslant C\delta^{-1}\left(\int_\Omega \left|e^{-\lambda(T+x\cdot\omega)}\partial_t u(T,x)\right|^2 dx + \int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-\lambda(t+x\cdot\omega)}\partial_\nu u\right|^2 d\sigma(x)dt\right) \\ C\delta^{-6}e^{2c\lambda}\left(\|\partial_\nu u\|_{L^2(G)}^2 + \left\|u_{|t=T}\right\|_{H^1(\Omega)}^2\right)\end{gathered} \tag{4.13}$$

where C depends on Ω, T, M, n, p. On the other hand, the Carleman estimate (4.2) and the fact that $\partial\Omega_{+,\varepsilon,\omega}\subset\partial\Omega_{+,\omega}$ imply

$$\begin{aligned}
&\int_{\Sigma_{+,\varepsilon,\omega}}\left|e^{-\lambda(t+x\cdot\omega)}\partial_\nu u\right|^2 d\sigma(x)dt+\int_\Omega\left|e^{-\lambda(T+x\cdot\omega)}\partial_t u(T,x)\right|^2dx\\
&\leqslant\varepsilon^{-1}\left(\int_{\Sigma_{+,\omega}}\left|e^{-\lambda(t+x\cdot\omega)}\partial_\nu u\right|^2\omega\cdot\nu(x)d\sigma(x)dt+\int_\Omega\left|e^{-\lambda(T+x\cdot\omega)}\partial_t u(T,x)\right|^2dx\right)\\
&\leqslant\frac{\varepsilon^{-1}C}{\lambda}\left(\int_Q\left|e^{-\lambda(t+x\cdot\omega)}(\partial_t^2-\Delta_x+q_1)u\right|^2dxdt+\int_{\Sigma_{-,\omega}}\left|e^{-\lambda(t+x\cdot\omega)}\partial_\nu u\right|^2|\omega\cdot\nu(x)|\,d\sigma(x)dt\right)\\
&\quad+\varepsilon^{-1}C\left(\lambda^3\int_\Omega e^{-2\lambda(T+x\cdot\omega)}\left|u_{|t=T}\right|^2dx+\lambda\int_\Omega e^{-2\lambda(T+x\cdot\omega)}\left|\nabla_x u_{|t=T}\right|^2dx\right)\\
&\leqslant\frac{\varepsilon^{-1}C}{\lambda}\left(\int_Q\left||q|^2\left(\chi_\delta(t,x,y)+|z|\right)^2\right|^2dxdt\right)+C\delta^{-6}e^{2c\lambda}(\|\partial_\nu u\|^2_{L^2(G)}+\|u_{t=T}\|^2_{H^1(\Omega)})
\end{aligned}$$

Combining this with (4.13), we obtain

$$\left|\int_Q qu_1u_2dxdt\right|^2\leqslant\frac{C}{\lambda}+C\delta^{-6}e^{2c\lambda}(\|\partial_\nu u\|^2_{L^2(G)}+\|u_{t=T}\|^2_{H^1(\Omega)})\tag{4.14}$$

with C depending only on Ω, T, G', M, n, p. On the other hand, we have

$$\int_Q qu_1u_2dxdt=\int_{\mathbb{R}^{1+n}}q(t,x)\chi_\delta^2(t,x,y)dxdt+\int_Q Z(t,x)dxdt$$

with $Z=q(z\chi_\delta+w\chi_\delta+zw)$. Then, in view of (3.11) and (4.5), an application of the Cauchy-Schwarz inequality yields

$$\left|\int_Q Z(t,x)dxdt\right|\leqslant C(\delta^{-2}\lambda^{-\frac12}+\delta^{-3}\lambda^{-1})$$

with C depending on Ω, T, G', M. Combining this estimate with (4.14), we obtain

$$|V_{\delta,q}(y)|^2\leqslant C\left(\delta^{-4}\lambda^{-1}+\delta^{-6}\lambda^{-2}+\delta^{-6}e^{2c\lambda}(\|\partial_\nu u\|^2_{L^2(G)}+\|u_{t=T}\|^2_{H^1(\Omega)})\right)$$

with

$$V_{\delta,q}(y)=\int_{\mathbb{R}^{1+n}}q(t,x)\delta^{-n}\varphi^2(\delta^{-1}(y-x-t\omega))dxdt=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}}q(t,x+t\omega)dt\right)\delta^{-n}\varphi^2(\delta^{-1}(y-x))dx,\quad y\in\mathbb{R}^n.$$

On the other hand, one can check that $\mathrm{supp}V_{\delta,q}\subset\{y:\ |y|\leqslant T+\mathrm{Diam}(\Omega)+1\}$ and from the previous estimate we get

$$\|V_{\delta,q}\|_{L^1(\mathbb{R}^n)}\leqslant C\left[\delta^{-2}\lambda^{-1/2}+\delta^{-3}\lambda^{-1}+\delta^{-3}e^{c\lambda}\left(\|\partial_\nu u\|_{L^2(G)}+\|u_{t=T}\|_{H^1(\Omega)}\right)\right]\tag{4.15}$$

In order to complete the proof of the lemma we only need to check that this estimate implies (4.7). For this purpose using the fact that $q\in W^{1,p}(Q)$ with $p>n+1$, by the Sobolev embedding theorem (e.g. [15, Theorem 1.4.4.1]) we have $q\in\mathcal{C}^\alpha(\overline{Q})$ with $\|q\|_{\mathcal{C}^\alpha(\overline{Q})}\leqslant CM$ with C depending on Ω, n, p and T. Moreover, applying (1.3), we deduce that for all $t\in\mathbb{R}$, $q(t,\cdot)\in\mathcal{C}^\alpha(\mathbb{R}^n)$. Thus, using the fact that

$$\mathcal{R}q(x,\omega)=\int_0^T q(t,x+t\omega)dt,$$

we deduce that $\mathcal{R}q(\cdot,\omega)\in\mathcal{C}^\alpha(\mathbb{R}^n)$ and $\mathrm{supp}\mathcal{R}q(\cdot,\omega)\subset\{x\in\mathbb{R}^n:\ |x|\leqslant\mathrm{Diam}(\Omega)+T\}$. Combining this with the fact that

$$V_{\delta,q}(y)=\int_{\mathbb{R}^n}\mathcal{R}q(y-\delta u,\omega)\varphi^2(u)du,$$

we obtain

$$\|V_{\delta,q}-\mathcal{R}q(\cdot,\omega)\|_{L^1(\mathbb{R}^n)}\leqslant C\delta^\alpha$$

with C depending on Ω, T, M, p and n. Combining this with (4.15) we deduce (4.7) by using the fact that

$$\begin{aligned}\|\partial_\nu u\|^2_{L^2(G)} + \|u_{t=T}\|^2_{H^1(\Omega)} &\leqslant \|B_{q_1} - B_{q_2}\|^2 \|(\tau_{0,1}u_2, \tau_{0,3}u_2)\|^2_{\mathcal{H}_F(\partial Q)} \\ &\leqslant C\|B_{q_1} - B_{q_2}\|^2 \|u_2\|^2_{H_\square(Q)} \\ &\leqslant C\|B_{q_1} - B_{q_2}\|^2 \|u_2\|^2_{L^2(Q)} \\ &\leqslant C\delta^{-4}e^{2c\lambda}\|B_{q_1} - B_{q_2}\|^2\end{aligned}$$

and by choosing $\delta = \lambda^{-\frac{1}{4+2\alpha}}$ and $d = 2c+1$. Here we have used (4.5) and the fact that

$$\|u_2\|^2_{H_\square(Q)} = (1 + \|q_2\|^2_{L^\infty(Q)})\|u_2\|^2_{L^2(Q)}.$$

This completes the proof of the lemma. $\square$

From now on, for all $r > 0$, we denote by B_r the set $B_r = \{z \in \mathbb{R}^{1+n} : \ |z| < r\}$. Let us recall the following result, which follows from [1, Theorem 3] (see also [42]), on the continuous dependence in the analytic continuation problem.

Proposition 3. *Let $\rho > 0$ and assume that $f : \ B_{2\rho} \subset \mathbb{R}^{1+n} \to \mathbb{C}$ is a real analytic function satisfying*

$$\left\|\partial^\beta f\right\|_{L^\infty(B_{2\rho})} \leqslant \frac{N\beta!}{(\rho\lambda)^{|\beta|}}, \quad \beta \in \mathbb{N}^{1+n}$$

for some $N > 0$ and $0 < \lambda \leqslant 1$. Further let $E \subset B_{\frac{\rho}{2}}$ be a measurable set with strictly positive Lebesgue measure. Then,

$$\|f\|_{L^\infty(B_\rho)} \leqslant C(N)^{(1-b)}\left(\|f\|_{L^\infty(E)}\right)^b,$$

where $b \in (0,1)$, $C > 0$ depend on λ, $|E|$ and ρ.

Armed with Lemma 2, we will use Proposition 3 to complete the proof of Theorem 1.

Proof of Theorem 1. We set $U = \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$. For all $\xi \in \mathbb{R}^n$ we introduce

$$a(\xi) = \inf_{\omega \in U} \xi\cdot\omega, \quad b(\xi) = \sup_{\omega\in U} \xi\cdot\omega.$$

Consider the set $E_1 = \{(\tau,\xi) : \ \xi \in \mathbb{R}^n, \ a(\xi) \leqslant \tau \leqslant b(\xi)\}$ and note that for all $(\tau,\xi) \in E_1$ there exists $\omega \in U$ such that $\tau = \xi\cdot\omega$. It is clear that

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathcal{R}q(x,\omega)e^{-ix\cdot\xi}dx = (2\pi)^{-n/2}\int_{\mathbb{R}^{n+1}} q(t,x)e^{-i(t\omega\cdot\xi + x\cdot\xi)}dtdx = (2\pi)\mathcal{F}(q)(\omega\cdot\xi,\xi), \ \xi \in \mathbb{R}^n,$$

where $\mathcal{F}(q)$ is the Fourier transform of q given by

$$\mathcal{F}(q)(\tau,\xi) = (2\pi)^{-(n+1)/2}\int_{\mathbb{R}^{n+1}} q(t,x)e^{-i(t\tau + x\cdot\xi)}dtdx.$$

Thus, we get

$$|\mathcal{F}(q)(\tau,\xi)| \leqslant (2\pi)^{-(n+1)/2}\sup_{\omega\in U}\|\mathcal{R}q(.,\omega)\|_{L^1(\mathbb{R}^n)}, \quad (\tau,\xi) \in E_1.$$

Combining this with (4.7) we obtain

$$|\mathcal{F}(q)(\tau,\xi)| \leqslant C\left(\lambda^{-\frac{\alpha}{4+2\alpha}} + e^{d\lambda}\|B_{q_1} - B_{q_2}\|\right), \quad (\tau,\xi) \in E_1. \tag{4.16}$$

We set for fixed $R > 0$, which will be made precise later, and $(\tau,\xi) \in \mathbb{R}^{1+n}$,

$$H(\tau,\xi) = \mathcal{F}(q)(R(\tau,\xi)) = (2\pi)^{-\frac{(1+n)}{2}}\int_{\mathbb{R}^{1+n}} q(t,x)e^{-iR(\tau,\xi)\cdot(t,x)}dxdt.$$

Since $\mathrm{supp} q \subset \overline{Q}$ and $0 \in \Omega$, H is real analytic and

$$\left|\partial^\beta H(\tau,\xi)\right| \leqslant C\frac{\|q\|_{L^1(Q)}R^{|\beta|}}{([\max(T,\mathrm{Diam}(\Omega))]^{-1})^{|\beta|}} \leqslant C\|q\|_{L^1(Q)}\frac{R^{|\beta|}}{\beta!([\max(T,\mathrm{Diam}(\Omega))]^{-1})^{|\beta|}}\beta!, \quad \beta \in \mathbb{N}^{1+n}$$

with C depending on T and Ω. Moreover, we have

$$\|q\|_{L^1(Q)} \leqslant 2M(T\,|\Omega|)^{1-\frac{1}{p}}$$

and one can check that

$$\frac{R^{|\beta|}}{\beta!} \leqslant e^{(1+n)R}.$$

Applying these estimates, we obtain

$$\left|\partial^\beta H(\tau,\xi)\right| \leqslant C\frac{e^{(1+n)R}\beta!}{([\max(T,\mathrm{Diam}(\Omega))]^{-1})^{|\beta|}},\quad \beta\in\mathbb{N}^{1+n} \tag{4.17}$$

with C depending on M, Ω, n, p and T. Set $\rho=[\max(T,\mathrm{Diam}(\Omega))]^{-1}+1$, $E=E_1\cap\{\zeta\in\mathbb{R}^{1+n}:\ |\zeta|<\min(\frac{\rho}{2},1)\}$ with $N=Ce^{(1+n)R}$ and $\lambda=\frac{[\max(T,\mathrm{Diam}(\Omega))]^{-1}}{\rho}$. In view of (4.17), we have

$$\left\|\partial^\beta H\right\|_{L^\infty(B_{2\rho})} \leqslant C\frac{e^{(1+n)R}\beta!}{([\max(T,\mathrm{Diam}(\Omega))]^{-1})^{|\beta|}} = \frac{N\beta!}{(\rho\lambda)^{|\beta|}},\quad \beta\in\mathbb{N}^{1+n}.$$

Since for all $(\tau,\xi)\in E_1$ we have $|(\tau,\xi)|^2\leqslant 2|\xi|^2$, one can check that

$$\mathcal{A}=\{(\tau,\xi):\ (\tau,\xi)\in E_1,\ \xi/|\xi|\in U,\ |\xi|<r\}\subset E$$

with $r=\frac{\min(\rho/2,1)}{\sqrt{2}}$. On the other hand, for all $(\tau,\xi)\in\mathcal{A}$, we have $b(\xi)=|\xi|$ and for any $\omega\in U\setminus\{\xi/|\xi|\}$ we have $\omega\cdot\xi<|\xi|=b(\xi)$. Therefore, for all $(\tau,\xi)\in\mathcal{A}$ we have $a(\xi)<b(\xi)$ and, by fixing $W=\{\xi\in\mathbb{R}^n:\ \xi/|\xi|\in U,\ |\xi|<r\}$, we get

$$|\mathcal{A}|=\int_{\mathcal{A}} d\tau d\xi=\int_W\int_{a(\xi)}^{b(\xi)} d\tau d\xi=\int_W (b(\xi)-a(\xi))d\tau d\xi>0.$$

Here we have used the fact that $a,b\in\mathcal{C}(\mathbb{R}^n)$ and the fact that $|W|>0$. Thus, we have $|E|>0$. Moreover, one can easily check that for all $(\tau,\xi)\in E_1$, $r>0$, $(r\tau,r\xi)\in E_1$. Then, since $E\subset B_{\frac{\rho}{2}}$, $0<\lambda<1$ and $\rho>1$, applying Proposition 3 to H we obtain

$$|\mathcal{F}(q)(R(\tau,\xi))|=|H(\tau,\xi)|\leqslant\|H\|_{L^\infty(B_\rho)}\leqslant Ce^{(1+n)R(1-b)}\left(\|H\|_{L^\infty(E)}\right)^b,\quad |(\tau,\xi)|<1,$$

where $C>0$ and $0<b<1$ depend only on Ω, T, M, F' and G'. But, estimate (4.16) implies that

$$|H(\tau,\xi)|^2=|\mathcal{F}(q)(R(\tau,\xi))|^2\leqslant C\left(\lambda^{-\frac{2\alpha}{4+2\alpha}}+e^{2d\lambda}\left\|B_{q_1}-B_{q_2}\right\|^2\right),\ (\tau,\xi)\in E$$

and we deduce

$$|\mathcal{F}(q)(\tau,\xi)|^2\leqslant Ce^{2(1+n)(1-b)R}\left(\lambda^{-\frac{2\alpha}{4+2\alpha}}+e^{2d\lambda}\left\|B_{q_1}-B_{q_2}\right\|^2\right)^b,\ |(\tau,\xi)|<R. \tag{4.18}$$

Note that

$$\|q\|_{H^{-1}(\mathbb{R}^{1+n})}^{\frac{2}{b}}\leqslant C\left(\int_{\mathbb{R}^{1+n}}(1+|(\tau,\xi)|^2)^{-1}|\mathcal{F}(q)(\tau,\xi)|^2d\xi d\tau\right)^{\frac{1}{b}}. \tag{4.19}$$

We shall make precise below,

$$\int_{B_R}(1+|(\tau,\xi)|^2)^{-1}|\mathcal{F}(q)(\tau,\xi)|^2d\xi d\tau$$

and

$$\int_{\mathbb{R}^{1+n}\setminus B_R}(1+|(\tau,\xi)|^2)^{-1}|\mathcal{F}(q)(\tau,\xi)|^2d\xi d\tau$$

separately. We start by examining the last integral. The Parseval-Plancherel theorem and the Sobolev embedding theorem imply

$$\begin{aligned}\int_{\mathbb{R}^3\setminus B_R}(1+|(\tau,\xi)|^2)^{-1}|\mathcal{F}(q)(\tau,\xi)|^2d\xi d\tau &\leqslant \frac{1}{R^2}\int_{\mathbb{R}^{1+n}\setminus B_R}|\mathcal{F}(q)(\tau,\xi)|^2d\xi d\tau\\ &\leqslant \frac{1}{R^2}\int_{\mathbb{R}^{1+n}}|\mathcal{F}(q)(\tau,\xi)|^2d\xi d\tau=\frac{1}{R^2}\int_{\mathbb{R}^{1+n}}|q(t,x)|^2dtdx\\ &\leqslant \frac{4(T\,|\Omega|)^{\frac{p-2}{p}}M^2}{R^2}.\end{aligned}$$

We end up getting that

$$\int_{\mathbb{R}^{1+n}\setminus B_R}(1+|(\tau,\xi)|^2)^{-1}|\mathcal{F}(q)(\tau,\xi)|^2d\xi d\tau \leqslant \frac{C}{R^2}. \tag{4.20}$$

Further, in light of (4.18), we get

$$\int_{B_R}(1+|(\tau,\xi)|^2)^{-1}|\mathcal{F}(q)(\tau,\xi)|^2d\xi d\tau \leqslant CR^{1+n}e^{2(1+n)(1-b)R}\left(\lambda^{-\frac{2\alpha}{4+2\alpha}}+e^{2d\lambda}\left\|B_{q_1}-B_{q_2}\right\|^2\right)^b, \tag{4.21}$$

upon eventually substituting C for some suitable algebraic expression of C.

Last, putting (4.20)–(4.21) together we find out that

$$\begin{aligned}\|q\|^{\frac{2}{b}}_{H^{-1}(\mathbb{R}^{1+n})} &\leqslant C\left(\frac{1}{R^2}+R^{1+n}e^{2(1+n)(1-b)R}\left(\lambda^{-\frac{2\alpha}{4+2\alpha}}+e^{2d\lambda}\left\|B_{q_1}-B_{q_2}\right\|^2\right)^b\right)^{\frac{1}{b}}\\ &\leqslant C\left(R^{-\frac{2}{b}}+\lambda^{-\frac{2\alpha}{4+2\alpha}}R^{\frac{n+1}{b}}e^{2(1+n)(\frac{1-b}{b})R}+R^{\frac{n+1}{b}}e^{2(1+n)(\frac{1-b}{b})R}e^{2d\lambda}\left\|B_{q_1}-B_{q_2}\right\|^2\right),\end{aligned} \tag{4.22}$$

for $\lambda>\lambda_2$ where the constant $C>0$ depends only on Ω, T, F', G', n, p and M. Here we have used the fact that $x\mapsto x^{\frac{1}{b}}$ is convex on $(0,+\infty)$ since $b\in(0,1)$. Now let $R_1>1$ be such that

$$R^{\frac{n+3}{b}}e^{2(1+n)R(\frac{1-b}{b})}>\lambda_2^{\frac{2\alpha}{4+2\alpha}},\quad R>R_1.$$

Then, choosing $\lambda^{\frac{2\alpha}{4+2\alpha}}=R^{\frac{n+3}{b}}e^{2(1+n)R(\frac{1-b}{b})}$ we have $\lambda>\lambda_2$ and $\lambda^{-\frac{2\alpha}{4+2\alpha}}R^{\frac{n+1}{b}}e^{2(1+n)(\frac{1-b}{b})R}=R^{-\frac{2}{b}}$. With this value of λ we obtain

$$\|q\|^{\frac{2}{b}}_{H^{-1}(\mathbb{R}^{1+n})}\leqslant C\left(R^{-\frac{2}{b}}+R^{\frac{n+1}{b}}e^{2(1+n)(\frac{1-b}{b})R}\exp\left(2dR^{\frac{(n+3)(4+2\alpha)}{2\alpha b}}e^{(4+2\alpha)(1+n)R(\frac{1-b}{\alpha b})}\right)\left\|B_{q_1}-B_{q_2}\right\|^2\right). \tag{4.23}$$

On the other hand, we have

$$\begin{aligned}&R^{\frac{n+1}{b}}e^{2(1+n)(\frac{1-b}{b})R}\exp\left(2dR^{\frac{(n+3)(4+2\alpha)}{2\alpha b}}e^{(4+2\alpha)(1+n)R(\frac{1-b}{\alpha b})}\right)\\ &\leqslant \exp\left(R^{\frac{n+1}{b}}+2(1+n)\left(\tfrac{1-b}{b}\right)R+2dR^{\frac{(n+3)(4+2\alpha)}{2\alpha b}}e^{(4+2\alpha)(1+n)R(\frac{1-b}{\alpha b})}\right)\\ &\leqslant \exp\left(e^{\frac{n+1}{b}R}+e^{2(1+n)(\frac{1-b}{b})R}+e^{[2d+\frac{(n+3)(4+2\alpha)}{2\alpha b}+(4+2\alpha)(1+n)(\frac{1-b}{\alpha b})]R}\right)\\ &\leqslant \exp\left(3e^{[\frac{n+1}{b}+2(1+n)(\frac{1-b}{b})+2d+\frac{(n+3)(4+2\alpha)}{2\alpha b}+(4+2\alpha)(1+n)(\frac{1-b}{\alpha b})]R}\right)\\ &\leqslant \exp\left(e^{[3+\frac{n+1}{b}+2(1+n)(\frac{1-b}{b})+2d+\frac{(n+3)(4+2\alpha)}{2\alpha b}+(4+2\alpha)(1+n)(\frac{1-b}{\alpha b})]R}\right).\end{aligned}$$

Setting $A=3+\frac{n+1}{b}+2(1+n)(\frac{1-b}{b})+2d+\frac{(n+3)(4+2\alpha)}{2\alpha b}+(4+2\alpha)(1+n)(\frac{1-b}{\alpha b})$, (4.23) leads to

$$\|q\|^{\frac{2}{b}}_{H^{-1}(\mathbb{R}^{1+n})}\leqslant C\left(R^{-\frac{2}{b}}+e^{e^{AR}}\left\|B_{q_1}-B_{q_2}\right\|^2\right),\quad R>R_1. \tag{4.24}$$

Set $\gamma=\|B_{q_1}-B_{q_2}\|$ and $\gamma^*=e^{-e^{AR_1}}$. For $\gamma\geqslant\gamma^*$ we have

$$\|q\|_{H^{-1}(Q)}\leqslant C\|q\|_{L^\infty(Q)}\leqslant \frac{2CM}{\gamma^*}\gamma. \tag{4.25}$$

For $0<\gamma<\gamma^*$, by taking $R=R_2=\frac{1}{A}\ln(|\ln\gamma|)$ in (4.24), which is permitted since $R_2>R_1$, we find out that

$$\|q\|_{H^{-1}(Q)}\leqslant \|q\|_{H^{-1}(\mathbb{R}^{1+n})}\leqslant C\ln(|\ln\gamma|)^{-1}\left(\ln(|\ln\gamma|)^{\frac{2}{b}}\gamma+A^{\frac{2}{b}}\right)^{\frac{b}{2}}.$$

Now, since $\sup_{0<\gamma\leqslant\gamma_*}\left(\ln(|\ln\gamma|)^{\frac{2}{b}}\gamma+A^{\frac{2}{b}}\right)^{\frac{b}{2}}$ is just another constant depending only on Ω, T, F', G', n, p and M, we obtain

$$\|q\|_{H^{-1}(Q)}\leqslant C\ln(|\ln\gamma|)^{-1},\quad 0<\gamma<\gamma^*.$$

Combining this estimate with (4.25) we deduce (1.4). □

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