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DETERMINATION OF A TIME-DEPENDENT COEFFICIENT FOR WAVE EQUATIONS FROM PARTIAL DATA

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ABSTRACT. We consider the stability in the inverse problem consisting of the determination of a time-dependent coefficient of order zero q , appearing in a Dirichlet initial-boundary value problem for a wave equation $\partial_t^2 u - \Delta u + q(t, x)u = 0$ in $Q = (0, T) \times \Omega$ with Ω a bounded C^2 domain of \mathbb{R}^n , $n \geq 3$, from partial observations on ∂Q . The observation is given by a boundary operator associated to the wave equation. Using suitable complex geometric optics solutions and a Carleman estimate, we prove a stability estimate in the determination of q from the boundary operator.

Keywords: Inverse problem, wave equation, scalar time-dependent potential, Carleman estimate.

Mathematics subject classification 2010 : 25R30.

CONTENTS

1. Introduction	2
1.1. Statement of the problem	2
1.2. Existing papers	2
1.3. Main result	3
1.4. Outline	4
2. Functional space	4
3. Complex geometric optics solutions	6
3.1. Solutions of PDE with constant coefficients	7
3.2. Construction of complex geometric optics solutions	9
4. Carleman estimate	11
5. Geometric optics solutions vanishing on part of the boundary	15
5.1. Weighted space	16
5.2. Intermediate result	16
5.3. Proof of Theorem 6	17
6. Stability estimate	18
Appendix	24
Density result in $H_{\square}(Q)$	24
Trace operator in $H_{\square}(Q)$	25
References	26

1. INTRODUCTION

1.1. Statement of the problem. In the present paper we consider Ω a C^2 bounded domain of \mathbb{R}^n , $n \geq 3$. We set $\Sigma = (0, T) \times \partial\Omega$ and $Q = (0, T) \times \Omega$ with $T > 0$. We introduce the following initial-boundary value problem (IBVP in short) for the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + q(t, x)u = 0, & \text{in } Q, \\ u(0, \cdot) = v_0, \quad \partial_t u(0, \cdot) = v_1, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma, \end{cases} \quad (1.1)$$

where the potential $q \in L^\infty(Q)$. We prove that problem (1.1) is well posed in some appropriate space. More precisely, in Section 2 we introduce the space $\mathcal{H}(\partial Q)$ and prove that, for $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$, problem (1.1) admits a unique weak solution $u \in L^2(Q)$ with

$$\|u\|_{L^2(Q)} \leq C \|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)}.$$

We associate to (1.1) the boundary operator

$$B_q : (g, v_0, v_1) \mapsto (\partial_\nu u|_\Sigma, u|_{t=T}, \partial_t u|_{t=T})$$

with u the solution of (1.1) and ν the outward unit normal vector to $\partial\Omega$. Here u is the solution of (1.1) and, for u sufficiently smooth, $\partial_\nu u(x) = \nabla u(x) \cdot \nu(x)$. See also Section 2 for a rigorous definition of this operator. Consider, for all $y \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$, the set

$$\partial\Omega_{+,y} = \{x \in \partial\Omega : \nu(x) \cdot y > 0\}, \quad \partial\Omega_{-,y} = \{x \in \partial\Omega : \nu(x) \cdot y \leq 0\}$$

and for $\varepsilon > 0$

$$\partial\Omega_{+,\varepsilon,y} = \{x \in \partial\Omega : \nu(x) \cdot y > \varepsilon\}, \quad \partial\Omega_{-,\varepsilon,y} = \{x \in \partial\Omega : \nu(x) \cdot y \leq \varepsilon\}$$

with ν the outward unit normal vector to $\partial\Omega$. Note also that

$$\partial\Omega_{+,\varepsilon,-y} = \{x \in \partial\Omega : \nu(x) \cdot y < -\varepsilon\}, \quad \partial\Omega_{-,\varepsilon,-y} = \{x \in \partial\Omega : \nu(x) \cdot y \geq -\varepsilon\}.$$

We introduce

$$\Sigma_{\pm,y} = (0, T) \times \partial\Omega_{\pm,y}, \quad \Sigma_{\pm,\varepsilon,y} = (0, T) \times \partial\Omega_{\pm,\varepsilon,y}$$

and the closed subspace $F_{y,\varepsilon}$ of $\mathcal{H}(\partial Q)$ defined by

$$F_{y,\varepsilon} = \{(g, v_0, v_1) \in \mathcal{H}(\partial Q) : v_0 = 0, \text{ supp } g \subset \Sigma_{-,\varepsilon,-y}\}.$$

Let $\omega_0 \in \mathbb{S}^{n-1}$, $\varepsilon_0 > 0$. The main purpose of this paper is to prove a stability estimate in the determination of the time-dependent potential q from the partial boundary operator

$$B_q^* = B_{q,\omega_0,\varepsilon_0}^* : F_{\omega_0,\varepsilon_0} \ni (g, 0, v_1) \mapsto (\partial_\nu u|_{\Sigma_{-,\varepsilon_0,\omega_0}}, u|_{t=T}). \quad (1.2)$$

Physically speaking, our inverse problem consists of determining properties such as density of an inhomogeneous medium, that evolves over time, by probing it with disturbances generated on the boundary and at initial time. The data is the response of the medium to these disturbances, measured on the boundary and at final time, and the purpose is to recover the function which measures the property of the medium.

1.2. Existing papers. In recent years the problem of recovering coefficients for hyperbolic equations from boundary measurements has attracted many attention. Many authors have considered this problem with an observation given by the reduced boundary operator

$$\Lambda_q : g \mapsto \partial_\nu u|_\Sigma,$$

where u solves (1.1) with $v_0 = v_1 = 0$. This operator is usually called the Dirichlet to Neumann map (DN map in short). In [27], the authors proved that the DN map uniquely determines the time-independent potential in a wave equation. Isakov [17] considered the determination of a coefficient of order zero and a damping coefficient. Note that all these results are concerned with measurements on the whole boundary. The uniqueness by local DN map has been considered by [10]. The stability estimate in the case where the DN map is considered on the whole lateral boundary was treated by Stefanov and Uhlmann [32]. The uniqueness

and Hölder stability estimate in a subdomain were established by Isakov and Sun [18] and, assuming that the coefficients are known in a neighborhood of the boundary, Bellassoued, Choulli and Yamamoto [3] proved a log-type stability estimate in the case where the Neumann data are observed in an arbitrary subdomain of the boundary. In some recent work [21] extended the results of [27] to determination of large class of time-independent coefficient of order zero in an infinite cylindrical domain, also called cylindrical waveguide, and he proved that only measurements on a bounded subset are required for the determination of some class of coefficients including periodic coefficients and compactly supported coefficients.

Let us also mention that the method using Carleman inequalities was first considered by Bukhgeim and Klivanov [4]. For the application of Carleman estimate to the problem of recovering time-independent coefficients for hyperbolic equations we refer to [2], [16] and [20].

All the above mentioned results are concerned only with time-independent coefficients. Several authors considered the problem of determining time-dependent coefficients for hyperbolic equations. In [30], Stefanov proved unique determination of a time-dependent potential for the wave equation from scattering data. The result of [30] is equivalent to the consideration of the problem with boundary measurements (see also [31] where this problem is explicitly addressed). In [28], Ramm and Sjöstrand considered the problem of determining the time-dependent coefficient q from the DN map Λ_q associated to (1.1). For this purpose, they considered the problem (1.1) on the infinite time-space cylindrical domain $\mathbb{R} \times \Omega$ instead of Q ($t \in \mathbb{R}$ instead of $0 < t < T < \infty$). Then, with some additional assumptions [28] proved a result of uniqueness. In [26], Rakesh and Ramm considered the same problem at finite time on Q , with $T > \text{Diam}(\Omega)$, and they proved a result of uniqueness for the determination of q restricted to the subset S of Q , consisting of the lines which make 45 degree with the t -axis and which meet the planes $t = 0$ and $t = T$ outside \bar{Q} , from the DN map Λ_q . Applying a result of unique continuation due to [34], Eskin [11] proved that the DN map uniquely determines time-dependent coefficients that are analytic wrt the time variable t . In some recent work, [29] extended the result of [28] to more general coefficients and he proved a result of stability for compactly supported coefficients provided T is sufficiently large. One of the main point in the strategy of these authors consists of using geometric optics solutions concentrate near lines in order to recover the X-ray transform of the coefficient q from the DN map Λ_q . For time dependent coefficients this approach requires measurements on the infinite time-space cylindrical domain $\mathbb{R} \times \Omega$, otherwise one can only expect the determination of the coefficients restricted to a subdomain of Q when the coefficients in consideration are not analytic wrt t . Indeed, even with the knowledge of B_q restricted to zero initial data $v_0 = v_1 = 0$, from domain of dependence argument there is no hope to determine q on the whole domain Q . In contrast to this approach, Isakov [17] used complex geometric optics similar to the one used by [33] for elliptic equations, and he proved results of density of products of solutions for different PDE including wave equations. Applying these results, in Theorem 4.2 of [17], Isakov established a result of uniqueness for a time-dependent coefficient of order zero. For problem (1.1) this last result can be stated as uniqueness in the determination of q , on the whole domain Q , from the boundary operator B_q (see also [6] and [8] for results of stability for parabolic and Schrödinger equations). In the present paper, we prove that the knowledge of the partial boundary operator B_q^* is sufficient for the determination of the time-dependent potential q on the whole domain Q . Moreover, we derive a stability estimate for this inverse problem. In contrast to the result of [17], which seems to be the only result of determination of a time-dependent coefficient, that is not analytic wrt to t , on the whole domain Q , in the present paper we consider only initial data (v_0, v_1) with $v_0 = 0$ and Dirichlet boundary condition g supported on some part of Σ (which, roughly speaking, corresponds to half of the boundary) and we measure $u|_{t=T}$ and $\partial_\nu u$ on some part of Σ (which, roughly speaking, corresponds to the other half of the boundary).

We also mention that [6], [7] and [12] considered the problem of determining a time-dependent coefficient for parabolic equations and they derived stability estimate for this problem.

1.3. Main result. In order to express the main result of this paper we first remark (see Section 2) that for every $q_1, q_2 \in L^\infty(Q)$ the operator

$$B_{q_1}^* - B_{q_2}^* : F_{\omega_0, \varepsilon_0} \rightarrow L^2(\Sigma_{-, \varepsilon_0, \omega_0}) \times H^1(\Omega)$$

is bounded. Then our main result can be stated as follows.

Theorem 1. *Let $q_1, q_2 \in W^{1,\infty}(Q)$. Assume that*

$$\|q_1\|_{W^{1,\infty}(Q)} + \|q_2\|_{W^{1,\infty}(Q)} \leq M.$$

Then, for all $\omega_0 \in \mathbb{S}^{n-1}$ and $\varepsilon_0 > 0$ we have

$$\|q_1 - q_2\|_{L^2(Q)} \leq h(\|B_{q_1}^* - B_{q_2}^*\|) \quad (1.3)$$

with

$$h(\gamma) = \begin{cases} C \frac{\gamma}{\gamma_*}, & \gamma \geq \gamma_*, \\ C \ln(|\ln \gamma|)^{-\frac{1}{2}}, & 0 < \gamma < \gamma_*, \\ 0, & \gamma = 0. \end{cases}$$

Here $\|B_{q_1}^ - B_{q_2}^*\|$ stands for the norm of $B_{q_1}^* - B_{q_2}^*$ as an element of $\mathcal{B}(F_{\omega, \varepsilon_0}, L^2(\Sigma_{-, \varepsilon_0, \omega}) \times H^1(\Omega))$. Moreover, C is a positive constant depending on $M, \Omega, T, \varepsilon_0, \omega_0$ and $\gamma_* = e^{-e^{AR_2}}$, with A and R_2 two constant introduced in Section 5 which depend on $M, T, \Omega, \varepsilon_0, \omega_0$.*

Let us observe that the stability estimate (1.3) can be improved into a log-type stability estimate if we replace B_q^* by the full boundary operator B_q . Indeed, by combining the results of Section 3 with arguments of [6] (see subsection 3.6) and [8], one can prove a log-type stability estimate in the determination of q from B_q .

Let us remark that in this paper we only treat the case $n \geq 3$. Nevertheless, we believe that with additional technical arguments one can extend our result to the case $n = 2$. For $n = 2$, the main difficulty comes from the choice for $\zeta_j, j = 1, 2$, in Proposition 4.

The main tools in our analysis are suitable complex geometric optics (CGO in short) solutions and Carleman estimates. More precisely, in this paper we adapt to the wave equation some arguments used by many authors for elliptic equations (e.g. [5], [13], [19], [25]). This approach seems quite natural since we want to determine a coefficient that depends on every variable of our equation.

1.4. Outline. This paper is organized as follows. In Section 2 we treat the direct problem. We show that problem (1.1) is well posed in some appropriate space and we define the boundary operator B_q associated to this problem. In Section 3, using some results of [6], [14] and [15], we build suitable CGO solutions associated to (1.1) without condition on ∂Q . In Section 4, we establish a Carleman estimate for the wave equation with linear weight. In Section 5, we use the Carleman estimate introduced in Section 4 to build CGO solutions associated to (1.1) that vanish on parts of ∂Q . More precisely, we build CGO u which are solutions of (1.1) with $(g, v_0, v_1) \in F_{\omega_0, \varepsilon_0}$. In Section 6 we prove Theorem 1. We prove also some auxiliary results in the appendix.

2. FUNCTIONAL SPACE

In this section we study the IBVP (1.1). We define the space $\mathcal{H}(\partial Q)$ and its topology. We define also the boundary operator B_q in some appropriate spaces. We first introduce the space

$$J = \{u \in L^2(Q) : (\partial_t^2 - \Delta)u = 0\}$$

and topologize it as a closed subset of $L^2(Q)$. We work with the space

$$H_{\square}(Q) = \{u \in L^2(Q) : \square u = (\partial_t^2 - \Delta)u \in L^2(Q)\},$$

with the norm

$$\|u\|_{H_{\square}(Q)}^2 = \|u\|_{L^2(Q)}^2 + \|(\partial_t^2 - \Delta)u\|_{L^2(Q)}^2.$$

Here and in all this paper we denote by \square the differential operator defined by $\square u = (\partial_t^2 - \Delta)u$. Repeating some arguments of Theorem 6.4 in chapter 2 of [23] we prove in the appendix (see Theorem 7) that $H_{\square}(Q)$ embedded continuously into the closure of $\mathcal{C}^{\infty}(\bar{Q})$ in the space

$$K_{\square}(Q) = \{u \in H^{-1}(0, T; L^2(\Omega)) : \square u = (\partial_t^2 - \Delta)u \in L^2(Q)\}$$

topologized by the norm

$$\|u\|_{K_{\square}(Q)}^2 = \|u\|_{H^{-1}(0,T;L^2(\Omega))}^2 + \|(\partial_t^2 - \Delta)u\|_{L^2(Q)}^2.$$

Then, following Theorem 6.5 in chapter 2 of [23], we prove in the appendix that the maps

$$\mathcal{T}_0 w = (w|_{\Sigma}, w|_{t=0}, \partial_t w|_{t=0}), \quad \mathcal{T}_1 w = (\partial_\nu w|_{\Sigma}, w|_{t=T}, \partial_t w|_{t=T}), \quad w \in \mathcal{C}^\infty(\overline{Q}),$$

can be extended continuously to $\mathcal{T}_0 : H_{\square}(Q) \rightarrow H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$, $\mathcal{T}_1 : H_{\square}(Q) \rightarrow H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$ (see Proposition 7). Therefore, we can define

$$\mathcal{H}(\partial Q) = \{\mathcal{T}_0 u : u \in H_{\square}(Q)\} \subset H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega).$$

Following [25], in order to define an appropriate topology on $\mathcal{H}(\partial Q)$ we consider the restriction of \mathcal{T}_0 to the space J .

Proposition 1. *The restriction of \mathcal{T}_0 to J is one to one and onto.*

Proof. Let $u, v \in J$ with $\mathcal{T}_0 u = \mathcal{T}_0 v$. Then $w = u - v$ solves

$$\begin{cases} \partial_t^2 w - \Delta w = 0, & (t, x) \in Q, \\ w|_{t=0} = \partial_t w|_{t=0} = 0, \\ w|_{\Sigma} = 0 \end{cases}$$

and the uniqueness of solutions of this IBVP implies that $w = 0$. Thus, the restriction of \mathcal{T}_0 to J is one to one. Now let $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$. There exists $F \in H_{\square}(Q)$ such that $\mathcal{T}_0 F = (g, v_0, v_1)$. Consider the IBVP

$$\begin{cases} \partial_t^2 v - \Delta v = -(\partial_t^2 - \Delta)F, & (t, x) \in Q, \\ v|_{t=0} = \partial_t v|_{t=0} = 0, \\ v|_{\Sigma} = 0. \end{cases}$$

Since $-(\partial_t^2 - \Delta)F \in L^2(Q)$, from the theory introduced in Section 8 of Chapter 3 of [23] we deduce that this IBVP admits a unique solution $v \in \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; H_0^1(\Omega))$. Then, $u = v + F \in L^2(Q)$ satisfies $(\partial_t^2 - \Delta)u = 0$ and $\mathcal{T}_0 u = \mathcal{T}_0 v + \mathcal{T}_0 F = (g, v_0, v_1)$. Thus \mathcal{T}_0 is onto. \square

We set \mathcal{P}_0 the inverse of $\mathcal{T}_0 : J \rightarrow \mathcal{H}(\partial Q)$ and we define the norm of $\mathcal{H}(\partial Q)$ by

$$\|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)} = \|\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)}.$$

We are now in position to state the well posedness of the IBVP (1.1).

Proposition 2. *Let $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$ and $q \in L^\infty(Q)$. Then the IBVP (1.1) admits a unique solution $u \in L^2(Q)$ satisfying*

$$\|u\|_{L^2(Q)} \leq C \|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)} \quad (2.1)$$

and the boundary operator $B_q : (g, v_0, v_1) \mapsto \mathcal{T}_1 u$ is a bounded operator from $\mathcal{H}(\partial Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$.

Proof. We split u into two term $u = v + \mathcal{P}_0(g, v_0, v_1)$ where v solves

$$\begin{cases} \partial_t^2 v - \Delta v + qv = -q\mathcal{P}_0(g, v_0, v_1), & (t, x) \in Q, \\ v|_{t=0} = \partial_t v|_{t=0} = 0, \\ v|_{\Sigma} = 0. \end{cases} \quad (2.2)$$

Since $\mathcal{P}_0(g, v_0, v_1) \in L^2(Q)$, the IBVP (2.2) admits a unique solution $v \in \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; H_0^1(\Omega))$ (e.g. Section 8 of Chapter 3 of [23]) satisfying

$$\|v\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|v\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \leq C \|-q\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)} \leq C \|q\|_{L^\infty(Q)} \|\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)}. \quad (2.3)$$

Therefore, $u = v + \mathcal{P}_0(g, v_0, v_1)$ is the unique solution of (1.1) and estimate (2.3) implies (2.1). Now let us show the last part of the proposition. For this purpose fix $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$ and consider u the

solution of (1.1). Note first that $u \in L^2(Q)$ and $(\partial_t^2 - \Delta)u = -qu \in L^2(Q)$. Thus $u \in H_{\square}(Q)$ and $\mathcal{T}_1 u \in H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$ with

$$\|\mathcal{T}_1 u\|^2 \leq C^2 \|u\|_{H_{\square}(Q)}^2 = C^2 (\|u\|_{L^2(Q)}^2 + \|qu\|_{L^2(Q)}^2) \leq C^2 (1 + \|q\|_{L^\infty(Q)}^2) \|u\|_{L^2(Q)}^2.$$

Combining this with (2.1) we deduce that B_q is a bounded operator from $\mathcal{H}(\partial Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$. \square

Consider the operator $B_{q_1} - B_{q_2}$ for $q_1, q_2 \in L^\infty(Q)$. We have the following regularity result.

Proposition 3. *Let $q_1, q_2 \in L^\infty(Q)$. Then the operator $B_{q_1} - B_{q_2}$ is a bounded operator from $\mathcal{H}(\partial Q)$ to $L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega)$.*

Proof. Let u_1, u_2 be respectively the unique solution of the IBVP (1.1) for $q = q_1$ and $q = q_2$. Then $u = u_1 - u_2$ solves

$$\begin{cases} \partial_t^2 u - \Delta u + q_1 u &= (q_2 - q_1)u_2, & (t, x) \in Q, \\ u|_{t=0} = \partial_t u|_{t=0} &= 0, \\ u|_{\Sigma} &= 0. \end{cases}$$

Since $(q_2 - q_1)u_2 \in L^2(Q)$, in view of Theorem A.2 in [3] (see also Theorem 2.1 in [22] for $q = 0$), $u \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; H_0^1(\Omega))$ with $\partial_\nu u \in L^2(\Sigma)$. Moreover we have the following energy estimate

$$\|u\|_{C^1([0, T]; L^2(\Omega))} + \|u\|_{C([0, T]; H_0^1(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} \leq C \|q_1 - q_2\|_{L^\infty(Q)} \|u_2\|_{L^2(Q)}.$$

Note that in Theorem A.2 of [3] the authors consider only the case q is independent of t but their arguments still work when q is time-dependent. Combining this estimate with (2.1), we deduce that $\mathcal{T}_1 u \in L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega)$ with

$$\|\mathcal{T}_1 u\|_{L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega)} \leq C \|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)},$$

where C depends on Ω, T and $M \geq \|q_1\|_{L^\infty(Q)} + \|q_2\|_{L^\infty(Q)}$. Finally, we complete the proof by recalling that

$$\mathcal{T}_1 u = \mathcal{T}_1 u_1 - \mathcal{T}_1 u_2 = (B_{q_1} - B_{q_2})(g, v_0, v_1).$$

\square

According to Proposition 2 and 3, for every $q_1, q_2 \in L^\infty(Q)$, the partial boundary operators $B_{q_j}^*$ are well defined as bounded operator from $F_{\omega_0, \varepsilon_0}$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega_{-, \varepsilon_0, \omega_0})) \times H^{-2}(\Omega)$. Moreover the operator $B_{q_1}^* - B_{q_2}^*$ is bounded from $F_{\omega_0, \varepsilon_0}$ to $L^2(\Sigma_{-, \varepsilon_0, \omega_0}) \times H^1(\Omega)$.

3. COMPLEX GEOMETRIC OPTICS SOLUTIONS

The goal of this section is to build CGO u associated to the equation

$$\partial_t^2 u - \Delta u + q(t, x)u = 0 \quad \text{on } Q.$$

More precisely we consider solutions of this equation of the form

$$u = e^{\zeta_1 \cdot (t, x)} (1 + w(t, x)) \tag{3.1}$$

with $u \in H^2(Q)$. Here $\zeta_1 \in \mathbb{C}^{1+n}$ and it is chosen in such way that $(\partial_t^2 - \Delta)e^{\zeta_1 \cdot (t, x)} = 0$. Moreover, ζ_1 depends on some parameter $r > 0$ and the remainder term w in the asymptotic expansion of u wrt r satisfies

$$\|w\|_{L^2(Q)} \leq \frac{C}{r}$$

with $C > 0$ independent of r . In order to build such CGO, we first introduce some well known results of Hörmander about solutions of PDE's with constant coefficients of the form $P(D)u = f$ on Q with $P \in \mathbb{C}[X_0, X_1, \dots, X_n]$ a polynomial with complex coefficient and $D = -i(\partial_t, \partial_x)$.

3.1. Solutions of PDE with constant coefficients. We start this subsection by recalling some properties of solutions of PDE's of the form $P(D)u = f$ with constant coefficients. Let $P \in \mathbb{C}[X_0, X_1, \dots, X_n]$ and \tilde{P} defined by

$$\tilde{P} = \left(\sum_{\alpha \in \mathbb{N}^{1+n}} \left| \partial_{(t,x)}^\alpha P(t,x) \right|^2 \right)^{\frac{1}{2}}.$$

Theorem 2. (Theorem 7.3.10, [14]) *For every $P \in \mathbb{C}[X_0, X_1, \dots, X_n]$ with $P \neq 0$ one can find a distribution of finite order $E_P \in D'(\mathbb{R}^{1+n})$ such that $P(D)E_P = \delta$.*

Such distributions E_P are called fundamental solutions of P . Note that

$$E_P * (P(D)u) = u, \quad u \in \mathcal{E}'(\mathbb{R}^{1+n}),$$

$$P(D)(E_P * f) = f, \quad f \in \mathcal{E}'(\mathbb{R}^{1+n}),$$

where $\mathcal{E}'(\mathbb{R}^{1+n})$ is the set of distributions with compact support. Thus, for all $f \in \mathcal{E}'(\mathbb{R}^{1+n})$, $u = E_P * f$ is a solution of $P(D)u = f$. Let us give some information about the regularity of such a solution. For this purpose we need the following definitions introduced in [15].

Definition 1. *A positive function κ defined in \mathbb{R}^{1+n} will be called a temperate weight function if there exist positive constants C and N such that*

$$\kappa(\xi + \eta) \leq C(1 + |\xi|)^N \kappa(\eta), \quad \xi, \eta \in \mathbb{R}^{1+n}.$$

The set of all such functions κ will be denoted by \mathcal{K} .

Notice that, for all $P \in \mathbb{C}[X_0, X_1, \dots, X_n]$, $\tilde{P} \in \mathcal{K}$.

Definition 2. *If $\kappa \in \mathcal{K}$ and $1 \leq p \leq \infty$, we denote by $B_{p,\kappa}$ the set of all temperate distribution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ such that \hat{u} is a function and*

$$\|u\|_{p,\kappa} = \left(\frac{1}{(2\pi)^{1+n}} \int_{\mathbb{R}^{1+n}} |\kappa(\xi) \hat{u}(\xi)|^p d\xi \right)^{\frac{1}{p}} < \infty.$$

When $p = \infty$ we shall interpret $\|u\|_{p,\kappa}$ as $\text{ess. sup} |\kappa(\xi) \hat{u}(\xi)|$. We denote by $B_{p,\kappa}^{loc}$ the set of $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ such that for all $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n})$ we have $\chi u \in B_{p,\kappa}$.

Remark 1. *Let*

$$\kappa_1(\tau, \eta) = (1 + |(\tau, \eta)|^2)^{\frac{1}{2}}, \quad \tau \in \mathbb{R}, \quad \eta \in \mathbb{R}^n.$$

Then, in view of Example 10.1.2 of [15], one can easily show that $\kappa_1 \in \mathcal{K}$ and $B_{2,\kappa_1} = H^1(\mathbb{R}^{1+n})$.

Remark 2. *In view of Theorem 10.1.12 of [15], for $\kappa'_1, \kappa'_2 \in \mathcal{K}$, $\kappa = \kappa'_1 \cdot \kappa'_2$, $u_1 \in B_{p,\kappa'_1} \cap \mathcal{E}'(\mathbb{R}^{1+n})$ and $u_2 \in B_{\infty,\kappa'_2}$, we have $u_1 * u_2 \in B_{p,\kappa}$ and*

$$\|u_1 * u_2\|_{B_{p,\kappa}} \leq \|u_1\|_{B_{p,\kappa'_1}} \|u_2\|_{B_{\infty,\kappa'_2}}. \quad (3.2)$$

Theorem 3. (Theorem 10.21, [15]) *Every $P \in \mathbb{C}[X_0, X_1, \dots, X_n]$, with $P \neq 0$, has a fundamental solution $E_P \in B_{\infty,\tilde{P}}^{loc}$ such that $\frac{E_P}{\cosh(|(t,x)|)} \in B_{\infty,\tilde{P}}$ and*

$$\left\| \frac{E_P}{\cosh(|(t,x)|)} \right\|_{B_{\infty,\tilde{P}}} \leq C \quad (3.3)$$

with $C > 0$ a constant depending only on the degree of P .

Such a fundamental solution will be denoted by regular fundamental solution. Let us remark that in our construction of complex geometric optics solutions we need to consider an operator E such that $P(D)E = \text{Id}$ for some $P \in \mathbb{C}[X_0, X_1, \dots, X_n]$. Using the properties of regular fundamental solutions, Hörmander proved in Theorem 10.3.7 of [15] that such operator exists and it is a bounded operator of $L^2(X)$ for X a bounded open set of \mathbb{R}^{1+n} . In contrast to elliptic equations and parabolic equations (see Subsection 2.1 and 3.6 of [6]), we can not build CGO lying in $H^2(Q)$ by applying the result of Hörmander. What we can actually build from this result is CGO lying in $H^1(Q)$. Therefore, we need to extend the result of Hörmander in the following way.

Theorem 4. *Let $P \in \mathbb{C}[X_0, X_1, \dots, X_n]$ with $P \neq 0$. Then there exists an operator*

$$E : H^1(Q) \rightarrow H^1(Q)$$

such that:

- (1) $P(D)Ef = f$, $f \in H^1(Q)$,
- (2) for all $S \in \mathbb{C}[X_0, X_1, \dots, X_n]$ such that $\frac{\tilde{S}}{\tilde{P}}$ is bounded, we have $S(D)E \in B(H^1(Q))$, and

$$\|S(D)E\|_{B(H^1(Q))} \leq C \sup_{\xi \in \mathbb{R}^{1+n}} \frac{|S(\xi)|}{\tilde{P}(\xi)}, \quad k = 0, 1, \quad (3.4)$$

where $C > 0$ depends only on the degree of P , Ω and T .

Proof. Let $f \in H^1(Q)$. In view of Theorem 2.2 and 8.1 in Chapter 1 of [23], there exists an extension operator $\mathcal{E} \in B(H^1(Q), H^1(\mathbb{R}^{1+n}))$ such that $\mathcal{E}f|_Q = f$. Here we consider the extension operator \mathcal{E} introduced by [23]. Set $\chi \in C_0^\infty(\mathbb{R}^{1+n})$ and $R > 0$ such that $\chi = 1$ on a neighborhood of \bar{Q} and $\text{supp}\chi \subset B_R$ with B_R the ball of radius R and of center 0 of \mathbb{R}^{1+n} . Let E_P be a regular fundamental solution of P . Now consider the operator

$$E : f \mapsto (E_P * (\chi \mathcal{E}f))|_Q.$$

Clearly we have

$$P(D)E_P * (\chi \mathcal{E}f) = \chi \mathcal{E}f$$

and it follows that

$$P(D)Ef = (\chi \mathcal{E}f)|_Q = f$$

which proves (1). Now let us show (2). For this purpose, let $\psi \in C_0^\infty(\mathbb{R}^{1+n})$ be such that $\psi = 1$ on the closure of $B_R - B_R = \{x - y : x, y \in B_R\}$ and notice that

$$(E_P * (\chi \mathcal{E}f))|_Q = ((\psi E_P) * (\chi \mathcal{E}f))|_Q. \quad (3.5)$$

Note that

$$|\mathcal{F}(S(D)\psi E_P)| \leq \frac{|S(\xi)|}{\tilde{P}(\xi)} \tilde{P}(\xi) \left| \mathcal{F} \left(\psi \cosh(|(t, x)|) \frac{E_P}{\cosh(|(t, x)|)} \right) \right|.$$

Then, since $\psi \cosh(|(t, x)|) \in C_0^\infty(\mathbb{R}^{1+n})$, from Lemma 2.1 of [6] we deduce that

$$\psi \cosh(|(t, x)|) \frac{E_P}{\cosh(|(t, x)|)} \in B_{\infty, \tilde{P}}$$

and

$$\left\| \psi \cosh(|(t, x)|) \frac{E_P}{\cosh(|(t, x)|)} \right\|_{B_{\infty, \tilde{P}}} \leq C_1 \left\| \frac{E_P}{\cosh(|(t, x)|)} \right\|_{B_{\infty, \tilde{P}}} \leq C'$$

with $C' > 0$ a constant depending only on the degree of P and χ . It follows

$$\|S(D)\psi E_P\|_{B_{\infty, 1}} \leq C' \sup_{\xi \in \mathbb{R}^{1+n}} \frac{|S(\xi)|}{\tilde{P}(\xi)}.$$

In view of Remark 2, since $\chi\mathcal{E}f \in H^1(\mathbb{R}^{1+n}) = B_{2,\kappa_1}$ with κ_1 introduced in Remark 1, we have $S(D)(\psi E_P) * (\chi\mathcal{E}f) = (S(D)\psi E_P) * (\chi\mathcal{E}f) \in B_{2,\kappa_1}$ and

$$\begin{aligned} \|S(D)(\psi E_P) * (\chi\mathcal{E}f)\|_{H^1(\mathbb{R}^{1+n})} &= \|S(D)(\psi E_P) * (\chi\mathcal{E}f)\|_{B_{2,\kappa_1}} \\ &\leq \|S(D)\psi E_P\|_{B_{\infty,1}} \|\chi\mathcal{E}f\|_{H^1(\mathbb{R}^{1+n})} \\ &\leq C \sup_{\xi \in \mathbb{R}^{1+n}} \frac{|S(\xi)|}{\tilde{P}(\xi)} \|f\|_{H^1(Q)} \end{aligned}$$

with $C > 0$ a constant depending only on the degree of P , χ , Ω and T . Thus, in view of (3.5), we have $S(D)Ef \in H^1(Q)$ and

$$\|S(D)Ef\|_{H^1(Q)} \leq \|(\psi E_P) * (\chi\mathcal{E}f)\|_{H^1(Q)} \leq C \sup_{\xi \in \mathbb{R}^{1+n}} \frac{|S(\xi)|}{\tilde{P}(\xi)} \|f\|_{H^1(Q)}.$$

□

Armed with this result, we are now in position to build CGO of the form (3.1) lying in $H^2(Q)$.

3.2. Construction of complex geometric optics solutions. For every $1 \leq c \leq 2$ and $\omega \in \mathbb{S}^{n-1}$ we set

$$\xi_0 = \frac{1}{\sqrt{1+(1+c)^2}}, \quad \xi' = \frac{1+c}{\sqrt{1+(1+c)^2}}\omega, \quad \xi = (\xi_0, \xi'). \quad (3.6)$$

Note that here $\xi \in \mathbb{S}^n$ and we have

$$\frac{2}{\sqrt{10}} \leq |\xi'| \leq \frac{3}{\sqrt{5}}, \quad \frac{1}{\sqrt{10}} \leq \xi_0 \leq \frac{1}{\sqrt{5}}. \quad (3.7)$$

Now let us consider the following.

Proposition 4. *Let $1 \leq c \leq 2$ and $\omega \in \mathbb{S}^{n-1}$ and let $\xi \in \mathbb{S}^n$ be defined by (3.6). Then, for every $r > 0$, $(l, y) \in \mathbb{R} \times \mathbb{R}^n$ satisfying*

$$l\xi_0 - y \cdot \xi' = 0 = (l, y) \cdot (-\xi_0, \xi'), \quad (3.8)$$

$$r > \sqrt{\frac{\left| \left(\frac{l}{2}\right)^2 - \left(\frac{|y|}{2}\right)^2 \right|}{\left(1 - \frac{1}{(1+c)^2}\right) |\xi'|^2}}, \quad (3.9)$$

there exists $\theta \in \mathbb{R}^n$ such that for

$$\zeta_1 = -r\xi - \frac{i}{2}(l, y) - i(0, \theta), \quad (3.10)$$

$$\zeta_2 = r\xi - \frac{i}{2}(l, y) + i(0, \theta), \quad (3.11)$$

we have

$$\begin{cases} (\partial_t^2 - \Delta)e^{\zeta_j \cdot (t,x)} = 0, \quad j = 1, 2, \\ e^{\zeta_1 \cdot (t,x)} e^{\zeta_2 \cdot (t,x)} = e^{-i(l,y) \cdot (t,x)}. \end{cases} \quad (3.12)$$

Proof. Choose $\theta \in \mathbb{R}^n$ such that $\theta \cdot \eta = \theta \cdot \xi' = 0$. Notice that

$$(\partial_t^2 - \Delta)e^{\zeta_1 \cdot (t,x)} = \left[r^2(\xi_0^2 - |\xi'|^2) - \left(\frac{l}{2}\right)^2 + \left(\frac{|y|}{2}\right)^2 - ir(l, y) \cdot (-\xi_0, \xi') + |\theta|^2 \right] e^{\zeta_1 \cdot (t,x)}.$$

Applying (3.6) and (3.8), we obtain

$$(\partial_t^2 - \Delta)e^{\zeta_1 \cdot (t,x)} = \left[-r^2 \left(1 - \frac{1}{(1+c)^2}\right) |\xi'|^2 - \left(\frac{l}{2}\right)^2 + \left(\frac{|y|}{2}\right)^2 + |\theta|^2 \right] e^{\zeta_1 \cdot (t,x)}$$

and, in view of (3.9), we can choose

$$|\theta|^2 = r^2 \left(1 - \frac{1}{(1+c)^2} \right) |\xi'|^2 + \left(\frac{l}{2} \right)^2 - \left(\frac{|y|}{2} \right)^2 > 0.$$

Then, we obtain

$$(\partial_t^2 - \Delta) e^{\zeta_1 \cdot (t,x)} = 0.$$

In the same way we prove that

$$(\partial_t^2 - \Delta) e^{\zeta_2 \cdot (t,x)} = \left[-r^2 \left(1 - \frac{1}{(1+c)^2} \right) |\xi'|^2 - \left(\frac{l}{2} \right)^2 + \left(\frac{|y|}{2} \right)^2 + |\theta|^2 \right] e^{\zeta_2 \cdot (t,x)} = 0$$

and we deduce (3.12). \square

Proposition 5. *Let $q \in W^{1,\infty}(Q)$, $1 \leq c \leq 2$, $\omega \in \mathbb{S}^{n-1}$, $\xi \in \mathbb{S}^n$ be defined by (3.6) and ζ_1 be defined by (3.10). Then, there exists $r_0 > 1$ such that for $r \geq r_0$ the equation $\partial_t^2 u - \Delta u + qu = 0$ admits a solution $u \in H^2(Q)$ of the form (3.1) with*

$$\|w\|_{H^1(Q)} \leq \frac{C}{r}, \quad (3.13)$$

where C and r_0 depend on Ω , T , $M \geq \|q\|_{W^{1,\infty}(Q)}$. Moreover, for $|(l, y)| \leq Br$, this solution u satisfies the estimate

$$\|u\|_{H^2(Q)} \leq C e^{\delta r}, \quad r > r_0. \quad (3.14)$$

Here C depends on Ω , T , $M \geq \|q\|_{W^{1,\infty}(Q)}$, B and δ depends on Ω and T .

Proof. First notice that, in view of (3.12), w should be a solution of

$$\partial_t^2 w - \Delta w + 2\zeta_1 \cdot (\partial_t, -\nabla_x) w = -q(1+w). \quad (3.15)$$

Set $P \in \mathbb{C}[X_0, X_1, \dots, X_n]$ defined by

$$P(\mu, \eta) = |\eta|^2 - \mu^2 + 2i\zeta_1 \cdot (\mu, -\eta), \quad \mu \in \mathbb{R}, \eta \in \mathbb{R}^n.$$

In view of Theorem 4, there exists $E \in B(H^1(Q))$ such that

$$P(D)EF = (\partial_t^2 - \Delta + 2\zeta_1 \cdot (\partial_t, -\nabla_x))EF = F, \quad F \in H^1(Q). \quad (3.16)$$

Moreover, for every $S \in \mathbb{C}[X_0, X_1, \dots, X_n]$ such that $\frac{\tilde{S}}{P}$ is bounded, we have $S(D)E \in \mathcal{B}(H^1(Q))$ and

$$\|S(D)E\|_{\mathcal{B}(H^1(Q))} \leq K \sup_{(\mu, \eta) \in \mathbb{R}^{n+1}} \frac{|S(\mu, \eta)|}{\tilde{P}(\mu, \eta)}, \quad (3.17)$$

where $K > 0$ depends only on Ω and T . Applying estimate (3.17) for $S = 1$ we deduce

$$\|E\|_{\mathcal{B}(H^1(Q))} \leq K \sup_{(\mu, \eta) \in \mathbb{R}^{n+1}} \frac{1}{\tilde{P}(\mu, \eta)} \leq K \sup_{(\mu, \eta) \in \mathbb{R}^{n+1}} \frac{1}{|\partial_\eta P(\mu, \eta)|}.$$

But $\partial_\eta P(\mu, \eta) = 2\eta + 2i(r\xi' + i(\frac{y}{2} + \theta))$ and $|\partial_\eta P(\mu, \eta)| \geq |\Im \partial_\eta P(\mu, \eta)| = 2|\xi'|r$. Therefore, using (3.7), we find

$$\|E\|_{\mathcal{B}(H^1(Q))} \leq \frac{K}{2|\xi'|r} \leq \frac{K\sqrt{10}}{4r}. \quad (3.18)$$

Now consider the map

$$\mathcal{G}: H^1(Q) \rightarrow H^1(Q),$$

$$F \mapsto -E[q(1+F)].$$

Combining estimate (3.18) with a standard fixed point argument, we deduce that there exists $r_0 > 0$ such that for $r \geq r_0$ the map \mathcal{G} admits a unique fix point $w \in H^1(Q)$ satisfying (3.13). Since w satisfies

$w = -E[q(1+w)]$, from (3.16) we deduce that it is a solution of (3.15). It remains to show that $w \in H^2(Q)$ and prove estimate (3.14). For this purpose, note that

$$\sup_{(\mu,\eta) \in \mathbb{R}^{n+1}} \frac{|\eta|}{|\tilde{P}(\mu,\eta)|} \leq \sup_{(\mu,\eta) \in \mathbb{R}^{n+1}} \frac{|\eta|}{|\partial_\eta P(\mu,\eta)|} = \sup_{(\mu,\eta) \in \mathbb{R}^{n+1}} \frac{|\eta|}{|2\eta + 2i(r\xi' + i(\frac{y}{2} + \theta))|} \leq \max\left(1, \frac{\sqrt{10}|\zeta_1|}{2r}\right),$$

$$\sup_{(\mu,\eta) \in \mathbb{R}^{n+1}} \frac{|\mu|}{|\tilde{P}(\mu,\eta)|} \leq \sup_{(\mu,\eta) \in \mathbb{R}^{n+1}} \frac{|\mu|}{|\partial_\mu P(\mu,\eta)|} = \sup_{(\mu,\eta) \in \mathbb{R}^{n+1}} \frac{|\mu|}{|2\mu + 2i(r\xi_0 + \frac{i}{2})|} \leq \max\left(1, \frac{\sqrt{10}|\zeta_1|}{r}\right).$$

Here we have used (3.7). Combining these estimates with (3.17), we deduce that $\partial_t w, \nabla_x w \in H^1(Q)$ which implies that $w \in H^2(Q)$ and we have

$$\|w\|_{H^2(Q)} \leq C \left(1 + \frac{2|\zeta_1|}{2|\xi'|r} + \frac{2|\zeta_1|}{2|\xi_0|r}\right) (1 + \|w\|_{H^1(Q)}) \leq C \left(1 + \frac{|\zeta_1|}{r}\right)$$

with C a generic constant depending on Ω, T and $M \geq \|q\|_{W^{1,\infty}(Q)}$. Therefore, u defined by (3.1) is an $H^2(Q)$ -solution of $\partial_t^2 u - \Delta u + qu = 0$ satisfying

$$\|u\|_{H^2(Q)} \leq C \left(1 + \frac{|\zeta_1|}{r}\right) (1 + |\zeta_1|)^2 \sup_{(t,x) \in Q} e^{r\xi \cdot (t,x)}$$

which implies (3.14). \square

4. CARLEMAN ESTIMATE

This section is devoted to the proof of a Carleman estimate with linear weight. For this purpose we fix $\xi = (\xi_0, \xi') \in \mathbb{S}^n$ satisfying (3.6). The main result of this section can be stated as follows.

Theorem 5. *Let $q \in L^\infty(Q)$ and $u \in C^2(\overline{Q})$. If u satisfies the condition*

$$u|_\Sigma = 0, \quad u|_{t=0} = \partial_t u|_{t=0} = 0 \tag{4.1}$$

then the estimate

$$\begin{aligned} & r \int_\Omega e^{-2r\xi \cdot (T,x)} |\partial_t u|_{t=T}|^2 dx + r \int_{\Sigma_{+,w}} e^{-2r\xi \cdot (t,x)} |\partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt + r^2 \int_Q e^{-2r\xi \cdot (t,x)} |u|^2 dx dt \\ & \leq C \left(\int_Q e^{-2r\xi \cdot (t,x)} |(\partial_t^2 - \Delta + q)u|^2 dx dt + r^5 \int_\Omega e^{-2r\xi \cdot (T,x)} |u|_{t=T}|^2 dx + r^3 \int_\Omega e^{-2r\xi \cdot (T,x)} |\omega \cdot \nabla_x u|_{t=T}|^2 dx \right) \\ & \quad + Cr \int_{\Sigma_{-,w}} e^{-2r\xi \cdot (t,x)} |\partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \end{aligned} \tag{4.2}$$

holds true for $r \geq r_1 > 1$ with C and r_1 depending only on Ω, T and $M \geq \|q\|_{L^\infty(Q)}$. If u satisfies the condition

$$u|_\Sigma = 0, \quad u|_{t=T} = \partial_t u|_{t=T} = 0 \tag{4.3}$$

then the estimate

$$\begin{aligned} & r \int_\Omega e^{2r\xi \cdot (0,x)} |\partial_t u|_{t=0}|^2 dx + r \int_{\Sigma_{-,w}} e^{2r\xi \cdot (t,x)} |\partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt + r^2 \int_Q e^{2r\xi \cdot (t,x)} |u|^2 dx dt \\ & \leq C \left(\int_Q e^{2r\xi \cdot (t,x)} |(\partial_t^2 - \Delta + q)u|^2 dx dt + r^5 \int_\Omega e^{-2r\xi \cdot (0,x)} |u|_{t=0}|^2 dx + r^3 \int_\Omega e^{2r\xi \cdot (0,x)} |\omega \cdot \nabla_x u|_{t=0}|^2 dx \right) \\ & \quad + Cr \int_{\Sigma_{+,w}} e^{2r\xi \cdot (t,x)} |\partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \end{aligned} \tag{4.4}$$

holds true for $r \geq r_1 > 0$.

In order to prove these results, we introduce the weighted operator

$$P_s = e^{-s\xi \cdot (t,x)} (\partial_t^2 - \Delta) e^{s\xi \cdot (t,x)}, \quad s \in \mathbb{R}.$$

Then, we consider the following Carleman estimate associated to $P_{\pm r}, r > 0$.

Lemma 1. *Let $v \in C^2(\overline{Q})$ and $r > 1$. If v satisfies the condition*

$$v|_{\Sigma} = 0, \quad v|_{t=0} = \partial_t v|_{t=0} = 0 \quad (4.5)$$

then the estimate

$$\begin{aligned} & r \int_{\Omega} |\partial_t v|_{t=T}|^2 + r \int_{\Sigma_{+, \omega}} |\partial_\nu v|^2 |\omega \cdot \nu(x)| d\sigma(x) dt + r^2 \int_Q |v|^2 dx dt \\ & \leq C \left(\int_Q |P_r v|^2 dx dt + r^3 \int_{\Omega} |\omega \cdot \nabla_x v|_{t=T}|^2 dx + r \int_{\Sigma_{-, \omega}} |\partial_\nu v|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \right) \end{aligned} \quad (4.6)$$

holds true for C depending only on Ω and T . If v satisfies the condition

$$v|_{\Sigma} = 0, \quad v|_{t=T} = \partial_t v|_{t=T} = 0 \quad (4.7)$$

then the estimate

$$\begin{aligned} & r \int_{\Omega} |\partial_t v|_{t=0}|^2 dx + r \int_{\Sigma_{-, \omega}} |\partial_\nu v|^2 |\omega \cdot \nu(x)| d\sigma(x) dt + r^2 \int_Q |v|^2 dx dt \\ & \leq C \left(\int_Q |P_{-r} v|^2 dx dt + r^3 \int_{\Omega} |\omega \cdot \nabla_x v|_{t=0}|^2 dx + r \int_{\Sigma_{+, \omega}} |\partial_\nu v|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \right) \end{aligned} \quad (4.8)$$

holds true.

Proof. Since the differential operator P_s , $s \in \mathbb{R}$, is real valued, without loss of generality we can assume that v is real valued. We start with the proof of (4.6). Note first that P_r can be decomposed into $P_r = Q_1 + Q_2$ with $Q_1 = \partial_t^2 - \Delta + r^2(\xi_0^2 - |\xi'|^2)$, $Q_2 = r\xi_0 \partial_t - r\xi' \cdot \nabla_x$. Therefore we have

$$\|Q_2 v\|_{L^2(Q)}^2 + 2 \langle Q_1 v, Q_2 v \rangle \leq \|P_r v\|_{L^2(Q)}^2.$$

Here we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(Q)$. In view of this estimate, (4.6) follows from

$$\begin{aligned} & C_1 \left(r \int_{\Omega} |\partial_t v|_{t=T}|^2 + r \int_{\Sigma_{+, \omega}} |\partial_\nu v|^2 |\omega \cdot \nu(x)| d\sigma(x) dt + r^2 \int_Q |v|^2 dx dt \right) \\ & \leq \|Q_2 v\|_{L^2(Q)}^2 + 2 \langle Q_1 v, Q_2 v \rangle \\ & \quad + C_2 \left(r^3 \int_{\Omega} |\omega \cdot \nabla_x v|_{t=T}|^2 dx + r \int_{\Sigma_{-, \omega}} |\partial_\nu v|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \right) \end{aligned} \quad (4.9)$$

with $C_1, C_2 > 0$ depending only on Ω and T . We will prove the above estimate by applying suitably the Green formula on Q . We start by decomposing the following terms

$$\begin{aligned} & \langle Q_1 v, Q_2 v \rangle \\ & = r\xi_0 \int_Q \partial_t^2 v \partial_t v - r\xi_0 \int_Q \Delta v \partial_t v + r^3 \xi_0 (\xi_0^2 - |\xi'|^2) \int_Q v \partial_t v - r \int_Q \partial_t^2 v \xi' \cdot \nabla v \\ & \quad + r \int_Q \Delta v \xi' \cdot \nabla v - r^3 (\xi_0^2 - |\xi'|^2) \int_Q v \xi' \cdot \nabla v \\ & = A + B + C + D + E + F. \end{aligned}$$

For A: integrating wrt $t \in (0, T)$ we get

$$\int_Q \partial_t^2 v \partial_t v = \frac{1}{2} \int_Q \partial_t (\partial_t v)^2 = \frac{1}{2} \int_{\Omega} (\partial_t v|_{t=T})^2 dx.$$

For B: applying the Green formula wrt $x \in \Omega$ we find

$$\int_Q \Delta v \partial_t v = - \int_Q \nabla v \partial_t \nabla v = -\frac{1}{2} \int_Q \partial_t |\nabla v|^2 = -\frac{1}{2} \int_{\Omega} |\nabla v|_{t=T}|^2.$$

For C: integrating wrt $t \in (0, T)$ we obtain

$$\int_Q \partial_t v v = \frac{1}{2} \int_Q \partial_t (v)^2 = \frac{1}{2} \int_{\Omega} (v|_{t=T})^2 dx.$$

For D: integrating by parts wrt $t \in (0, T)$ and applying the Green formula wrt $x \in \Omega$ we get

$$\begin{aligned} \int_Q \partial_t^2 v \xi' \cdot \nabla v &= - \int_Q \partial_t v \xi' \cdot \nabla \partial_t v + \int_\Omega \partial_t v|_{t=T} (\xi' \cdot \nabla v|_{t=T}) \\ &= -\frac{1}{2} \int_0^T \int_\Omega \operatorname{div}_x ((\partial_t v)^2 \xi') dx dt + \int_\Omega \partial_t v|_{t=T} (\xi' \cdot \nabla v|_{t=T}) \\ &= -\frac{1}{2} \int_0^T \int_{\partial\Omega} (\partial_t v)^2 \nu(x) \cdot \xi' d\sigma(x) dt + \int_\Omega \partial_t v|_{t=T} (\xi' \cdot \nabla v|_{t=T}) \\ &= \int_\Omega (\partial_t v|_{t=T}) (\xi' \cdot \nabla v|_{t=T}). \end{aligned}$$

For E and F: applying a well known argument of [5] (see the proof of Lemma 2.1 in [5]) we obtain

$$\begin{aligned} \int_Q \Delta v \xi' \cdot \nabla v &= \frac{1}{2} \int_\Sigma |\partial_\nu v|^2 \xi' \cdot \nu(x) d\sigma(x) dt, \\ \int_Q v \xi' \cdot \nabla v &= 0. \end{aligned}$$

Combining all these results, we deduce that

$$\begin{aligned} \langle Q_1 v, Q_2 v \rangle &= \frac{r\xi_0}{2} \int_\Omega (\partial_t v|_{t=T})^2 dx + \frac{r\xi_0}{2} \int_\Omega |\nabla v|_{t=T}|^2 dx + \frac{r^3(\xi_0^2 - |\xi'|^2)\xi_0}{2} \int_\Omega (v|_{t=T})^2 dx \\ &\quad - r \int_\Omega (\partial_t v|_{t=T}) (\xi' \cdot \nabla v|_{t=T}) dx + \frac{|\xi'|r}{2} \int_\Sigma |\partial_\nu v|^2 \omega \cdot \nu(x) d\sigma(x) dt. \end{aligned} \tag{4.10}$$

For the fourth term on the rhs, applying the Cauchy-Schwarz inequality we get

$$\left| -r \int_\Omega (\partial_t v|_{t=T}) (\xi' \cdot \nabla v|_{t=T}) dx \right| \leq \frac{r\xi_0^2}{8} \int_\Omega |\partial_t v|_{t=T}|^2 + \frac{32r}{\xi_0^2} \int_\Omega |\xi' \cdot \nabla v|_{t=T}|^2 dx.$$

On the other hand, in view of (4.5), an application of some suitable Poincaré inequality (see [9] vol. 2 pp 125-126 and Proposition 2.2 in [5]) yields

$$\int_\Omega (v|_{t=T})^2 dx \leq C \int_\Omega |\xi' \cdot \nabla v|_{t=T}|^2 dx$$

with C depending on Ω . Therefore, in view of (3.7), the third term on the rhs of (4.10) can be majored by

$$\left| \frac{r^3(\xi_0^2 - |\xi'|^2)\xi_0}{2} \int_\Omega (v|_{t=T})^2 dx \right| \leq Cr^3 \int_\Omega |\omega \cdot \nabla v|_{t=T}|^2 dx$$

with C a generic constant depending only on Ω . Now let us consider the following Poincaré inequality.

Lemma 2. *Let $\theta = (\theta_0, \theta') \in \mathbb{R} \times \mathbb{R}^n$ be such that $\theta \in \mathbb{S}^n$ and $\theta_0 > 0$. Then, for all $w \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ satisfying $w|_{t=0} = 0$, we have*

$$\|w\|_{L^2(Q)} \leq \left(\frac{T}{\theta_0} \right)^2 \|\theta \cdot \nabla_{t,x} w\|_{L^2(Q)}. \tag{4.11}$$

For instance we admit the result of this lemma, whose proof is postponed to the end of the present demonstration. Recall that $Q_2 v = r\theta \cdot \nabla_{t,x} v$ with $\theta = (\xi_0, -\xi') \in \mathbb{S}^n$ and $\xi_0 > 0$. Thus, applying Lemma 2 we obtain

$$\frac{r^2}{C} \int_Q |v|^2 dx dt \leq \|Q_2 v\|_{L^2(Q)}^2$$

with $C > 0$ depending on T . Here we have used (3.7). Combining all these estimates with (4.10) we deduce (4.9), for $r > 1$, and by the same way (4.6). Now let us consider (4.8). For this purpose note that for v

satisfying (4.7), w defined by $w(t, x) = v(T - t, x)$ satisfies (4.5). Thus applying (4.6) to w with ξ' replaced by $-\xi'$ we obtain (4.8). \square

Now that we have completed the proof of Lemma 1 let us show Lemma 2.

Proof of Lemma 2. We first assume that $w \in C^\infty([0, T]; C_0^\infty(\Omega))$ with w equal to zero on a neighborhood of $t = 0$. Let w_1 be defined by

$$\begin{cases} w_1(t, x) = w(t, x), & (t, x) \in Q, \\ w_1(t, x) = 0, & (t, x) \in (-\infty, T) \times \mathbb{R}^n \setminus Q. \end{cases}$$

Since $w = 0$ on a neighborhood of $t = 0$, one can check that $w_1 \in C^\infty((-\infty, T) \times \mathbb{R}^n)$. Fix $(t, x) \in (-\infty, T) \times \mathbb{R}^n$ and consider the map

$$h(s) = w_1((t, x) + s\theta), \quad s \in (-\infty, 0).$$

Note that h is C^1 in $(-\infty, 0)$ and since $w_1|_{t \leq 0} = 0$ we have $h(s) = 0$ for $s \leq -\frac{T}{\theta_0}$. Therefore, we find

$$w_1((t, x) + s\theta) = h(s) = \int_{-\frac{T}{\theta_0}}^s h'(\tau) d\tau = \int_{-\frac{T}{\theta_0}}^s \theta \cdot \nabla_{t,x} w_1((t, x) + \tau\theta) d\tau, \quad -\frac{T}{\theta_0} < s < 0.$$

Then, we get

$$|w_1((t, x) + s\theta)|^2 = \left| \int_{-\frac{T}{\theta_0}}^s \theta \cdot \nabla_{t,x} w_1((t, x) + \tau\theta) d\tau \right|^2, \quad -\frac{T}{\theta_0} < s < 0$$

and applying the Cauchy Schwarz inequality we find

$$|w_1((t, x) + s\theta)|^2 \leq \left(s + \frac{T}{\theta_0}\right) \int_{-\frac{T}{\theta_0}}^s |\theta \cdot \nabla_{t,x} w_1((t, x) + \tau\theta)|^2 d\tau \leq \frac{T}{\theta_0} \int_{-\frac{T}{\theta_0}}^s |\theta \cdot \nabla_{t,x} w_1((t, x) + \tau\theta)|^2 d\tau.$$

Integrating this last expression wrt $(t, x) \in (-\infty, T - s\theta_0) \times \mathbb{R}^n$ and applying the Fubini theorem, we obtain

$$\int_{-\infty}^{T-s\theta_0} \int_{\mathbb{R}^n} |w_1((t, x) + s\theta)|^2 dx dt \leq \frac{T}{\theta_0} \int_{-\frac{T}{\theta_0}}^s \int_{-\infty}^{T-s\theta_0} \int_{\mathbb{R}^n} |\theta \cdot \nabla_{t,x} w_1((t, x) + \tau\theta)|^2 dx dt d\tau, \quad -\frac{T}{\theta_0} < s < 0. \quad (4.12)$$

For the lhs of this inequality, making the substitution $(u, y) = (t, x) + s\theta$ we obtain

$$\int_{-\infty}^{T-s\theta_0} \int_{\mathbb{R}^n} |w_1((t, x) + s\theta)|^2 dx dt = \int_{-\infty}^T \int_{\mathbb{R}^n} |w_1(u, y)|^2 dy du = \int_Q |w|^2 dx dt.$$

For the rhs of (4.12), making the substitution $(u, y) = (t, x) + \tau\theta$, for $\tau < s$, we obtain

$$\int_{-\infty}^{T-s\theta_0} \int_{\mathbb{R}^n} |\theta \cdot \nabla_{t,x} w_1((t, x) + \tau\theta)|^2 dx dt = \int_{-\infty}^{T+(\tau-s)\theta_0} \int_{\mathbb{R}^n} |\theta \cdot \nabla_{t,x} w_1(u, y)|^2 dy du$$

then, using the fact that $T + (\tau - s)\theta_0 < T$, we get

$$\int_{-\infty}^{T-s\theta_0} \int_{\mathbb{R}^n} |\theta \cdot \nabla_{t,x} w_1((t, x) + \tau\theta)|^2 dx dt \leq \int_{-\infty}^T \int_{\mathbb{R}^n} |\theta \cdot \nabla_{t,x} w_1(t, x)|^2 dx dt \leq \int_Q |\theta \cdot \nabla_{t,x} w(t, x)|^2 dx dt.$$

Combining these estimates with (4.12) we obtain

$$\int_Q |w|^2 dx dt \leq \frac{T}{\theta_0} \int_{-\frac{T}{\theta_0}}^s \int_Q |\theta \cdot \nabla_{t,x} w(t, x)|^2 dx dt d\tau \leq \left(\frac{T}{\theta_0}\right)^2 \int_Q |\theta \cdot \nabla_{t,x} w(t, x)|^2 dx dt.$$

By density (e.g. pp 10 of [24]), we deduce that this estimate holds for all $w \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ satisfying $w|_{t=0} = 0$. \square

In light of Lemma 1, we are now in position to prove Theorem 5.

Proof of Theorem 5. Let us first consider the case $q = 0$. Note that for u satisfying (4.1), $v = e^{-r\xi \cdot (t,x)}u$ satisfies (4.5). Moreover, we have

$$\int_Q e^{-2r\xi \cdot (t,x)} |(\partial_t^2 - \Delta)u|^2 dxdt = \int_Q |P_r v|^2 dxdt$$

and from (4.1)

$$\partial_\nu v|_\Sigma = e^{-r\xi \cdot (t,x)} \partial_\nu u|_\Sigma.$$

Finally, using the fact that

$$\partial_t u = \partial_t (e^{r\xi \cdot (t,x)} v) = r\xi_0 u + e^{r\xi \cdot (t,x)} \partial_t v, \quad \omega \cdot \nabla_x v = e^{-r\xi \cdot (t,x)} (\omega \cdot \nabla_x u - r|\xi'|u),$$

we obtain

$$\begin{aligned} \int_\Omega e^{-2r\xi \cdot (T,x)} |\partial_t u|_{t=T}|^2 dx &\leq 2 \int_\Omega |\partial_t v|_{t=T}|^2 dx + 2r^2 \int_\Omega e^{-2r\xi \cdot (T,x)} |u|_{t=T}|^2 dx, \\ \int_\Omega |\omega \cdot \nabla v|_{t=T}|^2 dx &\leq 2r^2 \int_\Omega e^{-2r\xi \cdot (T,x)} |u|_{t=T}|^2 dx + 2 \int_\Omega e^{-2r\xi \cdot (T,x)} |\omega \cdot \nabla u|_{t=T}|^2 dx. \end{aligned}$$

Thus, applying the Carleman estimate (4.6) to v , we deduce (4.2). For $q \neq 0$, we have

$$|\partial_t^2 u - \Delta u|^2 = |\partial_t^2 u - \Delta u + qu - qu|^2 \leq 2 |(\partial_t^2 - \Delta + q)u|^2 + 2 \|q\|_{L^\infty(Q)}^2 |u|^2$$

and hence if we choose $r_1 > 2C \|q\|_{L^\infty(Q)}^2$, replacing C by

$$C_1 = \sup_{r \geq r_1} \frac{Cr^2}{r^2 - 2C \|q\|_{L^\infty(Q)}^2} < \infty,$$

we deduce (4.2) from the same estimate when $q = 0$. Using similar arguments, we prove (4.4). \square

Remark 3. Note that, by density, estimate (4.2) can be extended to function $u \in \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; H^1(\Omega))$ satisfying (4.5), $(\partial_t^2 - \Delta)u \in L^2(Q)$ and $\partial_\nu u \in L^2(\Sigma)$.

5. GEOMETRIC OPTICS SOLUTIONS VANISHING ON PART OF THE BOUNDARY

In this section we fix $q \in L^\infty(Q)$. The goal of this section is to use the Carleman estimate (4.4) in order to build solutions $u \in H_\square(Q)$ to

$$\begin{cases} (\partial_t^2 - \Delta + q(t,x))u = 0 & \text{in } Q, \\ u|_{t=0} = 0, \\ u|_{\Sigma_{-, \varepsilon, -\omega}} = 0, \end{cases} \quad (5.1)$$

of the form

$$u(t,x) = e^{\zeta_2 \cdot (t,x)} (1 + z(t,x)), \quad (t,x) \in Q, \quad (5.2)$$

where ξ is defined by (3.6), $\zeta_2 \in \mathbb{C}^{1+n}$ is defined by (3.11) and satisfy (3.12), $z \in e^{-\zeta_2 \cdot (t,x)} H_\square(Q)$ fulfills

$$z(t,x) = -1, \quad (t,x) \in \Sigma_{-, \varepsilon, -\omega} \cup \{0\} \times \Omega,$$

$$\|z\|_{L^2(Q)} \leq Cr^{-\frac{1}{2}}. \quad (5.3)$$

The main result of this section can be stated as follows.

Theorem 6. Let $q \in L^\infty(Q)$. For all $r \geq r_1$, with r_1 the constant of Theorem 5, there is a solution $u \in H_\square(Q)$ of (5.1) of the form (5.2) with z satisfying (5.3). Moreover, we have

$$\|\mathcal{T}_0 u\|_{\mathcal{H}(\partial Q)} \leq Ce^{\delta r} \quad (5.4)$$

with C and δ depending on Ω , T and $M \geq \|q\|_{L^\infty(Q)}$, ε and ω .

In order to prove existence of such solutions of (5.1) we need some preliminary tools and an intermediate result.

5.1. Weighted space. In this subsection we give the definition of some weighted spaces. We set $s \in \mathbb{R}$ and we denote by γ the function defined on $\partial\Omega$ by

$$\gamma(x) = |\omega \cdot \nu(x)|, \quad x \in \partial\Omega.$$

We introduce the spaces $L_s(Q)$, $L_s(\Omega)$, and for all non negative measurable function h on $\partial\Omega$ the spaces $L_{s,h,\pm}$ defined respectively by

$$L_s(Q) = e^{s\xi \cdot (t,x)} L^2(Q), \quad L_s(\Omega) = e^{s\xi \cdot (0,x)} L^2(\Omega), \quad L_{s,h,\pm} = \{f : e^{s\xi \cdot (t,x)} h(x) f \in L^2(\Sigma_{\pm,\omega})\}$$

with the associated norm

$$\begin{aligned} \|u\|_s &= \left(\int_Q e^{2s\xi \cdot (t,x)} |u|^2 dx dt \right)^{\frac{1}{2}}, \quad u \in L_s(Q), \\ \|u\|_{s,0} &= \left(\int_\Omega e^{2s\xi \cdot (0,x)} |u|^2 dx \right)^{\frac{1}{2}}, \quad u \in L_s(\Omega), \\ \|u\|_{s,h,\pm} &= \left(\int_{\Sigma_{\pm,\omega}} e^{2s\xi \cdot (t,x)} h(x) |u|^2 d\sigma(x) dt \right)^{\frac{1}{2}}, \quad u, v \in L_{s,h,\pm}. \end{aligned}$$

5.2. Intermediate result. We set the space

$$\mathcal{D} = \{v \in C^2(\bar{Q}) : v|_\Sigma = 0, v|_{t=T} = \partial_t v|_{t=T} = v|_{t=0} = 0\}$$

and, in view of Theorem 5, applying the Carleman estimate (4.4) to any $f \in \mathcal{D}$ we obtain

$$r \|f\|_r + r^{\frac{1}{2}} \|\partial_t f|_{t=0}\|_{r,0} + r^{\frac{1}{2}} \|\partial_\nu f\|_{r,\gamma,-} \leq C(\|(\partial_t^2 - \Delta + q)f\|_r + \|\partial_\nu f\|_{r,r\gamma,+}), \quad r \geq r_1. \quad (5.5)$$

We introduce also the space

$$\mathcal{M} = \{((\partial_t^2 - \Delta + q)v, \partial_\nu v|_{\Sigma_{+,\omega}}) : v \in \mathcal{D}\}$$

and we think of \mathcal{M} as a subspace of $L_r(Q) \times L_{r,r\gamma,+}$. We consider the following intermediate result.

Lemma 3. *Given $r \geq r_1$, with r_1 the constant of Theorem 5, and*

$$v \in L_{-r}(Q), \quad v_- \in L_{-r,\gamma^{-1,-}}, \quad v_0 \in L_{-r}(\Omega),$$

there exists $u \in L_{-r}(Q)$ such that:

- 1) $(\partial_t^2 - \Delta + q)u = v$,
- 2) $u|_{\Sigma_{-,\omega}} = v_-$, $u|_{t=0} = v_0$,
- 3) $\|u\|_{-r} \leq C \left(r^{-1} \|v\|_{-r} + r^{-\frac{1}{2}} \|v_-\|_{-r,\gamma^{-1,-}} + r^{-\frac{1}{2}} \|v_0\|_{-r,0} \right)$ with C depending on Ω, T ,
 $M \geq \|q\|_{L^\infty(Q)}$.

Proof. Define a linear function K on \mathcal{M} by

$$K[(\square + q)f, \partial_\nu f|_{\Sigma_{+,\omega}}] = \langle f, v \rangle_{L^2(Q)} - \langle \partial_\nu f, v_- \rangle_{L^2(\Sigma_{-,\omega})} + \langle \partial_t f|_{t=0}, v_0 \rangle_{L^2(\Omega)}, \quad f \in \mathcal{D}.$$

Using (5.5), for all $f \in \mathcal{D}$, we obtain

$$\begin{aligned} & |K[(\square + q)f, \partial_\nu f|_{\Sigma_{+,\omega}}]| \\ & \leq \|f\|_r \|v\|_{-r} + \|\partial_\nu f\|_{r,\gamma,-} \|v_-\|_{-r,\gamma^{-1,-}} + \|\partial_t f|_{t=0}\|_{r,0} \|v_0\|_{-r,0} \\ & \leq r^{-1} \|v\|_{-r} (r \|f\|_r) + r^{-\frac{1}{2}} \|v_-\|_{-r,\gamma^{-1,-}} \left(r^{\frac{1}{2}} \|\partial_\nu f\|_{r,\gamma,-} \right) + r^{-\frac{1}{2}} \|v_0\|_{-r,0} \left(r^{\frac{1}{2}} \|\partial_t f|_{t=0}\|_{r,0} \right) \\ & \leq C \left(r^{-1} \|v\|_{-r} + r^{-\frac{1}{2}} \|v_-\|_{-r,\gamma^{-1,-}} + r^{-\frac{1}{2}} \|v_0\|_{-r,0} \right) \left(\|(\square + q)f\|_r + \|\partial_\nu f\|_{r,r\gamma,+} \right) \\ & \leq 2C \left(r^{-1} \|v\|_{-r} + r^{-\frac{1}{2}} \|v_-\|_{-r,\gamma^{-1,-}} + r^{-\frac{1}{2}} \|v_0\|_{-r,0} \right) \|((\square + q)f, \partial_\nu f|_{\Sigma_{+,\omega}})\|_{L_r(Q) \times L_{r,r\gamma,+}} \end{aligned}$$

with C the constant of (5.5). Thus, applying the Hahn Banach theorem we deduce that K can be extended to a continuous linear form, also denoted by K , on $L_r(Q) \times L_{r,r\gamma,+}$ satisfying

$$\|K\| \leq C \left(r^{-1} \|v\|_{-r} + r^{-\frac{1}{2}} \|v_-\|_{-r,\gamma^{-1},-} + r^{-\frac{1}{2}} \|v_0\|_{-r,0} \right). \quad (5.6)$$

Thus, there exists

$$(u, u_+) \in L_{-r}(Q) \times L_{r,(r\gamma)^{-1},+}$$

such that for all $f \in \mathcal{D}$ we have

$$K[(\square + q)f, \partial_\nu f|_{\Sigma_{+,\omega}}] = \langle (\square + q)f, u \rangle_{L^2(Q)} - \langle \partial_\nu f, u_+ \rangle_{L^2(\Sigma_{+,\omega})}.$$

Therefore, for all $f \in \mathcal{D}$ we have

$$\begin{aligned} & \langle (\square + q)f, u \rangle_{L^2(Q)} - \langle \partial_\nu f, u_+ \rangle_{L^2(\Sigma_{+,\omega})} \\ &= \langle f, v \rangle_{L^2(Q)} - \langle \partial_\nu f, v_- \rangle_{L^2(\Sigma_{-,\omega})} + \langle \partial_t f|_{t=0}, v_0 \rangle_{L^2(\Omega)}. \end{aligned} \quad (5.7)$$

Note first that, since $L_{\pm r}(Q)$ embedded continuously into $L^2(Q)$, we have $u \in L^2(Q)$. Therefore, taking $f \in \mathcal{C}_0^\infty(Q)$ shows 1). For condition 2), using the fact that $L_{\pm r}(Q)$ embedded continuously into $L^2(Q)$ we deduce that $u \in H_\square(Q)$ and we can define the trace $u|_\Sigma$ and $u|_{t=0}$. Thus, allowing $f \in \mathcal{D}$ to be arbitrary shows that $u|_{\Sigma_{-,\omega}} = v_-$, $u|_{t=0} = v_0$ and $u|_{\Sigma_{+,\omega}} = -u_+$. Finally, condition 3) follows from the fact that

$$\|u\|_{-r} \leq \|K\| \leq C \left(r^{-1} \|v\|_{-r} + r^{-\frac{1}{2}} \|v_-\|_{-r,\gamma^{-1},-} + r^{-\frac{1}{2}} \|v_0\|_{-r,0} \right).$$

□

Armed with this lemma we are now in position to prove Theorem 6.

5.3. Proof of Theorem 6. Note first that z must satisfy

$$\begin{cases} z \in L^2(Q) \\ (\partial_t^2 - \Delta + q)(e^{\zeta_2 \cdot (t,x)} z) = -q(t,x)e^{\zeta_2 \cdot (t,x)} & \text{in } Q, \\ z(0,x) = -1 & x \in \Omega, \\ z(t,x) = -1 & (t,x) \in \Sigma_{+,\varepsilon,-\omega}. \end{cases} \quad (5.8)$$

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\text{supp} \psi \cap \partial\Omega \subset \{x \in \partial\Omega : \nu(x) \cdot \omega < -\frac{\varepsilon}{2}\}$ satisfying $\psi = 1$ on $\partial\Omega_{+,\varepsilon,-\omega}$. Choose $v_-(t,x) = -e^{\zeta_2 \cdot (t,x)} \psi(x)$, $(t,x) \in \Sigma_{-,\omega}$. Since $v_-(t,x) = 0$ for $\nu(x) \cdot \omega \geq -\frac{\varepsilon}{2}$ we have $v_- \in L_{-r,\gamma^{-1},-}$. Fix also $v(t,x) = -q(t,x)e^{\zeta_2 \cdot (t,x)}$ and $v_0(x) = -e^{\zeta_2 \cdot (0,x)}$, $(t,x) \in Q$. From Lemma 3, we deduce that there exists $w \in H_\square(Q)$ such that

$$\begin{cases} (\partial_t^2 - \Delta + q)w = v(t,x) = -q(t,x)e^{\zeta_2 \cdot (t,x)} & \text{in } Q, \\ w(0,x) = v_0(x) = -e^{\zeta_2 \cdot (0,x)} & x \in \Omega, \\ w(t,x) = v_-(t,x) = -e^{\zeta_2 \cdot (t,x)} \psi(x) & (t,x) \in \Sigma_{-,\omega}. \end{cases}$$

Then, for $z = e^{-\zeta_2 \cdot (t,x)} w$ condition (5.8) will be fulfilled. Moreover, combining condition 3) of Lemma 3 with the fact that

$$(\zeta_2 - r\xi) \cdot (t,x) \in i\mathbb{R}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

we deduce (5.3). Using the fact that $e^{\zeta_2 \cdot (t,x)} z = w \in H_\square(Q)$, we deduce that u defined by (5.2) is lying in $H_\square(Q)$ and is a solution of (5.1) with $\mathcal{T}_0 u \in F_{\omega,\varepsilon}$. Moreover, in view of estimate (5.3), we have

$$\begin{aligned} \|\mathcal{T}_0 u\|_{\mathcal{H}(\partial Q)} &\leq C(\|u\|_{L^2(Q)} + \|(\partial_t^2 - \Delta)u\|_{L^2(Q)}) \\ &\leq C(\|u\|_{L^2(Q)} + \|qu\|_{L^2(Q)}) \leq C e^{\delta r} \end{aligned}$$

with C depending on Ω , T , M , ω , ε , and $\delta = \sup_{(t,x) \in Q} |(t,x)|$. This last estimate implies (5.4). □

6. STABILITY ESTIMATE

This section is devoted to the proof of Theorem 1. We start with an intermediate result. From now on we set $q = q_2 - q_1$ on Q and we assume that $q = 0$ on $\mathbb{R}^{1+n} \setminus Q$. Using the Carleman estimate introduced in the previous section and the geometric optics solutions of Section 3 and Section 5, we obtain the following.

Lemma 4. *Assume that the condition of Theorem 1 are fulfilled. Let $\varepsilon = \frac{\varepsilon_0}{2}$ and $\omega \in \{z \in \mathbb{S}^{n-1} : |z - \omega_0| \leq \varepsilon\}$. Then, there exists $r_2 \geq \max(r_0, r_1)$, with r_0 introduced in Proposition 5 and r_1 the constant of Theorem 5, such that for all $r > r_2$, $\xi \in \mathbb{S}^n$ of the form (3.6) with $1 \leq c \leq 2$, $(l, y) \in \mathbb{R} \times \mathbb{R}^n$ satisfying (3.8), (3.9) and $|(l, y)| \leq Br$, we have*

$$\left| \int_{\mathbb{R}^{1+n}} q(t, x) e^{-i(l, y) \cdot (t, x)} dt dx \right|^2 \leq C \left(\frac{1}{r} + e^{dr} \|B_{q_1}^* - B_{q_2}^*\|^2 \right) \quad (6.1)$$

with d, C depending only on $\Omega, M, T, \varepsilon_0, B, \omega_0$.

Proof. Let $\zeta_j, j = 1, 2$, be defined by (3.10), (3.11) and satisfy (3.12). According to Proposition 5, we can introduce

$$u_1(t, x) = e^{\zeta_1 \cdot (t, x)} (1 + w(t, x)), \quad (t, x) \in Q,$$

where $u_1 \in H^2(Q)$ satisfies $\partial_t^2 u_1 - \Delta u_1 + q_1 u_1 = 0$ and w satisfies (3.13). Moreover, in view of Theorem 6, we consider $u_2 \in H_{\square}(Q)$ solution of (5.1) with $q = q_2$ of the form

$$u_2(t, x) = e^{\zeta_2 \cdot (t, x)} (1 + z(t, x)), \quad (t, x) \in Q$$

with z satisfying (5.3), such that $\mathcal{T}_0 u_2 \in F_{\varepsilon, \omega}$ and u_2 fulfills (5.4). Let w_1 be the solution of

$$\begin{cases} \partial_t^2 w_1 - \Delta w_1 + q_1 w_1 = 0 & \text{in } Q, \\ \mathcal{T}_0 w_1 = \mathcal{T}_0 u_2, & \end{cases} \quad (6.2)$$

Then, $u = w_1 - u_2$ solves

$$\begin{cases} \partial_t^2 - \Delta u + q_1 u = (q_2 - q_1) u_2 & \text{in } Q, \\ u(0, x) = \partial_t u(0, x) = 0 & \text{on } \Omega \\ u = 0 & \text{on } \Sigma, \end{cases} \quad (6.3)$$

and since $(q_2 - q_1) u_2 \in L^2(Q)$, in view of Theorem A.2 in [3], we deduce that $u \in C^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; H_0^1(\Omega))$ with $\partial_\nu u \in L^2(\Sigma)$. Moreover we have

$$\|u\|_{C^1([0, T]; L^2(\Omega))} + \|u\|_{\mathcal{C}([0, T]; H_0^1(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} \leq 2CM \|u_2\|_{L^2(Q)}.$$

Applying the Green formula wrt $x \in \Omega$ and integration by parts wrt $t \in (0, T)$, we find

$$\begin{aligned} \int_Q q u_2 u_1 dx dt &= \int_Q (\partial_t^2 - \Delta + q_1) u u_1 dx dt \\ &= \int_Q u (\partial_t^2 - \Delta + q_1) u_1 - \int_\Sigma \partial_\nu u u_1 d\sigma(x) dt \\ &\quad + \int_\Omega \partial_t u(T, x) u_1(T, x) dx - \int_\Omega u(T, x) \partial_t u_1(T, x) dx \\ &= - \int_\Sigma \partial_\nu u u_1 d\sigma(x) dt + \int_\Omega \partial_t u(T, x) u_1(T, x) dx - \int_\Omega u(T, x) \partial_t u_1(T, x) dx \\ &= - \int_{\Sigma_{+, \varepsilon, \omega}} \partial_\nu u u_1 d\sigma(x) dt - \int_{\Sigma_{-, \varepsilon, \omega}} \partial_\nu u u_1 d\sigma(x) dt \\ &\quad + \int_\Omega \partial_t u(T, x) u_1(T, x) dx - \int_\Omega u(T, x) \partial_t u_1(T, x) dx. \end{aligned} \quad (6.4)$$

In view of estimate (3.13), we have

$$\|w\|_{L^2(\Sigma)} \leq C \|w\|_{L^2(0, T; H^{\frac{1}{2}}(\Omega))} \leq C \|w\|_{H^1(Q)} \leq C$$

with C depending on Ω , T and M . Applying this estimate and the Cauchy Schwarz inequality, we obtain

$$\begin{aligned} \left| \int_{\Sigma_{\pm, \varepsilon, \omega}} \partial_\nu u u_1 d\sigma(x) dt \right| &\leq \int_{\Sigma_{\pm, \varepsilon, \omega}} \left| \partial_\nu u e^{-r\xi \cdot (t, x)} (1 + w) \right| dt d\sigma(x) \\ &\leq C \left(\int_{\Sigma_{\pm, \varepsilon, \omega}} \left| e^{-r\xi \cdot (t, x)} \partial_\nu u \right|^2 d\sigma(x) dt \right)^{\frac{1}{2}} \end{aligned}$$

for some C depending only on Ω , T and M . In the same way, we have

$$\|w|_{t=T}\|_{L^2(\Omega)} \leq C \|w\|_{H^{\frac{1}{2}}(0, T; L^2(\Omega))} \leq C \|w\|_{H^1(Q)} \leq C$$

and from the proof of Proposition 5 we get

$$\|\partial_t w|_{t=T}\|_{L^2(\Omega)} \leq C \|w\|_{H^{\frac{3}{2}}(0, T; L^2(\Omega))} \leq C \|w\|_{H^2(Q)} \leq C(|\zeta_1| + 1) \leq Cr$$

with C depending on Ω , T and M . Thus, we obtain

$$\begin{aligned} \left| \int_{\Omega} \partial_t u(T, x) u_1(T, x) dx \right| &\leq C \left(\int_{\Omega} \left| e^{-r\xi \cdot (T, x)} \partial_t u(T, x) \right|^2 dx \right)^{\frac{1}{2}}, \\ \left| \int_{\Omega} u(T, x) \partial_t u_1(T, x) dx \right| &\leq Cr \left(\int_{\Omega} \left| e^{-r\xi \cdot (T, x)} u(T, x) \right|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

In view of these estimates and (6.4), we have

$$\begin{aligned} &\left| \int_Q q u_2 u_1 dx dt \right|^2 \\ &\leq C \left(\int_{\Omega} \left| e^{-r\xi \cdot (T, x)} \partial_t u(T, x) \right|^2 dx + r^2 \int_{\Omega} \left| e^{-r\xi \cdot (T, x)} u(T, x) \right|^2 dx + \int_{\Sigma_{\pm, \varepsilon, \omega}} \left| e^{-r\xi \cdot (t, x)} \partial_\nu u \right|^2 d\sigma(x) dt \right). \end{aligned} \quad (6.5)$$

On the other hand, in views of Remark 3, the Carleman estimate (4.2) and the fact that $\partial\Omega_{+, \varepsilon, \omega} \subset \partial\Omega_{+, \omega}$ imply

$$\begin{aligned} &r \int_{\Sigma_{+, \varepsilon, \omega}} \left| e^{-r\xi \cdot (t, x)} \partial_\nu u \right|^2 d\sigma(x) dt + r \int_{\Omega} \left| e^{-r\xi \cdot (T, x)} \partial_t u(T, x) \right|^2 dx \\ &\leq \varepsilon^{-1} \left(r \int_{\Sigma_{+, \omega}} \left| e^{-r\xi \cdot (t, x)} \partial_\nu u \right|^2 \omega \cdot \nu(x) d\sigma(x) dt + r \int_{\Omega} \left| e^{-r\xi \cdot (T, x)} \partial_t u(T, x) \right|^2 dx \right) \\ &\leq \varepsilon^{-1} C \left(\int_Q \left| e^{-r\xi \cdot (t, x)} (\partial_t^2 - \Delta + q_1) u \right|^2 dx dt + r \int_{\Sigma_{-, \omega}} \left| e^{-r\xi \cdot (t, x)} \partial_\nu u \right|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \right) \\ &\quad + \varepsilon^{-1} C \left(r^5 \int_{\Omega} e^{-2r\xi \cdot (T, x)} |u|_{t=T}^2 dx + r^3 \int_{\Omega} e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}^2 dx \right) \\ &\leq \varepsilon^{-1} C \left(\int_Q \left| e^{-r\xi \cdot (t, x)} q u_2 \right|^2 dx dt + r \int_{\Sigma_{-, \omega}} \left| e^{-r\xi \cdot (t, x)} \partial_\nu u \right|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \right) \\ &\quad + \varepsilon^{-1} C \left(r^5 \int_{\Omega} e^{-2r\xi \cdot (T, x)} |u|_{t=T}^2 dx + r^3 \int_{\Omega} e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}^2 dx \right) \\ &\leq \varepsilon^{-1} C \left(\int_Q |q|^2 (1 + |z|)^2 dx dt + r \int_{\Sigma_{-, \omega}} \left| e^{-r\xi \cdot (t, x)} \partial_\nu u \right|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \right) \\ &\quad + \varepsilon^{-1} C \left(r^5 \int_{\Omega} e^{-2r\xi \cdot (T, x)} |u|_{t=T}^2 dx + r^3 \int_{\Omega} e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}^2 dx \right) \\ &\leq \varepsilon^{-1} C \left(8M^2 T |\Omega| + 8M^2 \|z\|_{L^2(Q)}^2 + r \int_{\Sigma_{-, \omega}} \left| e^{-r\xi \cdot (t, x)} \partial_\nu u \right|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \right) \\ &\quad + \varepsilon^{-1} C \left(r^5 \int_{\Omega} e^{-2r\xi \cdot (T, x)} |u|_{t=T}^2 dx + r^3 \int_{\Omega} e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}^2 dx \right). \end{aligned}$$

Using estimate (5.3) and the fact that $\Sigma_{-, \omega} \subset \Sigma_{-, \varepsilon, \omega}$, $|\omega \cdot \nu(x)| \leq 1$, we obtain

$$\begin{aligned}
& r \int_{\Sigma_{+, \varepsilon, \omega}} |e^{-r\xi \cdot (t, x)} \partial_\nu u|^2 d\sigma(x) dt + r \int_\Omega |e^{-r\xi \cdot (T, x)} \partial_t u(T, x)|^2 dx \\
& \leq C \left(1 + r \int_{\Sigma_{-, \varepsilon, \omega}} |e^{-r\xi \cdot (t, x)} \partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \right) \\
& \quad + C \left(r^5 \int_\Omega e^{-2r\xi \cdot (T, x)} |u|_{t=T}|^2 dx + r^3 \int_\Omega e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}|^2 dx \right) \\
& \leq C \left(1 + r \int_{\Sigma_{-, \varepsilon, \omega}} |e^{-r\xi \cdot (t, x)} \partial_\nu u|^2 d\sigma(x) dt \right) \\
& \quad + C \left(r^5 \int_\Omega e^{-2r\xi \cdot (T, x)} |u|_{t=T}|^2 dx + r^3 \int_\Omega e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}|^2 dx \right),
\end{aligned}$$

where C depends only on Ω , T , ε_0 , ω_0 and M . Therefore, we have

$$\begin{aligned}
& \int_{\Sigma_{+, \varepsilon, \omega}} |e^{-r\xi \cdot (t, x)} \partial_\nu u|^2 d\sigma(x) dt + \int_\Omega |e^{-r\xi \cdot (T, x)} \partial_t u(T, x)|^2 dx \\
& \leq C \left(r^{-1} + \int_{\Sigma_{-, \varepsilon, \omega}} |e^{-r\xi \cdot (t, x)} \partial_\nu u|^2 d\sigma(x) dt + r^4 \int_\Omega e^{-2r\xi \cdot (T, x)} |u|_{t=T}|^2 dx + r^2 \int_\Omega e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}|^2 dx \right).
\end{aligned}$$

Combining this with (6.5), we obtain

$$\begin{aligned}
& \left| \int_Q q u_1 u_2 dx dt \right|^2 \\
& \leq C \left(r^{-1} + \int_{\Sigma_{-, \varepsilon, \omega}} |e^{-r\xi \cdot (t, x)} \partial_\nu u|^2 d\sigma(x) dt + r^4 \int_\Omega e^{-2r\xi \cdot (T, x)} |u|_{t=T}|^2 dx + r^2 \int_\Omega e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}|^2 dx \right)
\end{aligned} \tag{6.6}$$

with C depending only on Ω , T , ε_0 , ω_0 and M . On the other hand, (3.12) implies

$$\int_Q q u_1 u_2 dx dt = \int_{\mathbb{R}^{1+n}} q(t, x) e^{-i(l, y) \cdot (t, x)} dx dt + \int_Q Z e^{-i(l, y) \cdot (t, x)} dx dt$$

with $Z(t, x) = z + w + zw$. Then, in view of (3.13) and (5.3), an application of the Cauchy-Schwarz inequality yields

$$\left| \int_Q Z e^{-i(l, y) \cdot (t, x)} dx \right| \leq C r^{-\frac{1}{2}}$$

with C depending on Ω , T , ε_0 , ω_0 and M . Combining this estimate with (6.6), we obtain

$$\begin{aligned}
& \left| \int_{\mathbb{R}^{1+n}} q(t, x) e^{-i(l, y) \cdot (t, x)} dx dt \right|^2 \\
& \leq C \left(r^{-1} + \int_{\Sigma_{-, \varepsilon, \omega}} |e^{-r\xi \cdot (t, x)} \partial_\nu u|^2 d\sigma(x) dt + r^4 \int_\Omega e^{-2r\xi \cdot (T, x)} |u|_{t=T}|^2 dx + r^2 \int_\Omega e^{-2r\xi \cdot (T, x)} |\omega \cdot \nabla_x u|_{t=T}|^2 dx \right) \\
& \leq C \left[\frac{1}{r} + e^{d_1 r} \left(\|\partial_\nu u\|_{L^2(\Sigma_{-, \varepsilon, \omega})}^2 + \|u|_{t=T}\|_{H^1(\Omega)}^2 \right) \right]
\end{aligned} \tag{6.7}$$

with $C, d_1 > 0$ depending only on Ω , T , ε_0 , ω_0 and M . Using the fact that $\Sigma_{-, \varepsilon, \omega} \subset \Sigma_{-, \varepsilon_0, \omega_0}$ and

$$\text{supp } g \subset \Sigma_{-, \varepsilon, -\omega} \subset \Sigma_{-, \varepsilon_0, -\omega_0}, \quad v_0 = 0$$

with $(g, v_0, v_1) = \mathcal{T}_0 u_2$, we deduce that

$$(\partial_\nu u|_{\Sigma_{-, \varepsilon_0, \omega_0}}, u|_{t=T}) = (B_{q_1}^* - B_{q_2}^*) \mathcal{T}_0 u_2.$$

Then, we have

$$\begin{aligned} \|\partial_\nu u\|_{L^2(\Sigma_{-, \varepsilon, \omega})}^2 + \|u|_{t=T}\|_{H^1(\Omega)}^2 &\leq \|\partial_\nu u\|_{L^2(\Sigma_{-, \varepsilon_0, \omega_0})}^2 + \|u|_{t=T}\|_{H^1(\Omega)}^2 \\ &\leq \|(B_{q_1}^* - B_{q_2}^*)\mathcal{T}_0 u_2\|_{L^2(\Sigma_{-, \varepsilon_0, \omega_0}) \times H^1(\Omega)}^2 \end{aligned}$$

and from (5.4) we deduce

$$\|\partial_\nu u\|_{L^2(\Sigma_{-, \varepsilon, \omega})}^2 + \|u|_{t=T}\|_{H^1(\Omega)}^2 \leq \|B_{q_1}^* - B_{q_2}^*\|^2 \|\mathcal{T}_0 u_2\|_{\mathcal{H}(\partial Q)}^2 \leq C \|B_{q_1}^* - B_{q_2}^*\|^2 e^{2\delta r}$$

with C depending only on Ω , T , ε_0 , ω_0 and M . Combining this estimate with (6.7), we obtain (6.1). \square

From now on, for all $s > 0$, we denote by B_s the set $B_s = \{z \in \mathbb{R}^{1+n} : |z| < s\}$. Let us recall the following result, which follows from Theorem 3 in [1] (see also [35]), on the continuous dependence in the analytic continuation problem.

Proposition 6. *Let $\rho > 0$ and assume that $f : B_{2\rho} \subset \mathbb{R}^{1+n} \rightarrow \mathbb{R}$ is a real analytic function satisfying*

$$\|\partial^\beta f\|_{L^\infty(B_{2\rho})} \leq \frac{N\beta!}{(\rho\lambda)^{|\beta|}}, \quad \beta \in \mathbb{N}^{1+n}$$

for some $N > 0$ and $0 < \lambda \leq 1$. Further let $E \subset B_{\frac{\rho}{2}}$ be a measurable set with strictly positive measure. Then,

$$\|f\|_{L^\infty(B_\rho)} \leq C(N)^{(1-b)} \left(\|f\|_{L^\infty(E)} \right)^b,$$

where $b \in (0, 1)$, $C > 0$ depend on λ , $|E|$ and ρ .

Armed with Lemma 4, we will use Proposition 6 to complete the proof of Theorem 1.

Proof of Theorem 1. Without lost of generality, we can assume that $0 \in \Omega$. Consider the set

$$G = \{\xi \in \mathbb{S}^n : \xi \text{ of the form (3.6) with } 1 < c < 2, \omega \in \{z \in \mathbb{S}^{n-1} : |z - \omega_0| < \varepsilon\}\}.$$

One can easily check that G is an open set of \mathbb{S}^n . Introduce the set

$$E_1 = \{\eta \in \mathbb{R}^{1+n} : \text{there exists } \xi = (\xi_0, \xi') \in G \text{ such that } \eta \cdot (-\xi_0, \xi') = 0\}$$

and note that estimate (6.1) holds for all $(l, y) \in E_1$, $r \geq r_2$, such that condition (3.9) is fulfilled and $|(l, y)| \leq Br$ for some $B > 0$. Let us observe that in view of (3.6) and (3.9), (3.9) will be fulfilled if

$$|(l, y)| \leq 2 \inf_{1 \leq c \leq 2} \left(\left(1 - \frac{1}{(1+c)^2} \right) \frac{(1+c)^2}{1+(1+c)^2} \right)^{\frac{1}{2}} r = Br$$

with

$$B = 2 \min_{1 \leq c \leq 2} \left(\frac{(1+c)^2 - 1}{1+(1+c)^2} \right)^{\frac{1}{2}} > 0.$$

From now on we fix this value of B . Next we set for fixed $R > 0$, which will be made precise later, and $(l, y) \in \mathbb{R}^{1+n}$,

$$H(l, y) = \mathcal{F}(q)(R(l, y)) = (2\pi)^{-\frac{(1+n)}{2}} \int_{\mathbb{R}^{1+n}} q(t, x) e^{-iR(l, y) \cdot (t, x)} dx dt.$$

Since $\text{supp } q \subset \bar{Q}$ and $0 \in \Omega$, H is analytic and

$$|\partial^\beta H(l, y)| \leq C \frac{\|q\|_{L^1(Q)} R^{|\beta|}}{([\max(T, \text{Diam}(\Omega))]^{-1})^{|\beta|}} \leq C \|q\|_{L^1(Q)} \frac{R^{|\beta|}}{\beta! ([\max(T, \text{Diam}(\Omega))]^{-1})^{|\beta|}} \beta!, \quad \beta \in \mathbb{N}^{1+n}$$

with C depending on T and Ω . Moreover, we have

$$\|q\|_{L^1(Q)} \leq 2MT |\Omega|$$

and one can check that

$$\frac{R^{|\beta|}}{\beta!} \leq e^{(1+n)R}.$$

Applying these estimates, we obtain

$$|\partial^\beta H(l, y)| \leq C \frac{e^{(1+n)R} \beta!}{([\max(T, \text{Diam}(\Omega))]^{-1})^{|\beta|}}, \quad \beta \in \mathbb{N}^{1+n} \quad (6.8)$$

with C depending on M , Ω and T . Set $\rho = [\max(T, \text{Diam}(\Omega))]^{-1} + 1$, $E = E_1 \cap \{x \in \mathbb{R}^{1+n} : |x| < \min(\frac{\rho}{2}, 1)\}$ with $N = Ce^{(1+n)R}$ and $\lambda = \frac{[\max(T, \text{Diam}(\Omega))]^{-1}}{\rho}$. In view of (6.8), we have

$$\|\partial^\beta H\|_{L^\infty(B_{2\rho})} \leq C \frac{e^{(1+n)R} \beta!}{([\max(T, \text{Diam}(\Omega))]^{-1})^{|\beta|}} = \frac{N\beta!}{(\rho\lambda)^{|\beta|}}, \quad \beta \in \mathbb{N}^{1+n}.$$

Using the fact that G is an open set of \mathbb{S}^n and $tE_1 = E_1$ for any $t > 0$, one can check that $|E| > 0$. Then, since $E \subset B_{\frac{\rho}{2}}$, $0 < \lambda < 1$ and $\rho > 1$, applying Proposition 6 to H we obtain

$$|\mathcal{F}(q)(R(l, y))| = |H(l, y)| \leq \|H\|_{L^\infty(B_\rho)} \leq Ce^{(1+n)R(1-b)} \left(\|H\|_{L^\infty(E)}\right)^b, \quad |(l, y)| < 1,$$

where $C > 0$ and $0 < b < 1$ depend only on Ω , T , M , ω_0 and ε_0 . But, for $R < Br$ estimate (6.1) implies that

$$|H(l, y)|^2 = |\mathcal{F}(q)(R(l, y))|^2 \leq C \left(\frac{1}{r} + e^{dr} \|B_{q_1}^* - B_{q_2}^*\|^2\right), \quad (l, y) \in E$$

and we deduce

$$|\mathcal{F}(q)(l, y)|^2 \leq Ce^{2(1+n)(1-b)R} \left(\frac{1}{r} + e^{dr} \|B_{q_1}^* - B_{q_2}^*\|^2\right)^b, \quad |(l, y)| < R. \quad (6.9)$$

Note that

$$\|q\|_{H^{-1}(\mathbb{R}^{1+n})}^{\frac{2}{b}} \leq C \left(\int_{\mathbb{R}^{1+n}} (1 + |(l, y)|^2)^{-1} |\mathcal{F}(q)(l, y)|^2 dy dl\right)^{\frac{1}{b}}. \quad (6.10)$$

We shall make precise below,

$$\int_{B_R} (1 + |(l, y)|^2)^{-1} |\mathcal{F}(q)(l, y)|^2 dy dl$$

and

$$\int_{\mathbb{R}^{1+n} \setminus B_R} (1 + |(l, y)|^2)^{-1} |\mathcal{F}(q)(l, y)|^2 dldy$$

separately. We start by examining the last integral. The Parseval-Plancherel theorem implies

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus B_R} (1 + |(l, y)|^2)^{-1} |\mathcal{F}(q)(l, y)|^2 dy dl &\leq \frac{1}{R^2} \int_{\mathbb{R}^{1+n} \setminus B_R} |\mathcal{F}(q)(l, y)|^2 dy dl \\ &\leq \frac{1}{R^2} \int_{\mathbb{R}^{1+n}} |\mathcal{F}(q)(l, y)|^2 dy dl = \frac{1}{R^2} \int_{\mathbb{R}^{1+n}} |q(t, x)|^2 dt dx \\ &\leq \frac{4T |\Omega| M^2}{R^2}. \end{aligned}$$

We end up getting that

$$\int_{\mathbb{R}^{1+n} \setminus B_R} (1 + |(l, y)|^2)^{-1} |\mathcal{F}(q)(l, y)|^2 dy dl \leq \frac{C}{R^2}. \quad (6.11)$$

Further, in light of (6.9), we get

$$\int_{B_R} (1 + |(l, y)|^2)^{-1} |\mathcal{F}(q)(l, y)|^2 dy dl \leq CR^{1+n} e^{2(1+n)(1-b)R} \left(\frac{1}{r} + e^{dr} \|B_{q_1}^* - B_{q_2}^*\|^2\right)^b, \quad r > r_2, \quad (6.12)$$

upon eventually substituting C for some suitable algebraic expression of C .

Last, putting (6.11)–(6.12) together we find out that

$$\begin{aligned} \|q\|_{H^{-1}(\mathbb{R}^{1+n})}^{\frac{2}{b}} &\leq C \left(\frac{1}{R^2} + R^{1+n} e^{2(1+n)(1-b)R} \left(\frac{1}{r} + e^{dr} \|B_{q_1}^* - B_{q_2}^*\|^2 \right)^b \right)^{\frac{1}{b}} \\ &\leq C \left(R^{-\frac{2}{b}} + \frac{R^{\frac{n+1}{b}} e^{2(1+n)(\frac{1-b}{b})R}}{r} + R^{\frac{n+1}{b}} e^{2(1+n)(\frac{1-b}{b})R} e^{dr} \|B_{q_1}^* - B_{q_2}^*\|^2 \right), \end{aligned} \quad (6.13)$$

for $1 < R < Br$ and $r > r_2$ where the constant $C > 0$ depends only on $\Omega, T, \varepsilon_0, \omega_0$ and M . Here we use the fact that $x \mapsto x^{\frac{1}{b}}$ is convex on $(0, +\infty)$ since $b \in (0, 1)$. Now let $R_1 > 1$ be such that

$$R < BR^{\frac{n+3}{b}} e^{2(1+n)R(\frac{1-b}{b})}, \quad R > R_1.$$

Choose $r = R^{\frac{n+3}{b}} e^{2(1+n)R(\frac{1-b}{b})}$ for $R > R_2$, with $R_2 > R_1$ such that

$$R^{\frac{n+3}{b}} e^{2(1+n)R(\frac{1-b}{b})} > r_2, \quad R > R_2.$$

Then, we have $r > r_2$, $R < Br$ and $\frac{R^{\frac{n+1}{b}} e^{2(1+n)R(\frac{1-b}{b})}}{r} = R^{-\frac{2}{b}}$. With this value of r we obtain

$$\|q\|_{H^{-1}(\mathbb{R}^{1+n})}^{\frac{2}{b}} \leq C \left(R^{-\frac{2}{b}} + R^{\frac{n+1}{b}} e^{2(1+n)(\frac{1-b}{b})R} \exp \left(dR^{\frac{n+3}{b}} e^{2(1+n)R(\frac{1-b}{b})} \right) \|B_{q_1}^* - B_{q_2}^*\|^2 \right). \quad (6.14)$$

On the other hand, we have

$$\begin{aligned} R^{\frac{n+1}{b}} e^{2(1+n)(\frac{1-b}{b})R} \exp \left(dR^{\frac{n+3}{b}} e^{2(1+n)R(\frac{1-b}{b})} \right) &\leq \exp \left(R^{\frac{n+1}{b}} + 2(1+n) \left(\frac{1-b}{b} \right) R + dR^{\frac{n+3}{b}} e^{2(1+n)R(\frac{1-b}{b})} \right) \\ &\leq \exp \left(e^{\frac{n+1}{b}R} + e^{2(1+n)(\frac{1-b}{b})R} + e^{[d + \frac{n+3}{b} + 2(1+n)(\frac{1-b}{b})]R} \right) \\ &\leq \exp \left(3e^{[\frac{n+1}{b} + 2(1+n)(\frac{1-b}{b}) + d + \frac{n+3}{b} + 2(1+n)(\frac{1-b}{b})]R} \right) \\ &\leq \exp \left(e^{[3 + \frac{n+1}{b} + 2(1+n)(\frac{1-b}{b}) + d + \frac{n+3}{b} + 2(1+n)(\frac{1-b}{b})]R} \right). \end{aligned}$$

Setting $A = 3 + \frac{n+1}{b} + 2(1+n)(\frac{1-b}{b}) + d + \frac{n+3}{b} + 2(1+n)(\frac{1-b}{b})$, (6.14) leads to

$$\|q\|_{H^{-1}(\mathbb{R}^{1+n})}^{\frac{2}{b}} \leq C \left(R^{-\frac{2}{b}} + e^{e^{AR}} \|B_{q_1}^* - B_{q_2}^*\|^2 \right), \quad R > R_2. \quad (6.15)$$

Set $\gamma = \|B_{q_1}^* - B_{q_2}^*\|$ and $\gamma^* = e^{-e^{AR_2}}$. For $\gamma \geq \gamma^*$ we have

$$\|q\|_{H^{-1}(Q)} \leq C \|q\|_{L^\infty(Q)} \leq \frac{2CM}{\gamma^*} \gamma. \quad (6.16)$$

For $0 < \gamma < \gamma^*$, by taking $R = R_3 = \frac{1}{A} \ln(|\ln \gamma|)$ in (6.15), which is permitted since $R_3 > R_2$, we find out that

$$\|q\|_{H^{-1}(Q)} \leq \|q\|_{H^{-1}(\mathbb{R}^{1+n})} \leq C \ln(|\ln \gamma|)^{-1} \left(\ln(|\ln \gamma|)^{\frac{2}{b}} \gamma + A^{\frac{2}{b}} \right)^{\frac{b}{2}}.$$

Now, since $\sup_{0 < \gamma \leq \gamma^*} \left(\ln(|\ln \gamma|)^{\frac{2}{b}} \gamma + A^{\frac{2}{b}} \right)^{\frac{b}{2}}$ is just another constant depending only on $\Omega, T, \varepsilon_0, \omega_0$ and M , we obtain

$$\|q\|_{H^{-1}(Q)} \leq C \ln(|\ln \gamma|)^{-1}, \quad 0 < \gamma < \gamma^*. \quad (6.17)$$

By interpolation we find

$$\|q\|_{L^2(Q)} \leq C \|q\|_{H^1(Q)}^{\frac{1}{2}} \|q\|_{H^{-1}(Q)}^{\frac{1}{2}} \leq C (M \sqrt{|\Omega| T})^{\frac{1}{2}} \|q\|_{H^{-1}(Q)}^{\frac{1}{2}}$$

with C depending only on Ω and T . Combining this estimate with (6.17), we deduce (1.3). \square

APPENDIX

In this appendix we prove that the space $\mathcal{C}^\infty(\overline{Q})$ is dense in $H_\square(Q)$ in some appropriate sense and we show that the maps \mathcal{T}_0 and \mathcal{T}_1 can be extended continuously on these spaces. Without loss of generality we consider only these spaces for real valued functions. The results of this section are well known, nevertheless we prove them for sake of completeness.

Density result in $H_\square(Q)$. Let us first recall the definition of $K_\square(Q)$:

$$K_\square(Q) = \{u \in H^{-1}(0, T; L^2(\Omega)) : \square u = (\partial_t^2 - \Delta)u \in L^2(Q)\}$$

with the norm

$$\|u\|_{K_\square(Q)}^2 = \|u\|_{H^{-1}(0, T; L^2(\Omega))}^2 + \|\square u\|_{L^2(Q)}^2.$$

The goal of this subsection is to prove the following.

Theorem 7. $H_\square(Q)$ embedded continuously into the closure of $\mathcal{C}^\infty(\overline{Q})$ wrt $K_\square(Q)$.

Proof. Let N be a continuous linear form on $K_\square(Q)$ satisfying

$$Nf = 0, \quad f \in \mathcal{C}^\infty(\overline{Q}). \quad (6.18)$$

In order to show the required density result we will prove that this condition implies that

$$N|_{H_\square(Q)} = 0.$$

By considering the application $u \mapsto (u, \square u)$ we can identify $K_\square(Q)$ to a subspace of $H^{-1}(0, T; L^2(\Omega)) \times L^2(Q)$. Then applying the Hahn Banach theorem we deduce that N can be extended to a continuous linear form on $H^{-1}(0, T; L^2(\Omega)) \times L^2(Q)$. Therefore, there exist $h_1 \in H_0^1(0, T; L^2(\Omega))$, $h_2 \in L^2(Q)$ such that

$$N(u) = \langle u, h_1 \rangle_{H^{-1}(0, T; L^2(\Omega)), H_0^1(0, T; L^2(\Omega))} + \langle \square u, h_2 \rangle_{L^2(Q)}, \quad u \in K_\square(Q).$$

Now let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded \mathcal{C}^∞ domain such that $\overline{\Omega} \subset \mathcal{O}$ and fix $Q_\varepsilon = (-\varepsilon, T + \varepsilon) \times \mathcal{O}$ with $\varepsilon > 0$. Let \tilde{h}_j be the extension of h_j on \mathbb{R}^{1+n} by 0 outside of Q for $j = 1, 2$. In view of (6.18) we have

$$\left\langle f, \tilde{h}_1 \right\rangle_{L^2(\mathcal{O})} + \left\langle (\partial_t^2 - \Delta)f, \tilde{h}_2 \right\rangle_{L^2(\mathcal{O})} = N(f|_Q) = 0, \quad f \in \mathcal{C}_0^\infty(Q_\varepsilon).$$

Thus, in the sense of distribution we have

$$\square \tilde{h}_2 = -\tilde{h}_1, \quad \text{on } Q_\varepsilon.$$

Moreover, since $\tilde{h}_2 = 0$ on $\mathbb{R}^{1+n} \setminus \overline{Q} \supset \partial Q_\varepsilon$, we deduce that \tilde{h}_2 solves

$$\begin{cases} \partial_t^2 \tilde{h}_2 - \Delta \tilde{h}_2 = -\tilde{h}_1, & \text{in } Q_\varepsilon, \\ \tilde{h}_2(-\varepsilon, x) = \partial_t \tilde{h}_2(-\varepsilon, x) = 0, & x \in \mathcal{O} \\ \tilde{h}_2(t, x) = 0, & (t, x) \in (-\varepsilon, T + \varepsilon) \times \partial \mathcal{O}. \end{cases}$$

But, since $h_1 \in H_0^1(0, T; L^2(\Omega))$, we have $\tilde{h}_1 \in H_0^1(-\varepsilon, T + \varepsilon; L^2(\mathcal{O}))$ and we deduce from Theorem 2.1 in Chapter 5 of [24] that this IBVP admits a unique solution lying in $H^2(Q_\varepsilon)$. Therefore, $\tilde{h}_2 \in H^2(Q_\varepsilon)$. Combining this with the fact that $\tilde{h}_2 = 0$ on $Q_\varepsilon \setminus Q$, we deduce that $h_2 \in H_0^2(Q)$, with $H_0^2(Q)$ the closure of $\mathcal{C}_0^\infty(Q)$ in $H^2(Q)$, and that $\square h_2 = -h_1$ on Q . Thus, for every $u \in H_\square(Q)$ we have

$$\langle \square u, h_2 \rangle_{L^2(Q)} = \langle \square u, h_2 \rangle_{H^{-2}(Q), H_0^2(Q)} = \langle u, \square h_2 \rangle_{L^2(Q)} = -\langle u, h_1 \rangle_{L^2(Q)}.$$

Here we use the fact that $H_\square(Q) \subset L^2(Q)$. Then it follows that

$$N(u) = \langle u, h_1 \rangle_{L^2(Q)} - \langle u, h_1 \rangle_{L^2(Q)} = 0, \quad u \in H_\square(Q).$$

From this last result we deduce that $H_\square(Q)$ is contained into the closure of $\mathcal{C}^\infty(\overline{Q})$ wrt $K_\square(Q)$. Combining this with the fact that $H_\square(Q)$ embedded continuously into $K_\square(Q)$ we deduce the required result. \square

Trace operator in $H_{\square}(Q)$. In this subsection we extend the trace maps \mathcal{T}_0 and \mathcal{T}_1 into $H_{\square}(Q)$ by duality in the following way.

Proposition 7. *The maps*

$$\mathcal{T}_0 w = (\mathcal{T}_0^1 w, \mathcal{T}_0^2 w, \mathcal{T}_0^3 w) = (w|_{\Sigma}, w|_{t=0}, \partial_t w|_{t=0}), \quad w \in \mathcal{C}^{\infty}(\overline{Q}),$$

$$\mathcal{T}_1 w = (\mathcal{T}_1^1 w, \mathcal{T}_1^2 w, \mathcal{T}_1^3 w) = (\partial_{\nu} w|_{\Sigma}, w|_{t=T}, \partial_t w|_{t=T}), \quad w \in \mathcal{C}^{\infty}(\overline{Q}),$$

can be extended continuously to $\mathcal{T}_0 : H_{\square}(Q) \rightarrow H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$,
 $\mathcal{T}_1 : H_{\square}(Q) \rightarrow H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$.

Proof. It is well known that the trace maps

$$u \mapsto (u|_{\partial\Omega}, \partial_{\nu} u|_{\partial\Omega})$$

can be extended continuously to a bounded operator from $H^2(\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ which is onto. Therefore, there exists a bounded operator $R : H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^2(\Omega)$ such that

$$R[h_1, h_2]|_{\partial\Omega} = h_1, \quad \partial_{\nu} R[h_1, h_2]|_{\partial\Omega} = h_2, \quad (h_1, h_2) \in H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

Fix $g \in H_0^3(0, T; H^{\frac{1}{2}}(\partial\Omega))$ and choose $G(t, \cdot) = R(0, g(t, \cdot))$. One can check that $G \in H_0^3(0, T; H^2(\Omega))$ and

$$\|G\|_{H^3(0, T; H^2(\Omega))} \leq \|R\| \|g\|_{H^3(0, T; H^{\frac{1}{2}}(\partial\Omega))}. \quad (6.19)$$

Applying twice the Green formula we obtain

$$\int_{\Sigma} v g d\sigma(x) dt = \int_Q \square v G dx - \int_Q v \square G dx, \quad v \in \mathcal{C}^{\infty}(\overline{Q}).$$

But $\square G \in H_0^1(0, T; H^2(\Omega))$, and we have

$$\langle \mathcal{T}_0^1 v, g \rangle_{H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega)), H_0^3(0, T; H^{\frac{1}{2}}(\partial\Omega))} = \langle \square v, G \rangle_{L^2(Q)} - \langle v, \square G \rangle_{H^{-1}(0, T; L^2(\Omega)), H_0^1(0, T; L^2(\Omega))}.$$

Then, using (6.19) and the Cauchy Schwarz inequality, for all $v \in \mathcal{C}^{\infty}(\overline{Q})$, we obtain

$$|\langle \mathcal{T}_0^1 v, g \rangle| \leq \|\square v\|_{L^2(Q)} \|G\|_{L^2(Q)} + \|v\|_{H^{-1}(0, T; L^2(\Omega))} \|\square G\|_{H_0^1(0, T; L^2(\Omega))}$$

$$\leq C \|v\|_{K_{\square}(Q)} \|g\|_{H^3(0, T; H^{\frac{1}{2}}(\partial\Omega))}$$

which, combined with the density result of Proposition 7, implies that $\mathcal{T}_0^1 : v \mapsto v|_{\Sigma}$ extend continuously to a bounded operator from $H_{\square}(Q)$ to $H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega))$. In a same way we prove that

$$\mathcal{T}_1^1 v = \partial_{\nu} v|_{\Sigma}, \quad v \in \mathcal{C}^{\infty}(\overline{Q})$$

extend continuously to a bounded operator from $H_{\square}(Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega))$.

Now let us consider the operators \mathcal{T}_i^j , $i = 0, 1$, $j = 2, 3$. We start with

$$\mathcal{T}_0^2 : v \mapsto v|_{t=0}, \quad v \in \mathcal{C}^{\infty}(\overline{Q}).$$

Let $h \in H_0^2(\Omega)$ and fix $H(t, x) = t\psi(t)h(x)$ with $\psi \in \mathcal{C}_0^{\infty}(-T, \frac{T}{2})$ satisfying $0 \leq \psi \leq 1$ and $\psi = 1$ on $[-\frac{T}{3}, \frac{T}{3}]$. Then, using the fact that $\psi = 1$ on a neighborhood of $t = 0$, we deduce that

$$H|_{\Sigma} = \partial_{\nu} H|_{\Sigma} = H|_{t=0} = \square H|_{t=0} = \square H|_{t=T} = 0, \quad \partial_t H|_{t=0} = h.$$

Therefore, $\square H \in H_0^1(0, T; L^2(\Omega))$ and repeating the above arguments, for all $v \in \mathcal{C}^{\infty}(\overline{Q})$, we obtain the representation

$$\langle \mathcal{T}_0^2 v, h \rangle_{H^{-2}(\Omega), H_0^2(\Omega)} = \langle v, \square H \rangle_{H^{-1}(0, T; L^2(\Omega)), H_0^1(0, T; L^2(\Omega))} - \langle H, \square v \rangle_{L^2(Q)}.$$

Then, we prove by density that \mathcal{T}_0^2 extend continuously to $\mathcal{T}_0^2 : H_{\square}(Q) \rightarrow H^{-2}(\Omega)$.

Now, let us consider

$$\mathcal{T}_0^3 : v \mapsto \partial_t v|_{t=0}, \quad v \in C^\infty(\overline{Q}).$$

Let $\varphi \in H_0^4(\Omega)$ and fix

$$\Phi(t, x) = \psi(t)\varphi(x) + \frac{\psi(t)t^2 \Delta \varphi(x)}{2}.$$

Then, Φ satisfies

$$\Phi|_\Sigma = \partial_\nu \Phi|_\Sigma = \partial_t \Phi|_{t=0} = 0, \quad \Phi|_{t=0} = \varphi.$$

Moreover, we have $\square \Phi \in H^1(0, T; L^2(\Omega))$ with

$$(\partial_t^2 - \Delta)\Phi|_{t=0} = -\Delta\varphi + \Delta\varphi = 0, \quad (\partial_t^2 - \Delta)\Phi|_{t=T} = 0,$$

which implies that $\square \Phi \in H_0^1(0, T; L^2(\Omega))$. Therefore, repeating the above arguments we obtain the representation

$$\langle \mathcal{T}_0^3 v, \varphi \rangle_{H^{-4}(\Omega), H_0^4(\Omega)} = \langle \square v, \Phi \rangle_{L^2(Q)} - \langle v, \square \Phi \rangle_{H^{-1}(0, T; L^2(\Omega)), H_0^1(0, T; L^2(\Omega))}$$

and we deduce that \mathcal{T}_0^3 extends continuously to $\mathcal{T}_0^3 : H_\square(Q) \rightarrow H^{-4}(\Omega)$. In a same way, one can check that

$$\mathcal{T}_1^2 : v \mapsto v|_{t=T}, \quad \mathcal{T}_1^3 : v \mapsto \partial_t v|_{t=T} \quad v \in C^\infty(\overline{Q})$$

extend continuously to $\mathcal{T}_1^2 : H_\square(Q) \rightarrow H^{-2}(\Omega)$ and $\mathcal{T}_1^3 : H_\square(Q) \rightarrow H^{-4}(\Omega)$. \square

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