Determination of a time-dependent coefficient for wave equations from partial data

Yavar Kian

DETERMINATION OF A TIME-DEPENDENT COEFFICIENT FOR WAVE EQUATIONS FROM PARTIAL DATA

YAVAR KIAN

CPT, UMR CNRS 7332, Aix Marseille Université, 13288 Marseille, France,
and Université de Toulon, 83957 La Garde, France.
E-mail: yavar.kian@univ-amu.fr

Abstract. We consider the stability in the inverse problem consisting of the determination of a time-dependent coefficient of order zero q, appearing in a Dirichlet initial-boundary value problem for a wave equation $\partial_t^2 u - \Delta u + q(t,x)u = 0$ in $Q = (0,T) \times \Omega$ with Ω a bounded C^2 domain of $\mathbb{R}^n$, $n \geqslant 3$, from partial observations on ∂Q. The observation is given by a boundary operator associated to the wave equation. Using suitable complex geometric optics solutions and a Carleman estimate with linear weight, we prove a stability estimate in the determination of q from the boundary operator.

Keywords: Inverse problem, wave equation, scalar time-dependent potential, Carleman estimate.

Mathematics subject classification 2010 : 25R30.

Contents

1. Introduction

1.1. **Statement of the problem.** In the present paper we consider Ω a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n \geqslant 3$. We set $\Sigma = (0,T) \times \partial\Omega$ and $Q = (0,T) \times \Omega$ with $T > 0$. We introduce the following initial-boundary value problem (IBVP in short) for the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + q(t,x)u = 0, & \text{in } Q \\ u(0,\cdot) = v_0, \quad \partial_t u(0,\cdot) = v_1, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma. \end{cases} \tag{1.1}$$

where the potential $q \in L^\infty(Q)$. We prove that problem (1.1) is well posed in some appropriate space. More precisely, in Section 2 we introduce the space $\mathcal{H}(\partial Q)$ and prove that, for $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$, problem (1.1) admits a unique weak solution $u \in L^2(Q)$ with

$$\|u\|_{L^2(Q)} \leqslant C \|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)}.$$

We associate to (1.1) the boundary operator

$$B_q : \ (g, v_0, v_1) \mapsto (\partial_\nu u_{|\Sigma}, u_{|t=T}, \partial_t u_{|t=T})$$

with u the solution of (1.1) and ν the outward unit normal vector to $\partial\Omega$. Here u is the solution of (1.1) and, for u sufficiently smooth, $\partial_\nu u(x) = \nabla u(x) \cdot \nu(x)$. See also Section 2.1 for a rigorous definition of this operator. Consider, for all $y \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n : \ |y| = 1\}$, the set

$$\partial\Omega_{+,y} = \{x \in \partial\Omega : \ \nu(x) \cdot y > 0\}, \quad \partial\Omega_{-,y} = \{x \in \partial\Omega : \ \nu(x) \cdot y \leqslant 0\}$$

and for $\varepsilon > 0$

$$\partial\Omega_{+,\varepsilon,y} = \{x \in \partial\Omega : \ \nu(x) \cdot y > \varepsilon\}, \quad \partial\Omega_{-,\varepsilon,y} = \{x \in \partial\Omega : \ \nu(x) \cdot y \leqslant \varepsilon\}$$

with ν the outward unit normal vector to $\partial\Omega$. Note also that

$$\partial\Omega_{+,\varepsilon,-y} = \{x \in \partial\Omega : \ \nu(x) \cdot y < -\varepsilon\}, \quad \partial\Omega_{-,\varepsilon,-y} = \{x \in \partial\Omega : \ \nu(x) \cdot y \geqslant -\varepsilon\}.$$

We introduce

$$\Sigma_{\pm,y} = (0,T) \times \partial\Omega_{\pm,y}, \quad \Sigma_{\pm,\varepsilon,y} = (0,T) \times \partial\Omega_{\pm,\varepsilon,y}$$

and the closed subspace $F_{y,\varepsilon}$ of $\mathcal{H}(\partial Q)$ defined by

$$F_{y,\varepsilon} = \{(g, v_0, v_1) \in \mathcal{H}(\partial Q) : \ v_0 = 0, \ \text{supp} g \subset \Sigma_{-,\varepsilon,-y}\}.$$

Let $\omega_0 \in \mathbb{S}^{n-1}$, $\varepsilon_0 > 0$. The main purpose of this paper is to prove a stability estimate in the determination of the time-dependent potential q from the partial boundary operator

$$B_q^* = B_{q,\varepsilon_0,y_0}^* : F_{\omega_0,\varepsilon_0} \ni (g, 0, v_1) \mapsto (\partial_\nu u_{|\Sigma_{-,\varepsilon_0,\omega_0}}, u_{|t=T}). \tag{1.2}$$

Physically speaking, our inverse problem consists of determining properties such as density of an inhomogeneous medium, that evolve over time, by probing it with disturbances generated on the boundary and at initial time. The data is the response of the medium to these disturbances, measured on the boundary and at final time, and the purpose is to recover the function which measures the property of the medium.

1.2. **Existing papers.** In recent years the problem of recovering coefficients for hyperbolic equations from boundary measurements has attracted many attention. Many authors have considered this problem with an observation given by the reduced boundary operator

$$\Lambda_q : \ g \mapsto \partial_\nu u_{|\Sigma},$$

where u solves (1.1) with $v_0 = v_1 = 0$. This operator is usually called the Dirichlet to Neumann map (DN map in short). In [26], the authors proved that the DN map uniquely determines the time-independent potential in a wave equation. Isakov [16] considered the determination of a coefficient of order zero and a damping coefficient. Note that all these results are concerned with measurements on the whole boundary. The uniqueness by local DN map has been considered by [9]. The stability estimate in the case where the DN map is considered on the whole lateral boundary was treated by Stefanov and Uhlmann [29]. The uniqueness

and Hölder stability estimate in a subdomain were established by Isakov and Sun [17] and, assuming that the coefficients are known in a neighborhood of the boundary, Bellassoued, Choulli and Yamamoto [3] proved a log-type stability estimate in the case where the Neumann data are observed in an arbitrary subdomain of the boundary. In some recent work [20] extended the results of [26] to determination of large class of time-independent coefficient of order zero in an infinite cylindrical domain, also called cylindrical waveguide, and he proved that only measurements on a bounded subset are required for the determination of some class of coefficients including periodic coefficients and compactly supported coefficients.

Let us also mention that the method using Carleman inequalities was first considered by Bukhgeim and Klibanov [4]. For the application of Carleman estimate to the problem of recovering time-independent coefficients for hyperbolic equations we refer to [2], [15] and [19].

All the above mentioned results are concerned only with time-independent coefficients. Several authors considered the problem of determining time-dependent coefficients for hyperbolic equations. In [27], the authors considered the problem of determining the time-dependent coefficient q from the DN map Λ_q associated to (1.1). For this purpose, they considered the problem (1.1) on the infinite time-space cylindrical domain $\mathbb{R}\times\Omega$ instead of Q ($t\in\mathbb{R}$ instead of $0<t<T<\infty$). Then, with some additional assumptions [27] proved a result of uniqueness. Using similar arguments, [25] considered the same problem at finite time on Q, with $T>\mathrm{Diam}(\Omega)$, and they proved a result of uniqueness for the determination of q restricted to the subset S of Q, consisting of the lines which make 45 degree with the t-axis and which meet the planes $t=0$ and $t=T$ outside $\overline{Q}$, from the DN map Λ_q. Using a result of unique continuation due to [31], Eskin [10] proved that the DN map uniquely determines time-dependent coefficients that are analytic wrt the time variable t. In some recent work, [28] extended the result of [27] to more general coefficients and he proved a result of stability for compactly supported coefficients provided T is sufficiently large. One of the main point in the strategy of these authors consists of using geometric optics solutions concentrate near lines in order to recover the X-ray transform of the coefficient q from the DN map Λ_q. For time dependent coefficient this approach requires measurements on the infinite time-space cylindrical domain $\mathbb{R}\times\Omega$, otherwise one can only expect the determination of the coefficients restricted to a subdomain of Q when the coefficients in consideration are not analytic wrt t. Indeed, even with the knowledge of B_q restricted to zero initial data $v_0=v_1=0$, from domain of dependence argument there is no hope to determine q on the whole domain Q. In contrast to this approach, Isakov [16] used complex geometric optics similar to the one used by [30] for elliptic equations, and he proved a result of uniqueness in the determination of q on the whole domain Q from the boundary operator B_q (see also [6] and [8] for results of stability for parabolic and Schrödinger equations). In the present paper, we prove that the knowledge of the partial boundary operator B_q^* is sufficient for the determination of the time-dependent potential q on the whole domain Q. Moreover, we derive a stability estimate for this inverse problem. In contrast to the result of [16], which seems to be the only result of determination of a time-dependent coefficient, that is not analytic wrt to t, on the whole domain Q, in the present paper we consider only initial data (v_0,v_1) with $v_0=0$ and Dirichlet boundary condition g supported on some part of Σ (which, roughly speaking, corresponds to half of the boundary) and we measure $u_{|t=T}$ and $\partial_\nu u$ on some part of Σ (which, roughly speaking, corresponds to the other half of the boundary). Therefore, we restricted both our data and measurements.

We also mention that [6], [7] and [11] considered the problem of determining a time-dependent coefficient for parabolic equations and they derived stability estimate for this problem.

1.3. **Main result.** In order to express the main result of this paper we first remark (see Section 2) that for every $q_1,\ q_2\in L^\infty(Q)$ the operator

$$B_{q_1}^*-B_{q_2}^*:F_{\omega_0,\varepsilon_0}\to L^2(\Sigma_{-,\varepsilon_0,\omega_0})\times H^1(\Omega)$$

is bounded. Then our main result can be stated as follows.

Theorem 1. *Let $q_1,\ q_2\in W^{1,\infty}(Q)$. Assume that*

$$\|q_1\|_{W^{1,\infty}(Q)}+\|q_2\|_{W^{1,\infty}(Q)}\leqslant M.$$

Then, for all $\omega_0 \in \mathbb{S}^{n-1}$ *and* $\varepsilon_0 > 0$ *we have*

$$\|q_1 - q_2\|_{L^2(Q)} \leqslant h\left(\left\|B^*_{q_1} - B^*_{q_2}\right\|\right) \tag{1.3}$$

with

$$h(\gamma) = \begin{cases} C\frac{\gamma}{\gamma^*}, & \gamma \geqslant \gamma^*, \\ C\ln(|\ln\gamma|)^{-\frac{1}{2}}, & 0 < \gamma < \gamma^*, \\ 0, & \gamma = 0. \end{cases}$$

Here $\left\|B^*_{q_1} - B^*_{q_2}\right\|$ *stands for the norm of* $B^*_{q_1} - B^*_{q_2}$ *as an element of* $\mathcal{B}(F_{\omega,\varepsilon_0}, L^2(\Sigma_{-,\varepsilon_0,\omega}) \times H^1(\Omega))$. *Moreover* C *is a positive constant depending on* M, Ω, T, ε_0, ω_0 *and* $\gamma_* = e^{-e^{AR_1}}$, *with* A *and* R_1 *two constant introduced in Section 5 which depend on* M, T, Ω, ε_0, ω_0.

Let us observe that the stability estimate (1.3) can be improved into a log-type stability estimate if we replace B^*_q by the full boundary operator B_q. Indeed, by combining the results of Section 3 with arguments of [6] (see subsection 3.6) and [8], one can prove a log-type stability estimate in the determination of q from B_q.

Let us remark that in this paper we only treat the case $n \geqslant 3$. Nevertheless, we believe that with additional technical arguments one can extend our result to the case $n = 2$. For $n = 2$, the main difficulty comes from the choice for ζ_j, $j = 1, 2$, in Proposition 4.

The main tools in our analysis are suitable complex geometric optics (CGO in short) solutions and Carleman estimates with linear weight. More precisely, in this paper we adapt to the wave equation some arguments used by many authors for elliptic equations (see [5], [12], [18], [24]). This approach seems quite natural since we want to determine a coefficient that depends on every variable of our equation.

1.4. **Outline.** This paper is organized as follows. In Section 2 we treat the direct problem. We show that problem (1.1) is well posed in some appropriate space and we define the boundary operator B_q associated to this problem. In Section 3, using some results of [6], [13] and [14], we build suitable CGO solutions associated to (1.1) without condition on ∂Q. In Section 4, we establish a Carleman estimate for the wave equation with linear weight. In Section 5, we use the Carleman estimate introduced in Section 4 to build CGO solutions associated to (1.1) that vanish on parts of ∂Q. More precisely, we build CGO u which are solutions of (1.1) with $(g, v_0, v_1) \in F_{\omega_0,\varepsilon_0}$. In Section 6 we prove Theorem 1. We prove also some auxiliary results in the appendix.

2. Functional space

In this section we study the IBVP (1.1). We define the space $\mathcal{H}(\partial Q)$ and its topology. We define also the boundary operator B_q in some appropriate spaces. We first introduce the space

$$J = \{u \in L^2(Q) : \ (\partial_t^2 - \Delta)u = 0\}$$

and topologize it as a closed subset of $L^2(Q)$. We work with the space

$$H_\Box(Q) = \{u \in L^2(Q) : \ \Box u = (\partial_t^2 - \Delta)u \in L^2(Q)\},$$

with the norm

$$\|u\|^2_{H_\Box(Q)} = \|u\|^2_{L^2(Q)} + \left\|(\partial_t^2 - \Delta)u\right\|^2_{L^2(Q)}.$$

Here and in all this paper we denote by $\Box$ the differential operator defined by $\Box u = (\partial_t^2 - \Delta)u$. Repeating some arguments of Theorem 6.4 in chapter 2 of [22] we prove in the appendix (see Theorem 7) that $H_\Box(Q)$ embedded continuously into the closure of $\mathcal{C}^\infty(\overline{Q})$ in the space

$$K_\Box(Q) = \{u \in H^{-1}(0,T;L^2(\Omega)) : \ \Box u = (\partial_t^2 - \Delta)u \in L^2(Q)\}$$

topologized by the norm

$$\|u\|^2_{K_\Box(Q)} = \|u\|^2_{H^{-1}(0,T;L^2(\Omega))} + \left\|(\partial_t^2 - \Delta)u\right\|^2_{L^2(Q)}.$$

Then, following Theorem 6.5 in chapter 2 of [22], we prove in the appendix that the maps

$$\mathcal{T}_0 w = (w_{|\Sigma}, w_{|t=0}, \partial_t w_{|t=0}), \quad \mathcal{T}_1 w = (\partial_\nu w_{|\Sigma}, w_{|t=T}, \partial_t w_{|t=T}), \quad w \in \mathcal{C}^\infty(\overline{Q}),$$

can be extended continuously to $\mathcal{T}_0 : H_\Box(Q) \to H^{-3}(0,T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$, $\mathcal{T}_1 : H_\Box(Q) \to H^{-3}(0,T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$ (see Proposition 8). Therefore, we can define

$$\mathcal{H}(\partial Q) = \{\mathcal{T}_0 u : \ u \in H_\Box(Q)\} \subset H^{-3}(0,T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega).$$

Following [24], in order to define an appropriate topology on $\mathcal{H}(\partial Q)$ we consider the restriction of $\mathcal{T}_0$ to the space J.

Proposition 1. *The restriction of $\mathcal{T}_0$ to J is one to one and onto.*

Proof. Let $u, v \in J$ with $\mathcal{T}_0 u = \mathcal{T}_0 v$. Then $w = u - v$ solves

$$\begin{cases} \partial_t^2 w - \Delta w &= 0, \quad (t,x) \in Q, \\ w_{t=0} = \partial_t w_{|t=0} &= 0, \\ w_{|\Sigma} &= 0 \end{cases}$$

and the uniqueness of solutions of this IBVP implies that $w = 0$. Thus, the restriction of $\mathcal{T}_0$ to J is one to one. Now let $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$. There exists $F \in H_\Box(Q)$ such that $\mathcal{T}_0 F = (g, v_0, v_1)$. Consider the IBVP

$$\begin{cases} \partial_t^2 v - \Delta v &= -(\partial_t^2 - \Delta)F, \quad (t,x) \in Q, \\ v_{|t=0} = \partial_t v_{|t=0} &= 0, \\ v_{|\Sigma} &= 0, \end{cases}$$

Since $-(\partial_t^2 - \Delta)F \in L^2(Q)$, this IBVP admits a unique solution $v \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$. Then, $u = v + F \in L^2(Q)$ satisfies $(\partial_t^2 - \Delta)u = 0$ and $\mathcal{T}_0 u = \mathcal{T}_0 v + \mathcal{T}_0 F = (g, v_0, v_1)$. Thus $\mathcal{T}_0$ is onto. □

We set $\mathcal{P}_0$ the inverse of $\mathcal{T}_0 : J \to \mathcal{H}(\partial Q)$ and we define the norm of $\mathcal{H}(\partial Q)$ by

$$\|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)} = \|\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)}.$$

We are now in position to state the well possedness of the IBVP (1.1).

Proposition 2. *Let $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$ and $q \in L^\infty(Q)$. Then the IBVP (1.1) admits a unique solution $u \in L^2(Q)$ satisfying*

$$\|u\|_{L^2(Q)} \leqslant C \|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)} \tag{2.1}$$

and the boundary operator $B_q : (g, v_0, v_1) \mapsto \mathcal{T}_1 u$ is a bounded operator from $\mathcal{H}(\partial Q)$ to $H^{-3}(0,T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$.

Proof. We split u into two term $u = v + \mathcal{P}_0(g, v_0, v_1)$ where v solves

$$\begin{cases} \partial_t^2 v - \Delta v + qv &= -q\mathcal{P}_0(g, v_0, v_1), \quad (t,x) \in Q, \\ v_{|t=0} = \partial_t v_{|t=0} &= 0, \\ v_{|\Sigma} &= 0. \end{cases} \tag{2.2}$$

Since $\mathcal{P}_0(g, v_0, v_1) \in L^2(Q)$, the IBVP (2.2) admits a unique solution $v \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$ satisfying

$$\|v\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|v\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \leqslant C \|-q\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)} \leqslant C \|q\|_{L^\infty(Q)} \|\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)}. \tag{2.3}$$

Therefore, $u = v + \mathcal{P}_0(g, v_0, v_1)$ is the unique solution of (1.1) and estimate (2.3) implies (2.1).

Now let us show the last part of the proposition. For this purpose fix $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$ and consider u the solution of (1.1). Note first that $u \in L^2(Q)$ and $(\partial_t^2 - \Delta)u = -qu \in L^2(Q)$. Thus $u \in H_\Box(Q)$ and $\mathcal{T}_1 u \in H^{-3}(0,T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$ with

$$\|\mathcal{T}_1 u\|^2 \leqslant C^2 \|u\|^2_{H_\Box(Q)} = C^2(\|u\|^2_{L^2(Q)} + \|qu\|^2_{L^2(Q)}) \leqslant C^2(1 + \|q\|^2_{L^\infty(Q)}) \|u\|^2_{L^2(Q)}.$$

Combining this with (2.1) we deduce that B_q is a bounded operator from $\mathcal{H}(\partial Q)$ to $H^{-3}(0,T;H^{-\frac{3}{2}}(\partial\Omega))\times H^{-2}(\Omega)\times H^{-4}(\Omega)$. □

Consider the operator $B_{q_1}-B_{q_2}$ for $q_1,\ q_2\in L^\infty(Q)$. We have the following regularity result.

Proposition 3. *Let $q_1,\ q_2\in L^\infty(Q)$. Then the operator $B_{q_1}-B_{q_2}$ is a bounded operator from $\mathcal{H}(\partial Q)$ to $L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega)$.*

Proof. Let u_1,u_2 be respectively the unique solution of the IBVP (1.1) for $q=q_1$ and $q=q_2$. Then $u=u_1-u_2$ solves

$$\begin{cases} \partial_t^2 u-\Delta u+q_1u &= (q_2-q_1)u_2, \quad (t,x)\in Q,\\ u_{|t=0}=\partial_t u_{|t=0} &= 0,\\ u_{|\Sigma} &= 0. \end{cases}$$

Since $(q_2-q_1)u_2\in L^2(Q)$, in view of Theorem A.2 in [3] (see also Theorem 2.1 in [21] for $q=0$), $u\in\mathcal{C}^1([0,T];L^2(\Omega))\cap\mathcal{C}([0,T];H_0^1(\Omega))$ with $\partial_\nu u\in L^2(\Sigma)$. Moreover we have the following energy estimate

$$\|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))}+\|u\|_{\mathcal{C}([0,T];H_0^1(\Omega))}+\|\partial_\nu u\|_{L^2(\Sigma)}\leqslant C\left\|q_1-q_2\right\|_{L^\infty(Q)}\left\|u_2\right\|_{L^2(Q)}.$$

Note that in Theorem A.2 of [3] the authors consider only the case q is independent of t but their arguments still work when q is time-dependent. Combining this estimate with (2.1), we deduce that $\mathcal{T}_1u\in L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega)$ with

$$\|\mathcal{T}_1u\|_{L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega)}\leqslant C\left\|(g,v_0,v_1)\right\|_{\mathcal{H}(\partial Q)},$$

where C depends on Ω, T and $M\geqslant\|q_1\|_{L^\infty(Q)}+\|q_2\|_{L^\infty(Q)}$. Finally, we complete the proof by recalling that

$$\mathcal{T}_1u=\mathcal{T}_1u_1-\mathcal{T}_1u_2=(B_{q_1}-B_{q_2})(g,v_0,v_1).$$

□

According to Proposition 2 and 3, for every $q_1,\ q_2\in L^\infty(Q)$, the partial boundary operators $B_{q_j}^*$ are well defined as bounded operator from $F_{\omega_0,\varepsilon_0}$ to $H^{-3}(0,T;H^{-\frac{3}{2}}(\partial\Omega_{-,\varepsilon_0,\omega_0}))\times H^{-2}(\Omega)$. Moreover the operator $B_{q_1}^*-B_{q_2}^*$ is bounded from $F_{\omega_0,\varepsilon_0}$ to $L^2(\Sigma_{-,\varepsilon_0,\omega_0})\times H^1(\Omega)$.

3. Complex geometric optics solutions

The goal of this section is to build CGO u associated to the equation

$$\partial_t^2u-\Delta u+q(t,x)u=0\quad\text{on } Q.$$

More precisely we consider solutions of this equation of the form

$$u=e^{\zeta_1\cdot(t,x)}(1+w(t,x)) \tag{3.1}$$

with $u\in H^2(Q)$. Here $\zeta_1\in\mathbb{C}^{1+n}$ and it is chosen in such way that $(\partial_t^2-\Delta)e^{\zeta_1\cdot(t,x)}=0$. Moreover, ζ_1 depends on some parameter $r>0$ and the remainder term w in the asymptotic expansion of u wrt r satisfies

$$\|w\|_{L^2(Q)}\leqslant\frac{C}{r}$$

with $C>0$ independent of r. In order to build such CGO, we first introduce some well known results of Hörmander about solutions of PDE's with constant coefficients of the form $P(D)u=f$ on Q with $P\in\mathbb{C}[X_0,X_1,\ldots,X_n]$ a polynomial with complex coefficient and $D=-i(\partial_t,\partial_x)$.

3.1. **Solutions of PDE with constant coefficients.** We start this subsection by recalling some properties of solutions of PDE's of the form $P(D)u = f$ with constant coefficient. Let $P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ and $\tilde{P}$ defined by

$$\tilde{P} = \left(\sum_{\alpha \in \mathbb{N}^{1+n}} \left| \partial^{\alpha}_{(t,x)} P(t,x) \right|^2 \right)^{\frac{1}{2}}.$$

Theorem 2. (Theorem 7.3.10, [13])*For every $P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ with $P \neq 0$ one can find a distribution of finite order $E_P \in D'(\mathbb{R}^{1+n})$ such that $P(D)E_P = \delta$.*

Such distributions E_P are called fundamental solutions of P. Note that

$$E_P * (P(D)u) = u, \quad u \in \mathcal{E}'(\mathbb{R}^{1+n}),$$

$$P(D)(E_P * f) = f, \quad f \in \mathcal{E}'(\mathbb{R}^{1+n}),$$

where $\mathcal{E}'(\mathbb{R}^{1+n})$ is the set of distributions with compact support. Thus, for all $f \in \mathcal{E}'(\mathbb{R}^{1+n})$, $u = E_P * f$ is a solution of $P(D)u = f$. Let us give some information about the regularity of such a solution. For this purpose we need the following definitions introduced in [14].

Definition 1. *A positive function κ defined in $\mathbb{R}^{1+n}$ will be called a temperate weight function if there exist positive constants C and N such that*

$$\kappa(\xi + \eta) \leqslant C(1 + |\xi|)^N \kappa(\eta), \quad \xi, \eta \in \mathbb{R}^{1+n}.$$

The set of all such functions κ will be denoted by $\mathcal{K}$.

Notice that, for all $P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$, $\tilde{P} \in \mathcal{K}$.

Definition 2. *If $\kappa \in \mathcal{K}$ and $1 \leqslant p \leqslant \infty$, we denote by $B_{p,\kappa}$ the set of all temperate distribution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ such that $\hat{u}$ is a function and*

$$\|u\|_{p,\kappa} = \left(\frac{1}{(2\pi)^{1+n}} \int_{\mathbb{R}^{1+n}} |\kappa(\xi)\hat{u}(\xi)|^p \, d\xi \right)^{\frac{1}{p}} < \infty.$$

When $p = \infty$ we shall interpret $\|u\|_{p,\kappa}$ as ess. sup$|\kappa(\xi)\hat{u}(\xi)|$. We denote by $B^{loc}_{p,\kappa}$ the set of $u \in \mathcal{S}'(\mathbb{R}^{1+n})$ such that for all $\chi \in \mathcal{C}^{\infty}_0(\mathbb{R}^{1+n})$ we have $\chi u \in B_{p,\kappa}$.

Remark 1. *Let*

$$\kappa_1(\tau, \eta) = (1 + |(\tau, \eta)|^2)^{\frac{1}{2}}, \quad \tau \in \mathbb{R}, \ \eta \in \mathbb{R}^n.$$

Then, in view of Example 10.1.2 of [14]*, one can easily show that $\kappa_1 \in \mathcal{K}$ and $B_{2,\kappa_1} = H^1(\mathbb{R}^{1+n})$.*

Remark 2. *In view of Theorem 10.1.12 of* [14]*, for $\kappa_1', \kappa_2' \in \mathcal{K}$, $\kappa = \kappa_1' \cdot \kappa_2'$, $u_1 \in B_{p,\kappa_1'} \cap \mathcal{E}'(\mathbb{R}^{1+n})$ and $u_2 \in B_{\infty,\kappa_2'}$, we have $u_1 * u_2 \in B_{p,\kappa}$ and*

$$\|u_1 * u_2\|_{B_{p,\kappa}} \leqslant \|u_1\|_{B_{p,\kappa_1'}} \|u_2\|_{B_{\infty,\kappa_2'}}. \tag{3.2}$$

Theorem 3. (Theorem 10.21, [14]) *Every $P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$, with $P \neq 0$, has a fundamental solution $E_P \in B^{loc}_{\infty,\tilde{P}}$ such that $\frac{E_P}{\cosh(|(t,x)|)} \in B_{\infty,\tilde{P}}$ and*

$$\left\| \frac{E_P}{\cosh(|(t,x)|)} \right\|_{B_{\infty,\tilde{P}}} \leqslant C \tag{3.3}$$

with $C > 0$ a constant depending only on the degree of P.

Such a fundamental solution will be denoted by regular fundamental solution. Let us remark that in our construction of complex geometric optics solutions we need to consider an operator E such that $P(D)E = \mathrm{Id}$ for some $P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$. Using the properties of regular fundamental solutions, Hörmander proved in Theorem 10.3.7 of [14] that such operator exists and it is a bounded operator of $L^2(X)$ for X a bounded open set of $\mathbb{R}^{1+n}$. In contrast to elliptic equations and parabolic equations (see Subsection 2.1 and 3.6 of [6]), we can not build CGO lying in $H^2(Q)$ by applying the result of Hörmander. What we can actually build from this result is CGO lying in $H^1(Q)$. Therefore, we need to extend the result of Hörmander in the following way.

Theorem 4. *Let $P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ with $P \neq 0$. Then there exists an operator*

$$E: \ H^1(Q) \to H^1(Q)$$

such that:

(1) $P(D)Ef = f, \quad f \in H^1(Q)$,

(2) *for all $S \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ such that $\frac{\tilde{S}}{\tilde{P}}$ is bounded, we have $S(D)E \in B(H^1(Q))$, and*

$$\|S(D)E\|_{B(H^1(Q))} \leqslant C \sup_{\xi \in \mathbb{R}^{1+n}} \frac{|S(\xi)|}{\tilde{P}(\xi)}, \quad k = 0, 1, \tag{3.4}$$

where $C > 0$ depends only on the degree of P, Ω and T.

Proof of Theorem 4. Let $f \in H^1(Q)$. In view of Theorem 2.2 and 8.1 in Chapter 1 of [22], there exists an extension operator $p \in B\left(H^1(Q), H^1(\mathbb{R}^{1+n})\right)$ such that $p(f)_{|Q} = f$. Here we consider the extension operator p introduced by [22]. Set $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n})$ and $R > 0$ such that $\chi = 1$ on a neighborhood of $\overline{Q}$ and $\mathrm{supp}\chi \subset B_R$ with B_R the ball of radius R and of center 0 of $\mathbb{R}^{1+n}$. Let E_P be a regular fundamental solution of P. Now consider the operator

$$E : f \longmapsto (E_P * (\chi p(f)))_{|Q}.$$

Clearly we have

$$P(D)E_P * (\chi p(f)) = \chi p(f)$$

and it follows that

$$P(D)Ef = (\chi p(f))_{|Q} = f$$

which proves (1). Now let us show (2). For this purpose, let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n})$ be such that $\psi = 1$ on the closure of $B_R - B_R = \{x - y : \ x, y \in B_R\}$ and notice that

$$(E_P * (\chi p(f)))_{|Q} = ((\psi E_P) * (\chi p(f)))_{|Q}. \tag{3.5}$$

Note that

$$|\mathcal{F}(S(D)\psi E_P)| \leqslant \frac{|S(\xi)|}{\tilde{P}(\xi)} \tilde{P}(\xi) \left| \mathcal{F}\left(\psi \cosh(|(t,x)|) \frac{E_P}{\cosh(|(t,x)|)} \right) \right|$$

Then, since $\psi \cosh(|(t,x)|) \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n})$, from Lemma 2.1 of [6] we deduce that

$$\psi \cosh(|(t,x)|) \frac{E_P}{\cosh(|(t,x)|)} \in B_{\infty, \tilde{P}}$$

and

$$\left\| \psi \cosh(|(t,x)|) \frac{E_P}{\cosh(|(t,x)|)} \right\|_{B_{\infty,\tilde{P}}} \leqslant C_1 \left\| \frac{E_P}{\cosh(|(t,x)|)} \right\|_{B_{\infty,\tilde{P}}} \leqslant C'$$

with $C' > 0$ a constant depending only on the degree of P and χ. It follows

$$\|S(D)\psi E_P\|_{B_{\infty,1}} \leqslant C' \sup_{\xi \in \mathbb{R}^{1+n}} \frac{|S(\xi)|}{\tilde{P}(\xi)}.$$

In view of Remark 2, since $\chi p(f) \in H^1(\mathbb{R}^{1+n}) = B_{2,\kappa_1}$ with κ_1 introduced in Remark 1, we have $S(D)(\psi E_P) * (\chi p(f)) = (S(D)\psi E_P) * (\chi p(f)) \in B_{2,\kappa_1}$ and

$$\begin{aligned}\|S(D)(\psi E_P) * (\chi p(f))\|_{H^1(\mathbb{R}^{1+n})} &= \|S(D)(\psi E_P) * (\chi p(f))\|_{B_{2,\kappa_1}} \\ &\leqslant \|S(D)\psi E_P\|_{B_{\infty,1}} \|\chi p(f)\|_{H^1(\mathbb{R}^{1+n}))} \\ &\leqslant C \sup_{\xi\in\mathbb{R}^{1+n}} \frac{|S(\xi)|}{\tilde{P}(\xi)} \|f\|_{H^1(Q)}\end{aligned}$$

with $C > 0$ a constant depending only on the degree of P, χ, Ω and T. Thus, in view of (3.5), we have $S(D)Ef \in H^1(Q)$ and

$$\|S(D)Ef\|_{H^1(Q)} \leqslant \|(\psi E_P) * (\chi p(f))\|_{H^1(Q)} \leqslant C \sup_{\xi\in\mathbb{R}^{1+n}} \frac{|S(\xi)|}{\tilde{P}(\xi)} \|f\|_{H^1(Q)}\,.$$

□

Armed with this result, we are now in position to build CGO of the form (3.1) lying in $H^2(Q)$.

3.2. **Construction of complex geometric optics solutions.** For every $1 \leqslant c \leqslant 2$ and $\omega \in \mathbb{S}^{n-1}$ we set

$$\xi_0 = \frac{1}{\sqrt{1+(1+c)^2}}, \quad \xi' = \frac{1+c}{\sqrt{1+(1+c)^2}}\omega, \quad \xi = (\xi_0, \xi'). \tag{3.6}$$

Note that here $\xi \in \mathbb{S}^n$ and we have

$$\frac{2}{\sqrt{10}} \leqslant |\xi'| \leqslant \frac{3}{\sqrt{5}}, \quad \frac{1}{\sqrt{10}} \leqslant \xi_0 \leqslant \frac{1}{\sqrt{5}}. \tag{3.7}$$

Now let us consider the following.

Proposition 4. *Let $1 \leqslant c \leqslant 2$ and $\omega \in \mathbb{S}^{n-1}$ and let $\xi \in \mathbb{S}^n$ be defined by* (3.6)*. Then, for every $r > 0$, $(l, y) \in \mathbb{R} \times \mathbb{R}^n$ satisfying*

$$l\xi_0 - y \cdot \xi' = 0 = (l, y) \cdot (-\xi_0, \xi'), \tag{3.8}$$

$$r > \sqrt{\frac{\left|\left(\frac{l}{2}\right)^2 - \left(\frac{|y|}{2}\right)^2\right|}{\left(1 - \frac{1}{(1+c)^2}\right)|\xi'|^2}}, \tag{3.9}$$

there exists $\theta \in \mathbb{R}^n$ such that for

$$\zeta_1 = -r\xi - \frac{i}{2}(l, y) - i(0, \theta), \tag{3.10}$$

$$\zeta_2 = r\xi - \frac{i}{2}(l, y) + i(0, \theta), \tag{3.11}$$

we have

$$\begin{cases} (\partial_t^2 - \Delta)e^{\zeta_j\cdot(t,x)} = 0, \ j = 1, 2, \\ e^{\zeta_1\cdot(t,x)}e^{\zeta_2\cdot(t,x)} = e^{-i(l,y)\cdot(t,x)}. \end{cases} \tag{3.12}$$

Proof. Choose $\theta \in \mathbb{R}^n$ such that $\theta \cdot \eta = \theta \cdot \xi' = 0$. Notice that

$$(\partial_t^2 - \Delta)e^{\zeta_1\cdot(t,x)} = \left[r^2(\xi_0^2 - |\xi'|^2) - \left(\frac{l}{2}\right)^2 + \left(\frac{|y|}{2}\right)^2 - i(l, y) \cdot (-\xi_0, \xi') + |\theta|^2\right] e^{\zeta_1\cdot(t,x)}.$$

Applying (3.6) and (3.8), we obtain

$$(\partial_t^2 - \Delta)e^{\zeta_1\cdot(t,x)} = \left[-r^2\left(1 - \frac{1}{(1+c)^2}\right)|\xi'|^2 - \left(\frac{l}{2}\right)^2 + \left(\frac{|y|}{2}\right)^2 + |\theta|^2\right] e^{\zeta_1\cdot(t,x)}$$

and, in view of (3.9), we can choose

$$|\theta|^2 = r^2\left(1-\frac{1}{(1+c)^2}\right)|\xi'|^2 + \left(\frac{l}{2}\right)^2 - \left(\frac{|y|}{2}\right)^2 > 0.$$

Then, we obtain

$$(\partial_t^2 - \Delta)e^{\zeta_1\cdot(t,x)} = 0.$$

In the same way we prove that

$$(\partial_t^2 - \Delta)e^{\zeta_2\cdot(t,x)} = \left[-r^2\left(1-\frac{1}{(1+c)^2}\right)|\xi'|^2 - \left(\frac{l}{2}\right)^2 + \left(\frac{|y|}{2}\right)^2 + |\theta|^2\right]e^{\zeta_2\cdot(t,x)} = 0$$

and we deduce (3.12). □

Proposition 5. *Let $q \in W^{1,\infty}(Q)$, $1 \leqslant c \leqslant 2$, $\omega \in \mathbb{S}^{n-1}$, $\xi \in \mathbb{S}^n$ be defined by (3.6) and ζ_1 be defined by (3.10). Then, there exists $r_0 > 1$ such that for $r \geqslant r_0$ the equation $\partial_t^2 u - \Delta u + qu = 0$ admits a solution $u \in H^2(Q)$ of the form (3.1) with*

$$\|w\|_{H^1(Q)} \leqslant \frac{C}{r} \tag{3.13}$$

with C depending on Ω, T and $M \geqslant \|q\|_{W^{1,\infty}(Q)}$. Moreover, for $|(l,y)| \leqslant Br$, this solution u satisfies the estimate

$$\|u\|_{H^2(Q)} \leqslant Ce^{\delta r} \tag{3.14}$$

with C a constant depending on Ω, T, $M \geqslant \|q\|_{W^{1,\infty}(Q)}$, B and δ depending on Ω and T.

Proof. First notice that, in view of (3.12), w should be a solution of

$$\partial_t^2 w - \Delta w + 2\zeta_1\cdot(\partial_t, -\nabla_x)w = -q(1+w), \tag{3.15}$$

Set $P \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ defined by

$$P(\mu,\eta) = |\eta|^2 - \mu^2 + 2i\zeta_1\cdot(\mu,-\eta), \quad \mu \in \mathbb{R},\ \eta \in \mathbb{R}^n.$$

In view of Theorem 4, there exists $E \in B(H^1(Q))$ such that

$$P(D)EF = (\partial_t^2 - \Delta + 2\zeta_1\cdot(\partial_t,-\nabla_x))EF = F, \quad F \in H^1(Q). \tag{3.16}$$

Moreover, for every $S \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ such that $\frac{\tilde{S}}{\tilde{P}}$ is bounded, we have $S(D)E \in \mathcal{B}(H^1(Q))$ and

$$\|S(D)E\|_{\mathcal{B}(H^1(Q))} \leqslant K \sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{|S(\mu,\eta)|}{\tilde{P}(\mu,\eta)}, \tag{3.17}$$

where $K > 0$ depends only of Ω and T. Applying estimate (3.17) for $S = 1$ we deduce

$$\|E\|_{\mathcal{B}(H^1(Q))} \leqslant K \sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{1}{\tilde{P}(\mu,\eta)} \leqslant K \sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{1}{|\partial_\eta P(\mu,\eta)|}.$$

But $\partial_\eta P(\mu,\eta) = 2\eta + 2i(r\xi' + i(\frac{y}{2}+\theta))$ and $|\partial_\eta P(\mu,\eta)| \geqslant |\Im\partial_\eta P(\mu,\eta)| = 2|\xi'|\,r$. Therefore, using (3.7), we find

$$\|E\|_{\mathcal{B}(H^1(Q))} \leqslant \frac{K}{2|\xi'|\,r} \leqslant \frac{K\sqrt{10}}{4r} \tag{3.18}$$

Now consider the map

$$\begin{aligned} G: \quad H^1(Q) &\to H^1(Q), \\ F &\mapsto -E\left[q(1+F)\right]. \end{aligned}$$

Combining estimate (3.18) with a fixed point argument, we deduce that there exists $r_0 > 0$ such that for $r \geqslant r_0$ the map G admits a unique fix point $w \in H^1(Q)$ satisfying (3.13). Since w satisfies $w = -E\left[q(1+w)\right]$,

from (3.16) we deduce that it is a solution of (3.15). It remains to show that $w \in H^2(Q)$ and prove estimate (3.14). For this purpose, note that

$$\sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{|\eta|}{\left|\tilde{P}(\mu,\eta)\right|} \leqslant \sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{|\eta|}{|\partial_\eta P(\mu,\eta)|} = \sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{|\eta|}{\left|2\eta + 2i(r\xi' + i(\frac{y}{2}+\theta))\right|} \leqslant \max\left(1, \frac{\sqrt{10}\,|\zeta_1|}{2r}\right),$$

$$\sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{|\mu|}{\left|\tilde{P}(\mu,\eta)\right|} \leqslant \sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{|\mu|}{|\partial_\mu P(\mu,\eta)|} = \sup_{(\mu,\eta)\in\mathbb{R}^{n+1}} \frac{|\mu|}{\left|2\mu + 2i(r\xi_0 + \frac{il}{2}))\right|} \leqslant \max\left(1, \frac{\sqrt{10}\,|\zeta_1|}{r}\right).$$

Here we have used (3.7). Combining these estimates with (3.17), we deduce that $\partial_t w, \nabla_x w \in H^1(Q)$ which implies that $w \in H^2(Q)$ and we have

$$\|w\|_{H^2(Q)} \leqslant C\left(1 + \frac{2\,|\zeta_1|}{2\,|\xi'|\,r} + \frac{2\,|\zeta_1|}{2\,|\xi_0|\,r}\right)(1 + \|w\|_{H^1(Q)}) \leqslant C\left(1 + \frac{|\zeta_1|}{r}\right)$$

with C a generic constant depending on Ω, T and $M \geqslant \|q\|_{W^{1,\infty}(Q)}$. Therefore, u defined by (3.1) is an $H^2(Q)$-solution of $\partial_t^2 u - \Delta u + qu$ satisfying

$$\|u\|_{H^2(Q)} \leqslant C\left(1 + \frac{|\zeta_1|}{r}\right)(1 + |\zeta_1|)^2 \sup_{(t,x)\in Q} e^{r\xi\cdot(t,x)}$$

which implies (3.14). □

4. Carleman estimate with linear weight

This section is devoted to the proof of a Carleman estimate with linear weight. Fix $\xi = (\xi_0, \xi') \in \mathbb{S}^n$ satisfying (3.6) and consider the weighted operator

$$P_s = e^{-s\xi\cdot(t,x)}(\partial_t^2 - \Delta)e^{s\xi\cdot(t,x)}, \quad s \in \mathbb{R}.$$

The main result of this section can be stated as follows.

Theorem 5. *Let $v \in \mathcal{C}^2(\overline{Q})$ and $r > 0$. If u satisfies the condition*

$$v_{|\Sigma} = 0, \quad v_{|t=0} = \partial_t v_{|t=0} = 0 \tag{4.1}$$

then the estimate

$$\begin{aligned} &r\int_\Omega \left|\partial_t v_{|t=T}\right|^2 + r\int_{\Sigma_{+,\omega}} |\partial_\nu v|^2\,|\omega\cdot\nu(x)|\,d\sigma(x)dt + r^2\int_Q |v|^2\,dxdt \\ &\leqslant C\left(\int_Q |P_r v|^2\,dxdt + r^3\int_\Omega \left|\omega\cdot\nabla_x v_{|t=T}\right|^2 dx + r\int_{\Sigma_{-,\omega}} |\partial_\nu v|^2\,|\omega\cdot\nu(x)|\,d\sigma(x)dt\right) \end{aligned} \tag{4.2}$$

holds true for $r \geqslant r_1 > 0$ with C and r_1 depending only on Ω and T. If u satisfies the condition

$$v_{|\Sigma} = 0, \quad v_{|t=T} = \partial_t v_{|t=T} = 0 \tag{4.3}$$

then the estimate

$$\begin{aligned} &r\int_\Omega \left|\partial_t v_{|t=0}\right|^2 dx + r\int_{\Sigma_{-,\omega}} |\partial_\nu v|^2\,|\omega\cdot\nu(x)|\,d\sigma(x)dt + r^2\int_Q |v|^2\,dxdt \\ &\leqslant C\left(\int_Q |P_{-r} v|^2\,dxdt + r^3\int_\Omega \left|\omega\cdot\nabla_x v_{|t=0}\right|^2 dx + r\int_{\Sigma_{+,\omega}} |\partial_\nu v|^2\,|\omega\cdot\nu(x)|\,d\sigma(x)dt\right) \end{aligned} \tag{4.4}$$

holds true for $r \geqslant r_1 > 0$.

Proof. Since the differential operator P_s, $s \in \mathbb{R}$, is real valued, without lost of generality we can assume that u is real valued. We start with the proof of (4.2). Note first that P_r can be decomposed into $P_r = Q_1 + Q_2 + Q_3$ with $Q_1 = \partial_t^2 - \Delta + r^2(\xi_0^2 - |\xi'|^2)$, $Q_2 = r\xi_0\partial_t$, $Q_3 = -r\xi'\cdot\nabla_x$. Therefore we have

$$\|Q_3 v\|_{L^2(Q)}^2 + 2\langle Q_1 v, Q_2 v\rangle + 2\langle Q_1 v, Q_3 v\rangle + 2\langle Q_2 v, Q_3 v\rangle \leqslant \|P_r v\|_{L^2(Q)}^2.$$

Here we denote by $\langle .,. \rangle$ the scalar product in $L^2(Q)$. In view of this estimate, (4.2) follows from

$$\begin{aligned} & C_1\left(r\int_\Omega \left|\partial_t v_{|t=T}\right|^2 + r\int_{\Sigma_{+,\omega}} |\partial_\nu v|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt + r^2\int_Q |v|^2\, dxdt\right) \\ & \leqslant \|Q_3 v\|^2_{L^2(Q)} + 2\sum_{i\neq j}\langle Q_i v, Q_j v\rangle \\ & \quad + C_2\left(r^3\int_\Omega \left|\omega\cdot\nabla_x v_{|t=T}\right|^2 dx + r\int_{\Sigma_{-,\omega}} |\partial_\nu v|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt\right) \end{aligned} \tag{4.5}$$

with $C_1, C_2 > 0$ depending only on Ω and T. We will prove the above estimate by applying suitably the Green formula on Q. We start by decomposing the following terms

$$\langle Q_1 v, Q_2 v\rangle = r\xi_0\int_Q \partial_t^2 v\partial_t v - r\xi_0\int_Q \Delta v\partial_t v + r^3\xi_0(\xi_0^2 - |\xi'|^2)\int_Q v\partial_t v = A + B + C,$$

$$\langle Q_1 v, Q_3 v\rangle = -r\int_Q \partial_t^2 v\xi'\cdot\nabla v + r\int_Q \Delta v\xi'\cdot\nabla v - r^3(\xi_0^2 - |\xi'|^2)\int_Q v\xi'\cdot\nabla v = D + E + F,$$

$$\langle Q_2 v, Q_3 v\rangle = -r^2\xi_0\int_Q \partial_t v\xi'\cdot\nabla v = G.$$

For A: integrating wrt $t\in(0,T)$ we get

$$\int_Q \partial_t^2 v\partial_t v = \frac{1}{2}\int_Q \partial_t(\partial_t v)^2 = \frac{1}{2}\int_\Omega (\partial_t v_{|t=T})^2 dx.$$

For B: applying the Green formula wrt $x\in\Omega$ we find

$$\int_Q \Delta v\partial_t v = -\int_Q \nabla v\partial_t\nabla v = -\frac{1}{2}\int_Q \partial_t |\nabla v|^2 = -\frac{1}{2}\int_\Omega \left|\nabla v_{|t=T}\right|^2.$$

For C: integrating wrt $t\in(0,T)$ we obtain

$$\int_Q \partial_t v v = \frac{1}{2}\int_Q \partial_t(v)^2 = \frac{1}{2}\int_\Omega (v_{|t=T})^2 dx.$$

For D: integrating by parts wrt $t\in(0,T)$ and applying the Green formula wrt $x\in\Omega$ we get

$$\begin{aligned} \int_Q \partial_t^2 v\xi'\cdot\nabla v &= -\int_Q \partial_t v\xi'\cdot\nabla\partial_t v + \int_\Omega \partial_t v_{|t=T}(\xi'\cdot\nabla v_{|t=T}) \\ &= -\frac{1}{2}\int_0^T\int_\Omega \mathrm{div}_x((\partial_t v)^2\xi')dxdt + \int_\Omega \partial_t v_{|t=T}(\xi'\cdot\nabla v_{|t=T}) \\ &= -\frac{1}{2}\int_0^T\int_{\partial\Omega} (\partial_t v)^2\nu(x)\cdot\xi' d\sigma(x)dt + \int_\Omega \partial_t v_{|t=T}(\xi'\cdot\nabla v_{|t=T}) \\ &= \int_\Omega (\partial_t v_{|t=T})(\xi'\cdot\nabla v_{|t=T}). \end{aligned}$$

For E and F: applying a well known argument of [BU] (see the proof of Lemma 2.1 in [BU]) we obtain

$$\int_Q \Delta v\xi'\cdot\nabla v = \frac{1}{2}\int_\Sigma |\partial_\nu v|^2\, \xi'\cdot\nu(x)d\sigma(x)dt,$$

$$\int_Q v\xi'\cdot\nabla v = 0.$$

For G: integrating by parts wrt $t \in (0,T)$ and applying the Green formula wrt $x \in \Omega$ we find

$$\int_Q \partial_t v \xi' \cdot \nabla v = \int_\Omega (v_{|t=T})(\xi' \cdot \nabla v_{|t=T}) - \int_Q v \xi' \cdot \nabla \partial_t v = \int_\Omega (v_{|t=T})(\xi' \cdot \nabla v_{|t=T}) - \int_Q \xi' \cdot \nabla v \partial_t v$$

and it follows

$$\int_Q \partial_t v \xi' \cdot \nabla v = \frac{1}{2} \int_\Omega (v_{|t=T})(\xi' \cdot \nabla v_{|t=T}).$$

Combining all these results, we deduce that

$$\begin{aligned} &\langle Q_1 v, Q_2 v \rangle + \langle Q_1 v, Q_3 v \rangle + \langle Q_2 v, Q_3 v \rangle \\ &= \tfrac{r\xi_0}{2} \textstyle\int_\Omega (\partial_t v_{|t=T})^2 dx + \tfrac{r\xi_0}{2} \int_\Omega \left|\nabla v_{|t=T}\right|^2 dx + \tfrac{r^3(\xi_0^2 - |\xi'|^2)\xi_0}{2} \int_\Omega (v_{|t=T})^2 dx \qquad (4.6) \\ &\quad -r \textstyle\int_\Omega (\partial_t v_{|t=T})(\xi' \cdot \nabla v_{|t=T}) dx + \frac{r}{2} \int_\Sigma |\partial_\nu v|^2 \xi' \cdot \nu(x) d\sigma(x) dt - \frac{r^2\xi_0}{2} \int_\Omega (v_{|t=T})(\xi' \cdot \nabla v_{|t=T}) dx. \end{aligned}$$

For the fourth term on the rhs, applying the Cauchy-Schwarz inequality we get

$$\left| -r \int_\Omega (\partial_t v_{|t=T})(\xi' \cdot \nabla v_{|t=T}) dx \right| \leqslant \frac{r\xi_0^2}{8} \int_\Omega \left|\partial_t v_{|t=T}\right|^2 + \frac{32r}{\xi_0^2} \int_\Omega \left|\xi' \cdot \nabla v_{|t=T}\right|^2 dx.$$

On the other hand, in view of (4.1), an application of some suitable Poincarré inequality (see [DL] vol. 2 pp 125-126 and Proposition 2.2 in [BU]) yields

$$\int_\Omega (v_{|t=T})^2 dx \leqslant C \int_\Omega \left|\xi' \cdot \nabla v_{|t=T}\right|^2 dx$$

with C depending on $|\xi'|$ (which depends only on c according to (3.6)) and Ω. Then, using (3.7), we can replace C by a constant depending only on Ω. Therefore, the last term on the rhs of (4.6) can be majored by

$$\left| -\frac{r^2\xi_0}{2} \int_\Omega (v_{|t=T})(\xi' \cdot \nabla v_{|t=T}) dx \right| \leqslant C \frac{r^2\xi_0}{2} \int_\Omega \left|\xi' \cdot \nabla v_{|t=T}\right|^2 dx$$

and, in a same way, for the third term on the rhs of (4.6) we get

$$\left| \frac{r^3(\xi_0^2 - |\xi'|^2)\xi_0}{2} \int_\Omega (v_{|t=T})^2 dx \right| \leqslant C r^3 \frac{(|\xi'|^2 - \xi_0^2)\xi_0}{2} \int_\Omega \left|\xi' \cdot \nabla v_{|t=T}\right|^2 dx = C r^3 \int_\Omega \left|\xi' \cdot \nabla v_{|t=T}\right|^2 dx$$

with C a generic constant depending only on Ω. Moreover, using (4.1) and our Poincarré inequality we get

$$\int_Q |v|^2 dt dx \leqslant C \int_Q |\xi' \cdot \nabla v|^2 dt dx$$

with $C > 0$ depending on Ω, which implies

$$\frac{r^2}{C} \int_Q |v|^2 dt dx \leqslant \|Q_3 v\|^2_{L^2(Q)}.$$

Combining all these estimates with (4.6) we deduce (4.5), for $r > 1$, and by the same way (4.2). Now let us consider (4.4). For this purpose note that for v satisfying (4.3), w defined by $w(t,x) = v(T-t,x)$ satisfies (4.1). Thus applying (4.2) to w with ξ' replaced by $-\xi'$ we obtain (4.4). □

A direct consequence of this result is the following Carlemann estimate for the wave equation with potential.

Corollary 1. *Let $q \in L^\infty(Q)$. Then, there exists $r_2 > 0$ and $C > 0$ such that for all $u \in \mathcal{C}^2(\overline{Q})$ satisfying (4.1), and all $r \geqslant r_2$, we have*

$$\begin{aligned} & r\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\partial_t u_{|t=T}\right|^2 dx + r\int_{\Sigma_{+,\omega}} e^{-2r\xi\cdot(t,x)}\left|\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt + r^2\int_Q e^{-2r\xi\cdot(t,x)}\left|u\right|^2 dxdt \\ & \leqslant C\left(\int_Q e^{-2r\xi\cdot(t,x)}\left|(\partial_t^2-\Delta+q)u\right|^2 dxdt + r^3\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx\right) \\ & \quad + Cr\int_{\Sigma_{-,\omega}} e^{-2r\xi\cdot(t,x)}\left|\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt. \end{aligned} \tag{4.7}$$

Here r_2 and C depend on Ω, T and $M \geqslant \|q\|_{L^\infty(\Omega)}$.

Proof. Let us first consider the case $q = 0$. Note first that for u satisfying (4.1), $v = e^{-r\xi\cdot(t,x)}u$ satisfies also (4.1). Moreover, we have

$$\int_Q e^{-2r\xi\cdot(t,x)}\left|(\partial_t^2-\Delta)u\right|^2 dxdt = \int_Q |P_r v|^2\, dxdt$$

and from (4.1)

$$\partial_\nu v_{|\Sigma} = e^{-r\xi\cdot(t,x)}\partial_\nu u_{|\Sigma}.$$

Finally, applying (4.1) and the Poincarré inequality used in Theorem 5 we get

$$\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\partial_t u_{|t=T}\right|^2 dx \leqslant 2\int_\Omega \left|\partial_t v_{|t=T}\right|^2 dx + Cr^2\int_\Omega \left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx$$

with C depending on Ω. Thus, applying the Carleman estimate (4.2) to v, we deduce (4.7). For $q \neq 0$, we have

$$\left|\partial_t^2 u - \Delta u\right|^2 = \left|\partial_t^2 u - \Delta u + qu - qu\right|^2 \leqslant 2\left|(\partial_t^2-\Delta+q)u\right|^2 + 2\|q\|^2_{L^\infty(Q)}|u|^2$$

and hence if we choose $r_2 > 2C\|q\|^2_{L^\infty(Q)}$, replacing C by

$$C_1 = \sup_{r\geqslant r_2}\frac{Cr^2}{r^2 - 2C\|q\|^2_{L^\infty(Q)}} < \infty,$$

we deduce (4.7) from the same estimate when $q = 0$. □

Remark 3. *Note that, by density, estimate (4.7) can be extended to function $u \in \mathcal{C}^1([0,T];L^2(\Omega)) \cap \mathcal{C}([0,T];H^1(\Omega))$ satisfying (4.1), $(\partial_t^2-\Delta)u \in L^2(Q)$ and $\partial_\nu u \in L^2(\Sigma)$.*

5. Geometric optics solutions vanishing on part of the boundary

In this section we fix $q \in L^\infty(Q)$. The goal of this section is to use the Carleman estimate (4.4) in order to build solutions $u \in H_\Box(Q)$ to

$$\begin{cases} (\partial_t^2 - \Delta + q(t,x))u = 0 & \text{in } Q, \\ u_{|t=0} = 0, \\ u_{|\Sigma_{-,\varepsilon,-\omega}} = 0, \end{cases} \tag{5.1}$$

of the form

$$u(t,x) = e^{r\xi\cdot(t,x)}\left(e^{(\zeta_2 - r\xi)\cdot(t,x)} + z(t,x)\right), \quad (t,x) \in Q, \tag{5.2}$$

where ξ is defined by (3.6), $\zeta_2 \in \mathbb{C}^{1+n}$ is defined by (3.11) and satisfy (3.12), $z \in e^{-r\xi\cdot(t,x)}H_\Box(Q)$ fulfills

$$z(t,x) = -e^{(\zeta_2-r\xi)\cdot(t,x)}, \quad (t,x) \in \Sigma_{-,\varepsilon,-\omega} \cup \{0\}\times\Omega,$$

$$\|z\|_{L^2(Q)} \leqslant Cr^{-\frac{1}{2}}. \tag{5.3}$$

The main result of this section can be stated as follows.

Theorem 6. *Let $q \in L^\infty(Q)$. There exists $r_3 > 0$ such that for all $r \geqslant r_3$ there is a solution $u \in H_\Box(Q)$ of (5.1) of the form (5.2) with z satisfying (5.3). Moreover, we have*

$$\|\mathcal{T}_0 u\|_{\mathcal{H}(\partial Q)} \leqslant C e^{\delta r} \tag{5.4}$$

with C and δ depending on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$, ε and ω.

In order to prove existence of such solutions of (5.1) we need some preliminary tools and two intermediate results.

5.1. **Weighted space.** We first recall that

$$P_s = e^{-s\xi\cdot(t,x)}(\partial_t^2 - \Delta)e^{s\xi\cdot(t,x)}, \quad (t,x) \in Q, \ s \in \mathbb{R}$$

and we consider the space

$$H_{P_s}(Q) = \{v \in L^2(Q) : \ P_s v \in L^2(Q)\}$$

with the norm

$$\|v\|^2_{H_{P_s}(Q)} = \|v\|^2_{L^2(Q)} + \|P_s v\|^2_{L^2(Q)}.$$

One can easily check that

$$H_{P_s}(Q) = e^{-s\xi\cdot(t,x)} H_\Box(Q) := \{e^{-s\xi\cdot(t,x)} v : \ v \in H_\Box(Q)\}.$$

Set

$$\mathcal{H}_s(\partial Q) = \{e^{-s\xi\cdot(t,x)}(g, v_0, v_1 - s\xi_0 v_0) : \ (g, v_0, v_1) \in \mathcal{H}(\partial Q)\}$$

with the norm

$$\|(h, w_0, w_1)\|_{\mathcal{H}_s(\partial Q)} = \left\| e^{s\xi\cdot(t,x)}(h, w_0, w_1 + s\xi_0 w_0) \right\|_{\mathcal{H}(\partial Q)}.$$

Then, we can extend

$$\mathcal{T}_0 : v \mapsto (v_{|\Sigma}, v_{|t=0}, \partial_t v_{|t=0})$$

continuously to $\mathcal{T}_0 : H_{P_s}(Q) \longrightarrow \mathcal{H}_s(\partial Q)$.

5.2. **Construction of special solutions by Carleman estimate.** Applying the Carleman estimate (4.4) to $v \in \mathcal{C}^2(\overline{Q})$ satisfying (4.3) and $v_{|t=0} = 0$, for $r > r_2$, we obtain

$$\begin{aligned} & r^{\frac{1}{2}} \left\| \sqrt{|\omega\cdot\nu(x)|}\partial_\nu v \right\|_{L^2(\Sigma_{-,\omega})} + r^{\frac{1}{2}} \left\| \partial_t v_{|t=0} \right\|_{L^2(\Omega)} + r \|v\|_{L^2(Q)} \\ & \leqslant C \left(\|P_{-\tau} v\|_{L^2(Q)} + \tau^{\frac{1}{2}} \left\| \sqrt{|\omega\cdot\nu(x)|}\partial_\nu v \right\|_{L^2(\Sigma_{+,\omega})} \right) \end{aligned} \tag{5.5}$$

with C depending only on Ω, T. Therefore, estimate (5.5) holds for every $v \in \mathcal{C}^2(\overline{Q})$ satisfying $v_{|\Sigma} = 0$ and $v_{|t=T} = \partial_t v_{|t=T} = v_{|t=0} = 0$. Define

$$\mathcal{D} = \{v \in \mathcal{C}^2(\overline{Q}) : \ v_{|\Sigma} = 0, \ v_{|t=T} = \partial_t v_{|t=T} = v_{|t=0} = 0\}$$

and set

$$M_r = \{(P_{-r} f, \partial_\nu f_{|\Sigma_{+,\omega}}) : \ f \in \mathcal{D}\}.$$

We think of M_r as a subspace of $L^2(Q) \times L^2(r\omega\cdot\nu(x) d\sigma(x) dt, \Sigma_{+,\omega})$. Let $\mathcal{M}_r$ denote the orthogonal projection onto the closure of M_r in this Hilbert space. We then have the following result.

Proposition 6. *Given $r > r_2$ and*

$$v \in L^2(Q), \quad v_- \in L^2\left(\frac{d\sigma(x)dt}{|\omega\cdot\nu(x)|}, \Sigma_{-,\omega} \right), \quad v_0 \in L^2(\Omega)$$

there exists a unique $u \in L^2(Q)$ such that:
1) $P_r u = v$,

2) $\|u\|_{L^2(Q)} \leqslant C\left(r^{-1}\|v\|_{L^2(Q)} + r^{-\frac{1}{2}}\left\|\frac{v_-}{\sqrt{|\omega\cdot\nu(x)|}}\right\|_{L^2(\Sigma_{-,\omega})} + r^{-\frac{1}{2}}\|v_0\|_{L^2(\Omega)}\right)$ *with* C *depending on* Ω *and* T,
3) $u_{|\Sigma_{-,\omega}} = v_-,\ u_{|t=0} = v_0$,
4) $\mathcal{M}_r(u, -u_{|\Sigma_{+,\omega}}) = (u, -u_{|\Sigma_{+,\omega}})$.

Proof. Define a linear function L on M_r by

$$L((P_{-r}f, \partial_\nu f_{|\Sigma_{+,\omega}})) = \langle f, v\rangle_{L^2(Q)} - \langle \partial_\nu f, v_-\rangle_{L^2(\Sigma_{-,\omega})} + \left\langle \partial_t f_{|t=0}, v_0\right\rangle_{L^2(\Omega)}, \quad f \in \mathcal{D}.$$

Let $L = 0$ on the orthogonal complement of M_r. Using (5.5), for all $f \in \mathcal{D}$, we obtain

$$\begin{aligned}
&\left|L((P_{-r}f, \partial_\nu f_{|\Sigma_{+,\omega}}))\right| \\
&\leqslant \|f\|_{L^2(Q)}\|v\|_{L^2(Q)} + \left\|\sqrt{|\omega\cdot\nu(x')|}\partial_\nu f\right\|_{L^2(\Sigma_{-,\omega})}\left\|\frac{v_-}{\sqrt{|\omega\cdot\nu(x')|}}\right\|_{L^2(\Sigma_{-,\omega})} + \left\|\partial_t f_{|t=0}\right\|_{L^2(\Omega)}\|v_0\|_{L^2(\Omega)} \\
&\leqslant r^{-1}\|v\|_{L^2(Q)}\left(r\|f\|_{L^2(Q)}\right) + r^{-\frac{1}{2}}\left\|\frac{v_-}{\sqrt{|\omega\cdot\nu(x)|}}\right\|_{L^2(\Sigma_{-,\omega})}\left(r^{\frac{1}{2}}\left\|\sqrt{|\omega\cdot\nu(x)|}\partial_\nu f\right\|_{L^2(\Sigma_{-,\omega})}\right) \\
&\quad r^{-\frac{1}{2}}\|v_0\|_{L^2(\Omega)}\left(r^{\frac{1}{2}}\left\|\partial_t f_{|t=0}\right\|_{L^2(\Omega)}\right) \\
&\leqslant C\left(r^{-1}\|v\|_{L^2(Q)} + r^{-\frac{1}{2}}\left\|\frac{v_-}{\sqrt{|\omega\cdot\nu(x)|}}\right\|_{L^2(\Sigma_{-,\omega})} + r^{-\frac{1}{2}}\|v_0\|_{L^2(\Omega)}\right) \\
&\quad \times\left(\|P_{-r}f\|_{L^2(Q)} + r^{\frac{1}{2}}\left\|\sqrt{|\omega\cdot\nu(x)|}\partial_\nu f\right\|_{L^2(\Sigma_{+,\omega})}\right)
\end{aligned}$$

with C the constant of (5.5). Since the rhs of the last inequality is precisely the norm of $(P_{-r}f, \partial_\nu f_{|\Sigma_{+,\omega}})$ in $L^2(Q)\times L^2(r\omega\cdot\nu(x)d\sigma(x)dt, \Sigma_{+,\omega})$, L can be extended uniquely to a bounded linear form on $L^2(Q)\times L^2(r\omega\cdot\nu(x)d\sigma(x)dt, \Sigma_{+,\omega})$ with

$$\|L\| \leqslant C\left(r^{-1}\|v\|_{L^2(Q)} + r^{-\frac{1}{2}}\left\|\frac{v_-}{\sqrt{|\omega\cdot\nu(x)|}}\right\|_{L^2(\Sigma_{-,\omega})} + r^{-\frac{1}{2}}\|v_0\|_{L^2(\Omega)}\right). \tag{5.6}$$

Thus, there exists

$$(u, u_+) \in L^2(Q)\times L^2\left(r^{-1}|\omega\cdot\nu(x)|^{-1}d\sigma(x)dt, \Sigma_{+,\omega}\right)$$

such that for all $f \in \mathcal{D}$ we have

$$L((P_{-r}f, \partial_\nu f_{|\Sigma_{+,\omega}})) = \langle P_{-r}f, u\rangle_{L^2(Q)} - \langle \partial_\nu f, u_+\rangle_{L^2(\Sigma_{+,\omega})}.$$

Therefore, for all $f \in \mathcal{D}$ we have

$$\begin{aligned}
&\langle P_{-r}f, u\rangle_{L^2(Q)} - \langle \partial_\nu f, u_+\rangle_{L^2(\Sigma_{+,\omega})} \\
&= \langle f, v\rangle_{L^2(Q)} - \langle \partial_\nu f, v_-\rangle_{L^2(\Sigma_{-,\omega})} + \left\langle \partial_t f_{|t=0}, v_0\right\rangle_{L^2(\Omega)}
\end{aligned} \tag{5.7}$$

and estimate (5.6) implies

$$\|u\|_{L^2(Q)} \leqslant \|L\| \leqslant C\left(r^{-1}\|v\|_{L^2(Q)} + r^{-\frac{1}{2}}\left\|\frac{v_-}{\sqrt{|\omega\cdot\nu(x)|}}\right\|_{L^2(\Sigma_{-,\omega})} + r^{-\frac{1}{2}}\|v_0\|_{L^2(\Omega)}\right).$$

Taking $f \in \mathcal{C}_0^\infty(Q)$ shows that $P_r u = v \in L^2(Q)$ and $u \in H_{P_r}(Q)$. We have proved statement 1) and 2), now let us consider 3). Allowing $f \in \mathcal{D}$ to be arbitrary shows that $u_{|\Sigma_{-,\omega}} = v_-$, $u_{|t=0} = v_0$ and $u_{|\Sigma_{+,\omega}} = -u_+$. Finally, condition 4) follows from the fact that $L = 0$ in the orthogonal complement of M_r and this last condition guaranties the uniqueness of such u. □

Armed with this proposition we are now in position to prove Theorem 6. For this purpose, we introduce the operator

$$R_r :\ L^2(Q)\times L^2\left(\frac{d\sigma(x)dt}{|\omega\cdot\nu(x)|}, \Sigma_{-,\omega}\right)\times L^2(\Omega) \to L^2(Q)$$

such that for any $(v, v_-, v_0) \in L^2(Q) \times L^2\left(\frac{d\sigma(x)dt}{|\omega\cdot\nu(x)|}, \Sigma_{-,\omega}\right) \times L^2(\Omega)$, $u = R_r(v, v_-, v_0)$ is the unique element of $L^2(Q)$ satisfying conditions 1)-4) of Proposition 6. In particular condition 2) implies that

$$\|R_r\|_{B\left(\ L^2(Q)\times L^2\left(\frac{d\sigma(x)dt}{|\omega\cdot\nu(x)|},\Sigma_{-,\omega}\right)\times L^2(\Omega), L^2(Q)\right)} \leqslant Cr^{-\frac{1}{2}}, \quad r \geqslant r_2.$$

5.3. **Proof of Theorem 6.** Note first that z must satisfy

$$\begin{cases} z \in L^2(Q) & \\ P_r z = -q(t,x)\left(e^{(\zeta_2 - r\xi)\cdot(t,x)} + z(t,x)\right) & \text{in } Q, \\ z(0,x) = -e^{(\zeta_2 - r\xi)\cdot(0,x)} & \text{on } \Omega, \\ z(t,x) = -e^{(\zeta_2 - r\xi)\cdot(t,x)}. & \text{on } \Sigma_{+,\varepsilon,-\omega} \end{cases} \tag{5.8}$$

Choose $\psi \in \mathcal{C}_0^2(\{x \in \partial\Omega :\ \nu(x)\cdot\omega < -\frac{\varepsilon}{2}\})$ satisfying $\psi = 1$ on $\partial\Omega_{+,\varepsilon,-\omega}$ and set h_r defined by

$$h_r(t,x) = -\psi(x)e^{(\zeta_2 - r\xi)\cdot(t,x)}.$$

Since $h_r(t,x) = 0$ for $\nu\cdot\omega \geqslant -\frac{\varepsilon}{2}$ we have

$$h_r \in L^2\left(\frac{dx_1 d\sigma(x')}{|\omega\cdot\nu(x')|}, \Sigma_{-,\omega}\right).$$

Now let us introduce the map

$$G_r : L^2(Q) \ni F \mapsto R_r\left(-q(t,x)\left(e^{(\zeta_2 - r\xi)\cdot(t,x)} + F(t,x)\right), h_r, -e^{(\zeta_2 - r\xi)\cdot(0,x)}\right) \in L^2(Q).$$

We have $G_rF_1 - G_rF_2 = R_r\left[-q(F_1 - F_2), 0, 0\right]$ and statement 2) of Proposition 6 implies

$$\|G_rF_1 - G_rF_2\|_{L^2(Q)} \leqslant \frac{C}{r}\|q\|_{L^\infty(Q)}\|F_1 - F_2\|_{L^2(Q)}.$$

Moreover, since

$$(\zeta_2 - r\xi)\cdot(t,x) \in i\mathbb{R}, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,$$

we have

$$\|h_r\|_{L^2\left(\frac{d\sigma(x)dt}{|\omega\cdot\nu(x')|},\Sigma_{-,\omega}\right)} \leqslant C', \quad \left\|q(t,x)e^{(\zeta_2 - r\xi)\cdot(t,x)}\right\|_{L^2(Q)} \leqslant M\sqrt{T\,|\Omega|} \leqslant C', \tag{5.9}$$

$$\left\|e^{(\zeta_2 - r\xi)\cdot(0,x)}\right\|_{L^2(\Omega)} \leqslant C' \tag{5.10}$$

with C' depending only on Ω, T, M, ω and ε. Therefore, in view of statement 2) of Proposition 6, there exists $r_3 > 2CM + 1$ such that, for $r \geqslant r_3$, G_r is a contraction that admits a unique fixed point z. Then Proposition 6 implies that

$$z = R_r\left(-q(t,x)\left(e^{(\zeta_2 - r\xi)\cdot(t,x)} + z(t,x)\right), h_r, -e^{(\zeta_2 - r\xi)\cdot(0,x)}\right)$$

satisfies (5.8). From statement 2) of Proposition 6 and (5.9), we deduce that for all $r > r_3$

$$\|z\|_{L^2(Q)} = \left\|R_r\left(-q(t,x)\left(e^{(\zeta_2 - r\xi)\cdot(t,x)} + z\right), h_r, -e^{(\zeta_2 - r\xi)\cdot(0,x)}\right)\right\|_{L^2(Q)} \leqslant \frac{\|z\|_{L^2(Q)}}{2} + CC'r^{-1} + CC'r^{-\frac{1}{2}}$$

which implies (5.3). In view of (5.8), we have $z \in H_{P_r}(Q)$ which implies that u defined by (5.2) is lying in $H_\Box(Q)$ and is a solution of (5.1) with $\mathcal{T}_0 u \in F_{\omega,\varepsilon}$. Moreover, in view of estimate (5.3), we have

$$\begin{aligned} \|\mathcal{T}_0 u\|_{\mathcal{H}(\partial Q)} &\leqslant C(\|u\|_{L^2(Q)} + \left\|(\partial_t^2 - \Delta)u\right\|_{L^2(Q)}) \\ &\leqslant C(\|u\|_{L^2(Q)} + \|qu\|_{L^2(Q)}) \leqslant Ce^{\delta r} \end{aligned}$$

with C depending on ω, M, ξ, ε, and $\delta = \sup_{(t,x)\in Q}|(t,x)|$. This last estimate implies (5.4). □

6. Stability estimate

This section is devoted to the proof of Theorem 1. We start with an intermediate result. From now on we set $q = q_2 - q_1$ on Q and we assume that $q = 0$ on $\mathbb{R}^{1+n} \setminus Q$. Using the Carleman estimate introduced in the previous section and the geometric optics solutions of Section 3 and Section 5, we obtain the following.

Lemma 1. *Assume that the condition of Theorem 1 are fulfilled. Let $\varepsilon = \frac{\varepsilon_0}{2}$ and $\omega \in \{z \in \mathbb{S}^{n-1} : \ |z - \omega_0| \leqslant \varepsilon\}$. Then, there exists $r_4 \geqslant \max(r_0, r_1, r_2, r_3)$, with r_0 introduced in Proposition 5, r_1 the constant of Theorem 5, r_2 the constant introduced in Corollary 1 and r_3 introduced in Theorem 6, such that for all $r > r_4$, $\xi \in \mathbb{S}^n$ of the form (3.6) with $1 \leqslant c \leqslant 2$, $(l, y) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ satisfying (3.8), (3.9) and $|(l, y)| \leqslant Br$, we have*

$$\left| \int_{\mathbb{R}^{1+n}} q(t,x) e^{-i(l,y)\cdot(t,x)} dt dx \right|^2 \leqslant C \left(\frac{1}{r} + e^{dr} \left\| B^*_{q_1} - B^*_{q_2} \right\|^2 \right) \tag{6.1}$$

with d, C depending only on Ω, M, T, ε_0, B, ω_0.

Proof. Let ζ_j, $j = 1, 2$, be defined by (3.10), (3.11) and satisfy (3.12). According to Proposition 5, we can introduce

$$u_1(t,x) = (1 + w(t,x))\, e^{\zeta_1 \cdot (t,x)}, \ (t,x) \in Q,$$

where $u_1 \in H^2(Q)$ satisfies $\partial_t^2 u_1 - \Delta u_1 + q_1 u_1 = 0$ and w satisfies (3.13). Moreover, in view of Theorem 6, we consider $u_2 \in H_\square(Q)$ solution of (5.1) with $q = q_2$ of the form

$$u_2(t,x) = e^{r\xi\cdot(t,x)} \left(e^{(\zeta_2 - r\xi)\cdot(t,x)} + z(t,x) \right), \quad (t,x) \in Q$$

with z satisfying (5.3), such that $\mathcal{T}_0 u_2 \in F_{\varepsilon,\omega}$ and u_2 fulfills (5.4). Let w_1 be the solution of

$$\begin{cases} \partial_t^2 w_1 - \Delta w_1 + q_1 w_1 = 0 \quad \text{in } Q, \\ \mathcal{T}_0 w_1 = \mathcal{T}_0 u_2, \end{cases} \tag{6.2}$$

Then, $u = w_1 - u_2$ solves

$$\begin{cases} \partial_t^2 - \Delta u + q_1 u = (q_2 - q_1) u_2 & \text{in } \Omega', \\ u = 0 & \text{on } \Sigma, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{on } \Omega \end{cases} \tag{6.3}$$

and since $(q_2 - q_1)u_2 \in L^2(Q)$, in view of Theorem A.2 in [BCY], we deduce that $u \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$ with $\partial_\nu u \in L^2(\Sigma)$. Moreover we have

$$\|u\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|u\|_{\mathcal{C}([0,T];H_0^1(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} \leqslant 2CM \|u_2\|_{L^2(Q)}.$$

Applying the Green formula in $x \in \Omega$ and integration by parts in $t \in (0,T)$, we find

$$\begin{aligned} \int_Q q u_2 u_1 dx dt &= \int_Q (\partial_t^2 - \Delta + q_1) u u_1 dx dt \\ &= \int_Q u(\partial_t^2 - \Delta + q_1) u_1 - \int_\Sigma \partial_\nu u u_1 d\sigma(x) dt \\ &\quad + \int_\Omega \partial_t u(T,x) u_1(T,x) dx - \int_\Omega u(T,x) \partial_t u_1(T,x) dx \\ &= -\int_\Sigma \partial_\nu u u_1 d\sigma(x) dt + \int_\Omega \partial_t u(T,x) u_1(T,x) dx - \int_\Omega u(T,x) \partial_t u_1(T,x) dx \\ &= -\int_{\Sigma_{+,\varepsilon,\omega}} \partial_\nu u u_1 d\sigma(x) dt - \int_{\Sigma_{-,\varepsilon,\omega}} \partial_\nu u u_1 d\sigma(x) dt \\ &\quad + \int_\Omega \partial_t u(T,x) u_1(T,x) dx - \int_\Omega u(T,x) \partial_t u_1(T,x) dx. \end{aligned} \tag{6.4}$$

In view of estimate (3.13), we have

$$\|w\|_{L^2(\Sigma)} \leqslant C \|w\|_{L^2(0,T;H^{\frac{1}{2}}(\Omega))} \leqslant C \|w\|_{H^1(Q)} \leqslant C$$

with C depending on Ω, T and M. Applying this estimate and the Cauchy Schwarz inequality, we obtain

$$\begin{aligned}\left|\int_{\Sigma_{\pm,\varepsilon,\omega}} \partial_\nu u u_1 d\sigma(x)dt\right| &\leqslant \int_{\Sigma_{\pm,\varepsilon,\omega}} \left|\partial_\nu u e^{-r\xi\cdot(t,x)}(1+w)\right| dtd\sigma(x) \\ &\leqslant C\left(\int_{\Sigma_{\pm,\varepsilon,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 d\sigma(x)dt\right)^{\frac{1}{2}}\end{aligned} \tag{6.5}$$

for some C depending only on Ω, T and M. In the same way, we have

$$\left\|w_{|t=T}\right\|_{L^2(\Omega)} \leqslant C\left\|w\right\|_{H^{\frac{1}{2}}(0,T;L^2(\Omega))} \leqslant C\left\|w\right\|_{H^1(Q)} \leqslant C$$

and we deduce

$$\left|\int_\Omega \partial_t u(T,x)u_1(T,x)dx\right| \leqslant C\left(\int_\Omega \left|e^{-r\xi\cdot(T,x)}\partial_t u(T,x)\right|^2 dx\right)^{\frac{1}{2}}. \tag{6.6}$$

Moreover, in views of Remark 3, the Carleman estimate (4.7) and the fact that $\partial\Omega_{+,\varepsilon,\omega} \subset \partial\Omega_{+,\omega}$ imply

$$\begin{aligned}&r\int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 d\sigma(x)dt + r\int_\Omega \left|e^{-r\xi\cdot(T,x)}\partial_t u(T,x)\right|^2 dx \\ &\leqslant \varepsilon^{-1}r\int_{\Sigma_{+,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 \omega\cdot\nu(x)d\sigma(x)dt + r\int_\Omega \left|e^{-r\xi\cdot(T,x)}\partial_t u(T,x)\right|^2 dx \\ &\leqslant \varepsilon^{-1}C\left(\int_Q \left|e^{-r\xi\cdot(t,x)}(\partial_t^2-\Delta+q_1)u\right|^2 dxdt + r\int_{\Sigma_{-,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt\right) \\ &\quad +\varepsilon^{-1}Cr^3\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx \\ &\leqslant \varepsilon^{-1}C\left(\int_Q \left|e^{-r\xi\cdot(t,x)}qu_2\right|^2 dxdt + r\int_{\Sigma_{-,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt\right) \\ &\quad +\varepsilon^{-1}Cr^3\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx \\ &\leqslant \varepsilon^{-1}C\left(\int_Q |q|\,(1+|z|)^2 dxdt + r\int_{\Sigma_{-,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt\right) \\ &\quad +\varepsilon^{-1}Cr^3\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx \\ &\leqslant \varepsilon^{-1}C\left(8M^2T\,|\Omega| + 2\left\|z\right\|^2_{L^2(Q)} + r\int_{\Sigma_{-,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt\right) \\ &\quad +\varepsilon^{-1}Cr^3\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx.\end{aligned}$$

Using estimate (5.3) and the fact that $\Sigma_{-,\omega} \subset \Sigma_{-,\varepsilon,\omega}$, $|\omega\cdot\nu(x)| \leqslant 1$, we obtain

$$\begin{aligned}&r\int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 d\sigma(x)dt + r\int_\Omega \left|e^{-r\xi\cdot(T,x)}\partial_t u(T,x)\right|^2 dx \\ &\leqslant C\left(1 + r\int_{\Sigma_{-,\varepsilon,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 |\omega\cdot\nu(x)|\, d\sigma(x)dt + r^3\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx\right) \\ &\leqslant C\left(1 + r\int_{\Sigma_{-,\varepsilon,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 d\sigma(x)dt + r^3\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx\right),\end{aligned}$$

where C depends only on Ω, T, ε_0, ω_0 and M. Therefore, we have

$$\begin{aligned}&\int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 d\sigma(x)dt + \int_\Omega \left|e^{-r\xi\cdot(T,x)}\partial_t u(T,x)\right|^2 dx \\ &\leqslant C\left(r^{-1} + \int_{\Sigma_{-,\varepsilon,\omega}} \left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 d\sigma(x)dt + r^2\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx\right).\end{aligned}$$

Combining this with (6.4), we obtain

$$\left|\int_Q qu_1u_2dxdt\right|^2 \leqslant C\left(r^{-1}+\int_{\Sigma_{-,\varepsilon,\omega}}\left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 d\sigma(x)dt+r^2\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx\right) \tag{6.7}$$

with C depending only on Ω, T, ε_0, ω_0 and M. On the other hand, (3.12) implies

$$\int_Q qu_1u_2dxdt=\int_{\mathbb{R}^{1+n}} q(t,x)e^{-i(l,y)\cdot(t,x)}dxdt+\int_Q Zdxdt$$

with

$$Z(t,x)=ze^{(\zeta_1+r\xi)\cdot(t,x)}(1+w)+we^{-i(l,y)\cdot(t,x)},\quad (t,x)\in Q.$$

Using the fact that $(\zeta_1+r\xi)\cdot(t,x)\in i\mathbb{R}$ we deduce that

$$|Z|\leqslant|z|\,(1+|w|)+|w|\,.$$

Then, in view of (3.13) and (5.3), an application of the Cauchy-Schwarz inequality yields

$$\left|\int_Q Zdx\right|\leqslant Cr^{-\frac{1}{2}}$$

with C depending on Ω, T, ε_0, ω_0 and M. Combining this estimate with (6.7), we obtain

$$\begin{aligned}&\left|\textstyle\int_{\mathbb{R}^{1+n}} q(t,x)e^{-i(l,y)\cdot(t,x)}dxdt\right|^2\\&\leqslant C\left(r^{-1}+\textstyle\int_{\Sigma_{-,\varepsilon,\omega}}\left|e^{-r\xi\cdot(t,x)}\partial_\nu u\right|^2 d\sigma(x)dt+r^2\int_\Omega e^{-2r\xi\cdot(T,x)}\left|\omega\cdot\nabla_x u_{|t=T}\right|^2 dx\right)\\&\leqslant C\left[\tfrac{1}{r}+e^{dr}\left(\|\partial_\nu u\|^2_{L^2(\Sigma_{-,\varepsilon,\omega})}+\left\|u_{|t=T}\right\|^2_{H^1(\Omega)}\right)\right]\end{aligned} \tag{6.8}$$

with $C,d>0$ depending only on Ω, T, ε_0, ω_0 and M. Using the fact that $\Sigma_{-,\varepsilon,\omega}\subset\Sigma_{-,\varepsilon_0,\omega_0}$ and

$$\mathrm{supp}g\subset\Sigma_{-,\varepsilon,-\omega}\subset\Sigma_{-,\varepsilon_0,-\omega_0},\quad v_0=0$$

with $(g,v_0,v_1)=\mathcal{T}_0u_2$, we deduce that

$$(\partial_\nu u_{|\Sigma_{-,\varepsilon_0,\omega}},u_{|t=T})=(B^*_{q_1}-B^*_{q_2})\mathcal{T}_0u_2.$$

Then, we have

$$\begin{aligned}\|\partial_\nu u\|^2_{L^2(\Sigma_{-,\varepsilon,\omega})}+\left\|u_{|t=T}\right\|^2_{H^1(\Omega)}&\leqslant\|\partial_\nu u\|^2_{L^2(\Sigma_{-,\varepsilon_0,\omega_0})}+\left\|u_{|t=T}\right\|^2_{H^1(\Omega)}\\&\leqslant\left\|(B^*_{q_1}-B^*_{q_2})\mathcal{T}_0u_2\right\|^2_{L^2(\Sigma_{-,\varepsilon_0,\omega_0})\times H^1(\Omega)}\end{aligned}$$

and from (5.4) we deduce

$$\|\partial_\nu u\|^2_{L^2(\Sigma_{-,\varepsilon,\omega})}+\left\|u_{|t=T}\right\|^2_{H^1(\Omega)}\leqslant\left\|B^*_{q_1}-B^*_{q_2}\right\|^2\|\mathcal{T}_0u_2\|^2_{\mathcal{H}(\partial Q)}\leqslant C\left\|B^*_{q_1}-B^*_{q_2}\right\|^2e^{2dr}$$

with C and d depending only on Ω, T, ε_0, ω_0 and M. Combining this estimate with (6.8), we obtain (6.1). □

From now on, for all $s>0$, we denote by D_s the set $D_s=\{z\in\mathbb{R}^{1+n}:\ |z|<s\}$. Let us recall the following result, which follows from Theorem 3 in [1] (see also [32]), on the continuous dependence in the analytic continuation problem.

Proposition 7. *Let $\rho>0$ and assume that $f:\ D_{2\rho}\subset\mathbb{R}^{1+n}\to\mathbb{R}$ is a real analytic function satisfying*

$$\left\|\partial^\beta f\right\|_{L^\infty(D_{2\rho})}\leqslant\frac{N\beta!}{(\rho l)^{|\beta|}},\quad \beta\in\mathbb{N}^{1+n}$$

for some $N > 0$ and $0 < l \leqslant 1$. Further let $E \subset D_{\frac{\rho}{2}}$ be a measurable set with strictly positive measure. Then,

$$\|f\|_{L^\infty(D_\rho)} \leqslant C(N)^{(1-b)} \left(\|f\|_{L^\infty(E)}\right)^b,$$

where $b \in (0,1)$, $C > 0$ depend on l, $|E|$ and ρ.

Armed with Lemma 1, we will use Proposition 7 to complete the proof of of Theorem 1.

Proof of Theorem 1. Without lost of generality, we can assume that $0 \in \Omega$. Consider the set

$$G = \{\xi \in \mathbb{S}^n : \ \xi \text{ of the form (3.6) with } 1 < c < 2, \ \omega \in \{z \in \mathbb{S}^{n-1} : \ |z - \omega_0| < \varepsilon\}\}.$$

One can easily check that G is an open set of $\mathbb{S}^n$. Introduce the set

$$E_1 = \{\eta \in \mathbb{R}^{1+n} : \text{ there exists } \xi = (\xi_0, \xi') \in G \text{ such that } \eta \cdot (-\xi_0, \xi') = 0\}$$

and note that estimate (6.1) holds for all $(l, y) \in E_1$, $r \geqslant r_4$, such that condition (3.9) is fulfilled and $|(l, y)| \leqslant Br$ for some $B > 0$. Let us observe that in view of (3.6) and (3.9), (3.9) will be fulfilled if

$$|(l, y)| \leqslant 2 \inf_{1 \leqslant c \leqslant 2} \left(\left(1 - \frac{1}{(1+c)^2}\right) \frac{(1+c)^2}{1 + (1+c)^2} \right)^{\frac{1}{2}} r = Br$$

with

$$B = 2 \min_{1 \leqslant c \leqslant 2} \left(\frac{(1+c)^2 - 1}{1 + (1+c)^2} \right)^{\frac{1}{2}} > 0.$$

From now on we fix this value of B. Next we set for fixed $R > 0$, which will be made precise later, and $(l, y) \in \mathbb{R}^{1+n}$,

$$H(l, y) = \mathcal{F}(q)(R(l, y)) = (2\pi)^{-\frac{(1+n)}{2}} \int_{\mathbb{R}^{1+n}} q(t, x) e^{-iR(l,y)\cdot(t,x)} dx dt.$$

Since $\text{supp} q \subset \overline{Q}$ and $0 \in \Omega$, H is analytic and

$$\left|D^\beta H(l, y)\right| \leqslant C \frac{\|q\|_{L^1(Q)} R^{|\beta|}}{([T\text{Diam}(\Omega)]^{-1})^{|\beta|}} \leqslant C \|q\|_{L^2(\omega)} \frac{R^{|\beta|}}{\beta!([T\text{Diam}(\Omega)]^{-1})^{|\beta|}} \beta!$$

with C depending on T and Ω. Moreover, we have

$$\|q\|_{L^2(Q)} \leqslant 2M\sqrt{T|\Omega|}$$

and one can check that

$$\frac{R^{|\beta|}}{\beta!} \leqslant e^{2R}.$$

Applying these estimates, we obtain

$$\left|D^\beta H(l, y)\right| \leqslant C \frac{e^{2R}\beta!}{([T\text{Diam}(\Omega)]^{-1})^{|\beta|}} \tag{6.9}$$

with C depending on M, Ω and T. Set $\rho = [T\text{Diam}(\Omega)]^{-1} + 1$, $E = E_1 \cap \{x \in \mathbb{R}^{1+n} : \ |x| < \frac{\rho}{2}\}$ with $N = Ce^{2R}$ and $l = \frac{[T\text{Diam}(\Omega)]^{-1}}{\rho}$. In view of (6.9), we have

$$\left\|D^\beta H(l, y)\right\|_{L^\infty(D_{2\rho})} \leqslant C \frac{e^{2R}\beta!}{([T\text{Diam}(\Omega)]^{-1})^{|\beta|}} = \frac{N\beta!}{(\rho l)^{|\beta|}}.$$

Using the fact that G is an open set of $\mathbb{S}^n$ and $tE_1 = E_1$ for any $t > 0$, one can check that $|E| > 0$. Then, since $E \subset D_{\frac{\rho}{2}}$, $0 < l < 1$ and $\rho > 1$, applying Proposition 7 to H we obtain

$$|\mathcal{F}(q)(R(l, y))| = |H(l, y)| \leqslant \|H\|_{L^\infty(D_\rho)} \leqslant Ce^{2R(1-b)} \left(\|H\|_{L^\infty(E)}\right)^b, \quad |(l, y)| < 1,$$

where $C>0$ and $0<b<1$ depend only of Ω, T, M, ω_0 and ε_0. But, for $R<Br$ estimate (6.1) implies that

$$|H(l,y)|^2=|\mathcal{F}(q)(R(l,y))|^2\leqslant C\left(\frac{1}{r}+e^{dr}\left\|B_{q_1}^*-B_{q_2}^*\right\|^2\right),\ (l,y)\in E$$

and we deduce

$$|\mathcal{F}(q)(l,y)|^2\leqslant Ce^{4(1-b)R}\left(\frac{1}{r}+e^{dr}\left\|B_{q_1}^*-B_{q_2}^*\right\|^2\right)^b,\ |(l,y)|<R. \tag{6.10}$$

Note that

$$\|q\|_{H^{-1}(\mathbb{R}^{1+n})}^{\frac{2}{b}}\leqslant C\left(\int_{\mathbb{R}^{1+n}}(1+|(l,y)|^2)^{-1}|\mathcal{F}(q)(l,y)|^2dydl\right)^{\frac{1}{b}}. \tag{6.11}$$

We shall make precise below,

$$\int_{D_R}(1+|(l,y)|^2)^{-1}|\mathcal{F}(q)(l,y)|^2dydl$$

and

$$\int_{\mathbb{R}^{1+n}\setminus D_R}(1+|(l,y)|^2)^{-1}|\mathcal{F}(q)(l,y)|^2dldy$$

separately. We start by examining the last integral. The Parseval-Plancherel theorem implies

$$\begin{aligned}
&\int_{\mathbb{R}^3\setminus D_R}(1+|(l,y)|^2)^{-1}|\mathcal{F}(q)(l,y)|^2dydl\\
\leqslant\ &\frac{1}{R^2}\int_{\mathbb{R}^{1+n}\setminus D_R}|\mathcal{F}(q)(l,y)|^2dydl\\
\leqslant\ &\frac{1}{R^2}\int_{\mathbb{R}^{1+n}}|\mathcal{F}(q)(l,y)|^2dydl=\frac{1}{R^2}\int_{\mathbb{R}^{1+n}}|q(t,x)|^2dtdx\\
\leqslant\ &\frac{8T\left|\Omega\right|M^2}{R^2}.
\end{aligned}$$

We end up getting that

$$\int_{\mathbb{R}^{1+n}\setminus D_R}(1+|(l,y)|^2)^{-1}|\mathcal{F}(q)(l,y)|^2dydl\leqslant\frac{C}{R^2}. \tag{6.12}$$

Further, in light of (6.10), we get

$$\int_{D_R}|\mathcal{F}(q)(l,y)|^2dydl\leqslant CR^{1+n}e^{4(1-b)R}\left(\frac{1}{r}+e^{dr}\left\|B_{q_1}^*-B_{q_2}^*\right\|^2\right)^b,\ r>r_4, \tag{6.13}$$

upon eventually substituting C for some suitable algebraic expression of C.

Last, putting (6.12)–(6.13) together we find out that

$$\begin{aligned}
\|q\|_{H^{-1}(\mathbb{R}^{1+n})}^{\frac{2}{b}}&\leqslant C\left(\frac{1}{R^2}+R^{1+n}e^{4(1-b)R}\left(\frac{1}{r}+e^{dr}\left\|B_{q_1}^*-B_{q_2}^*\right\|^2\right)^b\right)^{\frac{1}{b}}\\
&\leqslant C\left(R^{-\frac{2}{b}}+\frac{R^{\frac{n+1}{b}}e^{4(\frac{1-b}{b})R}}{r}+R^{\frac{n+1}{b}}e^{4(\frac{1-b}{b})R}e^{dr}\left\|B_{q_1}^*-B_{q_2}^*\right\|^2\right),
\end{aligned} \tag{6.14}$$

for $1<R<Br$ and $r>r_4$ where the constant $C>0$ depends only on Ω, T, ε_0, ω_0 and M. Here we use the fact that $x\mapsto x^{\frac{1}{b}}$ is convex on $\mathbb{R}^+$ since $b\in(0,1)$. Now let $R_1>1$ be such that

$$R<BR^{\frac{n+3}{b}}e^{4R(\frac{1-b}{b})},\quad R>R_1.$$

Then, set $r_5=\max(R_1^{\frac{n+3}{b}}e^{4R_1(\frac{1-b}{b})},r_4)$. Choosing $r=R^{\frac{n+3}{b}}e^{4R(\frac{1-b}{b})}$, for $R>R_2$ with $r_5=R_2^{\frac{n+3}{b}}e^{4R_2(\frac{1-b}{b})}$, such that $\frac{R^{\frac{n+1}{b}}e^{4R(\frac{1-b}{b})}}{r}=R^{-\frac{2}{b}}$ and $R<Br$, we proceed

$$\|q\|^{\frac{2}{b}}_{H^{-1}(\mathbb{R}^{1+n})} \leqslant C\left(R^{-\frac{2}{b}} + R^{\frac{n+1}{b}} e^{4(\frac{1-b}{b})R} \exp\left(dR^{\frac{n+3}{b}} e^{4R(\frac{1-b}{b})}\right) \left\|\Lambda^*_{V_2,\theta} - \Lambda^*_{V_1,\theta}\right\|^2\right). \tag{6.15}$$

It is not hard to check that

$$R^{\frac{n+1}{b}} e^{4R(\frac{1-b}{b})} \exp\left(dR^{\frac{n+3}{b}} e^{4R(\frac{1-b}{b})}\right) \leqslant \exp\left(e^{[d+\frac{n+1}{b}+4(\frac{1-b}{b})+\frac{n+3}{b}+4(\frac{1-b}{b})]R}\right), \quad R > R_2.$$

Setting $A = \frac{2n+12-8b+db}{b}$ (6.15) leads to

$$\|q\|^{\frac{2}{b}}_{H^{-1}(\mathbb{R}^{1+n})} \leqslant C\left(R^{-\frac{2}{b}} + e^{e^{AR}} \left\|B^*_{q_1} - B^*_{q_2}\right\|^2\right), \quad R > R_2. \tag{6.16}$$

Set $\gamma = \left\|B^*_{q_1} - B^*_{q_2}\right\|$ and $\gamma^* = e^{-e^{AR_2}}$. For $\gamma \geqslant \gamma^*$ we have

$$\|q\|_{H^{-1}(Q)} \leqslant C\|q\|_{L^\infty(Q)} \leqslant \frac{2CM}{\gamma^*}\gamma. \tag{6.17}$$

For $0 < \gamma < \gamma^*$, by taking $R = R_3 = \frac{1}{A}\ln(|\ln\gamma|)$ in (6.16), which is permitted since $R_3 > R_2$, we find out that

$$\|q\|_{H^{-1}(Q)} \leqslant \|q\|_{H^{-1}(\mathbb{R}^{1+n})} \leqslant C\ln(|\ln\gamma|)^{-1}\left(\ln(|\ln\gamma|)^{\frac{2}{b}}\gamma + A^{-\frac{2}{b}}\right)^{\frac{b}{2}}.$$

Now, since $\sup_{0<\gamma\leqslant\gamma_*}\left(\ln(|\ln\gamma|)^{\frac{2}{b}}\gamma + A^{-\frac{2}{b}}\right)^{\frac{b}{2}}$ is just another constant depending only on Ω, T, ε_0, ω_0 and M, we obtain

$$\|q\|_{H^{-1}(Q)} \leqslant C\ln(|\ln\gamma|)^{-1}, \quad 0 < \gamma < \gamma^*. \tag{6.18}$$

By interpolation we find

$$\|q\|_{L^2(Q)} \leqslant C\|q\|^{\frac{1}{2}}_{H^1(Q)}\|q\|^{\frac{1}{2}}_{H^{-1}(Q)} \leqslant C(M\sqrt{|\Omega|\,T})^{\frac{1}{2}}\|q\|^{\frac{1}{2}}_{H^{-1}(Q)}$$

with C depending only on Ω and T. Combining this estimate with (6.18), we deduce (1.3). □

Appendix

In this appendix we prove that the space $\mathcal{C}^\infty(\overline{Q})$ is dense in $H_\Box(Q)$ in some appropriate sense and we show that the maps $\mathcal{T}_0$ and $\mathcal{T}_1$ can be extended continuously on these spaces. The results of this section are well known, nevertheless we prove them for sake of completeness.

Density result in $H_\Box(Q)$. Let us first recall the definition of $K_\Box(Q)$:

$$K_\Box(Q) = \{u \in H^{-1}(0,T;L^2(\Omega)) :\ \Box u = (\partial_t^2 - \Delta)u \in L^2(Q)\}$$

with the norm

$$\|u\|^2_{K_\Box(Q)} = \|u\|^2_{H^{-1}(0,T;L^2(\Omega))} + \|\Box u\|^2_{L^2(Q)}.$$

The goal of this subsection is to prove the following.

Theorem 7. *$H_\Box(Q)$ embedded continuously into the closure of $\mathcal{C}^\infty(\overline{Q})$ wrt $K_\Box(Q)$.*

Proof. Let N be a continuous linear form on $K_\Box(Q)$ satisfying

$$N(f) = 0, \quad f \in \mathcal{C}^\infty(\overline{Q}). \tag{6.19}$$

In order to show the required density result we will prove that this condition implies that $N_{|H_\Box(Q)} = 0$.

By considering the application $u \mapsto (u, \Box u)$ we can identify $K_\Box(Q)$ to a subspace of $H^{-1}(0,T;L^2(\Omega)) \times L^2(Q)$. Then applying the Hahn Banach theorem we deduce that N can be extended to a continuous linear form on $H^{-1}(0,T;L^2(\Omega)) \times L^2(Q)$. Therefore, there exist $h_1 \in H^1_0(0,T;L^2(\Omega))$, $h_2 \in L^2(Q)$ such that

$$N(u) = \langle u, h_1\rangle_{H^{-1}(0,T;L^2(\Omega)),H^1_0(0,T;L^2(\Omega))} + \langle \Box u, h_2\rangle_{L^2(Q)}, \quad u \in K_\Box(Q).$$

Now let $\mathcal{O}\subset\mathbb{R}^n$ be a bounded $\mathcal{C}^\infty$ domain such that $\overline{\Omega}\subset\mathcal{O}$ and fix $Q_\varepsilon=(-\varepsilon,T+\varepsilon)\times\mathcal{O}$ with $\varepsilon>0$. Let $\tilde{h}_j$ be the extension of h_j on $\mathbb{R}^{1+n}$ by 0 outside of Q for $j=1,2$. In view of (6.19) we have

$$\left\langle f,\tilde{h}_1\right\rangle_{L^2(\mathcal{O})}+\left\langle(\partial_t^2-\Delta)f,\tilde{h}_2\right\rangle_{L^2(\mathcal{O})}=N(f_{|Q})=0,\quad f\in\mathcal{C}_0^\infty(Q_\varepsilon).$$

Thus, in the sense of distribution we have

$$\Box\tilde{h}_2=-\tilde{h}_1,\quad \text{on } Q_\varepsilon.$$

Moreover, since $\tilde{h}_2=0$ on $\mathbb{R}^{1+n}\setminus\overline{Q}\supset\partial Q_\varepsilon$, we deduce that $\tilde{h}_2$ solves

$$\left\{\begin{array}{rcll}\partial_t^2\tilde{h}_2-\Delta\tilde{h}_2 &=& -\tilde{h}_1, & \text{in } Q_\varepsilon,\\ \tilde{h}_2(-\varepsilon,x)=\partial_t\tilde{h}_2(-\varepsilon,x) &=& 0, & x\in\mathcal{O}\\ \tilde{h}_2(t,x) &=& 0, & (t,x)\in(-\varepsilon,T+\varepsilon)\times\partial\mathcal{O}.\end{array}\right.$$

But, since $h_1\in H_0^1(0,T;L^2(\Omega))$, we have $\tilde{h}_1\in H_0^1(-\varepsilon,T+\varepsilon;L^2(\mathcal{O}))$ and we deduce from Theorem 2.1 in Chapter 5 of [23] that this IBVP admits a unique solution lying in $H^2(Q_\varepsilon)$. Therefore, $\tilde{h}_2\in H^2(Q_\varepsilon)$. Combining this with the fact that $\tilde{h}_2=0$ on $Q_\varepsilon\setminus Q$, we deduce that $h_2\in H_0^2(Q)$, with $H_0^2(Q)$ the closure of $\mathcal{C}_0^\infty(Q)$ in $H^2(Q)$, and that $\Box h_2=-h_1$ on Q. Thus, for every $u\in H_\Box(Q)$ we have

$$\langle\Box u,h_2\rangle_{L^2(Q)}=\langle\Box u,h_2\rangle_{H^{-2}(Q),H_0^2(Q)}=\langle u,\Box h_2\rangle_{L^2(Q)}=-\langle u,h_1\rangle_{L^2(Q)}.$$

Here we use the fact that $H_\Box(Q)\subset L^2(Q)$. Then it follows that

$$N(u)=\langle u,h_1\rangle_{L^2(Q)}-\langle u,h_1\rangle_{L^2(Q)}=0,\quad u\in H_\Box(Q).$$

From this last result we deduce that $H_\Box(Q)$ is contained into the closure of $\mathcal{C}^\infty(\overline{Q})$ wrt $K_\Box(Q)$. Combining this with the fact that $H_\Box(Q)$ embedded continuously into $K_\Box(Q)$ we deduce the required result. □

Trace operator in $H_\Box(Q)$. In this subsection we extend the trace maps $\mathcal{T}_0$ and $\mathcal{T}_1$ into $H_\Box(Q)$ by duality in the following way.

Proposition 8. *The maps*

$$\mathcal{T}_0w=(\mathcal{T}_0^1w,\mathcal{T}_0^2w,\mathcal{T}_0^3w)=(w_{|\Sigma},w_{|t=0},\partial_tw_{|t=0}),\quad w\in\mathcal{C}^\infty(\overline{Q}),$$

$$\mathcal{T}_1w=(\mathcal{T}_1^1w,\mathcal{T}_1^2w,\mathcal{T}_1^3w)=(\partial_\nu w_{|\Sigma},w_{|t=T},\partial_tw_{|t=T}),\quad w\in\mathcal{C}^\infty(\overline{Q}),$$

can be extended continuously to $\mathcal{T}_0:H_\Box(Q)\to H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega))\times H^{-2}(\Omega)\times H^{-4}(\Omega)$, $\mathcal{T}_1:H_\Box(\Omega)\to H^{-3}(0,T;H^{-\frac{3}{2}}(\partial\Omega))\times H^{-2}(\Omega)\times H^{-4}(\Omega)$.

Proof. It is well known that the trace maps

$$u\mapsto(u_{|\partial\Omega},\partial_\nu u_{|\partial\Omega})$$

can be extended continuously to a bounded operator from $H^2(\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ which is onto. Therefore, there exists a bounded operator $R:H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)\to H^2(\Omega)$ such that

$$R[h_1,h_2]_{|\partial\Omega}=h_1,\quad \partial_\nu R[h_1,h_2]_{|\partial\Omega}=h_2,\quad (h_1,h_2)\in H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega).$$

Fix $g\in H_0^3(0,T;H^{\frac{1}{2}}(\partial\Omega))$ and choose $G(t,.)=R(0,g(t,.))$. One can check that $G\in H_0^3(0,T;H^2(\Omega))$ and

$$\|G\|_{H^3(0,T;H^2(\Omega))}\leqslant\|R\|\,\|g\|_{H^3(0,T;H^{\frac{1}{2}}(\partial\Omega))}.\tag{6.20}$$

Applying twice the Green formula we obtain

$$\int_\Sigma vgd\sigma(x)dt=\int_Q\Box vGdx-\int_Qv\Box Gdx,\quad v\in\mathcal{C}^\infty(\overline{Q}).$$

But $\Box G\in H_0^1(0,T;H^2(\Omega)$, and we have

$$\left\langle\mathcal{T}_0^1v,g\right\rangle_{H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega)),H_0^3(0,T;H^{\frac{1}{2}}(\partial\Omega))}=\langle\Box v,G\rangle_{L^2(Q)}-\langle v,\Box G\rangle_{H^{-1}(0,T;L^2(\Omega)),H_0^1(0,T;L^2(\Omega))}.$$

Then, using (6.20) and the Cauchy Schwarz inequality, for all $v \in \mathcal{C}^\infty(\overline{Q})$, we obtain

$$\left|\left\langle \mathcal{T}_0^1 v, g\right\rangle\right| \leqslant \|\Box v\|_{L^2(Q)} \|G\|_{L^2(Q)} + \|v\|_{H^{-1}(0,T;L^2(\Omega))} \|\Box G\|_{H_0^1(0,T;L^2(\Omega))}$$

$$\leqslant C \|v\|_{K_\Box(Q)} \|g\|_{H^3(0,T;H^{\frac{1}{2}}(\partial\Omega))}$$

which, combined with the density result of Proposition 7, implies that $\mathcal{T}_0^1 : \ v \mapsto v_{|\Sigma}$ extend continuously to a bounded operator from $H_\Box(Q)$ to $H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega))$. In a same way we prove that

$$\mathcal{T}_1^1 v = \partial_\nu v_{|\Sigma}, \quad v \in \mathcal{C}^\infty(\overline{Q})$$

extend continuously to a bounded operator from $H_\Box(Q)$ to $H^{-3}(0,T;H^{-\frac{3}{2}}(\partial\Omega))$.

Now let us consider the operators $\mathcal{T}_i^j$, $0 \leqslant i \leqslant 1$, $2 \leqslant j \leqslant 3$. We consider first

$$\mathcal{T}_0^2 : v \longmapsto v_{|t=0}, \quad v \in \mathcal{C}^\infty(\overline{Q}).$$

Let $h \in H_0^2(\Omega)$ and fix $H(t,x) = t\psi(t)h(x)$ with $\psi \in \mathcal{C}_0^\infty(-T, \frac{T}{2})$ satisfying $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ on $[-\frac{T}{3}, \frac{T}{3}]$. Then, using the fact that $\psi = 1$ on a neighborhood of $t = 0$, we deduce that

$$H_{|\Sigma} = \partial_\nu H_{|\Sigma} = H_{|t=0} = \Box H_{|t=0} = \Box H_{|t=T} = 0, \quad \partial_t H_{|t=0} = h.$$

Therefore, $\Box H \in H_0^1(0,T;L^2(\Omega))$ and repeating the above arguments, for all $v \in \mathcal{C}^\infty(\overline{Q})$, we obtain the representation

$$\left\langle \mathcal{T}_0^2 v, h\right\rangle_{H^{-2}(\Omega),H_0^2(\Omega)} = \left\langle v, \Box H\right\rangle_{H^{-1}(0,T;L^2(\Omega)),H_0^1(0,T;L^2(\Omega))} - \left\langle H, \Box v\right\rangle_{L^2(Q)}.$$

Then, we prove by density that $\mathcal{T}_0^2$ extend continuously to $\mathcal{T}_0^2 : H_\Box(Q) \longrightarrow H^{-2}(\Omega)$.

Now, let us consider

$$\mathcal{T}_0^3 : v \longmapsto \partial_t v_{|t=0}, \quad v \in \mathcal{C}^\infty(\overline{Q}).$$

Let $\varphi \in H_0^4(\Omega)$ and fix

$$\Phi(t,x) = \psi(t)\varphi(x) + \frac{\psi(t)t^2\Delta\varphi(x)}{2}.$$

Then, Φ satisfies

$$\Phi_{|\Sigma} = \partial_\nu \Phi_{|\Sigma} = \partial_t \Phi_{|t=0} = 0, \quad \Phi_{|t=0} = \varphi.$$

Moreover, we have $\Box\Phi \in H^1(0,T;L^2(\Omega))$ with

$$(\partial_t^2 - \Delta)\Phi_{|t=0} = -\Delta\varphi + \Delta\varphi = 0, \quad (\partial_t^2 - \Delta)\Phi_{|t=T} = 0$$

which implies that $\Box\Phi \in H_0^1(0,T;L^2(\Omega))$. Therefore, repeating the above arguments we obtain the representation

$$\left\langle \mathcal{T}_0^3 v, \varphi\right\rangle_{H^{-4}(\Omega),H_0^4(\Omega)} = \left\langle \Box v, \Phi\right\rangle_{L^2(Q)} - \left\langle v, \Box\Phi\right\rangle_{H^{-1}(0,T;L^2(\Omega)),H_0^1(0,T;L^2(\Omega))}$$

and we deduce that $\mathcal{T}_0^3$ extends continuously to $\mathcal{T}_0^3 : H_\Box(Q) \longrightarrow H^{-4}(\Omega)$. In a same way, one can check that

$$\mathcal{T}_1^2 : v \longmapsto v_{|t=T}, \ \mathcal{T}_1^3 : v \longmapsto \partial_t v_{|t=T} \quad v \in \mathcal{C}^\infty(\overline{Q}).$$

extend continuously to $\mathcal{T}_1^2 : H_\Box(Q) \longrightarrow H^{-2}(\Omega)$ and $\mathcal{T}_1^3 : H_\Box(Q) \longrightarrow H^{-4}(\Omega)$. □

References

[1] J. Apraiz, L. Escauriaza, *Null-control and measurable sets*, ESAIM: Control, Optimisation and Calculus of Variations, **19** (2013), 239-254.

[2] M. Bellassoued, *Uniqueness and stability in determining the speed of propagation of second-order hyperbolic equation with variable coefficients*, Appl. Anal., **83** (2004), 983-1014.

[3] M. Bellassoued, M. Choulli and M. Yamamoto, *Stability estimate for an inverse wave equation and a multidimensional Borg-Levinson theorem*, J. Diff. Equat., **247**(2) (2009), 465-494.

[4] A.L. Bukhgeim and M.V. Klibanov, *Global uniqueness of class of multidimensional inverse problems*, Sov. Math. Dokl., **24** (1981), 244-247.

[5] A. L. Bukhgeim and G. Uhlmann, *Recovering a potential from partial Cauchy data*, Commun. Partial Diff. Eqns., **27** (2002), no 3-4, 653-668.

[6] M. Choulli, *Une introduction aux problèmes inverses elliptiques et paraboliques*, Mathématiques et Applications, Vol. 65, Springer-Verlag, Berlin, 2009.

[7] M. Choulli and Y. Kian, *Stability of the determination of a time-dependent coefficient in parabolic equations*, MCRF, **3** (2) (2013), 143-160.

[8] M. Choulli, Y.Kian and E. Soccorsi, *Determining the time dependent external potential from the DN map in a periodic quantum waveguide*, arXiv:1306.6601.

[9] G. Eskin, *A new approach to hyperbolic inverse problems*, Inverse Problems, **22** no. 3 (2006), 815-831.

[10] G. Eskin, *Inverse hyperbolic problems with time-dependent coefficients*, Commun. Partial Diff. Eqns., **32** (11) (2007), 1737-1758.

[11] P. Gaitan and Y. Kian, *A stability result for a time-dependent potential in a cylindrical domain*, Inverse Problems, **29** (6) (2013), 065006.

[12] H. Heck and J-N. Wang, *Stability estimate for the inverse boundary value problem by partial Cauchy data*, Inverse Problems, **22** (2006), 1787-1797.

[13] L. Hörmander, " The Analysis of linear partial differential operators", Vol I, Springer, (1983).

[14] L. Hörmander, "The Analysis of linear partial differential operators", Vol II, Springer, (1983).

[15] O.Yu. Imanuvilov and M. Yamamoto, *Global uniqueness and stability in determining coefficients of wave equations*, Commun. Partial Diff. Eqns, **26** (2001), 1409-1425.

[16] V. Isakov, *Completness of products of solutions and some inverse problems for PDE*, J. Diff. Equat., **92** (1991), 305-316.

[17] V. Isakov and Z. Sun, *Stability estimates for hyperbolic inverse problems with local boundary data*, Inverse Problems, **8** (1992), 193-206.

[18] C.E. Kenig, J. Sjöstrand, G. Uhlmann, *The Calderon problem with partial data*, Ann. of Math., **165** (2007), 567-591.

[19] M.V. Klibanov, *Inverse problems and Carleman estimates*, Inverse Problems, **8** (1992), 575-596.

[20] Y. Kian, *Stability of the determination of a coefficient for the wave equation in an infinite wave guide*, arXiv:1305.4888.

[21] I. Lasiecka, J.-L. Lions and R. Triggiani, , *Non homogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pures Appl., **65** (1986), 149-192.

[22] J-L. Lions and E. Magenes, "Problèmes aux limites non homogènes et applications", Vol. I, Dunod, Paris 1968.

[23] J-L. Lions and E. Magenes, "Problèmes aux limites non homogènes et applications", Vol. II, Dunod, Paris 1968.

[24] A. Nachman and B. Street, *Reconstruction in the Calderón problem with partial data*, Commun. Partial Diff. Eqns., **35** (2010), 375-390.

[25] Rakesh and A. G. Ramm, *Property C and an Inverse Problem for a Hyperbolic Equation*, J. Math. Anal. Appl., **156** (1991), 209-219.

[26] Rakesh and W. Symes, *Uniqueness for an inverse problem for the wave equation*, Commun. Partial Diff. Eqns., **13** (1) (1988), 87-96.

[27] A. G. Ramm and J. Sjöstrand, *An inverse problem of the wave equation*, Math. Z., **206** (1991), 119-130.

[28] R. Salazar, *Determination of time-dependent coefficients for a hyperbolic inverse problem*, Inverse Problems, **29** (9) (2013), 095015.

[29] P. Stefanov and G. Uhlmann, *Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media*, J. Funct. Anal., **154** (1998), 330-358.

[30] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), 153-169.

[31] D. Tataru, *Unique continuation for solutions to PDE*, Commun. Partial Diff. Eqns., **20** (1995), 855-84.

[32] S. Vessella, *A continuous dependence result in the analytic continuation problem*, Forum math., **11** (1999), no. 6, 695-703.