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Michèle LODAY-RICHAUD

**DIVERGENT SERIES AND
DIFFERENTIAL EQUATIONS**

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DIVERGENT SERIES AND DIFFERENTIAL EQUATIONS

Michèle LODAY-RICHAUD

Abstract. — The aim of these notes is to develop the various known approaches to the summability of a class of series that contains all divergent series solutions of ordinary differential equations in the complex field. We split the study into two parts: the first and easiest one deals with the case when the divergence depends only on one parameter, the level k also said critical time, and is called *k-summability*; the second one provides generalizations to the case when the divergence depends on several (but finitely many) levels and is called *multi-summability*. We prove the coherence of the definitions and their equivalences and we provide some applications.

A key role in most of these theories is played by Gevrey asymptotics. The notes begin with a presentation of these asymptotics and their main properties. To help readers that are not familiar with these concepts we provide a survey of sheaf theory and cohomology of sheaves. We also state the main properties of linear ordinary differential equations connected with the subject we are dealing with, including a sketch algorithm to compute levels and various formal invariants of linear differential equations as well as a chapter on irregularity and index theorems. A chapter is devoted to tangent-to-identity germs of diffeomorphisms in $\mathbb{C}, 0$ as an application of the cohomological point of view of summability.

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CHAPTER 1

INTRODUCTION

Divergent series may diverge in many various ways. When a divergent series issues from a natural problem it must satisfy specific constraints restricting thus the range of possibilities. What we mean, here, by natural problem is a problem formulated in terms of a particular type of equations such as differential equations, ordinary or partial, linear or non-linear, difference equations, q -difference equations and so on . . .

Much has been done in the last decades towards the understanding of the divergence of natural series, their classification and how they can be related to analytic solutions of the natural problem. The question of “summing” divergent series dates back long ago. Famous are the works of Euler and later of Borel, Poincaré, Birkhoff, Hardy and their school until the 1920’s. After a long period of inactivity, the question knew exploding developments in the 1970’s and 1980’s with the introduction by Y. Sibuya and B. Malgrange of the cohomological point of view followed by works of J.-P. Ramis, J. Écalle and many others.

In these lecture notes, we focus on the best known class of divergent series, a class motivated by the study of solutions of ordinary linear differential equations with complex meromorphic coefficients at 0 (for short, differential equations) to which they all belong. It is well-known (Cauchy-Lipschitz Theorem) that series solutions of differential equations at an ordinary point are convergent defining so analytic solutions in a neighborhood of 0 in \mathbb{C} . At a singular point one must distinguish between regular singular points where all formal solutions are convergent (*cf.* [Was76, Thm. 5.3] for instance) and irregular singular points where the formal solutions are divergent in general; several examples of divergent series are presented and commented throughout

the text. The strong point with formal solutions is that they are “easily” computed; at least, there exist algorithms to compute them, whatever the order of the linear differential equation. Nonetheless, one wishes to find actual solutions near such singular points and to understand their behavior.

The idea underlying a theory of summation is to build a tool that transforms formal solutions into unique well-defined actual solutions. Roughly speaking, it is natural to ask that the former ones be linked to the latter ones by an asymptotic condition; in other words, that the formal solutions be Taylor series of the actual solutions in a generalized sense. Only convergent series have an asymptotic function on a full neighborhood of 0 in \mathbb{C} ; otherwise, the asymptotics are required on sectors with vertex 0. Uniqueness is essential to go back and forth and to guaranty good, well-defined properties. The problem is now fully solved for the class under consideration in several equivalent ways providing thus several equivalent *theories of summation* or *theories of summability*. Some methods provide necessary and sufficient conditions for a series to be summable, some others provide explicit formulæ. Each method has its own interest; none is the best and their variety must be thought as an enrichment of our means to solve problems. The theories here considered depend on parameters called levels or critical times. The simplest case with only one level k is called *k-summability* (actually, “simpler than the simplest” is the case when $k = 1$). The case of several levels k_1, k_2, \dots, k_ν is called *multisummability* or, to be precise, *(k₁, k₂, ..., k_ν)-summability*.

At first sight, since the singular points of differential equations are isolated, one could discuss the interest of such a procedure, for, one can approach as close as wished the singular points with the Cauchy-Lipschitz Theorem at the neighboring ordinary points. However, such an approach does not allow a good understanding of the singularities; even numerically, the usual numerical procedures stop being efficient when approaching a singular point, not providing thus even an idea of the behavior at the singular point. On the contrary, a good understanding of the singularity by means of a theory of summation permits a numerical calculation of solutions and of their invariants in most cases.

Chapter 2 deals with asymptotics in the complex domain, ordinary (also called Poincaré asymptotics) and Gevrey asymptotics. The presentation is classical and comes with five examples of divergent series (not all solutions

of differential equations) that will be commented throughout the text. The chapter contains also a proof of the Borel-Ritt Theorem in Poincaré and in Gevrey asymptotics and a proof of the Cauchy-Heine Theorem in classical form.

In chapter 3 we introduce the language of sheaves and rudiments in Čech cohomology. The sheaves \mathcal{A} , \mathcal{A}_s , $\mathcal{A}^{<0}$ and $\mathcal{A}^{\leq -k}$ of germs of various types of asymptotic functions that are at the core of what follows, are carefully defined. Cohomological versions of the Borel-Ritt Theorem and of the Cauchy-Heine Theorem are made explicit.

Chapter 4 contains basic recalls in the theory of ordinary linear differential equations: comparison of equations and systems with Deligne's Cyclic vector lemma, the viewpoint of \mathcal{D} -modules, equivalence of equations or systems, formal and meromorphic classifications, Newton polygons and calculation of the formal invariants in the case of equations, Main Asymptotic Existence Theorem in sheaf form and in classical form. We end the chapter with the construction of infinitesimal neighborhoods of singularities of differential equations.

Chapter 5 is devoted to index theorems for ordinary linear differential operators in various spaces with an application to the irregularity of operators.

In chapter 6 we develop four different approaches to k -summability (that is, summability depending on a unique level k) and we prove their equivalence: Ramis k -summability, Ramis-Sibuya k -summability, Borel-Laplace summation with a proof of Nevanlinna's Theorem and wild-summability, that is, by means of wild analytic continuation in the infinitesimal neighborhood of 0. Follow some applications: Maillet-Ramis Theorem, sufficient conditions for the k -summability of solutions of differential equations, their resurgence in the sense of J. Écalle, and Martinet-Ramis Tauberian Theorems. In each case, we chose the approach that seemed to us to be the most convenient.

Chapter 7 deals with tangent-to-identity germs of diffeomorphisms that are formally conjugated to the translation (by 1). It is meant as an application of Ramis-Sibuya Theorem to prove the 1-summability of the conjugacy map. A proof of the Birkhoff-Kimura sectorial normalization Theorem is provided. A careful study by means of Borel and Laplace transforms will be find in [Sau].

In chapter 8 we develop six different approaches to multisummability and we prove their equivalence: an asymptotic definition generalizing Ramis k -summability, Malgrange-Ramis summability generalizing Ramis-Sibuya k -summability, summation by iterated Laplace integrals and accelero-summation

generalizing the Borel-Laplace summation, Balser's decomposition into sums and the wild-multisummability in the infinitesimal neighborhood of 0. Some applications to differential equations and Tauberian Theorems are given.

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CHAPTER 2

ASYMPTOTIC EXPANSIONS IN THE COMPLEX DOMAIN

2.1. Generalities

We consider functions of a complex variable x and their asymptotic expansions at a given point x_0 of the Riemann sphere. Without loss of generality we assume that $x_0 = 0$ although for some examples classically studied at infinity we keep $x_0 = \infty$. Indeed, asymptotic expansions at infinity reduce to asymptotic expansions at 0 after the change of variable $x \mapsto z = 1/x$ and asymptotic expansions at $x_0 \in \mathbb{C}$ after the change of variable $x \mapsto t = x - x_0$. The point 0 must belong to the closure of the domain where the asymptotics are studied. In general, we consider sectors with vertex 0, or germs of such sectors when the radius approaches 0. The sectors are drawn either in the complex plane \mathbb{C} , precisely, in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (the functions are then *univalent*) or on the Riemann surface of the logarithm $\check{\mathbb{C}}$ (the functions are *multivalent* or given in terms of polar coordinates).

Notations 2.1.1. — We denote by

▷ $\Delta = \Delta_{\alpha,\beta}(R)$ the open sector with vertex 0 made of all points $x \in \mathbb{C}$ satisfying $\alpha < \arg(x) < \beta$ and $0 < |x| < R$;

▷ $\bar{\Delta} = \bar{\Delta}_{\alpha,\beta}(R)$ its closure in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or in $\check{\mathbb{C}}$ (0 is always excluded) and we use the term closed sector;

▷ $\mathcal{O}(\Delta)$ the set of all holomorphic functions on Δ .

Definition 2.1.2. — A sector $\Delta_{\alpha',\beta'}(R')$ is said to be a proper sub-sector of (or to be properly included in) the sector $\Delta_{\alpha,\beta}(R)$ and one denotes

$$\Delta_{\alpha',\beta'}(R') \Subset \Delta_{\alpha,\beta}(R)$$

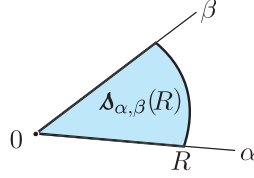


Figure 1

if its closure $\bar{\Delta}_{\alpha',\beta'}(R')$ in \mathbb{C}^* or $\check{\mathbb{C}}$ is included in $\Delta_{\alpha,\beta}(R)$.

Thus, the notation $\Delta_{\alpha',\beta'}(R') \Subset \Delta_{\alpha,\beta}(R)$ means $\alpha < \alpha' < \beta' < \beta$ and $R' < R$.

2.2. Poincaré asymptotics

Poincaré asymptotic expansions, or for short, asymptotic expansions, are expansions in the basic sense of Taylor expansions providing successive approximations of a function. Unless otherwise mentioned we consider functions of a complex variable and asymptotic expansions in the complex domain and this allows us to use the methods of complex analysis. As we will see, the properties of asymptotic expansions in the complex domain may differ quite a little bit of those in the real domain.

In what follows Δ denotes an open sector with vertex 0 either in \mathbb{C}^* or in $\check{\mathbb{C}}$, the Riemann surface of the logarithm.

2.2.1. Definition. —

Definition 2.2.1. — A function $f \in \mathcal{O}(\Delta)$ is said to admit a series $\sum_{n \geq 0} a_n x^n$ as asymptotic expansion (or to be asymptotic to the series) on a sector Δ if for all proper sub-sector $\Delta' \Subset \Delta$ of Δ and all $N \in \mathbb{N}$, there exists a constant $C > 0$ such that the following estimate holds for all $x \in \Delta'$:

$$\left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq C |x|^N.$$

The constant $C = C_{N,\Delta'}$ depends on N and Δ' but no condition is required on the nature of this dependence.

The technical condition “for all $\Delta' \Subset \Delta$ ” plays a fundamental role of which we will take benefit soon (cf. Rem. 2.2.10).

Observe that the definition includes infinitely many estimates in each of which N is fixed. The conditions have nothing to do with the convergence or

the divergence of the series as N goes to infinity. For $N = 1$ the condition says that f can be continuously continued at 0 on \mathcal{A} . For $N = 2$ it says that the function f has a derivative at 0 on \mathcal{A} and more generally for any N , that f has a “Taylor expansion” of order N . As in the case of a real variable, asymptotic expansions of functions of a complex variable, when they exist, are unique and they satisfy the same algebraic rules on sums, products, anti-derivatives and compositions. The proofs are similar and we leave them to the reader. The main difference between the real and the complex case lies in the behavior with respect to derivation (cf. Prop. 2.2.9 and Rem. 2.2.10).

Notations 2.2.2. — We denote by

▷ $\bar{\mathcal{A}}(\mathcal{A})$ the set of functions of $\mathcal{O}(\mathcal{A})$ admitting an asymptotic expansion at 0 on \mathcal{A} ;

▷ $\bar{\mathcal{A}}^{<0}(\mathcal{A})$ the sub-set of functions of $\bar{\mathcal{A}}(\mathcal{A})$ asymptotic to zero at 0 on \mathcal{A} . Such functions are called flat functions at 0 on \mathcal{A} ;

▷ $T = T_{\mathcal{A}} : \bar{\mathcal{A}}(\mathcal{A}) \rightarrow \mathbb{C}[[x]]$ the map assigning to each $f \in \bar{\mathcal{A}}(\mathcal{A})$ its asymptotic expansion at 0 on \mathcal{A} .

Due to the uniqueness of the asymptotic expansion, the map $T_{\mathcal{A}}$ is well defined and is called the *Taylor map* on \mathcal{A} (cf. Exa. 2.2.3 below). Due to the algebraic properties of asymptotic expansions the sets $\bar{\mathcal{A}}(\mathcal{A})$ and $\bar{\mathcal{A}}^{<0}(\mathcal{A})$ are naturally endowed with a structure of vector spaces and the Taylor map is a linear map with kernel $\bar{\mathcal{A}}^{<0}(\mathcal{A})$. Proposition 2.2.9 below will improve this result. We notice that $\bar{\mathcal{A}}^{<0}(\mathcal{A})$ is not 0: exponentials of various powers of x provide examples of non-zero functions of $\bar{\mathcal{A}}^{<0}(\mathcal{A})$ for any \mathcal{A} . For instance, if $\mathcal{A} = \{x; |\arg(x)| < \pi/2\}$, the function $\exp(-1/x)$ belongs to $\bar{\mathcal{A}}^{<0}(\mathcal{A})$; if $\mathcal{A} = \{x; |\arg(x)| < \pi\}$, the function $\exp(-1/\sqrt{x})$ where \sqrt{x} stands for the principal determination of $x^{1/2}$ belongs to $\bar{\mathcal{A}}^{<0}(\mathcal{A})$.

2.2.2. Examples. —

Example 2.2.3 (A trivial example: convergent series)

Let \mathcal{A} be a punctured disc D^* around 0 (i.e., a sector of opening $> 2\pi$ in \mathbb{C}). If f is an analytic function on D then f is asymptotic to its Taylor series at 0 on D^* . Reciprocally, if f is an analytic function on D^* that has an asymptotic expansion at 0 on D^* then, f is bounded near 0 and according to the removable singularity Theorem, f is analytic on all of D .

Example 2.2.4 (A fundamental example: the Euler function)

Consider the Euler equation

$$(1) \quad x^2 \frac{dy}{dx} + y = x.$$

Looking for a power series solution one finds the unique series

$$(2) \quad \tilde{E}(x) = \sum_{n \geq 0} (-1)^n n! x^{n+1}$$

called the *Euler series*. The Euler series is clearly divergent for all $x \neq 0$ and thus, it does not provide an analytic solution near 0 by Cauchy summation.

However, an actual solution can be found by applying the Lagrange method on \mathbb{R}^+ ; notice that 0 is a singular point of the equation and the Lagrange method must be applied on a domain (*i.e.*, a connected open set) containing no singular point (\mathbb{R}^+ is connected, open in \mathbb{R} and does not contain 0). Among the infinitely many solutions given by the method we choose the only one which is bounded as x tend to 0^+ ; it reads

$$E(x) = \int_0^x \exp\left(-\frac{1}{t} + \frac{1}{x}\right) \frac{dt}{t} = \int_0^{+\infty} \frac{e^{-\xi/x}}{1+\xi} d\xi$$

and is not only a solution on \mathbb{R}^+ but also a well defined solution on $\Re(x) > 0$.

Actually, the function E is asymptotic to the Euler series \tilde{E} on $\{x \in \mathbb{C}; \Re(x) > 0\}$.

A proof works as follows: writing

$$\frac{1}{1+\xi} = \sum_{n=0}^{N-2} (-1)^n \xi^n + (-1)^{N-1} \frac{\xi^{N-1}}{1+\xi}$$

and using $\int_0^{+\infty} u^n e^{-u} du = \Gamma(1+n)$, we get the relation

$$E(x) = \sum_{n=0}^{N-2} (-1)^n \Gamma(1+n) x^{n+1} + (-1)^{N-1} \int_0^{+\infty} \frac{\xi^{N-1} e^{-\xi/x}}{1+\xi} d\xi$$

and we are left to bound the integral remainder term.

Choose $0 < \delta < \pi/2$ and consider the (unlimited) proper sub-sector

$$\Delta_\delta = \{x; |\arg(x)| < \pi/2 - \delta\}$$

of the half-plane $\Delta = \{x; \Re(x) > 0\}$.

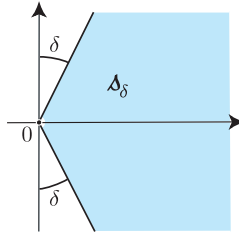


Figure 2

For all $x \in \mathcal{A}_\delta$, we can write

$$\begin{aligned} \left| E(x) - \sum_{n=0}^{N-2} (-1)^n n! x^{n+1} \right| &\leq \int_0^{+\infty} \xi^{N-1} e^{-\Re(\xi/x)} d\xi \\ &\leq \int_0^{+\infty} \xi^{N-1} e^{-\xi \sin(\delta)/|x|} d\xi \\ &= \frac{|x|^N}{(\sin \delta)^N} \int_0^{+\infty} u^{N-1} e^{-u} du = C|x|^N \end{aligned}$$

with $C = \Gamma(N)/(\sin \delta)^N$. This proves that the function $E(x)$ is asymptotic to the Euler series $\tilde{E}(x)$ at 0 on the half plane \mathcal{A} . Observe that the constant C does not depend on x but it depends on N and \mathcal{A}_δ and it tends to infinity as δ tends to 0. Thus, the estimate is no longer valid on the whole sector $\mathcal{A} = \{x; \Re(x) > 0\}$.

If we slightly turn the line of integration to the line d_θ with argument θ then, the same calculation stays valid and provides a function $E_\theta(x)$ with the same asymptotic expansion on the half plane bisected by d_θ . Due to Cauchy's Theorem, $E_\theta(x)$ is the analytic continuation of $E(x)$. Denote by $E(x)$ the largest analytic continuation of the initial function $E(x)$ by such a method. Its domain of definition is easily determined: we can rotate the line d_θ as long as it does not meet the pole $\xi = -1$ of the integrand, *i.e.*, we can rotate it from $\theta = -\pi$ to $\theta = +\pi$. We get so an analytic continuation of the initial function E on the sector

$$\mathcal{A}_E = \{x \in \tilde{\mathcal{C}}; -3\pi/2 < \arg(x) < +3\pi/2\}$$

of the universal cover $\tilde{\mathcal{C}}$ of \mathbb{C}^* . On such a sector, $E(x)$ is asymptotic to the Euler series $\tilde{E}(x)$.

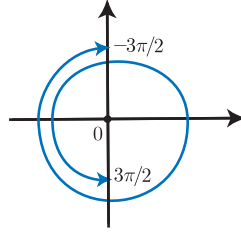


Figure 3

With this construction we are given on $\{x \in \mathbb{C}^*; \Re(x) < 0\}$ two determinations $E^+(x)$ and $E^-(x)$ of $E(x)$ when the direction θ approaches $+\pi$ and $-\pi$ respectively. Let us observe the following two facts:

▷ The determinations $E^+(x)$ and $E^-(x)$ are distinct since, otherwise, the function $E(x)$ would be analytic at 0 and the Euler series $\tilde{E}(x)$ would be convergent.

▷ Although $E(x)$ admits an analytic continuation as a solution of the Euler equation on all of the universal cover $\tilde{\mathcal{C}}$ of \mathbb{C}^* (Cauchy-Lipschitz Theorem) it stops having an asymptotic expansion on any sector \mathcal{A} larger than \mathcal{A}_E (*i.e.*, $\mathcal{A}_E \subsetneq \mathcal{A}$). Indeed, the two determinations $E^+(x)$ and $E^-(x)$ satisfy the relation (see [LR90] or the calculation of the variation of $E(x)$ in Remark 2.5.3)

$$(3) \quad E^+(x) - E^-(x) = 2\pi i e^{1/x}.$$

Thus, $E^+(x)$ can be continued through the negative imaginary axis by setting $E^+(x) = E(x) + 2\pi i e^{1/x}$ and symmetrically for $E^-(x)$ through the positive imaginary axis. Any asymptotic condition fails since $e^{1/x}$ is unbounded at 0 when $\Re(x)$ is positive. Such a phenomenon of discontinuity of the asymptotics is called *Stokes phenomenon* (see end of Rem. 2.5.3 and Sect. 4.3).

The function $E(x)$ is called the *Euler function*. Unless otherwise specified we consider it as a function defined on $\{x \in \mathbb{C}; |\arg(x)| < 3\pi/2\}$.

Example 2.2.5 (A classical example: the exponential integral)

The exponential integral $\text{Ei}(x)$ is the function given by

$$(4) \quad \text{Ei}(x) = \int_x^{+\infty} e^{-t} \frac{dt}{t}.$$

The integral being well defined on horizontal lines avoiding 0 the function $\text{Ei}(x)$ is well defined and analytic on the plane \mathbb{C} slit along the real non positive axis.

Let us first determine its asymptotic behavior at the origin 0 on the right half plane $\Delta = \{x; \Re(x) > 0\}$. For this, we start with the asymptotic expansion of its derivative $\text{Ei}'(x) = -e^{-x}/x$. Taylor expansion with integral remainder for e^{-x} gives

$$e^{-x} = \sum_{n=0}^{N-1} (-1)^n \frac{x^n}{n!} + (-1)^N \frac{x^N}{(N-1)!} \int_0^1 (1-u)^{N-1} e^{-ux} du$$

and then, since $\Re(-ux) < 0$,

$$\left| \text{Ei}'(x) + \frac{1}{x} + \sum_{n=1}^{N-1} (-1)^n \frac{x^{n-1}}{n!} \right| \leq \frac{|x|^{N-1}}{N!}.$$

We see that a negative power of x occurs with a logarithm as anti-derivative. Integrating between $\varepsilon > 0$ and x and making ε tend to 0 we obtain

$$\left| \text{Ei}(x) + \ln(x) + \gamma + \sum_{n=1}^{N-1} (-1)^n \frac{x^n}{n \cdot n!} \right| \leq \frac{|x|^N}{N!} \quad \text{with } \gamma = -\lim_{x \rightarrow 0^+} (\text{Ei}(x) + \ln(x)).$$

To fit our definition of an asymptotic expansion we must consider the function $\text{Ei}(x) + \ln(x)$. By extension, one says that $\text{Ei}(x)$ has the asymptotic expansion

$$-\ln(x) - \gamma - \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \cdot n!}.$$

We leave as an exercise the fact that γ is indeed the Euler constant $\lim_{n \rightarrow +\infty} \sum_{p=1}^n 1/p - \ln(n)$. Notice that, this time, we did not need to shrink the sector Δ .

Look now what happens at infinity. Instead of calculating the asymptotic expansion of $\text{Ei}(z)$ at infinity from its definition we notice that the function $y(x) = e^{1/x} \text{Ei}(1/x)$ is the Euler function $f(x)$. Hence, it has on Δ at 0 the same asymptotics as $f(x)$. Turning back to the variable $z = 1/x \simeq \infty$ we can state that $e^z \text{Ei}(z)$ has the series $\sum_{n \geq 0} (-1)^n n!/z^{n+1}$ as asymptotic expansion at infinity on Δ . By extension, one says that $\text{Ei}(z)$ is asymptotic to $e^{-z} \sum_{n \geq 0} (-1)^n n!/z^{n+1}$ on Δ at infinity.

Example 2.2.6 (A generalized hypergeometric series ${}_3F_0$)

We consider a generalized hypergeometric equation with given values of the parameters, say,

$$(5) \quad D_{3,1}y \equiv \left\{ z \left(z \frac{d}{dz} + 4 \right) - z \frac{d}{dz} \left(z \frac{d}{dz} + 1 \right) \left(z \frac{d}{dz} - 1 \right) \right\} y = 0.$$

The equation has an irregular singular point at infinity and a unique series solution

$$(6) \quad \tilde{g}(z) = \frac{1}{z^4} \sum_{n \geq 0} \frac{(n+2)!(n+3)!(n+4)!}{2!3!4!n!} \frac{1}{z^n}.$$

Using the standard notation for the hypergeometric series, the series \tilde{g} reads

$$\tilde{g}(z) = z^{-4} {}_3F_0 \left(\{3, 4, 5\} \middle| \frac{1}{z} \right).$$

By abuse of language, we will also call \tilde{g} an hypergeometric series.

One can check that the equation admits, for $-3\pi < \arg(z) < +\pi$, the solution

$$g(z) = \frac{1}{2\pi i 2!3!4!} \int_{\gamma} \Gamma(1-s)\Gamma(-s)\Gamma(-1-s)\Gamma(4+s) e^{i\pi s} z^s ds$$

where γ is a path from $-3 - i\infty$ to $-3 + i\infty$ along the line $\Re(s) = -3$. This follows from the fact that the integrand $G(s, z)$ satisfies the order one difference equation deduced from $D_{3,1}$ by applying a Mellin transform. We leave the proof to the reader. Instead, let us check that the integral is well defined. The integrand $G(s, z)$ being continuous along γ we just have to check the behavior of $G(s, z)$ as s tends to infinity along γ . An asymptotic expansion of $\Gamma(t + iu)$ for $t \in \mathbb{R}$ fixed and $u \in \mathbb{R}$ large is given by (see [BH86, p. 83]):

$$(7) \quad \Gamma(t + iu) = |u|^{t-\frac{1}{2}} e^{-\frac{\pi}{2}|u|} \left\{ \sqrt{2\pi} e^{i\frac{\pi}{2}(t-\frac{1}{2}) \operatorname{sgn}(u) - iu} |u|^{iu} \right\} \{1 + O(1/u)\}.$$

It follows that $G(t + iu, z)$ satisfies

$$(8) \quad |G(t + iu, z)| = (2\pi)^2 |u|^{-2t+2} |z|^t e^{-2\pi|u| - \pi u - u \arg(z)} \{1 + O(1/u)\}.$$

The exponent of the exponential being negative for $-3\pi < \arg(z) < \pi$ the integral is convergent and it defines an analytic function $g(z)$.

Let us prove that the function $g(z)$ is asymptotic to $\tilde{g}(z)$ at infinity on the sector $\mathcal{A} = \{z; -3\pi < \arg(z) < +\pi\}$. For this, consider a path

$$\gamma_{n,p} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \quad (n, p \in \mathbb{N}^*)$$

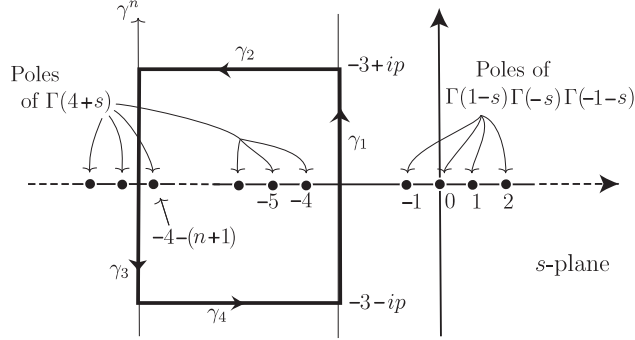
as drawn on Fig. 4.

The path $\gamma_{n,p}$ encloses the poles $s_m = -4 - m$ for $m = 0, \dots, n+1$ of $G(s, z)$ and the residues are $\operatorname{Res}(G(s, z); s = -4 - m) = (2+m)!(3+m)!(4+m)! z^{-4-m}/m! = 2!3!4! a_m$. Indeed, $\Gamma(4+s)$ has a simple pole at $s = -4 - m$ and reads

$$\Gamma(4 + (-4 - m + t)) = \Gamma(-m + t) = \frac{(-1)^m}{m!} t^{-1} + O(1)$$

while all other factors of G are non-zero analytic functions. We deduce that

$$\frac{1}{2\pi i 2!3!4!} \int_{\gamma_{n,p}} G(s, z) ds = \frac{1}{z^4} \sum_{m=0}^{n+1} \frac{a_m}{z^m}.$$

Figure 4. Path $\gamma_{n,p}$

Formula (8) implies the estimate

$$|G(t + i\varepsilon p, z)| \leq Cp^{2n+5} e^{-(2\pi + \varepsilon\pi + \varepsilon \arg(z))p}, \quad \varepsilon = \pm 1,$$

valid for $|z| > 1$ all along $\gamma_2 \cup \gamma_4$, the constant C depending on n and z but not on p . This shows that the integral along $\gamma_2 \cup \gamma_4$ tends to zero as p tends to infinity and consequently, we obtain

$$g(z) = \frac{1}{z^4} \sum_{m=0}^{n+1} \frac{a_m}{z^m} + g_n(z)$$

where $g_n(z) = \frac{1}{2\pi i} \int_{\gamma^n} G(s, z) ds$ and $\gamma^n = \{s \in \mathbb{C}; \Re(s) = -4 - n - \frac{3}{2}\}$ oriented upwards.

For any (small enough) $\delta > 0$ consider the proper sub-sector Δ_δ of Δ defined by

$$\Delta_\delta = \{z \in \mathbb{C}; |z| > 1 \text{ and } -3\pi + \delta < \arg(z) < \pi - \delta\}.$$

For $z \in \Delta_\delta$ and $s = -4 - n - \frac{3}{2} + iu \in \gamma^n$, the factor z^s satisfies

$$|z^s| \leq \frac{1}{|z|^{4+n+\frac{3}{2}}} \cdot \begin{cases} e^{-u(\pi-\delta)} & \text{if } u < 0, \\ e^{u(3\pi-\delta)} & \text{if } u > 0. \end{cases}$$

and using again formula (7) we obtain

$$\left| G\left(-4 - n - \frac{3}{2} + iu, z\right) \right| \leq \frac{\text{Const}_{n,\delta}}{|z|^{(4+n)+1}} |u|^{13+n} e^{-|u|\delta}.$$

Hence, there exists a constant $C = C(n, \delta)$ depending on n and δ but not on z such that

$$\left| g(z) - \frac{1}{z^4} \sum_{m=0}^{n+1} \frac{a_m}{z^m} \right| = |g_n(z)| \leq \frac{C}{|z|^{(4+n)+1}} \quad \text{for all } z \in \Delta_\delta.$$

Rewriting this estimate in the form

$$\left| g(z) - \frac{1}{z^4} \sum_{m=0}^n \frac{a_m}{z^m} \right| = \left| g_n(z) + \frac{a_{n+1}}{z^{(4+n)+1}} \right| \leq \frac{C + |a_{n+1}|}{|z|^{(4+n)+1}} \quad \text{for all } z \in \Delta_\delta$$

we satisfy Definition 2.2.1 for g at the order $4 + n$.

With this method we do not know how the constant C depends on n but we know that $|a_{n+1}|$ grows like $(n!)^2$ and then $C + |a_{n+1}|$ itself grows at least like $(n!)^2$.

Example 2.2.7 (A series solution of a mild difference equation)

Consider the order one difference equation

$$(9) \quad h(z+1) - 2h(z) = \frac{1}{z}.$$

A difference equation is said to be *mild* when its companion system, here

$$\begin{bmatrix} y_1(z+1) \\ y_2(z+1) \end{bmatrix} = \begin{bmatrix} 2 & 1/z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}$$

has an invertible leading term; in our case, $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible. The term “mild” and its contrary “wild” were introduced by M. van der Put and M. Singer [vdPS97].

Let us look at what happens at infinity. By identification, we see that equation (9) has a unique power series solution in the form $\tilde{h}(z) = \sum_{n \geq 1} h_n/z^n$. The coefficients h_n are defined by the recurrence relation

$$h_n = \sum_{\substack{m+p=n \\ m, p \geq 1}} (-1)^p h_m \frac{(m+p-1)!}{(m-1)! p!}$$

starting from the initial value $h_1 = -1$. It follows that the sequence h_n is alternate and satisfies

$$|h_n| \geq n|h_{n-1}|.$$

Consequently, $|h_n| \geq n!$ and the series \tilde{h} is divergent. Actually the recurrence relation can be solved as follows. Consider the Borel transform

$$\hat{h}(\zeta) = \sum_{n \geq 1} h_n \frac{\zeta^{n-1}}{(n-1)!}$$

of \tilde{h} (cf. Def. 6.3.1). It satisfies the Borel transformed equation $e^{-\zeta} \hat{h}(\zeta) - 2\hat{h}(\zeta) = 1$ and then $\hat{h}(\zeta) = 1/(e^{-\zeta} - 2)$. Its Taylor series at 0 reads

$$T_0 \hat{h}(\zeta) = \sum_{n \geq 0} \frac{(-1)^{n+1}}{n!} \sum_{p \geq 0} \frac{p^n}{2^{p+1}} \zeta^n$$

which implies that $h_n = (-1)^n \sum_{p \geq 0} p^{n-1}/2^{p+1}$. Again, we see that the series \tilde{h} is divergent since $|h_n| \geq n^{n-1}/2^{n+1}$.

We claim that the function

$$h(z) = \int_0^{+\infty} \hat{h}(\zeta) e^{-z\zeta} d\zeta$$

is asymptotic to $\tilde{h}(z)$ at infinity on the sector $\mathcal{A} = \{z; \Re(z) > 0\}$ (right half-plane). Indeed, choose $N \in \mathbb{N}$ and a proper sub-sector $\mathcal{A}_\delta = \{z; -\frac{\pi}{2} + \delta < \arg(z) < \frac{\pi}{2} - \delta\}$ of \mathcal{A} . From the Taylor expansion with integral remainder of $\hat{h}(\zeta)$ at 0

$$\hat{h}(\zeta) = \sum_{n=1}^N h_n \frac{\zeta^{n-1}}{(n-1)!} + \frac{\zeta^N}{(N-1)!} \int_0^1 (1-t)^{N-1} \hat{h}^{(N)}(\zeta t) dt$$

we obtain

$$h(z) = \sum_{n=1}^N \frac{h_n}{z^n} + \int_0^{+\infty} \frac{\zeta^N}{(N-1)!} \int_0^1 (1-t)^{N-1} \hat{h}^{(N)}(\zeta t) dt e^{-z\zeta} d\zeta.$$

To bound $\widehat{h}^{(N)}(\zeta t)$ we use the Cauchy Integral Formula

$$\widehat{h}^{(N)}(\zeta t) = \frac{N!}{2\pi i} \int_{C_{\zeta t}} \frac{\widehat{h}(u)}{(u - \zeta t)^{N+1}} du$$

where $C_{\zeta t}$ denotes the circle with center ζt , radius $1/2$, oriented counterclockwise. For $t \in [0, 1]$ and $\zeta \in [0, +\infty[$ then ζt is non negative and $\Re(u) \geq -1/2$ when u runs over any $C_{\zeta t}$. Hence, we obtain

$$|\widehat{h}^{(N)}(\zeta t)| \leq \frac{N!2^N}{2 - e^{1/2}} \quad \text{and} \quad \left| \int_0^1 (1-t)^{N-1} \widehat{h}^{(N)}(\zeta t) dt \right| \leq \frac{(N-1)!2^N}{2 - e^{1/2}}.$$

Finally, from the identity above we can conclude that, for all $z \in \mathcal{A}_\delta$,

$$(10) \quad \left| h(z) - \sum_{n=1}^N \frac{h_n}{z^n} \right| \leq \frac{2^N}{2 - e^{1/2}} \int_0^{+\infty} \zeta^N |e^{-\zeta z}| d\zeta = \frac{C}{|z|^{N+1}}$$

with $C = \frac{1}{2 - e^{1/2}} N! \frac{2^N}{(\sin \delta)^{N+1}}$.

This proves that $h(z)$ is asymptotic to $\widetilde{h}(z)$ at infinity on \mathcal{A} .

Example 2.2.8 (A series solution of a wild difference equation)

Consider the order one inhomogeneous wild difference equation

$$(11) \quad \frac{1}{z} \ell(z+1) + \left(1 + \frac{1}{z}\right) \ell(z) = \frac{1}{z}.$$

An identification of terms of equal power shows that it admits a unique series solution

$$\widetilde{\ell}(z) = \sum_{n \geq 1} \ell_n z^{-n}$$

whose coefficients ℓ_n are given by the recurrence relation

$$\ell_{n+1} = -2\ell_n - \sum_{\substack{m+p=n \\ m \geq 1, p \geq 1}} (-1)^p \ell_m \frac{(m+p-1)!}{p!(m-1)!}$$

from the initial value $\ell_1 = 1$. It follows that the sequence $(\ell_n)_{n \geq 1}$ is alternate and satisfies

$$|\ell_{n+1}| \geq (n-1)|\ell_{n-1}|.$$

Hence, $\ell_{2n} \geq 2^n (n-1)!$ for all n and consequently, the series is divergent. The Borel transform $\widehat{\ell}(\zeta)$ of the series $\widetilde{\ell}(z)$ satisfies the equation

$$\int_0^\zeta e^{-\xi} \widehat{\ell}(\xi) d\xi + \int_0^\zeta \widehat{\ell}(\xi) d\xi + \widehat{\ell}(\xi) = 1$$

equivalent to the two conditions $\widehat{\ell}(0) = 1$ and $\widehat{\ell}'(\zeta) = (-e^{-\zeta} - 1)\widehat{\ell}(\zeta)$. Hence,

$$\widehat{\ell}(\zeta) = \frac{1}{e} e^{-\zeta + e^{-\zeta}}.$$

We leave as an exercise to prove that the Laplace integral $\int_0^{+\infty} \widehat{\ell}(\zeta) e^{-z\zeta} d\zeta$ is a solution of (11) asymptotic to $\widetilde{\ell}(z)$ at infinity on the sector $\Re(z) > -1$ (Follow the same method as in the previous exercise and estimate the constant C).

2.2.3. Algebras of asymptotic functions. — Recall that \mathcal{A} denotes a given open sector with vertex 0 in $\mathbb{C} \setminus \{0\}$ or in the Riemann surface of the logarithm $\tilde{\mathbb{C}}$. Unless otherwise mentioned we refer to the usual derivation d/dx and to Notations 2.2.2.

Proposition 2.2.9 (Differential algebra and Taylor map)

▷ The set $\bar{\mathcal{A}}(\mathcal{A})$ endowed with the usual algebraic operations and the usual derivation d/dx is a differential algebra.

▷ The Taylor map $T = T_{\mathcal{A}} : \bar{\mathcal{A}}(\mathcal{A}) \rightarrow \mathbb{C}[[x]]$ is a morphism of differential algebras with kernel $\bar{\mathcal{A}}^{<0}(\mathcal{A})$.

Proof. — Due to the algebraic rules on asymptotic expansions $\bar{\mathcal{A}}(\mathcal{A})$ is a sub-algebra of $\mathcal{O}(\mathcal{A})$. We are left to prove that $\bar{\mathcal{A}}(\mathcal{A})$ is stable under derivation with respect to x and that the Taylor map $T_{\mathcal{A}}$ commutes with derivation.

Let $f \in \bar{\mathcal{A}}(\mathcal{A})$ have an asymptotic expansion $T_{\mathcal{A}}f(x) = \sum_{n \geq 0} a_n x^n$. Since f belongs to $\mathcal{O}(\mathcal{A})$ it admits a derivative $f' \in \mathcal{O}(\mathcal{A})$. Moreover, for all $\mathcal{A}' \Subset \mathcal{A}$ and all $N \geq 0$, there exists $C > 0$ such that, for all $x \in \mathcal{A}'$,

$$\left| f(x) - \sum_{n=0}^N a_n x^n \right| \leq C|x|^{N+1}$$

and we want to prove that for all $\mathcal{A}'' \Subset \mathcal{A}$, for all $N > 0$, there exists $C' > 0$ such that, for all $x \in \mathcal{A}''$,

$$\left| f'(x) - \sum_{n=0}^{N-1} (n+1)a_{n+1} x^n \right| \leq C'|x|^N.$$

Fix $N > 0$ and consider the function $g(x) = f(x) - \sum_{n=0}^N a_n x^n$.

We must prove that the condition

- for all $\mathcal{A}' \Subset \mathcal{A}$, there exists $C > 0$ such that $|g(x)| \leq C|x|^{N+1}$ for all $x \in \mathcal{A}'$ implies the condition
- for all $\mathcal{A}'' \Subset \mathcal{A}$, there exists $C' > 0$ such that $|g'(x)| \leq C'|x|^N$ for all $x \in \mathcal{A}''$.

Given $\mathcal{A}'' \Subset \mathcal{A}$, choose a sector \mathcal{A}' such that $\mathcal{A}'' \Subset \mathcal{A}' \Subset \mathcal{A}$ (see Fig 5) and let δ be so small that, for all $x \in \mathcal{A}''$, the closed disc $\bar{B}(x, |x|\delta)$ centered at x with radius $|x|\delta$ be contained in \mathcal{A}' . Denote by γ_x the boundary of $\bar{B}(x, |x|\delta)$.

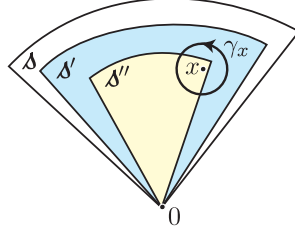


Figure 5

By assumption, for all $t \in \bar{B}(x, |x|\delta)$ and, especially, for all $t \in \gamma_x$ the function g satisfies $|g(t)| \leq C|t|^{N+1}$. We deduce from Cauchy's integral formula $g'(x) = \frac{1}{2\pi i} \int_{\gamma_x} \frac{g(t)}{(t-x)^2} dt$ that, for all $x \in \delta''$, the derivative g' satisfies

$$|g'(x)| \leq \frac{1}{2\pi} \max_{t \in \gamma_x} |g(t)| \frac{2\pi |x| \delta}{(|x| \delta)^2} \leq \frac{C}{|x| \delta} (|x| (1 + \delta))^{N+1} = C' |x|^N$$

with $C' = C(1 + \delta)^{N+1}/\delta$. Hence, the result. \square

Remarks 2.2.10. — Let us insist on the role of Cauchy's integral formula.

▷ The proof of Proposition 2.2.9 does require that the estimates in Definition 2.2.1 be satisfied for all $\delta' \Subset \delta$ instead of δ itself. Otherwise, we could not apply Cauchy's integral formula and we could not assert anymore that the algebra $\bar{\mathcal{A}}(\delta)$ is differential. In such a case, algebras of asymptotic functions would not be suitable to handle solutions of differential equations.

▷ Theorem 2.2.9 is no longer valid in real asymptotics, where Cauchy's integral formula does not hold, as it is shown by the following counter-example.

The function $f(x) = e^{-1/x} \sin(e^{1/x})$ is asymptotic to 0 (the null series) on \mathbb{R}^+ at 0. Its derivative $f'(x) = \frac{1}{x^2} e^{-1/x} \sin(e^{1/x}) - \frac{1}{x^2} \cos(e^{1/x})$ has no limit at 0 on \mathbb{R}^+ and then no asymptotic expansion. This proves that the set of real analytic functions admitting an asymptotic expansion at 0 on \mathbb{R}^+ is not a differential algebra.

The following proposition provides, in particular, a proof of the uniqueness of the asymptotic expansion, if any exists.

Proposition 2.2.11. — A function f belongs to $\bar{\mathcal{A}}(\delta)$ if and only if f belongs to $\mathcal{O}(\delta)$ and a sequence $(a_n)_{n \in \mathbb{N}}$ exists such that

$$\frac{1}{n!} \lim_{\substack{x \rightarrow 0 \\ x \in \delta'}} f^{(n)}(x) = a_n \quad \text{for all } \delta' \Subset \delta.$$

Proof. — The *only if* part follows from Proposition 2.2.9. To prove the *if* part consider $\mathcal{A}' \Subset \mathcal{A}$. For all x and $x_0 \in \mathcal{A}'$, f admits the Taylor expansion with integral remainder

$$f(x) - \sum_{n=0}^{N-1} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n = \int_{x_0}^x \frac{1}{(N-1)!} (x-t)^{N-1} f^{(N)}(t) dt.$$

Notice that we cannot write such a formula for $x_0 = 0$ since 0 does not even belong to the definition set of f . However, by assumption, the limit of the left hand side as x_0 tends to 0 in \mathcal{A}' exists; hence, the limit of the right hand side exists and we can write

$$f(x) - \sum_{n=0}^{N-1} a_n x^n = \int_0^x \frac{1}{(N-1)!} (x-t)^{N-1} f^{(N)}(t) dt.$$

Then,

$$\begin{aligned} \left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| &\leq \frac{1}{(N-1)!} \left| \int_0^x (x-t)^{N-1} f^{(N)}(t) dt \right| \\ &\leq \frac{|x|^N}{N!} \sup_{t \in \mathcal{A}'} |f^{(N)}(t)| \leq C |x|^N, \end{aligned}$$

the constant $C = \frac{1}{N!} \sup_{t \in \mathcal{A}'} |f^{(N)}(t)|$ being finite by assumption. Hence, the conclusion \square

2.3. Gevrey asymptotics

When working with differential equations for instance, it appears easily that the conditions required in Poincaré asymptotics are too weak to fit some natural requests, say for instance, to provide asymptotic functions that are solutions of the equation when the asymptotic series themselves are solution or, better, to set a 1-to-1 correspondence between the series solution and their asymptotic expansion. A precise answer to these questions is found in the theories of summation (*cf.* Chaps. 6 and 8). A first step towards that aim is given by strengthening Poincaré asymptotics into Gevrey asymptotics.

From now on, we are given $k > 0$ and we denote its inverse by

$$s = 1/k$$

When $k > 1/2$ then $\pi/k < 2\pi$ and the sectors of the critical opening π/k to be further considered may be seen as sectors of \mathbb{C}^* itself; otherwise, they must be considered as sectors of the universal cover $\tilde{\mathbb{C}}$ of \mathbb{C}^* . In general,

depending on the problem, we may assume that $k > 1/2$ after performing a change of variable (ramification) $x = t^p$ with a large enough $p \in \mathbb{N}$.

Recall that, unless otherwise specified, we denote by $\mathfrak{A}, \mathfrak{A}', \dots$ open sectors in \mathbb{C}^* or $\check{\mathbb{C}}$ and that the notation $\mathfrak{A}' \Subset \mathfrak{A}$ means that the closure of the sector \mathfrak{A}' in \mathbb{C}^* or $\check{\mathbb{C}}$ lies in \mathfrak{A} (cf. Def. 2.1.2).

2.3.1. Gevrey series. —

Definition 2.3.1 (Gevrey series of order s or of level k)

A series $\sum_{n \geq 0} a_n x^n$ is of Gevrey type of order s (in short, s -Gevrey) if there exist constants $C > 0, A > 0$ such that the coefficients a_n satisfy

$$|a_n| \leq C(n!)^s A^n \quad \text{for all } n.$$

The constants C and A do not depend on n . Equivalently, a series $\sum_{n \geq 0} a_n x^n$ is s -Gevrey if the series $\sum_{n \geq 0} a_n x^n / (n!)^s$ converges.

Notation 2.3.2. — We denote by $\mathbb{C}[[x]]_s$ the set of s -Gevrey series.

Observe that the spaces $\mathbb{C}[[x]]_s$ are filtered as follows:

$$\mathbb{C}\{x\} = \mathbb{C}[[x]]_0 \subset \mathbb{C}[[x]]_s \subset \mathbb{C}[[x]]_{s'} \subset \mathbb{C}[[x]]_\infty = \mathbb{C}[[x]]$$

for all s, s' satisfying $0 < s < s' < +\infty$.

Comments 2.3.3 (On the examples of chapter 1)

▷ A convergent series (cf. Exa. 2.2.3) is a 0-Gevrey series.

▷ The Euler series $\tilde{E}(x)$ (cf. Exa. 2.2.4) is 1-Gevrey and hence s -Gevrey for any $s > 1$.

It is s -Gevrey for no $s < 1$.

▷ The hypergeometric series ${}_3F_0(1/z)$ (cf. Exa. 2.2.6) is 2-Gevrey and s -Gevrey for no $s < 2$.

▷ The series $\tilde{h}(z)$ (cf. Exa. 2.2.7) is 1-Gevrey. Indeed, it is at least 1-Gevrey since $|h_n| \geq n!$ and it is at most 1-Gevrey since its Borel transform at infinity converges.

▷ From the fact that $|\ell_{2n+1}| \geq 2^n n!$ and $|\ell_{2n}| \geq 2^n (n-1)!$ we know that, if the series $\tilde{\ell}(z)$ (cf. Exa. 2.2.8) is of Gevrey type then it is at least 1/2-Gevrey. From the fact that its Borel transform is convergent it is of Gevrey type and at most 1-Gevrey. Note however that its Borel transform is an entire function and consequently, $\tilde{\ell}(z)$ could be less than 1-Gevrey.

Proposition 2.3.4. — $\mathbb{C}[[x]]_s$ is a differential sub-algebra of $\mathbb{C}[[x]]$ stable under composition.

Proof. — $\mathbb{C}[[x]]_s$ is clearly a sub-vector space of $\mathbb{C}[[x]]$. We have to prove that it is stable under product, derivation and composition.

▷ *Stability of $\mathbb{C}[[x]]_s$ under product.* — Consider two s -Gevrey series $\sum_{n \geq 0} a_n x^n$ and $\sum_{n \geq 0} b_n x^n$ satisfying, for all n and for positive constants A, B, C and K , the estimates

$$|a_n| \leq C(n!)^s A^n \quad \text{and} \quad |b_n| \leq K(n!)^s B^n.$$

Their product is the series $\sum_{n \geq 0} c_n x^n$ where $c_n = \sum_{p+q=n} a_p b_q$. Then,

$$|c_n| \leq CK \sum_{p+q=n} (p!)^s (q!)^s A^p B^q \leq CK(n!)^s (A+B)^n.$$

Hence the result.

▷ *Stability of $\mathbb{C}[[x]]_s$ under derivation.* — Given an s -Gevrey series $\sum_{n \geq 0} a_n x^n$ satisfying $|a_n| \leq C(n!)^s A^n$ for all n , its derivative $\sum_{n \geq 0} b_n x^n$ satisfies

$$|b_n| = (n+1)|a_{n+1}| \leq (n+1)C((n+1)!)^s A^{n+1} \leq C'(n!)^s A'^n$$

for convenient constants $A' > A$ and $C' \geq C$. Hence the result.

▷ *Stability of $\mathbb{C}[[x]]_s$ under composition* [Gev18]. — Let $\tilde{f}(x) = \sum_{p \geq 1} a_p x^p$ and $\tilde{g}(y) = \sum_{n \geq 0} b_n y^n$ be two s -Gevrey series. The composition $\tilde{g} \circ \tilde{f}(x) = \sum_{n \geq 0} c_n x^n$ provides a well-defined power series in x . From the hypothesis, there exist constants $h, k, a, b > 0$ such that, for all p and n , the coefficients of the series \tilde{f} and \tilde{g} satisfy respectively $|a_p| \leq h(p!)^s a^p$ and $|b_n| \leq k(n!)^s b^n$.

Faà di Bruno's formula allows us to write

$$n! c_n = \sum_{\underline{m} \in I_n} N(\underline{m}) |\underline{m}|! b_{|\underline{m}|} \prod_{j=1}^n (j! a_j)^{m_j}$$

where I_n stands for the set of non-negative n -tuples $\underline{m} = (m_1, m_2, \dots, m_n)$ satisfying the condition $\sum_{j=1}^n j m_j = n$, where $|\underline{m}| = \sum_{j=1}^n m_j$ and the coefficient $N(\underline{m})$ is a positive integer depending neither on \tilde{f} nor on \tilde{g} . Using the Gevrey hypothesis and the condition $\sum_{j=1}^n j m_j = n$, we can then write

$$n! |c_n| \leq k a^n \sum_{\underline{m} \in I_n} N(\underline{m}) |\underline{m}|!^{1+s} (hb)^{|\underline{m}|} \left(\prod_{j=1}^n j!^{m_j} \right)^{1+s}.$$

As clearly $|\underline{m}| \leq n$ and $N(\underline{m}) \leq N(\underline{m})^{1+s}$, with $B = \max(hb, 1)$, we obtain

$$n!|c_n| \leq k (aB)^n \sum_{\underline{m} \in I_n} \left(N(\underline{m}) |\underline{m}|! \prod_{j=1}^n j!^{m_j} \right)^{1+s}$$

and then, from the inequality $\sum_{i=1}^K X_i^{1+s} \leq (\sum_{i=1}^K X_i)^{1+s}$ for non-negative s and X_i 's, the estimate

$$n!|c_n| \leq k (aB)^n \left(\sum_{\underline{m} \in I_n} N(\underline{m}) |\underline{m}|! \prod_{j=1}^n j!^{m_j} \right)^{1+s}.$$

Now, applying Faà di Bruno's formula to the case of the series $\tilde{f}(x) = x/(1-x)$ and $\tilde{g}(x) = 1/(1-x)$, implying thus $\tilde{g} \circ \tilde{f}(x) = 1 + x/(1-2x)$, we get the relation

$$\sum_{\underline{m} \in I_n} N(\underline{m}) |\underline{m}|! \prod_{j=1}^n (j!)^{m_j} = \begin{cases} 2^{n-1} n! & \text{when } n \geq 1 \\ 1 & \text{when } n = 0; \end{cases}$$

hence, a fortiori,

$$\sum_{\underline{m} \in I_n} N(\underline{m}) |\underline{m}|! \prod_{j=1}^n (j!)^{m_j} \leq 2^n n!$$

and we can conclude that

$$|c_n| \leq k (n!)^s (2^{1+s} aB)^n$$

for all $n \in \mathbb{N}$, which ends the proof. \square

One has actually the more general result stated in Proposition 2.3.6 below.

Definition 2.3.5. — A series $\tilde{g}(y_1, \dots, y_r) = \sum_{n_1, \dots, n_r \geq 0} b_{n_1, \dots, n_r} y_1^{n_1} \dots y_r^{n_r}$ is said to be (s_1, \dots, s_r) -Gevrey if there exist positive constants C, M_1, \dots, M_r such that, for all n -tuple (n_1, \dots, n_r) of non-negative integers, the series satisfies an estimate of the form

$$\left| b_{n_1, \dots, n_r} \right| \leq C (n_1!)^{s_1} \dots (n_r!)^{s_r} M_1^{n_1} \dots M_r^{n_r}.$$

It is said to be s -Gevrey when $s_1 = \dots = s_r = s$.

Proposition 2.3.6. — Let $\tilde{f}_1(x), \tilde{f}_2(x), \dots, \tilde{f}_r(x)$ be s -Gevrey series without constant term and let $\tilde{g}(y_1, \dots, y_r)$ be an s -Gevrey series in r variables.

Then, the series $\tilde{g}(\tilde{f}_1(x), \dots, \tilde{f}_r(x))$ is an s -Gevrey series.

Since the expression of the n^{th} derivative of $\tilde{g}(\tilde{f}_1(x), \dots, \tilde{f}_r(x))$ has the same form as in the case of $\tilde{g}(\tilde{f}(x))$ the proof is identical to the one for $\tilde{g}(\tilde{f}(x))$ and we leave it as an exercise.

The result is, *a fortiori*, true when \tilde{g} or some of the \tilde{f}_j 's are analytic. The fact that $\mathbb{C}[[x]]_s$ be stable by product (and composition of course) can then be seen as a consequence of that proposition.

2.3.2. Algebras of Gevrey asymptotic functions. —

Definition 2.3.7 (Gevrey asymptotics of order s)

A function $f \in \mathcal{O}(\mathcal{A})$ is said to be Gevrey asymptotic of order s (for short, s -Gevrey asymptotic) to a series $\sum_{n \geq 0} a_n x^n$ on \mathcal{A} if for any proper sub-sector $\mathcal{A}' \Subset \mathcal{A}$ there exist constants $C_{\mathcal{A}'} > 0$ and $A_{\mathcal{A}'} > 0$ such that, the following estimate holds for all $N \in \mathbb{N}^*$ and $x \in \mathcal{A}'$:

$$(12) \quad \left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq C_{\mathcal{A}'} (N!)^s A_{\mathcal{A}'}^N |x|^N$$

A series which is the s -Gevrey asymptotic expansion of a function is said to be an s -Gevrey asymptotic series.

Notation 2.3.8. — We denote by $\overline{\mathcal{A}}_s(\mathcal{A})$ the set of functions admitting an s -Gevrey asymptotic expansion on \mathcal{A} .

Given an open arc I of S^1 , let $\mathcal{A}_I(R)$ denote the sector based on I with radius R . Since there is no possible confusion, we also denote the set of germs of functions admitting an s -Gevrey asymptotic expansion on a sector based on I by

$$\overline{\mathcal{A}}_s(I) = \varinjlim_{R \rightarrow 0} \overline{\mathcal{A}}_s(\mathcal{A}_I(R)).$$

The constants $C_{\mathcal{A}'}$ and $A_{\mathcal{A}'}$ may depend on \mathcal{A}' ; they do not depend on $N \in \mathbb{N}^*$ and $x \in \mathcal{A}'$. Gevrey asymptotics differs from Poincaré asymptotics by the fact that the dependence on N of the constant $C_{N, \mathcal{A}'}$ (cf. Def. 2.2.1) has to be of Gevrey type.

Comments 2.3.9 (On the examples of chapter 1)

The calculations in Section 2.2.2 show the following Gevrey asymptotic properties:

- ▷ The Euler function $E(x)$ is 1-Gevrey asymptotic to the Euler series $\tilde{E}(x)$ on any (germ at 0 of) half-plane bisected by a line d_θ with argument θ such that $-\pi < \theta < +\pi$. It is then 1-Gevrey asymptotic to $\tilde{E}(x)$ at 0 on the full sector $-3\pi/2 < \arg(x) < +3\pi/2$.
- ▷ Up to an exponential factor the exponential integral has the same properties on germs of half-planes at infinity.

▷ The generalized hypergeometric series $\tilde{g}(z)$ of Example 2.2.6 is 2-Gevrey and we stated that the function $g(z)$ is asymptotic in the rough sense of Poincaré to $\tilde{g}(z)$ on the half-plane $\Re(z) > 0$ at infinity. We will see (cf. Com. 6.2.7) that the function $g(z)$ is actually 1/2-Gevrey asymptotic to $\tilde{g}(z)$. Our computations in Sect. 2.2.2 do not allow us to state yet such a fact since we did not determine how the constant C depends on N .

▷ The function $h(z)$ of Example 2.2.7 was proved to be 1-Gevrey asymptotic to the series $\tilde{h}(z)$ (cf. Estim. (10) on the right half-plane $\Re(z) > 0$ at infinity.

▷ The function $\ell(z)$ of Example 2.2.8 satisfies the same estimate (10) as $h(z)$ on the sector $\mathcal{S}' = \{-\pi/2 + \delta < \arg(z) < \pi/2 - \delta\}$, for $(0 < \delta < \pi/2)$, with a constant C which can be chosen equal to $C = e^{-1/2+e^{1/2}} N! 2^N / (\sin \delta)^{N+1}$. The function $\ell(z)$ is then 1-Gevrey asymptotic to the series $\tilde{\ell}(z)$ on the right half-plane $\Re(z) > 0$ at infinity.

Proposition 2.3.10. — *An s -Gevrey asymptotic series is an s -Gevrey series.*

Proof. — Suppose the series $\sum_{n \geq 0} a_n x^n$ is the s -Gevrey asymptotic series of a function f on \mathcal{S} . For all N , the result follows from Condition (12) applied twice to

$$a_N x^N = \left(f(x) - \sum_{n=0}^{N-1} a_n x^n \right) - \left(f(x) - \sum_{n=0}^N a_n x^n \right). \quad \square$$

Proposition 2.3.11. — *A function $f \in \bar{\mathcal{A}}(\mathcal{S})$ belongs to $\bar{\mathcal{A}}_s(\mathcal{S})$ if and only if for all $\mathcal{S}' \in \mathcal{S}$ there exist constants $C'_{\mathcal{S}'} > 0$ and $A'_{\mathcal{S}'} > 0$ such that the following estimate holds for all $N \in \mathbb{N}$ and $x \in \mathcal{S}'$:*

$$(13) \quad \left| \frac{d^N f}{dx^N}(x) \right| \leq C'_{\mathcal{S}'} (N!)^{s+1} A'_{\mathcal{S}'}{}^N.$$

Proof. — Prove that Condition (13) implies Condition (12). — Like in the proof of Prop. 2.2.11, write Taylor's formula with integral remainder:

$$f(x) - \sum_{n=0}^{N-1} a_n x^n = \int_0^x \frac{1}{(N-1)!} (x-t)^{N-1} f^{(N)}(t) dt = -\frac{1}{N!} \int_0^x f^{(N)}(t) d(x-t)^N$$

and conclude that

$$\left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq \frac{1}{N!} \sup_{t \in \mathcal{S}'} \left| \frac{d^N f}{dx^N}(t) \right| \cdot |x|^N \leq C'_{\mathcal{S}'} (N!)^s A'_{\mathcal{S}'}{}^N |x|^N.$$

Prove that Condition (12) implies Condition (13). — Like in the proof of Prop. 2.2.9, attach to any $x \in \mathcal{S}'$ a circle γ_x centered at x with radius $|x|\delta$, the constant δ being chosen so small that γ_x be contained in \mathcal{S} and apply Cauchy's

integral formula:

$$\frac{d^N f}{dx^N}(x) = \frac{N!}{2\pi i} \int_{\gamma_x} f(t) \frac{dt}{(t-x)^{N+1}} = \frac{N!}{2\pi i} \int_{\gamma_x} \left(f(t) - \sum_{n=0}^{N-1} a_n t^n \right) \frac{dt}{(t-x)^{N+1}}$$

since the N^{th} derivative of a polynomial of degree $N-1$ is 0. Hence,

$$\begin{aligned} \left| \frac{d^N f}{dx^N}(x) \right| &\leq \frac{N!}{2\pi} C_{\mathcal{A}'}(N!)^s A_{\mathcal{A}'}^N \left| \int_{\gamma_x} \frac{|t|^N}{|t-x|^{N+1}} dt \right| \\ &\leq \frac{1}{2\pi} C_{\mathcal{A}'}(N!)^{s+1} A_{\mathcal{A}'}^N \frac{|x|^N (1+\delta)^N}{|x|^{N+1} \delta^{N+1}} 2\pi \delta |x| \\ &= C_{\mathcal{A}'}(N!)^{s+1} A_{\mathcal{A}'}'^N \quad \text{with} \quad A_{\mathcal{A}'}' = A_{\mathcal{A}'} \left(1 + \frac{1}{\delta}\right). \end{aligned}$$

Proposition 2.3.12 (Differential algebra and Taylor map)

The set $\bar{\mathcal{A}}_s(\mathcal{A})$ is a differential \mathbb{C} -algebra and the Taylor map $T_{\mathcal{A}}$ restricted to $\bar{\mathcal{A}}_s(\mathcal{A})$ induces a morphism of differential algebras

$$T = T_{s,\mathcal{A}} : \bar{\mathcal{A}}_s(\mathcal{A}) \longrightarrow \mathbb{C}[[x]]_s$$

with values in the algebra of s -Gevrey series.

Proof. — Let $\mathcal{A}' \in \mathcal{A}$. Suppose f and g belong to $\bar{\mathcal{A}}_s(\mathcal{A})$ and satisfy on \mathcal{A}'

$$\left| \frac{d^N f}{dx^N}(x) \right| \leq C(N!)^{s+1} A^N \quad \text{and} \quad \left| \frac{d^N g}{dx^N}(x) \right| \leq C'(N!)^{s+1} A'^N.$$

The product fg belongs to $\bar{\mathcal{A}}(\mathcal{A})$ (cf. Prop. 2.2.9) and its derivatives satisfy

$$\left| \frac{d^N(fg)}{dx^N}(x) \right| \leq \sum_{p=0}^N C_N^p \frac{d^p f}{dx^p}(x) \frac{d^{N-p} g}{dx^{N-p}}(x) \leq CC'(N!)^{s+1} (A + A')^N.$$

The fact that the range $T_{s,\mathcal{A}}(\bar{\mathcal{A}}_s(\mathcal{A}))$ be included in $\mathbb{C}[[x]]_s$ follows from Proposition 2.3.10. \square

Observe now the effect of a change of variable $x = t^r$, $r \in \mathbb{N}^*$. Clearly, if a series $\tilde{f}(x)$ is Gevrey of order s (level k) then the series $\tilde{f}(t^r)$ is Gevrey of order s/r (level kr). What about the asymptotics?

Let $\mathcal{A} =]\alpha, \beta[\times]0, R[$ be a sector in $\tilde{\mathcal{C}}$ (the directions α and β are not given modulo 2π) and let $\mathcal{A}_{/r} =]\alpha/r, \beta/r[\times]0, R^{1/r}[$ so that as the variable t runs over $\mathcal{A}_{/r}$ the variable $x = t^r$ runs over \mathcal{A} . From Definition 2.3.7 we can state:

Proposition 2.3.13 (Gevrey asymptotics in an extension of the variable)

The following two assertions are equivalent:

- (i) the function $f(x)$ is s -Gevrey asymptotic to the series $\tilde{f}(x)$ on \mathcal{A} ;
- (ii) the function $g(t) = f(t^r)$ is s/r -Gevrey asymptotic to $\tilde{g}(t) = \tilde{f}(t^r)$ on \mathcal{A}/r .

Way back, given an s' -Gevrey series $\tilde{g}(t)$, the series $\tilde{f}(x) = \tilde{g}(x^{1/r})$ exhibits, in general, fractional powers of x . To keep working with series of integer powers of x one may use rank reduction as follows [LR01]. One can uniquely decompose the series $\tilde{g}(t)$ as a sum

$$\tilde{g}(t) = \sum_{j=0}^{r-1} t^j \tilde{g}_j(t^r)$$

where the terms $\tilde{g}_j(t^r)$ are entire power series in t^r . Set $\omega = e^{2\pi i/r}$ and $x = t^r$. The series $\tilde{g}_j(x)$ are given, for $j = 0, \dots, r-1$, by the relations

$$rt^j \tilde{g}_j(t^r) = \sum_{\ell=0}^{r-1} \omega^{\ell(r-j)} \tilde{g}(\omega^\ell t).$$

For $j = 0, \dots, r-1$, let $\mathcal{A}_{/r}^j$ denote the sector

$$\mathcal{A}_{/r}^j =](\alpha + 2j\pi)/r, (\beta + 2j\pi)/r[\times]0, R^{1/r}[$$

so that as t runs through $\mathcal{A}/r = \mathcal{A}_{/r}^0$ then $\omega^j t$ runs through $\mathcal{A}_{/r}^j$ and $x = t^r$ runs through \mathcal{A} .

From the previous relations and Proposition 2.3.13 we can state:

Corollary 2.3.14 (Gevrey asymptotics and rank reduction)

The following two assertions are equivalent:

- (i) for $\ell = 0, \dots, r-1$ the series $\tilde{g}(t)$ is an s' -Gevrey asymptotic series on $\mathcal{A}_{\ell/r}$ (in the variable t);
- (ii) for $j = 0, \dots, r-1$ the r -rank reduced series $\tilde{g}_j(x)$ is an $s'r$ -Gevrey asymptotic series on \mathcal{A} (in the variable $x = t^r$).

With these results we might limit the study of Gevrey asymptotics to small values of s ($s \leq s_0$) or to large ones ($s \geq s_1$) at convenience.

2.3.3. Flat s -Gevrey asymptotic functions. — In this section we address the following question: to characterize the functions that are both s -Gevrey asymptotic and flat on a given sector \mathcal{A} . To this end, we introduce the notion of exponential flatness.

Definition 2.3.15. — A function f is said to be exponentially flat of order k (or k -exponentially flat) on a sector \mathcal{S} if, for any proper subsector $\mathcal{S}' \Subset \mathcal{S}$ of \mathcal{S} , there exist constants K and $A > 0$ such that the following estimate holds for all $x \in \mathcal{S}'$:

$$(14) \quad |f(x)| \leq K \exp\left(-\frac{A}{|x|^k}\right).$$

The constants K and A may depend on \mathcal{S}' .

Notation 2.3.16. — We denote the set of k -exponentially flat functions on \mathcal{S} by

$$\bar{\mathcal{A}}^{\leq -k}(\mathcal{S}).$$

Proposition 2.3.17. — Let \mathcal{S} be an open sector. The functions which are s -Gevrey asymptotically flat on \mathcal{S} are the k -exponentially flat functions, i.e.,

$$\bar{\mathcal{A}}_s(\mathcal{S}) \cap \bar{\mathcal{A}}^{<0}(\mathcal{S}) = \bar{\mathcal{A}}^{\leq -k}(\mathcal{S}) \quad (\text{recall } s = 1/k).$$

Proof. — \triangleright Let $f \in \bar{\mathcal{A}}_s(\mathcal{S}) \cap \bar{\mathcal{A}}^{<0}(\mathcal{S})$ and prove that $f \in \bar{\mathcal{A}}^{\leq -k}(\mathcal{S})$.

It is, here, more convenient to write Condition (12) in the following equivalent form: for all $\mathcal{S}' \Subset \mathcal{S}$, there exist $A > 0, C > 0$ such that the estimate

$$|f(x)| \leq CN^{N/k}(A|x|)^N = C \exp\left(\frac{N}{k} \ln(N(A|x|)^k)\right)$$

holds for all N and all $x \in \mathcal{S}'$ (with possibly new constants A and C).

For x fixed, look for a lower bound of the right hand side of this estimate as N runs over \mathbb{N} . The derivative $\varphi'(N) = \ln(N(A|x|)^k) + 1$ of the function

$$\varphi(N) = N \ln(N(A|x|)^k)$$

seen as a function of a real variable $N > 0$ vanishes at $N_0 = 1/(e(A|x|)^k)$ and φ reaches its minimal value $\varphi(N_0) = -N_0$ at that point. Taking into account the monotonicity of φ , for instance to the right of N_0 , we can assert that

$$\inf_{N \in \mathbb{N}} \varphi(N) \leq \varphi(N_0 + 1) = \varphi(N_0) \left(1 + \frac{1}{N_0} \left(1 - (1 + N_0) \ln\left(1 + \frac{1}{N_0}\right)\right)\right).$$

Substituting this value of N_0 as a function of x in φ , we can write $\varphi(N_0 + 1) = \varphi(N_0)\psi(x)$

where $\psi(x)$ is a bounded function on \mathcal{S} . Hence, there exists a constant $C' > 0$ such that $|f(x)| \leq C' \exp\left(-\frac{a}{|x|^k}\right)$ with $a = \frac{1}{keA^k} > 0$ independent of $x \in \mathcal{S}'$.

This proves that f belongs to $\bar{\mathcal{A}}^{\leq -k}(\mathcal{S})$.

▷ Let $f \in \bar{\mathcal{A}}^{\leq -k}(\mathcal{A})$ and prove that $f \in \bar{\mathcal{A}}_s(\mathcal{A}) \cap \bar{\mathcal{A}}^{<0}(\mathcal{A})$.

The hypothesis is now: for all $\mathcal{A}' \Subset \mathcal{A}$, there exist $A > 0, C > 0$ such that an estimate

$$|f(x)| \leq C \exp\left(-\frac{A}{|x|^k}\right)$$

holds for all $x \in \mathcal{A}'$. Hence, for any N , the estimate

$$|f(x)| \cdot |x|^{-N} \leq C \exp\left(-\frac{A}{|x|^k}\right) |x|^{-N}.$$

For N fixed, look for an upper bound of the right hand side of this estimate as $|x|$ runs over \mathbb{R}^+ . Let $\psi(|x|) = \exp\left(-\frac{A}{|x|^k}\right) |x|^{-N}$. Its logarithmic derivative

$$\frac{\psi'(|x|)}{\psi(|x|)} = -\frac{N}{|x|} + \frac{Ak}{|x|^{k+1}}$$

vanishes for $Ak/|x|^k = N$ and ψ reaches its maximum value at that point.

Thus, $\max_{|x|>0} \psi(|x|) = \exp\left(-\frac{N}{k}\right) \left(\frac{N}{Ak}\right)^{N/k}$ and there exists constants $a = (eAk)^{-1/k}$ and $C > 0$ such that, for all $N \in \mathbb{N}$ and $x \in \Sigma'$ the function f satisfies

$$|f(x)| \leq CN^{N/k} (a|x|)^N.$$

Hence, f belongs to $\bar{\mathcal{A}}_s(\mathcal{A}) \cap \bar{\mathcal{A}}^{<0}(\mathcal{A})$. □

2.4. The Borel-Ritt Theorem

With any asymptotic function $f \in \bar{\mathcal{A}}(\mathcal{A})$ over a sector \mathcal{A} the Taylor map $T_{\mathcal{A}}$ associates a formal series $\tilde{f} = T_{\mathcal{A}}(f)$. We address now the converse problem: is any formal series the Taylor series of an asymptotic function over a given sector \mathcal{A} ? The theorem below states that the answer is yes for any open sector \mathcal{A} with finite radius in \mathbb{C}^* or $\check{\mathbb{C}}$ in Poincaré asymptotics. In case the series is s -Gevrey an s -Gevrey asymptotic function always exists when the opening of the sector \mathcal{A} is small enough but we will see on examples that it might not exist for a too wide \mathcal{A} . Notice that the Taylor series of a function $f \in \bar{\mathcal{A}}(\mathbb{C}^*)$ is necessarily convergent by the removable singularity Theorem of Riemann. And thus, when \mathcal{A} is included in \mathbb{C}^* , it cannot be a full neighborhood of 0 in \mathbb{C}^* .

Theorem 2.4.1 (Borel-Ritt). — *Let $\mathcal{A} \neq \mathbb{C}^*$ be an open sector of \mathbb{C}^* or of the Riemann surface of logarithm $\check{\mathbb{C}}$ with finite radius R .*

(i) (Poincaré asymptotics) *The Taylor map $T_{\mathcal{A}} : \bar{\mathcal{A}}(\mathcal{A}) \rightarrow \mathbb{C}[[x]]$ is onto.*

(ii) (Gevrey asymptotics) *Suppose \mathcal{A} has opening $|\mathcal{A}| \leq \pi/k$. Then, the Taylor map $T_{s,\mathcal{A}} : \overline{\mathcal{A}}_s(\mathcal{A}) \rightarrow \mathbb{C}[[x]]_s$ is onto. Recall $s = 1/k$.*

Proof. — (i) *Poincaré asymptotics.* — Various proofs exist. The one presented here can be found in [Mal95]. For simplicity, begin with the case of a sector in \mathbb{C}^* .

▷ Case when \mathcal{A} lies in \mathbb{C}^* . Modulo rotation it is sufficient to consider the case when $\mathcal{A} = \mathcal{A}_{-\pi,+\pi}(R)$ is the disc of radius R slit on the real negative axis.

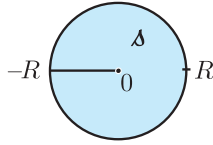


Figure 6

Given any series $\sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$ we look for a function $f \in \overline{\mathcal{A}}(\mathcal{A})$ with Taylor series $T_{\mathcal{A}} f = \sum_{n \geq 0} a_n x^n$. To this end, one introduces functions $\beta_n(x) \in \mathcal{O}(\mathcal{A})$ satisfying the two conditions

$$(1) : \sum_{n \geq 0} a_n \beta_n(x) x^n \in \mathcal{O}(\mathcal{A}) \quad \text{and} \quad (2) : T_{\mathcal{A}} \beta_n(x) \equiv 1 \quad \text{for all } n \geq 0.$$

Such functions exist: consider, for instance, the functions $\beta_0 \equiv 1$ and, for $n \geq 1$, $\beta_n(x) = 1 - \exp(-b_n/\sqrt{x})$ with positive b_n and \sqrt{x} the principal determination of the square root.

In view to Condition (1), observe that since $1 - e^z = -\int_0^z e^t dt$ then $|1 - e^z| < |z|$ for $\Re(z) < 0$. This implies $|\beta_n(x)| \leq b_n/\sqrt{|x|}$ for all $x \in \mathcal{A}$ and $n \geq 1$ and then,

$$|a_n \beta_n(x) x^n| \leq |a_n| b_n |x|^{n-1/2} \leq |a_n| b_n R^{n-1/2}.$$

Now, choose b_n such that the series $\sum_{n \geq 1} |a_n| b_n R^{n-1/2}$ be convergent. Then, the series $\sum_{n \geq 0} a_n \beta_n(x) x^n$ converges normally on \mathcal{A} and its sum $f(x) = \sum_{n \geq 0} a_n \beta_n(x) x^n$ is holomorphic on \mathcal{A} .

To prove Condition (2), consider any proper sub-sector $\mathcal{A}' \Subset \mathcal{A}$ of \mathcal{A} and $x \in \mathcal{A}'$. Then, for any $N \geq 1$, we can write

$$\left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq \left| \sum_{n=0}^{N-1} a_n (\beta_n(x) - 1) x^n \right| + |x|^N \sum_{n \geq N} |a_n \beta_n(x) x^{n-N}|.$$

The first summand is a finite sum of terms all asymptotic to 0 and then, is majorized by $C'|x|^N$, for a convenient positive constant C' . The second

summand is majorized by

$$|x|^N \left(2|a_N| + \sum_{n \geq N+1} |a_n| b_n R^{n-1/2-N} \right).$$

Choosing $C = C' + 2|a_N| + \sum_{n \geq N+1} |a_n| b_n R^{n-1/2-N}$ provides a positive constant C (independent of x but depending on N and \mathcal{A}') such that

$$\left| f(x) - \sum_{n=0}^N a_n x^n \right| \leq C|x|^N \quad \text{for all } x \in \mathcal{A}'.$$

This ends the proof in this case.

▷ General case when \mathcal{A} lies in $\tilde{\mathcal{C}}^*$. — It is again sufficient to consider the case of a sector of the form $\mathcal{A} = \{x \in \tilde{\mathcal{C}}^*; |\arg(x)| < k\pi, 0 < |x| < R\}$ where $k \in \mathbb{N}^*$. The same proof can be applied after replacing \sqrt{x} by a convenient power x^α of x so that $\Re(x^\alpha)$ is positive for all x in \mathcal{A} and taking $b_n \equiv 1$ for all $n \leq \alpha$.

(ii) *Gevrey asymptotics*

Let $\tilde{f}(x) \in \mathbb{C}[[x]]_s$ be an s -Gevrey series which, up to a polynomial, we may assume to be of the form $\tilde{f}(x) = \sum_{n \geq k} a_n x^n$. It is sufficient to consider a sector \mathcal{A} of opening π/k (as always, $k = 1/s$) and by means of a rotation, we can then assume that \mathcal{A} is an open sector bisected by the direction $\theta = 0$ with opening π/k ; we denote by R its radius. We must find a function $f \in \bar{\mathcal{A}}_s(\mathcal{A})$, s -Gevrey asymptotic to \tilde{f} over \mathcal{A} .

The proof used here is based on the Borel and the Laplace transforms which will be at the core of Borel-Laplace summation in Section 6.3.

Since $\tilde{f}(x)$ is an s -Gevrey series (cf. Def. 2.3.1) its k -Borel transform⁽¹⁾

$$\hat{f}(\xi) = \sum_{n \geq k} \frac{a_n}{\Gamma(n/k)} \xi^{n-k}$$

is a convergent series⁽²⁾ and we denote by $\varphi(\xi)$ its sum. The adequate Laplace transform to “invert” the k -Borel transform (as a function $\varphi(\xi)$, not as a series $\hat{f}(\xi)$) in the direction $\theta = 0$ would be the k -Laplace transform

$$\mathcal{L}_k(\varphi)(x) = \int_0^{+\infty} \phi(\zeta) e^{-\zeta/x^k} d\zeta$$

⁽¹⁾ See Sect. 6.3.1. The k -Borel transform of a series $\sum_{n \geq k} a_n x^n$ is the usual Borel transform of the series $\sum_{n \geq k} a_n X^{n/k}$ with respect to the variable $X = x^k$ and expressed in the variable $\xi = \zeta^s$.

⁽²⁾ Although, when k is not an integer, the series $\hat{f}(\xi)$ is not a series in integer powers of ξ it becomes so after factoring by ξ^{-k} . We mean here that the power series $\xi^k \hat{f}(\xi)$ is convergent.

where $\zeta = \xi^k$ and $\phi(\zeta) = \varphi(\zeta^{1/k})$. However, although the series $\widehat{f}(\xi)$ is convergent, its sum $\varphi(\xi)$ cannot be analytically continued along \mathbb{R}^+ up to infinity in general. So, we choose $b > 0$ belonging to the disc of convergence of $\widehat{f}(\xi)$ and we consider a truncated k -Laplace transform

$$(15) \quad f^b(x) = \int_0^{b^k} \phi(\zeta) e^{-\zeta/x^k} d\zeta$$

instead of the full Laplace transform $\mathcal{L}_k(\varphi)(x)$. Lemma 2.4.2 below shows that the function $f = f^b$ answers the question. \square

Lemma 2.4.2 (Truncated Laplace transform). — *With notations and conditions as above, and especially Δ being an open sector bisected by $\theta = 0$ with opening π/k , the truncated k -Laplace transform $f^b(x)$ of the sum $\varphi(\xi)$ of the k -Borel transform of $\widetilde{f}(x)$ in direction $\theta = 0$ is s -Gevrey asymptotic to $\widetilde{f}(x)$ on Δ (with $s = 1/k$ as usually).*

Proof. — Given $0 < \delta < \pi/2$ and $R' < R$, consider the proper sub-sector of Δ defined by $\Delta_\delta = \{x; |\arg(x)| < \pi/(2k) - \delta/k \text{ and } |x| < R'\}$. For $x \in \Delta_\delta$ we can write

$$\begin{aligned} f^b(x) - \sum_{n=k}^{N-1} a_n x^n &= \int_0^{b^k} \sum_{n \geq k} \frac{a_n}{\Gamma(n/k)} \zeta^{(n/k)-1} e^{-\zeta/x^k} d\zeta \\ &\quad - \sum_{n=k}^{N-1} \frac{a_n}{\Gamma(n/k)} \int_0^{+\infty} \zeta^{(n/k)-1} e^{-\zeta/x^k} d\zeta. \end{aligned}$$

Since $\Re(x^k) > 0$ then, $|\zeta^{(n/k)-1} e^{-\zeta/x^k}| \leq b^{n-k}$ for all $\zeta \in [0, b^k]$. Consequently, the series $\sum_{n \geq k} \frac{a_n}{\Gamma(n/k)} \zeta^{(n/k)-1} e^{-\zeta/x^k}$ converges normally on $[0, b^k]$ and we can permute sum and integral. Hence,

$$\begin{aligned} f^b(x) - \sum_{n=k}^{N-1} a_n x^n &= \sum_{n \geq N} \frac{a_n}{\Gamma(n/k)} \int_0^{b^k} \zeta^{(n/k)-1} e^{-\zeta/x^k} d\zeta \\ &\quad - \sum_{n=k}^{N-1} \frac{a_n}{\Gamma(n/k)} \int_{b^k}^{+\infty} \zeta^{(n/k)-1} e^{-\zeta/x^k} d\zeta. \end{aligned}$$

However, $|\zeta/b^k|^{(n/k)-1} \leq |\zeta/b^k|^{(N/k)-1}$ both when $|\zeta| \leq b^k$ and $n \geq N$ and when $|\zeta| \geq b^k$ and $n < N$ and then,

$$\begin{aligned}
\left| f^b(x) - \sum_{n=k}^{N-1} a_n x^n \right| &\leq \sum_{n \geq N} \frac{|a_n|}{\Gamma(n/k)} \int_0^{b^k} b^{n-N} |\zeta|^{(N/k)-1} e^{-\zeta \Re(1/x^k)} d\zeta \\
&\quad + \sum_{n=k}^{N-1} \frac{|a_n|}{\Gamma(n/k)} \int_{b^k}^{+\infty} \text{idem} \\
&\leq \sum_{n \geq k} \frac{|a_n|}{\Gamma(n/k)} b^{n-N} \int_0^{+\infty} |\zeta|^{(N/k)-1} e^{-\zeta \sin(\delta)/|x|^k} d\zeta \\
&= \sum_{n \geq k} \frac{|a_n|}{\Gamma(n/k)} b^{n-N} \frac{|x|^N}{(\sin \delta)^{N/k}} \int_0^{+\infty} u^{(N/k)-1} e^{-u} du \\
&= \sum_{n \geq k} \frac{|a_n|}{\Gamma(n/k)} b^{n-N} \frac{|x|^N}{(\sin \delta)^{N/k}} \Gamma(N/k) = C \Gamma(N/k) A^N |x|^N
\end{aligned}$$

where $A = \frac{1}{b(\sin \delta)^{1/k}}$ and $C = \sum_{n \geq k} \frac{|a_n|}{\Gamma(n/k)} b^n < +\infty$. The constants A and C depend on δ and on the choice of b but are independent of x . This achieves the proof. \square

Comment 2.4.3 (On the Euler series (Exa. 2.2.4))

The proof of the Borel-Ritt Theorem provides infinitely many functions asymptotic to the Euler series $\tilde{E}(x) = \sum_{n \geq 0} (-1)^n n! x^{n+1}$ at 0 on the sector $\mathcal{A} = \{x; |\arg(x)| < 3\pi/2\}$. For instance, the following family provides infinitely many such functions:

$$F_a(x) = \sum_{n \geq 0} (-1)^n n! \left(1 - e^{-a/((n!)^2 x^{1/3})}\right) x^{n+1}, \quad a > 0.$$

We saw in Example 2.2.4 that the Euler function $E(x) = \int_0^{+\infty} \frac{e^{-\xi/x}}{1+\xi} d\xi$ is both solution of the Euler equation and asymptotic to the Euler series on \mathcal{A} . We claim that it is the unique function with these properties. Indeed, suppose E_1 be another such function. Then, the difference $E(x) - E_1(x)$ would be both asymptotic to the null series 0 on \mathcal{A} and solution of the homogeneous associated equation $x^2 y' + y = 0$. However, the equation $x^2 y' + y = 0$ admits no such solution on \mathcal{A} but 0. Hence, $E = E_1$ and the infinitely many functions given by the proof of the Borel-Ritt Theorem do not satisfy the Euler equation in general.

Taking into account Props. 2.2.9, 2.3.12 and 2.3.17 we can reformulate the Borel-Ritt Theorem 2.4.1 as follows.

Corollary 2.4.4. — *The set $\bar{\mathcal{A}}^{<0}(\mathcal{A})$ of flat functions on \mathcal{A} and the set $\bar{\mathcal{A}}^{\leq -k}(\mathcal{A})$ of k -exponentially flat functions on \mathcal{A} are differential ideals of $\bar{\mathcal{A}}(\mathcal{A})$*

and $\bar{\mathcal{A}}_s(\mathcal{A})$ respectively. The sequences

$$0 \rightarrow \bar{\mathcal{A}}^{<0}(\mathcal{A}) \rightarrow \bar{\mathcal{A}}(\mathcal{A}) \xrightarrow{T_{\mathcal{A}}} \mathbb{C}[[x]] \rightarrow 0$$

and, when $|\mathcal{A}| \leq \pi/k$,

$$0 \rightarrow \bar{\mathcal{A}}^{\leq -k}(\mathcal{A}) \rightarrow \bar{\mathcal{A}}_s(\mathcal{A}) \xrightarrow{T_{s,\mathcal{A}}} \mathbb{C}[[x]]_s \rightarrow 0$$

are exact sequences of morphisms of differential algebras.

The Borel-Ritt Theorem implies the classical Borel Theorem in the real case providing thus a new proof of it.

Corollary 2.4.5 (Classical Borel Theorem). —

Any formal power series $\sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$ is the Taylor series at 0 of a C^∞ -function of a real variable x .

Proof. — Apply the Borel-Ritt Theorem on a sector \mathcal{A}' containing \mathbb{R}^+ and on a sector \mathcal{A}'' containing \mathbb{R}^- . The two functions so obtained glue together at 0 into a C^∞ -function in a neighborhood of 0 in \mathbb{R} . \square

2.5. The Cauchy-Heine Theorem

In this section we are given:

- ▷ a sector $\dot{\mathcal{A}} = \mathcal{A}_{\alpha,\beta}(R)$ with vertex 0 in \mathbb{C}^* ;
- ▷ a point x_0 in $\dot{\mathcal{A}}$ and the straight path $\gamma =]0, x_0]$ in $\dot{\mathcal{A}}$;
- ▷ a function $\varphi \in \bar{\mathcal{A}}^{<0}(\dot{\mathcal{A}})$ flat at 0 on $\dot{\mathcal{A}}$.

Definition 2.5.1. — One defines the Cauchy-Heine integral associated with φ and x_0 , to be the function

$$f(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(t)}{t-x} dt.$$

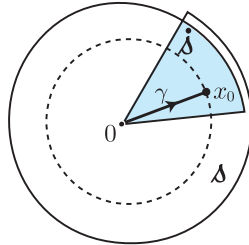


Figure 7

Denote by:

▷ $\mathcal{A} = \mathcal{A}_{\alpha, \beta + 2\pi}(R)$ a sector with vertex 0 in the Riemann surface of logarithm overlapping on \mathcal{A} ;

▷ θ_0 the argument of x_0 satisfying $\alpha < \theta_0 < \beta$;

▷ $\mathcal{D}_\gamma = \mathcal{A}_{\theta_0, \theta_0 + 2\pi}(|x_0|)$ the disc of radius $|x_0|$ slit along γ ;

▷ $\mathcal{A}' = \mathcal{A} \cap \{|x| < |x_0|\} = \mathcal{A}_{\alpha, \beta}(|x_0|)$;

▷ $\mathcal{A}'' = \mathcal{A} \cap \{|x| < |x_0|\} = \mathcal{A}_{\alpha, \beta + 2\pi}(|x_0|)$.

The Cauchy-Heine integral determines a well-defined and analytic function f on \mathcal{D}_γ . By Cauchy's Theorem, Cauchy-Heine integrals associated with different points x_0 and x_1 in \mathcal{A} differ by $\frac{1}{2\pi i} \int_{x_0 x_1} \frac{\varphi(t)}{t-x} dt$, an analytic function on a neighborhood of 0.

Theorem 2.5.2 (Cauchy-Heine). — *With notations and conditions as before and especially, φ flat on \mathcal{A} , the Cauchy-Heine integral $f(x) = \frac{1}{2\pi i} \int_\gamma \frac{\varphi(t)}{t-x} dt$ has the following properties:*

1. *The function f can be analytically continued from \mathcal{D}_γ to \mathcal{A}' ; we also use the term Cauchy-Heine integral when referring to this analytic continuation which we keep denoting by f .*

2. *The function f belongs to $\bar{\mathcal{A}}(\mathcal{A}')$.*

3. *Its Taylor series at 0 on \mathcal{A}' reads*

$$T_{\mathcal{A}'} f(x) = \sum_{n \geq 0} a_n x^n \quad \text{with} \quad a_n = \frac{1}{2\pi i} \int_\gamma \frac{\varphi(t)}{t^{n+1}} dt.$$

4. *Its variation $\text{var} f(x) = f(x) - f(xe^{2\pi i})$ is equal to $\varphi(x)$ for all $x \in \mathcal{A}'$.*

5. *If, in addition, φ belongs to $\bar{\mathcal{A}}^{\leq -k}(\mathcal{A})$ then, f belongs to $\bar{\mathcal{A}}_s(\mathcal{A}')$ with the above Taylor series, i.e., if φ is k -exponentially flat on \mathcal{A} then, f is s -Gevrey asymptotic to the above series $\sum_{n \geq 0} a_n x^n$ on \mathcal{A}' (recall $s = 1/k$).*

Proof. — The five steps can be proved as follows.

1. — Consider, for instance, the function f for values of x on the left of γ . To analytically continue this “branch” of the function f to the right of γ it suffices to deform the path γ by pushing it to the right keeping its endpoints 0 and x_0 fixed. This allows us to go up to the boundary $\arg(x) = \alpha$ of \mathcal{A}' . We can similarly continue the “branch” of the function f defined for values of x on the right of γ up to the boundary $\arg(x) = \beta + 2\pi$ of \mathcal{A}' .

2–3. — We have to prove that, for all subsector $\mathcal{A}'' \Subset \mathcal{A}'$, the function f satisfies the asymptotic estimates of Definition 2.2.1.

▷ Suppose first that $\overline{\mathcal{A}''} \cap \gamma = \emptyset$. Writing

$$\frac{1}{t-x} = \sum_{n=0}^{N-1} \frac{x^n}{t^{n+1}} + \frac{x^N}{t^N(t-x)}$$

as in Example 2.2.4, we get

$$f(x) = \sum_{n=0}^{N-1} a_n x^n + \frac{x^N}{2\pi i} \int_{\gamma} \frac{\varphi(t)}{t^N(t-x)} dt.$$

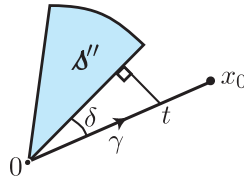


Figure 8

Given $x \in \mathcal{A}''$, then $|t-x| \geq \text{dist}(t, \mathcal{A}'') = |t| \sin(\delta)$ for all $t \in \gamma$ and so

$$(16) \quad \left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq C|x|^N$$

where the constant $C = \frac{1}{2\pi} \left| \int_{\gamma} \frac{|\varphi(t)|}{|t|^{N+1} \sin(\delta)} dt \right|$ is finite (the integral converges since φ is flat at 0 on γ) and depends on N and \mathcal{A}'' , but is independent of $x \in \mathcal{A}''$.

▷ Suppose now that $\overline{\mathcal{A}''} \cap \gamma \neq \emptyset$. Push homotopically γ into a path made of the union of a segment $\gamma_1 =]0, x_1]$ and a curve γ_2 , say a circular arc, joining x_1 to x_0 without meeting $\overline{\mathcal{A}''}$ as shown on the figure. The integral splits into two parts $f_1(x)$ and $f_2(x)$.

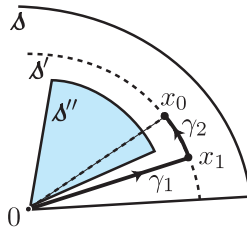


Figure 9

The term $f_1(x)$ belongs to the previous case and is then asymptotic to

$$\sum_{n \geq 0} \frac{1}{2\pi i} \int_{\gamma_1} \frac{\varphi(t)}{t^{n+1}} dt x^n \quad \text{on } \mathcal{S}''.$$

The term $f_2(x)$ defines an analytic function on the disc $|x| < |x_0|$ and is asymptotic to its Taylor series $\sum_{n \geq 0} \frac{1}{2\pi i} \int_{\gamma_2} \frac{\varphi(t)}{t^{n+1}} dt x^n$. Hence, the result.

4. — Given $x \in \mathcal{S}'$ compute the variation of f at x . Recall that $x \in \mathcal{S}'$ means that x belongs to the first sheet of \mathcal{S}' . So, as explained in the proof of point 1, to evaluate $f(x)$ we might have to push homotopically the path γ to the right into a path γ' . When x lies to the left of γ we can keep $\gamma' = \gamma$. To evaluate $f(xe^{2\pi i})$ we might have to push homotopically the path γ to the left into a path γ'' taking $\gamma'' = \gamma$ when x lies to the right of γ .

The concatenation of γ' and $-\gamma''$ generates a path Γ in \mathcal{S} enclosing x and since the function $\varphi(t)/(t-x)$ is meromorphic on \mathcal{S} we obtain by the Cauchy's Residue Theorem:

$$(\mathfrak{h}\bar{\Gamma})f(x) = f(x) - f(xe^{2\pi i}) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(t)}{t-x} dt = \text{Res} \left(\frac{\varphi(t)}{t-x}, t=x \right) = \varphi(x)$$

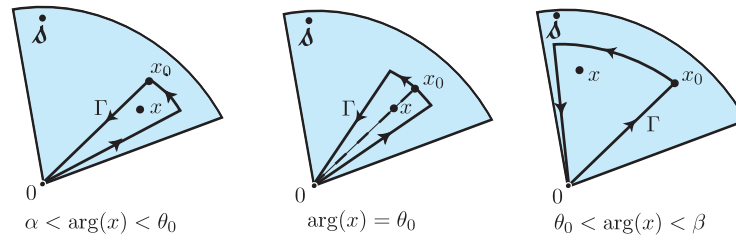


Figure 10

5. — Given $\mathcal{S}'' \Subset \mathcal{S}'$ suppose that the function φ satisfies

$$|\varphi(x)| \leq K \exp(-A/|x|^k) \quad \text{on } \mathcal{S}''.$$

Consider the case when $\overline{\mathcal{S}''} \cap \gamma = \emptyset$. Then, the constant C in estimate (16) satisfies

$$C \leq \frac{K}{2\pi} \left| \int_{\gamma} \frac{\exp(-A/|t|^k)}{|t|^{N+1}} dt \right| \leq C' A^{-N/k} \Gamma(N/k)$$

with a constant $C' > 0$ independent of N . The case when $\overline{\mathcal{S}''} \cap \gamma \neq \emptyset$ is treated similarly by deforming the path γ as in points 2–3. Hence, $f(x)$ is s -Gevrey asymptotic to the series $\sum_{n \geq 0} a_n x^n$ on \mathcal{S}' . \square

Comments 2.5.3 (On the Euler function (Exa. 2.2.4))

Set $\dot{\mathcal{A}} = \mathcal{A}_{\alpha, \beta}(\infty)$ with $\alpha = -3\pi/2$ and $\beta = -\pi/2$ and $\mathcal{A} = \mathcal{A}_{\alpha, \beta + 2\pi}(\infty)$. Let $E(x)$ denote the Euler function as in Example 2.2.4 and, given θ , let d_θ denote the half line issuing from 0 with direction θ .

▷ The variation of the Euler function E on $\dot{\mathcal{A}}$ is given by

$$\begin{aligned} \text{var } E(x) &= \int_{d_{-\pi+\varepsilon}} \frac{e^{-\xi/x}}{1+\xi} d\xi - \int_{d_{\pi-\varepsilon}} \frac{e^{-\xi/x}}{1+\xi} d\xi \quad (\varepsilon \text{ small enough}) \\ &= -2\pi i \operatorname{Res}\left(\frac{e^{-\xi/x}}{1+\xi}, \xi = -1\right) \quad (\text{Cauchy's Residue Thm.}) \\ &= -2\pi i e^{1/x}. \end{aligned}$$

▷ Apply the Cauchy-Heine Theorem by choosing the 1-exponentially flat function $\varphi(x) = -2\pi i e^{1/x}$ on $\dot{\mathcal{A}}$ and a point $x_0 \in \dot{\mathcal{A}}$, for instance $x_0 = -r$ real negative. Denote $\mathcal{A}' = \mathcal{A} \cap \{|x| < |x_0|\}$ and $\dot{\mathcal{A}}' = \dot{\mathcal{A}} \cap \{|x| < |x_0|\}$. The Cauchy-Heine Theorem provides a function f which, as the Euler function, belongs to $\bar{\mathcal{A}}_1(\mathcal{A}')$ with variation $\varphi(x)$ on $\dot{\mathcal{A}}'$.

We claim that E and f differ by an analytic function near 0. Indeed, the Taylor series of f on \mathcal{A}' reads $\sum_{n \geq 0} a_{n, x_0} x^n$ with coefficients

$$a_{n, x_0} = - \int_0^{x_0} \frac{e^{1/t}}{t^{n+1}} dt = (-1)^{n-1} \int_{1/r}^{+\infty} u^{n-1} e^{-u} du$$

while the Taylor coefficients a_n of the Euler function E are given by $a_0 = 0$ and for $n \geq 1$ by $a_n = \lim_{r \rightarrow +\infty} a_{n, x_0}$. Since a_{0, x_0} has no limit as r tends to $+\infty$ we consider, instead of f , the function

$$f(x) - a_{0, x_0} = - \int_0^{-r} \left(\frac{1}{t-x} - \frac{1}{t} \right) e^{1/t} dt = \int_{1/r}^{+\infty} \frac{x e^{-u}}{1+ux} du.$$

Suppose $x = |x| e^{i\theta}$. Then, the Euler function at x can be defined by the integral

$$E(x) = \int_{d_\theta} \frac{e^{-\xi/x}}{1+\xi} d\xi = \int_0^{+\infty} \frac{x e^{-u}}{1+ux} du$$

and

$$E(x) - f(x) = -a_{0, x_0} + x \int_0^{1/r} \frac{e^{-u}}{1+ux} du$$

which is an analytic function on the disc $|x| < r = |x_0|$.

This property will follow from a general argument of uniqueness given by Watson's Lemma 6.1.3. Indeed, the function $f(x) - a_{0, x_0} + x \int_0^{1/r} \frac{e^{-u}}{1+ux} du$ has the Euler series as asymptotic expansion and is 1-Gevrey asymptotic to the Euler series $\tilde{E}(x)$ on \mathcal{A}' . Then, it must be equal to the Euler function $E(x)$ which has the same properties.

▷ *Stokes phenomenon.* — The Euler function $E(x)$ is also solution of the homogeneous linear differential equation

$$\mathcal{E}_0(y) \equiv x^3 \frac{d^2 y}{dx^2} + (x^2 + x) \frac{dy}{dx} - y = 0$$

deduced from the Euler equation (1) by dividing it by x and then, differentiating once. Since the equation has no singular point but 0 (and infinity) the Cauchy-Lipschitz Theorem allows one to analytically continue the Euler function along any path which avoids 0 and then in particular, outside of the sector $-3\pi/2 < \arg(x) < +3\pi/2$. However, when crossing the lateral boundaries of this sector the Euler function $E(x)$ stops being asymptotic to the Euler series at 0; it even stops having an asymptotic expansion since, from the variation formula above (*cf.* also the end of Exa. 2.2.4), one has now to take into account an exponential term which is unbounded. This phenomenon is known under the name of *Stokes phenomenon*. It is at the core of the meromorphic classification of linear differential equations (*cf.* Sect. 4.3).

Exercise 2.5.4. — Study the asymptotics at 0 of the function

$$F(x) = \int_0^{+\infty} \frac{e^{-\xi/x}}{\xi^2 + 3\xi + 2} d\xi.$$

and its analytic continuation. Compute its variation.

CHAPTER 3

SHEAVES AND ČECH COHOMOLOGY WITH AN INSIGHT INTO ASYMPTOTICS

In this chapter, we recall some definitions and results used later and some examples, about sheaves and Čech cohomology. For more precisions we refer to the classical literature (*cf.* [God58], [Ten75], [Ive86] for instance).

3.1. Presheaves and sheaves

Sheaves are the adequate tool to handle objects defined by local conditions without having to make explicit how large is the domain of validity of the conditions. They are mainly used as a bridge from local to global properties. It is convenient to start with the weaker concept of presheaves which we usually denote with an overline.

3.1.1. Presheaves. — Let us start with the definition of presheaves with values in the category of sets and continue with the case of various subcategories (for the definition of a category, see for instance [God58, Sect. 1.7]).

Definition 3.1.1 (Presheaf). — A presheaf (of sets) $\overline{\mathcal{F}}$ over a topological space X called the base space is defined by the following data:

(i) to any open set U of X there is a set $\overline{\mathcal{F}}(U)$ whose elements are called sections of $\overline{\mathcal{F}}$ on U ;

(ii) to any couple of open sets $V \subseteq U$ there is a map $\rho_{V,U} : \overline{\mathcal{F}}(U) \rightarrow \overline{\mathcal{F}}(V)$ called restriction map satisfying the two conditions:

$$\triangleright \rho_{U,U} = \text{id}_U \text{ for all } U,$$

$$\triangleright \rho_{W,V} \circ \rho_{V,U} = \rho_{W,U} \text{ for all open sets } W \subseteq V \subseteq U.$$

In the language of categories, a presheaf of sets over X is then a *contravariant functor* from the category of open subsets of X into the category of sets.

Unless otherwise specified, we assume that X does not reduce to one element.

The names “section” and “restriction map” take their origin in Example 3.1.2 below which, with the notion of *espace étalé* (cf. Def. 3.1.10), will become a reference example.

Example 3.1.2 (A fundamental example). — Let F be a topological space and $\pi : F \rightarrow X$ a continuous map. A presheaf $\bar{\mathcal{F}}$ is associated with F and π as follows: for all open set U in X one defines $\bar{\mathcal{F}}(U)$ as the set of *sections* of π on U , i.e., continuous maps $s : U \rightarrow F$ such that $\pi \circ s = \text{id}_U$. The restriction maps $\rho_{V,U}$ for $V \subseteq U$ are defined by $\rho_{V,U}(s) = s|_V$.

Example 3.1.3 (Constant presheaf). — Given any set (or group, vector space, etc...) C , the constant presheaf $\bar{\mathcal{C}}_X$ over X is defined by $\bar{\mathcal{C}}_X(U) = C$ for all open set U in X and the maps $\rho_{V,U} = \text{id}_C : C \rightarrow C$ as restriction maps.

Example 3.1.4 (An exotic example). — Given any marked set with more than one element, say $(X = \mathbb{C}, 0)$, one defines a presheaf $\bar{\mathcal{G}}$ over X as follows: $\bar{\mathcal{G}}(X) = X$ and $\bar{\mathcal{G}}(U) = \{0\}$ when $U \neq X$; all the restriction maps are equal to the null maps except $\rho_{X,X}$ which is the identity on X .

Below, we consider presheaves with values in a category \mathcal{C} equipped with an algebraic structure. We assume moreover that, in \mathcal{C} , there exist products, the terminal objects are the singletons, the isomorphisms are the bijective morphisms. The same conditions will apply to the sheaves we consider later on.

Definition 3.1.5. — A presheaf over X with values in a category \mathcal{C} is a presheaf of sets satisfying the following two conditions:

- (iii) For all open set U of X the set $\bar{\mathcal{F}}(U)$ is an object of the category \mathcal{C} ;
- (iv) For any couple of open sets $V \subseteq U$ the map $\rho_{V,U}$ is a morphism in \mathcal{C} .

In the next chapters, we will mostly be dealing with presheaves or sheaves of modules, in particular, of Abelian groups or vector spaces, and presheaves or sheaves of differential \mathbb{C} -algebras, i.e., presheaves or sheaves with values in a category of modules, Abelian groups, or vector spaces and presheaves or sheaves with values in the category of differential \mathbb{C} -algebras.

Definition 3.1.6 (Morphism of presheaves). — Given $\bar{\mathcal{F}}$ and $\bar{\mathcal{G}}$ two presheaves over X with values in a category \mathcal{C} , a morphism $f : \bar{\mathcal{F}} \rightarrow \bar{\mathcal{G}}$ is a collection, for all open sets U of X , of morphisms

$$f(U) : \bar{\mathcal{F}}(U) \longrightarrow \bar{\mathcal{G}}(U)$$

in the category \mathcal{C} which are compatible with the restriction maps, i.e., such that the diagrams

$$\begin{array}{ccc} \bar{\mathcal{F}}(U) & \xrightarrow{f(U)} & \bar{\mathcal{G}}(U) \\ \rho_{V,U} \downarrow & & \downarrow \rho'_{V,U} \\ \bar{\mathcal{F}}(V) & \xrightarrow{f(V)} & \bar{\mathcal{G}}(V) \end{array}$$

commute ($\rho_{V,U}$ and $\rho'_{V,U}$ denote the restriction maps in $\bar{\mathcal{F}}$ and $\bar{\mathcal{G}}$ respectively).

Definition 3.1.7. — A morphism f of presheaves is said to be injective or surjective when all morphisms $f(U)$ are injective or surjective.

The morphisms of presheaves from $\bar{\mathcal{F}}$ into $\bar{\mathcal{G}}$ form a set, precisely, they form a subset of $\prod_{U \subseteq X} \text{Hom}(\bar{\mathcal{F}}(U), \bar{\mathcal{G}}(U))$. Composition of morphisms in the category \mathcal{C} induces composition of morphisms of presheaves over X with values in \mathcal{C} . It follows that presheaves over X with values in \mathcal{C} form themselves a category.

When \mathcal{C} is Abelian, the category of presheaves over X with values in \mathcal{C} is also Abelian. In particular, one can talk of an *exact sequence of presheaves*

$$\cdots \rightarrow \bar{\mathcal{F}}_{j-1} \xrightarrow{f_j} \bar{\mathcal{F}}_j \xrightarrow{f_{j+1}} \bar{\mathcal{F}}_{j+1} \rightarrow \cdots$$

which means that the following sequence is exact for all open set U :

$$\cdots \rightarrow \bar{\mathcal{F}}_{j-1}(U) \xrightarrow{f_j(U)} \bar{\mathcal{F}}_j(U) \xrightarrow{f_{j+1}(U)} \bar{\mathcal{F}}_{j+1}(U) \rightarrow \cdots$$

The category of modules over a given ring, hence also the category of Abelian groups and the category of vector spaces, are Abelian. They admit the trivial module $\{0\}$ as terminal object.

The category of rings, and in particular, the category of differential \mathbb{C} -algebras, is not Abelian. Although the quotient of a ring \mathcal{A} by a subring \mathcal{J} is not a ring in general, this becomes true when \mathcal{J} is an ideal and allows one to consider short exact sequences $0 \rightarrow \bar{\mathcal{J}} \rightarrow \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}}/\bar{\mathcal{J}} \rightarrow 0$ of presheaves of rings or of differential \mathbb{C} -algebras.

Definition 3.1.8 (Stalk). — Given a presheaf $\bar{\mathcal{F}}$ over X and $x \in X$, the stalk of $\bar{\mathcal{F}}$ at x is the direct limit

$$\bar{\mathcal{F}}_x = \varinjlim_{U \ni x} \bar{\mathcal{F}}(U),$$

the limit being taken on the filtrant set of the open neighborhoods of x in X ordered by inclusion. The elements of $\bar{\mathcal{F}}_x$ are called germs of sections of $\bar{\mathcal{F}}$ at x .

Let us first recall what is understood by the terms *direct limit* and *filtrant*.

▷ The *direct limit*

$$E = \varinjlim_{\alpha \in I} (E_\alpha, f_{\beta, \alpha})$$

of a direct family $(E_\alpha, f_{\beta, \alpha} : E_\alpha \rightarrow E_\beta \text{ for } \alpha \leq \beta)$ (i.e., it is required that the set of indices I be ordered and right filtrant which means that given $\alpha, \beta \in I$ there exists $\gamma \in I$ greater than both α and β ; moreover, the morphisms must satisfy $f_{\alpha, \alpha} = \text{id}_\alpha$ and $f_{\gamma, \beta} \circ f_{\beta, \alpha} = f_{\gamma, \alpha}$ for all $\alpha \leq \beta \leq \gamma$) is the quotient of the sum $F = \bigsqcup_{\alpha \in I} E_\alpha$ of the spaces E_α by the equivalence relation \mathcal{R} : for $x \in E_\alpha$ and $y \in E_\beta$, one says that

$$x \mathcal{R} y \text{ if there exists } \gamma \text{ such that } \gamma \geq \alpha, \gamma \geq \beta \text{ and } f_{\gamma, \alpha}(x) = f_{\gamma, \beta}(y).$$

In the case of a stalk here considered, the maps $f_{\beta, \alpha}$ are the restriction maps $\rho_{V, U}$.

▷ *Filtrant* means here that, given any two neighborhoods of x , there exists a neighborhood smaller than both of them. Their intersection, for example, provides such a smaller neighborhood.

Thus, a germ φ at x is an equivalence class of sections under the equivalence relation: given two open sets U and V of X containing x , two sections $s \in \bar{\mathcal{F}}(U)$ and $t \in \bar{\mathcal{F}}(V)$ are equivalent if and only if there is an open set $W \subseteq U \cap V$ containing x such that $\rho_{W, U}(s) = \rho_{W, V}(t)$.

By abuse and for simplicity, we allow us to say “the germ φ at x ” when φ is an element of $\bar{\mathcal{F}}(U)$ with $U \ni x$ identifying so the element φ in the equivalence class to the equivalence class itself.

Given $s \in \bar{\mathcal{F}}(U)$ and $t \in \bar{\mathcal{F}}(V)$ one should be aware of the fact that the equality of the germs $s_x = t_x$ for all $x \in U \cap V$ does not imply the equality of the sections themselves on $U \cap V$.

A counter-example is given by taking the sections $s \equiv 0$ and $t \equiv 1$ whose germs are everywhere 0 in Example 3.1.4.

Also, it is worth to notice that a consistent collection of germs for all $x \in U$ does not imply the existence of a section $s \in \overline{\mathcal{F}}(U)$ inducing the given germs at each $x \in U$. *Consistent* means here that any section $v \in \overline{\mathcal{F}}(V)$ representing a given germ at x induces the neighboring germs: there exists an open sub-neighborhood $V' \subseteq V \subseteq U$ of x where the given germs are all represented by v .

A counter-example is given by the constant presheaf $\overline{\mathcal{C}}_X$ when X is disconnected. Consider, for instance, $X = \mathbb{R}^*$, $C = \mathbb{R}$ and the collection of germs $s_x = 0$ for $x < 0$ and $s_x = 1$ for $x > 0$. The presheaf $\overline{\mathcal{A}}$ defined in Section 3.1.5 will provide another example.

Such inconveniences are circumvented by restricting the notion of presheaf to the stronger notion of sheaf given just below.

3.1.2. Sheaves. —

Definition 3.1.9 (Sheaf). — *A presheaf $\overline{\mathcal{F}}$ over X is a sheaf (we denote it then by \mathcal{F}) if, for all open set U of X , the following two properties hold:*

1. *If two sections s and σ of $\mathcal{F}(U)$ agree on an open covering $\mathcal{U} = \{U_j\}_{j \in J}$ of U (i.e., if they satisfy $\rho_{U_j, U}(s) = \rho_{U_j, U}(\sigma)$ for all j) then $s = \sigma$.*
2. *Given any consistent family of sections $s_j \in \mathcal{F}(U_j)$ on an open covering $\mathcal{U} = \{U_j\}_{j \in J}$ of U there exists a section $s \in \mathcal{F}(U)$ gluing all the s_j 's (i.e., such that for all j , $\rho_{U_j, U}(s) = s_j$).*

Consistent means here that, for all i, j , the restrictions of s_i and s_j agree on $U_i \cap U_j$, i.e., $\rho_{U_i \cap U_j, U_i}(s_i) = \rho_{U_i \cap U_j, U_j}(s_j)$.

The presheaf $\overline{\mathcal{F}}$ of Example 3.1.2 is a sheaf. In Example 3.1.4 Condition 1 fails. In the case of the constant presheaf over a disconnected base space X (cf. Exa. 3.1.3) and in the case of the presheaf $\overline{\mathcal{A}}$ in the next section Condition 2 fails.

It follows from the axioms of sheaves that $\mathcal{F}(\emptyset)$ is a terminal object. Thus, $\mathcal{F}(\emptyset) = \{0\}$ when \mathcal{F} is a sheaf of modules, Abelian groups and vector spaces or of differential \mathbb{C} -algebras.

When \mathcal{F} is a sheaf of modules the restriction maps are linear and Condition 1 reduces to: *a section which is zero in restriction to a covering $\mathcal{U} = \{U_j\}$ is the null section.*

3.1.3. From presheaves to sheaves: espaces étalés. — With any presheaf $\bar{\mathcal{F}}$ there is a sheaf \mathcal{F} canonically associated as follows. Consider the space $F = \bigsqcup_{x \in X} \bar{\mathcal{F}}_x$ (disjoint union of the stalks of $\bar{\mathcal{F}}$) and endow it with the following topology: a set $\Omega \subseteq F$ is open in F if, for all open set U of X and all section $s \in \bar{\mathcal{F}}(U)$, the set of all elements $x \in U$ such that the germ s_x of s at x belong to Ω is open in X .

Given $s \in \bar{\mathcal{F}}(U)$ where U is an open subset of X , consider the map $\tilde{s} : U \rightarrow F$ defined by $\tilde{s}(x) = s_x$. Denote by π the projection map $\pi : F \rightarrow X$, $s_x \mapsto \pi(s_x) = x$. The topology on F is the less fine for which \tilde{s} is continuous for all U and s , and the topology induced on the stalks $\bar{\mathcal{F}}_x = \pi^{-1}(x)$ is the discrete topology. The sets $\tilde{s}(U)$ are open in F and the maps \tilde{s} satisfy $\pi(\tilde{s}(x)) = x$ for all $x \in U$. It follows that π is a local homeomorphism.

Definition 3.1.10 (Espace étalé, associated sheaf)

▷ The topological space F is called the espace étalé over X associated with $\bar{\mathcal{F}}$.

▷ The sheaf \mathcal{F} associated with the presheaf $\bar{\mathcal{F}}$ is the sheaf of continuous sections of $\pi : F \rightarrow X$ as defined in Example 3.1.2.

Example 3.1.11 (Constant sheaf). — The espace étalé associated with the constant presheaf $\bar{\mathcal{C}}_X$ in Example 3.1.3 is the topological space $X \times C$ endowed with the topology product of the given topology on X and of the discrete topology on C . Whereas the sections of $\bar{\mathcal{C}}_X$ are the constant functions over X , the sections of the associated sheaf \mathcal{C}_X are all locally constant functions. The sheaf \mathcal{C}_X is commonly called *the constant sheaf over X with stalk C* . Since there is no possible confusion one calls it too, *the constant sheaf C over X* using the same notation for the sheaf and its stalks.

The maps $i(U)$ given, for all open subsets U of X , by

$$i(U) : \bar{\mathcal{F}}(U) \rightarrow \mathcal{F}(U), \quad s \mapsto \tilde{s}$$

define a morphism i of presheaves. These maps may be neither injective (failure of condition 1 in Def. 3.1.9. See Exa. 3.1.4) nor surjective (failure of Condition 2 in Def. 3.1.9. See Exa. 3.1.11 or 3.1.22). One can check that the morphism i is injective when Condition 1 of sheaves (*cf.* Def. 3.1.9) is satisfied and that it is surjective when both Conditions 1 and 2 are satisfied, and so, we can state

Proposition 3.1.12. — *The morphism i is an isomorphism of presheaves if and only if $\bar{\mathcal{F}}$ is a sheaf.*

In all cases, i induces an isomorphism between the stalks $\overline{\mathcal{F}}_x$ and \mathcal{F}_x at any point $x \in X$.

The morphism of presheaves i satisfies the following universal property:

Suppose \mathcal{G} is a sheaf; then, any morphism of presheaves $\overline{\psi} : \overline{\mathcal{F}} \rightarrow \mathcal{G}$ can be factored uniquely through the sheaf \mathcal{F} associated with $\overline{\mathcal{F}}$, *i.e.*, there exists a unique morphism ψ such that the following diagram commutes:

$$\begin{array}{ccc} \overline{\mathcal{F}} & \xrightarrow{\overline{\psi}} & \mathcal{G} \\ & \searrow i & \nearrow \psi \\ & \mathcal{F} & \end{array}$$

From the fact that, when $\overline{\mathcal{F}}$ is itself a sheaf, the morphism i is an isomorphism of presheaves between any presheaf and its associated sheaf, one can always think of a sheaf as being the sheaf of the sections of an espace étalé $F \xrightarrow{\pi} X$. From that viewpoint, it makes sense to consider sections over any subset W of X , open or not, and also to define any section as a collection of germs. Not any collection of germs is allowed. Indeed, if $\varphi \in \overline{\mathcal{F}}(W_x)$ represents the germ s_x on a neighborhood W_x of x then, for the section $s : W \rightarrow F$ to be continuous at x , the germs $s_{x'}$ for x' close to x must also be represented by φ . The set $\mathcal{F}(W)$ of the sections of a sheaf \mathcal{F} over a subset W of X is widely denoted by $\Gamma(W; \mathcal{F})$.

Recall the following definition (see end of Sect. 3.1.1 and Def. 3.1.9).

Definition 3.1.13 (Consistency). —

▷ A family of sections $s_j \in \mathcal{F}(W_j)$ is said to be consistent if, when $W_i \cap W_j$ is not empty, the restrictions of s_i and s_j to $W_i \cap W_j$ coincide.

▷ A family of germs is said to be consistent if any germ generates its neighbors.

One can state:

Proposition 3.1.14. — Given \mathcal{F} a sheaf over X and W any subset of X , open or not, a family of germs $(s_x)_{x \in W}$ is a section of \mathcal{F} over W if and only if it is consistent.

Definition 3.1.15. — Let \mathcal{F} be the sheaf associated with a presheaf $\overline{\mathcal{F}}$. We define a local section of \mathcal{F} to be any section of the presheaf $\overline{\mathcal{F}}$.

Considering representatives of the germs s_x of a section $s \in \Gamma(W; \mathcal{F})$, Proposition 3.1.14 can be reformulated as follows.

Proposition 3.1.16. — *Let \mathcal{F} be the sheaf associated with a presheaf $\overline{\mathcal{F}}$ over X and let W be any subset of X , open or not. Sections of \mathcal{F} over W can be seen as consistent collections of local sections $s_j \in \overline{\mathcal{F}}(U_j)$ with U_j open in X and $W \subseteq \bigcup_j U_j$.*

Clearly, such collections are not unique.

When W is not open the inclusion $W \subseteq \bigcup_j U_j$ is proper and the section lives actually on a larger open set (the size of which depends not only on W but both on W and the section).

3.1.4. Morphisms of sheaves. —

Definition 3.1.17 (Sheaf morphism). — *A morphism of sheaves is just a morphism of presheaves.*

With this definition, Proposition 3.1.12 has the following corollary.

Corollary 3.1.18. — *Let \mathcal{F} be a sheaf and \mathcal{F}' its associated sheaf when considered as a presheaf. Then, \mathcal{F} and \mathcal{F}' are isomorphic sheaves.*

Given two sheaves \mathcal{F} and \mathcal{F}' over X , let $F \xrightarrow{\pi} X$ and $F' \xrightarrow{\pi'} X'$ be their respective espace étalé. From the identification of a sheaf to its espace étalé a morphism $f : \mathcal{F} \rightarrow \mathcal{F}'$ of sheaves can be seen as a continuous map, which can also be denoted safely by f , between the associated espaces étalés with the condition that the following diagram commute:

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ \pi \searrow & & \swarrow \pi' \\ & X & \end{array}$$

Like presheaves, sheaves with values in a given category \mathcal{C} and their morphisms form a category which is Abelian when \mathcal{C} is also Abelian. The category of sheaves and the category of espaces étalés with values in a given category \mathcal{C} are equivalent.

Definition 3.1.19. — *A morphism $f : \mathcal{F} \rightarrow \mathcal{F}'$ of sheaves over X is said to be injective (resp. surjective, resp. an isomorphism) if, for any $x \in X$, the stalk map $f_x : \mathcal{F}_x \rightarrow \mathcal{F}'_x$ is injective (resp. surjective, resp. bijective).*

When a morphism $f : \mathcal{F} \rightarrow \mathcal{F}'$ is injective then, for all open subset U of X , the map $f(U) : \mathcal{F}(U) \rightarrow \mathcal{F}'(U)$ is injective. However, the fact that f be surjective does not imply the surjectivity of the maps $f(U)$ for all U ; hence, a surjective morphism of sheaves is not necessarily surjective as a morphism of presheaves, the converse being, of course, true since the functor direct limit is exact.

Example 3.1.20. — Take for \mathcal{F} the sheaf of germs of holomorphic functions on $X = \mathbb{C}^*$ and for \mathcal{F}' the subsheaf (see Def. 3.1.21 below) of the non-vanishing functions. The map $f : \varphi \mapsto \exp \circ \varphi$ is a morphism from \mathcal{F} to \mathcal{F}' which is surjective as a morphism of sheaves since the logarithm exists locally on \mathbb{C}^* . However, the logarithm is not defined as a univariate function on all of \mathbb{C}^* and so, the map f is not a surjective morphism of presheaves. For instance, the identical function $\text{Id} : x \mapsto x$ cannot be written in the form $\text{Id} = f(\varphi)$ for any φ in $\mathcal{F}(\mathbb{C}^*)$ or more generally, any φ in $\mathcal{F}(U)$ as soon as U is not simply connected in \mathbb{C}^* .

Definition 3.1.21. — A sheaf \mathcal{F} over X is a subsheaf of a sheaf \mathcal{G} over X if, for all open set U , it satisfies the conditions

- ▷ $\mathcal{F}(U) \subseteq \mathcal{G}(U)$,
- ▷ the inclusion map $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$ commute to the restriction maps.

The inclusion $j : \mathcal{F} \hookrightarrow \mathcal{G}$ is an injective morphism of sheaves.

3.1.5. Sheaves \mathcal{A} of asymptotic and \mathcal{A}_s of s -Gevrey asymptotic functions over S^1 . — The sheaves \mathcal{A} and \mathcal{A}_s of asymptotic functions we introduce in this section play a fundamental role in what follows.

▷ *Topology of the base space S^1 .* — The base space S^1 is the circle of directions from 0. One should consider it as the boundary of the real blow up $\check{\mathbb{C}}$ of 0 in \mathbb{C} , *i.e.*, as the boundary $S^1 \times \{0\}$ of the space of polar coordinates $(\theta, r) \in S^1 \times [0, \infty[$.

For simplicity, we denote S^1 for $S^1 \times \{0\}$.

The map $\pi : \check{\mathbb{C}} \rightarrow \mathbb{C}$ defined by $\pi(\theta, r) = re^{i\theta}$ sends S^1 to 0 and $\check{\mathbb{C}} \setminus S^1$ homeomorphically to \mathbb{C}^* . A basis of open sets of S^1 is given by the arcs $I =]\theta_0, \theta_1[$ seen as the direct limit of the domains $\check{\Delta} = I \times]0, R[$ in $\check{\mathbb{C}}$ as R tends to 0. Such domains are identified via π to sectors $\Delta = \{x = re^{i\theta}; \theta_0 < \theta < \theta_1 \text{ and } 0 < r < R\}$ of \mathbb{C}^* .

▷ *The presheaf $\bar{\mathcal{A}}$ over S^1 .* — Given an open arc $I =]\theta_0, \theta_1[$ we denote by $\Delta_{I,R} = I \times]0, R[$ a sector based on I with radius R . The sections of $\bar{\mathcal{A}}$ over

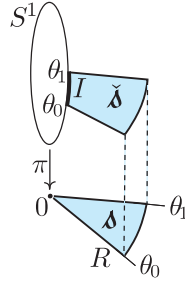


Figure 1

I are given by

$$\bar{\mathcal{A}}(I) = \varinjlim_{R \rightarrow 0} \bar{\mathcal{A}}(\Delta_{I,R}).$$

Suppose an element of $\bar{\mathcal{A}}(I)$ is represented by two functions $\varphi \in \bar{\mathcal{A}}(\Delta_{I,R})$ and $\psi \in \bar{\mathcal{A}}(\Delta_{I,R})$ on the same sector $\Delta_{I,R}$. This means that there exists a subsector $\Delta_{I,R'}$ of $\Delta_{I,R}$ on which φ and ψ coincide. By analytic continuation, we conclude that $\varphi = \psi$ on all of $\Delta_{I,R}$.

Choosing as restriction maps the usual restriction of functions, this defines a presheaf of differential \mathbb{C} -algebras. The example below shows that such a presheaf is not a sheaf.

Example 3.1.22. — Consider the lacunar series (see [Rud87, Hadamard’s Thm. 16.6 and Exa. 16.7])

$$f_1(x) = \sum_{n \geq 0} a_n(x-1)^{2^n} \quad \text{with } a_n = \exp(-2^{n/2}).$$

Since $\limsup_{n \rightarrow +\infty} |a_n|^{2^{-n}} = 1$ its radius of convergence as a series in powers of $x-1$ is equal to 1. We know from a theorem of Hadamard that its natural domain of holomorphy is the open disc $D = \{x \in \mathbb{C}; |x-1| < 1\}$. The series of the derivatives of any order (starting from order 0) converge uniformly on the closed disc \bar{D} . The function f_1 admits then an asymptotic expansion at 0 on any sector included in \bar{D} .

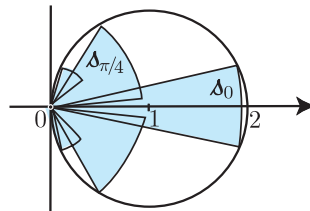


Figure 2

Consider now the arc $I =]-\frac{\pi}{2}, \frac{\pi}{2}[$ of S^1 . To any $\theta \in I$ there is a sector $\Delta_\theta = I_\theta \times]0, R_\theta[$ on which f_1 is well defined and belongs to $\overline{\mathcal{A}}(\Delta_\theta)$. However, as θ approaches $\pm\frac{\pi}{2}$ the radius R_θ tends to 0 and there is no sector $\Delta = I \times]0, R[$ with $R > 0$ on which f_1 is even defined. Thus, Condition 2 of Definition 3.1.9 fails on $U = I$.

▷ *The sheaf \mathcal{A} over S^1 .* — The sheaf \mathcal{A} of asymptotic functions over S^1 is the sheaf associated with the presheaf $\overline{\mathcal{A}}$. A section of \mathcal{A} over an interval I is defined by a collection of asymptotic functions $f_j \in \overline{\mathcal{A}}(\Delta_j)$ on $\Delta_j = I_j \times]0, R_j[$ where $\{I_j\}$ is an open covering of I and $R_j \neq 0$ for all j . The sheaf \mathcal{A} is a sheaf of differential \mathbb{C} -algebras.

▷ *The subsheaf $\mathcal{A}^{<0}$ of flat germs.* — Given an open sector $\Delta_{I,R} = I \times]0, R[$ (cf. Notations. 2.2.2), we define

$$\overline{\mathcal{A}}^{<0}(I) = \varinjlim_{R \rightarrow 0} \overline{\mathcal{A}}^{<0}(\Delta_{I,R}).$$

The set $\overline{\mathcal{A}}^{<0}(I)$ is a subset of $\overline{\mathcal{A}}(I)$. Considering the restriction maps $\rho_{J,I}$ of the presheaf $\overline{\mathcal{A}}(I)$ restricted to $\overline{\mathcal{A}}^{<0}(I)$ we obtain a presheaf $I \mapsto \overline{\mathcal{A}}^{<0}(I)$ over S^1 . The associated sheaf is denoted by $\mathcal{A}^{<0}$ and is a subsheaf of \mathcal{A} over S^1 .

▷ *The Taylor map.* — The Taylor map $T_{\Delta_{I,R}} : \overline{\mathcal{A}}(\Delta_{I,R}) \rightarrow \mathbb{C}[[x]]$ induces a map

$$T : \mathcal{A} \rightarrow \mathbb{C}[[x]]$$

also called *Taylor map* which is a morphism of sheaves of \mathbb{C} -differential algebras with kernel $\mathcal{A}^{<0}$. Thus, $\mathcal{A}^{<0}$ is a subsheaf of ideals of \mathcal{A} .

▷ *The sheaf \mathcal{A}_s over S^1 .* — Similarly, one defines a presheaf $\overline{\mathcal{A}}_s$ over S^1 by setting

$$\overline{\mathcal{A}}_s(I) = \varinjlim_{R \rightarrow 0} \overline{\mathcal{A}}_s(\Delta_{I,R})$$

for the set of (equivalence classes of) s -Gevrey asymptotic functions on a sector based on I . Its associated sheaf is denoted by \mathcal{A}_s .

▷ *The sheaf $\mathcal{A}^{\leq -k}$ over S^1 .* — One also defines a presheaf by setting

$$\overline{\mathcal{A}}^{\leq -k}(I) = \varinjlim_{R \rightarrow 0} \overline{\mathcal{A}}^{\leq -k}(\Delta_{I,R})$$

and $\mathcal{A}^{\leq -k}$ denotes the associated sheaf over S^1 . According to Proposition 2.3.17, the presheaf $\overline{\mathcal{A}}^{\leq -k}$ is a sub-presheaf of $\overline{\mathcal{A}}_s$, and then, $\mathcal{A}^{\leq -k}$ is a subsheaf of \mathcal{A}_s , precisely, the subsheaf of s -Gevrey flat germs.

The Taylor map $T : \mathcal{A} \rightarrow \mathbb{C}[[x]]$ induces a Taylor map

$$T = T_s : \mathcal{A}_s \longrightarrow \mathbb{C}[[x]]_s$$

which is a morphism of sheaves of \mathbb{C} -differential algebras with kernel $\mathcal{A}^{\leq -k}$. Thus, $\mathcal{A}^{\leq -k}$ is a subsheaf of ideals of \mathcal{A}_s .

3.1.6. Quotient sheaves and exact sequences. — From now on, unless otherwise specified, we suppose that all the sheaves or presheaves we consider are sheaves or presheaves of *Abelian groups* (or, more generally, sheaves or presheaves with values in an Abelian category \mathcal{C}). Recall that such sheaves or presheaves and their morphisms form themselves an Abelian category which will allow us to talk of exact sequences of sheaves.

Given a sheaf \mathcal{G} with values in \mathcal{C} and a subsheaf \mathcal{F} one defines a presheaf by setting $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$ for all open set U of the base space X , the restriction maps being induced by those of \mathcal{G} .

Condition 1 of sheaves is always satisfied (for a proof see [Mal95, Annexe 1] for instance) while Condition 2 fails in general (*cf.* Exa. 3.1.24).

Definition 3.1.23. — *One defines the quotient sheaf $\mathcal{H} = \mathcal{G}/\mathcal{F}$ to be the sheaf over X associated with the presheaf*

$$U \longmapsto \mathcal{G}(U)/\mathcal{F}(U) \quad \text{for all open set } U \text{ of } X$$

with restriction maps induced by those of \mathcal{G} .

If \mathcal{F} and \mathcal{G} are sheaves of Abelian groups or of vector spaces so is the quotient \mathcal{H} . If \mathcal{G} is a sheaf of algebras and \mathcal{F} a subsheaf of ideals, then \mathcal{H} is a sheaf of algebras.

As noticed at the end of Section 3.1.3, the fact that the quotient presheaf satisfies Condition 1 of sheaves (Def. 3.1.9) means that the natural map

$$\mathcal{G}(U)/\mathcal{F}(U) \rightarrow \mathcal{H}(U)$$

is injective. If Condition 2 were also satisfied then this natural map would be surjective. However, this is not true, in general, as shown by the Example 3.1.24 below.

Example 3.1.24. — (Quotient sheaf and Euler equation) We saw in Example 2.2.4 that the Euler equation

$$x^2 \frac{dy}{dx} + y = x \tag{1}$$

admits an actual solution $E(x) = \int_0^{+\infty} \frac{e^{-\xi/x}}{1+\xi} d\xi$ which is asymptotic to the Euler series $\tilde{E}(x) = \sum_{n \geq 0} (-1)^n n! x^{n+1}$ on the sector $-\frac{3\pi}{2} < \arg(x) < +\frac{3\pi}{2}$.

Consider the homogeneous version of the Euler equation

$$\mathcal{E}_0 y \equiv x^3 \frac{d^2 y}{dx^2} + (x^2 + x) \frac{dy}{dx} - y = 0.$$

Recall that one obtains the equation $\mathcal{E}_0 y = 0$ by dividing equation(1) by x and differentiating. In any direction, $\mathcal{E}_0 y = 0$ admits a two dimensional \mathbb{C} -vector space of solutions spanned by $e^{1/x}$ and $E(x)$.

Following P. Deligne we denote by \mathcal{V} the sheaf over S^1 of the germs of solutions of (\mathcal{E}_0) having an asymptotic expansion at 0 and we denote by \mathcal{V}_θ the stalk of \mathcal{V} in a direction θ . The sheaf \mathcal{V} is a sheaf of vector spaces and a subsheaf of \mathcal{A} seen as a sheaf of vector spaces. Since $E(x)$ has an asymptotic expansion in all directions $-\pi/2 < \theta < 3\pi/2$ and $e^{1/x}$ has an asymptotic expansion (equal to 0) on $\Re(x) < 0$ we can assert that

$$\dim_{\mathbb{C}} \mathcal{V}_\theta = \begin{cases} 2 & \text{if } +\pi/2 < \theta < 3\pi/2, \\ 1 & \text{if } -\pi/2 \leq \theta \leq +\pi/2. \end{cases}$$

Denote by $\mathcal{V}^{<0} = \mathcal{V} \cap \mathcal{A}^{<0}$ the subsheaf of flat germs of \mathcal{V} . We observe that $\mathcal{V}(S^1) = \{0\}$ and $\mathcal{V}^{<0}(S^1) = \{0\}$, hence the quotient $\mathcal{V}(S^1)/\mathcal{V}^{<0}(S^1) = \{0\}$.

A global section of the quotient sheaf $\mathcal{V}/\mathcal{V}^{<0}$ is a collection of solutions over an open covering of S^1 which agree on the intersections up to flat solutions. The solution E induces such a global section while $e^{1/x}$ does not. Thus, the space of global sections $\Gamma(S^1; \mathcal{V}/\mathcal{V}^{<0})$ has dimension $\dim_{\mathbb{C}} \Gamma(S^1; \mathcal{V}/\mathcal{V}^{<0}) = 1$. This shows that the quotient sheaf $\mathcal{V}/\mathcal{V}^{<0}$ is different from the quotient presheaf. The quotient sheaf $\mathcal{V}/\mathcal{V}^{<0}$ is isomorphic to the constant sheaf \mathbb{C} as a sheaf of \mathbb{C} -vector spaces.

Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves with values in \mathcal{C} over the same base space X . Let $\rho_{V,U}$ and $\rho'_{V,U}$ denote the restriction maps in \mathcal{F} and \mathcal{G} respectively. One can define the presheaves $\overline{Ker}(f)$, $\overline{Im}(f)$ and $\overline{Coker}(f)$ over X with values in \mathcal{C} by setting

- ▷ for $\overline{Ker}(f) : U \mapsto \ker(f(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U))$ for all open set $U \subseteq X$ with restriction maps $r_{V,U} = \rho_{V,U}|_{\ker(f(U))}$;
- ▷ for $\overline{Im}(f) : U \mapsto f(\mathcal{F}(U))$ with restriction maps $r'_{V,U} = \rho'_{V,U}|_{f(\mathcal{F}(U))}$;
- ▷ for $\overline{Coker}(f) : U \mapsto \mathcal{G}(U)/f(\mathcal{F}(U))$ with restriction maps canonically induced from $\rho'_{V,U}$ on the quotient.

So defined, $\overline{Ker}(f)$ and $\overline{Im}(f)$ appear as sub-presheaves of \mathcal{F} and \mathcal{G} respectively, $\overline{Coker}(f)$ as a quotient of \mathcal{G} . For a definition by a universal property we refer to the classical literature.

One can check that the presheaf $\overline{Ker}(f)$ is actually a sheaf (precisely, a subsheaf of \mathcal{F}). Hence, the definition:

Definition 3.1.25. — *The sheaves kernel, image and cokernel of a morphism of sheaves f can be defined as follows.*

▷ The kernel $\text{Ker}(f)$ of the sheaf morphism f , is the sheaf defined by

$$U \longmapsto \ker(f(U)) \quad \text{for all open set } U \subseteq X$$

with the restriction maps $\rho_{V,U}|_{\ker(f(U))}$.

▷ The image $\text{Im}(f)$ and the cokernel $\text{Coker}(f)$ of the sheaf morphism f , are the sheaves respectively associated with the presheaves $\overline{\text{Im}}(f)$ and $\overline{\text{Coker}}(f)$.

The sheaves $\text{Coker}(f)$ and $\text{Im}(f)$ are respectively a quotient and a kernel:

$$\text{Coker}(f) = \mathcal{G}/\text{Im}(f), \quad \text{Im}(f) = \text{Ker}(\mathcal{G} \rightarrow \text{Coker}(f))$$

where $\mathcal{G} \rightarrow \text{Coker}(f)$ stands for the canonical quotient map.

Definition 3.1.26. — *Exactness of sequences of presheaves and of sheaves are defined by the following non-equivalent conditions:*

▷ A sequence of presheaves $\overline{\mathcal{F}} \xrightarrow{\overline{f}} \overline{\mathcal{G}} \xrightarrow{\overline{g}} \overline{\mathcal{H}}$ is said to be exact when

$$\overline{\text{Im}}(\overline{f}(U)) = \overline{\text{Ker}}(\overline{g}(U)) \quad \text{for all open set } U \subseteq X.$$

▷ A sequence of sheaves $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ is said to be exact when

$$\text{Im}(f_x) = \text{Ker}(g_x) \quad \text{for all } x \in X.$$

▷ A sequence $\cdots \rightarrow \mathcal{F}_{n-1} \xrightarrow{f_n} \mathcal{F}_n \xrightarrow{f_{n+1}} \mathcal{F}_{n+1} \rightarrow \cdots$ of presheaves or sheaves is exact when each subsequence $\mathcal{F}_{n-1} \xrightarrow{f_n} \mathcal{F}_n \xrightarrow{f_{n+1}} \mathcal{F}_{n+1}$ is exact.

A sequence of sheaves can be seen as a sequence of presheaves. One can show that exactness as a sequence of presheaves implies exactness as a sequence of sheaves the converse being false in general. Precisely, to a short (hence to any) exact sequence of presheaves $0 \rightarrow \overline{\mathcal{F}} \xrightarrow{\overline{f}} \overline{\mathcal{G}} \xrightarrow{\overline{g}} \overline{\mathcal{H}} \rightarrow 0$ there corresponds canonically the exact sequence of sheaves $0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \rightarrow 0$. Reciprocally, an exact sequence $0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \rightarrow 0$ of sheaves can be seen as a sequence of presheaves but, in general, only the truncated sequence $0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ is exact as a sequence of presheaves.

Let Presh_X and Sh_X denote respectively the categories of presheaves and sheaves over X with values in a given Abelian category \mathcal{C} . In the language of categories the properties above are formulated as follows.

▷ The functor of sheafification $\text{Presh}_X \rightarrow \text{Sh}_X$ is exact.

▷ The functor of inclusion $\text{Sh}_X \hookrightarrow \text{Presh}_X$ is only left exact.

3.1.7. The Borel-Ritt Theorem revisited. — By construction, $\bar{\mathcal{A}}^{<0}(I)$ and $\bar{\mathcal{A}}^{\leq-k}(I)$ are the kernels of the Taylor maps

$$T_I : \bar{\mathcal{A}}(I) \longrightarrow \mathbb{C}[[x]] \quad \text{and} \quad T_{s,I} : \bar{\mathcal{A}}_s(I) \longrightarrow \mathbb{C}[[x]]_s$$

respectively for any open arc I of S^1 . Hence, the sequences

$$0 \rightarrow \bar{\mathcal{A}}^{<0} \rightarrow \bar{\mathcal{A}} \xrightarrow{T} \mathbb{C}[[x]] \quad \text{and} \quad 0 \rightarrow \bar{\mathcal{A}}^{\leq-k} \rightarrow \bar{\mathcal{A}}_s \xrightarrow{T_s} \mathbb{C}[[x]]_s$$

are exact sequences of presheaves and they generate the exact sequences of sheaves of differential algebras

$$0 \rightarrow \mathcal{A}^{<0} \rightarrow \mathcal{A} \xrightarrow{T} \mathbb{C}[[x]] \quad \text{and} \quad 0 \rightarrow \mathcal{A}^{\leq-k} \rightarrow \mathcal{A}_s \xrightarrow{T_s} \mathbb{C}[[x]]_s.$$

The Borel-Ritt Theorem 2.4.1 allows one to complete these sequences into short exact sequences as follows.

Corollary 3.1.27 (Borel-Ritt). — *The sequences*

$$(18) \quad 0 \rightarrow \mathcal{A}^{<0} \rightarrow \mathcal{A} \xrightarrow{T} \mathbb{C}[[x]] \rightarrow 0,$$

$$(19) \quad 0 \rightarrow \mathcal{A}^{\leq-k} \rightarrow \mathcal{A}_s \xrightarrow{T_s} \mathbb{C}[[x]]_s \rightarrow 0$$

are exact sequences of sheaves of differential \mathbb{C} -algebras over S^1 . Equivalently, the quotient sheaves $\mathcal{A}/\mathcal{A}^{<0}$ and $\mathcal{A}_s/\mathcal{A}^{\leq-k}$ are isomorphic via the Taylor map to the constant sheaves $\mathbb{C}[[x]]$ and $\mathbb{C}[[x]]_s$ respectively, as sheaves of differential \mathbb{C} -algebras.

With this approach, the surjectivity of T or T_s means that, given any series and any direction there exist a sector containing the direction and a function asymptotic on it to the given series. We cannot not claim that the sector can be chosen to be arbitrarily wide.

Observe that (18) and (19) are not exact sequences of presheaves over S^1 . Indeed, the range of the Taylor map $T : \mathcal{A}(S^1) \rightarrow \mathbb{C}[[x]]$, as well as the range of $T_s : \mathcal{A}_s(S^1) \rightarrow \mathbb{C}[[x]]_s$, is made of convergent series and, consequently, these maps are not surjective.

3.1.8. Change of base space: direct image, restriction and extension by 0. — The following definition makes sense since for f continuous and U open in Y the set $f^{-1}(U)$ is open in X .

Definition 3.1.28 (Direct image). — *Let $f : X \rightarrow Y$ be a continuous map. With any sheaf \mathcal{F} over X one can associate a sheaf $f_*\mathcal{F}$ over Y called its direct image by setting*

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \quad \text{for all open set } U \text{ in } Y,$$

with restriction maps

$$\rho_{*V,U}(s_*) = \rho_{f^{-1}(V),f^{-1}(U)}(s) \quad \text{for all open sets } V \subseteq U \text{ in } Y.$$

When \mathcal{F} is a sheaf of Abelian groups, vector spaces, etc. . . , so is its direct image $f_*\mathcal{F}$.

To a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves over X there corresponds a morphism of sheaves $\varphi_* : f_*\mathcal{F} \rightarrow f_*\mathcal{G}$ over Y defined by

$$s_* \in f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \mapsto \varphi(s_*) \in \mathcal{G}(f^{-1}(U)) = f_*\mathcal{G}(U).$$

The functor direct image is left exact. Thus, to an exact sequence

$$0 \rightarrow \mathcal{F}' \xrightarrow{u} \mathcal{F}' \xrightarrow{v} \mathcal{F}''$$

there corresponds the exact sequence

$$0 \rightarrow f_*\mathcal{F}'' \xrightarrow{u_*} f_*\mathcal{F}' \xrightarrow{v_*} f_*\mathcal{F}''.$$

We suppose now that X is a subspace of Y with inclusion $j : X \hookrightarrow Y$ and that we are given \mathcal{G} a sheaf over Y . The restriction of \mathcal{G} into a sheaf over X is fully natural in terms of espaces étalés. We denote by $\pi : G \rightarrow Y$ the espace étalé associated with \mathcal{G} .

Definition 3.1.29 (Restriction). —

The sheaf \mathcal{G} restricted to X is the sheaf $\mathcal{G}|_X$ with espace étalé

$$\pi|_{\pi^{-1}(X)} : \pi^{-1}(X) \rightarrow X.$$

The definition makes sense since as $\pi : G \rightarrow Y$ is a local homeomorphism so is $\pi|_{\pi^{-1}(X)} : \pi^{-1}(X) \rightarrow X$. The restricted sheaf can also be seen as the inverse image of \mathcal{G} by the inclusion map j , a viewpoint which we won't develop here.

As in the previous section we consider now sheaves of Abelian groups and we denote by 0 the neutral element. With $X \xrightarrow{j} Y$ let \mathcal{F} and \mathcal{F}' be sheaves over X and Y respectively.

Definition 3.1.30 (Extension). —

▷ A sheaf \mathcal{F}' is an extension of a sheaf \mathcal{F} if its restriction $\mathcal{F}'|_X$ to X is isomorphic to \mathcal{F} .

▷ An extension \mathcal{F}' of \mathcal{F} is an extension by 0 if, for all $y \in Y \setminus X$, the stalk \mathcal{F}'_y is 0. (Equivalently, $\mathcal{F}'|_{Y \setminus X}$ is the constant sheaf 0.)

Definition 3.1.31 (Support of a section). —

The support of a section $s \in \Gamma(U; \mathcal{F})$ is the subset of U where s does not vanish:

$$\text{supp}(s) = \{x \in U; s_x \neq 0\}.$$

Example 3.1.32. — Let \mathcal{E} be the sheaf of \mathbb{C} -vector spaces generated over S^1 by the function $e^{1/x}$. The sheaf \mathcal{E} is isomorphic to the constant sheaf with stalk \mathbb{C} over S^1 . Let $e(x)$ be the class of $e^{1/x}$ in the quotient sheaf $\mathcal{E}/\mathcal{E}^{<0}$ where $\mathcal{E}^{<0} = \mathcal{E} \cap \mathcal{A}^{<0}$. Thus, $e(x) = 0$ for $\Re(x) < 0$ and the support of e is the arc $-\pi/2 \leq \arg(x) \leq \pi/2$, a closed subset of S^1 .

The support $\text{supp}(s)$ is always a closed subset of U , for, if a germ s_x is 0 then, there is an open neighborhood V_x of x on which s_x is represented by the 0 function generating thus the germs 0 on a neighborhood of x .

Recall that a subset X of Y is said to be *locally closed in Y* if any point $x \in X$ admits in Y a neighborhood $V_Y(x)$ such that its intersection $V_Y(x) \cap X$ is closed in $V_Y(x)$. This is equivalent to saying that there exist X_1 open in Y and X_2 closed in Y such that $X = X_1 \cap X_2$.

Definition 3.1.33 (Sheaf $j_!\mathcal{F}$). —

Suppose X is locally closed in Y and denote by $j : X \hookrightarrow Y$ the inclusion map of X in Y . Given \mathcal{F} a sheaf of Abelian groups over X one defines the sheaf $j_!\mathcal{F}$ over Y by setting, for all open U of Y ,

$$j_!\mathcal{F}(U) = \{s \in \Gamma(X \cap U; \mathcal{F}); \text{supp}(s) \text{ is closed in } U\}$$

with restriction maps induced by those of $j_*\mathcal{F}$ (of which $j_!\mathcal{F}$ is a subsheaf).

One can check that $j_!\mathcal{F}$ is a sheaf; it is then clearly a subsheaf of $j_*\mathcal{F}$ and there is a canonical inclusion $j_!\mathcal{F} \hookrightarrow j_*\mathcal{F}$. Moreover, $j_!\mathcal{F}$ is an extension of \mathcal{F} by 0. When X is closed in Y then the two sheaves coincide: $j_!\mathcal{F} = j_*\mathcal{F}$. Unlike the functor j_* which is only left exact, the functor $j_!$ is exact.

The extension of sheaves by 0 provides a characterization of locally closed subspaces as follows: *X is locally closed in Y if and only if, for any sheaf \mathcal{F} over X , there is a unique extension of \mathcal{F} to Y by 0 (cf. [Ten75] Thm. 3.8.6).*

Example 3.1.34 ($j_* \neq j_!$). — As an illustration consider the sheaf \mathcal{E}' generated as a sheaf of \mathbb{C} -vector spaces by $e^{1/x}$ over the punctured disc $D^* = \{x \in \mathbb{C}; 0 < |x| < 1\}$ and consider the inclusion $j : D^* \hookrightarrow \mathbb{C}$.

The direct image $j_*\mathcal{E}'$ of \mathcal{E}' by j is a non-constant sheaf of \mathbb{C} -vector spaces. Indeed, for U a connected open set in \mathbb{C} , one has $j_*\mathcal{E}'(U) \simeq \mathbb{C}^n$ where n is the number of connected components of $U \cap D^*$.

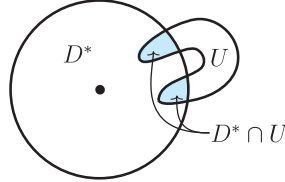


Figure 3

The stalks of $j_*\mathcal{E}'$ are given by

$$j_*\mathcal{E}'_x \simeq \begin{cases} \mathbb{C} & \text{if } x \in \overline{D^*}, \\ 0 & \text{otherwise,} \end{cases}$$

so that, in some way, the direct image $j_*\mathcal{E}'$ spreads \mathcal{E}' out, onto the closure of D^* . Thus, the direct image $j_*\mathcal{E}'$ is an extension of \mathcal{E}' but not an extension by 0.

On the contrary, the sheaf $j_!\mathcal{E}'$ is an extension of \mathcal{E}' by 0. It is well defined since D^* being open in \mathbb{C} is also locally closed in \mathbb{C} . This shows that $j_*\mathcal{E}' \neq j_!\mathcal{E}'$ and therefore, that the functors j_* and $j_!$ are different.

3.2. Čech cohomology

Let \mathcal{F} be a sheaf over a topological space X . We assume that \mathcal{F} is a *sheaf of Abelian groups*. The set $\Gamma(U; \mathcal{F})$ of sections of \mathcal{F} over a $U \subset X$ is then naturally endowed with a structure of Abelian group and $\mathcal{F}(\emptyset) = \{0\}$, the trivial Abelian group 0. Unless otherwise specified, all the coverings we consider are coverings by open sets.

3.2.1. Čech cohomology of a covering \mathcal{U} . — Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X .

Denote $U_{i,j} = U_i \cap U_j$, $U_{i,j,k} = U_i \cap U_j \cap U_k$, and so on...

Definition 3.2.1. — One defines the Čech complex of \mathcal{F} associated with the covering \mathcal{U} to be the differential complex

$$\begin{aligned} 0 \rightarrow \prod_{i_0} \Gamma(U_{i_0}; \mathcal{F}) &\xrightarrow{d_0} \prod_{i_0, i_1} \Gamma(U_{i_0, i_1}; \mathcal{F}) \xrightarrow{d_1} \\ &\dots \xrightarrow{d_{n-1}} \prod \Gamma(U_{i_0, \dots, i_n}; \mathcal{F}) \xrightarrow{d_n} \prod \Gamma(U_{i_0, \dots, i_{n+1}}; \mathcal{F}) \xrightarrow{d_{n+1}} \dots \end{aligned}$$

where, for all n , the map d_n is defined by

$$d_n : f = (f_{i_0, \dots, i_n}) \mapsto g = (g_{i_0, \dots, i_{n+1}})$$

where

$$g_{i_0, \dots, i_{n+1}} = \sum_{\ell=0}^{n+1} (-1)^\ell f_{i_0, \dots, i_{\ell-1}, \widehat{i_\ell}, i_{\ell+1}, \dots, i_{n+1}} \Big|_{U_{i_0, \dots, i_{n+1}}}$$

the hat over i_ℓ indicating that the index i_ℓ is omitted.

Each term of the complex is an Abelian group.

The maps d_n are morphisms of Abelian groups. Consequently, the image $\text{im } d_n$ and the kernel $\ker d_n$ are Abelian groups. For all n , the maps d_n are “differentials” which, in this context, means that $d_n \circ d_{n-1} = 0$ and thus, $\text{im } d_{n-1} \subset \ker d_n$ and the quotients $\ker d_n / \text{im } d_{n-1}$ are Abelian groups.

Definition 3.2.2. — One calls

▷ n -cochains of \mathcal{U} (with values) in \mathcal{F} the elements of the Abelian group

$$\mathcal{C}^n(\mathcal{U}; \mathcal{F}) = \prod \Gamma(U_{i_0, \dots, i_n}; \mathcal{F}),$$

▷ n -cocycles of \mathcal{U} (with values) in \mathcal{F} the elements of the Abelian group

$$\mathcal{Z}^n(\mathcal{U}; \mathcal{F}) = \ker d_n,$$

▷ n -coboundaries of \mathcal{U} (with values) in \mathcal{F} the elements of the Abelian group

$$\mathcal{B}^n(\mathcal{U}; \mathcal{F}) = \text{im } d_{n-1},$$

▷ n -th Čech cohomology group of \mathcal{U} (with values) in \mathcal{F} the Abelian group

$$H^n(\mathcal{U}; \mathcal{F}) = \mathcal{Z}^n(\mathcal{U}; \mathcal{F}) / \mathcal{B}^n(\mathcal{U}; \mathcal{F}) = \ker d_n / \text{im } d_{n-1}.$$

In particular, $H^0(\mathcal{U}; \mathcal{F}) \simeq \Gamma(X; \mathcal{F})$ the set of global sections of \mathcal{F} over X .

Definition 3.2.3 (Refinement of a covering). — A covering $\mathcal{V} = \{V_j\}_{j \in J}$ is said to be finer than the covering $\mathcal{U} = \{U_i\}_{i \in I}$, and we denote $\mathcal{V} \succeq \mathcal{U}$, if any element in \mathcal{V} is contained in at least one element of \mathcal{U} .

Equivalently, one can say that there exists a map

$$\sigma : J \longrightarrow I \quad \text{such that } V_j \subset U_{\sigma(j)} \text{ for all } j \in J.$$

Such a map is called inclusion map or simplicial map.

With the simplicial map σ are naturally associated the maps

$$\sigma_n^* : \mathcal{C}^n(\mathcal{U}; \mathcal{F}) \longrightarrow \mathcal{C}^n(\mathcal{V}; \mathcal{F}), \quad f = (f_{i_0, \dots, i_n}) \longmapsto \sigma_n^* f = (F_{j_0, \dots, j_n})$$

given by

$$F_{j_0, \dots, j_n} = f_{\sigma(j_0), \dots, \sigma(j_n)} \Big|_{V_{j_0, \dots, j_n}}.$$

The family $\sigma^* = (\sigma_n^*)$ defines a morphism of Čech complexes and induces, for all n , a morphism of groups

$$\mathfrak{S}^n(\mathcal{V}, \mathcal{U}) : H^n(\mathcal{U}; \mathcal{F}) \longrightarrow H^n(\mathcal{V}; \mathcal{F}).$$

It turns out that these latter homomorphisms are independent of the choice of the simplicial map σ . The case when $n = 1$ has the following specificity:

Proposition 3.2.4. — *When $n = 1$, the morphism*

$$\mathfrak{S}^1(\mathcal{V}, \mathcal{U}) : H^1(\mathcal{U}; \mathcal{F}) \longrightarrow H^1(\mathcal{V}; \mathcal{F}) \quad \text{is injective.}$$

We refer to [Ten75, Thm. 4.15, p. 148] .

3.2.2. Čech cohomology of the space X . — The preceding section suggests to take the direct limit (cf. Def. p. 40) of the groups $H^n(\mathcal{U}; \mathcal{F})$ using the maps $\mathfrak{S}(\mathcal{V}, \mathcal{U})$ as the coverings become finer and finer. Indeed, the coverings of X endowed with fineness form an ordered, right filtrant⁽¹⁾ “set” and the maps $\mathfrak{S}^n(\mathcal{V}, \mathcal{U})$ satisfy, for all n , the conditions:

- ▷ $\mathfrak{S}^n(\mathcal{U}, \mathcal{U}) = \text{Id}$ for all \mathcal{U} ,
- ▷ $\mathfrak{S}^n(\mathcal{W}, \mathcal{V}) \circ \mathfrak{S}^n(\mathcal{V}, \mathcal{U}) = \mathfrak{S}^n(\mathcal{W}, \mathcal{U})$ for all $\mathcal{W} \succeq \mathcal{V} \succeq \mathcal{U}$

providing thus a direct system $(H^n(\mathcal{U}; \mathcal{F}), \mathfrak{S}^n(\mathcal{V}, \mathcal{U}))$ of Abelian groups.

The only problem is that coverings of a topological space do not form a set. That difficulty can be circumvented by limiting the considered coverings to those that are indexed by a given convenient set, *i.e.*, a set of indices large enough to allow arbitrarily fine coverings. In the cases we consider any countable set is convenient, say, \mathbb{N} or \mathbb{Z} . Actually, for $X = S^1$, we may consider coverings with a finite number of open sets since there exists finite coverings of S^1 that are arbitrarily fine. From now on, we assume that the coverings are indexed by subsets J of \mathbb{N} .

Another trick due to R. Godement consists in considering only the coverings $\{U_x\}$ indexed by the points $x \in X$ with the condition $x \in U_x$ (cf. [God58, Sect. 5.8, p. 223]). Hence, the following definition:

Definition 3.2.5. — *The n -th Čech cohomology group of the space X (with values) in \mathcal{F} is the direct limit of the cohomology groups $H^n(\mathcal{U}; \mathcal{F})$, the limit*

⁽¹⁾ “Right filtrant” means here that to each finite family $\mathcal{U}_1, \dots, \mathcal{U}_p$ of open coverings of X there is a covering \mathcal{V} finer than all of them.

being taken over coverings ordered with fineness. One denotes

$$H^n(X; \mathcal{F}) = \varinjlim_{\mathcal{U}} H^n(\mathcal{U}; \mathcal{F}).$$

When X is a manifold and $n > \dim X$ there exists arbitrarily fine coverings without intersections $n + 1$ by $n + 1$ and then, $H^n(X; \mathcal{F}) = 0$. The canonical isomorphism $H^0(X; \mathcal{F}) \simeq \Gamma(X; \mathcal{F})$ is valid without restriction.

The following two results are useful.

Theorem 3.2.6 (Leray's Theorem). — *Given \mathcal{U} an acyclic covering of X which is either closed and locally finite or open then,*

$$H^n(\mathcal{U}; \mathcal{F}) = H^n(X; \mathcal{F}) \quad \text{for all } n.$$

Acyclic means that $H^n(U_i; \mathcal{F}) = 0$ for all $U_i \in \mathcal{U}$ and all $n \geq 1$.

We refer to [God58, Thm. 5.2.4, Cor. p. 209], (case \mathcal{U} closed and locally finite) and to [God58, Thm. 5.4.1, Cor. p. 213] (case \mathcal{U} open).

Theorem 3.2.7. — *To any short exact sequence of sheaves of Abelian groups over X*

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{H} \rightarrow 0$$

there is a long exact sequence of cohomology

$$\begin{aligned} 0 \rightarrow H^0(X; \mathcal{G}) &\rightarrow H^0(X; \mathcal{F}) \rightarrow H^0(X; \mathcal{H}) \\ &\xrightarrow{\delta_0} H^1(X; \mathcal{G}) \rightarrow H^1(X; \mathcal{F}) \rightarrow H^1(X; \mathcal{H}) \\ &\xrightarrow{\delta_1} H^2(X; \mathcal{G}) \rightarrow H^2(X; \mathcal{F}) \rightarrow H^2(X; \mathcal{H}) \\ &\xrightarrow{\delta_2} \dots \end{aligned}$$

The maps $\delta_0, \delta_1, \dots$ are called *coboundary maps*. For their general definition, see the references above.

3.2.3. The Borel-Ritt Theorem and cohomology. — We know from Corollary 3.1.27 that the sheaves $\mathcal{A}/\mathcal{A}^{<0}$ and $\mathcal{A}_s/\mathcal{A}^{\leq -k}$ are constant sheaves with stalks $\mathbb{C}[[x]]$ and $\mathbb{C}[[x]]_s$ respectively. Their global sections

$$\Gamma(S^1; \mathcal{A}/\mathcal{A}^{<0}) \equiv H^0(S^1; \mathcal{A}/\mathcal{A}^{<0}) \quad \text{and} \quad \Gamma(S^1; \mathcal{A}_s/\mathcal{A}^{\leq -k}) \equiv H^0(S^1; \mathcal{A}_s/\mathcal{A}^{\leq -k})$$

are then also respectively isomorphic to $\mathbb{C}[[x]]$ and $\mathbb{C}[[x]]_s$ and we can state the following corollary of the Borel-Ritt Theorem.

Corollary 3.2.8 (Borel-Ritt). — *The Taylor map induces the following isomorphisms:*

$$H^0(S^1; \mathcal{A}/\mathcal{A}^{<0}) \simeq \mathbb{C}[[x]], \quad H^0(S^1; \mathcal{A}_s/\mathcal{A}^{\leq -k}) \simeq \mathbb{C}[[x]]_s.$$

We can synthesize:

$\left. \begin{array}{l} \text{formal series} \\ \tilde{f}(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]] \end{array} \right\}$	\iff	$\left\{ \begin{array}{l} \text{(equivalence class of a)} \\ \text{0-cochain } (f_j)_{j \in J} \text{ over } S^1 \\ \text{with values in } \mathcal{A} \text{ and} \\ \text{coboundary } (f_j - f_\ell)_{j, \ell \in J} \\ \text{with values in } \mathcal{A}^{<0} \end{array} \right.$
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The components $f_j(x)$ of the 0-cochains are all asymptotic to $\tilde{f}(x)$.

$\left. \begin{array}{l} \text{s-Gevrey series} \\ \tilde{f}(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]_s \end{array} \right\}$	\iff	$\left\{ \begin{array}{l} \text{(equivalence class of a)} \\ \text{0-cochain } (f_j)_{j \in J} \text{ over } S^1 \\ \text{with values in } \mathcal{A}_s \text{ and} \\ \text{coboundary } (f_j - f_\ell)_{j, \ell \in J} \\ \text{with values in } \mathcal{A}^{\leq -k} \end{array} \right.$
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The components $f_j(x)$ of the 0-cochains are all s -Gevrey asymptotic to $\tilde{f}(x)$. Due to Proposition 2.3.17 it would actually be sufficient to ask for the coboundary to be with values in $\mathcal{A}^{<0}$. This latter equivalence will be improved in Corollary 6.2.2.

3.2.4. The case when $X = S^1$ and the Cauchy-Heine Theorem. —

Since, in what follows, we will mostly be dealing with sheaves over S^1 , it is worth developing this case further. With $X = S^1$ things are often made simpler by the fact that S^1 is a manifold of dimension 1. On another hand, one has to take into account the fact that S^1 has a non-trivial π_1 .

Definition 3.2.9 (Good covering). — *An open covering $\mathcal{I} = (I_j)_{j \in J}$ of S^1 is said to be a good covering if*

- ▷ *it is finite with $|J| = p$ elements,*
- ▷ *its elements I_j are connected (i.e., open arcs of S^1),*
- ▷ *it has thickness ≤ 2 (i.e., no 3-by-3 intersections),*

▷ when $p = 2$ its two open arcs I_1 and I_2 are proper arcs of S^1 so that $I_1 \cap I_2$ is made of two disjoint open arcs which we denote by \dot{I}_1 and \dot{I}_2 ; when $p \geq 3$ its open arcs I_j can be indexed by the cyclic group $\mathbb{Z}/p\mathbb{Z}$ so that $\dot{I}_j := I_j \cap I_{j+1} \neq \emptyset$ and $I_j \cap I_\ell = \emptyset$ as soon as $|j - \ell| > 1$ modulo p .

The definition implies that open arcs of a good covering are not nested. The family of the arcs \dot{I}_j is sometimes called the *nerve of the covering* \mathcal{I} .

The case $p = 1$, *i.e.*, the case of coverings of S^1 by just one arc, is worth to consider. These unique arcs cannot be proper arcs of S^1 : one has to introduce overlapping arcs *i.e.*, arcs of the universal cover of S^1 of length $> 2\pi$. Such coverings are widely used to make proofs simpler by using the additivity of 1-cocycles. A typical example is given by the Cauchy-Heine Theorem (Thm. 2.5.2 and Cor. 3.2.14 below).

Definition 3.2.10 (Elementary good covering). — An open covering $\mathcal{I} = \{I\}$ with only one overlapping open arc $I =]\alpha, \beta + 2\pi[$ and nerve $\dot{I} =]\alpha, \beta[\subsetneq S^1$ is called an elementary good covering.

Example 3.2.11 (The Euler series as a 0-cochain)

The Euler series $\tilde{f}(x)$, which belongs to $\mathbb{C}[[x]]_1$, can be seen as a 0-cochain as follows. Consider the covering $\mathcal{I} = \{I_1, I_2\}$ of S^1 made of the arcs

$$I_1 =] - 3\pi/2, +\pi/2[\quad \text{and} \quad I_2 =] - \pi/2, +3\pi/2[.$$

The elements of \mathcal{I} intersect over the two arcs

$$\dot{I}_1 = \{x \in S^1 ; \Re(x) < 0\} \quad \text{and} \quad \dot{I}_2 = \{x \in S^1 ; \Re(x) > 0\}.$$

The corresponding 0-cochain to consider is the pair $(f_1(x), f_2(x))$ made of the restrictions of the Euler function $f(x)$ to I_1 and I_2 respectively. Both $f_1(x)$ and $f_2(x)$ are sections of \mathcal{A}_1 . The coboundary (\dot{f}_1, \dot{f}_2) is given by

$$\dot{f}_1(x) = f_1(x) - f_2(x) = 2\pi i e^{1/x} \quad \text{on} \quad \dot{I}_1 \quad \text{and} \quad \dot{f}_2(x) = f_2(x) - f_1(x) = 0 \quad \text{on} \quad \dot{I}_2$$

and has values in $\mathcal{A}^{\leq -1}$.

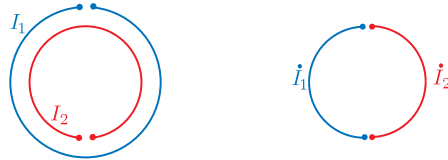


Figure 4

Since the component \dot{f}_2 is trivial one could also consider a branch covering made of the unique arc $I =] - 3\pi/2, +3\pi/2[$ overlapping on $\dot{I} = \{x \in S^1; \Re(x) < 0\}$.

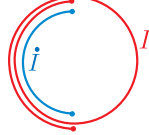


Figure 5

The (branched) 0-cochain $f(x)$ is 1-Gevrey asymptotic to $\tilde{f}(x)$ on I and its coboundary $f^+(x) - f^-(x) = 2\pi i \exp(1/x)$ is a section of $\mathcal{A}^{\leq -1}$ over \dot{I} .

Given a good covering $\mathcal{I} = \{I_j\}$ of S^1 , a 1-cochain is a family

$$f_{j,\ell} \in \Gamma(I_j \cap I_\ell; \mathcal{F}) \quad \text{for } j \text{ and } \ell \in \mathbb{Z}/p\mathbb{Z}.$$

The 1-cocycle conditions $\dot{f}_{j,k} + \dot{f}_{k,\ell} = \dot{f}_{j,\ell}$ on $I_j \cap I_k \cap I_\ell$ for all j, k, ℓ are empty since so are the 3-by-3 intersections; consequently, any 1-cochain is a 1-cocycle. Taking into account the necessary conditions $f_{j,j} = 0$ and $f_{k,j} = -f_{j,k}$ on 1-cocycles, a 1-cocycle can thus be seen as any collection $(\dot{f}_j \in \Gamma(\dot{I}_j; \mathcal{F}))$ for $j \in \mathbb{Z}/p\mathbb{Z}$.

By linearity, a 1-cocycle $(\dot{f}_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$ can be decomposed into a sum $\sum_{j \in \mathbb{Z}/p\mathbb{Z}} \dot{\varphi}_j$ where $\dot{\varphi}_j$ is the 1-cocycle over the covering \mathcal{I} having all trivial components (equal to 0, the neutral element) but the j^{th} equal to \dot{f}_j . Fix j and consider the elementary good covering \mathcal{I}_j whose nerve is \dot{I}_j and the 1-cocycle $\dot{f}_j \in \Gamma(\dot{I}_j; \mathcal{F})$. The covering \mathcal{I} is finer than \mathcal{I}_j . We identify the 1-cocycles $\dot{\varphi}_j$ and \dot{f}_j and we say that the 1-cocycle $\dot{\varphi}_j$ can be lifted into the elementary 1-cocycle \dot{f}_j .

Proposition 3.2.12. — *There exist arbitrarily fine good coverings of S^1*

Consequently, when \mathcal{F} is a sheaf over S^1 , to determine $H^1(S^1; \mathcal{F})$ it suffices to consider good coverings.

Example 3.2.13. — **(Euler equation and cohomology)** We consider the elementary good covering $\mathcal{I} = \{I\}$ of S^1 defined by the overlapping interval $I =] - 3\pi/2, +3\pi/2[$ with self-intersection $\dot{I} =] - 3\pi/2, -\pi/2[$ and we consider the sheaf \mathcal{V} of asymptotic solutions of the Euler equation (cf. Exa. 3.1.24). A 1-cocycle of \mathcal{I} in \mathcal{V} is a section $\dot{\varphi}(x) = af(x) + be^{-1/x}$ over \dot{I} with arbitrary constants a and b in \mathbb{C} . There is no 1-cocycle condition. The 0-cochains are of the form $cf(x)$ over I , with $c \in \mathbb{C}$ an arbitrary constant and they generate the 1-coboundaries $c(f(xe^{2\pi i}) - f(x)) = 2\pi ice^{-1/x}$.

The cohomological class of $\dot{\varphi}$ is then uniquely represented by $af(x)$ for $-3\pi/2 < \arg(x) < -\pi/2$. Hence, $H^1(\mathcal{I}; \mathcal{V})$ is a vector space of dimension one, isomorphic to \mathbb{C} .

Given \mathcal{J} a covering of S^1 finer than \mathcal{I} we have seen (cf. Prop. 3.2.4) that the map $\mathfrak{S}_{\mathcal{J}, \mathcal{I}} : H^1(\mathcal{I}; \mathcal{V}) \rightarrow H^1(\mathcal{J}; \mathcal{V})$ is injective. Let us check that it is surjective on the example of $\mathcal{J} = \{J_1, J_2\}$ for $J_1 =]-\pi/4, 5\pi/4[$ and $J_2 =]-5\pi/4, \pi/4[$.

We set $\dot{J}_1 =]3\pi/4, 5\pi/4[$ and $\dot{J}_2 =]-\pi/4, \pi/4[$.

A 1-cocycle $(\dot{\varphi}_1, \dot{\varphi}_2)$ over the covering \mathcal{J} is cohomologous to $(\dot{\varphi}_1 + \dot{\varphi}_2, 0)$ via the 0-cochain $(0, \dot{\varphi}_2)$ where we keep denoting by $\dot{\varphi}_2$ the continuation of $\dot{\varphi}_2$ to \dot{J}_2 . However, $\dot{\varphi} = \dot{\varphi}_1 + \dot{\varphi}_2$ can be continued to \dot{I} (we keep denoting by $\dot{\varphi}$ the continuation) and therefore, the 1-cocycle $(\dot{\varphi}_1 + \dot{\varphi}_2, 0)$ lifts up into the 1-cocycle $\dot{\varphi}$ of the covering \mathcal{I} .

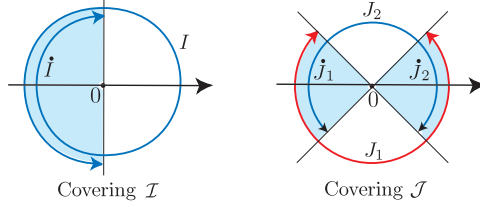


Figure 6

The proof extends to any good covering \mathcal{J} finer than \mathcal{I} by induction on the number of connected 2-by-2 intersections. We can conclude that $H^1(\mathcal{I}; \mathcal{V}) = H^1(S^1; \mathcal{V})$.

The same result can be seen as a consequence of the theorem of Leray (Thm. 3.2.6) after showing that \mathcal{I} is acyclic for \mathcal{V} .

In the case when $X = S^1$ the long exact sequence of cohomology of Theorem 3.2.7 reduces to

$$\begin{aligned} 0 \rightarrow H^0(S^1; \mathcal{G}) \rightarrow H^0(S^1; \mathcal{F}) \rightarrow H^0(S^1; \mathcal{H}) \\ \xrightarrow{\delta_0} H^1(S^1; \mathcal{G}) \rightarrow H^1(S^1; \mathcal{F}) \rightarrow H^1(S^1; \mathcal{H}) \rightarrow 0. \end{aligned}$$

The coboundary map δ_0 is defined as follows: The sheaf \mathcal{H} is the quotient of \mathcal{F} by \mathcal{G} . A 0-cocycle in $H^0(S^1; \mathcal{H})$ is a collection of $f_i \in \Gamma(I_i; \mathcal{F})$ such that $f_i - f_j$ belong to $\Gamma(I_{i,j}; \mathcal{G})$ for all i, j . There corresponds the 1-cocycle $(g_{i,j} = f_i - f_j)_{i,j}$ of \mathcal{I} with values in \mathcal{G} . To different representatives f'_i of the 0-cocycle there correspond a cohomologous 1-cocycle $(g'_{i,j} = f'_i - f'_j)_{i,j}$ of \mathcal{I} with values in \mathcal{G} ; hence, an element of $H^1(\{I_i\}; \mathcal{G})$ and consequently, an element of $H^1(S^1; \mathcal{G})$.

The Cauchy-Heine Theorem (Thm. 2.5.2) can be reformulated as a cohomological condition as follows.

Corollary 3.2.14 (Cauchy-Heine). —

(i) *The natural map $H^1(S^1, \mathcal{A}^{<0}) \rightarrow H^1(S^1, \mathcal{A})$ is the null map.*

(ii) *The natural map $H^1(S^1, \mathcal{A}^{\leq -k}) \rightarrow H^1(S^1, \mathcal{A}_s)$ is the null map.*

Proof. — (i) It suffices to prove the assertion for any good covering. Given a covering \mathcal{I} of S^1 there is a natural map from $H^1(\mathcal{I}; \mathcal{A}^{<0})$ into $H^1(\mathcal{I}; \mathcal{A})$ (cohomologous 1-cocycles of $H^1(\mathcal{I}; \mathcal{A}^{<0})$ are also cohomologous in $H^1(\mathcal{I}; \mathcal{A})$). By linearity, it suffices to consider the case of an elementary good covering $\mathcal{I} = \{I\}$ with self-intersection \dot{I} (cf. Def. 3.2.10). The Cauchy-Heine Theorem as stated in Thm. 2.5.2 says that a 1-cocycle of $H^1(\mathcal{I}; \mathcal{A}^{<0})$ is a coboundary in $H^1(\mathcal{I}; \mathcal{A})$, that is, it is cohomologous to the trivial 1-cocycle 0 in $H^1(\mathcal{I}; \mathcal{A})$.

(ii) Same proof by replacing $\mathcal{A}^{<0}$ by $\mathcal{A}^{\leq -k}$ and \mathcal{A} by \mathcal{A}_s . \square

Although the maps are zero maps, far from being null spaces, $H^1(S^1, \mathcal{A})$ and $H^1(S^1, \mathcal{A}_s)$ are huge spaces.

CHAPTER 4

LINEAR ORDINARY DIFFERENTIAL EQUATIONS: BASIC FACTS AND INFINITESIMAL NEIGHBORHOODS OF IRREGULAR SINGULARITIES

In this chapter, we first gather some basic facts on linear ordinary differential equations. Our aim is not to be exhaustive (in particular, we omit most of the proofs) but to provide the useful material to better understand series solutions of differential equations and examples. We end the chapter with the construction of infinitesimal neighborhoods for the singularities of solutions of linear differential equations at an irregular singular point in the spirit of the infinitesimal neighborhoods of algebraic geometry. The adequacy of such neighborhoods to characterize the summability properties of the formal solutions of a given differential equation is presented in Chapters 6 and 8 (Defs. 6.4.1 and 8.7.1).

Consider a linear differential operator of order n

$$(20) \quad D = b_n(x) \frac{d^n}{dx^n} + b_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + b_0(x) \quad \text{where } b_n(x) \neq 0$$

with analytic coefficients at $x = 0$. Unless otherwise specified, we assume that the coefficients b_n, b_{n-1}, \dots, b_0 do not vanish simultaneously at $x = 0$. When the coefficients b_n, b_{n-1}, \dots, b_0 are polynomials in x their maximal degree is called the *degree of D* .

4.1. Equation versus system

With the differential equation $Dy = 0$, setting

$$Y = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix},$$

is associated its companion system $\Delta Y = 0$ defined by the n -dimensional order one differential operator

$$\Delta = \frac{d}{dx} - B(x) \quad \text{where} \quad B(x) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & 0 & 1 \\ -\frac{b_0}{b_n} & \cdots & \cdots & -\frac{b_{n-1}}{b_n} \end{bmatrix}.$$

Reciprocally, the question is to determine if and how one can put a given system in companion form.

Definition 4.1.1 (Gauge transformation). — *Given a system of dimension n with meromorphic coefficients $\Delta Y \equiv \frac{dY}{dx} - B(x)Y = 0$ a gauge transformation is a linear change of the unknown variables $Z = TY$ with T invertible in a sense to be made precise. In the case when T belongs to $\text{GL}(n, \mathbb{C}\{x\}[1/x])$ the gauge transformation T is said to be meromorphic; in the case when T belongs to $\text{GL}(n, \mathbb{C}[[x]][1/x])$ it is said to be formal (meromorphic).*

A gauge transformation $Z = TY$ changes the system $\Delta Y = 0$ into the differential system ${}^T\Delta Z = 0$ with

$${}^T\Delta = T\Delta T^{-1} = \frac{d}{dx} - \frac{dT}{dx}T^{-1} - TBT^{-1}.$$

When T is meromorphic (resp. formal), so is ${}^T\Delta$; however, ${}^T\Delta$ may be meromorphic for some formal T . We can now answer the question.

Proposition 4.1.2 ((Deligne's) Cyclic vector lemma)

To any system $\Delta Y = 0$ with meromorphic coefficients there is a meromorphic gauge transformation $Z = TY$ such that the transformed system ${}^T\Delta = 0$ is in companion form.

The formulation in terms of cyclic vectors (*cf.* Rem. 4.2.6) is due to P. Deligne [Del70, Lem II.1.3] although more algorithmic proofs already existed [Cop36], [Jac37]. The companion form is obtained by differential elimination. Despite the fact that the program is short and simple it is not (at least, not yet) available in computer algebra systems such as Mathematica or Maple (see [BCLR03] for a sketched algorithm and references; see also [Ram84, Thm. 1.6.16]). As a consequence of the Cyclic vector lemma, theoretical properties can be proved equally on equations or systems (as long as these properties stay unchanged under meromorphic gauge transformations). To perform calculations one could, in principle using the algorithm, go from equations to systems

and reciprocally at convenience. Actually, these algorithms are usually very “expensive” and used sparingly.

4.2. The viewpoint of \mathcal{D} -modules

The notion of differential module, or equivalently, of \mathcal{D} -module generalizes the notion of order one differential system in an abstract setting free of coordinates. From this viewpoint, the gauge transformations and the meromorphic or formal equivalence arise naturally.

Suppose we are given a differential field (K, ∂) . Precisely, for our purpose, we suppose that K is either the field $\mathbb{C}\{x\}[1/x]$ of meromorphic series at 0 or the field $\mathbb{C}[[x]][1/x]$ of the formal ones. The derivation is $\partial = d/dx$. The constant subfield C of K , *i.e.*, the set of the elements $f \in K$ satisfying $\partial f = 0$, is $C = \mathbb{C}$ and the C -vector space of the derivations of K has dimension 1 and generator ∂ .

4.2.1. \mathcal{D} -modules and order one differential systems. —

Definition 4.2.1. — A differential module⁽¹⁾ (M, ∇) of rank n over K is a K -vector space M of dimension n equipped with a map

$$\nabla : M \longrightarrow M,$$

called connection, which satisfies the two conditions:

- (i) ∇ is additive;
- (ii) ∇ satisfies the Leibniz rule $\nabla(fm) = \partial f \cdot m + f\nabla(m)$ for all $f \in K$ and $m \in M$.

We may observe that ∇ is also \mathbb{C} -linear. Indeed, when $f \in \mathbb{C}$ is a constant the Leibniz rule reads $\nabla(fm) = f\nabla(m)$.

The link with differential systems is as follows.

Choose a K -basis $\underline{e} = [e_1 \ e_2 \ \cdots \ e_n]$ of M and let

$$[\varepsilon_1 \ \varepsilon_2 \ \cdots \ \varepsilon_n] = -[e_1 \ e_2 \ \cdots \ e_n]B \quad \text{with } B \in \text{gl}(n, K)$$

be its image by ∇ (the minus sign is introduced to fit the usual notations for systems and has no special meaning). The connection ∇ is fully determined by the matrix B . Indeed, let $y = \sum_{j=1}^n y_j e_j$ be any element of M . In matrix notation, we write $y = \underline{e}Y$ where $Y = {}^t[y_1 \ \cdots \ y_n]$ is the column matrix of the

⁽¹⁾ In French, one says “un vectoriel à connexion”.

components of y in the basis \underline{e} . Then, applying the Leibniz rule, we see that ∇y is uniquely determined by

$$\nabla y = \underline{e}(\partial Y - BY).$$

Thus, with the connection ∇ and the K -basis \underline{e} is naturally associated the differential operator $\Delta = \partial - B$ of order one and dimension n .

Definition 4.2.2. — Let (M_1, ∇_1) and (M_2, ∇_2) be differential modules.

(i) A morphism of differential modules from (M_1, ∇_1) to (M_2, ∇_2) is a K -linear map $\mathcal{T} : M_1 \rightarrow M_2$ which commutes to the connections ∇_1 and ∇_2 , i.e., such that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{\mathcal{T}} & M_2 \\ \nabla_1 \downarrow & & \downarrow \nabla_2 \\ M_1 & \xrightarrow{\mathcal{T}} & M_2 \end{array}$$

(ii) A morphism \mathcal{T} is an isomorphism if \mathcal{T} is bijective.

Denote by n_1 and n_2 the rank of (M_1, ∇_1) and (M_2, ∇_2) respectively. Choose K -basis \underline{e}_1 and \underline{e}_2 of M_1 and M_2 and denote by Δ_1 and Δ_2 the differential system operators associated with ∇_1 and ∇_2 in the basis \underline{e}_1 and \underline{e}_2 respectively. Denote by T the matrix of \mathcal{T} in these basis. The definition says that \mathcal{T} is a morphism if T satisfies the relation

$$\Delta_2 T = T \Delta_1.$$

It says that \mathcal{T} is an isomorphism if, in addition, $n_1 = n_2$ and the matrix T is invertible so that the condition may be written

$$\Delta_1 = T^{-1} \Delta_2 T$$

and is also valid for T^{-1} in the form $\Delta_1 T^{-1} = T^{-1} \Delta_2$; hence, the commutation of the diagram with $T : M_1 \rightarrow M_2$ replaced by $T^{-1} : M_2 \rightarrow M_1$. We recognize the formula linking the operators Δ_1 and Δ_2 under the gauge transformation \mathcal{T} (cf. Def. refgauge). Suppose $M_1 = M_2 =: M$. An invertible K -morphism \mathcal{T} is just a change of K -basis in M . Therefore, to the connection ∇ there are the infinitely many system operators $T^{-1} \Delta T$ associated with all $T \in \text{GL}(n, K)$ and it is natural to set the following definition.

Definition 4.2.3. — Two differential system operators $\Delta_1 = \partial - B_1$ and $\Delta_2 = \partial - B_2$ are said to be K -equivalent if there exists a gauge transformation T in $\mathrm{GL}(n, K)$ such that

$$\Delta_1 = T^{-1}\Delta_2T.$$

When $K = \mathbb{C}\{x\}[1/x]$ is the field of meromorphic series the systems are said to be meromorphically equivalent. When $K = \mathbb{C}[[x]][1/x]$ is the field of formal meromorphic series they are said to be formally equivalent or formally meromorphically equivalent.

In modern language, we should say K -similar but the old denomination K -equivalent is still in common use.

The condition is clearly an equivalence relation: indeed, any system operator Δ satisfies $\Delta = I^{-1}\Delta I$; if $\Delta_1 = T^{-1}\Delta_2T$ then $\Delta_2 = S^{-1}\Delta_1S$ with $S = T^{-1}$; if $\Delta_1 = T^{-1}\Delta_2T$ and $\Delta_2 = S^{-1}\Delta_3S$ then $\Delta_1 = (ST)^{-1}\Delta_3(ST)$. With this definition, a differential module can be identified to an equivalence class of systems.

Denote by $\mathcal{D} = K[\partial]$ the ring of differential operators on K , *i.e.*, the ring of polynomials in ∂ with coefficients in K satisfying the non-commutative rule

$$\partial x = x\partial + 1.$$

Let us now show how a differential module can be identified to a \mathcal{D} -module, *i.e.*, a module over the ring \mathcal{D} in the classical sense. For this, we go to a dual approach as follows.

Consider \mathcal{D}^n as a left \mathcal{D} -module and denote by $\underline{\varepsilon} = [\varepsilon_1 \cdots \varepsilon_n]$ its canonical \mathcal{D} -basis. Given a n -dimensional system operator $\Delta = \partial - B$ with coefficients in K we make it act linearly on \mathcal{D}^n to the right by setting

$$\sum_{j=1}^n P_j \varepsilon_j \longmapsto [P_1 \cdots P_n] \Delta = [P_1 \partial \cdots P_n \partial] - [P_1 \cdots P_n] B.$$

The cokernel $\mathcal{D}^n / \mathcal{D}^n \Delta$ has a natural structure of left \mathcal{D} -module (but no natural structure of right-module over \mathcal{D}) and has rank n (its dimension as K -vector space). Denote by $M \equiv \mathcal{D}^n / \mathcal{D}^n \Delta$ this K -vector space of dimension n . The images in the cokernel of the n elements $\varepsilon_1, \dots, \varepsilon_n$ — which we keep denoting by $\varepsilon_1, \dots, \varepsilon_n$ — of the canonical \mathcal{D} -basis $\underline{\varepsilon}$ form a K -basis of M . On another hand, the operator ∂ acting on M to the left defines a connection on M : indeed, it acts additively and satisfies the Liebniz rule. The question remains to determine which class of systems it represents. From the relation $\partial - B = 0$

in M we deduce that, for all $j = 1, \dots, n$, the components of $\partial\varepsilon_j$ in the basis $\underline{\varepsilon}$ are given by the j^{th} row of the matrix B . Hence,

$$\partial[\varepsilon_1\varepsilon_2\cdots\varepsilon_n] = [\varepsilon_1\varepsilon_2\cdots\varepsilon_n]^t B.$$

And we can conclude that the system operator associated with the connection ∂ is the adjoint $\Delta^* = \partial + {}^t B$ of Δ . We can state:

Proposition 4.2.4. — *Given a differential system operator $\Delta = \partial - B$ with coefficients B in K the pair $(M = \mathcal{D}^n/\mathcal{D}^n\Delta, \partial)$ defines a differential module of rank n over K with connection $\partial = \nabla^*$ adjoint to Δ .*

From now on, we may talk of the differential module $\mathcal{D}^n/\mathcal{D}^n\Delta$, the connection $\nabla = \partial$ being understood. With this result we can identify left \mathcal{D} -modules and differential modules equipped with a K -basis. Observe, in particular, that a morphism or an isomorphism

$$\phi : \mathcal{D}^n/\mathcal{D}^n\Delta_1 \longrightarrow \mathcal{D}^n/\mathcal{D}^n\Delta_2$$

in the sense of Definition 4.2.2 is a morphism or an isomorphism of \mathcal{D} -modules in the classical sense and reciprocally.

Proposition 4.2.5. — *Two system operators $\Delta_1 = \partial - B_1$ and $\Delta_2 = \partial - B_2$ with coefficients in K are K -equivalent if and only if the \mathcal{D} -modules $\mathcal{D}^n/\mathcal{D}^n\Delta_1$ and $\mathcal{D}^n/\mathcal{D}^n\Delta_2$ are isomorphic.*

Proof. — We have to prove that two differential systems $\Delta_1 Y = 0$ and $\Delta_2 Y = 0$ on one hand and their adjoints $\Delta_1^* Y = 0$ and $\Delta_2^* Y = 0$ on the other hand are simultaneously K -equivalent. To this end, consider fundamental solutions \mathcal{Y}_1 and \mathcal{Y}_2 of $\Delta_1 Y = 0$ and $\Delta_2 Y = 0$ respectively in any convenient extension of K (for instance, the formal fundamental solutions given by Thm. 4.3.1). The systems $\Delta_1 Y = 0$ and $\Delta_2 Y = 0$ are equivalent if and only if there exists a gauge transformation $T \in \text{GL}(n, K)$ such that $\Delta_1 = T^{-1}\Delta_2 T$ or equivalently $\mathcal{Y}_2 = T\mathcal{Y}_1$. This latter relation is equivalent to the relation ${}^t\mathcal{Y}_2^{-1} = {}^tT^{-1}{}^t\mathcal{Y}_1^{-1}$. Hence the result since ${}^t\mathcal{Y}_1^{-1}$ and ${}^t\mathcal{Y}_2^{-1}$ are fundamental solutions of the adjoints equations $\Delta_1^* Y = 0$ and $\Delta_2^* Y = 0$ respectively. \square

Remark 4.2.6. — Let us end this section with a remark on the Cyclic vector lemma (Prop. 4.1.2). In a differential module (M, ∇) of rank n one calls *cyclic vector* any vector $e \in M$ such that the n vectors $e, \nabla e, \dots, \nabla^{n-1}e$ form a K -basis of M . In such a basis, the matrix of the connection ∇ reads in the

form

$$B_{\nabla} = \begin{bmatrix} 0 & \cdots & 0 & a_{n-1} \\ 1 & & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_0 \end{bmatrix}.$$

Let Δ be a system of dimension n with coefficients in K and consider the \mathcal{D} -module $\mathcal{D}^n/\mathcal{D}^n\Delta$. In a cyclic basis \underline{e} the system Δ admits $-{}^tB_{\nabla}$ as matrix which is a companion form (cf. Sect. 4.1) but, *stricto sensu*, the minus signs in the sup-diagonal of 1's. One can cancel these minus signs by taking the basis $(e, -\partial e, \dots, (-1)^{n-1}\partial^{n-1}e)$.

4.2.2. \mathcal{D} -modules and differential operators of order n . — The aim of this section is to describe the K -equivalence of order n linear differential operators with coefficients in K . Consider a single linear differential operator

$$D = \partial^n + b_{n-1}(x)\partial^{n-1} + \cdots + b_0(x), \quad b_0, \dots, b_{n-1} \in K.$$

The operator D acts linearly on \mathcal{D} by multiplication to the right. Its cokernel $\mathcal{D}/\mathcal{D}D$ has a natural structure of left \mathcal{D} -module. The pair $(\mathcal{D}/\mathcal{D}D, \partial)$ defines a differential module of rank n . Again, by abuse, we talk of the differential module $\mathcal{D}/\mathcal{D}D$, the connection ∂ being understood.

Proposition 4.2.7. — *Let Δ be the companion system operator of D (cf. Sect. 4.1). Then, the \mathcal{D} -modules $\mathcal{D}/\mathcal{D}D$ and $\mathcal{D}^n/\mathcal{D}^n\Delta$ are isomorphic.*

Proof. — Consider the map

$$U : \mathcal{D}^n \longrightarrow \mathcal{D}, \quad (\delta_1 \cdots \delta_n) \longmapsto \delta_1 + \delta_2\partial + \cdots + \delta_n\partial^{n-1}$$

and the map $V : \mathcal{D}^n \rightarrow \mathcal{D}$, projection over the last component, defined by

$$(\delta_1 \cdots \delta_n) \longmapsto \delta_n.$$

The maps U and V are \mathcal{D} -linear; the diagram

$$\begin{array}{ccc} \mathcal{D}^n & \xrightarrow{\cdot\Delta} & \mathcal{D}^n \\ \downarrow V & & \downarrow U \\ \mathcal{D} & \xrightarrow{\cdot D} & \mathcal{D} \end{array}$$

commutes and it can be completed into the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{D}^n & \xrightarrow{\cdot\Delta} & \mathcal{D}^n & \longrightarrow & \mathcal{D}^n/\mathcal{D}^n\Delta \longrightarrow 0 \\ & & \downarrow V & & \downarrow U & & \downarrow u \\ 0 & \longrightarrow & \mathcal{D} & \xrightarrow{\cdot D} & \mathcal{D} & \longrightarrow & \mathcal{D}/\mathcal{D}D \longrightarrow 0. \end{array}$$

The quotient map u does exist. It is left \mathcal{D} -linear and surjective since U is also left \mathcal{D} -linear and surjective. On the other hand, the modules $\mathcal{D}^n/\mathcal{D}^n\Delta$ and $\mathcal{D}/\mathcal{D}D$ have equal ranks. Therefore, u is an isomorphism of K -vector spaces and in particular, is injective. \square

From Propositions 4.2.7 and 4.2.5 we may set the following definition.

Definition 4.2.8 (equivalent operators). — *Two linear differential operators D_1 and D_2 of order n are said to be K -equivalent if the \mathcal{D} -modules $\mathcal{D}/\mathcal{D}D_1$ and $\mathcal{D}/\mathcal{D}D_2$ are isomorphic.*

Let us now make explicit the equivalence of order n linear differential operators in the spirit of Definition 4.2.3.

Recall that $\mathcal{D} = K[\partial]$ is a non commutative ring with non-commutation relations generated by $\partial x = x\partial + 1$. In the ring \mathcal{D} there is an euclidian division on the right and on the left. Consequently, any left or right ideal is principal and any two differential operators have a greatest common divisor on the left (denoted by lgcd) and on the right (rgcd) as well as a least common multiple on the left (llcm) and on the right (rlcm). These gcd and lcm are uniquely determined by adding the condition that they are monic polynomials, which we do.

The counterpart for a differential operator $D \in \mathcal{D}$ of a gauge transformation for a system involves a transformation \mathcal{T}_A , with $A \in \mathcal{D}$, of the form

$$\mathcal{T}_A(D) = \text{llcm}(D, A)A^{-1}.$$

By this, we mean that we take the lcm of D and A on the left and we divide it by A on the right (this is possible since, by definition, A can be factored on the right in any llcm involving A). In other words, $\mathcal{T}_A(D)$ is the factor of smallest degree we must multiply A on the left to obtain a left multiple of D . Notice that such a factor is unique due to the uniqueness of $\text{llcm}(D, A)$ as a monic polynomial.

Proposition 4.2.9. — *The differential operators D_1 and $D_2 \in \mathcal{D}$ are K -equivalent if and only if there exists $A \in \mathcal{D}$ prime to D_2 to the right such*

that

$$D_1 = \mathcal{T}_A(D_2).$$

We may notice that, as A and D_2 are prime, the operators D_2 and $\mathcal{T}_A(D_2)$ have the same order.

Proof. — By definition, the K -equivalence of D_1 and D_2 means that there is an isomorphism of \mathcal{D} -modules

$$\varphi : \mathcal{D}/\mathcal{D}D_1 \longrightarrow \mathcal{D}/\mathcal{D}D_2.$$

As a morphism of \mathcal{D} -modules the map φ is well defined by

$$\varphi(1 + \mathcal{D}D_1) = A + \mathcal{D}D_2.$$

For any $L \in \mathcal{D}$, one has then $\varphi(L + \mathcal{D}D_1) = LA + \mathcal{D}D_2$. Since $\varphi(D_1) = 0$ there exists $L_1 \in \mathcal{D}$ such that $D_1A = L_1D_2$. Conversely, any A such that there is an L_1 satisfying $D_1A = L_1D_2$ determines a morphism of \mathcal{D} -modules from $\mathcal{D}/\mathcal{D}D_1$ into $\mathcal{D}/\mathcal{D}D_2$ by setting $\varphi(1 + \mathcal{D}D_1) = A + \mathcal{D}D_2$.

The injectivity of φ means that the condition $\varphi(L) = 0$, *i.e.*, $LA = PD_2$ for a certain $P \in \mathcal{D}$, implies $L = QD_1$ with $Q \in \mathcal{D}$. Hence, to any relation $LA = PD_2$ there is $Q \in \mathcal{D}$ such that $PD_2 = QD_1A$, that is to say, any left common multiple of A and D_2 is a left multiple of D_1A . Otherwise said, D_1A is the lcm of A and D_2 and then,

$$D_1 = \mathcal{T}_A(D_2).$$

Let us now express the surjectivity of φ . This amounts to the fact that there exists $L \in \mathcal{D}$ such that $\varphi(L + \mathcal{D}D_1) = 1 + \mathcal{D}D_2$, which means that there is $P \in \mathcal{D}$ such that $LA + PD_2 = 1$. This is a Bézout relation for A and D_2 on the right which means that A and D_2 are prime on the right. \square

4.3. Classifications

We denote by $\tilde{K} = \mathbb{C}[[x]][1/x]$ the field of all meromorphic series at 0 either convergent or not and by $K = \mathbb{C}\{x\}[1/x]$ the subfield of the convergent ones. We consider linear differential systems or equations with coefficients in K , *i.e.*, with convergent meromorphic coefficients.

The formal classification of linear differential systems or equations is the classification under \tilde{K} -equivalence (*cf.* Def 4.2.3 and Prop. 4.2.9). The meromorphic⁽²⁾ classification is the classification under K -equivalence.

⁽²⁾ We use the term *meromorphic* in the sense of convergent meromorphic. Otherwise, we specify *formal meromorphic* or simply *formal*.

In this section, we sketch the main theoretical results on the formal and the meromorphic classes of systems or equations. In the case of equations we also sketch the practical algorithms based on Newton polygons to compute the formal invariants.

4.3.1. The case of systems. — Denote by $'$ the derivation with respect to x , writing Y' instead of $\frac{dY}{dx}$, and consider an order one linear differential system

$$(21) \quad \Delta Y \equiv Y' - B(x)Y = 0$$

with meromorphic coefficients (*i.e.*, $B(x) \in g\ell(n, K)$).

Recall (*cf.* Sect. 4.1) that a gauge transformation $Z = TY$ changes the differential system $Y' - B(x)Y = 0$ into the differential system $Z' - {}^T B(x)Z = 0$ with

$${}^T B = T' T^{-1} + T B T^{-1}.$$

When $T(x)$ is meromorphic (we denote $T \in \mathbb{G} = \text{GL}(n, \mathbb{C}\{x\}[1/x])$) the matrix ${}^T B(x)$ is also meromorphic. But the matrix ${}^T B(x)$ may also be convergent for some divergent $T(x)$. We denote by $\tilde{\mathbb{G}}(B)$ the set of formal meromorphic gauge transformations $T \in \text{GL}(n, \mathbb{C}[[x]][1/x])$ such that ${}^T B(x)$ is convergent. The set $\tilde{\mathbb{G}}(B)$ contains \mathbb{G} . While \mathbb{G} is a group, $\tilde{\mathbb{G}}(B)$ is not. The *meromorphic class* of the system is its orbit under the gauge transformations in \mathbb{G} while its *formal class* is its (larger) orbit under those in $\tilde{\mathbb{G}}(B)$.

4.3.1.1. Formal classification. — The formal classification of n -dimensional meromorphic linear differential systems is performed by selecting, in each class, a system of a special form called a *normal form*. There exist algorithms to fully calculate a normal form of any given system (*cf.* end of Sect. 4.3.2.3).

Theorem 4.3.1 (Formal fundamental solution and normal form)

1. To any system (21) : $Y' = B(x)Y$ there is a formal fundamental solution (*i.e.*, a matrix of n linearly independent formal solutions) of the form

$$\mathcal{Y}(x) = \tilde{F}(x) x^L e^{Q(1/x)}$$

where

$\triangleright Q(1/x) = \bigoplus_{j=1}^J q_j(1/x) I_{n_j}$ (assume the q_j 's are distinct) is a diagonal matrix satisfying $Q(0) = 0$; its diagonal entries are polynomials in $1/x$ or in a fractional power $1/t = 1/x^{1/p}$ of $1/x$; the notation I_{n_j} stands for the identity

matrix of dimension n_j . The smallest possible number p is called the degree of ramification of the system, $e^{Q(1/x)}$ the irregular part of $\mathcal{Y}(x)$ and the q_j 's the determining polynomials.

▷ $L \in \mathfrak{gl}(n, \mathbb{C})$ is a constant matrix called the matrix of the exponents of formal monodromy.

▷ $\tilde{F}(x) \in \mathrm{GL}(n, \mathbb{C}[[x]][1/x])$ is an invertible formal meromorphic matrix.

2. The matrix $\mathcal{Y}_0(x) = x^L e^{Q(1/x)}$ is a (formal) fundamental solution of a system

$$Y' = B_0(x)Y$$

with polynomial coefficients in x and $1/x$. The system $Y' = B_0(x)Y$ is formally equivalent to the initial system $Y' = B(x)Y$ via the formal gauge transformation $\tilde{F}(x)$ (hence, $B(x) = \tilde{F}B_0(x)$) and it is called a normal form of the given system $Y' = B(x)Y$. The fundamental matrix $\mathcal{Y}_0(x)$ is called a normal solution.

A normal solution exhibits all formal invariants. However, the normal form and the normal solution are not unique: indeed, given $P \in \mathrm{GL}(n, \mathbb{C})$ any permutation matrix or any matrix commuting with $Q(1/x)$, the matrix

$$P \mathcal{Y}_0(x) P^{-1} = x^{PLP^{-1}} e^{PQ(1/x)P^{-1}}$$

is also a normal solution associated with the normal form $Y' = {}^P B_0(x)Y$ since a fundamental solution of the given system $Y' = B(x)Y$ reads in the form

$$\mathcal{Y}(x) P^{-1} = (\tilde{F}(x) P^{-1}) P \mathcal{Y}_0(x) P^{-1}$$

and $\tilde{F}(x) P^{-1}$ still belongs to $\mathrm{GL}(n, \mathbb{C}[[x]][1/x])$. In the unramified case (*i.e.*, with ramification degree $p = 1$), a minimal full set of formal invariants is given by the diagonal matrix $Q(1/x)$ of the determining polynomials up to permutation and by the invariants of similarity of L (eigenvalues and size of the corresponding irreducible Jordan blocks). In the ramified case (*i.e.*, with ramification degree $p > 1$) the situation is a little more intricate : given a determining polynomial in the variable $t' = x^{1/p'}$ (*i.e.*, with ramification degree p') any element in its orbit under the action of the Galois group of the ramification $t'^{p'} = x$ is also a determining polynomial and any of them equally characterizes the orbit. In other words, a minimal set of invariants is well determined by one polynomial in each orbit jointly with the invariants of similarity of L .

Any normal form is meromorphically equivalent to $Y' = B_0(x)Y$. That's why, sometimes, one generalizes the definition by calling normal form any system meromorphically equivalent to $Y' = B_0(x)Y$.

The theorem of formal classification was first proved in a weaker form, called Hukuhara-Turrittin Theorem, in which the given system $Y' = B(x)Y$ is considered as a system in the ramified variable $t = x^{1/p}$ (p the degree of ramification of the system) allowing, thus, gauge transformations in a finite extension of the initial variable x (cf. [DMR07, p. 104, Thm. (4.2.1)] and also [Was76, HS99]). Stated as above it was first proved by W. Balser, W. Jurkat and D.A. Lutz [BJL79]. A simpler proof and an expression of the normal form in terms of rank reduced systems built on the minimal set of invariants can be found in [LR01].

Let us now state some definitions associated with the formal invariants.

Choose a formal fundamental solution of System (21):

$$(22) \quad \mathcal{Y}(x) = \tilde{F}(x)x^L e^{Q(1/x)} \quad \text{with} \quad Q = \bigoplus_{j=1}^J q_j(1/x)I_{n_j} \quad \text{and distinct } q_j\text{'s}$$

and the normal form $Y' = B_0(x)Y$ with fundamental solution $\mathcal{Y}_0(x) = x^L e^{Q(1/x)}$.

Definition 4.3.2 (Stokes arcs). —

(i) Let $q \in \mathbb{C}[1/x]$ be a polynomial of degree $k > 0$ in the variable $1/x$. We call Stokes arc associated with $e^{q(1/x)}$ (in short, with q) the closure of any arc of S^1 of length π/k made of directions where $e^{q(1/x)}$ is flat.

(ii) In the case of ramified polynomials $q \in \mathbb{C}[1/x^{1/p}]$, $p \in \mathbb{N}^*$, Stokes arcs can be defined similarly w.r.t. the variable $t = x^{1/p}$ on the corresponding p -sheet cover of S^1 . When the fractional degree of q is over $1/2$ we call Stokes arcs of q their projection on S^1 by the map $t \mapsto x = t^p$. Otherwise, the projections are onto S^1 and one has to keep working with the variable t in the p -sheet cover.

(iii) The Stokes arcs of a linear differential equation or system are the Stokes arcs associated with all its determining polynomials.

Example 4.3.3. — Suppose a determining polynomial of system (21) be given by

$$q(1/x) = -1/x^{2/3}.$$

Then, the polynomials $jq(1/x)$ and $j^2q(1/x)$ (where $j^3 = 1$) are also determining polynomials of system (21). A fundamental solution of the system in the variable $t = x^{1/3}$ contains the three exponentials e^{-1/t^2} , e^{-j/t^2} and e^{-j^2/t^2} to which correspond the six Stokes arcs defined by $-\pi/4 \leq \arg(t) \leq +\pi/4 \pmod{\pi/3}$ and $3\pi/4 \leq \arg(t) \leq 5\pi/4 \pmod{\pi/3}$. By projection of these six arcs on the circle S^1 of directions in the variable x we obtain the two Stokes arcs defined by $-3\pi/4 \leq \arg(x) \leq +3\pi/4 \pmod{\pi}$, each one associated with the three polynomials.

The matrix $\tilde{F}(x)$ satisfies the homological system

$$(23) \quad \frac{dF}{dx} = B(x)F - FB_0(x).$$

which is a linear differential system in the entries of F and which admits the polynomials $q_\ell - q_j$ for $j, \ell = 1, \dots, J$ as determining polynomials and so, we can state:

Proposition 4.3.4. — *The Stokes arcs of the homological system (23) are the Stokes arcs associated with all polynomials $q_\ell - q_j$ for $1 \leq j \neq \ell \leq J$.*

Split the matrix $\tilde{F}(x)$ into column-blocks corresponding to the block-structure of Q (for $j = 1, \dots, J$, the matrix $\tilde{F}_j(x)$ has n_j columns):

$$\tilde{F}(x) = [\tilde{F}_1(x) \tilde{F}_2(x) \cdots \tilde{F}_J(x)].$$

Definition 4.3.5 (Stokes arcs of $\tilde{F}_j(x)$). — *We call Stokes arcs of $\tilde{F}_j(x)$ the Stokes arcs associated with the polynomials $q_\ell - q_j$ for $1 \leq \ell \leq J$, $\ell \neq j$.*

The Stokes arcs of the homological system are the Stokes arcs of all $\tilde{F}_j(x)$.

Definition 4.3.6 (Levels, anti-Stokes directions)

We call

(i) levels of system (21) the degrees of the determining polynomials $q_\ell - q_j$ for $1 \leq j \neq \ell \leq J$, of the homological system (23);

(ii) anti-Stokes direction associated with

$$(24) \quad (q_\ell - q_j)(1/x) = -a_{\ell,j} / x^k (1 + o(1/x)) \neq 0$$

any direction along which the exponential $e^{q_\ell - q_j}$ has maximal decay, i.e., any direction $\theta = \arg(a_{\ell,j})/k \pmod{2\pi/k}$ along which $-a_{\ell,j}/x^k$ is real negative;

(iii) anti-Stokes directions of system (21) the anti-Stokes directions associated with all determining polynomials $(q_\ell - q_j)(1/x) \neq 0$ of the homological system (23);

(iv) levels of $\tilde{F}_j(x)$ the degrees of the polynomials $q_\ell - q_j$ for $1 \leq \ell \leq J$, $\ell \neq j$;

(v) anti-Stokes directions of $\tilde{F}_j(x)$ the anti-Stokes directions associated with the polynomials $q_\ell - q_j$ for $1 \leq \ell \leq J$, $\ell \neq j$.

Observe that 0 is not a level since $q_\ell \neq q_j$ for all $\ell \neq j$ and the polynomials q contain no constant term. Notice, in the right hand side of (24), the minus sign which we would not introduce if we worked at infinity.

The anti-Stokes directions of a system are the middle points of the Stokes arcs of its homological system. The denomination “anti-Stokes directions” is not universal: sometimes, one calls such directions “Stokes directions” while to us, the Stokes directions are the oscillating lines of the exponentials $e^{q_\ell - q_j}$.

It is worth to notice that it is always possible to permute the columns of a formal fundamental solution by writing it

$$\mathcal{Y}(x) = \tilde{F}(x) P x^{P^{-1}LP} e^{P^{-1}Q(1/x)P}$$

with P the chosen permutation. It is also always possible to normalize a given eigenvalue of L , say λ_1 , and a given determining polynomial, say q_1 , to zero by the change of variable $Y \mapsto x^{-\lambda_1} e^{-q_1} Y$ in the initial system (and at the same time, in its normal form). The Stokes arcs and the levels of $\tilde{F}_1(x)$ are then the Stokes arcs and the degrees of the determining polynomials q_j themselves.

4.3.1.2. Meromorphic classification. — The meromorphic classification proceeds differently than the formal one since it’s hopeless to exhibit (and, a fortiori, to calculate), in each meromorphic class, a system of a special form analogous to the normal form of the formal classification. Theoretically, the meromorphic classes are well identified as non-Abelian 1-cohomology classes. In practice, the meromorphic classes are identified via a finite number of matrices of a special form called *Stokes matrices*. Contrary to normal solutions, the Stokes matrices do not depend algebraically on the system; they are, in general, deeply transcendental with respect to the coefficients of the system. Some algorithms exist to calculate numerical approximations of the Stokes matrices in some “simple” situations but, yet, none is efficient in the very general case.

Since the meromorphic classification refines the formal meromorphic one it is convenient, without any loss, to restrict the classification to a given formal class with normal form $Y' = B_0(x)Y$ and normal solution $\mathcal{Y}_0(x) = x^L e^{Q(1/x)}$.

Any system $Y' = B(x)Y$ in the formal class of $Y' = B_0(x)Y$ satisfies, by definition, a relation $B(x) = \tilde{F}B_0(x)$ for a convenient formal gauge transformation $\tilde{F}(x)$ but such a gauge transformation $\tilde{F}(x)$ is not unique in general: one

has $\tilde{F}_1 B_0 = \tilde{F}_2 B_0$ if and only if $\tilde{F}_2^{-1} \tilde{F}_1 B_0 = B_0$, *i.e.*, if and only if there exists a gauge transformation T which leaves invariant the normal form $Y' = B_0(x)Y$ and such that $\tilde{F}_1(x) = \tilde{F}_2(x)T(x)$. Notice that $T(x)$ acts on $\tilde{F}_2(x)$ to the right. The gauge transformations T for which ${}^T B_0 = B_0$ form a group.

Definition 4.3.7. — *The group $\mathbb{G}_0(B_0) \subset \tilde{\mathbb{G}}(B_0)$ of the gauge transformations T for which ${}^T B_0 = B_0$ is called the group of isotropies or group of invariance of the normal form $Y' = B_0(x)Y$.*

The group $\mathbb{G}_0(B_0)$ is in general small, even trivial, and it is easily determined in each particular case: it is made of all matrices $T(x)$ such that there exists a matrix $C \in \text{GL}(n, \mathbb{C})$ satisfying $T(x)\mathcal{Y}_0(x) = \mathcal{Y}_0(x)C$; this corresponds to constant block-diagonal matrices C commuting with $Q^{(3)}$ and such that $x^L C x^{-L}$ is meromorphic. In the case when all diagonal terms q_j in $Q(1/x)$ are distinct the group $\mathbb{G}_0(B_0)$ is made of all invertible constant diagonal matrices; if, in addition, we ask for tangent-to-identity transformations then the group reduces to the identity.

Examples 4.3.8. — Denote by I_j the identity matrix of dimension j and by J_j the irreducible nilpotent upper Jordan block of dimension j .

▷ Suppose the normal solution has the form

$$\mathcal{Y}_0(x) = x^{\lambda_1 I_1 \oplus (\lambda_2 I_3 + J_3)} e^{q_1 I_1 \oplus q_2 I_3}$$

where $0 < \Re(\lambda_1), \Re(\lambda_2) < 1$ and where $q_1 \neq q_2$ are polynomials in $1/x$. The invertible matrices C such that $\mathcal{Y}_0(x)C\mathcal{Y}_0(x)^{-1}$ is a meromorphic transformation are those which commute both to $e^{q_1 I_1 \oplus q_2 I_3}$ (this is a general fact) and to $x^{\lambda_1 I_1 \oplus (\lambda_2 I_3 + J_3)}$. One can check that this means that the matrix C has the form $C = C_1 \oplus C_2$ where $C_1 = cI_1$ with $c \in \mathbb{C}^*$ and $C_2 = c_1 I_3 + c_2 J_3 + c_3 J_3^2$ with $c_1, c_2, c_3 \in \mathbb{C}$ and $c_1 \neq 0$. All such constant matrices C form the group $\mathbb{G}_0(B_0)$.

▷ Suppose the normal solution has the form $\mathcal{Y}_0(x) = \bigoplus x^{L_j} e^{q_j I_{n_j}}$ with distinct q_j 's and matrices $L_j = \text{diag}(\lambda_{j,1}, \dots, \lambda_{j,n_j})$ with integer coefficients $\lambda_{j,1}, \dots, \lambda_{j,n_j} \in \mathbb{Z}$. Then, $C = \bigoplus C_j$ is any constant invertible block-diagonal matrix with C_j of dimension n_j and the elements of $\mathbb{G}_0(B_0)$ are the transformations of the form $T(x) = \bigoplus x^{L_j} C_j x^{-L_j}$. Their coefficients are polynomials in x and $1/x$.

The meromorphic classes of formal gauge transformations of a system, either a normal form or not, $Y' = B_0(x)Y$ are, by definition, the elements of the quotient $\mathbb{G} \setminus \tilde{\mathbb{G}}(B_0)$ of all formal meromorphic gauge transformations of $Y' = B_0 Y$ by the convergent ones to the left. The meromorphic classes of systems in the formal class of $Y' = B_0 Y$ are the quotient $\mathbb{G} \setminus \tilde{\mathbb{G}}(B_0) / \mathbb{G}_0(B_0)$

⁽³⁾ If $Q = \bigoplus q_j I_{n_j}$ then $C = \bigoplus C_j$ with matrices C_j of size n_j .

of the previous classifying set by the group $\mathbb{G}_0(B_0)$ of invariance of $Y' = B_0 Y$ to the right (recall that the isotropies act on gauge transformations to the right, *cf. supra*). Since the group $\mathbb{G}_0(B_0)$ can always be made explicit it is sufficient to describe the classifying set $\mathbb{G} \setminus \tilde{\mathbb{G}}(B_0)$ of gauge transformations⁽⁴⁾ of the normal form.

A first description of the meromorphic classes of gauge transformations was set up through a careful analysis of the Stokes phenomenon by Y. Sibuya [Sib77, Sib90] and by B. Malgrange [Mal79] (*cf. Coms. 2.5.3*). To state their result we need to introduce the sheaf $\Lambda^{<0}(B_0)$ over S^1 of germs of flat isotropies of the normal form $Y' = B_0(x)Y$: a germ $\varphi(x)$ in $\Lambda^{<0}(B_0)$ is a germ of $\mathrm{GL}(n, \mathcal{A})$ which is asymptotic to the identity⁽⁵⁾ and satisfies $\varphi B_0 = B_0$. The sheaf $\Lambda^{<0}(B_0)$ is a sheaf of non-commutative groups.

Theorem 4.3.9 (Malgrange-Sibuya). — *The classifying set $\mathbb{G} \setminus \tilde{\mathbb{G}}(B_0)$ is isomorphic to the first (non Abelian) cohomology set $H^1(S^1; \Lambda^{<0}(B_0))$.*

The map from $\mathbb{G} \setminus \tilde{\mathbb{G}}(B_0)$ into $H^1(S^1; \Lambda^{<0}(B_0))$ is abstractly given by the Main Asymptotic Existence Theorem (Cor. 4.4.4) while, way back, it is made explicit by means of Cauchy-Heine integrals.

Actually, meromorphic classes of gauge transformations can be given a simpler characterization as follows.

Let \mathfrak{A} be the set of anti-Stokes directions of the normal form $Y' = B_0(x)Y$ and denote by $\mathrm{Sto}_\alpha(B_0)$ the subgroup of the stalk $\Lambda_\alpha^{<0}(B_0)$ made of all germs of flat isotropies of $Y' = B_0(x)Y$ having maximal decay at α .

When $\alpha \notin \mathfrak{A}$ the group $\mathrm{Sto}_\alpha(B_0)$ is trivial (no flat isotropy has maximal decay but the identity). When $\alpha \in \mathfrak{A}$ the group $\mathrm{Sto}_\alpha(B_0)$ can be given a linear representation as follows: given a normal solution $\mathcal{Y}_0(x) = x^L e^{Q(1/x)}$ with $Q(1/x) = \bigoplus_{j=1}^J q_j(1/x)$ and distinct q_j 's choose a determination $\underline{\alpha}$ of α . Denote by $\mathcal{Y}_{0,\underline{\alpha}}(x)$ the function defined by $\mathcal{Y}_0(x)$ with that determination of the argument near the direction α . An element $\varphi_\alpha(x)$ of $\mathrm{Sto}_\alpha(B_0)$ is a flat transformation such that

$$\varphi_\alpha(x) \mathcal{Y}_{0,\underline{\alpha}}(x) = \mathcal{Y}_{0,\underline{\alpha}}(x) (I_n + C_\alpha)$$

for a unique constant invertible matrix $I_n + C_\alpha$.

This implies that $\varphi_\alpha(x) = x^L e^{Q(1/x)} (I_n + C_\alpha) e^{-Q(1/x)} x^{-L}$ with the given

⁽⁴⁾ D.G. Babbitt and V.S. Varadarajan [BV89] call them *meromorphic pairs* (B_0, \tilde{F}) .

⁽⁵⁾ *Flatness* must be understood, here, in the multiplicative meaning of *asymptotic to identity*.

choice of the argument near α . Denote by $C_\alpha = [C_\alpha^{(\ell,j)}]$ the decomposition of C_α into blocks fitting the structure of Q . Hence, the germ $\varphi_\alpha(x)$ reads

$$\varphi_\alpha(x) = x^L (I_n + C_\alpha^{(\ell,j)} e^{(q_\ell - q_j)(1/x)}) x^{-L}.$$

An exponential $e^{q(1/x)} = e^{-a/x^k(1+o(1/|x^k|))}$ has maximal decay in a direction $\underline{\alpha} \in S^1$ if and only if $-ae^{-ik\underline{\alpha}}$ is real negative (k might be fractional). Hence, $\varphi_\alpha(x)$ is flat in direction α if and only if, as soon as $e^{q_\ell(1/x) - q_j(1/x)}$ does not have maximal decay in direction $\underline{\alpha}$ the corresponding block $C_\alpha^{(\ell,j)}$ of C_α vanishes. In particular, for $j = \ell$, the exponential $e^{q_j - q_\ell}$ does not have maximal decay and the corresponding diagonal block $C_\alpha^{(j,j)}$ is zero; if $e^{q_j - q_\ell}$ has maximal decay in direction $\underline{\alpha}$ then $e^{q_j - q_\ell}$ has not and thus, if a block $C_\alpha^{(\ell,j)}$ is not equal to zero the symmetric block $C_\alpha^{(j,\ell)}$ is necessarily zero. This implies that the matrix $I_n + C_\alpha$ is unipotent. Reciprocally, any constant unipotent matrix with the necessary blocks of zeros characterizes a unique element of $\text{Sto}_\alpha(B_0)$. Consequently, $\text{Sto}_\alpha(B_0)$ has a natural structure of a *unipotent Lie group*.

The Malgrange-Sibuya Theorem has been improved by showing that in each 1-cohomology class there is a unique 1-cocycle of a special form called *the Stokes cocycle* which is constructible from any cocycle in its 1-cohomology class [LR94, Thm. II.2.1]; the uniqueness of the Stokes cocycle is further developed in [LR03].

Definition 4.3.10 (Stokes cocycle). — A Stokes cocycle is a 1-cocycle $(\varphi_\alpha)_{\alpha \in \mathfrak{A}}$ with the following properties: it is indexed by the set \mathfrak{A} of anti-Stokes directions and each component φ_α determines an element of $\text{Sto}_\alpha(B_0)$.

The set of Stokes cocycles can be identified to the finite product $\prod_{\alpha \in \mathfrak{A}_0} \text{Sto}_\alpha(B_0)$ and we can state:

Theorem 4.3.11 (Stokes cocycle). —

The classifying set $\mathbb{G} \setminus \tilde{\mathbb{G}}(B_0)$ is isomorphic to the product $\prod_{\alpha \in \mathfrak{A}_0} \text{Sto}_\alpha(B_0)$ of the Stokes groups associated with a normal form $Y' = B_0(x)Y$.

From this theorem the classifying set inherits a natural structure of a unipotent Lie group. For applications of this property we refer to [LR94].

Let $(\varphi_\alpha)_{\alpha \in \mathfrak{A}}$ be a Stokes cocycle associated with a gauge transformation $\tilde{F}(x)$ from the normal form $Y' = B_0(x)Y$ to a system $Y' = B(x)Y$ (hence, $\tilde{F}(x) \in \tilde{\mathbb{G}}(B_0)$). Let $\mathcal{Y}_0(x)$ be a normal solution. Choose a determination $\underline{\alpha}$ of the argument for all α (it is usually understood that all $\underline{\alpha}$ belong to a same interval $]2m\pi, 2(m+1)\pi[$) and denote by $\mathcal{Y}_{0,\underline{\alpha}}(x)$ the normal solution

with that choice of a determination of the argument. Finally, for all $\alpha \in \mathfrak{A}$, let the matrix $(I_n + C_{\underline{\alpha}})_{\alpha \in \mathfrak{A}}$ represent φ_α with respect to these choices.

Definition 4.3.12. — *The matrices $(I_n + C_{\underline{\alpha}})_{\alpha \in \mathfrak{A}}$ are called the Stokes matrices associated with the fundamental solution $\tilde{F}(x) \mathcal{Y}_{0,\underline{\alpha}}(x)$.*

Like Stokes cocycles, Stokes matrices characterize the meromorphic classes of gauge transformations: they form *a full free set of meromorphic invariants*.

The Stokes cocycle and the Stokes matrices are connected to the theory of summation (Chap. 6) as follows. Suppose we are given a formal fundamental solution $\mathcal{Y}(x) = \tilde{F}(x) \mathcal{Y}_0(x)$ at 0 and an anti-Stokes direction $\alpha \in \mathfrak{A}$ and denote by $F_\alpha^+(x)$ and $F_\alpha^-(x)$ the sums (k - or multisums) of $\tilde{F}(x)$ to the left and to the right of the direction α .

Theorem 4.3.13. — *The Stokes cocycle $(\varphi_\alpha)_{\alpha \in \mathfrak{A}}$ satisfy*

$$\varphi_\alpha = F_\alpha^+(x)^{-1} F_\alpha^-(x) \quad \text{for all } \alpha \in \mathfrak{A}.$$

The Stokes matrices $(I_n + C_{\underline{\alpha}})_{\alpha \in \mathfrak{A}}$ at 0 associated with $\tilde{F}(x) \mathcal{Y}_{0,\underline{\alpha}}(x)$ for a given determination $\underline{\alpha}$ of α satisfy

$$F_\alpha^-(x) \mathcal{Y}_{0,\underline{\alpha}}(x) = F_\alpha^+(x) \mathcal{Y}_{0,\underline{\alpha}}(x) (I_n + C_{\underline{\alpha}}) \quad \text{for all } \alpha \in \mathfrak{A}.$$

Formerly, one used to call Stokes matrices all matrices $I_n + C$ satisfying a condition of the type

$$F_j(x) \mathcal{Y}_{0,\underline{\alpha}}(x) = F_\ell(x) \mathcal{Y}_{0,\underline{\alpha}}(x) (I_n + C)$$

linking two overlapping asymptotic solutions, *i.e.*, any matrix representing a germ of isotropy $F_\ell(x)^{-1} F_j(x) = \mathcal{Y}_{0,\underline{\alpha}}(x) (I_n + C) \mathcal{Y}_{0,\underline{\alpha}}(x)^{-1}$, not necessarily a Stokes germ. This appeared to be not restrictive enough to easily characterize the meromorphic classes of systems or to exhibit good Galoisian properties: an example of a non-Galoisian “Stokes matrix” in the wide sense is given in [LR94, Sect. III.3.3.2]. Henceforward, we use the expression Stokes matrix in the restrictive sense of associated to a Stokes cocycle.

4.3.2. The case of equations. — The meromorphic and the formal equivalence of linear differential operators of order n were given in Definition 4.2.8 with a characterization in Proposition 4.2.9.

Like for systems the formal class of an equation is made explicit from a formal fundamental solution which can be read as the first row of a formal

fundamental solution of its companion system. Each such solution takes the form

$$\phi(x) x^\lambda e^{q(1/x)}$$

where the factors $\phi(x)$ are polynomials in $\ln(x)$ with formal series coefficients. Levels, Stokes arcs and anti-Stokes directions are defined similarly as for systems. The invariants are all the determining polynomials $q(1/x)$ with multiplicities, the corresponding exponents λ and the degrees in $\ln(x)$ of each associated $\phi(x)$. The meromorphic classes in a given formal class are also characterized by (adequate) Stokes matrices.

The formal invariants are much easier to determine for an equation than for a system. Below we sketch a procedure to follow for an equation.

4.3.2.1. Newton polygons. — Newton polygons are a very convenient tool to identify the formal invariants of a linear differential equation $Dy = 0$ at a singular point. By means of a change of variable any singular point can be moved to the origin 0. However, we state the definitions both at 0 and at infinity.

Consider a linear differential operator

$$D = b_n \frac{d^n}{dx^n} + b_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + b_0$$

with coefficients b_j that are either meromorphic series in x (for a study at $x = 0$) or in powers of $1/x$ (for a study at $x = \infty$). Temporarily, we do not need that the coefficients be convergent. The valuation of a power series $b(x) = \sum_{m \geq m_0} \beta_m x^m$ at the origin is denoted by $v_0(b)$ and defined as the smallest degree with respect to x of the non-zero monomials $\beta_m x^m$ of b ; thus, $v_0(b) = m_0$ when $\beta_{m_0} \neq 0$. The valuation of a power series $b(1/x) = \sum_{m \geq m_1} \beta_m / x^m$ at infinity is denoted by $v_\infty(b)$ and defined as the highest degree with respect to x of a non-zero monomial β_m / x^m of b ; thus, $v_\infty(b) = -m_1$ when $\beta_{m_1} \neq 0$. When b is a polynomial in x , then $v_\infty(b)$ is the degree of b with respect to x .

Definition 4.3.14 (Newton polygons). —

(i) Newton polygon at 0. — Suppose the coefficients b_j of D are formal or convergent meromorphic power series in x . With the operator D one associates in $\mathbb{R}^+ \times \mathbb{R}$ the set \mathcal{P}_D of marked points

$$\mathcal{P}_D = \{(j, v_0(b_j) - j); 0 \leq j \leq n\}.$$

The Newton polygon $\mathcal{N}_0(D)$ of D at 0 is the upper envelop in $\mathbb{R}^+ \times \mathbb{R}$ of the various attaching lines of \mathcal{P}_D with non-negative slopes.

(ii) Newton polygon at infinity. — Suppose the coefficients b_j of D are formal or convergent meromorphic power series in $1/x$. With the operator D one associates in $\mathbb{R}^+ \times \mathbb{R}$ the set \mathcal{P}_D of marked points

$$\mathcal{P}_D = \{(j, v_\infty(b_j) - j); 0 \leq j \leq n\}.$$

The Newton polygon $\mathcal{N}_\infty(D)$ of D at 0 is the lower envelop in $\mathbb{R}^+ \times \mathbb{R}$ of the various attaching lines of \mathcal{P}_D with non-positive slopes.

Equivalently, we can say that the Newton polygon at 0 is the intersection of the closed upper half-planes limited by the various attaching lines of \mathcal{P}_D with non-negative slopes while the Newton polygon at infinity is the intersection of the closed lower half-planes limited by the various attaching lines of \mathcal{P}_D with non-positive slopes.

One obtains the same Newton polygon when one enlarges the set of marked points to any points $(j, m - j)$ corresponding to a non-zero monomial $x^m \frac{d^j}{dx^j}$ in D or to the horizontal segments issuing from the points of \mathcal{P}_D backwards to the vertical axis.

Example 4.3.15. — Consider the operator $D = x^m \frac{d^j}{dx^j}$. Since x^m is both a meromorphic series in x and in $1/x$ it makes sense to determine both its Newton polygon at 0 and at infinity. There corresponds to D the unique marked point $(j, m - j)$ and the corresponding Newton polygons are as shown on Fig. 1.

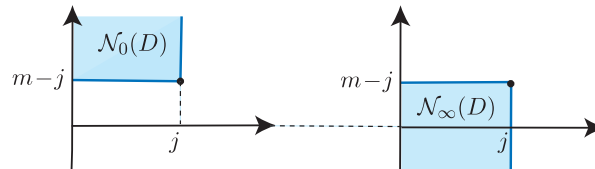


Figure 1

When D has polynomial coefficients one can define its Newton polygons both at 0 and at infinity.

Definition 4.3.16 (Full Newton polygon). — Suppose D has polynomial coefficients. The full Newton polygon $\mathcal{N}(D)$ is the intersection $\mathcal{N}_0(D) \cap \mathcal{N}_\infty(D)$ of the Newton polygons of D at 0 and at infinity.

For simplicity and when there is no ambiguity, we denote by $\mathcal{N}(D)$ anyone of these Newton polygons.

Example 4.3.17. — Here below are the full Newton polygons of the Euler operator $\mathcal{E} = x^2 \frac{d}{dx} + 1$, its homogeneous variant $\mathcal{E}_0 = x^3 \frac{d^2}{dx^2} + (x^2 + x) \frac{d}{dx} - 1$ and the hypergeometric operator $D_{3,1} = z(z \frac{d}{dz} + 4) - z \frac{d}{dz} (z \frac{d}{dz} + 1)(z \frac{d}{dz} - 1)$.

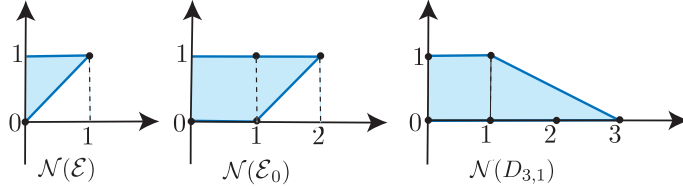


Figure 2

From now on, unless otherwise specified, we work at the origin 0, *i.e.*, we suppose that D has formal or convergent meromorphic coefficients at 0.

Proposition 4.3.18 (levels of D). — Suppose 0 is a singular point of D , *i.e.*, at least one of the coefficients b_j/b_n has a pole at 0.

- (i) The levels of D at 0 are the positive slopes of $\mathcal{N}_0(D)$.
- (ii) The point 0 is regular singular for D if and only if the Newton polygon $\mathcal{N}_0(D)$ has no non-zero slope.

Proposition 4.3.19. — Newton polygons satisfy the following properties.

- (i) Let $D_m = x^m D$, $m \in \mathbb{Z}$. The Newton polygon of D_m is the Newton polygon of D translated vertically by m .
- (ii) Let D_1 and D_2 be two linear differential operators meromorphic at 0. Then,

$$\mathcal{N}_0(D_1 D_2) = \mathcal{N}_0(D_1) + \mathcal{N}_0(D_2).$$

Proof. — Assertion (i) is elementary.

For a proof of (ii) we refer, for instance, to [DMR07, Lem. 1.4.1, p. 99]. \square

As a consequence of (i), we may define the Newton polygon of an equation $Dy = 0$ as being the Newton polygon of D up to vertical translation.

On the set $\mathbb{C}[[x]][1/x, d/dx]$ of linear differential operators at 0 it is convenient to introduce a *weight* (or 0-weight) w by setting

$$w\left(x^k \frac{d^j}{dx^j}\right) = k - j \quad \text{and} \quad w\left(\sum x^k \frac{d^j}{dx^j}\right) = \min_{k,j} w\left(x^k \frac{d^j}{dx^j}\right).$$

In particular, $w(x) = 1$, $w\left(\frac{d}{dx}\right) = -1$ and, to an operator D with weight $w(D) = w$, the product $x^{-w}D$ has weight 0. At our convenience, given a differential equation $Dy = 0$, we can then assume that D has weight 0.

Lemma 4.3.20. — Given $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, one has

$$\left(x^{k+1} \frac{d}{dx}\right)^j = x^{jk+j} \frac{d^j}{dx^j} + \sum_{\substack{1 \leq j'' < j \\ j' - j'' = jk}} c_{j', j''} x^{j'} \frac{d^{j''}}{dx^{j''}} \quad (c_{j', j''} \in \mathbb{C}).$$

Observe that all monomials in the right hand side have weight $j k$ and then, the whole expression in the left hand side has weight $j k$. The marked point associated with $x^{jk+j} d^j/dx^j$ is $A = (j, jk)$. The marked points associated with the monomials in the sum are (j', jk) with $1 \leq j' \leq j - 1$, hence points lying on the horizontal segment between A and the vertical coordinate axis.

Proof. — The formula in Lemma 4.3.20 is trivially true for $j = 1$. By Leibniz rule we obtain the commutation law $\frac{d}{dx} x^{k+1} = x^{k+1} \frac{d}{dx} + (k+1)x^k$, from which it follows that

$$\left(x^{k+1} \frac{d}{dx}\right)^2 = x^{2k+2} \frac{d^2}{dx^2} + (k+1)x^{2k+1} \frac{d}{dx}.$$

Hence the formula for $j = 2$. The general case is similarly obtained by recurrence. \square

Proposition 4.3.21. — Given a differential operator D in the variable x denote by D_z the operator deduced from D by the change of variable $x = 1/z$. Then, the Newton polygons $\mathcal{N}_0(D)$ and $\mathcal{N}_\infty(D_z)$ are symmetric with each other with respect to the horizontal coordinate axis.

Proof. — One has $\frac{d}{dz} = -x^2 \frac{d}{dx}$. From Lemma 4.3.20 we know that we can expand D in powers of the derivation $\delta = x^2 \frac{d}{dx}$ with weight $w(\delta) = +1$:

$$D = c_n(x)\delta^n + c_{n-1}(x)\delta^{n-1} + \cdots + c_0(x)$$

and the set \mathcal{P}_D of marked points is then given by $(j, v_0(c_j) + j)$ for $0 \leq j \leq n$. Now, the operator D_z reads

$$D_z = (-1)^n c_n(1/z) \frac{d^n}{dz^n} + (-1)^{n-1} c_{n-1}(1/z) \frac{d^{n-1}}{dz^{n-1}} + \cdots + c_0(1/z)$$

and the associated marked points are $(j, v_\infty(c_j(1/z)) - j) = (j, -v_0(c_j) - j)$. \square

4.3.2.2. Newton polygon and Borel transform. — We consider here the classical Borel transform \mathcal{B} (or 1-Borel transform) at 0 as defined in Section 6.3.1 below and we denote by ξ the variable in the Borel plane. We suppose that

D has polynomial coefficients in x and $1/x$. As previously, we can expand D in powers of $\delta = x^2 \frac{d}{dx}$:

$$D = c_n(x)\delta^n + c_{n-1}(x)\delta^{n-1} + \cdots + c_0(x).$$

We assume that the coefficients c_j are polynomials in $1/x$. If this were not the case, we would replace D by $x^{-N}D$ with N the degree of the c_j 's with respect to x .

Let $\Delta = \mathcal{B}(D)$ denote the operator deduced from D by Borel transform. Since $\mathcal{B}(\delta) = \xi$ and $\mathcal{B}(1/x) = \frac{d}{d\xi}$ (cf. Sect. 6.3.1) the operator Δ reads

$$\Delta = c_n\left(\frac{d}{d\xi}\right)\xi^n + c_{n-1}\left(\frac{d}{d\xi}\right)\xi^{n-1} + \cdots + c_0\left(\frac{d}{d\xi}\right)$$

and is then a linear differential operator with polynomial coefficients. The fact that D had coefficients polynomial in $1/x$ is a key point here. In the general case, due to the fact that $\mathcal{B}(fg) = \mathcal{B}(f) * \mathcal{B}(g)$, the Borel transform of a linear differential operator is a convolution operator. The proposition below is a corollary of [Mal91b, Thm. (1.4)].

Proposition 4.3.22. — *With normalization as above, the following two properties are equivalent:*

- (i) *the levels of D at 0 are ≤ 1 ;*
- (ii) *the levels of Δ at infinity are ≤ 1 .*

Proof. — Let $v = \min_j v_0(c_j) \leq 0$ be the minimal valuation of the coefficients of D at 0. This implies that all marked points associated with D at 0 are on the line issuing from $(0, v)$ with slope 1 (Recall that δ has weight 1) or over and that at least one of them belongs to the line. As a consequence, all levels of D are ≤ 1 if and only if the point $(n, v + n)$ of the line is a marked point, *i.e.*, if and only if $v_0(c_n) = v$.

To the other side, Δ has degree n and order $-v$. Similarly at 0, its Newton polygon at infinity has no slope > 1 if and only if the monomial $\xi^n \frac{d^{-v}}{d\xi^{-v}}$ does exist in Δ . And indeed, this is precisely what the condition $v_0(c_n) = v$ says. \square

4.3.2.3. Calculating the formal invariants. — We briefly sketch here how to calculate the formal invariants of a linear differential equation $Dy = 0$ with (formal) meromorphic coefficients at 0. Recall that the formal invariants at 0 of the equation are the determining polynomials $q(1/x)$ with multiplicities, the exponents of formal monodromy λ and how many logarithms are associated with.

▷ *Indicial equation.* — Suppose the Newton polygon $\mathcal{N}_0(D)$ has a horizontal side and consider the operator restricted to the marked points lying on that horizontal side. Up to a power of x to the left that operator reads

$$D_0 = \gamma_r x^r \frac{d^r}{dx^r} + \gamma_{r-1} x^{r-1} \frac{d^{r-1}}{dx^{r-1}} + \cdots + \gamma_1 x \frac{d}{dx} + \gamma_0, \quad \gamma_r, \dots, \gamma_0 \in \mathbb{C}.$$

The indicial equation is the equation in the variable λ obtained by writing that x^λ satisfies the equation $D_0 y = 0$, *i.e.*, denoting $[\lambda]_r = \lambda(\lambda-1)\dots(\lambda-r+1)$, the equation

$$\gamma_r [\lambda]_r + \gamma_{r-1} [\lambda]_{r-1} + \cdots + \gamma_1 [\lambda]_1 + \gamma_0 = 0.$$

Its roots λ_j (with multiplicities) are the exponents of factors x^λ associated with no exponential.

▷ *k-characteristic equation.* — Suppose the Newton polygon $\mathcal{N}_0(D)$ has a side with slope k and consider the differential operator restricted to the marked points lying on that side with slope k . This operator reads $x^{k'} D_k \frac{d^{s''}}{dx^{s''}}$ with

$$D_k = \left(c_s x^{s(k+1)} \frac{d^s}{dx^s} + c_{s-1} x^{(s-1)(k+1)} \frac{d^{s-1}}{dx^{s-1}} + \cdots + c_1 x^{k+1} \frac{d}{dx} + c_0 \right).$$

The k -characteristic equation is the equation obtained by writing that e^{-a/x^k} satisfies the equation $D_k y = 0$, *i.e.*,

$$c_s X^s + c_{s-1} X^{s-1} + \cdots + c_1 X + c_0 = 0.$$

Its roots (counted with multiplicities) are the dominant coefficients a of the exponentials $e^{-a/x^k + \cdots}$ times k . Differently said, they are equal to the dominant coefficients ak in the derivatives of the exponentials $e^{-a/x^k(1+0(1/x))}$.

▷ *Iterated characteristic equations.* — Once one has determined the dominant coefficient a in the exponentials the next coefficients including the factor x^λ attached to each exponential can be determined as follows. Select one root a and consider the differential operator $D_1 = D^{(-a/x^k)}$ deduced from D by the change of variable $y = e^{-a/x^k} Y$ (and simplifying by the factor e^{-a/x^k}). The Newton polygon $\mathcal{N}_0(D_1)$ may have no slope $k' < k$ and no horizontal side; in that case e^{-a/x^k} is the exponential we look for and it comes factored with no x^λ . It may have no slope $k' < k$ but a horizontal side; in that case, there exist terms of the form $x^\lambda e^{-a/x^k}$ for all root λ of the indicial equation of D_1 . The Newton polygon $\mathcal{N}_0(D_1)$ may also have non-zero slopes $k' < k$; in that case, one has to solve the k' -characteristic equations to determined the next

term in the exponential $e^{-a/x^k+\dots}$ and so on ... until all exponentials and associated x^λ are found.

▷ *Frobenius method.* — When the indicial equations have multiple roots modulo \mathbb{Z} there might exist logarithmic terms. To determine which terms appear with logarithms there exist a classical algorithm called *Frobenius algorithm*. Although the procedure is easy and natural from a theoretical viewpoint (it might be long and laborious in practice) we do not develop it here and we refer to the classical literature, for instance, [CL55, Sec 4.8].

When one knows all normal solutions, to complete them by formal series to get solutions of the initial equation $Dy = 0$ one proceeds by like powers identification.

All these algorithms have been implemented in MAPLE packages such as *Isolde* or *gfun*.

The case of systems is much more difficult to treat practically. There exists however algorithms to determine formal fundamental solutions. One can always apply the Cyclic vector algorithm (Sect. 4.1) and proceed as before. However, this way, there appear, in general, huge coefficients making the calculation heavy. It is then, in general, recommended to operate directly on the system itself. One method, which relies on Moser's rank, was developed by M. Barkatou and his group (*cf.* [BCLR03] for a sketched algorithm and references). A variant was developed by M. Miyake [Miy11].

4.4. The Main Asymptotic Existence Theorem

Consider a linear differential operator

$$D = \sum_{j=1}^n b_j(x) \frac{d^j}{dx^j}$$

with analytic coefficients at 0. The question here addressed is: is any formal solution of the equation $Dy = 0$ the asymptotic expansion of an asymptotic solution? A positive answer is given by the Main Asymptotic Existence Theorem (M.A.E.T.) either in Poincaré asymptotics or in Gevrey asymptotics.

In the case of Poincaré asymptotics the theorem, precisely Cor. 4.4.2, is mostly due to Hukuhara and Turritin with a complete proof by Wasow [Was76]. An extension to Gevrey asymptotics is given by B. Malgrange in [Mal91a, Append. 1] and to non linear operators [RS89] by J.-P. Ramis and Y. Sibuya.

The theorem roughly says that to a formal solution \tilde{f} of a differential equation (linear or non linear) there correspond actual solutions f asymptotic to \tilde{f} on various sectors. Given a direction, it is possible to determine from the equation itself a minimal opening of the sector on which such an asymptotic solution exists. However, these asymptotic solutions are, in general, neither unique nor given by explicit formulæ.

Theorem 4.4.1 (Main Asymptotic Existence Theorem)

The operator D acts linearly and surjectively on the sheaf $\mathcal{A}^{<0}$ and on the sheaves $\mathcal{A}^{\leq -k}$ for all $k > 0$.

In other words, the sequences

$$\mathcal{A}^{<0} \xrightarrow{D} \mathcal{A}^{<0} \longrightarrow 0 \quad \text{and} \quad \mathcal{A}^{\leq -k} \xrightarrow{D} \mathcal{A}^{\leq -k} \longrightarrow 0 \quad \text{for all } k > 0$$

are exact sequences of sheaves of \mathbb{C} -vector spaces. For the proof we refer to [Mal91a, Append 1; Thm. 1] where the theorem is stated and proved for all spaces $\mathcal{A}^{<k}$ and $\mathcal{A}^{\leq k}$ for all $k \in \mathbb{R}$ (see definitions in [Mal91a]).

The Main Asymptotic Existence Theorem implies the following corollary.

Corollary 4.4.2. — *Let $\tilde{f}(x) = \sum_{m \geq 0} a_m x^m$ be a power series solution of the differential equation $Dy = 0$.*

(i) *Given any direction $\theta \in S^1$, there exists a sector $\Delta_\theta = \Delta_{] \theta - \delta, \theta + \delta[}(R)$ and a function $f \in \bar{\mathcal{A}}(\Delta_\theta)$ such that*

- ▷ $Df(x) = 0$ for all $x \in \Delta_\theta$ (i.e., f is an analytic solution on Δ_θ),
- ▷ $T_{\Delta_\theta} f = \tilde{f}$ (i.e., f is asymptotic to \tilde{f} at 0 on Δ_θ).

(ii) *If the series $\tilde{f}(x)$ is Gevrey of order s then Δ_θ and $f(x)$ can be chosen so that $f(x)$ be s -Gevrey asymptotic to $\tilde{f}(x)$ on Δ_θ .*

Proof. — (i) The Borel-Ritt Theorem (cf. Thm. 2.4.1 (i)), provides for any sector \mathcal{S}' containing the direction θ , a function $g \in \bar{\mathcal{A}}(\mathcal{S}')$ with asymptotic expansion $T_{\mathcal{S}'} g = \tilde{f}$ on \mathcal{S}' . Since $T_{\mathcal{S}'}$ is a morphism of differential algebras, $T_{\mathcal{S}'} Dg = DT_{\mathcal{S}'} g = D\tilde{f} = 0$. Hence, the function Dg is flat: $Dg \in \bar{\mathcal{A}}^{<0}(\mathcal{S}')$. The Main Asymptotic Existence Theorem above applied to Dg in the direction θ provides a sector $\Delta_\theta \subset \mathcal{S}'$ containing the direction θ and a function $h \in \bar{\mathcal{A}}^{<0}(\Delta_\theta)$ such that $Dh = Dg$. The function $f = g - h$ satisfies the required conditions on Δ_θ .

(ii) When the series $\tilde{f}(x)$ is s -Gevrey the Borel-Ritt Theorem with Gevrey conditions (cf. Thm. 2.4.1.(ii)) provides a function $g \in \bar{\mathcal{A}}_s(\mathcal{S}')$ over some sector \mathcal{S}' containing the direction θ which is s -Gevrey asymptotic to \tilde{f} on \mathcal{S}' . Its

derivative Dg is asymptotic to $D\tilde{f}(x) = 0$ and, from Proposition 2.3.17, we can assert that Dg is k -exponentially flat on \mathcal{A}' . Hence, by the Main Asymptotic Existence Theorem, h belongs to $\overline{\mathcal{A}}^{\leq -k}$ and the conclusion follows as in the previous case. \square

Since the proof relies on the Borel-Ritt Theorem it does not provide the uniqueness of the asymptotic solutions.

The theorem does not make explicit the size of the sector \mathcal{A}_θ . When the series \tilde{f} is convergent the sector \mathcal{A}_θ can be chosen to be a full disc around 0 and $f(x)$ to be the sum of the series. The opening of a possible sector can be very different depending on the series and on the chosen direction θ . The analysis of the Stokes phenomenon of the differential equation shows that in any direction one can choose a sector of opening at least π/k for k the highest level of the equation.

Comments 4.4.3 (On the examples of section 2.2.2)

▷ Example 2.2.4. The Euler function is asymptotic to the Euler series on a sector of opening 3π and this sector is an asymptotic sector in any direction θ . The highest (and actually unique) level of the Euler equation is $k = 1$ and thus, the actual opening of 3π is larger than $\pi/k = \pi$. However, if we ask for a sector bisected by the direction θ the opening reduces to π in the direction $\theta = \pi$.

▷ Example 2.2.6. The hypergeometric function $g(z)$ is asymptotic to the hypergeometric series $\tilde{g}(z)$ on a sector of opening 4π while $\pi/k = 2\pi$ (the unique level of the hypergeometric equation $D_{3,1}y = 0$ is $k = 1/2$). The anti-Stokes (and singular) directions are the directions $\theta = 0 \pmod{2\pi}$ since the exponentials of a formal fundamental solution are $e^{\pm 2z^{1/2}}$. An asymptotic sector bisected by $\theta = 0$ has 2π as maximal opening.

▷ In the previous two examples there exists only one singular direction and the possible asymptotic sectors are much larger than the announced minimal value. Actually, considering two neighboring singular directions $\theta < \theta'$ an asymptotic sector always exists with opening $]\theta - \pi/(2k), \theta' + \pi/(2k)[$ for k the highest level of the equation. When the singular directions are irregularly distributed the asymptotic sectors are “irregularly” wide.

Let $\Delta Y \equiv Y' - B(x)Y = 0$ be a linear differential system of order 1 and dimension n with formal fundamental solution $\tilde{F}(x) x^L e^{Q(1/x)}$. The Main Asymptotic Existence Theorem 4.4.1 and its Corollary 4.4.2 remain valid for systems in the following form.

Corollary 4.4.4. — *The operator Δ acts surjectively in $(\mathcal{A}^{<0})^n$ and in $(\mathcal{A}^{\leq -k})^n$ for all $k > 0$ and consequently, it satisfies in all direction $\theta \in S^1$ the following properties:*

(i) *There exists a sector $\Delta_\theta = \Delta_{] \theta - \omega, \theta + \omega[}(R)$ and an invertible matrix function $F \in \text{GL}(n, \bar{\mathcal{A}}(\Delta_\theta))$ such that*

$$\begin{cases} \Delta(F(x) x^L e^{Q(1/x)}) = 0 & \text{for all } x \in \Delta_\theta, \\ T_{\Delta_\theta} F = \tilde{F} & (F \text{ is asymptotic to } \tilde{F} \text{ at } 0 \text{ on } \Delta_\theta). \end{cases}$$

(ii) *If an entry of \tilde{F} is s -Gevrey then the corresponding entry of F can be chosen to be s -Gevrey asymptotic on a convenient Δ_θ .*

Proof. — This extension to differential systems follows from the fact that each entry of $\tilde{F}(x)$ satisfies itself a linear differential equation with meromorphic coefficients deduced from the homological system (23): $F' = B F - F B_0$. \square

4.5. Infinitesimal neighborhoods of an irregular singular point

While algebraic functions have moderate growth the form of formal solutions given above and the Main Asymptotic Existence Theorem show that solutions of linear differential equations at an irregular singular point may exhibit exponential growth or decay. Infinitesimal neighborhoods of algebraic geometry are then insufficient to discriminate between the various solutions. We define below infinitesimal neighborhoods for irregular singularities of solutions of differential equations as suggested by P. Deligne in a letter to J.-P. Ramis dated 7/01/1986 [DMR07]. This approach is developed, with an application to index theorems, in [LRP97].

4.5.1. Infinitesimal neighborhoods associated with exponential order. — We begin with a concept related only to the exponential order of growth or decay of the singularity under consideration. This concept will show up to be slightly too poor for a good characterization of k -summable series but it is a necessary step, at least for clarity.

\triangleright *Base space X .* — From this viewpoint the infinitesimal neighborhood X of 0 in \mathbb{C} is defined as a full copy of \mathbb{C} compactified by the adjunction of a circle at infinity and endowed with a structural sheaf \mathcal{F} defined as below. For obvious reasons we represent the infinitesimal neighborhood of 0 as a compact disc in place of the origin 0 in \mathbb{C} . The “outside world” $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is not

affected by the construction and stays being endowed with the sheaf of germs of analytic functions.

▷ *Sheaf* $\mathcal{A}^{\leq k}$, $k > 0$. — Similar to the definition of k -exponentially flat functions in Section 3.1.5 one says that a function f has *exponential growth of order k on a sector \mathcal{A}* if, for any proper subsector $\mathcal{A}' \Subset \mathcal{A}$, there exist constants K and $A > 0$ such that the following estimate holds for all $x \in \mathcal{A}'$:

$$|f(x)| \leq K \exp\left(\frac{A}{|x|^k}\right).$$

The set of all functions with exponential growth of order k on \mathcal{A} is denoted by $\overline{\mathcal{A}}^{\leq k}(\mathcal{A})$ and one defines a sheaf $\mathcal{A}^{\leq k}$ over S^1 of germs with exponential growth of order k (or, with k -exponential growth) in a similar way as $\mathcal{A}^{\leq -k}$ (cf. Sect. 3.1.5).

▷ *Presheaf* $\overline{\mathcal{F}}$. — In view to define the sheaf \mathcal{F} it suffices to define the presheaf $\overline{\mathcal{F}}$ on a basis of open sets of X . We consider the following open sets (cf. Fig. 3):

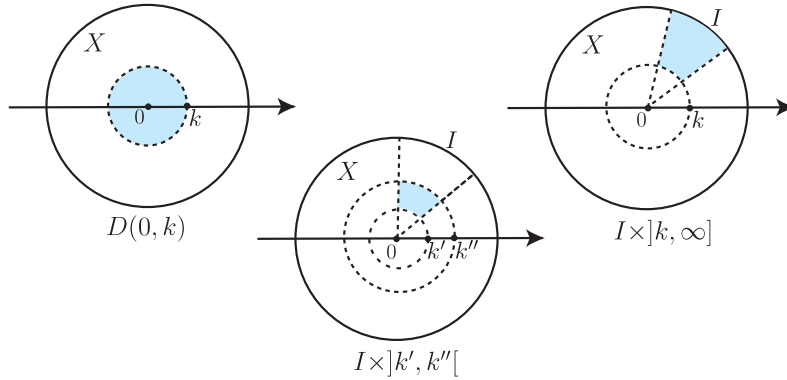


Figure 3. Basis of open sets in X

$\left\{ \begin{array}{l} \text{the discs } D(0, k) \text{ for all } k > 0, \\ \text{the (truncated) sectors } I \times]k', k''[= \{x = re^{i\theta}; \theta \in I \text{ and } 0 < k' < r < k''\}, \\ \text{the (truncated) sectors } I \times]k, \infty[= \{x = re^{i\theta}; \theta \in I \text{ and } 0 < k < r \leq \infty\}. \end{array} \right.$

and we set

$$\left\{ \begin{array}{l} \overline{\mathcal{F}}(D(0, k)) = \mathbb{C}[[x]]_s \quad (\text{recall } s = 1/k), \\ \overline{\mathcal{F}}(I \times]k', k''[) = H^0(I; \mathcal{A}^{\leq k'} / \mathcal{A}^{\leq -k''}), \\ \overline{\mathcal{F}}(I \times]k, \infty[) = H^0(I; \mathcal{A}^{\leq k}). \end{array} \right.$$

The restriction map between (truncated) sectors is the canonical restriction of functions and quotients. The restriction map from a disc $D(0, k)$ to a sector is made possible by the isomorphism between $\mathbb{C}[[x]]_s$ and $H^0(S^1; \mathcal{A}_s / \mathcal{A}^{\leq -k})$ following from the Borel-Ritt Theorem (*cf.* Seq. (19) and Cor. 3.2.8).

Notation 4.5.1. — From now on, a point $x = ke^{i\theta}$ with $k > 0$ is also denoted by its polar coordinates (θ, k) .

▷ *Sheaf \mathcal{F} .* — The sheaf \mathcal{F} over X is the sheaf associated with the presheaf $\overline{\mathcal{F}}$.

It is a sheaf of \mathbb{C} -algebras. The stalk \mathcal{F}_0 of \mathcal{F} at 0 is made of all Gevrey series. If useful, it could be extended to any series of $\mathbb{C}[[x]]$, the support of germs of non-Gevrey series having the point $\{0\}$ as support. To define the stalk at the other points (θ, k) we introduce the sheaves

$$\mathcal{A}^{\leq k-} = \varinjlim_{\varepsilon \rightarrow 0+} \mathcal{A}^{\leq k-\varepsilon} \quad \text{and} \quad \mathcal{A}^{\leq -k+} = \varinjlim_{\varepsilon \rightarrow 0+} \mathcal{A}^{\leq -(k+\varepsilon)}.$$

A germ f at θ belongs to $\mathcal{A}^{\leq k-}$ if there exist a sector \mathfrak{A} in \mathbb{C}^* containing the direction θ , an $\varepsilon > 0$, and constants $K, C > 0$ such that

$$|f(x)| \leq K \exp \frac{C}{|x|^{k-\varepsilon}} \quad \text{for all } x \in \mathfrak{A}.$$

A germ f is in $\mathcal{A}^{\leq -k+}$ if under the same conditions it satisfies

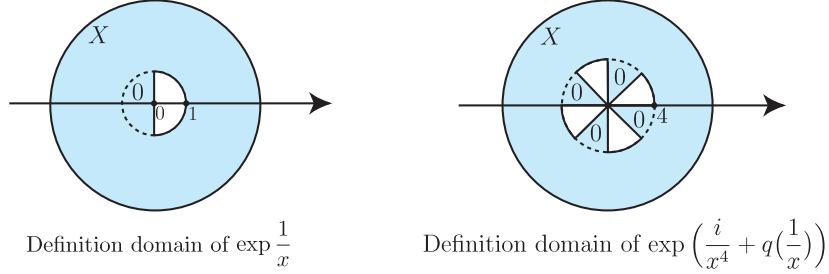
$$|f(x)| \leq K \exp -\frac{C}{|x|^{k+\varepsilon}} \quad \text{for all } x \in \mathfrak{A}.$$

The stalk of \mathcal{F} at (θ, k) is given by

$$\mathcal{F}_{(\theta, k)} = \mathcal{A}_\theta^{\leq k-} / \mathcal{A}_\theta^{\leq -k+}.$$

Example 4.5.2 (Definition domain and support of exponentials)

An exponential $\exp(-a/x^k + q(1/x))$ where q is a polynomial of degree less than k can be seen as a section of the complement of the closed disc $\overline{D}(0, k)$ since it has exponential growth of order less than k' for all $k' > k$.



On another hand, the exponential is flat on the k open sectors

$$|\arg(x) + \alpha/k| < \pi/(2k) \pmod{2\pi/k}$$

where α denotes the argument of a . The exponential can then be continued on all of these open sectors and is equal to 0 inside $D(0, k)$. It cannot be continued any further. Its support is the complement of the open disc $D(0, k)$ in its definition domain. The arcs on the circle of radius k limiting the sectors where the exponential is equal to zero are of length π/k . By analogy with the big points of algebraic geometry, their closure is called *k-big points associated with the exponential* $\exp(-a/x^k + q(1/x))$ (or with the polynomial $-a/x^k + q(1/x)$). On the picture are drawn two particular cases of the definition domain (open shadowed part) of an exponential. The big points are in dotted lines.

This example shows that the sheaf \mathcal{F} is in no way a coherent sheaf, hence its surname of “wild analytic” sheaf.

The following properties of the sheaf \mathcal{F} are elementary and their proof is left to the reader.

Proposition 4.5.3. — *The sheaf \mathcal{F} satisfies the following properties:*

1. The restriction $\mathcal{F}|_{S^1 \times \{k\}}$ of \mathcal{F} to the circle centered at 0 with radius k in X is isomorphic to the quotient sheaf $\mathcal{A}^{\leq k-} / \mathcal{A}^{\leq -k+}$ over S^1 .

2. Sections over an open disc:

$$H^0(D(0, k); \mathcal{F}) = \varprojlim_{\varepsilon \rightarrow 0+} \mathbb{C}[[x]]_{s+\varepsilon} = \bigcap_{\varepsilon > 0} \mathbb{C}[[x]]_{s+\varepsilon} =: \mathbb{C}[[x]]_{s+} \supsetneq \mathbb{C}[[x]]_s.$$

3. Sections over a closed disc:

$$H^0(\bar{D}(0, k); \mathcal{F}) = \varinjlim_{\varepsilon \rightarrow 0+} \mathbb{C}[[x]]_{s-\varepsilon} = \bigcup_{\varepsilon > 0} \mathbb{C}[[x]]_{s-\varepsilon} =: \mathbb{C}[[x]]_{s-} \subsetneq \mathbb{C}[[x]]_s.$$

4.5.2. Infinitesimal neighborhoods associated with exponential order and type. — As it follows from Proposition 4.5.3 the Gevrey space $\mathbb{C}[[x]]_s$ does not appear as a space of sections of \mathcal{F} over some disc or any other domain. To supply that gap we enrich the sheaf \mathcal{F} by taking into account both exponential order and exponential type.

For a given $k > 0$ we define extensions X^k, \mathcal{F}^k of X, \mathcal{F} as follows.

▷ *Base space X^k .* — While, building X , we replaced the origin 0 by a copy of \mathbb{C} compactified with a circle at infinity, we now replace the circle $S^1 \times \{k\}$ of radius k in X by a copy Y^k of $\mathbb{C}^* = S^1 \times]0, \infty[$ compactified by the two circles $S^1 \times \{0\}$ and $S^1 \times \{\infty\}$. Precisely, we glue the lower boundary $S^1 \times \{0\}$ of Y^k to the boundary of the disc $D(0, k)$ and the upper boundary $S^1 \times \{\infty\}$ of Y^k to the lower boundary of the complement $X \setminus D(0, k)$. As topological spaces X and X^k are isomorphic to \mathbb{C} .

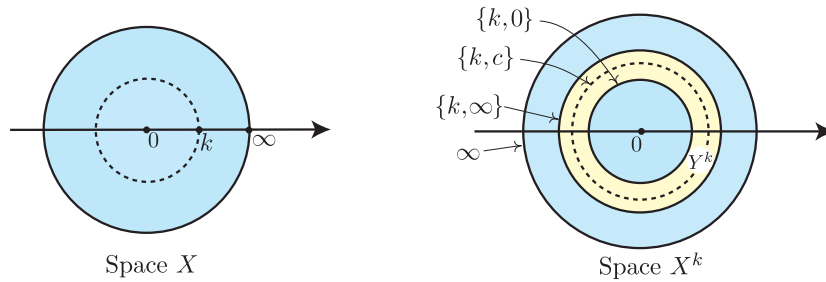


Figure 4

We denote by $(\theta, \{k, \rho\})$ the polar coordinates of the points of $\overline{Y^k}$. A basis of open sets in X^k is given by open discs centered at 0 and truncated sectors like in X .

▷ *Presheaf $\overline{\mathcal{F}}^k$.* — Given $c > 0$ we take into account the type c of exponentials of order k by introducing the subsheaf $\mathcal{A}^{\leq k, c^-}$ of $\mathcal{A}^{\leq k}$ and the subsheaf $\mathcal{A}^{\leq -k, c^+}$ of $\mathcal{A}^{\leq -k}$ over S^1 defined as follows.

A germ of $\mathcal{A}^{\leq k, c^-}$ at θ is a germ $f \in \mathcal{A}_\theta^{\leq k}$ which satisfies the condition: there exist an open sector \mathcal{A} containing the direction θ , an $\varepsilon > 0$ and a constant $K > 0$ such that

$$|f(x)| \leq K \exp \frac{c - \varepsilon}{|x|^k} \quad \text{for all } x \in \mathcal{A}.$$

A germ of $\mathcal{A}^{\leq -k, c^+}$ at θ is a germ $f \in \mathcal{A}_\theta^{\leq -k}$ which satisfies the condition: there exist an open sector \mathcal{A} containing the direction θ , an $\varepsilon > 0$ and a constant $K > 0$ such that

$$|f(x)| \leq K \exp -\frac{(c + \varepsilon)}{|x|^k} \quad \text{for all } x \in \mathcal{A}.$$

The space $\mathbb{C}[[x]]_{s, C}$ of series with fixed Gevrey order s and type C is the subspace of $\mathbb{C}[[x]]$ made of the series $\sum_{n \geq 0} a_n x^n$ whose coefficients satisfy an

estimate of the form $|a_N| \leq K(n!)^s C^{ns}$ for all $n \geq 0$ and a convenient $K > 0$. It is useful to introduce the spaces $\mathbb{C}[[x]]_{s,C^+} = \bigcap_{\varepsilon > 0} \mathbb{C}[[x]]_{s,C^+ \varepsilon}$. Thus, a series $\sum_{n \geq 0} a_n x^n$ belongs to $\mathbb{C}[[x]]_{s,C^+}$ if for all $\varepsilon > 0$ there exists $K > 0$ such that

$$|a_N| \leq K(n!)^s (C + \varepsilon)^{ns} \quad \text{for all } n \geq 0.$$

In view to construct the sheaf \mathcal{F}^k it suffices to define $\overline{\mathcal{F}}^k$ on a basis of open sets by setting:

$$\left\{ \begin{array}{l} \text{inside } X \setminus \overline{Y}^k, \text{ no change: } \overline{\mathcal{F}}^k = \overline{\mathcal{F}}, \\ \text{inside } Y^k : \quad \overline{\mathcal{F}}^k(I \times]\{k, c'\}, \{k, c''\}[) = H^0(I; \mathcal{A}^{\leq k, c'^-} / \mathcal{A}^{\leq -k, c''^+}), \\ \text{across } \partial Y^k : \quad \overline{\mathcal{F}}^k(D(0, \{k, c\})) = \mathbb{C}[[x]]_{s,(1/c)^+}, \text{ for } 0 < c < +\infty \\ \quad \overline{\mathcal{F}}^k(I \times]k', \{k, c\}[) = H^0(I; \mathcal{A}^{\leq k'} / \mathcal{A}^{\leq -k, c^+}), \\ \quad \overline{\mathcal{F}}^k(I \times]\{k, c\}, k''[) = H^0(I; \mathcal{A}^{\leq k, c^-} / \mathcal{A}^{\leq -k''}). \end{array} \right.$$

As for the presheaf $\overline{\mathcal{F}}$, the application of restriction in $\overline{\mathcal{F}}^k$ is defined on sectors by the natural restriction of functions and quotient. The restriction to an intersection of a sector and a disc is made consistent by the exact sequence

$$0 \rightarrow \mathcal{A}^{\leq -k, c^-} \rightarrow \mathcal{A}_{s,(1/c)^+} \xrightarrow{T} \mathbb{C}[[x]]_{s,(1/c)^+} \rightarrow 0,$$

analog to the Borel-Ritt exact sequence (19) [LRP97, Sect. 1]. In this sequence the notation \mathcal{A}_{s,C^+} stands for the following sheaf. A germ of \mathcal{A}_{s,C^+} at θ is a germ $f \in \mathcal{A}_\theta$ which satisfies the condition: there exist an open sector \mathfrak{A} containing the direction θ and a series $\sum_{n \geq 0} a_n x^n$ such that for all $\varepsilon > 0$ there is a constant $K > 0$ such that

$$\left| f(x) - \sum_{n=0}^{N-1} a_n x^n \right| \leq K (N!)^s |x|^N (C + \varepsilon)^{Ns} \quad \text{on } \mathfrak{A} \text{ for all } N \in \mathbb{N}.$$

▷ *Sheaf \mathcal{F}^k .* — The sheaf \mathcal{F}^k is the sheaf over X^k associated with the presheaf $\overline{\mathcal{F}}^k$. It is a sheaf of $\mathbb{C}\{x\}$ -modules and no longer a sheaf of \mathbb{C} -algebras since the product of two functions of $\mathcal{A}^{\leq k, c^-}$ belongs to $\mathcal{A}^{\leq k, (2c)^-}$ and not to $\mathcal{A}^{\leq k, c^-}$ in general. The stalk at a point $(\theta, \{k, c\})$ of \overline{Y}^k is given by

$$\mathcal{F}_{(\theta, \{k, c\})}^k = \begin{cases} \mathcal{A}_\theta^{\leq k^-} / \mathcal{A}_\theta^{\leq -k} & \text{if } c = 0, \\ \mathcal{A}_\theta^{\leq k, c^-} / \mathcal{A}_\theta^{\leq -k, c^+} & \text{if } 0 < c < +\infty, \\ \mathcal{A}_\theta^{\leq k} / \mathcal{A}_\theta^{\leq -k^+} & \text{if } c = +\infty. \end{cases}$$

Example 4.5.4 (Definition domain and support of exponentials)

An exponential $\exp(-a/x^k + q(1/x))$ where q is a polynomial of degree less than k and $a = Ae^{i\alpha}$, $A > 0$ is well defined in \bar{Y}^k for all $(\theta, \{k, \rho\})$ such that $-A \cos(\alpha - k\theta) < \rho$. When $\cos(\alpha - k\theta) \leq 0$ this means all points out of the “arch” $\rho = -A \cos(\alpha - k\theta)$. When $\cos(\alpha - k\theta) > 0$ this leads to no constraint on ρ ; moreover, the exponential is equal to 0 inside the arch $\rho = A \cos(\alpha - k\theta)$.

The k -big points associated with the exponential are now the closures of the arches $\rho < A \cos(\alpha - k\theta)$ and $\cos(\alpha - k\theta) > 0$ where the exponential vanishes in \bar{Y}^k .

In the example drawn in Figure 5 the definition domain of the exponential is the shadowed part of the infinitesimal neighborhood of 0. We indicated by “0” the open regions where the exponential vanishes. The big points are the closure of the arches (colored in orange-red in case of a colored copy) in which a small 0 is indicated.

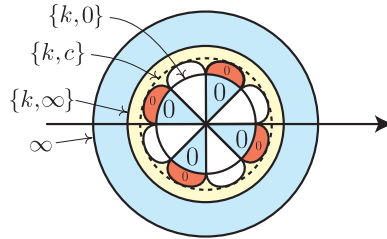


Figure 5. Definition domain of $\exp(\frac{i}{x^4} + q(\frac{1}{x}))$ (here, $k = 4$ and $c = 1$)

We can now see the space $\mathbb{C}[[x]]_s$ as a space of sections of the sheaf \mathcal{F}^k .

Proposition 4.5.5. — *Let $\bar{D}(0, \{k, 0\})$ be the closure in X^k of the disc $D(0, k)$ centered at 0 with radius k . Then,*

$$H^0(\bar{D}(0, \{k, 0\}); \mathcal{F}^k) = \mathbb{C}[[x]]_s.$$

Proof. — The equality follows from the fact that $\bar{D}(0, \{k, 0\}) = \bigcap_{c>0} D(0, \{k, c\})$ and that $H^0(D(0, \{k, c\}); \mathcal{F}^k) = \mathbb{C}[[x]]_{s, (1/c)^+}$. \square

4.5.3. More infinitesimal neighborhoods. — The previous construction can be repeated twice at levels $k = k_1$ and $k = k_2 > k_1$ or finitely many times at levels $k_1 < k_2 < \dots < k_\nu$. One obtains thus base spaces X^{k_1, k_2} or $X^{k_1, k_2, \dots, k_\nu}$ and sheaves \mathcal{F}^{k_1, k_2} or $\mathcal{F}^{k_1, k_2, \dots, k_\nu}$ in a trivial way. Such spaces are useful for handling multisummable series.

Exercise 4.5.6. — Consider a linear differential equation

$$(25) \quad Dy \equiv a_n(x) x^n \frac{d^n y}{dx^n} + a_{n-1}(x) x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(x) y = g(x)$$

with formal series coefficients $a_j(x) \in \mathbb{C}[[x]]$ and $g(x) \in \mathbb{C}[[x]][1/x]$. We suppose that $g(x)$ is non-zero and we denote by $p \in \mathbb{Z}$ its valuation.

Consider the operator $D' = g(x)^2 \frac{d}{dx} \frac{1}{g(x)} D$ so that the equation

$$(26) \quad D' y \equiv b_{n+1}(x) x^{n+1} \frac{d^{n+1} y}{dx^{n+1}} + b_n(x) x^n \frac{d^n y}{dx^n} + \cdots + b_0(x) y = 0$$

is the homogeneous form of equation (25).

Denote by $\mathcal{N}(D)$ and $\mathcal{N}(D')$ the Newton polygons at 0 of D and D' respectively, ℓ and ℓ' the lengths of their horizontal side and $\pi(\lambda)$ and $\pi'(\lambda)$ their indicial equations.

(a) Prove that $\ell' = \ell + 1$.

(b) Prove that $\pi'(\lambda) = C(\lambda - p)\pi(\lambda)$ for a convenient constant $C \neq 0$.

(c) When $\pi(\lambda)$ has no integer root conclude that equation (25) admits a solution in $\mathbb{C}[[x]][1/x]$. What happens when there exists $r \in \mathbb{Z}$ such that $\pi(r) = 0$?

Exercise 4.5.7. —

(1) Check that the function $F(x) = \int_0^{+\infty} \frac{e^{-\xi/x}}{\xi^2 + 3\xi + 2} d\xi$ of exercise (2.5.4) satisfies the linear differential equation

$$(27) \quad x^4 y''' + (2x^3 + 3x^2) y' + 2y = x$$

and explain the appearance of the exponential terms in the analytic continuation of $F(x)$ over the Riemann surface of logarithms.

Put equation (27) in homogeneous form

$$(28) \quad D_1 y = 0.$$

Draw its Newton polygon at 0 and write its characteristic and indicial equations.

Determine a fundamental set of formal solutions.

Write down the companion system of equation (28), a normal form and a formal fundamental solution.

Compute its Stokes matrix or matrices.

(2) Consider the linear differential equation

$$(29) \quad D_2 y \equiv \left(x^2(x+2) \frac{d}{dx} + 4(x+1) \right) \left(x^2 \frac{d^2}{dx^2} + (x^2+x) \frac{d}{dx} - 1 \right) y = 0$$

where the factor to the right is the homogeneous Euler operator \mathcal{E}_0 (cf. Exa. 3.1.24).

Show that $y = e^{2/x}$ satisfies $D_2 y = 0$ and conclude that the equations $D_1 y = 0$ and $D_2 y = 0$ admit a same normal form (*i.e.*, belong to the same formal class).

Compute the Stokes matrices of $D_2 y = 0$ and conclude that the equations $D_1 y = 0$ and $D_2 y = 0$ do not belong to the same meromorphic class.

CHAPTER 5

IRREGULARITY AND GEVREY INDEX THEOREMS FOR LINEAR DIFFERENTIAL OPERATORS

In this chapter, the results of the preceding sections are applied to prove index theorems for linear differential operators in the spaces of s -Gevrey series $\mathbb{C}[[x]]_s$ as well as in the space $\mathbb{C}[[x]]_\infty = \mathbb{C}[[x]]$ of formal series and the space $\mathbb{C}[[x]]_0 = \mathbb{C}\{x\}$ of convergent series. The existence and the value of the irregularity follow. We also sketch a method based on wild analytic continuation, *i.e.*, continuation in the infinitesimal neighborhood.

5.1. Introduction

A linear map $D : E \rightarrow E$ is said to have an index in E if it has finite dimensional kernel $\ker(D, E)$ and cokernel $\operatorname{coker}(D, E)$. If so, the index is defined as being the number

$$\chi(D, E) = \dim \ker(D, E) - \dim \operatorname{coker}(D, E).$$

An index is the Euler characteristic of the complex

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow E \xrightarrow{D} E \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

where D is placed in degree 0 or even. It meets then all algebraic properties of Euler characteristics. In case $\operatorname{coker}(D, E) = 0$ the index $\chi(D, E)$ gives the number of solutions of the equation $Dy = 0$ in E . More generally, one says that a linear morphism $L : E \rightarrow E'$ between two vector spaces E and E' has an index if its kernel and its cokernel have finite dimension, the index being again the difference of these dimensions.

From now on, we suppose that D is a linear differential operator

$$D = b_n(x) \frac{d^n}{dx^n} + b_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + b_1(x) \frac{d}{dx} + b_0(x)$$

where the coefficients $b_p(x)$ are convergent series at $0 \in \mathbb{C}$. The operator D is a linear operator in any of the spaces $\mathbb{C}[[x]]_s$ for $0 \leq s \leq +\infty$ and in any of the quotients $\mathbb{C}[[x]]_s/\mathbb{C}\{x\}$.

The irregularity of D was first defined by B. Malgrange in [Mal74] as follows.

Definition 5.1.1 (Irregularity). — *The irregularity of D at 0 is the index of D seen as a linear operator in the quotient $\mathbb{C}[[x]]/\mathbb{C}\{x\}$.*

It was proved in [Mal74] that D has an index both in $\mathbb{C}[[x]]$ and in $\mathbb{C}\{x\}$, the irregularity being then equal to $\chi(D, \mathbb{C}[[x]]) - \chi(D, \mathbb{C}\{x\})$. It was also proved the relation $\text{coker}(D, \mathbb{C}[[x]]/\mathbb{C}\{x\}) = 0$ which shows that the irregularity is the maximal number of divergent series solutions of the equations $Dy = g(x) \in \mathbb{C}\{x\}$ linearly independent modulo convergent ones. These indices were computed in terms of the coefficients $b_p(x)$ of D . The calculation of $\chi(D, \mathbb{C}[[x]])$ is elementary calculus (cf. Prop. 5.2.5 (i) below). The calculation of $\chi(D, \mathbb{C}\{x\})$ follows from an adequate application of Ascoli's Theorem.

By a similar analytical method, based on direct or projective limits of Banach spaces and compact perturbations of operators, J.-P. Ramis [Ram84] extended these indices to a large family of Gevrey series spaces: the Gevrey spaces $\mathbb{C}[[x]]_s$ as introduced above but also the Gevrey-Beurling spaces

$$\mathbb{C}[[x]]_{(s)} = \varprojlim_{C>0} \mathbb{C}[[x]]_{s,C} = \bigcap_{C>0} \mathbb{C}[[x]]_{s,C}$$

where

$$\mathbb{C}[[x]]_{s,C} = \left\{ \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]; \exists K > 0 \text{ such that } |a_n| \leq K(n!)^s C^{ns} \text{ for all } n \right\}$$

and the spaces $\mathbb{C}[[x]]_{s,C^+} = \bigcap_{\varepsilon>0} \mathbb{C}[[x]]_{s,C+\varepsilon}$ and $\mathbb{C}[[x]]_{s,C^-} = \bigcup_{\varepsilon>0} \mathbb{C}[[x]]_{s,C-\varepsilon}$.

B. Malgrange [Mal74] and J.-P. Ramis [Ram84] computed also the indices of D acting on the fraction fields of formal, convergent or Gevrey series. They proved that these indices differ by $-\chi(D, \mathbb{C}[[x]])$ from those of D acting in $\mathbb{C}[[x]]$, $\mathbb{C}\{x\}$ or $\mathbb{C}[[x]]_s$ for $0 < s < +\infty$. In particular, they are all computed in terms of the Newton polygon of D up to vertical translations and appear thus as formal meromorphic invariants of the equation. These indices are extended to systems by means of a cyclic vector.

A differential operator has no index in the spaces $\mathbb{C}[[x]]_{s,C}$ themselves in general. A counter-example (cf. [LRP97, p. 1420]) is given by the Euler

operator in $\mathbb{C}[[x]]_{1,1}$ as we prove below.

Proposition 5.1.2. — *The Euler operator*

$$\mathcal{E} = x^2 \frac{d}{dx} - 1 : \mathbb{C}[[x]]_{1,1} \longrightarrow \mathbb{C}[[x]]_{1,1}$$

has no index when acting in $\mathbb{C}[[x]]_{1,1}$ for, $\text{coker}(\mathcal{E}, \mathbb{C}[[x]]_{1,1})$ has infinite dimension.

Proof. — Check first that \mathcal{E} acts in $\mathbb{C}[[x]]_{1,1}$. Suppose $\sum_{n \geq 0} a_n x^n$ satisfy $|a_n| \leq Kn!$ for all n . Then,

$$\mathcal{E} \left(\sum_{n \geq 0} a_n x^n \right) = \sum_{n \geq 0} b_n x^n$$

where $b_n = (n-1)a_{n-1} - a_n$ satisfy $|b_n| \leq 2Kn!$ for all n and the series $\sum_{n \geq 0} b_n x^n$ belongs to $\mathbb{C}[[x]]_{1,1}$. Consider now the family of series of $\mathbb{C}[[x]]_{1,1}$

$$g_\alpha(x) = \sum_{n > 0} (n-1)! n^\alpha x^n, \quad 0 < \alpha < 1.$$

The unique series $\sum_{n \geq 0} c_n x^n$ solution of $\mathcal{E}(y) = g_\alpha(x)$ is given by $c_0 = 0$ and for $n > 0$ by $c_n = -(n-1)!(1^\alpha + 2^\alpha + \dots + n^\alpha)$. The coefficients c_n have an asymptotic behavior of the form

$$c_n = \frac{1}{\alpha+1} (n-1)! n^{1+\alpha} (1 + O(1/n)) = \frac{1}{\alpha+1} n! n^\alpha (1 + O(1/n))$$

with $\alpha > 0$ [Die80, p.119, Exer. 27 or p.305, Formula (7.5.1)]. Consequently, the series $\sum_{n \geq 0} c_n x^n$ does not belong to $\mathbb{C}[[x]]_{1,1}$ and $g_\alpha(x)$ does not belong to the range of \mathcal{E} . Any non trivial linear combination $\sum \lambda_j g_{\alpha_j}(x)$ has the same property.

To prove that $\text{coker}(\mathcal{E}, \mathbb{C}[[x]]_{1,1})$ has infinite dimension it suffices to prove that the g_α 's are linearly independent. To this end, suppose that the g_α 's satisfy a linear relation of the form $a_1 g_{\alpha_1} + a_2 g_{\alpha_2} + \dots + a_r g_{\alpha_r} = 0$. This means that $a_1 n^{\alpha_1} + a_2 n^{\alpha_2} + \dots + a_r n^{\alpha_r} = 0$ for all $n > 0$. Choose $n_0 \neq 1$. Applying the relation for $n = n_0, n_0^2, \dots, n_0^r$ provides the van der Monde system based on $(\lambda_1 = n_0^{\alpha_1}, \lambda_2 = n_0^{\alpha_2}, \lambda_3 = n_0^{\alpha_3}, \dots, \lambda_r = n_0^{\alpha_r})$:

$$\begin{cases} \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_r a_r = 0 \\ \lambda_1^2 a_1 + \lambda_2^2 a_2 + \dots + \lambda_r^2 a_r = 0 \\ \dots \\ \lambda_1^r a_1 + \lambda_2^r a_2 + \dots + \lambda_r^r a_r = 0 \end{cases}$$

to determine the coefficients a_1, a_2, \dots, a_r which are then all equal to 0. Hence the g_α 's are linearly independent and $\text{coker}(\mathcal{E}, \mathbb{C}[[x]]_{1,1})$ has infinite dimension. \square

5.2. Irregularity after Deligne-Malgrange and Gevrey index theorems

The proofs given in this section are due to B. Malgrange [Mal74] and P. Deligne [DMR07]) in letter to B. Malgrange, dated 22 août 1977. Introduce the following notations:

- ▷ \mathcal{V} is the sheaf over S^1 of germs of solutions of D ;
- ▷ $\mathcal{V}^{\leq k}$ the subsheaf of germs with exponential growth of order at most k ;
- ▷ $\mathcal{V}^{<0}$ the sheaf over S^1 of flat germs of solutions of D ; ⁽¹⁾
- ▷ $\mathcal{V}^{\leq -k}$ the subsheaf of germs with exponential decay of order at least k .

The sheaf $\mathcal{V}^{\leq k}$ is a subsheaf of $\mathcal{A}^{\leq k}$, the sheaf $\mathcal{V}^{\leq -k}$ a subsheaf of $\mathcal{A}^{\leq -k}$ and the sheaf $\mathcal{V}^{<0}$ a subsheaf of $\mathcal{A}^{<0}$. All these sheaves are sheaves of \mathbb{C} -vector spaces. The dimensions of the stalks of $\mathcal{V}^{<0}$ and $\mathcal{V}^{\leq -k}$ at $\theta \in S^1$ are denoted by

$$N^{<0}(\theta) = \dim \mathcal{V}_\theta^{<0} \quad \text{and} \quad N^{\leq -k}(\theta) = \dim \mathcal{V}_\theta^{\leq -k}.$$

Lemma 5.2.1. — *The sheaves $\mathcal{V}^{<0}$ and $\mathcal{V}^{\leq -k}$ for all $k > 0$ are piecewise constant. The functions $\theta \mapsto N^{<0}(\theta)$ and $\theta \mapsto N^{\leq -k}(\theta)$ are lower semi-continuous with jumps occurring only when entering or exiting a Stokes arc of D .*

Proof. — Let D' be a normal form of D . We denote by $\mathcal{V}', \mathcal{V}'^{\leq k}, \dots$ the sheaves associated with D' as $\mathcal{V}, \mathcal{V}^{\leq k}, \dots$ are associated with D . By the Main Asymptotic Existence Theorem (Thm. 4.4.2) the sheaves $\mathcal{V}^{<0}$ and $\mathcal{V}^{\leq -k}$ for all $k > 0$ are isomorphic to $\mathcal{V}'^{<0}$ and $\mathcal{V}'^{\leq -k}$ respectively and it is sufficient to prove the lemma for D' instead of D . The space of solutions of $D'y = 0$ is spanned by functions of the form $h(x)e^{q_j(1/x)}$ where $h(x)$ has moderate growth at $x = 0$ and is defined on the full germ of universal cover of \mathbb{C}^* at 0. Such functions belong to $\mathcal{V}^{<0}$ in a direction θ if and only if $e^{q_j(1/x)}$ belongs to $\mathcal{A}_\theta^{<0}$. If $q_j(1/x)$ appears with multiplicity m_j in a formal fundamental solution of $Dy = 0$ then the solutions of the form $h(x)e^{q_j(1/x)}$ generate a constant sheaf isomorphic to \mathbb{C}^{m_j} over the interior of each Stokes arc generated by $e^{q_j(1/x)}$

⁽¹⁾ The notations $\mathcal{A}^{<0}$ and $\mathcal{V}^{<0}$, in the continuation of the exponential case $\mathcal{A}^{\leq -k}$ and $\mathcal{V}^{\leq -k}$ for $k > 0$, are usually saved for germs with moderate growth.

and nothing else. The same result is valid for $\mathcal{V}^{\leq -k}$ by considering only the exponentials $e^{q_j(1/x)}$ of degree at least k . \square

Comments 5.2.2 (Sheaf of solutions of the Euler equation)

Here below are drawn the unique Stokes arc of the Euler equation (cf. Exa. 2.2.4) and the graph of the function $\theta \mapsto N^{\leq -1}(\theta)$. In this case, $N^{<0}(\theta) = N^{\leq -1}(\theta)$.

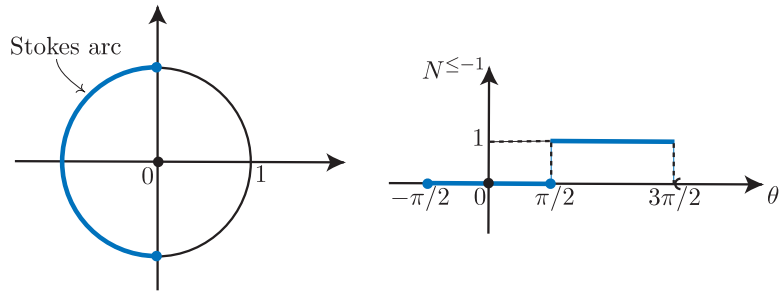


Figure 1

Theorem 5.2.3 (Deligne-Malgrange). — Any linear differential operator D with analytic coefficients satisfies the following properties.

1. $\ker(D, \mathbb{C}[[x]]_s/\mathbb{C}\{x\}) \simeq \begin{cases} H^1(S^1; \mathcal{V}^{<0}) & \text{for } s = +\infty, \\ H^1(S^1; \mathcal{V}^{\leq -k}) & \text{for } 0 < s = 1/k < +\infty; \end{cases}$
2. $\text{coker}(D, \mathbb{C}[[x]]_s/\mathbb{C}\{x\}) = 0$ for $0 < s \leq +\infty$;
3. $\dim H^1(S^1; \mathcal{V}^{<0}) = \frac{1}{2} \text{var}(N^{<0})$;
4. $\dim H^1(S^1; \mathcal{V}^{\leq -k}) = \frac{1}{2} \text{var}(N^{\leq -k})$ for all $k > 0$.

Proof. — 1.-2. Consider first the case $s = +\infty$. The long exact sequence of cohomology associated with the short exact sequence $0 \rightarrow \mathcal{A}^{<0} \rightarrow \mathcal{A} \rightarrow \mathcal{A}/\mathcal{A}^{<0} \rightarrow 0$ reads

$$\begin{array}{ccccccc}
 0 \rightarrow H^0(S^1; \mathcal{A}) & \longrightarrow & H^0(S^1; \mathcal{A}/\mathcal{A}^{<0}) & \longrightarrow & H^1(S^1; \mathcal{A}^{<0}) & \longrightarrow & H^1(S^1; \mathcal{A}) \\
 & & \wr & & \wr & & \searrow \swarrow \\
 & & \mathbb{C}\{x\} & & \mathbb{C}[[x]] & & 0
 \end{array}$$

for, $H^0(S^1; \mathcal{A}^{<0}) = 0$, $H^0(S^1; \mathcal{A}/\mathcal{A}^{<0})$ is isomorphic to $\mathbb{C}[[x]]$ by the Borel-Ritt Theorem (cf. Cor. 3.2.8) and the map $H^1(S^1; \mathcal{A}^{<0}) \rightarrow H^1(S^1; \mathcal{A})$ factors through 0 by the Cauchy-Heine Theorem (cf. Cor. 3.2.14). Hence,

$$H^1(S^1; \mathcal{A}^{<0}) \simeq \mathbb{C}[[x]]/\mathbb{C}\{x\}.$$

The Main Asymptotic Existence Theorem in sheaf form (Thm. 4.4.1) provides the short exact sequence $0 \rightarrow \mathcal{V}^{<0} \rightarrow \mathcal{A}^{<0} \xrightarrow{D} \mathcal{A}^{<0} \rightarrow 0$. The associated long exact sequence of cohomology reads

$$0 \rightarrow H^1(S^1; \mathcal{V}^{<0}) \rightarrow H^1(S^1; \mathcal{A}^{<0}) \xrightarrow{D} H^1(S^1; \mathcal{A}^{<0}) \rightarrow 0.$$

$$\begin{array}{ccc} & \wr & \wr \\ & \mathbb{C}[[x]]/\mathbb{C}\{x\} & \mathbb{C}[[x]]/\mathbb{C}\{x\} \end{array}$$

Hence, $\ker(D, \mathbb{C}[[x]]/\mathbb{C}\{x\}) \simeq H^1(S^1; \mathcal{V}^{<0})$ and $\text{coker}(D, \mathbb{C}[[x]]/\mathbb{C}\{x\}) \simeq 0$.

The case when $s < +\infty$ is proved similarly from the short exact sequences

$$0 \rightarrow \mathcal{A}^{\leq -k} \rightarrow \mathcal{A}_s \xrightarrow{T} \mathcal{A}_s/\mathcal{A}^{\leq -k} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{V}^{\leq -k} \rightarrow \mathcal{A}^{\leq -k} \xrightarrow{D} \mathcal{A}^{\leq -k} \rightarrow 0$$

using the Gevrey parts of the Borel-Ritt and the Cauchy-Heine Theorems.

3. Denote by α_ℓ for $\ell \in \mathbb{Z}/p\mathbb{Z}$, the boundary points of the Stokes arcs of D ordered cyclically on S^1 and by $i_\ell : \{\alpha_\ell\} \hookrightarrow S^1$ and $j_\ell :]\alpha_\ell, \alpha_{\ell+1}[\hookrightarrow S^1$ the canonical inclusions. Since S^1 is a real variety of dimension 1 the Euler characteristic⁽²⁾ of the sheaf $\mathcal{V}^{<0}$ satisfy

$$\chi(\mathcal{V}^{<0}) = \dim H^0(S^1; \mathcal{V}^{<0}) - \dim H^1(S^1; \mathcal{V}^{<0}).$$

Then, $\chi(\mathcal{V}^{<0}) = -\dim H^1(S^1; \mathcal{V}^{<0})$ since $\dim H^0(S^1; \mathcal{V}^{<0}) = 0$ (there exists no flat analytic function and, a fortiori, no flat solution all around 0 but the null function) and we are left to estimate the Euler characteristic of the sheaf $\mathcal{V}^{<0}$.

Consider the short exact sequence

$$0 \rightarrow \bigoplus_{\ell} j_{\ell!} \mathcal{V}^{<0} \rightarrow \mathcal{V}^{<0} \rightarrow \bigoplus_{\ell} i_{\ell*} \mathcal{V}^{<0} \rightarrow 0$$

The additivity of Euler characteristics allows us to write

$$\chi(\mathcal{V}^{<0}) = \sum_{\ell} \chi(j_{\ell!} \mathcal{V}^{<0}) + \sum_{\ell} \chi(i_{\ell*} \mathcal{V}^{<0}).$$

The space $H^0(S^1; j_{\ell!} \mathcal{V}^{<0})$ is 0 since $] \alpha_\ell, \alpha_{\ell+1}[$ is not a closed subset of S^1 (a germ at a point of the boundary is the null germ by definition and generates null germs in the neighborhood, hence all over $] \alpha_\ell, \alpha_{\ell+1}[$). The sheaf $j_{\ell!} \mathcal{V}^{<0}$ is a constant sheaf in restriction to $] \alpha_\ell, \alpha_{\ell+1}[$ and 0 outside. Hence, the space $H^1(S^1; j_{\ell!} \mathcal{V}^{<0}) \simeq H^1(] \alpha_\ell, \alpha_{\ell+1}[; j_{\ell!} \mathcal{V}^{<0})$ is isomorphic to the stalk of $\mathcal{V}^{<0}$ at any point α'_ℓ of $] \alpha_\ell, \alpha_{\ell+1}[$ and therefore, $\chi(j_{\ell!} \mathcal{V}^{<0}) = -\dim \mathcal{V}_{\alpha'_\ell}^{<0}$ for all ℓ .

⁽²⁾ The number $\chi(\mathcal{F}) = \sum (-1)^j \dim H^j(X; \mathcal{F})$ is, by definition, the Euler characteristic of a sheaf \mathcal{F} over a space X .

The space $H^1(S^1; i_{\ell*} \mathcal{V}^{<0})$ is 0 since the support of $i_{\ell*} \mathcal{V}^{<0}$ has dimension 0 and the space $H^0(S^1; i_{\ell*} \mathcal{V}^{<0})$ is isomorphic to the stalk $\mathcal{V}_{\alpha_\ell}^{<0}$ of $\mathcal{V}^{<0}$ at α_ℓ . Thus, $\chi(i_{\ell*} \mathcal{V}^{<0}) = \dim \mathcal{V}_{\alpha_\ell}^{<0}$ for all ℓ .

The number $\dim \mathcal{V}_{\alpha_\ell}^{<0} - \dim \mathcal{V}_{\alpha'_\ell}^{<0}$ is both the variation of $N^{<0}$ at α_ℓ and at $\alpha_{\ell+1}$. Hence, the $\frac{1}{2}$ in the formula $\sum_\ell \chi(j_{\ell!} \mathcal{V}^{<0}) + \sum_\ell \chi(i_{\ell*} \mathcal{V}^{<0}) = \frac{1}{2} \text{var}(N^{<0})$. This ends the proof of point 3.

4. The extension of the previous proof to the sheaf $\mathcal{V}^{\leq -k}$ is straightforward. □

Remark 5.2.4. — The half variation of $N^{<0}$ around S^1 is also the number of Stokes arcs of D counted with multiplicity (cf. Def. 4.3.2 and Exa. 4.3.3), or equivalently, the sum of the (possibly fractional) degrees of all of the exponentials of a formal fundamental solution of D . The half variation of $N^{\leq -k}$ around S^1 is the number of Stokes arcs of D with level at least k and counted with multiplicity, or equivalently, the sum of the degrees of all of the exponentials of degree at least k .

Corollary 5.2.5. — Let $0 < k_1 < k_2 < \dots < k_r < +\infty$ be the slopes of the Newton polygon of the linear differential operator D (i.e., the levels of D) and denote, as usually, $s_j = 1/k_j$ for $j = 1, \dots, r$. The operator D has an index in all spaces $\mathbb{C}[[x]]_s$ for $0 \leq s \leq +\infty$ with values

- (i) $\chi(D, \mathbb{C}[[x]]) = -$ lower ordinate of the Newton polygon $\mathcal{N}(D)$;
- (ii) $\chi(D, \mathbb{C}\{x\}) = \begin{cases} \chi(D, \mathbb{C}[[x]]) - \# \text{ Stokes arcs of any level, i.e.,} \\ -\text{lower ordinate of the vertical side in } \mathcal{N}(D); \end{cases}$
- (iii) $\chi(D, \mathbb{C}[[x]]_s) = \begin{cases} \chi(D, \mathbb{C}[[x]]) - \# \text{ Stokes arcs of level } < k_j, \text{ i.e.,} \\ -\text{lower ordinate of the side of slope } k_j \text{ in } \mathcal{N}(D) \\ \text{when } s \text{ satisfies } s_{j+1} < s \leq s_j \end{cases}$

where $\#$ stands for “the number of”. In particular, its irregularity $\text{irr}_0(D)$ satisfies

- (iv) $\text{irr}_0(D) =$ height of the Newton polygon $\mathcal{N}(D)$ of D (out of the vertical side).

Proof. — Denote by $v(b_p)$ the valuation of the coefficient $b_p(x)$ of d^p/dx^p in D and by $m = \inf(v(b_p) - p)$ the lower ordinate of the Newton polygon $\mathcal{N}(D)$ of D .

- (i) Prove that D has an index in $\mathbb{C}[[x]]$, equal to $-m$.

From the definition of m the valuation of b_p satisfies $v(b_p) \geq p + m$ for all p , the equality being reached on a non-empty set \mathcal{P} of indices p . Hence, the coefficient b_p reads $b_p(x) = \alpha_p x^{p+m} + A_p(x)$ with $v(A_p) > p + m$ for all p and the constant coefficient α_p is non-zero for $p \in \mathcal{P}$. For $r \geq -m$, we have

$$b_p(x) \frac{d^p}{dx^p} x^r = r(r-1) \cdots (r-p+1) \alpha_p x^{r+m} + \text{higher order terms.}$$

Hence,

$$D x^r = \beta_{r+m} x^{r+m} + B_{r+m}(x)$$

where $v(B_{r+m}) > r + m$. The constant $\beta_{r+m} = \sum_{p \in \mathcal{P}} r(r-1) \cdots (r-p+1) \alpha_p$ being a polynomial with respect to r is non-zero for $r \geq r_0$ large enough. Denote by \mathcal{M} the maximal ideal of $\mathbb{C}[[x]]$ (ideal generated by x). The previous calculation states that, for $r \geq r_0$, the operator D induces a morphism $D : \mathcal{M}^r \rightarrow \mathcal{M}^{r+m}$.

Prove that this morphism is an isomorphism. Let $g(x) = g_{r+m} x^{r+m} + G_{r+m}(x)$ with $v(G_{r+m}) > r + m$ be given. A series $f(x) = f_r x^r + F_r(x)$ with $v(F_r) > r$ satisfies the equation $Df = g$ if and only if $f_r = g_{r+m}/\beta_{r+m}$ and $DF_r = G_{r+m} + C_{r+m}$ for an adequate formal series C_{r+m} with valuation $v(C_{r+m}) > r + m$. The same reasoning applied to this new equation proves that the next term in $f(x)$ is also uniquely determined and so on by recurrence. This achieves the proof of the fact that $D : \mathcal{M}^r \rightarrow \mathcal{M}^{r+m}$ is an isomorphism.

Now, consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}^r & \longrightarrow & \mathbb{C}[[x]] & \longrightarrow & \mathbb{C}[[x]]/\mathcal{M}^r & \longrightarrow & 0 \\ & & \downarrow \sim & & \downarrow D & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}^{r+m} & \longrightarrow & \mathbb{C}[[x]] & \longrightarrow & \mathbb{C}[[x]]/\mathcal{M}^{r+m} & \longrightarrow & 0 \end{array}$$

The left vertical morphism has an index equal to 0. The spaces $\mathbb{C}[[x]]/\mathcal{M}^r$ and $\mathbb{C}[[x]]/\mathcal{M}^{r+m}$ being of finite dimension equal to r and $r + m$ respectively the right vertical morphism has an index equal to $-m$. We can conclude that the morphism in the middle has also an index $\chi(D, \mathbb{C}[[x]])$ and, by additivity of Euler characteristics, this index satisfies $-0 + \chi(D, \mathbb{C}[[x]]) - (-m) = 0$. Hence, the result.

(ii) To prove that D has an index in $\mathbb{C}\{x\}$ with the value given in the statement we consider the exact sequence

$$0 \rightarrow \mathbb{C}\{x\} \rightarrow \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]]/\mathbb{C}\{x\} \rightarrow 0.$$

Since D has an index both in $\mathbb{C}[[x]]$ and in $\mathbb{C}[[x]]/\mathbb{C}\{x\}$ (cf. Thm. 5.2.3) it has also an index in $\mathbb{C}\{x\}$ and the three indices satisfy the addition formula

$$\chi(D, \mathbb{C}\{x\}) - \chi(D, \mathbb{C}[[x]]) + \chi(D, \mathbb{C}[[x]]/\mathbb{C}\{x\}) = 0.$$

Hence, the result.

(iii) The same argument applied to the exact sequence

$$0 \rightarrow \mathbb{C}\{x\} \rightarrow \mathbb{C}[[x]]_s \rightarrow \mathbb{C}[[x]]_s/\mathbb{C}\{x\} \rightarrow 0$$

proves that D has an index in $\mathbb{C}[[x]]_s$ and provides the value given in the statement by the addition formula of Euler characteristics.

(iv) follows directly from (i) and (ii). □

Remarks 5.2.6. — The following remarks are straightforward.

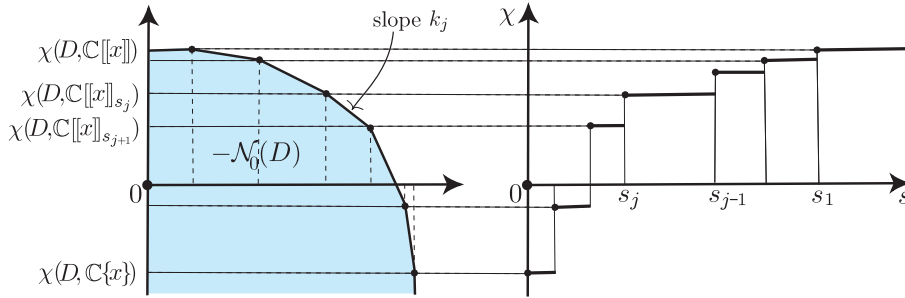


Figure 2. Newton polygon and the curve $s \rightarrow \chi(D, \mathbb{C}[[x]]_s)$

▷ The proof above shows that the cokernel of $D : \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]]$ can be generated by polynomials.

▷ The irregularity of D is zero if and only if the Newton polygon has only an horizontal edge. This corresponds, by definition, to a regular singular point at 0. Hence, the irregularity is zero if and only if 0 is a regular singular point.

▷ The function $s \mapsto \chi(D, \mathbb{C}[[x]]_s)$ is piecewise constant, increasing and left continuous.

▷ The indices $\chi(D, \mathbb{C}[[x]]_s), 0 \leq s \leq +\infty$, are not themselves formal meromorphic invariants since, in a gauge transformation, the Newton polygon is translated vertically. However, the irregularity is a meromorphic invariant as well as any difference of indices $\chi(D, \mathbb{C}[[x]]_s), 0 \leq s \leq +\infty$.

A consequence of the previous index theorem is the Maillet-Ramis Theorem. Maillet's Theorem asserts that the series solutions of a linear or non

linear differential equation are Gevrey of a certain order k . J.-P. Ramis made the theorem more precise in the linear case by proving that the possible k are the levels of the equation.

Theorem 5.2.7 (Maillet-Ramis Theorem). — *A series $\tilde{f}(x)$, solution of the differential equation $Dy = 0$ is either convergent or s -Gevrey where s can be chosen so that $k = 1/s$ be one of the levels $k_1 < k_2 < \dots < k_r$ of D associated with $\tilde{f}(x)$. Other values of k are not optimal.*

A series solution $\tilde{f}(x)$ is a formal solution of the form $\tilde{f}(x)e^0$ coming with a determining polynomial $q_j = 0$. The levels associated with $\tilde{f}(x)$ are the degrees of the polynomials $q_\ell - q_j = q_\ell$ for all $\ell \neq j$. They are, then, the non zero slopes of the Newton polygon $\mathcal{N}_0(D)$ of D at 0.

Proof. — Let $0 < s < +\infty$ and denote $k_0 = 0$, $k_\infty = +\infty$ and $s_j = 1/k_j$ for $j = 0, \dots, +\infty$. Since, for all s , $\text{coker}(D, \mathbb{C}[[x]]_s/\mathbb{C}\{x\}) = 0$ the number of independent solutions of D in $\mathbb{C}[[x]]_s/\mathbb{C}\{x\}$ is equal to the index $\chi(D, \mathbb{C}[[x]]_s/\mathbb{C}\{x\})$ of D in $\mathbb{C}[[x]]_s/\mathbb{C}\{x\}$. It is then constant for $s_{j+1} < s \leq s_j$, $j = 0, \dots, r$. Hence, a series solution is at least s_1 -Gevrey; if it is s -Gevrey with s satisfying $s_{j+1} < s \leq s_j$, it is also s_j -Gevrey and, in particular, if it is s -Gevrey with $s < s_r$ it is convergent. \square

Comments 5.2.8 (On the examples of Section 2.2.2)

▷ The Euler operator $\mathcal{E} = x^2 d/dx + 1$ and its homogeneous variant

$$\mathcal{E}_0 = x^3 \frac{d^2}{dx^2} + (x^2 + x) \frac{d}{dx} - 1$$

are singular irregular at 0. They have same indices and same irregularity as indicated on Fig. 3. Moreover, $\chi(\mathcal{E}, \mathbb{C}[[x]]_s) = \chi(\mathcal{E}, \mathbb{C}[[x]])$ for $s \geq 1$ and $\chi(\mathcal{E}, \mathbb{C}[[x]]_s) = \chi(\mathcal{E}, \mathbb{C}\{x\})$ for $s < 1$.

The unique non-zero slope of the Newton polygons $\mathcal{N}(\mathcal{E})$ and $\mathcal{N}(\mathcal{E}_0)$ is equal to 1 while the Euler series is 1-Gevrey and s -Gevrey for no $s < 1$ (cf. Com. 2.3.3).

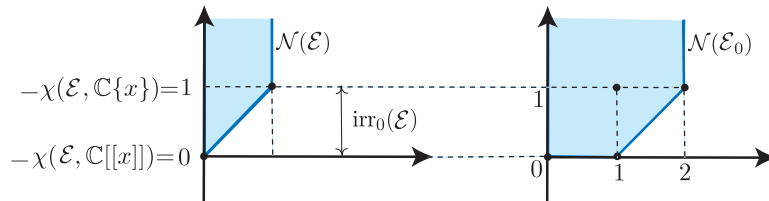


Figure 3

▷ The exponential integral function satisfies $\mathcal{E}i(y) = 0$ where $\mathcal{E}i$ is the operator

$$\mathcal{E}i = x \frac{d^2}{dx^2} + (x + 1) \frac{d}{dx}.$$

This operator is regular singular at 0 and irregular singular at infinity. One can check that the Newton polygon at 0 reduces to a horizontal slope. The picture below shows the Newton polygon $\mathcal{N}(\mathcal{E}i)$ at infinity. Recall that $\mathcal{N}(\mathcal{E}i)$ is the symmetric, with respect horizontal axis, of the Newton polygon at 0 of the operator $\mathcal{E}i$ after the change of variable $z = 1/x$. Hence the change of signs in the indices. The series $\mathcal{E}i(x)$ is 1-Gevrey.

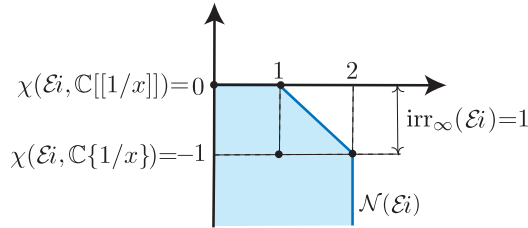


Figure 4

▷ The hypergeometric operator

$$D_{3,1} = z \left(z \frac{d}{dz} + 4 \right) - z \frac{d}{dz} \left(z \frac{d}{dz} + 1 \right) \left(z \frac{d}{dz} - 1 \right)$$

is irregular singular at infinity. Its Newton polygon at infinity has a slope 0 and a slope $-\frac{1}{2}$. Its indices and its irregularity at infinity are indicated on figure below.

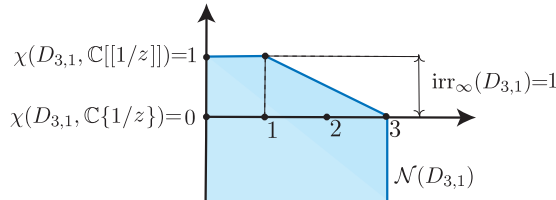


Figure 5

The unique non-zero slope of the Newton polygon $\mathcal{N}(D_{3,1})$ is $-1/2$ at infinity (hence $+1/2$ at 0 after the change of variable $x = 1/z$) and we saw that the hypergeometric series $\tilde{g}(x)$ is 2-Gevrey.

5.3. Wild analytic continuation and index theorems

We sketch here another method to compute a larger variety of index theorems for D . For more details we refer to [LRP97]. In that paper, indices are computed for D acting on a variety of spaces including the spaces considered before. The idea is to see each functional space as the 0-cohomology group of the sheaves $\mathcal{F}, \mathcal{F}^k$, and so on... on a convenient subset of the base space

X, X^k and so on ... The admissible subsets U that are considered are finite unions of truncated narrow sectors (for a technical reason, sectors are assumed to be closed on their lower boundary) and, possibly, of a small disc centered at 0. The index $\chi(D, \mathbb{C}[[x]])$ of D acting on the space of formal series $\mathbb{C}[[x]]$ is assumed to be known, for instance, from a calculation as before (Cor. 5.2.5 (i)). From the present viewpoint the situation is made more complicated by the fact that the base spaces are now varieties of real dimension 2. The cohomology groups $H^2(U; \mathcal{F}), H^2(U; \mathcal{F}^k)$ and so on ... are non-zero groups but the cohomology groups H^i for $i \geq 1$ satisfy (cf. [LRP97, Thms. 2.1 and 4.2]) the following sharp property.

Theorem 5.3.1. — *For $i \geq 1$, the linear maps*

$$D : H^i(U; \mathcal{F}) \longrightarrow H^i(U; \mathcal{F})$$

are isomorphisms for all admissible U .

The same result is valid for \mathcal{F} replaced by $\mathcal{F}^k, \mathcal{F}^{k_1, k_2}$ and so on ...

The technique is as follows. For small discs and narrow sectors (this means small enough to contain no big point associated with D) the calculation is elementary and based on the isomorphism between the spaces of solutions for D and for a normal form D' of D over such domains. For a union of narrow sectors or of a small disc and narrow sectors the calculation follows from the use of Mayer-Vietoris sequences as follows (cf. [LRP97, Lem. 3.5]).

Lemma 5.3.2. — *Let $U = U_1 \cup U_2$ where U_1 and U_2 are either open or closed subsets of U and suppose U, U_1, U_2 and $U_1 \cap U_2$ are admissible subsets.*

If D has an index in $H^0(U_1; \mathcal{F}), H^0(U_2; \mathcal{F})$ and in $H^0(U_1 \cap U_2; \mathcal{F})$ then, it has an index in $H^0(U; \mathcal{F})$ given by

$$\begin{aligned} \chi(D, H^0(U; \mathcal{F})) &= \chi(D, H^0(U_1; \mathcal{F})) + \chi(D, H^0(U_2; \mathcal{F})) \\ &\quad - \chi(D, H^0(U_1 \cap U_2; \mathcal{F})). \end{aligned}$$

The same result is valid with \mathcal{F} replaced by $\mathcal{F}^k, \mathcal{F}^{k_1, k_2}$ and so on ...

For a (non-exhaustive) list of indices which can be computed that way we refer to [LRP97]. Let us just mention that the list includes indices of D acting on k -summable series over any k -wide arc I (cf. Def. 6.1.2) or acting on multisummable series over any multi-arc (I_1, I_2, \dots) (cf. Sect. 8.7.1).

These indices are formal meromorphic invariants of D as long as U does not contain a small disc about 0. Otherwise, their difference with $\chi(D, \mathbb{C}[[x]])$ are formal meromorphic invariants.

CHAPTER 6

FOUR EQUIVALENT APPROACHES TO *K*-SUMMABILITY

Given a power series $\tilde{f}(x)$ at 0 we know from the Borel-Ritt Theorem that there are infinitely many functions asymptotic to $\tilde{f}(x)$ on any given sector with an arbitrary opening. However, when $\tilde{f}(x)$ satisfies an equation, these asymptotic functions do not satisfy the same equation in general. The Main Asymptotic Expansion Theorem fills in this gap on small enough sectors for series solutions of linear differential equations by asserting the existence of asymptotic solutions. However, the theorem does not guaranty uniqueness and consequently lets the situation under some indetermination.

The aim of a theory of summation on a given germ of sector (there might be some constraints on the size and the position of the sector) is to associate with any series an asymptotic function uniquely determined in a way as much natural as possible. What natural means depends on the category we want to consider. There is no known operator of summation applying to the algebra of all power series at one time and no hope towards such a universal tool. For the theory to apply to series solutions of differential equations an eligible request is that the summation operator be a morphism of differential algebras from an algebra of power series (containing the series under consideration) into an algebra of asymptotic functions (containing the corresponding asymptotic solutions). Both algebras must be chosen carefully and correspondingly. To sum series solutions of a difference equation one should look for a summation operator being a difference morphism; to sum basic series, for a summation operator being a q -morphism and so on ... The simplest example is given by the usual summation of convergent power series from the algebra of convergent series into the algebra of germs of analytic functions at 0. Such a summation operator is indeed a morphism of differential algebras.

This chapter deals with the simplest case of summability called k -summability which applies to some divergent series and it aims at being a detailed introduction to the subject. We present four approaches which show up to be equivalent characterizations of k -summable series. With each approach we discuss examples and we attach some applications fitting especially that viewpoint. We give extensive proofs for most of the results and we refer to the literature when the proofs are omitted. A good part of the chapter can be found in [Mal95] or, in [Bal94] and [Cos09] (mostly for the Borel-Laplace approach) and [LRP97] (for an approach through wild analytic continuation). More references can be found in these papers and books.

These questions were already widely considered by Euler. They have been developed at the end of the XIXth and the beginning of the XXth Century by mathematicians such as Borel, Hardy and al. A cohomological viewpoint brought them an impulse in the late 1970's and 1980's mostly with the works of Y. Sibuya, B. Malgrange, J.-P. Ramis, J. Martinet, W. Balsler and lately, C. Zhang for basic series, giving rise to the abstract notions of simple or multiple summability. An extension of Borel's approach was almost simultaneously developed by J. Écalle, B. Braaksma, G. Immink, . . . , giving rise to the theory of resurgence and integral formulæ applying to a variety of situations.

In the 1980's J.-P. Ramis and Y. Sibuya [RS89] (see also, [LR90]) answered negatively the Turrittin problem [Was76, p.326] by showing that series solutions of linear differential equations might be k -summable for no value of the parameter $k > 0$. They showed however that they are all, at worst, multisummable. The levels k_j entering the multisummability process are the levels of the equation (*cf.* Def. 4.3.6). In the case of series solutions of linear difference equations J. Écalle noticed that some series are neither k -summable nor multisummable. He showed that one has to introduce a new concept named k^+ -summability (*cf.* [É93], [Imm96]) for a simple level k as well as for multiple levels k 's.

6.1. First approach: Ramis k -summability

The problem we address now is to determine under which conditions the Taylor map

$$T_{s,I} : H^0(I; \mathcal{A}_s) \longrightarrow \mathbb{C}[[x]]_s$$

which, with a section of \mathcal{A}_s over an arc I of S^1 (or of its universal cover $\check{S}^1 \simeq \mathbb{R}$), associates its s -Gevrey asymptotic expansion, could be inverted as a

morphism of differential \mathbb{C} -algebras. The answer is far from being straightforward and requires some restrictions both on I and $\mathbb{C}[[x]]_s$. The first definition of k -summability we present here is based on constraints for the asymptotic conditions themselves (recall $k = 1/s$). It relies on the results of chapters 2 and 3.

Comment 6.1.1 (On the Euler function (Exa. 2.2.4))

Although the problem here addressed is independent of any equation, what can happen is well illustrated by the behavior of the solutions of the Euler equation.

We saw (*cf.* Coms. 2.3.9, p. 21) that $E(x)$ is 1-Gevrey asymptotic at 0 to the Euler series $\tilde{E}(x)$ on any sector Δ_I based on the arc $I =] -\frac{3\pi}{2}, \frac{3\pi}{2}[$. Denote by $E^-(x)$ and $E^+(x)$ the two branches of $E(x)$ on the half-plane $\Delta_{-\pi} = \{x; \Re(x) < 0\}$; these branches are the respective analytic continuations of $E_{-\pi+\varepsilon}(x)$ and $E_{+\pi-\varepsilon}(x)$ as $\varepsilon > 0$ tends to 0. The functions $E^-(x)$ and $E^+(x)$ are distinct. Indeed, if they were equal, $E(x)$ would be asymptotic to $\tilde{E}(x)$ all around 0 and this would imply that $\tilde{E}(x)$ be convergent. More precisely, by applying Cauchy's Residue Theorem, one can check [LR90] that

$$E^+(x) - E^-(x) = 2\pi i \exp(1/x)$$

The functions $E^-(x)$ and $E^+(x)$ are both 1-Gevrey asymptotic to $\tilde{E}(x)$ at 0 on the half-plane $\Delta_{-\pi}$, and indeed, $\exp(1/x)$ is 1-Gevrey asymptotic to 0 on $\Delta_{-\pi}$.

When the sector Δ is narrow, that is, when Δ is at most an open half-plane, then, $E(x)$ provides always a 1-Gevrey asymptotic solution on Δ . However, when $\Delta \subseteq \Delta_{-\pi}$, the two solutions $E^-(x)$ and $E^+(x)$, and hence all the solutions, are 1-Gevrey asymptotic to $\tilde{E}(x)$. Existence is guaranteed, uniqueness fails.

When the sector Δ is wide, that is, when Δ contains a closed half-plane, then, either Δ does not contain the closure $\bar{\Delta}_{-\pi}$ of $\Delta_{-\pi}$ and f provides the unique 1-Gevrey asymptotic solution on Δ , or Δ contains $\bar{\Delta}_{-\pi}$ and there is no 1-Gevrey asymptotic solution on Δ . Uniqueness is guaranteed, existence may fail.

In conclusion, there is no good size for an open sector Δ to guaranty both existence and uniqueness of s -Gevrey asymptotic solutions. We will see that this property remains valid for s -asymptotic functions, not necessarily solutions. Note also that the defect of uniqueness is an exponential function. More generally, flatness for solutions of linear differential equations is always related to exponential functions.

It is convenient to introduce the following definition.

Definition 6.1.2 (k -wide arc or sector). —

▷ An arc I (of S^1 or of its universal cover \tilde{S}^1) is said to be k -wide if it is bounded and either closed with opening $|I| \geq \frac{\pi}{k}$ or open with opening $|I| > \frac{\pi}{k}$.

▷ A sector Δ is said to be k -wide if it is based on a k -wide arc.

It follows from the Borel-Ritt Theorem (Thm. 2.4.1 (ii) and Cor. 2.4.4) that the Taylor map

$$T_{s,I} : H^0(I; \mathcal{A}_s) \longrightarrow \mathbb{C}[[x]]_s$$

is surjective when the arc I is open with length $|I| \leq \pi/k$ and *a fortiori*, when I is closed with length $|I| < \pi/k$. Schematically, we can write

$$\boxed{I \text{ open or closed but not } k\text{-wide} \implies T_{s,I} \text{ surjective}}$$

Consider now the injectivity of $T_{s,I}$. The example of the Euler function (*cf.* comment 6.1.1) shows that the Taylor map $T_{s,I}$ may be not injective, at least, when I is small. For all I , the kernel of $T_{s,I}$ is the space $H^0(I; \mathcal{A}^{\leq -k})$. Indeed, the left exactness of the functor $\Gamma(I; \cdot) = H^0(I; \cdot)$ applied to the short exact sequence

$$0 \rightarrow \mathcal{A}^{\leq -k} \rightarrow \mathcal{A}_s \xrightarrow{T_s} \mathbb{C}[[x]]_s \rightarrow 0 \quad (19)$$

implies exactness for the sequence

$$0 \rightarrow H^0(I; \mathcal{A}^{\leq -k}) \rightarrow H^0(I; \mathcal{A}_s) \xrightarrow{T_{s,I}} \mathbb{C}[[x]]_s.$$

A sufficient condition for $T_{s,I}$ to be injective is given by Watson's Lemma.

Theorem 6.1.3 (Watson's Lemma). — *Let \mathcal{A} be an open sector with opening $|\mathcal{A}| = \pi/k$ and suppose that $f \in \mathcal{O}(\mathcal{A})$ satisfies a global estimate of exponential type of order k on \mathcal{A} , i.e., there exist constants $C > 0$, $A > 0$ such that the following estimate holds for all $x \in \mathcal{A}$:*

$$|f(x)| \leq C \exp - \frac{A}{|x|^k}.$$

Then, f is identically equal to 0 on \mathcal{A} .

Roughly speaking, the lemma says: “under a global estimate of exponential order k on \mathcal{A} , the function f is too flat on a too wide sector to be possibly non 0”.

For a proof, among the many possible references, quote [Mal95, p. 174, Lem. 1.2.3.3] or to [Bal00, p. 75 Prop. 11].

In terms of sheaves Watson's Lemma translates as follows.

Corollary 6.1.4 (Watson's Lemma). — *The sections of $\mathcal{A}^{\leq -k}$ over any k -wide arc I are all trivial and consequently, the Taylor map $T_{s,I}$ is injective.*

Schematically,

$$k\text{-wide arc } I \implies H^0(I; \mathcal{A}^{\leq -k}) = 0 \implies T_{s,I} \text{ injective}$$

Proof. — It suffices to consider the case when I is compact. A section of $\mathcal{A}^{\leq -k}$ on I is represented by a finite and consistent collection of $f_j \in \bar{\mathcal{A}}^{\leq -k}(\mathcal{A}_j(R_j))$ where the sectors $\mathcal{A}_j(R_j)$ have radius R_j and cover the arc I . Let $R = \min_j(R_j)$. Then, the f_j 's glue together into a function $f \in \bar{\mathcal{A}}^{\leq -k}(\mathcal{A}(R))$ where $\mathcal{A}(R)$ denotes the sector $\mathcal{A}(R) = \bigcup_j \mathcal{A}_j(R_j) \cap \{|x| < R\}$. Since $\mathcal{A}(R)$ contains I it is wider than $\frac{\pi}{k}$ and Watson's Lemma applies. \square

Comments 6.1.5 (Exponentials and Watson's Lemma)

▷ Choose $\mathcal{A} = \{x; \pi/2 < \arg(x) < 3\pi/2\}$. Then, the exponential function $\exp(1/x)$ (which appears in the Euler example) belongs to $\bar{\mathcal{A}}^{\leq -1}(\mathcal{A})$. Although this function is not zero that's not contradictory with Watson's Lemma. Indeed, denote $\theta = \arg(x)$; the best global estimate for $\exp(1/x)$ on \mathcal{A} is $\sup_{x \in \mathcal{A}} |\exp(1/x)| = \sup_{\pi/2 < \theta < 3\pi/2} \exp(\cos \theta / |x|) = 1$ since $\cos \theta$ tends to 0 as θ tends to $\pm\pi/2$.

▷ On another hand, the exponential $\exp(1/x)$ satisfies Watson's estimate on any proper subsector of \mathcal{A} . This shows that Watson's Lemma is no more valid on a smaller sector; here, for $k = 1$ on a sector of opening less than π and for any $k > 0$, using an adequate exponential of order k , on a sector of opening less than π/k .

▷ *Euler series.* — We can now achieve our comment 6.1.1 and show that when \mathcal{A} is a sector containing $\bar{\mathcal{A}}_{-\pi}$ there exists no function (solution or not solution of the Euler equation) being 1-Gevrey asymptotic to the Euler series $\tilde{\mathbb{E}}(x)$ on \mathcal{A} . Indeed, suppose $\mathcal{A} =]\alpha, \beta[\times]0, R[$ with $\alpha < \pi/2 < 3\pi/2 < \beta$ and $f(x)$ be 1-Gevrey asymptotic to $\tilde{\mathbb{E}}(x)$ on \mathcal{A} . In restriction to $] \alpha, 3\pi/2[$, the function $f(x) - \mathbb{E}^+(x)$ is 1-Gevrey asymptotic to $\tilde{\mathbb{E}}(x) - \mathbb{E}^+(x) \equiv 0$; hence, it is 1-exponentially flat (cf. Prop. 2.3.17) on a 1-wide sector and we can conclude by Corollary 6.1.4 of Watson's Lemma that $f(x) = \mathbb{E}^+(x)$ on $] \alpha, 3\pi/2[$. Symmetrically, $f(x) = \mathbb{E}^-(x)$ on $] \pi/2, \beta[$. Hence, the contradiction since $\mathbb{E}^+ \neq \mathbb{E}^-$ on $] \pi/2, 3\pi/2[$.

The conditions on the arc I to insure either the injectivity or surjectivity of the Taylor map $T_{s,I}$ are complementary and there is no intermediate condition insuring both injectivity and surjectivity. In such a situation a natural solution proposed by J.-P. Ramis in the early 80's to get both injectivity and surjectivity consisted in choosing for I a k -wide arc and restricting the space $\mathbb{C}[[x]]_s$ of s -Gevrey series into a smaller space.

Suppose we are given a power series $\tilde{f}(x) = \sum_{n \geq 0} a_n x^n$ at 0.

Definition 6.1.6 (Ramis k -summability). —

▷ k -summability on a k -wide arc I (recall $s = 1/k$). — The series $\tilde{f}(x)$ is said to be k -summable on I if I is a k -wide arc and \tilde{f} belongs to the range of the Taylor map $T_{s,I}$, i.e., if there exists a section $f \in H^0(I; \mathcal{A}_s)$ which is s -Gevrey asymptotic to \tilde{f} on the large enough arc I .

▷ k -summability in a direction $\arg(x) = \theta$. — The series $\tilde{f}(x)$ is said to be k -summable in the direction θ if there exists a k -wide arc I bisected by θ on which $\tilde{f}(x)$ is k -summable.

▷ k -sum. — The function f above, which is uniquely determined when it exists, is called the k -sum of $\tilde{f}(x)$ on I or in the direction θ and we denote it by $f = \mathcal{S}_{k,I}(\tilde{f})$ or $f = \mathcal{S}_{k,\theta}(\tilde{f})$.

▷ k -summability. — The series $\tilde{f}(x)$ is said to be k -summable if it is k -summable in all directions but finitely many, called the singular directions.

Notation 6.1.7. — We denote by $\mathbb{C}\{x\}_{\{k,I\}}$ the set of all k -summable series on I and by $\mathbb{C}\{x\}_{\{k,\theta\}}$ the set of all k -summable series in direction θ .

Notice that $\mathbb{C}\{x\}_{\{k,\theta\}} = \mathbb{C}\{x\}_{\{k,I\}}$ for I the closed arc bisected by θ with length π/k .

Remark 6.1.8. — It follows from the definition that a series which is k -summable in all direction is necessarily convergent.

Comment 6.1.9 (On the examples of chapter 1)

From Sect. 2.2.2 we deduce:

▷ The Euler series $\tilde{E}(x)$ of Example 2.2.4 is 1-summable according to the definition above: precisely, it is 1-summable in all directions but the direction $\theta = \pi$.

▷ Since we have not yet proved that the hypergeometric series $\tilde{g}(z)$ of Example 2.2.6 is a 2-Gevrey asymptotic expansion we cannot conclude yet about its possible $\frac{1}{2}$ -summability.

▷ In Example 2.2.7, as for the Euler function, we can move the line of integration from \mathbb{R}^+ to the half-line d_θ with argument θ and get an estimate of the same type as Estimate (10) as long as $-\pi/2 < \theta < \pi/2$ (we leave that point as an exercise). This shows that the series $\tilde{h}(z)$ is 1-summable in all directions $-\pi/2 < \theta < \pi/2$.

▷ Similarly, one can show that the series $\tilde{\ell}(z)$ of Example 2.2.8 is 1-summable in all directions $-\pi/2 < \theta < \pi/2$.

The following proposition is straightforward.

Proposition 6.1.10. — With definitions as above, and especially $s = 1/k$, we can state:

(i) The sets $\mathbb{C}\{x\}_{\{k,I\}}$ of k -summable series on a k -wide arc I and $\mathbb{C}\{x\}_{\{k,\theta\}}$ of k -summable series in direction θ are differential subalgebras of the Gevrey series space $\mathbb{C}[[x]]_s$;

(ii) For I a k -wide arc of S^1 , the Taylor map

$$\Gamma(I, \mathcal{A}_s) \xrightarrow{T_{s,I}} \mathbb{C}\{x\}_{\{k,I\}}$$

is an isomorphism of differential \mathbb{C} -algebras with inverse the summation map $\mathcal{S}_{k,I}$.

As in Chapter 2 (cf. Prop. 2.3.13 and Cor. 2.3.14) let us now observe the effect of a change of variable $x = t^r$, $r \in \mathbb{N}^*$. Let $I = (\alpha, \beta)$ be a k -wide arc. In accordance with the notation $\delta_{/r}^j$ for sectors in Section 2.3.2, denote by $I_{/r}^j$ the arc

$$I_{/r}^j = ((\alpha + 2j\pi)/r, (\beta + 2j\pi)/r)$$

so that when $\theta' = \arg(t)$ runs over $I_{/r} = I_{/r}^0$ then $\theta'' = \arg(\omega^\ell t)$ runs over $I_{/r}^\ell$ and $\theta = \arg(x = t^r)$ runs over I . Observe that $I_{/r}^j$ is kr -wide.

Proposition 6.1.11 (k -summability in an extension of the variable)

The following two assertions are equivalent:

- (i) the series $\tilde{f}(x)$ is k -summable on I with k -sum $f(x)$;
- (ii) the series $\tilde{g}(t) = \tilde{f}(t^r)$ is kr -summable on $I_{/r}$ with kr -sum $g(t) = f(t^r)$.

Proof. — The equivalence is a direct consequence of Definition 6.1.6 of k -summability and of Proposition 2.3.13. \square

Given a series $\tilde{g}(t)$ recall (cf. Sect. 2.3.2) that r -rank reduction consists in replacing $\tilde{g}(t)$ by the r series $\tilde{g}_j(x)$, $j = 0, \dots, r-1$, defined by

$$\tilde{g}(t) = \sum_{j=0}^{r-1} t^j \tilde{g}_j(t^r)$$

and that the series $\tilde{g}_j(x)$ are given, for $j = 0, \dots, r-1$, by the relations

$$rt^j \tilde{g}_j(t^r) = \sum_{\ell=0}^{r-1} \omega^{\ell(r-j)} \tilde{g}(\omega^\ell t).$$

From Corollary 2.3.14 we can state:

Corollary 6.1.12 (k -summability and rank reduction)

The following two properties are equivalent:

- (i) for $\ell = 0, \dots, r-1$ the series $\tilde{g}(t)$ is k' -summable on $I_{/r}^\ell$ with k' -sum $g(t)$;
- (ii) for $j = 0, \dots, r-1$ the r -rank reduced series $\tilde{g}_j(x)$ is k'/r -summable on I with k'/r -sums $g_j(x)$ defined by the relation

$$rx^{j/r} g_j(x) = \sum_{\ell=0}^{r-1} \omega^{\ell(r-j)} g(\omega^\ell x^{1/r}), \quad x^{1/r} \in I_{0/r}.$$

In particular, a series $\tilde{g}(t)$ is k' -summable if and only if its associated r -rank reduced series are k'/r -summable.

With these results we may assume, without loss of generality, that k is small or large at convenience. In particular, we may assume that $k > 1/2$ so that closed arcs of length π/k are shorter than 2π and can be seen as arcs of S^1 .

6.2. Second approach: Ramis-Sibuya k -summability

Due to the quite simple integral formula defining the Euler function $f(x)$ we were able to prove, in accordance to Definition 6.1.6, that the Euler series $\tilde{E}(x)$ is 1-summable in all directions but the direction $\theta = \pi$. However, to check s -asymptoticity on k -wide arcs is not an easy task in general (we refer for instance to our other examples in Sect 2.2.2) and it is worth to look for equivalent conditions in different form.

In this section, we discuss an alternate definition of k -summability, stated in the early 80' by J.-P. Ramis and Y. Sibuya, which is based on series seen as 0-cochains. In order to work on S^1 we assume that $k > 1/2$. This assumption does not affect the generality of the purpose as explained at the end of the previous section.

6.2.1. Definition. — Let $\mathcal{I} = \{I_j\}_{j \in \mathbb{Z}/p\mathbb{Z}}$ be a “good” covering of S^1 (hence a covering without 3-by-3 intersections; cf. Def. 3.2.9). Its connected intersections 2-by-2 are the arcs $(\dot{I}_j = I_j \cap I_{j+1})^{(1)}$ and, given a sheaf \mathcal{F} over S^1 , a 1-cocycle of \mathcal{I} with values in \mathcal{F} is well defined by the data of functions $\dot{\varphi}_j \in \mathcal{F}(\dot{I}_j)$ for all $j \in \mathbb{Z}/p\mathbb{Z}$.

⁽¹⁾ There is an ambiguity with the notations when $p = 2$. In that case, the intersection $I_1 \cap I_2$ is made of two arcs which we denote by \dot{I}_1 and \dot{I}_2 .

Theorem 6.2.1 (Ramis-Sibuya Theorem). —

Suppose $\dot{\varphi} = (\dot{\varphi}_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$ is a 1-cocycle of \mathcal{I} with values in $\mathcal{A}^{\leq -k}$. Then, there exist 0-cochains $(f_j \in \Gamma(I_j; \mathcal{A}))_{j \in \mathbb{Z}/p\mathbb{Z}}$ of \mathcal{I} with coboundary $\dot{\varphi}$ and any such 0-cochain $(f_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$ takes actually its values in \mathcal{A}_s , i.e., $f_j \in \Gamma(I_j; \mathcal{A}_s)$ for all j (recall that $s = 1/k$).

The theorem says in particular that, under the condition that all the differences $-f_j + f_{j+1}$ are k -exponentially flat, all the f_j 's are s -Gevrey asymptotic to a same s -Gevrey formal series $\tilde{f}(x)$.

Proof. — When $\dot{\varphi}$ is trivial ($\dot{\varphi}_j = 0$ for all j) then $\dot{\varphi}$ is the coboundary of any analytic function. Conversely, given any 0-cochain (f_j) which, by means of a refinement if necessary, we can assume to be a 0-cochain over a good covering the condition that its coboundary is trivial, i.e., $-f_j + f_{j+1} = 0$ for all j , implies that the functions f_j glue together into an analytic function f . The function f_j are, in particular, s -Gevrey asymptotic to f on I_j for any $s > 0$.

When $\dot{\varphi}$ is elementary (i.e., only one of its components is non zero; Def. 3.2.10) a 0-cochain with values in \mathcal{A}_s and coboundary $\dot{\varphi}$ is given by the Cauchy-Heine Theorem 2.5.2 (ii). The general case follows by additivity of cocycles. In all cases, there exists then a 0-cochain $(f_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$ with values in \mathcal{A}_s and coboundary $\dot{\varphi}$. Let $(g_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$ be another 0-cochain of \mathcal{I} with coboundary $\dot{\varphi}$. Then, the 0-cochain $(g_j - f_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$ has a trivial coboundary and comes from an analytic function h : for all $j \in \mathbb{Z}/p\mathbb{Z}$, $g_j = f_j + h$ and then, like f_j , the function g_j belongs to $\mathcal{A}_s(I_j)$. \square

The Ramis-Sibuya Theorem admits the following corollary:

Corollary 6.2.2. — *The natural injection $\mathcal{A}_s \hookrightarrow \mathcal{A}$ induces an isomorphism*

$$H^0(S^1; \mathcal{A}_s / \mathcal{A}^{\leq -k}) \xrightarrow{i} H^0(S^1; \mathcal{A} / \mathcal{A}^{\leq -k})$$

and, consequently (cf. Cor. 3.1.27), the Taylor map induces an isomorphism

$$H^0(S^1; \mathcal{A} / \mathcal{A}^{\leq -k}) \simeq \mathbb{C}[[x]]_s.$$

We can thus improve the characterization of s -Gevrey series given in Section 3.2.3 into a characterization free of Gevrey estimates:

$$\boxed{\left. \begin{array}{l} s\text{-Gevrey series} \\ \tilde{f}(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]_s \end{array} \right\} \iff \left\{ \begin{array}{l} \text{(equivalence class of a)} \\ \text{0-cochain } (f_j) \text{ over } S^1 \\ \text{with values in } \mathcal{A} \\ \text{and coboundary } (f_j - f_\ell)_{j, \ell \in J} \\ \text{with values in } \mathcal{A}^{\leq -k} \end{array} \right.$$

This equivalence is a subsequent improvement with respect to the setting in Section 3.2.3 since to check that the 0-cochain is asymptotic in the sense of Poincaré is usually much simpler than to check its s -Gevrey asymptotics. While in Section 3.2.3 it was sufficient to ask for the coboundary to be with values in $\mathcal{A}^{<0}$ it is now essential that the coboundary took its values in $\mathcal{A}^{\leq -k}$.

Definition 6.2.3 (k -quasi-sum). —

Given $\tilde{f}(x)$ an s -Gevrey series, the element $\varphi_0 \in H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k})$ associated with $\tilde{f}(x)$ by the Taylor isomorphism of Corollary 6.2.2 is called the k -quasi-sum of $\tilde{f}(x)$. By extension, any 0-cochain (f_j) representing φ_0 is called a k -quasi-sum of $\tilde{f}(x)$.

With these results k -summability can be equivalently defined as follows.

Definition 6.2.4 (Ramis-Sibuya k -summability)

An s -Gevrey series $\tilde{f}(x)$ is said to be k -summable on a k -wide arc I with k -sum $f(x) \in H^0(I; \mathcal{A})$ if, in restriction to I , its k -quasi-sum φ_0 satisfies the condition

$$\varphi_{0|_I}(x) = f(x) \pmod{\mathcal{A}^{\leq -k}}.$$

Indeed, suppose φ_0 satisfies the condition above and let the 0-cochain (f_j) be a k -quasi-sum of $\tilde{f}(x)$. We know by Corollary 6.2.2 of the Ramis-Sibuya Theorem that all components f_j are s -Gevrey asymptotic to $\tilde{f}(x)$. Hence, the same is true for $f(x)$ on the k -wide arc I and $f(x)$ fits Definition 6.1.6. Conversely, a k -sum $f(x)$ of $\tilde{f}(x)$ in the sense of Definition 6.1.6, can be completed into a k -quasi-sum (f_j) of $\tilde{f}(x)$ using the Borel-Ritt Theorem 2.4.1(ii) and Proposition 2.3.17. Thus, we can reformulated Definition 6.2.4 by saying:

The s -Gevrey series $\tilde{f}(x)$ is k -summable on the k -wide arc I with k -sum $f(x)$ if there exists a k -quasi-sum of $\tilde{f}(x)$ containing $f(x)$ as component.

6.2.2. Applications to differential equations. — As before we consider a linear differential operator with analytic coefficients at 0:

$$D = b_n(x) \frac{d^n}{dx^n} + b_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + b_0(x) \quad \text{with } b_n(x) \neq 0.$$

▷ *The Maillet-Ramis Theorem* (Thm. 5.2.7) can be obtained as a consequence of the Ramis-Sibuya Theorem as follows. Recall its statement:

Given $\tilde{f}(x)$, solution of $Dy = 0$, then, either $\tilde{f}(x)$ is convergent or $\tilde{f}(x)$ is s -Gevrey and $k = 1/s$ is one of the levels $k_1 < k_2 < \cdots < k_r$ of D associated with $\tilde{f}(x)$, i.e., the non-zero slopes of the Newton polygon $\mathcal{N}_0(D)$ of D at 0. Other values of k are not optimal.

Proof. — Using the Main Asymptotic Existence Theorem (Cor. 4.4.2 (i)) we can associate with $\tilde{f}(x)$ a (non-unique) 0-cochain $(f_j(x))_{j \in \mathbb{Z}/p\mathbb{Z}}$ made of asymptotic solutions of the equation $Dy = 0$ over a good covering of S^1 . The coboundary $(-f_j(x) + f_{j+1}(x))$ is made of flat solutions; each such flat solution is equal to some linear combinations of all flat solutions of the equation. Now, solutions of the equation are flat if and only if the exponential factor they contain is flat. They are then flat of an order k which is one of the levels k_1, k_2, \dots, k_r . It follows that either the coboundary $(-f_j(x) + f_{j+1}(x))$ is trivial (or cohomologous to trivial via flat functions) and the series $\tilde{f}(x)$ is convergent or the coboundary takes its values in $\mathcal{A}^{\leq -k_j}$ for a certain index j and the series $\tilde{f}(x)$ is $1/k_j$ -Gevrey according to the Ramis-Sibuya Theorem (Thm. 6.2.1) and Proposition 2.3.10. If the series $\tilde{f}(x)$ were $1/k$ -Gevrey with $k_j < k < k_{j+1}$ then, by Corollary 4.4.2 (ii) of the Main Asymptotic Existence Theorem, the 0-cochain could be chosen with values in $\mathcal{A}_{1/k}$; by Proposition 2.3.10 its coboundary would be with values in $\mathcal{A}^{\leq -k}$ and since it is made of solutions it would necessary be with values in $\mathcal{A}^{\leq -k_{j+1}}$. If the series $\tilde{f}(x)$ were $1/k$ -Gevrey with $k_r < k$ then the coboundary would be trivial and $\tilde{f}(x)$ convergent. Hence, the optimality of k is reached among the levels k_1, k_2, \dots, k_r . \square

▷ *Summability properties*

Theorem 6.2.5. — *Let $\tilde{f}(x)$ be a solution of $Dy = 0$ and suppose the equation has a unique level k associated with $\tilde{f}(x)$ (cf. Def. 4.3.6). Then, $\tilde{f}(x)$ is k -summable.*

Proof. — Let θ be a direction which is not anti-Stokes for the equation $Dy = 0$ and I_θ be a k -wide arc centered at θ and containing no Stokes arc. Such a k -wide arc exists since, under the assumption of the unique level k , the Stokes arcs of $Dy = 0$ are the closed arcs of length π/k centered at the anti-Stokes directions.

As in the proof of the Maillet-Ramis Theorem above, consider a 0-cochain $(f_j(x))_{j \in \mathbb{Z}/p\mathbb{Z}}$ associated with $\tilde{f}(x)$ made of asymptotic solutions. If we prove that, restricted to I_θ its coboundary $(-f_j + f_{j+1})$ is cohomologous to trivial via flat (hence, k -exponentially flat) solutions then, the 0-cochain is cohomologous to a k -quasi sum of $\tilde{f}(x)$ on I_θ . This is proved in Lemma 6.2.6 below. Hence, the k -summability of the series $\tilde{f}(x)$ in all directions but, possibly, the finitely many anti-Stokes directions. \square

Lemma 6.2.6. — *Let $\mathcal{V}^{<0}$ denote the sheaf of germs of flat solutions of the equation $Dy = 0$ and suppose the arc I contains no Stokes arc. Then,*

$$H^1(I; \mathcal{V}^{<0}) = 0.$$

Proof. — Let $\mathcal{V}'^{<0}$ denote the sheaf of flat solutions of a normal equation $D_0y = 0$ associated with $Dy = 0$. The property is easily proved for $\mathcal{V}'^{<0}$ instead of $\mathcal{V}^{<0}$. Indeed, by linearity, it is sufficient to consider the case when $\mathcal{V}'^{<0}$ is of dimension at most 1 (there is only one non-zero determining polynomial q) and when the covering is an elementary good covering of I , say $\mathcal{I} = (I_1, I_2)$. A non-zero 1-cocycle $\dot{\varphi}(x) \in \Gamma(I_1 \cap I_2; \mathcal{V}'^{<0})$ is of the form $m(x)e^{\pm q(1/x)}$ where $m(x)$ has moderate growth (precisely, is a linear combination of products of powers x^λ and logarithm $\ln(x)$) and $e^{\pm q(1/x)}$ is flat over $I_1 \cap I_2$. Since I does not contains any Stokes arc associated with q the exponential function $e^{\pm q(1/x)}$ is flat on at least one of the two open sets I_1, I_2 . Suppose, for instance, it is flat on I_2 . Then, $\dot{\varphi}$ can be continued into a flat function $\phi_2 \in H^0(I_2; \mathcal{V}'^{<0})$ and the 1-cocycle $\dot{\varphi}$ is the coboundary of the 0-cochain $(f_1 = 0, f_2 = \phi_2)$ with values in $\mathcal{V}'^{<0}$.

Now, the result is also true for $\mathcal{V}^{<0}$ since, as a consequence of the Main Asymptotic Existence Theorem, the sheaves $\mathcal{V}^{<0}$ and $\mathcal{V}'^{<0}$ are isomorphic. \square

Comments 6.2.7 (On Examples 2.2.4, 2.2.5 and 2.2.6)

| Theorem 6.2.5 applied to these examples yields the following results:

▷ The Newton polygon $\mathcal{N}(\mathcal{E})$ at the origin 0 of the Euler equation (1) (cf. Exa. 2.2.4) and the Newton polygon at the origin 0 of the Euler equation $\mathcal{E}_0 y = 0$ in homogeneous form (cf. Exa. 3.1.24) are drawn below.

The non-zero slopes reduce to a unique slope equal to 1. This implies that the exponentials in the formal solutions are all of degree 1. The fact that the horizontal length of the side of slope 1 is 1 means that there is only one such exponential (including multiplicity). The fact that $\mathcal{N}(\mathcal{E}_0)$ has one horizontal slope of length 1 means that there exists a one dimensional space of formal series solution of $\mathcal{E}_0 y = 0$ (possibly factored by a complex power of x ; logarithms could also occur when the length is 2 and higher).

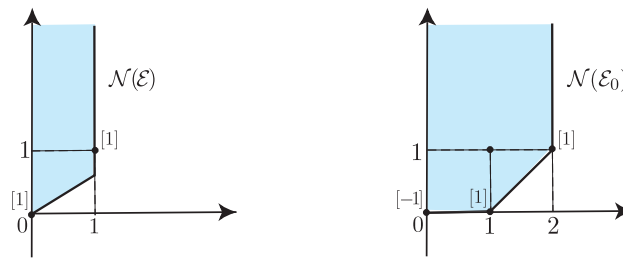


Figure 1. Numbers enclosed into brackets are the coefficients to take into account in the indicial and the characteristic equations.

The Euler series $\tilde{E}(x)$ is the unique, up to multiplication by a constant, series solution of the Euler equation in homogeneous form (Exa. 3.1.24). The exponent of the exponential is given by the characteristic equation associated with the slope 1, i.e., the equation $r+1 = 0$ with solution $r_0 = -1$. Hence, the exponential $e^{\int r_0/x^2} = e^{1/x}$. The unique associated anti-Stokes direction is $\theta = \pi$.

Theorem 6.2.5 allows us to assert what we were already able to prove directly on this very simple example: the Euler series $\tilde{E}(x)$ is 1-summable in all direction but the direction $\theta = \pi$.

▷ The exponential integral $Ei(z)$ has, at infinity, the same properties as the Euler function at 0 due to the formula $Ei(z) = e^{-z} E(1/z)$.

▷ The Newton polygon at infinity of the generalized hypergeometric equation $D_{3,1} y = 0$ (Eq. (2.2.6)) drawn below has a horizontal slope of length 1 and a slope 1/2 with horizontal length 2.

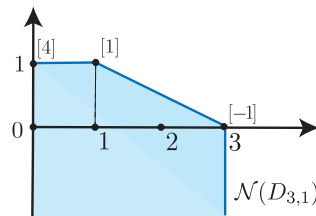


Figure 2

It follows that the equation has a one dimensional space of formal series solutions (space generated by the “hypergeometric” series $\tilde{g}(z)$); moreover, all exponentials are of degree $1/2$. The characteristic equation associated with the slope $1/2$ reads $r^2 - 1 = 0$ with solutions $r_{\pm 0} = \pm 1$ and the exponentials are $\exp(\int \pm z^{-1/2}) = \exp(\pm 2z^{1/2})$. The anti-Stokes directions are the directions $\theta = 2\pi \bmod 4\pi$. (We need to go to the Riemann surface of the logarithm since the slope is not an integer. After a ramification $z = t^2$ we could stay in the plane \mathbb{C} of the variable t : the anti-Stokes directions would become $\theta = \pi \in S^1$.) The indicial equation associated with the horizontal slope reads $r + 4 = 0$ with solution $r_0 = -4$. Hence the factor $1/z^4$ in $\tilde{g}(z)$.

Therefore, by Theorem 6.2.5, the series $\tilde{g}(z)$ is $1/2$ -summable with respect to the variable z with singular directions $\theta = 2\pi \pmod{4\pi}$, which we had not proved earlier.

Theorem 6.2.5 holds for systems.

Corollary 6.2.8. — *Let $dY/dx = B(x)Y$ be a differential system with a formal fundamental solution $\mathcal{Y}(x) = \tilde{F}(x)x^L e^{Q(1/x)}$ where $Q(1/x) = \bigoplus_{j=1}^J q_j(1/x)I_{n_j}$, the q_j 's being distinct. Split the matrix \tilde{F} into column-blocks fitting the structure of Q :*

$$\tilde{F}(x) = [\tilde{F}_1(x) \tilde{F}_2(x) \cdots \tilde{F}_J(x)]$$

(for $j = 1, \dots, J$, the matrix $\tilde{F}_j(x)$ has n_j columns). Suppose the degrees of the polynomials $q_\ell - q_j$ for $\ell \neq j$ are all equal to k . Then, the matrix $\tilde{F}_j(x)$ (i.e., its entries) is k -summable.

Recall that the matrix $\tilde{F}(x)$ satisfies the homological system

$$(23) \quad \frac{dF}{dx} = B(x)F - F B_0(x)$$

which admits the polynomials $q_\ell - q_j$ for $j, \ell = 1, \dots, J$ as determining polynomials. $B_0(x)$ stands for the matrix of the normal form $\frac{dY}{dx} = B_0(x)Y$ with fundamental solution $\mathcal{Y}_0(x) = x^L e^{Q(1/x)}$.

6.3. Third approach: Borel-Laplace summation

The third definition provides explicit k -sums in terms of k -Borel-Laplace integrals.

6.3.1. Definitions. — Due to its main role we first make explicit the classical Borel-Laplace summation which corresponds to level $k = 1$. Actually, the general case $k > 0$ can be reduced to $k = 1$ by setting $x = t^k$ and taking t as a new variable. However, this introduces non integer powers in general with connected problems. We prefer to keep working with the initial variable x .

Definition 6.3.1 (Classical Borel and Laplace transforms)

(i) The Borel transform \mathcal{B} of a series $\tilde{f}(x) = \sum_{n>0} a_n x^n$ is the power series

$$\hat{f}(\xi) = \sum_{n>0} a_n \xi^{n-1} / \Gamma(n);$$

(ii) Given a direction θ , the Laplace transform \mathcal{L}_θ of a function $\varphi(\xi)$ in direction θ is defined, when the integral exists, by

$$f(x) = \int_0^{e^{i\theta}\infty} \varphi(\xi) e^{-\xi/x} d\xi.$$

Although we do not need it in this chapter let us mention here that there exists a functional version of the Borel transform given, in each direction θ , by the integral

$$\mathcal{B}_\theta(f(x))(\xi) = \frac{1}{2\pi i} \int_{\gamma_\theta} f(x) e^{\xi/x} \frac{dx}{x^2}$$

where γ_θ denotes the inverse (image by $x \mapsto 1/x$) of a Hankel contour directed by the direction θ and oriented positively. (Let us observe that we need a contour that ends at 0 since the function is studied near the origin; if we worked at infinity we would use a Hankel contour itself). Using Hankel's formula for the gamma function we obtain $\mathcal{B}_\theta(x^n) = \xi^{n-1} / \Gamma(n)$ for all θ ; hence, the coherence with the definition of the formal Borel transform. Similarly, $\mathcal{L}_\theta(\xi^{n-1}) = \Gamma(n)x^n$. When there is no ambiguity we denote \mathcal{B} and \mathcal{L} instead of \mathcal{B}_θ and \mathcal{L}_θ .

Observe that the formal Borel transform applies to series without constant term. With the constant 1 it would be natural to associate the Dirac distribution δ at 0. This is necessary in certain situations, for instance, when one needs to work with convolution algebras (δ is then a neutral element). For our purpose, this is unnecessary and we assume that our series have no constant term.

Here are some of the basic actions of the Borel transform. We denote by $\varphi(\xi)$ both the Borel series of $\tilde{f}(x)$ and its sum and by ψ both the Borel series

of \tilde{g}) and its sum.

$$\begin{aligned} \frac{1}{x} \tilde{f}(x) &\xrightarrow{\mathcal{B}} \frac{d}{d\xi} \varphi(\xi) \quad (\text{assume } \tilde{f}(x)/x \text{ has no constant term}), \\ x^2 \frac{d}{dx} \tilde{f}(x) &\xrightarrow{\mathcal{B}} \xi \varphi(\xi), \\ \tilde{f}(x) \tilde{g}(x) &\xrightarrow{\mathcal{B}} \varphi * \psi(\xi) = \int_0^\xi \varphi(\xi - \eta) \psi(\eta) d\eta. \end{aligned}$$

and way back for the Laplace transform when it exists.

Proposition 6.3.2. — Suppose the Borel series $\hat{f}(\xi) = \sum_{n \geq 1} \frac{a_n}{\Gamma(n)} \xi^{n-1}$ converges and its sum $\varphi(\xi)$ can be analytically continued to an infinite sector $\mathcal{S} = \mathcal{S}_{\theta_1, \theta_2}(\infty)$ with exponential growth at infinity: there exist $A, K > 0$ such that

$$|\varphi(\xi)| \leq K \exp(A|\xi|) \quad \text{on } \mathcal{S}.$$

Then, for all $\theta \in]\theta_1, \theta_2[$, the Laplace integral

$$f_\theta(x) = \int_0^{e^{i\theta}\infty} \varphi(\xi) e^{-\xi/x} d\xi$$

exists and is analytic on the open disc $\mathcal{D}_\theta(A)$ with diameter $(0, e^{i\theta}/A)$ and the various f_θ glue together into an analytic function defined on $\bigcup_\theta \mathcal{D}_\theta(A)$ and, especially, on a sector $\mathcal{S}' = \mathcal{S}'_{\theta_1 - \pi/2, \theta_2 + \pi/2} \subset \bigcup_\theta \mathcal{D}_\theta(A)$ of opening greater than π .

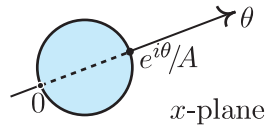


Figure 3. Borel disc $\mathcal{D}_\theta(A)$

Definition 6.3.3. — The disc $\mathcal{D}_\theta(A)$ is called a Borel disc in direction θ .

Proof. — Since $|\varphi(\xi) e^{-\xi/x}| \leq K \exp(-(\Re(e^{i\theta}/x) - A)|\xi|)$ the Laplace integral $f_\theta(x)$ exists and is analytic on the disc $\Re(e^{i\theta}/x) > A$, i.e., the open disc $\mathcal{D}_\theta(A)$ with diameter $(0, e^{i\theta}/A)$.

Consider two directions $\theta' < \theta''$ in $]\theta_1, \theta_2[$ such that, say, $\theta'' - \theta' \leq \pi/4 < \pi/2$ and apply Cauchy's Theorem to $\varphi(\xi) e^{-\xi/x}$ along the boundary C_R of

a sector of radius R limited by the lines $\theta = \theta'$ and $\theta = \theta''$ and oriented counterclockwise. Then,

$$\int_0^{\Re e^{i\theta'}} + \int_{\Re e^{i\theta'}}^{\Re e^{i\theta''}} - \int_0^{\Re e^{i\theta''}} \varphi(\xi) e^{-\xi/x} d\xi = 0.$$

However, denoting $x = |x| e^{i\omega}$ we can write

$$\begin{aligned} \left| \int_{\Re e^{i\theta'}}^{\Re e^{i\theta''}} \varphi(\xi) e^{-\xi/x} d\xi \right| &\leq \int_{\theta'}^{\theta''} K \exp(-(\Re(e^{i\theta}/x) - A)|\xi|) |\xi e^{i\theta}| d\theta \\ &= K \int_{\theta'}^{\theta''} \exp(-(\cos(\theta - \omega)/|x| - A)R) R d\theta. \end{aligned}$$

Choose $\theta' < \omega < \theta''$. Then, $|\theta - \omega| < \pi/4$ for θ from θ' to θ'' . The inequality becomes

$$\left| \int_{\Re e^{i\theta'}}^{\Re e^{i\theta''}} \varphi(\xi) e^{-\xi/x} d\xi \right| \leq K(\theta'' - \theta') R \exp(- (1/(\sqrt{2}|x|) - A)R)$$

and the integral tends to 0 as R tends to infinity as soon as $|x| < 1/(A\sqrt{2})$. Consequently, the Laplace integrals in directions θ' and θ'' coincide on the domain $\{|x| < 1/(A\sqrt{2}) \text{ and } \theta' < \arg(x) < \theta''\}$ and they are, then, analytic continuations of each other. \square

With this result we can set the following definition.

Definition 6.3.4 (Borel-Laplace summation). — A series $\tilde{f}(x) = \sum_{n>0} a_n x^n$ is said to be Borel-Laplace summable in a direction θ_0 if the following two conditions are satisfied:

(i) The Borel transform $\hat{f}(\xi) = \sum_{n>0} a_n \xi^{n-1}/\Gamma(n)$ of $\tilde{f}(x)$ is convergent, i.e., the series $\tilde{f}(x)$ is 1-Gevrey.

(ii) The sum $\varphi(\xi)$ of the Borel series $\hat{f}(\xi)$ of $\tilde{f}(x)$ can be analytically continued to a sector σ neighboring the direction θ_0 with exponential growth of order 1. We still denote by φ its analytic continuation.

When these conditions are satisfied, the Borel-Laplace sum of $\tilde{f}(x)$ in direction θ_0 is given by the Laplace integrals

$$f_\theta(x) = \int_0^{e^{i\theta}\infty} \varphi(\xi) e^{-\xi/x} d\xi \quad \text{for } \theta \in \sigma$$

gluing into an analytic function $f(x)$ defined (at least) on a sector bisected by θ_0 with opening larger than π .

This definition comes with explicit integral formulæ for the sum of the series. However, an explicit calculation of $\varphi(\xi)$ in terms of classical functions is, in general, out of reach or even impossible.

The domain of definition of $\varphi(\xi)$ must contain a disc centered at 0 and a sector $\sigma = \sigma_{] \theta_1, \theta_2 [}$ neighboring the direction θ_0 in the shape of a champagne cork as below: the Borel-Laplace sum $f(x)$ is analytic on the union $\bigcup_{\theta_1 < \theta < \theta_2} \mathcal{D}_\theta(A)$ of the Borel discs with diameter $(0, e^{i\theta}/A)$ for all direction θ in σ . The domain contains sectors $\delta_{] \theta'_1, \theta'_2 [}$ for any $\theta'_1 > \theta_1 - \pi/2$ and $\theta'_2 < \theta_2 + \pi/2$.

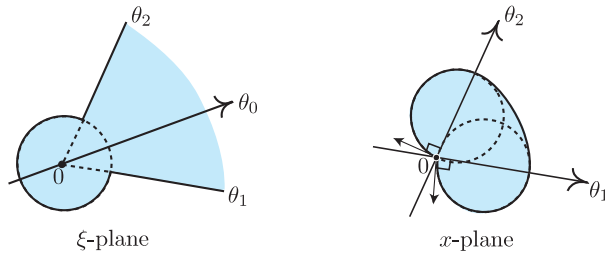


Figure 4

These definitions can be extended to any level $k > 0$ as follows. Denote temporarily by \mathcal{B}_1 or $\mathcal{B}_{1,\theta}$ the classical Borel operators defined as above and generally, by \mathcal{B}_k or $\mathcal{B}_{k,\theta}$ the k -Borel operators. Denote by $\mathcal{L}_{1,\theta}$ the Laplace operator and generally, by $\mathcal{L}_{k,\theta}$ the k -Laplace operator in direction θ . The operators of level k are transmuted from those of level one by means of ramifications according to the following schemes.

The k -Borel operators are defined by the following commutative diagram:

$$\begin{array}{ccc}
 f(x) & \xrightarrow{\mathcal{B}_{k,\theta}} & \mathcal{B}_{k,\theta}(f)(\xi) = \psi(\xi^k) \\
 \rho_k \downarrow & & \uparrow \rho_{1/k} \\
 f(t^{1/k}) & \xrightarrow{\mathcal{B}_{1,k\theta}} & \psi(\tau)
 \end{array}$$

where $\psi(\tau) = \frac{1}{2\pi i} \int_{\gamma_{k\theta}} f(t^{1/k}) e^{\tau/t} dt/t^2 = \frac{1}{2\pi i} \int_{\gamma'_\theta} f(x) e^{\tau/x^k} k dx/x^{k+1}$, the path γ'_θ being deduced from the ‘‘Hankel’’ contour $\gamma_{k\theta}$ by the ramification $t = x^k$.

The formal k -Borel transform is obtained by applying these formulæ to the monomials $f(x) = x^n$. One obtains $\mathcal{B}_k(x^n) = \xi^{n-k}/\Gamma(n/k)$. In accordance to the fact that the usual 1-Borel transform applies to series with valuation $k_0 \geq 1$, the k -Borel transform applies to series with valuation $k_0 \geq k$.

The k -Laplace operators are defined by the following commutative diagram:

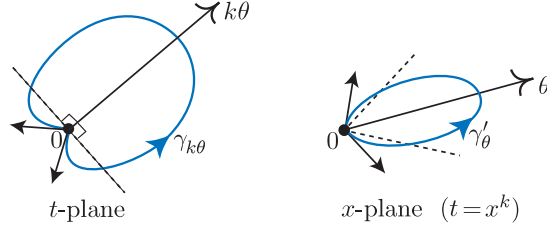


Figure 5. The “Hankel” contour $\gamma_{k\theta}$ compared to γ'_θ when $k > 1$

$$\begin{array}{ccc}
 \varphi(\xi) & \xrightarrow{\mathcal{L}_{k,\theta}} & \mathcal{L}_{k,\theta}(\varphi)(x) = g(x^k) \\
 \rho_k \downarrow & & \uparrow \rho_{1/k} \\
 \varphi(\tau^{1/k}) & \xrightarrow{\mathcal{L}_{1,k\theta}} & g(t).
 \end{array}$$

where $g(t) = \int_0^{e^{ik\theta}\infty} \varphi(\tau^{1/k}) e^{-\tau/t} d\tau$.

We can then state the following definitions generalizing for any $k > 0$ the classical definitions of the Borel and the Laplace transforms stated above with $k = 1$.

Definition 6.3.5 (k -Borel and k -Laplace transforms)

(i) The (formal) k -Borel transform of a series $\tilde{f}(x) = \sum_{n \geq k_0} a_n x^n$ with valuation $k_0 \geq k$ is the series

$$\hat{f}(\xi) = \sum_{n \geq k_0} a_n \frac{\xi^{n-k}}{\Gamma(n/k)}.$$

(ii) The k -Borel transform of a function $f(x)$ in a direction θ is defined, when the integral exists, by

$$\mathcal{B}_{k,\theta}(f)(\xi) = \frac{1}{2\pi i} \int_{\gamma'_\theta} f(x) e^{\xi^k/x^k} \frac{k dx}{x^{k+1}}.$$

with γ'_θ a Hankel-type contour as above.

(iii) The k -Laplace transform of a function $\varphi(\xi)$ in a direction θ is defined, when the integral exists, by

$$f(x) = \int_{\xi=0}^{e^{i\theta}\infty} \varphi(\xi) e^{-\xi^k/x^k} d(\xi^k).$$

Proposition 6.3.2 can be generalized to any level $k > 0$. The Borel discs must however be changed into Fatou petals (or Fatou flowers) defined in direction θ by conditions of the type $\Re(e^{i\theta}/x^k) > A$.

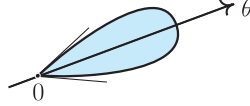


Figure 6. Fatou petal

Definition 6.3.6 (k -Borel-Laplace summation). — Let $k_0 \geq k$ be an integer.

A series $\tilde{f}(x) = \sum_{n \geq k_0} a_n x^n$ is said to be k -Borel-Laplace summable in a direction θ_0 if the following two conditions are satisfied:

(i) The k -Borel transform $\hat{f}(\xi) = \sum_{n \geq k_0} a_n \xi^{n-k} / \Gamma(n/k)$ of $\tilde{f}(x)$ is convergent, i.e., the series $\tilde{f}(x)$ is $1/k$ -Gevrey.

(ii) The sum $\varphi(\xi)$ of the Borel series of $\tilde{f}(x)$ can be analytically continued to a sector σ neighboring the direction θ_0 with exponential growth of order k . We keep denoting by φ its analytic continuation.

If these conditions are satisfied, the k -Borel-Laplace sum of $\tilde{f}(x)$ in direction θ_0 is given by the k -Laplace integrals

$$f_\theta(x) = \int_{\xi=0}^{e^{i\theta}\infty} \varphi(\xi) e^{-\xi^k/x^k} d(\xi^k) \quad \text{for } \theta \in \sigma$$

gluing into an analytic function $f(x)$ defined (at least) on a sector bisected by θ_0 with opening larger than π/k .

The natural question of the equivalence between k -summability and k -Borel-Laplace summability is studied in the next section (cf. Prop. 6.3.9).

6.3.2. Nevanlinna's Theorem and summability. — We begin with the proof of Nevanlinna's Theorem which solves the main step in the equivalence of k -summability and k -Borel-Laplace summability, precisely, the fact that k -summability implies k -Borel-Laplace summability.

Assume we are given a direction θ issuing from 0 which by means of a rotation we assume to be $\theta = 0$ and let us first describe the curves and domains we will be concerned with. We consider two copies of \mathbb{C} , one which we call the *Laplace plane* with coordinate x and the other one, called *Borel plane*, with coordinate ξ .

We fix $k > 0$ and $\gamma > 0$ and we introduce two new copies of \mathbb{C} with coordinates $Z = 1/x^k$ and $\zeta = \xi^k$ respectively.

▷ In the x -plane we consider

1. The sector $\Delta_0 = \{x \in \mathbb{C}; |x| < \gamma \text{ and } |\arg(x)| < \pi/2k\}$.
2. For any $\ell > 0$, the domain (*Fatou's petal* or *Borel disc* when $k = 1$) defined by

$$\Delta_\ell = \left\{ x \in \mathbb{C}; \Re\left(\frac{1}{x^k}\right) > \ell^k \text{ and } |\arg(x)| < \frac{\pi}{2k} \right\}.$$

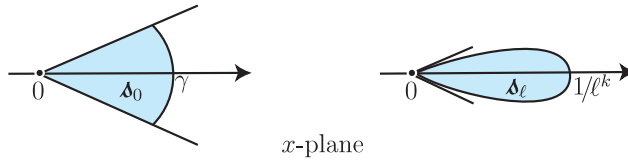


Figure 7

▷ In the Z -plane, we consider the images \mathcal{D}_0 and \mathcal{D}_ℓ of Δ_0 and Δ_ℓ respectively, by the map $Z = 1/x^k$. Hence, \mathcal{D}_ℓ is the half-plane $\{Z; \Re(Z) > \ell^k\}$ and \mathcal{D}_0 the half-plane $\{Z; \Re(Z) > 0\}$ but the half-disc $\{|Z| \leq 1/\gamma^k, \Re(Z) > 0\}$.

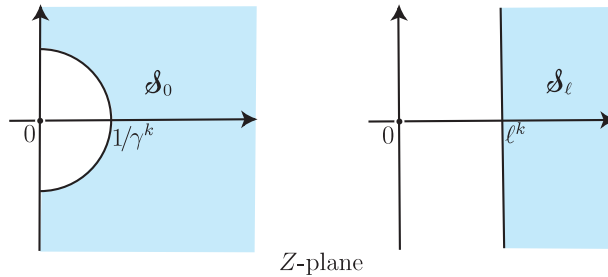


Figure 8

▷ In the ζ -plane, for $B > 0$, we consider the domain $\Sigma_B = D(0, B^k) \cup \Sigma'_B$ union of the open disc $D(0, B^k)$ with center 0 and radius B^k and of the set Σ'_B of points in \mathbb{C} at a distance less than B^k of the line $[B^k, +\infty[$.

▷ In the ξ -plane, we consider for $B > 0$, the domain $\sigma_B = D(0, B) \cup \sigma'_B$ union of the disc $D(0, B)$ with center 0 and radius B and of the image σ'_B of Σ'_B by the map $\xi = \zeta^{1/k}$ for the choice of the principal determination of the k^{th} -root.

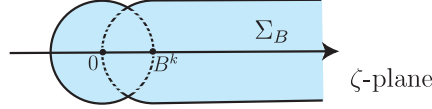


Figure 9

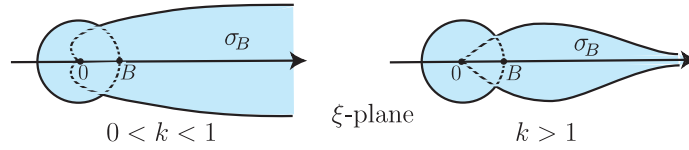


Figure 10

Theorem 6.3.7 (Nevanlinna's Theorem [Nev19, pp. 44–45])

Let $k > 0$.

Let $\tilde{f}(x) = \sum_{n \geq k_0} a_n x^n \in \mathbb{C}[[x]]$ be a power series at 0 with valuation $k_0 \geq k$ and let $\hat{f}(\xi) = \sum_{n \geq k_0} \frac{a_n}{\Gamma(n/k)} \xi^{n-k}$ denote its k -Borel transform.

Suppose $f(x) \in \mathcal{A}(\Delta_0)$ is asymptotic to $\tilde{f}(x)$ and satisfies global k -Gevrey estimates on Δ_0 : there exist constants $C, B > 0$ such that for all $x \in \Delta_0$ and $N \in \mathbb{N}^*$

$$(30) \quad \left| f(x) - \sum_{n=k_0}^{N-1} a_n x^n \right| \leq C \left(\frac{N}{k} \right)^{N/k} e^{-N/k} \frac{|x|^N}{B^N}.$$

Then, the k -Borel series $\hat{f}(\xi)$ is convergent and its sum $\varphi(\xi)$ can be analytically continued to the domain σ_B with exponential growth of order k at infinity: for any $B_\varepsilon = B - \varepsilon < B$ there exist constants $K, A > 0$ such that

$$|\varphi(\xi)| \leq K \exp(A|\xi|^k) \quad \text{for all } \xi \in \sigma_{B_\varepsilon}.$$

Moreover, the functions $f(x)$ and $\varphi(\xi)$ are k -Laplace and k -Borel transforms of each other: given $\ell > \ell_0 = \inf\{\ell; f \in \mathcal{O}(\Delta_\ell)\}$ they satisfy

$$(31) \quad f(x) = \int_0^{+\infty} \varphi(\xi) e^{-\xi^k/x^k} d(\xi^k) \quad (k\text{-Laplace transform of } \varphi),$$

$$(32) \quad \varphi(\xi) = \frac{1}{2\pi i} \int_{\Re(1/u^k) = \ell^k} f(u) e^{\xi^k/u^k} d(1/u^k) \quad \text{for all } \xi > 0.$$

Remark 6.3.8 (k -fine summability). — One should observe that Condition (30) is stronger than k -Gevrey asymptoticity of $f(x)$ to the series $\tilde{f}(x)$ on Δ_0 . Indeed, while Condition (30) is valid in restriction to any proper subsector

of \mathfrak{A}_0 (with same constants) implying thus k -asymptoticity on \mathfrak{A}_0 , conversely, the existence of estimates valid on any proper subsector of \mathfrak{A}_0 does not imply the existence of constants B and C valid on all of \mathfrak{A}_0 .

Note also that, due to Condition (31), when an analytic function f satisfying Condition (30) exists then it is unique. (Compare Watson's Lemma (Thm. 6.1.3) and Proposition 2.3.17). This comforts the fact that Condition (30) is stronger than k -Gevrey asymptoticity. Recall the example of the Euler function $E(x)$ (cf. Exa. 1 and Com. 2.3.9) that provides two functions (its two determinations) that are 1-Gevrey asymptotic to the Euler series $\tilde{E}(x)$ on the half-plane $\Re(x) < 0$.

With these results we see that Condition (30) is adequate to guaranty the existence of a unique well defined sum of $\tilde{f}(x)$ on \mathfrak{A}_0 with similar properties as a k -sum. And indeed, this corresponds to a notion called *k-fine summability* in the bisecting direction of \mathfrak{A}_0 which is weaker than k -summability in the same direction. By this, we mean that a k -summable series in a direction θ_0 is k -fine summable in direction θ_0 , the converse being false in general. For the case when $k = 1$ we refer to [Sau], Sect. "The fine Borel-Laplace summation" where the author uses Formula (31) as definition.

Proof of Theorem 6.3.7 We can check that any monomial x^n satisfies the theorem. Hence, we can assume that $\tilde{f}(x) = \sum_{n \geq k_0} a_n x^n$ has valuation $k_0 > k$.

Due to Condition (30) the series $\hat{f}(\xi)$ converges at 0 with radius at least B (cf. Prop. 2.3.10). Its sum $\varphi(\xi)$ defines then an analytic function on the disc $|\xi| < B$.

Set $Z = 1/x^k$ and $\zeta = \xi^k$ and denote by $Z^{1/k}$ and $\zeta^{1/k}$ the principal k^{th} -roots of Z and ζ . The function $F(Z) = f(1/Z^{1/k})$ is, by construction, analytic on the half-plane $\Pi_{\ell_0^k} = \{\Re(Z) > \ell_0^k\}$ where $0 \leq \ell_0 \leq 1/\gamma$.

Choose $\ell > \ell_0$ and set

$$\phi(\zeta) = \frac{1}{2\pi i} \int_{\ell^k + i\mathbb{R}} F(U) e^{\zeta U} dU \quad \text{for all } \zeta > 0.$$

This formula makes sense. Indeed, in the new variables, Condition (30) becomes: for all $N \geq k_0$ and all $Z \in \Pi_{\ell_0^k}$, the function $F(Z)$ satisfies

$$\begin{aligned} (33) \quad |R_N(Z)| &\equiv \left| F(Z) - \sum_{n=k_0}^{N-1} \frac{a_n}{Z^{n/k}} \right| \leq C \left(\frac{N}{k} \right)^{N/k} e^{-N/k} \frac{1}{(B|Z|^{1/k})^N} \\ &\leq C' \Gamma(N/k) \frac{1}{(B'|Z|^{1/k})^N} \quad (\text{using Stirling formula}) \end{aligned}$$

for any $B' < B$ jointly with a convenient $C' > 0$. In particular, there exist constants $M_0, M_1 > 0$ such that

$$(34) \quad |F(Z)| \leq \frac{M_1}{|Z|^{k_0/k}} \leq M_0 \quad \text{for all } Z \in \Pi_{\ell^k}$$

and since $k_0/k > 1$ this implies that $F(Z)$ belongs to $L^1(\ell^k + i\mathbb{R})$. Hence, its Fourier integral $\phi(\zeta)$ exists and is continuous with respect to $\zeta \in \mathbb{R}$.

We have to prove that:

1. The function $F(Z)$ can be written in the form

$$F(Z) = \int_0^{+\infty} \phi(\zeta) e^{-Z\zeta} d\zeta.$$

2. The Borel series $\widehat{f}(\xi)$ converges to $\phi(\xi^k)$ for $0 < \xi < B$; hence the analytic continuation of $\phi(\xi^k)$ by $\varphi(\xi)$ to the disc $|\xi| < B$ (0 might be a branch point for the series $\phi(\zeta)$ itself).

3. The function $\phi(\zeta)$ can be analytically continued to Σ'_B .

4. The function $\phi(\zeta)$ has exponential growth of order 1 at infinity on Σ'_B .

Prove now the four steps.

1. Given $Z \in \Pi_{\ell^k}$ we enclose it in a domain Ω limited by the vertical line at ℓ^k and an arc of a circle centered at 0 with radius R as drawn in figure 11.

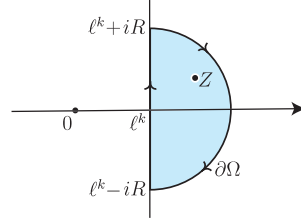


Figure 11

By Cauchy's integral formula we can write

$$F(Z) = \frac{1}{2\pi i} \int_{\partial\Omega} F(U) \frac{dU}{Z-U},$$

the boundary $\partial\Omega$ of Ω being oriented clockwise. From $|F(Z)| \leq M_1/|Z|^{k_0/k}$ (estimate (34)) we deduce that the integral along the half-circle tends to zero as R tends to infinity. Hence, $F(Z) = \frac{1}{2\pi i} \int_{\ell^k+i\mathbb{R}} F(U) \frac{dU}{Z-U}$. Write $\frac{1}{Z-U} =$

$\int_0^{+\infty} e^{(U-Z)\zeta} d\zeta$ so that

$$F(Z) = \frac{1}{2\pi i} \int_{\ell^k + i\mathbb{R}} F(U) \int_0^{+\infty} e^{(U-Z)\zeta} d\zeta dU.$$

Fubini's Theorem can be applied to the iterated integral since, using again the estimate (34) we obtain $|F(U)e^{-(Z-U)\zeta}| \leq \frac{M_1}{|U|^{k_0/k}} e^{-(\Re(Z)-\ell^k)\zeta}$ with $\Re(Z) - \ell^k > 0$. Hence, the followed formulae

$$F(Z) = \int_0^{+\infty} \phi(\zeta) e^{-Z\zeta} d\zeta \quad \text{with} \quad \phi(\zeta) = \frac{1}{2\pi i} \int_{\ell^k + i\mathbb{R}} F(U) e^{U\zeta} dU.$$

Moreover, $\phi(\zeta)$ is independent of $\ell > \ell_0$ (apply Cauchy's Theorem to $F(U)e^{U\zeta}$ along a rectangle with vertical sides at $\Re(Z) = \ell^k$ and $\Re(Z) = \ell'^k$, $\ell'^k \neq \ell^k$, and let the horizontal sides go to infinity).

2. From $F(U) = \sum_{n=k_0}^{N-1} a_n/U^{n/k} + R_N(U)$ and $\phi(\zeta) = \frac{1}{2\pi i} \int_{\ell^k + i\mathbb{R}} F(U) e^{U\zeta} dU$ for all $\zeta > 0$ we can write

$$\phi(\zeta) = \sum_{n=k_0}^{N-1} \frac{a_n}{\Gamma(n/k)} \zeta^{n/k} + \phi_N(\zeta) \quad \text{with} \quad \phi_N(\zeta) = \frac{1}{2\pi i} \int_{\ell^k + i\mathbb{R}} R_N(U) e^{U\zeta} dU.$$

By Condition (30), we have

$$|\phi_N(\zeta)| \leq \frac{C}{2\pi} \left(\frac{N}{k}\right)^{N/k} \frac{e^{-N/k}}{B^N} e^{\ell^k \zeta} \left| \int_{\ell^k + i\mathbb{R}} \frac{dU}{|U|^{N/k}} \right|$$

while

$$\ell^{N-k} \left| \int_{\ell^k + \mathbb{R}} \frac{dU}{|U|^{N/k}} \right| = \int_{\mathbb{R}} \frac{d\tau}{\sqrt{1+\tau^2}^{N/k}} \leq \int_{\mathbb{R}} \frac{d\tau}{\sqrt{1+\tau^2}^{k_0/k}} < +\infty.$$

Hence, there exists a constant $C_0 > 0$ such that

$$|\phi_N(\zeta)| \leq C_0 \left(\frac{N}{k}\right)^{N/k} e^{-N/k + \ell^k \zeta} \frac{1}{(\ell B)^N}.$$

Take $0 < \zeta < B^k$ and consider the right hand side as a function of ℓ .

The function $y(\ell) = \frac{e^{\ell^k \zeta}}{\ell^N}$ for $\ell > 0$ reaches its minimal value at $\ell_1 = \left(\frac{N}{k}\right)^{1/k} \frac{1}{\zeta^{1/k}}$ and $y(\ell_1) = \left(\frac{k}{N}\right)^{N/k} e^{N/k} \zeta^{N/k}$. Choose $n_0 = n_0(\zeta)$ so large that

$$\ell_1 = \left(\frac{n_0}{k}\right)^{1/k} \frac{1}{\zeta^{1/k}} > \ell_0 = \inf \{ \ell; f \in \mathcal{O}(\Delta_\ell) \}.$$

For $N \geq n_0(\zeta)$ and since $\phi_N(\zeta)$ does not depend on $\ell > \ell_0$, we can take $\ell = \ell_1$. Then, $\phi_N(\zeta)$ satisfies $|\phi_N(\zeta)| \leq C_0 \frac{\zeta^{N/k}}{B^N}$ and tends to 0 as N tends to infinity.

Hence, $\phi(\xi^k) = \sum_{n \geq k_0} \frac{a_n}{\Gamma(n/k)} \xi^n = \varphi(\xi)$ for $0 < \xi < B$ which proves that $\widehat{f}(\xi)$ is the Taylor series of $\phi(\xi^k)$ at 0 and that $\phi(\xi^k)$ can be analytically continued by $\varphi(\xi)$ to the disc $\{\xi; |\xi| < B\}$.

From now on, we denote $\varphi(\xi) = \phi(\xi^k)$.

3. Given $\zeta_0 \geq B^k$, prove that the Taylor series of $\phi(\zeta)$ at ζ_0 converges with radius B^k and converges to $\phi(\zeta)$ for ζ real.

Prove first that $\phi(\zeta)$ is infinitely derivable for $\zeta > 0$. Given $N \in \mathbb{N}^*$, let $m \in \mathbb{N}$ satisfy $k(N+1) < m \leq k(N+1) + 1$. Write

$$\phi(\zeta) = \sum_{n=k_0}^m \frac{a_n}{\Gamma(n/k)} \zeta^{n/k} + \frac{1}{2\pi i} \int_{\ell^k + i\mathbb{R}} R_{m+1}(U) e^{U\zeta} dU$$

and look at the ν^{th} derivative of the integrand for $1 \leq \nu \leq N$. From (33) we can write

$$(35) \quad \left| R_{m+1}(U) U^\nu e^{U\zeta} \right| \leq C' \Gamma((m+1)/k) \frac{e^{\ell^k \zeta}}{B'^{m+1}} |U|^{\nu-(m+1)/k} \leq \frac{C''}{|U|^{1+1/k}}$$

where the constant C'' is independent of ζ so long as ζ stays bounded. By Lebesgue's Theorem we can then conclude that $\phi(\zeta)$ can be derivated N times under the sign of integration for any $\zeta > 0$.

To estimate the N^{th} derivative when $\zeta \geq \zeta_0 \geq B^k$ we write

$$\frac{\partial^N}{\partial \zeta^N} \phi(\zeta) = J_N + I_N$$

where

$$J_N = \sum_{n=k_0}^m \frac{a_n}{\Gamma(n/k)} \frac{n}{k} \left(\frac{n}{k} - 1\right) \cdots \left(\frac{n}{k} - N + 1\right) \zeta^{n/k-N},$$

$$I_N = \frac{1}{2\pi i} \int_{\ell^k + i\mathbb{R}} R_{m+1}(U) U^N e^{U\zeta} dU.$$

From the Gevrey Condition (30) or (33) (*cf.* Prop. 2.3.10) and the fact that $|Z|^{1/k} \geq 1/\gamma$ it follows that

$$\frac{|a_n|}{\Gamma(n/k)} \leq \frac{C'}{B'^n} \left(\frac{\gamma}{B'} \frac{\Gamma((n+1)/k)}{\Gamma(n/k)} + 1 \right)$$

for all $n \in \mathbb{N}^*$, and since $\frac{\Gamma((n+1)/k)}{\Gamma(n/k)}$ behaves like $\left(\frac{n}{k}\right)^{1/k}$ as n tends to infinity we can conclude that, for all $B_\varepsilon = B - \varepsilon < B'$ there exists a constant $C'' > 0$ such that

$$\frac{|a_n|}{\Gamma(n/k)} \leq \frac{C''}{B_\varepsilon^n} \quad \text{for all } n \in \mathbb{N}^*.$$

This implies that

$$|J_N| \leq C'' \frac{m}{k} \left(\frac{m}{k} - 1\right) \cdots \left(\frac{m}{k} - N + 1\right) \zeta^{-N} \sum_{n=k_0}^m \left(\frac{\zeta^{1/k}}{B_\varepsilon}\right)^n$$

and, since $\sum_{n=k_0}^m \left(\frac{\zeta^{1/k}}{B_\varepsilon}\right)^n \leq m \frac{\zeta^{m/k}}{B_\varepsilon^m} \leq k(N+1+1/k) \frac{\zeta^{1+1/k}}{B_\varepsilon^{Nk+k+1}} \zeta^N$ if $B_\varepsilon^k \leq \zeta$, that

$$|J_N| \leq \frac{kC''}{B_\varepsilon^{k+1}} \Gamma\left(N+3+\frac{1}{k}\right) \frac{\zeta^{1+1/k}}{B_\varepsilon^{kN}} \quad \text{when } B_\varepsilon^k \leq \zeta.$$

From (35) and (33) we obtain

$$\begin{aligned} |I_N| &\leq \frac{1}{2\pi} \left| \int_{\ell^{k+i}\mathbb{R}} |R_{m+1}(U)| |U|^N e^{\ell^k \zeta} dU \right| \\ &\leq \frac{1}{2\pi} C' \Gamma((m+1)/k) \frac{e^{\ell^k \zeta}}{B'^{m+1}} \max\left(\frac{1}{\ell}, \frac{1}{\ell^2}\right) \int_{\mathbb{R}} \frac{dT}{\sqrt{1+T^2}^{1+1/k}} \\ &\leq C''' \Gamma(N+1+2/k) \frac{e^{\ell^k \zeta}}{B'^{kN}} \quad \text{for a convenient } C''' > 0 \\ &\leq C''' \Gamma(N+1+2/k) \frac{e^{\ell^k \zeta}}{B_\varepsilon^{kN}}. \end{aligned}$$

Recall that $k(N+1) < m \leq k(N+1)+1$ so that $1 < m - kN + 1 - k \leq 2$. Hence, the term $\max\left(\frac{1}{\ell}, \frac{1}{\ell^2}\right)$ and the power of the integrand.

Adding these two estimates we see that, for all $\varepsilon > 0$, there exists a constant $\alpha_\varepsilon > 0$ such that

$$(36) \quad \left| \frac{\partial^N \phi}{\partial \zeta^N}(\zeta) \right| \leq \alpha_\varepsilon \Gamma(N+3+2/k) \frac{e^{\ell^k \zeta}}{B_\varepsilon^{kN}} \quad \text{for all } \zeta \geq B_\varepsilon^k.$$

Hence, the Taylor series of $\phi(\zeta)$ at ζ_0 ,

$$\sum_{N \geq 0} \frac{1}{\Gamma(1+N)} \frac{\partial^N \phi}{\partial \zeta^N}(\zeta_0) (\zeta - \zeta_0)^N,$$

converges for $|\zeta - \zeta_0| < B_\varepsilon^k$. Making ε tend to 0, we can conclude that it converges for $|\zeta - \zeta_0| < B^k$ and consequently, the Taylor series of $\phi(\zeta)$ at ζ_0 has a radius of convergence at least equal to B^k .

To prove that this Taylor series converges to $\phi(\zeta)$ write the Taylor-Lagrange formulas

$$\begin{aligned} \phi(\zeta) &= \sum_{p=0}^{n-1} \frac{1}{\Gamma(1+p)} \frac{\partial^p \phi}{\partial \zeta^p}(\zeta_0) (\zeta - \zeta_0)^p + \psi_n(\zeta), \\ \psi_n(\zeta) &= \frac{1}{\Gamma(1+n)} \frac{\partial^n \phi}{\partial \zeta^n}(\zeta_0 + \theta(\zeta - \zeta_0)) (\zeta - \zeta_0)^n, \quad 0 < \theta < 1. \end{aligned}$$

For $\zeta_0 \geq B^k$ and $\zeta \geq B_\varepsilon^k$, then $\zeta_0 + \theta(\zeta - \zeta_0) > B_\varepsilon^k$ and we can apply the estimate (36) to $\frac{\partial^n \phi}{\partial \zeta^n}(\zeta_0 + \theta(\zeta - \zeta_0))$ so that

$$|\psi_n(\zeta)| \leq \alpha_\varepsilon \frac{\Gamma(n+3+2/k)}{\Gamma(1+n)} e^{\ell^k \max(\zeta, \zeta_0)} \frac{|\zeta - \zeta_0|^n}{B_\varepsilon^{kn}}$$

and tends to 0 as n tends to infinity as soon as $\max(B_\varepsilon^k, \zeta_0 - B_\varepsilon^k) < \zeta < \zeta_0 + B_\varepsilon^k$. Therefore, the sum of the Taylor series of ϕ at any $\zeta_0 \geq B^k$ coincides with $\phi(\zeta)$ on the interval $\max(B_\varepsilon^k, \zeta_0 - B_\varepsilon^k) < \zeta < \zeta_0 + B_\varepsilon^k$. This proves that $\phi(\zeta)$ admits an analytic continuation to $\Sigma'_{B_\varepsilon} \cap \{\Re(\zeta) > B_\varepsilon^k\}$.

Since the intervals $]0, B^k[$ and $] \max(B_\varepsilon^k, \zeta_0 - B_\varepsilon^k), \zeta_0 + B_\varepsilon^k[$ for $\zeta_0 = B^k$, for instance, overlap this analytic continuation fit the analytic continuation by $\varphi(\zeta^{1/k})$ on $D(0, B^k) \cap \Sigma'_{B_\varepsilon} \cap \{\Re(\zeta) > B_\varepsilon^k\}$.

Letting now ε tend to 0 allows us to extend the analytic continuation of $\phi(\zeta)$ up to Σ'_B . Hence, the analytic continuation of $\varphi(\xi)$ to the full domain σ_B .

4. Suppose $0 < B_\varepsilon < B$ be given. Since $\varphi(\xi)$ is analytic in the disc $D(0, B)$ it is bounded in the smaller disc $D(0, B_\varepsilon)$. Consequently, $\phi(\zeta)$ is bounded in $D(0, B_\varepsilon^k)$ and it suffices to prove the exponential estimate in the discs $D(\zeta_0, B_\varepsilon^k)$ for $\zeta_0 \geq B^k$.

The analytic continuation of ϕ to the disc $D(\zeta_0, B^k)$ is given by the Taylor series

$$\phi(\zeta) = \sum_{n \geq 0} \frac{1}{\Gamma(1+n)} \frac{\partial^n \phi}{\partial \zeta^n}(\zeta_0) (\zeta - \zeta_0)^n.$$

Apply estimate (36) to $\frac{\partial^n \phi}{\partial \zeta^n}(\zeta_0)$ with $\varepsilon' < \varepsilon$. It follows that, on the disc $D(\zeta_0, B_\varepsilon^k)$, the function ϕ satisfies

$$\begin{aligned} |\phi(\zeta)| &\leq \alpha_{\varepsilon'} \sum_{n \geq 0} \frac{\Gamma(n+3+2/k)}{\Gamma(1+n)} \frac{|\zeta - \zeta_0|^n}{B_{\varepsilon'}^{kn}} e^{\ell^k \zeta_0} \\ &\leq \alpha_{\varepsilon'} \sum_{n \geq 0} \frac{\Gamma(n+3+2/k)}{\Gamma(1+n)} \left(\frac{B_\varepsilon^k}{B_{\varepsilon'}^k}\right)^n e^{\ell^k B_\varepsilon^k} e^{\ell^k \Re \zeta} < +\infty \end{aligned}$$

(write $\zeta_0 = (\zeta_0 - \Re(\zeta)) + \Re(\zeta)$ and $|\zeta_0 - \Re(\zeta)| < B_\varepsilon^k$). The estimate being valid for all $\ell > \ell_0$ we can conclude that there exist constants $K > 0$, $A > \ell_0^k$ such that

$$|\phi(\zeta)| \leq K e^{A|\zeta|} \quad \text{on } \Sigma'_{B_\varepsilon}.$$

Hence the result. \square

Proposition 6.3.9. — *k -Borel-Laplace summability in a given direction θ_0 is equivalent to k -summability in the direction θ_0 .*

Proof. — \triangleright *k-Borel-Laplace summability implies k-summability.*

Suppose we are given a *k*-Borel-Laplace summable series $\tilde{f}(x) = \sum_{n \geq k} a_n x^n$ in a direction θ_0 . This means that its *k*-Borel transform $\sum_{n \geq k} \frac{a_n}{\Gamma(n/k)} \xi^{n-k}$ converges; its sum $\varphi(\xi)$ can be analytically continued to a sector $\sigma_{] \theta_1, \theta_2[}$ containing the direction θ_0 where $\varphi(\xi)$ satisfies the following inequality for some positive constants *A* and *K*:

$$|\varphi(\xi)| \leq K \exp(A|\xi^k|).$$

By applying Cauchy's Theorem one proves (proof left to the reader) that the *k*-Borel-Laplace integrals

$$f_\beta(x) = \int_{\xi=0}^{\infty e^{i\beta}} \varphi(\xi) e^{-\xi^k/x^k} d(\xi^k)$$

associated with the various directions $\beta \in] \theta_1, \theta_2[$ glue into a Borel-Laplace sum $f(x)$ defined and analytic on the union of the Fatou flowers

$$\mathcal{D} = \bigcup_{\theta_1 < \theta < \theta_2} \{x; \Re(e^{i\theta}/x^k) > A\}$$

(recall that Fatou flowers are called Borel discs when $k+1$). We must prove that $f(x)$ is *s*-Gevrey asymptotic to $\tilde{f}(x)$ on a sector $\mathcal{A} \subset \mathcal{D}$ bisected by θ_0 with opening larger than π/k (recall $s = 1/k$).

Let $\beta \in] \theta_1, \theta_2[$ and $\mathcal{A}_\beta \subset \mathcal{A}$ a sector bisected by β with opening $(\pi - 2\delta)/k < \pi/k$ be given. Prove that, under the hypothesis

$$|\varphi(\xi)| \leq K \exp(A|\xi^k|) \quad \text{for } \arg(\xi) = \beta$$

there exist constants $K', A' > 0$ such that f_β satisfies

$$\left| f_\beta(x) - \sum_{n=k}^{N-1} a_n x^n \right| \leq K' \Gamma\left(\frac{N}{k}\right) A'^N |x|^N$$

for all $x \in \mathcal{A}_\beta$ and $N \in \mathbb{N}^*$ and that moreover, the constants K', A' are independent of β while they depend on the size of \mathcal{A}_β .

We normalize the situation to the case when $\beta = 0$ by means of the rotation $-\beta$ both in the x - and the ξ -plane: the direction β becomes $\beta^* = 0$ and the variables x and ξ become $x^* = x e^{-i\beta}$ and $\xi^* = \xi e^{-i\beta}$ so that $|x^*| = |x|$ and $|\xi^*| = |\xi|$; we set $f^*(x^*) = f(x^* e^{i\beta})$ and $\varphi^*(\xi^*) = \varphi(\xi^* e^{i\beta})$. We normalize to the case $k = 1$ by the change of variable $\zeta = \xi^{*k}$ (at the price of introducing non integer powers). The sector \mathcal{A}_β has become a sector \mathcal{A}_β^* bisected by $\theta^* = 0$

with opening $(\pi - 2\delta)/k$. Setting $\phi(\zeta) = \varphi(\zeta^{1/k} e^{i\beta})$ we can write

$$f^*(x^*) = \int_0^{+\infty} \phi(\zeta) e^{-\zeta/x^{*k}} d\zeta$$

and ϕ satisfies

$$|\phi(\zeta)| \leq K e^{A\zeta} \quad \text{for } \zeta > 0.$$

Choose $b^k > 0$ in the disc of convergence of ϕ and split the Laplace integral into $f^*(x^*) = f_1^*(x^*) + f_2^*(x^*)$ with

$$f_1^*(x^*) = \int_0^{b^k} \phi(\zeta) e^{-\zeta/x^{*k}} d\zeta \quad \text{and} \quad f_2^*(x^*) = \int_{b^k}^{+\infty} \phi(\zeta) e^{-\zeta/x^{*k}} d\zeta.$$

It follows from Lemma 2.4.2 that $f_1^*(x^*)$ is s -Gevrey asymptotic to the series $\tilde{f}^*(x^*) = \tilde{f}(x^* e^{i\beta})$ on \mathcal{A}_β^* with an estimate of the form

$$(37) \quad \left| f_1^*(x^*) - \sum_{n=k}^{N-1} a_n x^{*n} \right| \leq C_1^* \Gamma\left(\frac{N}{k}\right) A_1^{*N} |x|^N$$

where $A_1^* = 1/(b|\sin(\delta)|^{1/k})$ and $C_1^* = \sum_{n \geq k} \frac{|a_n|}{\Gamma(n/k)} b^n$.

We must also prove that $f_2^*(x^*)$ is s -Gevrey asymptotic to 0 with a global estimate on \mathcal{A}_β^* . To this end, observe that f_2^* satisfies

$$|f_2^*(x^*)| \leq K \int_{b^k}^{+\infty} e^{(A - \Re(1/x^{*k}))\zeta} d\zeta = \frac{K e^{Ab^k}}{\Re(1/x^{*k}) - A} e^{-b^k \Re(1/x^{*k})}.$$

When $x^* = |x^*| e^{i\theta}$ belongs to \mathcal{A}_β^* then $x^{*k} = |x^*|^k e^{ik\theta}$ belongs to a sector bisected by $\theta^* = 0$ with opening $\pi - 2\delta$. Then, $\cos(k\theta) > \sin(\delta)$ and x^{*k} satisfies $\Re(1/x^{*k}) > \sin(\delta)/|x|^k$. Hence,

$$|f_2^*(x^*)| \leq \frac{K e^{Ab^k}}{\sin(\delta)/|x|^k - A} e^{-(b^k \sin(\delta))/|x|^k}.$$

The factor $K e^{Ab^k} |x|^k / (\sin(\delta) - A|x|^k)$ is bounded on \mathcal{A}_β^* and then, there exists constants $A^* = b^k \sin(\delta)$ and $C^* > 0$ such that

$$|f_2^*(x^*)| \leq C^* e^{-A^*/|x|^k} \quad \text{for } x^* \in \mathcal{A}_\beta^*.$$

The constants depend on b and δ and not on β . From Proposition 2.3.17 we obtain that $f_2^*(x^*)$ is s -Gevrey asymptotic to 0 on \mathcal{A}_β^* and that there exist constants $C_2^*, A_2^* > 0$ depending on δ such that

$$(38) \quad |f_2^*(x^*)| \leq C_2^* \Gamma\left(\frac{N}{k}\right) A_2^{*N} |x|^N,$$

the estimate being valid for all $N \in \mathbb{N}$ and $x^* \in \mathcal{L}_\beta^*$ (cf. proof of Prop. 2.3.17). Hence, putting together (37) and (38), we can conclude that there exist constants $C', A' > 0$ such that for all $N \in \mathbb{N}^*$ and $x \in \mathcal{L}_\beta^*$, the function $f_\beta(x)$ satisfies

$$\left| f_\beta(x) - \sum_{n=k}^{N-1} a_n x^n \right| \leq C' \Gamma\left(\frac{N}{k}\right) A'^N |x|^N.$$

Since the constants do not depend on the direction β the estimate (38) is valid for $f(x)$ on the union $\mathcal{L} = \bigcup_{\theta_1 < \beta < \theta_2} \mathcal{L}_\beta^*$. Choosing $\delta < \min(\theta_0 - \theta_1, \theta_2 - \theta_0)$ implies that \mathcal{L} is a sector with opening larger than $\pi/(2k)$ on both sides of the direction θ_0 . Hence the result.

▷ *k-summability implies k-Borel-Laplace summability.*

Suppose we are given a k -summable series $\tilde{f}(x) = \sum_{n \geq k} a_n x^n$ in a direction θ_0 . This means that there exist an analytic function $f(x)$ and constants $B, C > 0$ such that

$$\left| f(x) - \sum_{n=k}^{N-1} a_n x^n \right| \leq C \Gamma\left(\frac{N}{k}\right) B^N |x|^N$$

on a sector $\mathcal{L} = \{x; |\arg(x) - \theta_0| < (\pi + 2\delta)/2k \text{ and } |x| < R\}$ bisected by θ_0 with opening greater than π/k . Given a direction β such that $|\beta - \theta_0| < \delta/k$ Nevanlinna's Theorem (Thm. 6.3.7) provides constants $A, K > 0$ such that the Borel transform $\varphi(\xi)$ is analytic and satisfies

$$|\varphi(\xi)| \leq K e^{A|\xi|^k}$$

on the domain $\sigma'_{B,\beta}$ equal to σ'_B rotated by an angle β (cf. preamble of Nevanlinna's Theorem). A rotation does not affect the constants and thus, A and K being independent of the direction β , the estimate is valid on $\sigma' = \bigcup_{|\beta - \theta_0| < \delta/k} \sigma'_{B,\beta}$ which contains a champaign cork neighborhood of the direction θ_0 . Hence, the series $\tilde{f}(x)$ is k -Borel-Laplace summable in direction θ_0 . \square

Comments 6.3.10 (On Examples 2.2.4, 2.2.7 and 2.2.8)

▷ The Borel transform $\varphi(\xi) = 1/(\xi + 1)$ of the Euler series $\tilde{E}(x)$ (example 2.2.4) has exponential growth of order one (and even less) in all direction. However, the sum of the Borel series cannot be continued up to infinity in the direction $\theta = \pi$ due to the pole $\xi = -1$ of φ and indeed, we saw that the Euler series $\tilde{E}(x)$ is 1-summable in all directions but the direction $\theta = \pi$.

▷ The Borel series $\widehat{h}(\zeta)$ of Example 2.2.7 has as sum the function $\varphi(\zeta) = 1/(e^{-\zeta} - 2)$ which has exponential growth of order 1 in all directions. However, it has a line of poles $\zeta_n = -\ln(2) + 2ni\pi$, $n \in \mathbb{Z}$. Hence, we can now conclude that the series $h(z)$ is 1-summable in all directions but $\theta = \arg(-\ln(2) + 2ni\pi)$ for all $n \in \mathbb{Z}$ and their closure $\theta = \pm\pi/2$. In particular, it is not 1-summable in the sense of Definition 6.1.6 (point 4) which requires 1-summability in all directions but finitely many. This shows that solutions of difference equations, even when they are mild, can be not summable.

▷ The Borel series $\widehat{\ell}(\zeta)$ of Example 2.2.8 has as sum the function $\varphi(\zeta) = e^{-\zeta + e^{-\zeta} - 1}$ which is an entire function with exponential growth of order 1 in all directions $\Re(\zeta) \geq 0$ and exponential growth of no order in the directions $\Re(\zeta) < 0$. Hence, the series $\widetilde{h}(z)$ is 1-summable in all directions $\Re(z) > 0$ and not 1-summable in the other directions.

6.3.3. Tauberian Theorems. — The Tauberian Theorems we have in mind wish at comparing various k -sums of a given series in a given direction when several ones exist (cf. [Mal95] Théorème 2.4.2.2, [Bal94] Thms. 2.1 and 2.2). We begin with the following result.

Theorem 6.3.11. — *Given numbers k_1, k_2 satisfying $0 < k_1 < k_2$ define κ_1 by*

$$1/\kappa_1 = 1/k_1 - 1/k_2.$$

Suppose we are given two closed arcs I_1 and \widehat{I}_1 with same middle point θ_0 and respective length $|I_1| = \pi/k_1$ and $|\widehat{I}_1| = \pi/\kappa_1$. Given a formal power series $\widetilde{f}(x) \in \mathbb{C}[[x]]$ denote by $\widehat{g}(\xi) = \mathcal{B}_{k_2}(\widetilde{f})(\xi)$ its k_2 -Borel transform (cf. Def. 6.3.1).

The following two assertions are equivalent.

(i) *The series $\widetilde{f}(x)$ is k_1 -summable on I_1 with k_1 -sum $f(x)$;*

(ii) *The series $\widehat{g}(\xi)$ is κ_1 -summable on \widehat{I}_1 and its κ_1 -sum $g(\xi)$ can be analytically continued to an unlimited open sector $\widehat{\sigma}$, containing $\widehat{I}_1 \times]0, +\infty[$, with exponential growth of order k_2 at infinity.*

Moreover, $\mathcal{B}_{k_2}(f)(\xi) = g(\xi)$ and $\mathcal{L}_{k_2}(g)(x) = f(x)$ in direction θ_0 and neighboring directions.

Proof. — The theorem being true (and empty) for monomials we can assume that the series $\widetilde{f}(x) = \sum_{n \geq k_0} a_n x^n$ has valuation $k_0 > k_2$.

▷ *Prove that (i) implies (ii).* We proceed as in the proof of Nevanlinna's Theorem. By assumption, the series $\widetilde{f}(x)$ has a k_1 -sum $f(x)$ on the closed arc I_1 , hence, on a larger open arc. Thus, there exists a closed arc I'_1 containing I_1 in its interior, there exist $r_0 > 0$ and constants $A, C > 0$ such that the

estimate

$$(39) \quad \left| f(x) - \sum_{n=k_0}^{N-1} a_n x^n \right| \leq C N^{N/k_1} A^N |x|^N$$

holds for all $N \in \mathbb{N}^*$ and all x in the sector $\mathcal{S}'_1 = I'_1 \times]0, r_0]$.

For convenience, we normalize the Borel transform \mathcal{B}_{k_2} into the classical Borel transform \mathcal{B}_1 of level 1. To this end, set $Z = 1/x^{k_2}$, $\zeta = \xi^{k_2}$ and $R_0 = 1/r_0^{k_2}$. In the coordinate Z , the sector $\mathcal{S}'_1 = I'_1 \times]0, r_0]$ is changed into a sector $\mathcal{S}_1 = J_1 \times [R_0, +\infty[$ with opening larger than π (indeed, $k_2 > k_1$). The series $\tilde{f}(x)$ becomes the series $\tilde{F}(Z) = \tilde{f}(1/Z^{1/k_2})$ and the function $f(x)$ the function $F(Z) = f(1/Z^{1/k_2})$. In the coordinate ζ , the function $g(\xi)$ becomes $G(\zeta) = g(\zeta^{1/k_2})$.

Suppose first that $1/k_1 < 2/k_2$ (hence, $k_2\pi/(2k_1) < \pi$). Recall that, by assumption, we have $1/k_2 < 1/k_1$ (hence, $k_2\pi/(2k_1) > \pi/2$). After performing a rotation to normalize the direction θ_0 to 0 we get the following picture (Fig. 12) where $J_1 = [-\omega_1, \omega_1]$ with $\omega_1 = k_2(\pi/(2k_1) + \varepsilon) /$, (suppose ε chosen so small that $\omega_1 < \pi$).

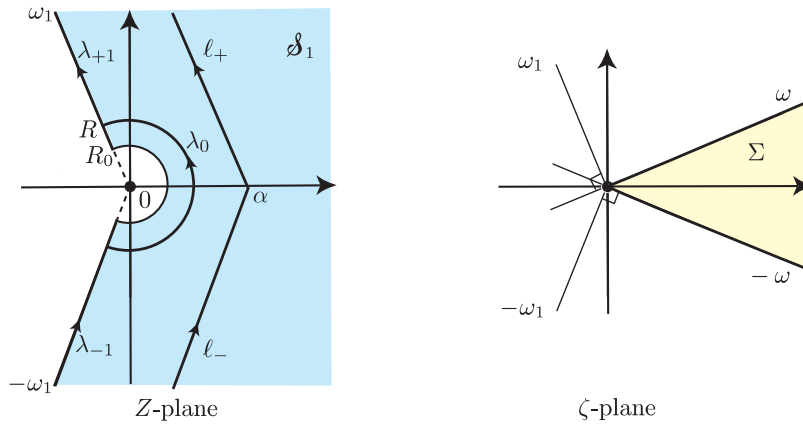


Figure 12

Denote by $g(\xi) = \mathcal{B}_{k_2, \theta_0}(f)(\xi)$ the k_2 -Borel transform of $f(x)$ in direction $\theta_0 = 0$.

The Borel path to define $G(\zeta) = g(\zeta^{1/k_2})$ can be chosen as the boundary

$$\partial\mathcal{S}_1 = \partial_{-1} \cup \partial_0 \cup \partial_{+1}$$

of \mathcal{S}_1 where ∂_0 denotes the part of the boundary which is a circular arc of radius R_0 and $\partial_{\pm 1}$ the two straight lines of $\partial\mathcal{S}_1$. Assume that $\partial\mathcal{S}_1$ is oriented as in

Fig. 12. From Cauchy's Theorem the path $\partial\mathcal{D}_1$ can equivalently be deformed into its homothetic $\lambda = \lambda_{-1} \cup \lambda_0 \cup \lambda_{+1}$ where λ_0 has radius $R > R_0$ or into a broken line $\ell = \ell_- \cup \ell_+$ passing through a large enough $\alpha > 0$ as shown in Fig. 12. Thus, γ being any of the Borel paths above, $G(\zeta)$ reads as

$$(40) \quad G(\zeta) = \frac{1}{2\pi i} \int_{\gamma} F(U) e^{\zeta U} dU.$$

It suffices to prove that

1. the function $G(\zeta)$ is defined and holomorphic on the unlimited open sector $\Sigma = J \times]0, +\infty[$ where $J =]-\omega, +\omega[$ with $\omega = k_2(\pi/(2\kappa_1) + \varepsilon)$;
2. the function $G(\zeta)$ has exponential growth of order one at infinity on Σ and has $F(Z)$ as Laplace transform;
3. there exist constants $C', A' > 0$ such that the following estimate holds for all N and all $\zeta \in \Sigma' = J' \times]0, 1/R_0[$ where $J' =]-\omega', +\omega'[$, $\omega' = \omega - k_2\varepsilon/2$:

$$(41) \quad \left| G(\zeta) - \sum_{n=k_0}^{N-1} \frac{a_n}{\Gamma(n/k_2)} \zeta^{n/k_2-1} \right| \leq C' N^{N/\kappa_1} A'^N |\zeta|^{N/k_2-1}.$$

Notice that $\Sigma' \Subset \Sigma$ (cf. Def. 2.1.2) and $\Sigma' \supseteq [-k_2\pi/(2\kappa_1), +k_2\pi/(2\kappa_1)] \times]0, 1/R_0[$.

1. In the variable Z , Estimate (39) reads

$$(42) \quad \left| F(Z) - \sum_{n=k_0}^{N-1} \frac{a_n}{Z^{n/k_2}} \right| \leq C N^{N/k_1} \frac{A^N}{|Z|^{N/k_2}} \quad \text{for all } Z \in \mathcal{D}_1$$

which, taking $N = k_0$, implies that there exists constants $M_0, M_1 > 0$ such that

$$(43) \quad |F(Z)| \leq \frac{M_1}{|Z|^{k_0/k_2}} \leq M_0, \quad \text{for all } Z \in \mathcal{D}_1.$$

In the integral of Formula (40) choose $\gamma = \partial\mathcal{D}_1$ and, for $j = \pm 1, 0$, denote

$$G_j(\zeta) = 1/(2\pi i) \int_{\partial_j} F(U) e^{\zeta U} dU$$

so that $G(\zeta) = G_{-1}(\zeta) + G_0(\zeta) + G_{+1}(\zeta)$.

The term $G_0(\zeta)$ is a Riemann integral and determines a holomorphic function for all ζ . Due to Estimate (43) and the fact that $k_0/k_2 > 1$ the function $F(U)$ is Lebesgue integrable on $\partial_{-1} \cup \partial_{+1}$ and, consequently, the functions $G_{\pm 1}(\zeta)$ are defined and holomorphic on the half-planes $\Re(\zeta e^{\pm i\omega_1}) < 0$ respectively. Thus,

the function $G(\zeta)$ is defined and holomorphic on the sector Σ , intersection of these two half-planes.

2. Denote $G_{\pm}(\zeta) = 1/(2\pi i) \int_{\ell_{\pm}} F(U) e^{\zeta U} dU$ so that $G(\zeta) = G_{-}(\zeta) + G_{+}(\zeta)$ for all $\zeta \in \Sigma$. Parameterizing the paths ℓ_{\pm} by $U = \alpha + u e^{\pm i\omega_1}$ we deduce from Estimate (43) that $G_{\pm}(\zeta)$ satisfies

$$|G_{\pm}(\zeta)| \leq M_1 \frac{e^{\alpha \Re(\zeta)}}{2\pi} \int_0^{+\infty} \frac{1}{|\alpha + u e^{\pm i\omega_1}|^{k_0/k_2}} du \quad \text{for all } \zeta \in \Sigma.$$

Thus, there exists a constant $c > 0$ such that $|G(\zeta)| \leq c e^{\alpha|\zeta|}$ for all $\zeta \in \Sigma$, and this proves the exponential growth of order 1 of $G(\zeta)$ at infinity on Σ .

Prove that the Laplace transform $\mathcal{L}(G)(Z)$ in direction $\theta_0 = 0$ is equal to $F(Z)$ on the half-plane $\{Z; \Re(Z) > R_0\}$. By definition, $\mathcal{L}(G)(Z)$ reads

$$\mathcal{L}(G)(Z) = \frac{1}{2\pi i} \int_0^{+\infty} \left(\int_{\partial \mathfrak{D}_1} F(U) e^{\zeta U} dU \right) e^{-\zeta Z} d\zeta$$

and the function $F(U) e^{\zeta(U-Z)}$ is in $L^1(\partial \mathfrak{D}_1 \times \mathbb{R}^+)$ when $\Re(Z) > R_0$. Indeed, parameterizing $\partial_{\pm 1}$ by $U = (R_0 + V) e^{\pm i\omega_1}$ and ∂_0 by $U = R_0 e^{i\theta}$ provides the estimates

$$\left| F(U) e^{\zeta(U-Z)} \right| \leq \begin{cases} M_1 e^{\zeta(R_0 \cos \omega_1 - \Re(Z))} / |R_0 + V|^{k_0/k_2} & \text{for } (U, \zeta) \in \partial_{\pm} \times \mathbb{R}^+ \\ M_0 e^{\zeta(R_0 - \Re(Z))} & \text{for } (U, \zeta) \in \partial_0 \times \mathbb{R}^+. \end{cases}$$

By Fubini's Theorem we can then write

$$\begin{aligned} \mathcal{L}(G)(Z) &= \frac{1}{2\pi i} \int_{\partial \mathfrak{D}_1} \left(F(U) \int_0^{+\infty} e^{\zeta(U-Z)} d\zeta \right) dU \\ &= \frac{1}{2\pi i} \int_{\partial \mathfrak{D}_1} -\frac{F(U)}{U-Z} dU \quad (\partial \mathfrak{D}_1 \text{ turns negatively around } Z) \\ &= F(Z) \quad (\text{by Cauchy's formula}). \end{aligned}$$

3. Use now the path λ with a radius R to be made explicit later. We want to estimate the quantity

$$\left| G(\zeta) - \sum_{n=k_0}^{N-1} \frac{a_n}{\Gamma(n/k_2)} \zeta^{n/k_2-1} \right| \leq Q_{-1} + Q_0 + Q_{+1}$$

where $Q_j = 1/(2\pi) \left| \int_{\lambda_j} (F(U) - \sum_{k_0}^{N-1} a_n/U^{n/k_2}) e^{\zeta U} dU \right|$, $j = \pm 1, 0$ for all $\zeta \in \Sigma'$.

To estimate $Q_{\pm 1}$ use inequality (42) to write

$$Q_{\pm 1} \leq \frac{C}{2\pi} N^{N/k_1} A^N \int_R^{+\infty} \frac{e^{\Re(\zeta U)}}{|U|^{N/k_2}} d|U|$$

where $\Re(\zeta U) = |\zeta U| \cos(\theta' \pm \omega_1)$ and $|\theta' = \arg(\zeta)| \leq \omega'$. For $\zeta = |\zeta| e^{i\theta'}$ in Σ then $\theta' + \omega_1$ satisfy $\pi/2 < |\theta' \pm \omega_1| < 3\pi/2$. Since Σ' is a proper subsector of Σ there exists $c' > 0$ such that $\cos(\theta' + \omega_1) \leq -c'$ for all $\zeta \in \Sigma'$ and therefore, $\Re(\zeta U) \leq -c'|\zeta U|$ for all $\zeta \in \Sigma'$ and $U \in \lambda_{\pm 1}$. Using this estimate and the change of variable $V = c'|\zeta||U|$ in the latter integral we obtain

$$\begin{aligned} Q_{\pm 1} &\leq \frac{C}{2\pi} N^{N/k_1} A^N (c'|\zeta|)^{N/k_2-1} \int_{c'|\zeta|R}^{+\infty} \frac{e^{-V}}{V^{N/k_2}} dV \\ &\leq \frac{C}{2\pi} N^{N/k_1} A^N (c'|\zeta|)^{N/k_2-1} \frac{1}{(c'|\zeta|R)^{N/k_2}} \int_0^{+\infty} e^{-V} dV \\ &= \frac{C}{2\pi} N^{N/k_1} A^N (c'|\zeta|)^{-1} \frac{1}{R^{N/k_2}}. \end{aligned}$$

For each $\zeta \in \Sigma'$ choose $R = N/|\zeta|$ (then, $R > R_0$) and denote by C'_1 the constant $C'_1 = C/(2\pi c')$. It follows that Σ' , $Q_{\pm 1}$ satisfies on Σ' the estimate

$$Q_{\pm 1} \leq C'_1 N^{N/\kappa_1} A^N |\zeta|^{N/k_2-1} \quad (\text{recall } 1/\kappa_1 = 1/k_1 - 1/k_2).$$

To estimate Q_0 parameterize λ_0 by $U = R e^{i\theta}$ with $R = N/|\zeta|$ to obtain

$$\begin{aligned} Q_0 &\leq \frac{C}{2\pi} N^{N/k_1} A^N \frac{1}{R^{N/k_2-1}} \int_{-\omega_1}^{\omega_1} e^{\Re(N e^{i\theta})} d\theta \\ &\leq \frac{C}{2\pi} N^{N/\kappa_1} N A^N |\zeta|^{N/k_2-1} e^N 2\omega_1. \end{aligned}$$

Choosing $A'_2 > A e$ (so that $N A^N e^N < \text{Cst. } A'_2{}^N$) and $C'_2 = \text{Cst. } C\omega_1/\pi$ we obtain

$$Q_0 \leq C'_2 N^{N/\kappa_1} A'_2{}^N |\zeta|^{N/k_2-1} \quad \text{on } \Sigma'.$$

By adding these estimates and choosing $A' = A'_2$ and $C' = 2C'_1 + C'_2$ it follows that Estimate (41) is satisfied for all $N \geq 1$ and all $\zeta \in \Sigma'$.

Suppose now that $1/k_1 \geq 2/k_2$ (hence, $k_2\pi/(2k_1) \geq \pi$).

We observe that when ω_1 passes the value π the expression of ω is changed from $\omega = -\pi/2 + \omega_1 = k_2(\pi/(2\kappa_1) + \varepsilon)$ to $\omega = 3\pi/2 - \omega_1 = \pi - k_2(\pi/(2\kappa_1) + \varepsilon)$. Hence, as ω_1 increases through the value π (also k_2/k_1 and k_2/κ_1 increase) the value of ω first increases up to $\pi/2$ (when $\omega_1 = \pi$) and then decreases. The sector Σ is no more large enough to prove the κ_1 -summability of $g(\xi)$. We can pass through that difficulty by breaking \mathcal{J}_1 into finitely many subsectors of opening less than 2π . To this end, choose some directions θ_j and closed arcs $J_{1,j}$ of length less than 2π whose interiors make a covering of J_1 . From Cauchy's Theorem the Borel transforms of $F(Z)$ at the various directions θ_j are analytic continuations of each others and we can apply the previous proof

to each arc $J_{1,j}$ taking now the Borel transform in direction θ_j . This ends the proof of that part.

▷ *Prove that (ii) implies (i).*

Again we can restrict the study to the case when $1/k_1 < 2/k_2$ (hence, $k_2\pi/(2k_1) < \pi$). We use the same notations as before.

Assume Conditions 1, 2 and 3 are satisfied. Denote by a the type of exponential growth of $G(\zeta)$ on Σ' and, up to increasing the value of R_0 (hence, up to shrinking the interval $]0, 1/R_0[$), suppose $R_0 > a$.

Choose a direction θ_1 in Σ' (i.e., $|\theta_1| \leq \omega'$) and set

$$F_{\theta_1}(Z) = \int_0^{+\infty e^{i\theta_1}} G(\zeta) e^{-\zeta Z} d\zeta.$$

Condition 2 says that $G(\zeta)$ has exponential growth, say, of type a and it follows that the above definition of $F_{\theta_1}(Z)$ defines a holomorphic function $F_{\theta_1}(Z)$ on the half-plane $\Re(e^{i\theta_1}Z) > a$ bisected by $-\theta_1$ at the distance a of 0. By Cauchy's Theorem the functions $F_{\theta_1}(Z)$ for the various values of $\theta_1 \in \Sigma'$ are analytic continuations from each other and we denote by $F(Z)$ the function they define on the open sector \mathfrak{S}_1 union of the half-planes associated with all $\theta_1 \in \Sigma'$. Observe that, since the opening of Σ' is larger than $k_2\pi/k_1$, the opening of \mathfrak{S}_1 is larger than $k_2\pi/k_1$.

By means of a rotation we can assume that $\theta_1 = 0$ and use Estimate (41) for $\zeta > 0$. Given $0 < \beta < \varepsilon/2$, denote by Π_β the sector

$$\Pi_\beta = \{Z; |\arg Z| \leq \pi/2 - \beta \text{ and } \Re(Z) \geq R_0 > a\}.$$

Notice that the condition on β implies that the sector \mathfrak{S}'_1 union of the Π_β 's associated with the various directions $\theta_1 \in \Sigma'$ has opening more than $k_2\pi/k_1$ (and this is also the case for $\mathfrak{S}'_1 \Subset \mathfrak{S}_1$).

Prove that Estimate (42):

$$\left| F(Z) - \sum_{n=k_0}^{N-1} \frac{a_n}{Z^{n/k_2}} \right| \leq C N^{N/k_1} \frac{A^N}{|Z|^{N/k_2}}$$

(there exist $A, C > 0$) holds for all N and all Z in Π_β , for, the constants involved are valid for any choice of θ_1 in Σ' .

Since $\int_0^{+\infty} \zeta^{n/k_2-1} e^{-\zeta Z} d\zeta = \Gamma(n/k_2)/Z^{n/k_2}$ we can write

$$F(Z) - \sum_{n=k_0}^{N-1} \frac{a_n}{Z^{n/k_2}} = \int_0^{+\infty} \left(G(\zeta) - \sum_{n=k_0}^{N-1} \frac{a_n}{\Gamma(n/k_2)} \zeta^{n/k_2-1} \right) e^{-\zeta Z} d\zeta$$

and then,

$$\left| F(Z) - \sum_{n=k_0}^{N-1} \frac{a_n}{Z^{n/k_2}} \right| \leq P_1 + P_2 + P_3$$

where

$$\begin{cases} P_1 = \int_0^{1/R_0} \left| G(\zeta) - \sum_{n=k_0}^{N-1} \frac{a_n}{\Gamma(n/k_2)} \zeta^{n/k_2-1} \right| |e^{-\zeta Z}| d\zeta, \\ P_2 = \int_{1/R_0}^{+\infty} |G(\zeta) e^{-\zeta Z}| d\zeta, \\ P_3 = \sum_{n=k_0}^{N-1} \frac{|a_n|}{\Gamma(n/k_2)} \int_{1/R_0}^{+\infty} |\zeta^{n/k_2-1} e^{-\zeta Z}| d\zeta. \end{cases}$$

From Estimate (41) we obtain, on Π_β ,

$$\begin{aligned} P_1 &\leq C' A'^N N^{N/\kappa_1} \int_0^{+\infty} \zeta^{N/k_2-1} e^{-\zeta \Re(Z)} d\zeta \\ &= C' \frac{A'^N}{\Re(Z)^{N/k_2}} N^{N/\kappa_1} \Gamma(N/k_2) \\ &\leq C' \frac{A'^N}{(|Z| \cos(\beta))^{N/k_2}} N^{N/\kappa_1} \Gamma(N/k_2) \quad \text{since } \Re(Z) \geq |Z| \cos \beta \text{ on } \Pi_\beta \\ &\leq C_1 \frac{A_1^N}{|Z|^{N/k_2}} N^{N/\kappa_1} N^{N/k_2} = C_1 \frac{A_1^N}{|Z|^{N/k_2}} N^{N/k_1} \quad \text{for larger constants } A_1, C_1 > 0. \end{aligned}$$

From Condition 2 we obtain, on Π_β ,

$$\begin{aligned} P_2 &\leq c \int_{1/R_0}^{+\infty} e^{(a-\Re(Z))\zeta} d\zeta \\ &\leq \frac{c e^{-(\Re(Z)-a)/R_0}}{(\Re(Z) - a)} \\ &\leq \frac{c e^{a/R_0}}{R_1 - a} \cdot \frac{n^n e^{-n} R_0^n}{(|Z| \cos \alpha)^n} \quad \text{for all } n > 0 \text{ and using } \Re(Z) \geq |Z| \cos \beta \\ &\leq C_2 \frac{A_2^N}{|Z|^{N/k_2}} N^{N/k_1} \end{aligned}$$

by taking $n = N/k_2$, $A_2 = (R_0/(ek_2 \cos \beta))^{1/k_2}$ and $C_2 = ce^{a/R_0}/(R_1 - a)$ and using $N^{1/k_2} < N^{1/k_1}$.

From estimate (41) we deduce (see Prop.2.3.10) that there exist constants A'' and $C'' > 0$ such that, for all n ,

$$(44) \quad \left| \frac{a_n}{\Gamma(n/k_2)} \right| \leq C'' n^{n/\kappa_1} A''^n.$$

It follows that P_3 satisfies

$$\begin{aligned} P_3 &\leq \sum_{n=k_0}^{N-1} C'' n^{n/\kappa_1} A''^n \int_{1/R_0}^{+\infty} \frac{1}{R_0^{n/k_2-1}} (R_0 \zeta)^{N/k_2-1} e^{-\zeta \Re(Z)} d\zeta \quad \text{since } R_0 \zeta \geq 1 \\ &\leq C'' N \max(A''^N, A''^{k_0}) R_0^{(N-n)/k_2} N^{N/\kappa_1} \frac{\Gamma(N/k_2)}{(|Z| \cos \beta)^{N/k_2}} \\ &\leq C_3 \frac{A_3^N}{|Z|^{N/k_2}} N^{N/k_1} \quad \text{as before with large enough constants } A_3 \text{ and } C_3. \end{aligned}$$

Adding these three estimates we obtain

$$(45) \quad \left| F(Z) - \sum_{n=k_0}^{N-1} \frac{a_n}{Z^{n/k_2}} \right| \leq C \frac{A^N}{|Z|^{N/k_2}} N^{N/k_1} \quad \text{on } \Pi_\alpha$$

by setting $A = \max(A_1, A_2, A_3)$ and $C = C_1 + C_2 + C_3$. The constants A and C are independent of $\theta_1 \in \Sigma'$. Henceforth, estimate (45) is valid for all $Z \in \mathcal{D}'_1$ and this proves the k_1 -summability of $f(x)$ in direction θ_0 since the opening of \mathcal{D}'_1 is larger than $k_2\pi/k_1$. This achieves the proof of the theorem. \square

The Tauberian Theorems of J. Martinet and J.-P. Ramis [MR89, Prop. 4.3]⁽²⁾ are easy corollaries of this theorem.

Corollary 6.3.12 (Martinet-Ramis Tauberian Theorem 1)

Let $0 < k_1 < k_2$ and let $I_1 \supseteq I_2$ be, respectively, a k_1 -wide and a k_2 -wide arc of S^1 . Set $s_2 = 1/k_2$.

If a series $\tilde{f}(x)$ is both s_2 -Gevrey and k_1 -summable on I_1 then it is k_2 -summable on I_2 and the two sums agree on I_2 .

Observe that the assertion is not trivial since, according to Definition 6.1.6, being k_2 -summable on I_2 , compared to being k_1 -summable on I_1 , is a stronger condition to be satisfied on the smaller arc I_2 .

Proof. — It is sufficient to prove the theorem when I_1 and I_2 are closed of length π/k_1 and π/k_2 respectively. Let θ_1 and θ_2 be the bisecting directions of I_1 and I_2 ; they satisfy $|\theta_1 - \theta_2| \leq \pi/k_1 - \pi/k_2 = \pi/k$. A k_1 -sum $f_1(x)$ of $\tilde{f}(x)$ exists on a larger open arc $I_{1,\varepsilon}$ containing I_1 and lives in a sector $\Delta_{1,\varepsilon}$ based on $I_{1,\varepsilon}$. By Theorem 6.3.11 the k_2 -Borel transform $g(\xi)$ of $f_1(x)$ in direction θ_1 lives on an unbounded sector σ of opening $\pi/k + \varepsilon$ bisected by θ_1 and has there exponential growth of order k_2 . Moreover, $g(\xi)$ is the unique function s -Gevrey asymptotic to $\widehat{g}(\xi)$ on σ since the opening of σ is

⁽²⁾ Caution: the notation s in that article corresponds to our κ_1 .

larger than $s\pi = \pi/k$. On another hand, since $\tilde{f}(x)$ is s_2 -Gevrey, the formal Borel transform $\widehat{g}(\xi)$ of $\tilde{f}(x)$ is convergent. Its sum in the usual sense and the unique s -Gevrey asymptotic function $g(\xi)$ must necessarily agree. Denote by σ_c the union of the sector σ with the disc of convergence of $\widehat{g}(\xi)$ and keep the notation $g(\xi)$ for the function $g(\xi)$ continued to σ_c . The domain σ_c contains the direction θ_2 . Using Definition 6.3.6 we can then conclude that $\tilde{f}(x)$ is k_2 -summable in direction θ_2 .

In addition, from Theorem 6.3.11 we know that $f_1(x)$ is the analytic continuation of the k_2 -Laplace transform of $g(\xi)$ in direction θ_1 . On another hand, it follows from Cauchy's Theorem that the Laplace transforms of $g(\xi)$ in directions θ_1 and θ_2 are analytic continuations from each other. Hence, the k_2 -sum $f_2(x)$ of $\tilde{f}(x)$ coincides with $f_1(x)$ on I_2 and the proof is achieved. \square

Corollary 6.3.13 (Martinet-Ramis Tauberian Theorem 2)

Let $0 < k_1 < k_2$ and $s_2 = 1/k_2$.

If a series $\tilde{f}(x)$ is both k_1 -summable and s_2 -Gevrey then it is convergent.

In particular, if $\tilde{f}(x)$ is k_1 - and k_2 -summable for $k_1 \neq k_2$ then, $\tilde{f}(x)$ is convergent.

Proof. — Any closed arc of length π/k_2 can be included in an arc of length π/k_1 on which $\tilde{f}(x)$ is k_1 -summable. Henceforth, by the previous corollary, $\tilde{f}(x)$ is k_2 -summable in all directions and it follows that it is convergent (cf. Rem. 6.1.8). \square

Example 6.3.14 (Leroy series). — The Leroy series $\tilde{L}(x) = \sum_{n \geq 0} (-1)^n n! x^{2n+2}$ is the series deduced from the Euler series $\tilde{E}(x) = \sum_{n \geq 0} (-1)^n n! x^{n+1}$ (cf. Exa. 2.2.4) by substituting x^2 for x . The Leroy series is thus divergent and $1/2$ -Gevrey ($k_2 = 2, s_2 = 1/2$). It satisfies the Leroy equation

$$x^3 y' + 2y = 2x^2.$$

Show that $\tilde{L}(x)$ is both 1- and 2-summable in the directions $|\theta| < \pi/4 \pmod{\pi}$ with same sums. We choose the directions $|\theta| < \pi/4$, the case when $|\theta - \pi| < \pi/4$ being similar. From the 1-summability of the Euler series $\tilde{E}(x)$ in directions $|\theta| < \pi$ we deduce that $\tilde{L}(x)$ is 2-summable in any direction θ satisfying $|\theta| < \pi/2$ with a 2-sum $E(x^2)$ defined on $|\arg(x)| < 3\pi/4$. In particular, $E(x^2)$ is $1/2$ - and then also 1-Gevrey asymptotic to $\tilde{L}(x)$ on $|\arg(x)| < 3\pi/4$. Since the sector $|\arg(x)| < 3\pi/4$ is wider than π this shows that $E(x^2)$ is also the (unique) 1-sum of $\tilde{L}(x)$ on $|\arg(x)| < 3\pi/4$ (cf. Def. 6.1.6). We have thus showed that the Leroy series satisfies the Tauberian Theorem 6.3.12 taking $k_1 = 1, k_2 = 2, I_1 = I_2 = \{\theta; |\theta| < 3\pi/4\}$ and that, moreover, the 1-sum and the 2-sum when they both exist, agree.

Show that $\tilde{L}(x)$ is not 1-summable in any direction θ satisfying $\pi/4 \leq \theta \leq 3\pi/4 \pmod{\pi}$. Indeed, suppose $\pi/4 < \theta < 3\pi/4 \pmod{\pi}$. The 1-Borel transform $\mathcal{B}_1 \tilde{L}(\xi)$ of $\tilde{L}(x)$ satisfies an equation obtained from the Leroy equation by substituting ξ (multiplication by ξ) for $x^2 d/dx$ and $d/d\xi$ (derivation w.r.t ξ) for $1/x$ and so, it satisfies the equation

$$\xi Y + 2 \frac{dY}{d\xi} = 2.$$

After noticing that $\mathcal{B}_1 \tilde{L}(0) = 0$ we observe that $\mathcal{B}_1 \tilde{L}(\xi)$ is the Taylor series of the entire function $\Phi(\xi) = \exp(-\xi^2/4) \int_0^\xi e^{t^2/4} dt$ solution of the same equation. The function $\Phi(\xi)$ has exponential growth of order exactly 2 in direction θ (since $\pi/4 < \theta < 3\pi/4$). Hence, it cannot be applied a Laplace transform and the series $\tilde{L}(x)$ is not 1-summable in direction θ , and therefore, not 1-summable in all directions $\pi/4 \leq \theta \leq 3\pi/4 \pmod{\pi}$. This property is coherent with the Tauberian Theorem 6.3.13 since, as the series $\tilde{L}(x)$ is divergent, it cannot be both 1- and 2- summable in almost all directions.

6.3.4. Borel-Laplace summability and summable-resurgence. — We saw in Theorem 6.2.5 that solutions of linear differential equations with a unique level k are k -summable in any non anti-Stokes direction. In this section, we investigate deeper properties of such solutions called resurgence and summable-resurgence in the case when $k = 1$. These notions of resurgence and summable-resurgence are precisely defined and developed in [Sau, this volume] and in [Sau05] in the case of a one-dimensional lattice of singular points for the Borel transform; see also [CNP93]. They were introduced by J. Écalle [É81, É85] in a very general setting. They apply to a wide class of series, among which solutions of non linear differential equations or of difference equations. For a different approach in the linear differential case, hence in the case of a finite arbitrary set of singular points for the Borel transform, we refer to [LRR11].

The aim of this section is to show that the solutions of any linear differential equations with the unique level one are summable-resurgent.

Let D be a linear differential operator with meromorphic (convergent) coefficients and let us expand it with respect to the derivation $\delta = x^2 \frac{d}{dx}$:

$$D = b_n(x)\delta^n + b_{n-1}(x)\delta^{n-1} + \dots + b_0(x).$$

We suppose that its Newton polygon $\mathcal{N}_0(D)$ at 0 has slopes 0 and 1 and that $\tilde{f}(x)$ is a series solution of the equation $Dy = 0$. The other formal solutions are either log-series $\tilde{f}_j(x)x^{\lambda_j}$ or log-exp-series $\tilde{f}_j(x)x^{\lambda_j} e^{q_j(1/x)}$ where $\tilde{f}_j(x) \in \mathbb{C}[[x]][\ln x]$, $\lambda_j \in \mathbb{C}$ and $q_j(1/x) = -a_j/x$, $a_j \neq 0$.

Definition 6.3.15. — The coefficients a_j of the (non-zero) determining polynomials $q_j(1/x)$ of the equation $Dy = 0$ are called Stokes values of $Dy = 0$ associated with $\tilde{f}(x)$.

These Stokes values indicate the anti-Stokes directions associated with $\tilde{f}(x)$ as well as with any other log-series solution $\tilde{f}_j(x) x^{\lambda_j} \ln^p(x)$.

From Theorem 6.2.5 we know that $\tilde{f}(x)$ is 1-summable in all directions but, possibly, anti-Stokes directions. From the Borel-Laplace viewpoint this means that, given a non anti-Stokes direction d_θ , its associated Borel series $\widehat{f}(\xi)$ is convergent and can be continued to a sector neighboring d_θ with exponential growth of order 1.

We prove below that such properties can be extended to a much larger domain.

▷ Suppose first that the coefficients $b_j(x)$ of D are algebraic (*i.e.*, polynomials in x and $1/x$). If the coefficients b_j contain polynomial terms in x of maximal degree $N > 0$ we replace D by the operator $D' = x^{-N}D$. The operator D' reads

$$D' = B_n(1/x)\delta^n + B_{n-1}(1/x)\delta^{n-1} + \cdots + B_0(1/x)$$

with coefficients $B_j(1/x) = x^{-N}b_j(x)$ that are polynomials in $1/x$. Denote by v their maximal degree in $1/x$ and set $B_j(1/x) = (1/x^v)(\gamma_j + o(x))$. By Borel transform, D' is changed into the linear differential operator (*cf.* Sect. 4.3.2.2)

$$\Delta' = B_n\left(\frac{d}{d\xi}\right)\xi^n + B_{n-1}\left(\frac{d}{d\xi}\right)\xi^{n-1} + \cdots + B_0\left(\frac{d}{d\xi}\right)$$

and $\widehat{f}(\xi)$ satisfies the equation $\Delta'\widehat{y} = 0$.

Lemma 6.3.16. — The set \mathcal{S} of the singular points of the equation $\Delta'\widehat{y} = 0$ is the set of the Stokes values associated with \widehat{f} in the equation $Dy = 0$.

Proof. — Since the Newton polygon at 0 of D (and D') has the two slopes 0 and 1 we must have $b_0 = 0$ and $b_n \neq 0$ (*cf.* proof of Prop. 4.3.22). The operator Δ' has order v and, since $\frac{d^k}{d\xi^k}\xi^\ell = \xi^\ell \frac{d^k}{d\xi^k} +$ “lower order terms” and $b_0 = 0$, it reads

$$\Delta' = \left(\sum_{j=1}^n \gamma_j \xi^j\right) \frac{d^v}{d\xi^v} + \text{“lower order terms”}.$$

Hence, the singular points of the equation $\Delta'\widehat{y} = 0$ are the zeroes of the polynomial $\sum_{j=1}^n \gamma_j \xi^j = 0$ which obviously vanishes for $\xi = 0$. However, this polynomial is also, up to a power of ξ , the 1-characteristic polynomial of D and we saw (*cf.* Sect. 4.3.2.3) that the non-zero Stokes values associated

with $\tilde{f}(x)$ are the roots of the various characteristic polynomials. In our case, since the Newton polygon of D has no other slopes than 0 and 1, there is no other characteristic polynomial than the 1-characteristic polynomial and we can conclude that the singular points of the equation $\Delta'\hat{y} = 0$ are all the Stokes values of the equation $Dy = 0$ including 0. \square

It follows from the Cauchy-Lipschitz Theorem that $\hat{f}(\xi)$ can be analytically continued along any path drawn in \mathbb{C} which avoids the (finite) set \mathcal{S} of singular points of the equation $\Delta'\hat{y} = 0$. The domain to which extend the convergent series $\hat{f}(\xi)$ is then the Riemann surface, named $\mathcal{R}_{\mathcal{S}}$, which is made of (the terminal end of) all homotopy classes in $\mathbb{C} \setminus \mathcal{S}$ of paths issuing from 0 and bypassing all points of \mathcal{S} . Only homotopically trivial paths are allowed to turn back to 0. The surface $\mathcal{R}_{\mathcal{S}}$ looks very much like the universal cover of $\mathbb{C} \setminus \mathcal{S}$ but the fact that 0 is not a branch point in the first sheet (we always start with a convergent power series $\hat{f}(\xi)$).

This property is named *resurgence* and we can state:

Lemma 6.3.17. — *The series $\tilde{f}(x)$ is resurgent with singular support the set \mathcal{S} of Stokes values associated with $\tilde{f}(x)$ in the equation $Dy = 0$, i.e., its Borel transform $\hat{f}(\xi)$ is convergent and can be analytically continued to the surface $\mathcal{R}_{\mathcal{S}}$.*

Now, consider a sector $\delta_{\mathcal{R}_{\mathcal{S}}}$ on $\mathcal{R}_{\mathcal{S}}$ with image $I \times]R, +\infty[$ in \mathbb{C} where the arc $I = \{\theta_1 < \theta < \theta_2\}$ is a bounded arc of directions and suppose it contains no singular point of the equation ($(I \times]R, +\infty[) \cap \mathcal{S} = \emptyset$). One reaches $\delta_{\mathcal{R}_{\mathcal{S}}}$ from 0 following a \mathcal{C}^1 -path γ of finite length in $\mathbb{C} \setminus \mathcal{S}$.

We keep denoting by $\hat{f}(\xi)$ the analytic continuation of $\hat{f}(\xi)$ to $\mathcal{R}_{\mathcal{S}}$.

Lemma 6.3.18. — *The series $\tilde{f}(x)$ is summable-resurgent on $\mathcal{R}_{\mathcal{S}}$, i.e., its Borel transform $\hat{f}(\xi)$ has exponential growth of order 1 at infinity on any sector $\delta_{\mathcal{R}_{\mathcal{S}}}$ with bounded opening on $\mathcal{R}_{\mathcal{S}}$.*

Proof. — This is a direct consequence of Proposition 4.3.22. If the opening of $\delta_{\mathcal{R}_{\mathcal{S}}}$ were not bounded we could turn infinitely many times around ∞ adding exponential terms at each turn. Hence, the necessity to bound the opening of $\delta_{\mathcal{R}_{\mathcal{S}}}$. \square

▷ Now, turn back to the general case when D has meromorphic, not necessarily algebraic, coefficients.

Theorem 6.3.19. — *Let D be a linear differential operator with meromorphic (convergent) coefficients at 0 and suppose D has the unique level $k = 1$. Then, any series $\tilde{f}(x)$ solution of $Dy = 0$ is summable-resurgent with singular support the Stokes values associated with $\tilde{f}(x)$ in the equation $Dy = 0$ (cf. Def. 6.3.15).*

Proof. — The Algebraisation Theorem of Birkhoff (see [Bir09] or [Sib90, thm. 3.3.1]) says that to any linear differential operator D with meromorphic coefficients at 0 there exists a meromorphic transformation which changes D into an operator D' with polynomial coefficients. The two equations $Dy = 0$ and $D'y = 0$ have the same determining polynomials, hence the same set \mathcal{S} of Stokes values associated with a series $\tilde{f}(x)$ in $Dy = 0$ or associated with its image after meromorphic transformation in $D'y = 0$. The operator D' is relevant of Lemma 6.3.18. The Borel transform of a convergent series is an entire function with exponential growth of order 1 at infinity.

We thus have to prove that, if a function $\varphi(\xi)$ is defined on all of $\mathcal{R}_{\mathcal{S}}$ with exponential growth of order 1 at infinity and $g(\xi)$ is an entire function with exponential growth of order 1 at infinity then $g * \varphi$ is well defined on all of $\mathcal{R}_{\mathcal{S}}$ and has exponential growth of order 1 at infinity.

Since g and φ are both analytic near the origin 0 their convolution product is well defined near 0 by the integral $g * \varphi(\xi) = \int_0^\xi g(\xi - t)\varphi(t) dy$. Given any path γ from 0 to ξ in $\mathbb{C} \setminus \mathcal{S}$ (but its starting point 0), the integral $\int_\gamma g(\xi - t)\varphi(t) dy$ is well defined and determines the analytic continuation of $g * \varphi$ along γ . Changing the path γ into a homotopic one does not affect the result according to Cauchy's Theorem. Hence, $g * \varphi$ is well defined on all of $\mathcal{R}_{\mathcal{S}}$. The fact that this function has exponential growth of order 1 at infinity follows from the fact that the convolution of exponentials $e^{A\xi}$ and $e^{B\xi}$ of order 1 is itself a combination of exponentials of order 1. \square

Remark 6.3.20. — The previous theorem is valid for *all series* appearing in a formal fundamental solution of a linear differential equation with meromorphic coefficients and unique level $k = 1$ even those that come with a complex power of x or in a formal-log sum. This follows from the fact that they are themselves solution of a (another) linear differential equation of the same type. This is easily seen on systems: in a formal fundamental solution

$\tilde{F}(x)x^L e^Q(1/x)$ the factor $\tilde{F}(x)$ is solution of the homological system, itself with meromorphic coefficients and unique level 1.

6.4. Fourth approach: wild analytic continuation

The fourth definition of k -summability deals with wild analytic continuation, that is, continuation of the series in the infinitesimal neighborhood of 0 (cf. Sect. 4.5).

6.4.1. k -summability. — A Gevrey series is a germ at 0 of the sheaf \mathcal{F} . We call *wild analytic continuation* any of its continuations as sections of \mathcal{F} . A series $\tilde{f}(x)$ which is k -summable on a k -wide arc I can be wild analytically continued to a domain containing the disc $D(0, k)$ and the sector $\{(\theta, k'); \theta \in I \text{ and } 0 < k' \leq +\infty\}$. These conditions are not quite sufficient to characterize k -summable series on I since the set of global sections of \mathcal{F} over the open disc $D(0, k)$ is isomorphic to

$$\mathbb{C}[[x]]_{s+} := \varprojlim_{\varepsilon \rightarrow 0+} \mathbb{C}[[x]]_{s+\varepsilon} = \bigcap_{\varepsilon > 0} \mathbb{C}[[x]]_{s+\varepsilon} \supset \mathbb{C}[[x]]_s$$

and is thus, bigger than $\mathbb{C}[[x]]_s$. As for the set of global sections of \mathcal{F} over the closed disc $\bar{D}(0, k)$, it is isomorphic to

$$\mathbb{C}[[x]]_{s-} := \varinjlim_{\varepsilon \rightarrow 0+} \mathbb{C}[[x]]_{s-\varepsilon} = \bigcup_{\varepsilon > 0} \mathbb{C}[[x]]_{s-\varepsilon} \subset \mathbb{C}[[x]]_s$$

and smaller than $\mathbb{C}[[x]]_s$ (cf. Prop. 4.5.3). The right domain lies in between $D(0, k)$ and $\bar{D}(0, k)$. It can be made explicit in the sheaf space (X^k, \mathcal{F}^k) since, indeed, the set of global sections of \mathcal{F}^k over the closure $\bar{D}(0, \{k, 0\})$ in X^k of the open disc $D(0, \{k, 0\})$ is isomorphic to $\mathbb{C}[[x]]_s$ (cf. Prop. 4.5.5).

We can then state the following new definition of k -summability:

Definition 6.4.1 (k -summability). — Let I be a k -wide arc of S^1 (cf. Def. 6.1.2). A series $\tilde{f}(x) = \sum_{n \geq 0} a_n x^n$ is k -summable on I if it can be wild analytically continued to a domain containing the closed disc $\bar{D}(0, \{k, 0\})$ and the sector $I \times]0, +\infty]$. We call such a domain a k -sector in X^k .

The definition above being the exact translation of Ramis-Sibuya definition of k -summability it is equivalent to all previous definitions of k -summability.

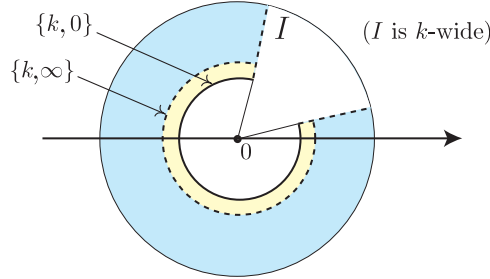


Figure 13. Domain for a k -sum in X^k (in white)

6.4.2. Applications. — 1. *Tauberian Theorems* — Let us revisit the Martinet-Ramis Tauberian Theorem 1 (Cor. 6.3.12) from the viewpoint of wild analytic continuation.

Consider a k_1 -summable series $\tilde{f}(x)$ on I_1 . In the viewpoint of wild analytic continuation this property translates in the space $(X^{k_1}, \mathcal{F}^{k_1})$ (cf. Sect. 4.5.2 p. 93) as the condition that the series $\tilde{f}(x)$ admits a continuation as a section of the sheaf \mathcal{F}^{k_1} to the k_1 -sector $\mathcal{A}_{k_1, I_1} = \overline{D}(0, \{k_1, 0\}) \cup (I_1 \times]0, +\infty])$ (see Fig. 13). The fact that it is k_2 -summable on I_2 has a similar interpretation in the space $(X^{k_2}, \mathcal{F}^{k_2})$. To interpret both we need to work in the space $(X^{k_1, k_2}, \mathcal{F}^{k_1, k_2})$.

The Tauberian Theorem says that, given $k_1 < k_2$ and $I_2 \subset I_1$, the fact that $\tilde{f}(x)$ be k_1 -summable on I_1 (i.e., that it can be continued to the k_1 -sector \mathcal{A}_{k_1, I_1}) and that it be also s_2 -Gevrey (i.e., that it can be continued to the disc $D_{k_2} = \overline{D}(0, \{k_2, 0\})$) implies that it can be continued to the k_2 -sector \mathcal{A}_{k_2, I_2} . Clearly, \mathcal{A}_{k_2, I_2} is included in $\mathcal{A}_{k_1, I_1} \cup D_{k_2}$.

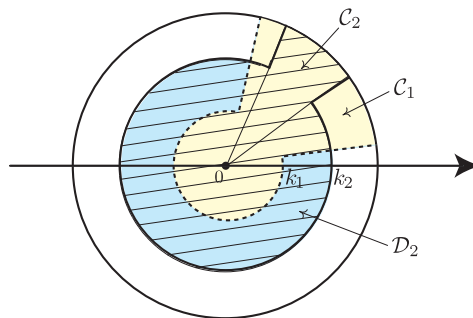


Figure 14. \mathcal{A}_{k_1, I_1} (in yellow) and \mathcal{A}_{k_2, I_2} (hachured) in X^{k_1, k_2}

The theorem asserts that, on the intersection $\Delta_{k_1, I_1} \cap D_{k_2}$, the two continuations agree. This is the case on D_{k_1} since there is a unique continuation of $\tilde{f}(x)$ to D_{k_1} (cf. R-S Cor. 6.2.2 p. 119). The compatibility of the two continuations on $(D_{k_2} \setminus D_{k_1}) \cap \Delta_{k_1, I_1}$ means that their difference belongs to the space $H^0(I_1, \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2})$. The Relative Watson's Lemma below (Thm. 8.2.1 p. 175) asserts that such a space reduces to the null section. Hence, the two continuations agree and define a k_2 -sum of $\tilde{f}(x)$ on I_2 .

2. Functions of k -summable series

Proposition 6.4.2. — *Let be given a k -wide arc I and r series $\tilde{f}_1(x), \dots, \tilde{f}_r(x)$ that are k -summable on I with k -sums $f_1(x), \dots, f_r(x)$ respectively.*

Assume that $\tilde{f}_1(0) = \dots = \tilde{f}_r(0) = 0$.

If $g(x, y_1, \dots, y_r)$ is an analytic function on a neighborhood of 0 in \mathbb{C}^{r+1} then, the series $g(x, \tilde{f}_1(x), \dots, \tilde{f}_r(x))$ is k -summable on I with k -sum $g(x, f_1(x), \dots, f_r(x))$.

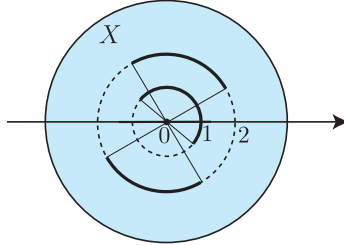
Proof. — According to Proposition 2.3.6 the expression $g(x, \tilde{f}_1(x), \dots, \tilde{f}_r(x))$ determines a well-defined s -Gevrey series, hence, a germ at 0 of the sheaf \mathcal{F}^k which can be continued to the closed disc $\bar{D}(0, \{k, 0\})$.

The series $\tilde{f}_1(x), \dots, \tilde{f}_r(x)$ being k -summable on I and vanishing at $x = 0$ can be continued to the sector $I \times]0, +\infty]$ with values in an arbitrary small neighborhood of 0. The function g being holomorphic on a neighborhood of 0 the series $g(x, \tilde{f}_1(x), \dots, \tilde{f}_r(x))$ can also be continued to the sector $I \times]0, +\infty]$ with analytic continuation $g(x, f_1(x), \dots, f_r(x))$. \square

3. Summability of solutions of differential equations

Let $\tilde{f}(x)$ be a series solution of a linear differential equation (or system). The fences to the wild analytic continuation of $\tilde{f}(x)$ in X or in any space X^k, X^{k_1, k_2}, \dots are the big points of the exponentials $(\exp(q_j(1/x)))_{j \in J}$ appearing in a formal fundamental solution. Indeed, when a direction passes a big point it exits the definition domain of the associated exponential and flat terms become undefined.

Recall that, in X , the big points associated with an exponential of degree k are the closed arcs of length π/k bisected by the anti-Stokes directions of the exponential (directions of maximal decay) and lying on the circle of radius k in X . In X^k they are arches based on the previous arcs. Below, are drawn the big points of two exponentials, one of degree 1 and one of degree 2 in X .



A series solution of a linear differential equation where only these two exponentials appear is k -summable on any k -sector containing none of these big points. Also, it is (k_1, k_2) -summable on any (k_1, k_2) -sector containing none of these big points. In particular, one can check easily that it is $(1, 2)$ -summable in almost all directions (here, all directions but the three anti-Stokes directions).

When $\tilde{f}(x)$ is a series solution of a non linear differential equation then the same procedure applies to the linearized equation along $\tilde{f}(x)$.

CHAPTER 7

TANGENT-TO-IDENTITY DIFFEOMORPHISMS AND BIRKHOFF NORMALISATION THEOREM

7.1. Introduction

This chapter deals with the conjugacy of tangent-to-identity germs of diffeomorphisms at 0. It aims at showing another example (not solution of a differential equation) where the Gevrey cohomological analysis is also efficient.

We consider the by-now classical case of a germ of “translation”

$$g : x \mapsto g(x) = \frac{x}{1+x}.$$

As a homography, g is defined over the whole Riemann sphere $\overline{\mathbb{C}}$. In the chart of infinity, setting $z = 1/x$ and $G(z) = 1/g(x)$, the germ g reads

$$G : z \mapsto G(z) = z + 1$$

hence, the name of translation.

Convention. — As previously, we denote by x the coordinate about 0 and by $z = 1/x$ the coordinate about infinity. We denote by the same letter a given germ in the chart of 0 and in the chart at infinity, using a small letter at 0 and the corresponding capital one at infinity.

In this context, the formal and meromorphic gauge transformations of the classification of linear differential systems are replaced by formal and convergent tangent-to-identity diffeomorphisms $\tilde{h}(x) = x + \sum_{n \geq 2} c_n x^n$ acting on g by conjugacy, that is, by changing g into $\tilde{h}^{-1} \circ g \circ \tilde{h}$.

Definition 7.1.1. — A germ f is formally conjugated (or analytically conjugated) to g if there exists $\tilde{h}(x) = x + \sum_{n \geq 2} c_n x^n$ formal (or convergent) satisfying the conjugacy equation

$$(46) \quad \tilde{h} \circ f = g \circ \tilde{h}.$$

One can check that such an \tilde{h} exists if and only if f has the form

$$f(x) = x - x^2 + x^3 + \sum_{n \geq 4} a_n x^n$$

and

$$\tilde{h}(x) = x + \sum_{p \geq 2} c_p x^p$$

is unique after c_2 is fixed, say, to $c_2 = 0$. In the chart of infinity, setting $z = 1/x$ and $F(z) = 1/f(x)$, the condition reads

$$F(z) = z + 1 + \sum_{n \geq 2} \frac{A_n}{z^n} \quad (\text{observe } A_1 = 0)$$

and \tilde{H} is unique in the form

$$\tilde{H}(z) = z + \sum_{p \geq 1} \frac{C_p}{z^p}.$$

Notice that, in its formal class, g has the particularity of being best behaved with respect to iteration and, thus, plays the role of a *normal form*.

From now on, the diffeomorphisms $\tilde{h}(x) = x + \sum_{p \geq 2} c_p x^p$ by which we conjugate are supposed to satisfy $c_2 = 0$. We denote the group of the so normalized germs of formal tangent-to-identity diffeomorphisms of \mathbb{C} at 0, endowed with composition, by

$$\tilde{\mathbb{G}} = \left\{ x + \sum_{n \geq 3} c_n x^n \in \mathbb{C}[[x]] \right\}$$

and the subgroup of convergent germs of $\tilde{\mathbb{G}}$ by

$$\mathbb{G} = \left\{ x + \sum_{n \geq 3} c_n x^n \in \mathbb{C}\{x\} \right\}.$$

With this normalization, a conjugacy map \tilde{h} when one exists is unique. It might be divergent although f and g are both convergent. One can prove, for instance, that a sufficient condition for \tilde{h} to be divergent is that f be an entire function.

As for linear differential systems the analytic classification of the conjugacy classes of diffeomorphisms is performed inside each formal class with a given normal form, here g . Our aim is not to longly develop that classification but to give a proof of the main point in the given example of the translation g , that is to say, to prove that the conjugacy maps \tilde{h} of g are 1-summable series.

A natural approach consists in analyzing the Borel transform of $\tilde{h}(x)$ following so J. Écalle [É74] (cf. D. Sauzin, section 14, this volume). We choose to develop here a sectorial approach due to Kimura [Kim71, Thm. 6.1] (see also [Bir39, première partie, § 5]); the proof is based on the Ramis-Sibuya Theorem (Thm. 6.2.1) after constructing an adequate 1-quasi-sum.

We consider the following sheaves:

▷ $\mathcal{G} = \{f \in \mathcal{A}; Tf \in \tilde{\mathbb{G}}\}$ the subsheaf of \mathcal{A} made of (normalized) tangent-to-identity germs of diffeomorphisms (Recall that \mathcal{A} is the sheaf over S^1 of germs of asymptotic functions at 0; cf. Sect. 3.1.5) and

▷ $\mathcal{G}^{<0} = \{f \in \mathcal{G}; Tf = id\}$ the subsheaf of \mathcal{G} made of its flat germs (“flat” in a multiplicative context means “asymptotic to identity”).

Equipped with composition law, \mathcal{G} is a sheaf of non commutative groups and $\mathcal{G}^{<0}$ a subsheaf of groups.

Given $0 < \alpha < \pi$ we consider in the chart of infinity the sectors

$$\begin{aligned} \Delta_+(\alpha, R) &= \{z; -\alpha < \arg(z - R) < \alpha\}, \\ \Delta_-(\alpha, R) &= \{z; \pi - \alpha < \arg(z + R) < \pi + \alpha\}. \end{aligned}$$

$\Delta_+(\alpha, R)$ and $\Delta_-(\alpha, R)$ are symmetric to each other with respect $z = 0$.

We denote by $\delta_+(\alpha, R)$ and $\delta_-(\alpha, R)$ their image in the coordinate $x = 1/z$.

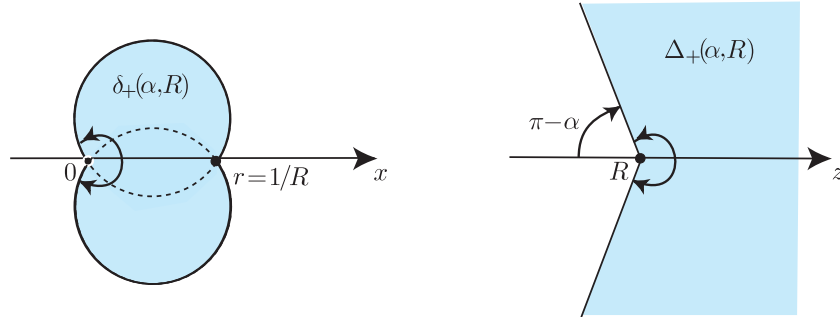


Figure 1

Given g a germ of diffeomorphism we denote its p^{th} power of compoition by

$$g^p = \underbrace{g \circ g \circ \cdots \circ g}_{p \text{ times}}.$$

7.2. Birkhoff-Kimura Sectorial Normalization

Although we state the theorem in a chart of 0 (coordinate x), as we are use to do it, it is worth, taking into account the very simple expression of $G(z)$, to perform the proof in the chart of infinity (coordinate z). Let us start with a technical lemma.

Lemma 7.2.1. — *Let $0 < \alpha_0 < \pi$ and $R_0 > 1$ be given. For all $m \in \mathbb{N}^*$, there exists a constant $c > 0$ which depends on α_0 and m but not on R_0 such that,*

$$\sum_{p \geq 0} \frac{1}{|z+p|^{m+1}} \leq \frac{c}{|z|^m} \quad \text{for all } z \in \Delta_+(\alpha_0, R_0).$$

Proof. — The proof is elementary. We compare the sum to an integral as soon as possible, *i.e.*, as soon as the general term of the series decreases and we estimate the extra terms.

Given $z \in \Delta_+(\alpha_0, R_0)$ let us denote by $p(z) + 1 \geq 0$ the smallest integer such that $\Re(z + p(z) + 1) > 0$. Notice that $p(z) \leq \max(0, |z| \cos(\pi - \alpha_0))$ and $|z| \geq R_0 \sin \alpha_0$ for all z in $\Delta_+(\alpha_0, R_0)$. We split the series into

$$\sum_{p \geq 0} \frac{1}{|z+p|^{m+1}} = \sum_{p=0}^{p(z)+1} \frac{1}{|z+p|^{m+1}} + \sum_{p \geq p(z)+2} \frac{1}{|z+p|^{m+1}}.$$

We claim first that $|z+p| \geq |z| \sin \alpha_0 > 0$ for all $p \in \mathbb{N}$ and $z \in \Delta_+(\alpha_0, R_0)$; for, $|z+p| \geq |z|$ when $\Re(z) \geq 0$ and $|z+p| \geq \Im(z) = |z| \sin \theta \geq |z| \sin \alpha_0$ when $\Re(z) < 0$ since then $\pi/2 < \theta < \alpha_0$. It follows that

$$\sum_{p=0}^{p(z)+1} \frac{1}{|z+p|^{m+1}} \leq \frac{1}{|z|^{m+1}} + \frac{p(z)+1}{(|z| \sin \alpha_0)^{m+1}} \leq \frac{c_1}{|z|^m}$$

for a constant c_1 depending on α_0 and m but not on $R_0 > 1$. Indeed, we have

$$\frac{1}{|z|} + \frac{p(z)+1}{|z|(\sin \alpha_0)^{m+1}} \leq \frac{1}{R_0 \sin \alpha_0} + \frac{\cos(\pi - \alpha_0)}{(\sin \alpha_0)^{m+1}} + \frac{1}{R_0(\sin \alpha_0)^{m+2}}$$

and, since $R_0 > 1$, we can choose $c_1 = 3/(\sin \alpha_0)^{m+2}$.

Starting from $p = p(z) + 1$ the function $p \mapsto \frac{1}{|z+p|^{m+1}}$ decreases and we have

$$\begin{aligned} \sum_{p \geq p(z)+2} \frac{1}{|z+p|^{m+1}} &\leq \int_{p(z)+1}^{+\infty} \frac{dp}{|z+p|^{m+1}} = \int_0^{+\infty} \frac{dq}{|z+p(z)+1+q|^{m+1}} \\ &= \frac{1}{|z+p(z)+1|^m} \int_0^{+\infty} \frac{dr}{(1+r)^{m+1}} \leq \frac{c_2}{|z|^m} \end{aligned}$$

for a constant $c_2 = 1/(\sin \alpha_0)^m$; indeed, $|z + p(z) + 1| \geq |z| \sin \alpha_0$ and since $m \geq 1$, we can write

$$\int_0^{+\infty} \frac{dr}{(1+r)^{m+1}} \leq \int_0^{+\infty} \frac{dr}{(1+r)^2} = 1.$$

Hence, the result if one chooses the constant $c = c_1 + c_2$. \square

Theorem 7.2.2 (Birkhoff-Kimura sectorial normalization)

Let φ be a flat diffeomorphism over a proper sub-arc $I_{\alpha_0}^+ =]-\alpha_0, +\alpha_0[$ of S^1 (i.e., $0 < \alpha_0 < \pi$ and $\varphi \in H^0(I_{\alpha_0}^+; \mathcal{G}^{<0})$).

Then, the diffeomorphism $g_1 = \varphi \circ g$ belongs to $H^0(I_{\alpha_0}^+; \mathcal{G})$ and is uniquely conjugated to g via a section of $\mathcal{G}^{<0}$: there exists a unique $\phi_+ \in H^0(I_{\alpha_0}^+; \mathcal{G}^{<0})$ such that

$$\phi_+ \circ g_1 = g \circ \phi_+ \quad \text{on } I_{\alpha_0}^+.$$

Symmetrically, denote by $I_{\alpha_0}^- = [-\alpha_0 + \pi, \alpha_0 + \pi]$ the arc opposite to $I_{\alpha_0}^+$ on S^1 and suppose that $\varphi \in H^0(I_{\alpha_0}^-; \mathcal{G}^{<0})$. Then, there exists a unique $\phi_- \in H^0(I_{\alpha_0}^-; \mathcal{G}^{<0})$ such that

$$\phi_- \circ g_1 = g \circ \phi_- \quad \text{on } I_{\alpha_0}^-.$$

Proof. — We make the proof over $I_{\alpha_0}^+$. The proof on $I_{\alpha_0}^-$ is similar when applied to g^{-1} and g_1^{-1} . The fact that $g_1 = \varphi \circ g$ be a diffeomorphism on $I_{\alpha_0}^+$ and have a Taylor expansion is clear since so do φ and g . Its Taylor expansion is equal to $Tg_1 = T\varphi \circ Tg = \text{id} \circ Tg = Tg$. Turn now to the variable z and denote by the corresponding capital letters the diffeomorphisms in the chart of infinity.

Given $\alpha < \alpha_0$ choose $\alpha_1 \in]\alpha, \alpha_0[$ and $R_1 > 1$ so that $G_1(z)$ be well defined on $\Delta_+(\alpha_1, R_1)$. Denoting $K(z) = G_1(z) - G(z)$ and $\phi_+(z) = z + \psi_+(z)$ the condition $\phi_+ \circ g_1 = g \circ \phi_+$ becomes $K(z) + \psi_+ \circ G_1(z) - \psi_+(z) = 0$. A solution will be given by $\psi_+(z) = \sum_{p \geq 0} K \circ G_1^p(z)$ if we prove that the series $\sum_{p \geq 0} K \circ G_1^p(z)$ converges to a holomorphic function asymptotic to 0 at infinity.

\triangleright Given $R > R_1 + 2$, the function $K(z)$ being asymptotic to 0 on $\Delta_+(\alpha_1, R_1)$ it satisfies: for all $m \in \mathbb{N}$, there exists $a > 0$ such that

$$(47) \quad |K(z)| \leq \frac{a}{|z|^{m+1}} \quad \text{on } \Delta_+(\alpha, R_1 + 1) \supset \Delta_+(\alpha, R - 1).$$

The constant a depends on m, α_1 and R_1 but not on R . Below, $R > R_1 + 2$ will be chosen conveniently large.

▷ Prove that, there exists a constant $A \geq a$ and independent of R such that,

$$(48) \quad \sup_{\substack{|z-z'| \leq \sin \alpha \\ z \in \Delta_+(\alpha, R)}} |K(z')| \leq \frac{A}{|z|^{m+1}}.$$

Indeed, the conditions $z \in \Delta_+(\alpha, R)$ and $|z - z'| \leq \sin \alpha$ imply $z' \in$

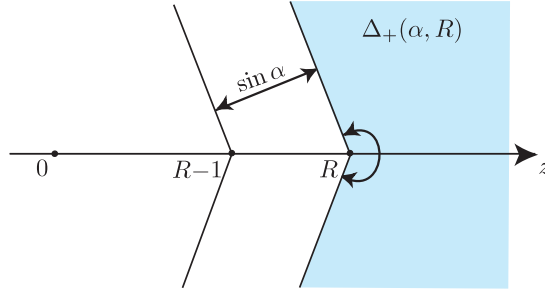


Figure 2

$\Delta_+(\alpha, R - 1)$. We can then apply Condition (47) to yield

$$|K(z')| \leq \frac{a}{|z'|^{m+1}} \leq \frac{a}{(|z| - \sin \alpha)^{m+1}} \leq \frac{A}{|z|^{m+1}}$$

with $A = a \left(\frac{R_1}{R_1 - 1} \right)^{m+1}$ since

$$\sup_{z \in \Delta_+(\alpha, R)} \frac{|z|}{|z| - \sin \alpha} \leq \sup_{|z| \geq R_1 \sin \alpha} \frac{|z|}{|z| - \sin \alpha} = \frac{R_1}{R_1 - 1}.$$

▷ Choose R so large that, for all $m \geq 1$,

$$(49) \quad A \sum_{p \geq 0} \frac{1}{|z + p|^{m+1}} \leq \sin \alpha \quad \text{for all } z \in \Delta_+(\alpha, R).$$

Such a choice is possible. Indeed, from Lemma 7.2.1 applied to $\Delta_+(\alpha_1, R_1)$, we obtain

$$A \sum_{p \geq 0} \frac{1}{|z + p|^{m+1}} \leq \frac{Ac}{|z|^m}$$

and the constant Ac does not depend on R . Now, $\max_{z \in \Delta_+(\alpha, R)} \frac{Ac}{|z|^m} = \frac{Ac}{(R \sin \alpha)^m}$. Assuming R large enough so that $R \sin \alpha > 1$ then, $\frac{Ac}{(R \sin \alpha)^m} \leq \frac{Ac}{R \sin \alpha}$ which is independent of m and can be made arbitrarily small by choosing R large.

▷ Prove by induction on p that, for all $p \geq 1$ and all $z \in \Delta_+(\alpha, R)$,

$$(50) \quad |G_1^p(z) - (z + p)| \leq A \sum_{q=0}^{p-1} \frac{1}{|z + q|^{m+1}}$$

(recall the notation $g^p = \underbrace{g \circ g \circ \cdots \circ g}_p$ times).

When $p = 1$, the inequality reads $K(z) \leq A/|z|^{m+1}$ and follows from Condition (47) with $a < A$. Suppose Condition (50) valid up to p . Then, from Condition (49), we get $|G_1^p(z) - (z + p)| \leq \sin \alpha$. And this implies that:

- (i) $\lim_{p \rightarrow \infty} G_1^p(z) = \infty$ for all $z \in \Delta_+(\alpha, R)$;
- (ii) $G_1^p(z) \in \Delta_+(\alpha, R)$ since $z + p \in \Delta_+(\alpha, R + p) \subset \Delta_+(\alpha, R + 1)$;
- (iii) $|K \circ G_1^p(z)| \leq \frac{A}{|z + p|^{m+1}}$ (Estimate (48) applied to $z' = G_1^p(z)$ and $z + p$ for z).

Since $G_1^p(z) \in \Delta_+(\alpha, R)$ it can be applied $G_1 = G + K$. We can then write

$$G_1^{p+1}(z) = G(G_1^p(z)) + K(G_1^p(z)) = G_1^p(z) + 1 + K \circ G_1^p(z)$$

from which we deduce $G_1^{p+1}(z) - (z + p + 1) = G_1^p(z) - (z + p) + K \circ G_1^p(z)$. Applying the recurrence hypothesis and Condition (iii) at rank p we obtain

$$|G_1^{p+1}(z) - (z + p + 1)| \leq A \sum_{q=0}^{p-1} \frac{1}{|z + q|^{m+1}} + \frac{A}{|z + p|^{m+1}}$$

which is Condition (50) at rank $p + 1$.

▷ Conclude on ψ_+ . Condition (iii) for all p and $m \geq 1$ proves that the series

$$\sum_{p \geq 0} K \circ G_1^p(z)$$

converges uniformly on compact sets of $\Delta_+(\alpha, R)$. The functions $K \circ G_1^p$ being holomorphic, the sum $\Psi_+(z) = \sum_{p \geq 0} K \circ G_1^p(z)$ is holomorphic on $\Delta_+(\alpha, R)$. Moreover, for all $m \in \mathbb{N}^*$ and all $z \in \Delta_+(\alpha, R)$ (Recall that α and R do not

depend on m), there exist constants A and $c > 0$ such that

$$\begin{aligned} |\Psi_+(z)| &\leq \sum_{p \geq 0} \frac{A}{|z+p|^{m+1}} \quad (\text{Condition (iii)}) \\ &\leq \frac{Ac}{|z|^m} \quad (\text{Lemma 7.2.1 for } \Delta_+(\alpha, R)) \end{aligned}$$

which shows that $\psi_+(z)$ is asymptotic to 0 at infinity.

▷ Proving the uniqueness of the solution resumes to proving that the equation $\psi \circ G_1 - \psi = 0$ has a unique solution asymptotic to 0 on $\Delta_+(\alpha, R)$. And indeed, if we iterate the equation we obtain $\psi \circ G_1^p(z) - \psi(z) = 0$; letting p tend to infinity, we obtain $\psi(z) = \lim_{p \rightarrow +\infty} \psi \circ G_1^p(z) = \lim_{z' \rightarrow \infty} \psi(z')$ according to Condition (i).

Hence, $\psi(z) = 0$ for all $z \in \Delta_+(\alpha, R)$ and the proof is achieved. \square

Actually, Birkhoff [Bir39] and Kimura [Kim71] stated the theorem in the following form (see also [Mal82] and [É74]).

Corollary 7.2.3 (Birkhoff-Kimura). — *Consider the conjugacy equation*

$$(46) \quad \tilde{h} \circ f = g \circ \tilde{h}.$$

With notations as before, there exist unique diffeomorphisms $h_+ \in H^0(I_{\alpha_0}^+; \mathcal{G})$ and $h_- \in H^0(I_{\alpha_0}^-; \mathcal{G})$ such that

$$\begin{aligned} h_+ \circ f &= g \circ h_+ \quad \text{and} \quad T_0 h_+ = \tilde{h} \quad \text{on} \quad I_{\alpha_0}^+, \\ h_- \circ f &= g \circ h_- \quad \text{and} \quad T_0 h_- = \tilde{h} \quad \text{on} \quad I_{\alpha_0}^-. \end{aligned}$$

where $T_0 h$ stands for “Taylor expansion of h at 0”.

Proof. — Again, we develop the proof over $I_{\alpha_0}^+$. We denote by $\delta_+(\alpha, R)$ the image in the chart of 0 of the domain $\Delta_+(\alpha, R)$ as built in the proof of Theorem 7.2.2. The Borel-Ritt Theorem (Thm 2.4.1 (i)) provides a function h holomorphic on $\delta_+(\alpha, R)$ and with Taylor expansion $\text{Th}(x) = \tilde{f}(x)$ at 0. Consider the function $f_1 = h \circ f \circ h^{-1}$. It has an asymptotic expansion on $\delta_+(\alpha, R)$ given by $\tilde{h} \circ f \circ \tilde{h}^{-1} = g$ according to the conjugacy equation (46). Hence, there exists $\varphi = f_1 \circ g^{-1}$ which is flat and satisfies $f_1 = \varphi \circ g$.

Birkhoff-Kimura Theorem 7.2.2 applied to f_1 and g provides a ψ_+ asymptotic to 0 and satisfying $\psi_+ \circ f_1 = g \circ \psi_+$ and $h_+ = \psi_+ \circ h$ solves the problem on $\delta_+(\alpha, R)$. Uniqueness is proved similarly as for ψ_+ and is valid on $\delta(\alpha, R)$

for all $\alpha < \alpha_0$ (whereas R might depend on α). Hence the existence and uniqueness of h_+ as a section of the sheaf \mathcal{G} over $I_{\alpha_0}^+$.

Symmetrically, we prove the existence and uniqueness of h_- over $I_{\alpha_0}^-$ by the same method. \square

When $\alpha_0 > \pi/2$ the domains of definition of h_+ and h_- overlap across the two imaginary directions.

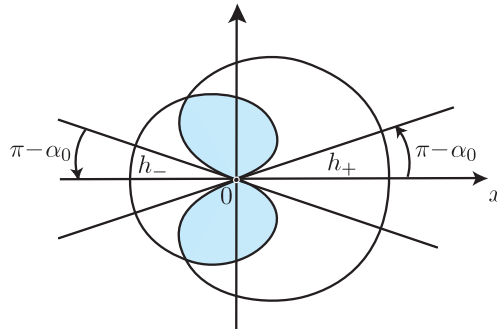


Figure 3

7.3. Invariance equation of g

The invariance equation

$$(51) \quad u \circ g = g \circ u$$

of g is a particular case of the conjugacy equation (46). Hence, it admits the unique solution $u = \text{Id}$ in $\tilde{\mathbb{G}}$ and, given $0 < \alpha_0 < \pi$, it admits a unique solution u_+ section of the sheaf \mathcal{G} over $I_{\alpha_0}^+ =]-\alpha_0, +\alpha_0[$ and a unique solution u_- over $I_{\alpha_0}^- =]\pi - \alpha_0, \pi + \alpha_0[$ asymptotic to g . But due to their uniqueness, since Id is a solution everywhere, then u_+ and u_- are both equal to Id . The situation is different in a neighborhood of the imaginary axis where there might exist non trivial germs of solutions. And indeed, the solution $h_-^{-1} \circ h_+$ might be non trivial depending on f .

In this section we study the behavior of germs of flat solutions near the two imaginary half axis.

Proposition 7.3.1. — *Let $\Delta_1 = \{|x| < r_1; \beta < \arg x < \pi - \beta\}$ with $0 < \beta < \pi/2$ be a sector with vertex 0 neighboring the positive imaginary axis. Any solution $u \in \mathcal{G}(\Delta_1)$ of the invariance equation (51) is exponentially*

flat of order 1 on \mathcal{A}_1 .

The same result holds on a sector $\mathcal{A}_2 = \{|x| < r_2; \beta - \pi < \arg x < -\beta\}$ neighboring the negative imaginary axis.

Proof. — It suffices to consider the case of \mathcal{A}_2 . Again, it is more convenient to work in the chart of infinity. The sector \mathcal{A}_2 is changed into

$$\mathcal{A}_2 = \left\{ |z| > R_2 = \frac{1}{r_2}; \beta < \arg z < \pi - \beta \right\},$$

the solution u is changed into U and we set $U = \text{Id} + V$. With these notations the invariance equation reads

$$(52) \quad V(z+1) = V(z)$$

whose solutions are the 1-periodic functions. Hence, to solutions $U \in \mathcal{G}(\mathcal{A}_2)$ there correspond functions V of the form $V(z) = \nu(e^{2\pi iz})$ that satisfy $\lim_{z \rightarrow \infty} V(z) = 0$.

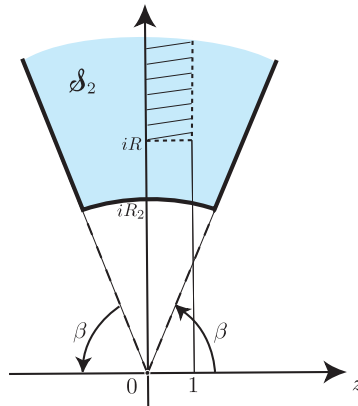


Figure 4

Consider, in \mathcal{A}_2 , a vertical half-stripe $[iR, 1 + iR[\times]iR, +i\infty[$ with width 1 (see Fig. 4, p. 168). It's easily checked that its image by the map $z \mapsto t = e^{2\pi iz}$ is a punctured disc Ω_2 centered at 0 in \mathbb{C} . Moreover, a fundamental system of neighborhoods of infinity in \mathcal{A}_2 is sent on a fundamental system of neighborhoods of 0. Hence, the condition $\lim_{\substack{z \rightarrow \infty \\ z \in \Sigma_1}} \nu(e^{2\pi iz}) = 0$ is equivalent to $\lim_{t \rightarrow 0} \nu(t) = 0$. Consequently, by the Inexisting Singularity Theorem, ν can be continued into a holomorphic function at 0.

Now, suppose ν is not identically 0. Then, it has a finite order, say k at 0 (denote $\nu(t) = O(t^k)$). This implies that $V(z) = O(e^{-2\pi k \Im(z)})$ as z tends

to infinity in \mathcal{D}_2 . However, on \mathcal{D}_2 , one has $\Im(z) > |z| \sin \beta$ and consequently, $V(z) = O(e^{-2\pi k \sin(\beta)|z|})$. Thus, V has (uniform) exponential decay of order one on \mathcal{D}_2 at infinity and so does the solution $u(x) = 1/(1/x + V(1/x))$ at $x = 0$ on \mathcal{D}_2 . \square

7.4. 1-summability of the conjugacy series \tilde{h}

Recall that the conjugacy equation

$$h \circ f = g \circ h \quad (46)$$

admits a unique formal solution $\tilde{h}(x)$ in $\tilde{\mathbb{G}}$. The 1-summability of \tilde{h} is now straightforward.

Theorem 7.4.1. — *The series $\tilde{h}(x)$ is 1-summable with singular directions the two imaginary half-axis.*

Proof. — Let $\pi/2 < \alpha_0 < \pi$ and consider the solutions $h_+(x) \in H^0(I_{\alpha_0}^-, \mathcal{G})$ and $h_-(x) \in H^0(I_{\alpha_0}^+, \mathcal{G})$ of Equation (46), asymptotic to $\tilde{h}(x)$ (cf. Cor. 7.2.3). The non Abelian 1-cocycle defined by $h_1 = h_-^{-1} \circ h_+$ on $] \pi - \alpha_0, \alpha_0 [$ and by $h_2 = h_+^{-1} \circ h_-$ on $] -\alpha_0, \alpha_0 - \pi [$ satisfies the invariance equation (51).

Denote $h_1 = \text{Id} + u_1$ and $h_2 = \text{Id} + u_2$. It follows from the previous section that u_1 and u_2 are exponentially flat of order one on $] \pi - \alpha_0, \alpha_0 [$ and $] -\alpha_0, \alpha_0 - \pi [$ respectively.

To apply the Ramis-Sibuya Theorem to $\tilde{h}(x)$ on the covering $\mathcal{I} = (I_{\alpha_0}^+, I_{\alpha_0}^-)$ of S^1 we must prove that the Abelian 1-cocycle equal to $h_+ - h_-$ on $] \pi - \alpha_0, \alpha_0 [$ and $h_- - h_+$ on $] -\alpha_0, \alpha_0 - \pi [$ is exponentially flat. And indeed, from the form $h_1 = h_-^{-1} \circ h_+ = \text{Id} + u_1$ and $h_2 = h_+^{-1} \circ h_- = \text{Id} + u_2$ of h_1 and h_2 we deduce that

$$\begin{cases} h_+ - h_- = h_- \circ u_1 & \text{on }] \pi - \alpha_0, \alpha_0 [, \\ h_- - h_+ = h_+ \circ u_2 & \text{on }] -\alpha_0, \alpha_0 - \pi [. \end{cases}$$

Hence, the Abelian 1-cocycle is exponentially flat of order one since so are u_1 and u_2 while h_- and h_+ are asymptotic to the identity.

Since α_0 can be chosen arbitrarily close to π we can conclude that the series $\tilde{h}(x)$ is 1-summable in all direction but the two imaginary half axis. \square

One proves that these cocycles are not trivial in general while $\tilde{h}(x)$ is divergent; the non Abelian 1-cocycle (h_1, h_2) classifies the analytic classes of diffeomorphisms $f(x)$ formally conjugated to $g(x)$.

CHAPTER 8

SIX EQUIVALENT APPROACHES TO MULTISUMMABILITY

8.1. Introduction and the Ramis-Sibuya series

We may observe that the examples of series given in the previous chapters that are solution of linear differential equations are all k -summable for a convenient value of k . In Theorem 6.2.5 sufficient conditions are stated for the k -summability of solutions of linear differential equations (k -summability must be understood there in its global meaning, that is, k -summability in almost all directions). Recall that Corollary 6.3.13 asserts that a series both k_1 - and k_2 -summable for two distinct values $k_1 \neq k_2$ of k is necessarily convergent. Though, such a result is no longer valid if one considers k_1 - and k_2 -summability in a given direction θ : as shown in Example 6.3.14, the Leroy series $\tilde{L}(x)$ is both 1- and 2-summable in all directions $\theta \in]-\pi/4, +\pi/4[\bmod \pi$.

A first natural question is to determine whether any series solution of a linear differential equation is k -summable for a convenient value of k . This question, known under the name of *Turrittin problem* although Turrittin after Trjitzinsky, Horn and al. formulated the question in different terms, received a negative answer by J.-P. Ramis and Y. Sibuya in 1984 (published later [RS89]) through a counter-example (*cf.* Exa. 8.1.1). A more intricate summation process called *multisummation* had become necessary.

The counter-example given by J.-P. Ramis and Y. Sibuya with a proof of the fact that the Ramis-Sibuya series is k -summable for no $k > 0$ is as follows.

Example 8.1.1 (Ramis-Sibuya series). — The Ramis-Sibuya series is the series

$$\widetilde{\text{RS}}(x) = \widetilde{\text{E}}(x) + \widetilde{\text{L}}(x)$$

sum of the Euler series $\tilde{E}(x) = \sum_{n \geq 0} (-1)^n n! x^{n+1}$ and of the Leroy series $\tilde{L}(x) = \sum_{n \geq 0} (-1)^n n! x^{2n+2}$ deduced from the Euler series by substituting x^2 for x (cf. Exa. 2.2.4 and 6.3.14). From the Euler equation $x^2 y' + y = x$ and the Leroy equation $x^3 y' + 2y = 2x^2$ one deduces that the Ramis-Sibuya series satisfies the Ramis-Sibuya equation

$$\text{RS}(y) = 4x + 2x^2 + 10x^3 - 3x^4$$

where the operator RS reads

$$\text{RS} = x^5(2-x)d^2/dx^2 + x^2(4+5x^2-2x^3)d/dx + 2(2-x+x^2).$$

It is worth to notice that the operator RS admits the following Newton polygon with the two slopes 1 and 2. We will see that this indicates that the series solution of the Ramis-Sibuya equation are all, at worst, (1,2)-summable as defined in the next sections.

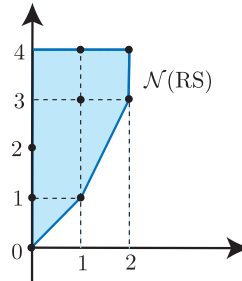


Figure 1. Newton polygon of the Ramis-Sibuya operator

Check that the series $\widetilde{\text{RS}}(x)$ is k -summable for no $k > 0$. Indeed, as we saw earlier (cf. Com. 6.1.9 and Sect. 2.2.2), the Euler series $\tilde{E}(x)$ is 1-summable in all direction but the direction $\theta = \pi$. As a consequence, the Leroy series $\tilde{L}(x)$ is 2-summable in all direction but the directions $\theta = \pm\pi/2$. We saw in Example 6.3.14 that the Leroy series, and then also the Ramis-Sibuya series, is 1-summable in the directions $\theta \in]-\pi/4, +\pi/4[\pmod{\pi}$ and in these directions only. In particular, the Ramis-Sibuya series is not 1-summable. On another hand, the Euler series $\tilde{E}(x)$, and then also the Ramis-Sibuya series, is 2-summable in no direction since its 2-Borel transform does not converge. In particular, the Ramis-Sibuya series is not 2-summable. Consider now a direction $\theta \in]\pi/4, 3\pi/4[\pmod{\pi}$ and show that $\widetilde{\text{RS}}(x)$ is k -summable for no other value of $k > 0$ in direction θ . This is the case for $k > 1$ (and in any direction) from the same argument as for $k = 2$: the k -Borel transform of $\widetilde{\text{RS}}(x)$ does not converge. Suppose there exists $k < 1$ such that $\widetilde{\text{RS}}(x)$ be k -summable in direction θ . Then, since $\widetilde{\text{RS}}(x)$ is a 1-Gevrey series, the Tauberian Theorem (Thm. 6.3.12) of Martinet-Ramis [MR89] (taking $k_1 = k < 1$ and $k_2 = 1$) would imply that $\widetilde{\text{RS}}(x)$ be 1-summable in direction θ . Hence, the contradiction and we can conclude that $\widetilde{\text{RS}}(x)$ is k -summable for no $k > 0$ since this is the case in all direction $\theta \in]\pi/4, 3\pi/4[\pmod{\pi}$.

Let us also sketch another proof of this latter fact which relies on the study of the Stokes phenomenon for $\widetilde{\text{RS}}(x)$ and makes no use of the Tauberian Theorem. A k -sum of $\widetilde{\text{RS}}(x)$ would be a solution of the Ramis-Sibuya equation (cf. Prop. 6.1.10). The space of solutions of the Ramis-Sibuya equation is the sum of the spaces of solutions of the Euler

equation and of the Leroy equation. The Stokes arcs are the arc $[\pi/2, 3\pi/2]$ inherited from $\tilde{E}(x)$ and the two arcs $[\pi/4, 3\pi/4] \bmod \pi$ inherited from $\tilde{L}(x)$. An arc centered at the chosen direction $\theta \in]\pi/4, 3\pi/4[\bmod \pi$ with length $> \pi/k > \pi$ contains a Stokes arc of $\tilde{L}(x)$ and then all solutions of the Leroy equation asymptotic to $\tilde{L}(x)$ must exhibit a Stokes phenomenon with non trivial Stokes automorphisms on that arc. Moreover, the discontinuities, *i.e.*, the Stokes automorphisms of the Leroy series being given in terms of exponentials of order 2 cannot be compensated by the Stokes automorphisms of the Euler series that are given in terms of an exponential of order 1. Consequently, there exists no solution of the Ramis-Sibuya equation asymptotic (and especially, k -asymptotic) to $\widetilde{RS}(x)$ on such an arc and we can again conclude that $\widetilde{RS}(x)$ is not k -summable in any direction $\theta \in]\pi/4, 3\pi/4[\bmod \pi$.

A natural candidate for the sum of $\widetilde{RS}(x)$ is the asymptotic function $E(x) + L(x)$ obtained by adding the 1-sum of $\tilde{E}(x)$ to the 2-sum of $\tilde{L}(x)$. But that choice is not so trivial as we will see soon. The (1,2)-summability of $\widetilde{RS}(x)$ following J. Écalle's approach is widely developed in [LR90].

This example shows that the set of k -summable series for all $k > 0$ is insufficient to embrace all series solutions of linear differential equations. Having in mind to sum solutions of linear differential equations another natural question is the following:

Is it possible to find a (in some way, minimal) set of series endowed with a summation process compatible with the various k -summation processes, that contains all solutions of linear differential equations?

From the example of the Ramis-Sibuya series we understand that any such set should contain the vector space $\sum_{0 < k \leq +\infty} \mathbb{C}\{x\}_k$ of k -summable series for all $k > 0$ ⁽¹⁾. Observe however, with the example of the 1-summable series $\tilde{f}(x) = x/\tilde{E}(x)$, that not all k -summable series are solutions of linear differential equations. Since the derivative and the product of solutions of linear differential equations satisfy themselves linear differential equations such a set should also contain the differential algebra $\text{Alg}_{k>0}$ generated by the spaces $\mathbb{C}\{x\}_k$ for all $k > 0$ including $k = +\infty$. And it results from the factorization theorem of solutions of linear differential systems [Ram85], [LR94, Thm. III.2.5] that the algebra $\text{Alg}_{k>0}$ suffices. As a homomorphism of differential algebras a summation operator \mathcal{S} on $\text{Alg}_{k>0}$, if it exists, is uniquely determined by its values on the spaces $\mathbb{C}\{x\}_k$ generating $\text{Alg}_{k>0}$. The problem lies in the *existence* of the operator. Indeed, the compatibility condition means that the restriction of \mathcal{S} to each space $\mathbb{C}\{x\}_k$ has to be the k -summation

⁽¹⁾ One could also limit the choice to rational $k > 0$ since all levels of linear differential equations are rational.

operator. Hence, given an element of $\text{Alg}_{k>0}$ in the form of a sum of products of k -summable series its sum is obtained by replacing each factor by its k -sum. The point is that an element in $\text{Alg}_{k>0}$ may have several decompositions into sums of products of k -summable series and showing that these decompositions all provide the same sum is not obvious. There exists no direct proof of that fact. A solution is found in developing independently a theory of summation called *multisummation* which extends the k -summation processes with existence and uniqueness of sums and showing that the elements of $\text{Alg}_{k>0}$ are summable in that theory (*cf.* Prop. 8.2.14).

The same question may be addressed in a given direction θ and the results are more precise. It was proved by W. Balsler in [Bal92a] that, under a weak restrictive condition on levels, any multisummable series in direction θ lives in the vector space $\sum_{k>0} \mathbb{C}\{x\}_{k,\theta}$ of k -summable series in direction θ for all $k > 0$. The decomposition is essentially unique (*cf.* Prop. 8.5.1) and again the sum obtained by adding the k -sums of each term coincide with the multisum in any other usual sense whatever the decomposition. However, when the series is multisummable (*i.e.*, multisummable in almost all direction) the decomposition depends on the chosen direction in general (*cf.* Sect. 8.5). One could think of this approach as a good numerical tool based only on simple summation processes. This is not the case since the decomposition into a sum is purely theoretical with no algorithm coming with.

The aim of this chapter is to describe in a general setting various definitions of multisummability. Of course, we look forward to the same properties as those of k -summation, *i.e.*, uniqueness, homomorphism of \mathbb{C} -differential algebras, . . . We also compare these various approaches to prove their equivalence. Comparison being not evaluation, our aim is not to grade the different approaches. None approach can be considered as being the best, none as being the worst. But any of them might be better than another one depending on the question to answer.

All along the chapter we use the Ramis-Sibuya series as our reference example.

8.2. First approach: asymptotic definition

In this section, we generalize the asymptotic approach of k -summability (*cf.* Sect. 6.1) to the case of several levels $k_1 < k_2 < \dots < k_\nu$. Watson's Lemma has to be replaced by the so-called *Relative Watson's Lemma* although the

Relative Watson's Lemma is not a parametric version of the classical Watson's Lemma.

8.2.1. Relative Watson's Lemma. — The Relative Watson's Lemma is due to B. Malgrange and J.-P. Ramis [MR92].

Theorem 8.2.1 (Relative Watson's Lemma). — *Let $0 < k_1 < k_2$ be given and let I be a k_1 -wide arc (cf. Def. 6.1.2). Then,*

$$H^0(I; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2}) = 0.$$

When the length $|I|$ of I is smaller than 2π the arc I may be supposed to belong to S^1 . Otherwise, it must be considered as an arc of the universal cover \mathbb{R} of S^1 . This latter case can be reduced to the first one by an adequate ramification of the variable x .

Compare Corollary 6.1.4 of Watson's Lemma. Instead of considering a k_1 -exponentially flat function on a k_1 -wide arc I one considers here a k_1 -exponentially flat 0-cochain with jumps (its 1-coboundary) small enough to be k_2 -exponentially flat. Roughly speaking, the theorem says that *the 0-cochain has too small jumps on a too large arc I to be not k_2 -exponentially flat itself.*

In [MR92] the lemma is stated for closed k_1 -wide arcs. It is equivalent to choose either closed or open k_1 -wide arcs. Indeed, if I is closed then an element of $H^0(I; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2})$ is represented by a 0-cochain that lives on a larger open arc I' . If the lemma is true for open arcs then the cochain is 0 in $H^0(I'; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2})$ and induces 0 in $H^0(I; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2})$. Conversely, suppose $H^0(I'; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2}) = 0$ for any closed k_1 -wide arc I' . Let I be an open k_1 -wide arc and $f = (f_j)_{j \in J}$ a 0-cochain in $H^0(I; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2})$ associated with a covering $\mathcal{I} = (I_j)_{j \in J}$ of I . Up to refining the covering \mathcal{I} we can assume that it is indexed by \mathbb{Z} and satisfies $I_j \cap I_\ell \neq \emptyset$ if $|j - \ell| = 1$ and $I_j \cap I_\ell = \emptyset$ otherwise (and thus, in particular, it has no 3-by-3 intersection), since there exists arbitrarily fine such coverings of I . Write I as an increasing union of closed k_1 -wide sub-arcs I'_ℓ . Due to the form of the covering \mathcal{I} any open arc I_j is contained in infinitely many closed arcs I'_ℓ ; choose one of them denoted by I'_{ℓ_j} . Then, the restriction of the 0-cochain f to I'_{ℓ_j} induces 0 in $H^0(I'_{\ell_j}; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2})$. This means, in particular, that f_j belongs to $\mathcal{A}^{\leq -k_2}(I_j)$. This being true for all $j \in \mathbb{Z}$ we can conclude that f induces 0 in $H^0(I; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2})$.

The Relative Watson's Lemma can be reformulated as follows.

Corollary 8.2.2. — *Under the conditions of the Relative Watson's Lemma the following natural map is injective:*

$$H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_2}) \longrightarrow H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_1}).$$

Proof. — Consider the short exact sequence

$$0 \longrightarrow \mathcal{A}^{\leq -k_1}/\mathcal{A}^{\leq -k_2} \longrightarrow \mathcal{A}/\mathcal{A}^{\leq -k_2} \longrightarrow \mathcal{A}/\mathcal{A}^{\leq -k_1} \longrightarrow 0.$$

The associated long exact sequence of cohomology over I provides the exact sequence

$$0 \longrightarrow H^0(I; \mathcal{A}^{\leq -k_1}/\mathcal{A}^{\leq -k_2}) \longrightarrow H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_2}) \longrightarrow H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_1}).$$

Hence, the equivalence of the Relative Watson's Lemma 8.2.1 and its Corollary 8.2.2. \square

One can find in [MR92] a direct proof of the Relative Watson's Lemma (see also [Mal95]). Instead of reproducing it we prefer to include a proof of the equivalence between the Tauberian Theorem 6.3.12 and the Relative Watson's Lemma 8.2.1 [MR92, Sect. 3 (ii)].

Lemma 8.2.3 (Malgrange-Ramis [MR92, Lemme (2.5)])

Let $1/2 < k_1 < k_2$ and a closed k_1 -wide arc I of S^1 be given.

To any h in $H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ there exist sections h' in $H^0(I; \mathcal{A})$ and h'' in $H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ such that

$$h = (h' \bmod \mathcal{A}^{\leq -k_2}) + h''|_I.$$

The notation $h' \bmod \mathcal{A}^{\leq -k_2}$ stands for the element of $H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ canonically induced by $h' \in H^0(I; \mathcal{A})$. Notice that, due to the condition $1/2 < k_1$, the arc I is less than 2π long. Observe that h'' , unlike h , exists all around S^1 . Roughly speaking the lemma says: *a section of $\mathcal{A}/\mathcal{A}^{\leq -k_2}$ over I can be continued into a section all over S^1 after "translation" by an adequate asymptotic function defined on I .*

Proof. — The section h can be represented as a finite 0-cochain $(h_j)_{j \in J}$ as follows.

Let $J = \{j_1, j_2, \dots, j_p\}$. The components h_j are functions in $\overline{\mathcal{A}}(\Delta_j)$ for some open sectors $\Delta_j = I_j \times]0, r_j[$ with vertex 0; the 2-by-2 intersections of these sectors satisfy the conditions $\Delta_j \cap \Delta_{j+1} \neq \emptyset$ and $\Delta_j \cap \Delta_\ell = \emptyset$ when $|j - \ell| > 1$ and we assume that the global arc $I_1 \cup I_2 \cup \dots \cup I_p$ is less than 2π wide, the union $I_2 \cup \dots \cup I_{p-1}$ is included in I while I_1 and I_p are not. Moreover, the differences $-h_j + h_{j+1}$ belong to $\overline{\mathcal{A}}^{\leq -k_2}(\Delta_j \cap \Delta_{j+1})$ for all $j \in J$. Complete the family $(\Delta_j)_{j \in J}$ into a covering $\mathfrak{J} = \{\Delta_j\}_{j \in J \cup K}$ of S^1 (denote also $\Delta_j = I_j \times]0, r_j[$

for $j \in K$) without 3-by-3 intersections and such that $(\cup_{j \in K} I_j) \cap I = \emptyset$. Consider the 1-cocycle $\dot{h} = (h_{j,j+1})_{j \in J \cup K}$ of \mathcal{D} with values in $\mathcal{A}^{\leq -k_2}$ defined by

$$h_{j,j+1}(x) = \begin{cases} -h_j(x) + h_{j+1}(x) & \text{if } j \text{ and } j+1 \in J \\ 0 & \text{otherwise.} \end{cases}$$

From the Ramis-Sibuya Theorem 6.2.1 and shrinking the sectors Δ_j if necessary, there exist functions $g_j(x)$ belonging to $\overline{\mathcal{A}}_{1/k_2}(\Delta_j)$ (cf. Not. 2.3.8) such that

$$h_{j,j+1} = -g_j + g_{j+1} \quad \text{for all } j \text{ and } j+1 \text{ in } J.$$

We obtain thus the equality $h_j(x) - g_j(x) = h_{j+1}(x) - g_{j+1}(x)$ on $\Delta_j \cap \Delta_{j+1}$ for all j and $j+1$ in J and the functions $h_j(x) - g_j(x)$ glue together into a section $h'(x)$ of $H^0(I; \mathcal{A})$. On another hand, by construction, the g_j 's for $j \in J \cup K$ determine an element $h''(x)$ of $H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ and we obtain

$$h' \bmod \mathcal{A}^{\leq -k_2} + h''_{|\Delta_j} = (h_j - g_j) + g_j = h_j \quad \text{on } \Delta_j \text{ for all } j \in J.$$

Hence, the result. \square

Proposition 8.2.4. — *The Tauberian Theorem (Thm. 6.3.12) and the Relative Watson's Lemma (Thm. 8.2.1) are equivalent.*

Proof. — Let $0 < k_1 < k_2$ and a closed k_1 -wide arc I be given. By means of a convenient ramification we may assume that the arc I is less than 2π long (which implies $k_1 > 1/2$) and this allows us to work on S^1 . As usually, we denote $s_1 = 1/k_1$ and $s_2 = 1/k_2$.

\triangleright Show that the Relative Watson's Lemma implies the Tauberian Theorem 6.3.12. Let the series $\tilde{f}(x)$ be both s_2 -Gevrey and k_1 -summable with sum $f(x)$ on a k_1 -wide arc I . We must prove that $\tilde{f}(x)$ is also k_2 -summable on I .

As a s_2 -Gevrey series and according to Corollary 6.2.2, the series $\tilde{f}(x)$ can be identified to an element of $H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_2})$, i.e., to a 0-cochain $g = (g_j)$ where g_j is asymptotic to $\tilde{f}(x)$ for all j and $g_i - g_j$ takes its values in $\mathcal{A}^{\leq -k_2}$ for all i, j . It follows from the Ramis-Sibuya Theorem 6.2.1 that $g_j(x)$ is actually s_2 -Gevrey (and hence also, s_1 -Gevrey) asymptotic to $\tilde{f}(x)$. The 0-cochain g induces canonically an element of $H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_1})$, thus characterizing $\tilde{f}(x)$ as a s_1 -Gevrey series. Since $f(x)$ is a k_1 -sum of $\tilde{f}(x)$ on I we deduce that $f = g \bmod \mathcal{A}^{\leq -k_1}$ on I (indeed, $f - g_{j|_I}$ is s_1 -Gevrey asymptotic to 0; cf. Prop. 2.3.17). From Corollary 8.2.2 of the Relative Watson's Lemma, it follows that $f = g \bmod \mathcal{A}^{\leq -k_2}$ on I which proves that $\tilde{f}(x)$ is k_2 -summable on I with k_2 -sum $f(x)$. Hence, the result.

▷ *Conversely, show that the Tauberian Theorem 6.3.12 implies the Relative Watson's Lemma.* Let $h(x)$ belong to $H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_2})$. The section $h(x)$ admits a canonical image in $H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_1})$: we specify $h(x) \bmod \mathcal{A}^{\leq -k_2}$ when $h(x)$ is seen as an element of $H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ and $h(x) \bmod \mathcal{A}^{\leq -k_1}$ to denote its canonical image in $H^0(I; \mathcal{A}/\mathcal{A}^{\leq -k_1})$. We must prove (cf. Cor. 8.2.2) that $h(x) = 0 \bmod \mathcal{A}^{\leq -k_1}$ on I implies $h(x) = 0 \bmod \mathcal{A}^{\leq -k_2}$ on I .

From Lemma 8.2.3 there exists $h' \in H^0(I; \mathcal{A})$ and $h'' \in H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ such that $h = (h' \bmod \mathcal{A}^{\leq -k_2}) + h''|_I$. By definition, $h''(x)$ can be seen as a 0-cochain $h'' = (h''_j)$ where the various components h''_j are asymptotic to a same Taylor series $\tilde{h}''(x)$ and $h_i - h_j$ takes its values in $\mathcal{A}^{\leq -k_2}$ for all i, j . And we know, from the Ramis-Sibuya Theorem 6.2.1 that the $h''_j(x)$'s are actually s_2 -asymptotic to $\tilde{h}''(x)$.

The assumption $h(x) = 0 \bmod \mathcal{A}^{\leq -k_1}$ on I implies $h''(x) = -h'(x) \bmod \mathcal{A}^{\leq -k_1}$ on I and the series $\tilde{h}''(x)$ is k_1 -summable on I with k_1 -sum $-h'(x)$ (h' is a true function; see Def.6.2.4). On another hand, by definition, $\tilde{h}''(x)$ is a s_2 -Gevrey series. By the Tauberian Theorem 6.3.12 the series $\tilde{h}''(x)$ is then k_2 -summable on I with the same sum $-h'(x)$. This implies the equality $h''(x) = -h'(x) \bmod \mathcal{A}^{\leq -k_2}$. Hence, $h(x) = 0 \bmod \mathcal{A}^{\leq -k_2}$ as followed. \square

8.2.2. Asymptotic definition of multisummability. — Towards the generalization of the asymptotic definition (cf. Def. 6.1.6, Sect. 6.1) of k -summability we proceed as follows. Begin with the case of two levels $k_1 < k_2$, that is, the case of (k_1, k_2) -summability.

Suppose we are given a series $\tilde{f}(x)$ and two arcs $I_1 \supseteq I_2$ of S^1 respectively k_1 - and k_2 -wide. Set as usually, $s_1 = 1/k_1$.

Definition 8.2.5 ((k_1, k_2)-summability). —

The series $\tilde{f}(x)$ is said to be (k_1, k_2) -summable on (I_1, I_2) with sum (f_1, f_2) if

- (i) $f_1(x)$ belongs to $H^0(I_1; \mathcal{A}_{s_1}/\mathcal{A}^{\leq -k_2})$;
- (ii) $f_2(x)$ belongs to $H^0(I_2; \mathcal{A}_{s_1})$ (thus, is a true asymptotic function on I_2);
- (iii) f_1 and f_2 agree on I_2 , i.e., $f_1|_{I_2} = f_2 \bmod \mathcal{A}^{\leq -k_2}$;
- (iv) f_1 and f_2 are s_1 -Gevrey asymptotic to $\tilde{f}(x)$ on I_1 and I_2 respectively:

$$T_{s_1, I_1} f_1(x) = T_{s_1, I_2} f_2(x) = \tilde{f}(x).$$

To be more precise the sum (f_1, f_2) is also called *multisum* or (k_1, k_2) -sum of $\tilde{f}(x)$ on (I_1, I_2) . Sometimes and especially when I_1 and I_2 have a same bisecting direction θ , one talks of f_2 as a (k_1, k_2) -sum of $\tilde{f}(x)$ on (I_1, I_2)

letting f_1 understood. In the latter case, one also says that f_2 is a (k_1, k_2) -sum of $\tilde{f}(x)$ in direction θ .

Remark 8.2.6. — Suppose I_1 and I_2 are closed arcs. Sections over I_1 or I_2 live then on larger open arcs. From Condition (ii), one can represent f_1 by a 0-cochain containing f_2 as a component. One can also choose a 0-cochain over an open covering of I_1 by arcs with no 3-by-3 intersection and no intersection 2-by-2 on I_2 . Thus, Definition 8.2.5 can be reformulated as follows:

The series $\tilde{f}(x)$ is (k_1, k_2) -summable on (I_1, I_2) if there exists a 0-cochain f , s_1 -Gevrey asymptotic to $\tilde{f}(x)$ on I_1 , that has no jump on I_2 and only k_2 -exponentially flat jumps on $I_1 \setminus I_2$.

The couple (f_1, f_2) where f_1 is the natural image of f in $H^0(I_1; \mathcal{A}_{s_1}/\mathcal{A}^{\leq -k_2})$ and f_2 its restriction to I_2 is a (k_1, k_2) -sum of $\tilde{f}(x)$ on (I_1, I_2) .

Recall that, in general, a 0-cochain which is s_1 -Gevrey asymptotic to a given series may have jumps (its coboundary) as large as k_1 -exponentially flat (Prop. 2.3.17). The condition that the jumps are k_2 -exponentially flat is strong and guaranties the uniqueness of the (k_1, k_2) -sum of $\tilde{f}(x)$ on (I_1, I_2) as we show below.

Watson's Lemma and the Relative Watson's Lemma give sense to the notion of (k_1, k_2) -summability by implying the uniqueness of (k_1, k_2) -sums.

Proposition 8.2.7 (Uniqueness of (k_1, k_2) -sums). —

The multisum (f_1, f_2) of $\tilde{f}(x)$ on (I_1, I_2) , when it exists, is unique.

Proof. — Suppose (f_1, f_2) and (f'_1, f'_2) are (k_1, k_2) -sums of $\tilde{f}(x)$ on (I_1, I_2) . By Prop. 2.3.17 the difference $f_1 - f'_1$ belongs to $H^0(I_1; \mathcal{A}^{\leq -k_1}/\mathcal{A}^{\leq -k_2})$ and the Relative Watson's Lemma (Thm. 8.2.1) implies that $f_1 - f'_1 = 0$. This, in turn, implies that $f_2 = f'_2 \bmod \mathcal{A}^{\leq -k_2}$ and from the classical Watson's Lemma (Thm. 6.1.3) that $f_2 = f'_2$ since I_2 is k_2 -wide. \square

Example 8.2.8. — The Ramis-Sibuya series $\widetilde{\text{RS}}(x)$ (Exa. refRSseries) is $(1, 2)$ -summable on (I_1, I_2) if and only if I_1 does not contain the Stokes arc of the Euler series $[\pi/2, 3\pi/2]$ and I_2 contains none of the two Stokes arcs of the Leroy series $[\pi/4, 3\pi/4] \bmod \pi$.

Make explicit the $(1, 2)$ -summability of $\widetilde{\text{RS}}(x)$ according to Definition 8.2.5 above in the case when, for instance, $I_1 = [0, \pi]$ and $I_2 = [0, \pi/2] \subset I_1$. Choose $0 < \varepsilon < \pi/4$ and consider the open covering \mathcal{I} of I_1 by the arcs $I'_1 =]\pi/2, \pi + \varepsilon[$ and $I'_2 =]-\varepsilon, \pi/2 + \varepsilon[$. Notice that $I'_1 \cap I_2 = \emptyset$ as mentioned to be a possible choice in Remark 8.2.6. Denote temporarily by $E(x)$ the determination of the Euler function defined on $-\varepsilon < \arg(x) < \pi + \varepsilon$. Denote by E'_1 and E'_2 the restrictions of E to I'_1 and I'_2 respectively. Clearly, the

0-cochain (E'_1, E'_2) of \mathcal{I} takes its values in \mathcal{A}_1 and has a trivial coboundary $E'_1 - E'_2 \equiv 0$. Denote by L'_1 and L'_2 the 2-sums of $L(x)$ on I'_1 and I'_2 respectively. The 0-cochain (L'_1, L'_2) of \mathcal{I} takes its values in $\mathcal{A}_{1/2}$ hence also in \mathcal{A}_1 and its coboundary $L'_1 - L'_2$ belongs to $H^0(I'_1 \cap I'_2; \mathcal{A}^{\leq -2})$. It follows that the 0-cochain $(E'_1 + L'_1, E'_2 + L'_2)$ determines an element f_1 of $H^0(I_1; \mathcal{A}_1/\mathcal{A}^{\leq -2})$. Denote by f_2 the element of $H^0(I_2; \mathcal{A}_1)$ defined by $E'_2 + L'_2$. Both f_1 and f_2 are 1-asymptotic to the Ramis-Sibuya series $\widetilde{\text{RS}}(x)$. The couple (f_1, f_2) is the $(1, 2)$ -sum of $\widetilde{\text{RS}}(x)$ on (I_1, I_2) .

Conversely, if I_1 contained the Stokes arc $[\pi/2, 3\pi/2]$ of the Euler series $\widetilde{\text{E}}(x)$ then we would have to use two different determinations of E on I_1 generating a non trivial coboundary with values in $\mathcal{A}^{\leq -1}$ and not in $\mathcal{A}^{\leq -2}$ and, thus, Condition (i) of Definition 8.2.5 would fail. If I_2 contained a Stokes arc $[\pi/4, 3\pi/4]$ or $[-3\pi/4, -\pi/4]$ of the Leroy series $\widetilde{\text{L}}(x)$ then we would have to split the 2-sum L'_2 of $\widetilde{\text{L}}(x)$ into a 0-cochain with non trivial coboundary on I_2 and Condition (ii) of Definition 8.2.5 would fail.

Let us now state the general case.

Definition 8.2.9 (multi-level \underline{k} and \underline{k} -multi-arc). —

▷ We call multi-level, and we denote by $\underline{k} = (k_1, k_2, \dots, k_\nu)$, any finite sequence of numbers k_1, k_2, \dots, k_ν satisfying the conditions

$$0 < k_1 < k_2 < \dots < k_\nu.$$

▷ We call \underline{k} -multi-arc, and we denote by $\underline{I} = (I_1, I_2, \dots, I_\nu)$, any sequence of arcs satisfying the conditions

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_\nu;$$

for $j = 1, \dots, \nu$, the arc I_j is k_j -wide (cf. Def. 6.1.2).

Observe that our levels are always ordered in increasing order: $k_1 < k_2 < \dots < k_\nu$ while some authors order them in decreasing order.

From now on, suppose we are given a series $\widetilde{f}(x)$, a multi-level $\underline{k} = (k_1, k_2, \dots, k_\nu)$ and a \underline{k} -multi-arc $\underline{I} = (I_1, I_2, \dots, I_\nu)$. As usually, we set $s_1 = 1/k_1$.

Definition 8.2.10 (multisummability). —

A series $\widetilde{f}(x)$ is said to be \underline{k} -summable on \underline{I} with \underline{k} -sum $\underline{f} = (f_1, f_2, \dots, f_\nu)$ if

- ▷ f_j belongs to $H^0(I_j; \mathcal{A}_{s_1}/\mathcal{A}^{\leq -k_{j+1}})$ for all $j = 1, 2, \dots, \nu - 1$;
- ▷ f_ν belongs to $H^0(I_\nu; \mathcal{A}_{s_1})$;
- ▷ the f_j 's are compatible:

$$f_{j|I_{j+1}} = f_{j+1} \bmod \mathcal{A}^{\leq -k_{j+1}} \text{ for all } j = 1, 2, \dots, \nu - 1;$$

▷ for all j , the section f_j is s_1 -Gevrey asymptotic to the series $\tilde{f}(x)$ on I_j :

$$T_{s_1, I_j} f_j(x) = \tilde{f}(x) \text{ for all } j = 1, 2, \dots, \nu.$$

In the case when the arcs I_1, I_2, \dots, I_ν are all bisected by a same direction θ then \underline{f} is called a \underline{k} -sum of $\tilde{f}(x)$ in direction θ . By abuse of language, one sometimes talks of f_ν as a \underline{k} -sum of $\tilde{f}(x)$ on \underline{I} , the components $f_1, f_2, \dots, f_{\nu-1}$ being understood.

Remark 8.2.11. — Remark 8.2.6 can be generalized as follows.

Suppose \underline{I} is a closed \underline{k} -multi-arc.

Definition 8.2.10 is equivalent to saying that

$\tilde{f}(x)$ is \underline{k} -summable on \underline{I} with \underline{k} -sum $\underline{f} = (f_1, f_2, \dots, f_\nu)$ if there exists a 0-cochain f on I_1 which is s_1 -Gevrey asymptotic to $\tilde{f}(x)$, which has no jump on I_ν and otherwise, has jumps that are at least k_ν -exponentially flat on $I_{\nu-1}$, $k_{\nu-1}$ -exponentially flat on $I_{\nu-2}, \dots$ and k_2 -exponentially flat on I_1 .

If so, the \underline{k} -sum \underline{f} is defined by taking as f_j , for $j = 1, 2, \dots, \nu - 1$, the natural image of f in $H^0(I_j; \mathcal{A}_{s_1} / \mathcal{A}^{\leq -k_{j+1}})$ and taking as f_ν the restriction of f to I_ν .

Proposition 8.2.12 (uniqueness). — The \underline{k} -sum $\underline{f}(x)$, when it exists, is unique.

Proof. — The proof proceeds as in the case of two levels (cf. Prop. 8.2.7). Suppose (f_1, f_2, \dots, f_ν) and $(f'_1, f'_2, \dots, f'_\nu)$ are \underline{k} -sums of $\tilde{f}(x)$ on \underline{I} . By Proposition 2.3.17 the difference $f_1 - f'_1$ belongs to $H^0(I_1; \mathcal{A}^{\leq -k_1} / \mathcal{A}^{\leq -k_2})$ and, I_1 being k_1 -wide, the Relative Watson's Lemma (cf. Thm. 8.2.1) implies that $f_1 - f'_1 = 0$. This, in turn, implies that $f_2 - f'_2$ belongs to $H^0(I_2; \mathcal{A}^{\leq -k_2} / \mathcal{A}^{\leq -k_3})$. Again, the Relative Watson's Theorem implies that $f_2 - f'_2 = 0$ and that $f_3 - f'_3$ belongs to $H^0(I_3; \mathcal{A}^{\leq -k_3} / \mathcal{A}^{\leq -k_4})$. And so on, until the ν^{th} step where $f_\nu - f'_\nu$ belongs to $H^0(I_\nu; \mathcal{A}^{\leq -k_\nu})$ and we conclude by the classical Watson's Lemma (Thm. 6.1.3) that $f_\nu = f'_\nu$ since I_ν is k_ν -wide. \square

Notation 8.2.13 (multisummable series). —

Given a multi-level \underline{k} and a \underline{k} -multi-arc \underline{I} we denote by

▷ $\mathbb{C}\{x\}_{\{\underline{k}, \underline{I}\}}$ the set of \underline{k} -summable series on \underline{I} ;

▷ $\mathbb{C}\{x\}_{\{\underline{k}, \theta\}}$ the set of \underline{k} -summable series in direction θ .

▷ $\text{Sum}_{\{\underline{k}, \underline{I}\}}$ the subset of $\prod_{j=1}^{\nu-1} H^0(I_j; \mathcal{A}_{s_1} / \mathcal{A}^{\leq -k_{j+1}}) \times H^0(I_\nu; \mathcal{A}_{s_1})$ made of the elements satisfying the compatibility condition of Definition 8.2.10 (iii).

▷ $\mathcal{S}_{\{\underline{k}, \underline{I}\}} : \mathbb{C}\{x\}_{\{\underline{k}, \underline{I}\}} \longrightarrow \text{Sum}_{\{\underline{k}, \underline{I}\}}$ the \underline{k} -summation operator on \underline{I} which to any \underline{k} -summable series on \underline{I} associates its unique \underline{k} -sum on \underline{I} according to Proposition 8.2.12.

▷ $\text{Sum}_{\{\underline{k}, \theta\}}$ and $\mathcal{S}_{\{\underline{k}, \theta\}}$ instead of $\text{Sum}_{\{\underline{k}, \underline{I}\}}$ and $\mathcal{S}_{\{\underline{k}, \underline{I}\}}$ in direction θ .

We leave as an exercise the proof of the following proposition generalizing Proposition 6.1.10.

Proposition 8.2.14. — Let $\underline{k} = (k_1, k_2, \dots, k_\nu)$ and $\underline{I} = (I_1, I_2, \dots, I_\nu)$ be a multi-level and a \underline{k} -multi-arc (cf. Def. 8.2.9).

(i) The set $\mathbb{C}\{x\}_{\{\underline{k}, \underline{I}\}}$ is a differential sub-algebra of the differential \mathbb{C} -algebra $\mathbb{C}[[x]]_{s_1}$ of s_1 -Gevrey series.

(ii) Let \underline{k}' be a multi-level extracted from \underline{k} and \underline{I}' the corresponding \underline{k}' -multi-arc extracted from \underline{I} .

Then, $\mathbb{C}\{x\}_{\{\underline{k}', \underline{I}'\}}$ is a differential sub-algebra of $\mathbb{C}\{x\}_{\{\underline{k}, \underline{I}\}}$.

In particular, the differential algebras $\mathbb{C}\{x\}_{\{k_j, I_j\}}$ of k_j -summable series on I_j for $j = 1, 2, \dots, \nu$ are differential sub-algebras of $\mathbb{C}\{x\}_{\{\underline{k}, \underline{I}\}}$.

(iii) The Taylor map

$$T_{s_1, \underline{I}} : \text{Sum}_{\{\underline{k}, \underline{I}\}} \longrightarrow \mathbb{C}\{x\}_{\{\underline{k}, \underline{I}\}}$$

is an isomorphism of differential \mathbb{C} -algebras with inverse the \underline{k} -summation operator $\mathcal{S}_{\underline{k}, \underline{I}}$ on \underline{I} .

Remark 8.2.15. — The previous proposition asserts that $\mathbb{C}\{x\}_{\{\underline{k}, \underline{I}\}}$ contains the differential algebra generated by the algebras $\mathbb{C}\{x\}_{\{k_j, I_j\}}$, $j = 1, 2, \dots, \nu$ of k_j -summable series on I_j . It will be shown in Section 8.5 that the two algebras are actually equal.

Although we do not provide an extensive proof of Proposition 8.2.14 let us observe how a k_j -summable series may be regarded as a \underline{k} -summable series. Consider the example of $j = \nu$, all cases being similar. Suppose $\tilde{f}(x)$ is k_ν -summable on I_ν . This means that there exists a function (its k_ν -sum) $f_\nu \in H^0(I_\nu; \mathcal{A}_{s_\nu})$ satisfying $T_{s_\nu, I_\nu} f_\nu(x) = \tilde{f}(x)$. This implies also that the series $\tilde{f}(x)$ is s_ν -Gevrey and the Borel-Ritt Theorem (Cor. 2.4.4) allows to complete the sum f_ν into a 0-cochain f'_ν over I_1 whose components are all s_ν -Gevrey asymptotic (hence, s_1 -Gevrey asymptotic) to $\tilde{f}(x)$ and its coboundary has values in $\mathcal{A}^{\leq -k_\nu}$. Recall that the sheaves $\mathcal{A}^{\leq -k}$ satisfies the inclusions $\mathcal{A}^{\leq -k_\nu} \subset \mathcal{A}^{\leq -k_\nu-1} \subset \dots \subset \mathcal{A}^{\leq -k_1}$. Thus, the 0-cochain f'_ν induces

canonically elements $f_1 \in H^0(I_1; \mathcal{A}_{s_1}/\mathcal{A}^{\leq -k_1})$, $f_2 \in H^0(I_2; \mathcal{A}_{s_1}/\mathcal{A}^{\leq -k_2})$ and so on... and $f_\nu \in H^0(I_\nu; \mathcal{A}_{s_1})$ defined on I_1, I_2, \dots, I_ν respectively and satisfying Def. 8.2.10. Thus, (f_1, f_2, \dots, f_ν) defines the \underline{k} -sum of $\tilde{f}(x)$ on \underline{I} .

To end this section let us mention the fact that Proposition 6.1.11 and its Corollary 6.1.12 remain valid if one replaces k -summability by multisummability.

Recall notations of Sections 2.3.2 and 6.1: given a series $\tilde{g}(t)$ we denote by \tilde{g}_j its r -rank reduced series defined for $j = 0, 1, \dots, r-1$, by $\tilde{g}(t) = \sum_{j=0}^{r-1} t^j \tilde{g}_j(t^r)$; given an arc $I = (\alpha, \beta)$ we denote by $I_{/r}^\ell$ the arc $((\alpha + 2\ell\pi)/r, (\beta + 2\ell\pi)/r)$.

We can state:

Proposition 8.2.16. — *Let $r > 1$ be an integer.*

- : (i) Extension of the variable. *A series $\tilde{f}(x)$ is \underline{k} -summable on $\underline{I} = (I_1, I_2, \dots, I_\nu)$ if and only if the series $\tilde{g}(x) = \tilde{f}(x^r)$ is $r\underline{k}$ -summable on $\underline{I}_{/r} = (I_{1/r}, I_{2/r}, \dots, I_{\nu/r})$.*
- : (ii) Rank reduction. *The series \tilde{g} is $r\underline{k}$ -summable on the arcs $I_{/r}^\ell$ for all $\ell = 0, 1, \dots, r-1$ if and only if the series \tilde{g}_j for $j = 0, 1, \dots, r-1$, are \underline{k} -summable on \underline{I} .*

Proof. — (i) Let $\underline{f}(x) = (f_1(x), f_2(x), \dots, f_\nu(x))$ be the \underline{k} -sum of $\tilde{f}(x)$ on \underline{I} . Then, $\underline{g}(x) = (g_1(x) = f_1(x^r), g_2(x) = f_2(x^r), \dots, g_\nu(x) = f_\nu(x^r))$ is the $r\underline{k}$ -sum of $\tilde{g}(x)$ on $\underline{I}_{/r}$.

(ii) Suppose the series $\tilde{g}_j(x)$ are all \underline{k} -summable on \underline{I} . By definition, these series satisfy $\tilde{g}(t) = \sum_{j=0}^{r-1} t^j \tilde{g}_j(t^r)$. From (i) and Proposition 8.2.14 it follows that $\tilde{g}(t)$ is $r\underline{k}$ -summable on the arcs corresponding to the various determinations of $t = x^{1/r}$. Conversely, use formula $rt^j \tilde{g}_j(t^r) = \sum_{\ell=0}^{r-1} \omega^{\ell(r-j)} \tilde{g}(\omega^\ell t)$ where $\omega = e^{2\pi i/r}$ to conclude that the series $\tilde{g}_j(t^r)$ are $r\underline{k}$ -summable on $I_{/r}^0$ and then, using (i) again, the series $\tilde{g}_j(x)$ are \underline{k} -summable on \underline{I} . \square

Proposition 8.2.16 allows us to assume k large or small at convenience. In what follows, the assumption $k > 1/2$ is quite often convenient.

8.3. Second approach: Malgrange-Ramis definition

For convenience, we assume any $k > 1/2$ so that arcs of length π/k are proper sub-arcs of the circle S^1 . This is always made possible by a convenient change of variable $x = t^r$ according to Proposition 8.2.16. Like in the

Ramis-Sibuya approach for k -summability, the aim is now to get rid of Gevrey asymptotics.

8.3.1. Definition. — Suppose we are given an s_1 -Gevrey series $\tilde{f}(x)$, a multi-level $\underline{k} = (k_1, k_2, \dots, k_\nu)$ and a \underline{k} -multi-arc $\underline{I} = (I_1, I_2, \dots, I_\nu)$. Denote by $\varphi_0 \in H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_1})$ the k_1 -quasi sum of $\tilde{f}(x)$.

Definition 8.3.1 (Malgrange-Ramis multisummability)

The series $\tilde{f}(x)$ is said to be \underline{k} -summable on \underline{I} with \underline{k} -sum $\underline{f} = (f_1, f_2, \dots, f_\nu)$ if

- ▷ f_j belongs to $H^0(I_j; \mathcal{A}/\mathcal{A}^{\leq -k_{j+1}})$ for all $j = 1, 2, \dots, \nu - 1$;
- ▷ f_ν belongs to $H^0(I_\nu; \mathcal{A})$;
- ▷ the f_j 's are compatible:

$$f_j|_{I_{j+1}} = f_{j+1} \bmod \mathcal{A}^{\leq -k_{j+1}} \text{ for all } j = 1, 2, \dots, \nu - 1;$$
- ▷ the f_j 's are compatible with the k_1 -quasi-sum φ_0 of $\tilde{f}(x)$:

$$\varphi_0|_{I_j} = f_j \bmod \mathcal{A}^{\leq -k_1} \text{ for all } j = 1, 2, \dots, \nu.$$

Proposition 8.3.2. — Definitions 8.2.10 and 8.3.1 of multisummability are equivalent (with same sums). In particular, Malgrange-Ramis \underline{k} -sum, when it exists, is unique.

Proof. — Suppose $\underline{f} = (f_1, f_2, \dots, f_\nu)$ is a $\underline{k} = (k_1, k_2, \dots, k_\nu)$ -sum of $\tilde{f}(x)$ on \underline{I} in the sense of Def. 8.3.1. By the Ramis-Sibuya Theorem 6.2.1 the k_1 -quasi-sum φ_0 of $\tilde{f}(x)$ belongs to $H^0(S^1; \mathcal{A}_{s_1}/\mathcal{A}^{\leq -k_1})$ and is asymptotic to $\tilde{f}(x)$. The compatibility Condition (iv) implies that the same is true for \underline{f} ; hence, Definition 8.2.10 is satisfied.

Conversely, suppose that \underline{f} is a \underline{k} -sum of $\tilde{f}(x)$ on \underline{I} following Definition 8.2.10. By the Borel-Ritt Theorem 2.4.1 a 0-cochain representing f_1 can be completed into a 0-cochain over S^1 representing φ_0 ; hence, satisfying Definition 8.3.1. \square

Example 8.3.3. — Let us go back to Example 8.2.8 in view to make explicit Malgrange-Ramis definition for the Ramis-Sibuya series $\widetilde{\text{RS}}(x)$ and prove its (1,2)-summability on the (1,2)-multi-arc $\underline{I} = (I_1, I_2)$ where $I_1 = [0, \pi]$ and $I_2 = [0, \pi/2] \subset I_1$. We want to represent the 1-quasi-sum of $\widetilde{\text{RS}}(x)$ by a 0-cochain with no jump on I_2 , flat jumps of exponential order at most 2 on I_1 and flat jumps of order at most 1 out of I_1 . To this end, choosing again $0 < \varepsilon < \pi/4$, we consider the open covering \mathcal{I}' of S^1 by $I'_1 =]\pi/2, 3\pi/2[$, $I'_2 =]-\varepsilon, \pi/2 + \varepsilon[$ and $I'_3 =]-\pi/2 - \varepsilon, 0[$. Denote by E^+ the determination of the Euler function $E(x)$ defined on $] -\pi/2, 3\pi/2[$ and by E^- its determination

on $] - 3\pi/2, \pi/2[$. Denote by L^+ the 2-sum of the Leroy series $\tilde{L}(x)$ on $] - 3\pi/4, 3\pi/4[$ and by L^- its 2-sum on $] \pi/4, 7\pi/4[$.

The 1-quasi-sum φ_0 of $\tilde{RS}(x)$ is represented by the 0-cochain of \mathcal{I}' defined as follows:

$$\begin{cases} RS_1(x) = E^+(x) + L^-(x) & \text{on } I'_1 \\ RS_2(x) = E^+(x) + L^+(x) & \text{on } I'_2 \\ RS_3(x) = E^-(x) + L^+(x) & \text{on } I'_3. \end{cases}$$

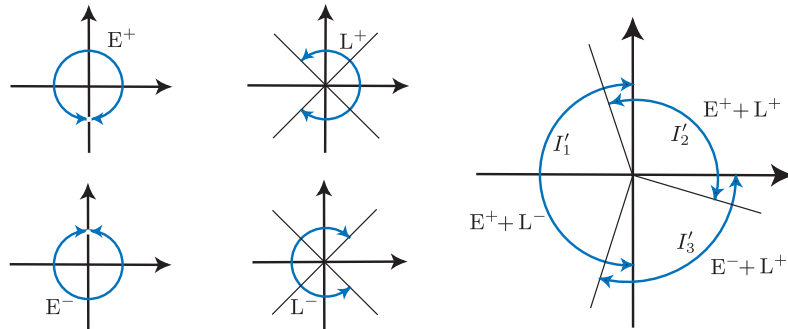


Figure 2

We observe that $RS_1 - RS_2 = L^- - L^+ = ce^{1/x^2}$ (c is the corresponding Stokes multiplier) is exponentially flat of order 2 on $I'_1 \cap I'_2$ while $RS_2 - RS_3$ and $RS_3 - RS_1$ are exponentially flat of order 1 on $I'_2 \cap I'_3$ and $I'_3 \cap I'_1$ respectively. The (1,2)-sum (f_1, f_2) of $\tilde{RS}(x)$ on (I_1, I_2) is given by the restriction of φ_0 to I_1 and I_2 respectively, *i.e.*, for f_1 by the 0-cochain (RS_1, RS_2) and for f_2 by $f_2 = RS_2$.

8.3.2. Application to differential equations. — In this section, we extend Theorem 6.2.5 to the case of several levels.

We consider a linear differential equation (or system) $Dy = 0$ with meromorphic coefficients at 0 and we suppose that the equation $Dy = 0$ has a series solution $\tilde{f}(x)$ with multi-level $\underline{k} = (k_1, k_2, \dots, k_\nu)$ (*cf.* Def. 4.3.6 (iv)). Recall that levels are ordered in increasing order: $k_1 < k_2 < \dots < k_\nu$.

Definition 8.3.4. — A \underline{k} -multi-arc $\underline{I} = (I_1, I_2, \dots, I_\nu)$ is said to be \underline{k} -generic (or simply, generic) for $\tilde{f}(x)$ if, for all j , the arc I_j contains no Stokes arc of level $\leq k_j$ associated with $\tilde{f}(x)$.

Theorem 8.3.5. — A series $\tilde{f}(x)$ as above, *i.e.*, solution of a differential equation with multi-level $\underline{k} = (k_1, k_2, \dots, k_\nu)$, is \underline{k} -summable on any \underline{k} -generic multi-arc \underline{I} .

Proof. — Recall that with no loss of generality we assume that $k_1 > 1/2$.

The Stokes phenomenon says that solutions asymptotic to $\tilde{f}(x)$ may be continued around 0 as long as no Stokes arc is passed. When a Stokes arc of one or several levels k_ℓ is passed to stay asymptotic to $\tilde{f}(x)$ the solution must be added k_ℓ -exponentially flat terms. Thus, an asymptotic solution on I_ν , given, for instance, by the Main Asymptotic Existence Theorem 4.4.2 may be continued by adding k_ν -exponentially flat terms over $I_{\nu-1}$, $k_{\nu-1}$ -exponentially flat terms on $I_{\nu-2}$, \dots , k_2 -exponentially flat terms on I_1 . Outside of I_1 it can be continued into a full 0-cochain over S^1 by allowing jumps k_1 -exponentially small. Hence, a \underline{k} -sum of $\tilde{f}(x)$ on \underline{I} . \square

8.4. Third approach: iterated Laplace integrals

The method is due to W. Balsler [Bal92b]. It proceeds by recursion and is based on the fact that a convenient Borel transform of the series is itself summable with conditions that we make explicit below. Among the known ones this approach is probably the best from a numerical view point to numerically evaluate multisums.

As previously, we develop first the case of two levels (k_1, k_2) .

Suppose we are given $0 < k_1 < k_2$ and a (k_1, k_2) -multi-arc (I_1, I_2) (cf. Def. 8.2.9) and set

$$\frac{1}{\kappa_1} = \frac{1}{k_1} - \frac{1}{k_2}.$$

Denote by θ_1, θ_2 the bisecting directions of I_1, I_2 and by $\widehat{I}_1, \widehat{I}_2$ arcs centered at θ_1, θ_2 with length $|\widehat{I}_1| = |I_1| - \pi/k_2 \geq \pi/\kappa_1$ and $|\widehat{I}_2| = |I_2| - \pi/k_2$ respectively. We assume that I_1 and \widehat{I}_1 , resp. I_2 and \widehat{I}_2 , are simultaneously open or closed⁽²⁾.

Definition 8.4.1 (**(k_1, k_2) -Li-summability**). — *A series $\tilde{f}(x)$ is said to be (k_1, k_2) -summable by Laplace iteration on (I_1, I_2) (in short, (k_1, k_2) -Li-summable on (I_1, I_2)) if its k_2 -Borel transform $\widehat{g}(\xi) = \mathcal{B}_{k_2}(\tilde{f})(\xi)$ satisfies the following two conditions:*

- ▷ $\widehat{g}(\xi)$ is κ_1 -summable on \widehat{I}_1

⁽²⁾ In case I_2 is closed of size $|I_2| = \pi/k_2$ then \widehat{I}_2 reduces to one point. Recall that, handling presheaves or sheaves, asymptotic conditions of sections over a closed set are valid on a convenient larger open set.

▷ its κ_1 -sum $g(\xi)$ can be analytically continued to an unlimited open sector Σ containing $\widehat{I}_2 \times]0, +\infty[$ with exponential growth of order k_2 at infinity.

The (k_1, k_2) -Li-sum $f(x)$ of $\widetilde{f}(x)$ on (I_1, I_2) is defined as

$$f^{\text{Li}}(x) = \mathcal{L}_{k_2, \theta}(g)(x)$$

for all direction $\theta \in \Sigma$ and corresponding x .

It follows from the definition, that the (k_1, k_2) -Li-sum of $\widetilde{f}(x)$ on (I_1, I_2) , when it exists, is unique.

Example 8.4.2. — (even part of the Ramis-Sibuya series)

Again, we illustrate the definition on the example of the Ramis-Sibuya series $\widetilde{\text{RS}}(x)$ (Exa. 8.1.1) for which $\underline{k} = (k_1, k_2) = (1, 2)$ and, again, we choose $\underline{I} = (I_1, I_2)$ with $I_1 = [0, \pi]$ and $I_2 = [0, \pi/2] \subset I_1$. Then, $\widehat{I}_1 = [\pi/4, 3\pi/4]$, $\widehat{I}_2 = \{\pi/4\}$ and $\kappa_1 = 2$ so that we must apply to $\widetilde{\text{RS}}(x)$ a 2-Borel transform. In order to make the calculations simpler we choose to use rank reduction of order two, *i.e.*, to perform the calculations separately on the even and the odd part of the series. Treat now the case of the even part and thus, consider the series $\widetilde{M}(x) = \widetilde{E}^0(x) + \widetilde{L}(x)$ where $\widetilde{E}^0(x) = \sum_{n \geq 1} -(2n-1)! x^{2n}$ is the even part of the Euler series $\widetilde{E}(x)$.

Look first at what happens to $\widetilde{E}^0(x)$ after a 2-Borel transform. The series $\widetilde{E}^0(x)$ satisfies the equation

$$(53) \quad Dy \equiv x^4 y'' + 2x^3 y' - y = x^2.$$

The homogeneous equation $Dy = 0$ admits the two linearly independent solutions $e^{1/x}$ and $e^{-1/x}$. It follows that the anti-Stokes directions for $\widetilde{E}^0(x)$ are the two real half-axis. To apply a 2-Borel transform one has to apply a ramification $x = t^{1/2}$ followed by a 1-Borel transform and the inverse ramification. So, set $x = t^{1/2}$. The series $\widetilde{E}^0(t^{1/2})$ is a series in integer powers of t (this is why we separated the even and the odd part of $\widetilde{\text{RS}}(x)$) and it satisfies the equation

$$4t^3 Y'' + 6t^2 Y' - Y = t.$$

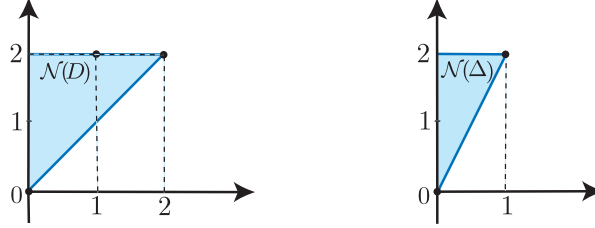
Its 1-Borel transform $\widehat{Y}(\tau)$ satisfies the equation

$$4\tau^2 \widehat{Y}' + (6\tau - 1)\widehat{Y} = 1$$

(Substitute τ for $t^2 d/dt$ and $d/d\tau$ for $1/t$. If we had not restricted the study to an even series, terms in $t^{1/2}$ would appear and this Borel equation would be a much more complicated convolution equation). With the inverse ramification $\tau = \xi^2$, the formal 2-Borel transform $\widehat{U}(\xi) = \widehat{Y}(\tau)$, of $\widetilde{E}^0(x)$, satisfies the equation

$$(54) \quad \Delta \widehat{U} \equiv 2\xi^3 \widehat{U}' + (6\xi^2 - 1)\widehat{U} = 1.$$

The Newton polygon of Δ has a unique slope, equal to 2, at 0 and a null slope at infinity.

Figure 3. Newton polygons of D (slope 1) and Δ (slope 2)

The 1-dimensional space of solutions of the homogeneous equation $\Delta\hat{U} = 0$ is generated by $\hat{u}(\xi) = e^{-1/(4\xi^2)}/\xi^3$. The anti-Stokes directions of equation (54) are then the two real half-axis and, in all other direction θ , the series $\hat{U}(\xi)$ is 2-summable (cf. Thm. 6.2.5). For the choice $\theta = \pi/2$ this means that $\hat{U}(\xi)$ is 2-summable on \hat{I}_1 . Hence, the first condition of Definition 8.4.1 is satisfied. On another hand, the 2-sum $U(\xi)$ of $\hat{U}(\xi)$ satisfies equation (54) which admits 0 and infinity as unique singular points. It can then be analytically continued up to infinity in the direction $\theta = \pi/4$ and neighboring ones, *i.e.*, to an unlimited open sector Σ containing $\hat{I}_2 \times]0, +\infty[$. From the Newton polygon of Δ we know that all solutions of (54) have moderate growth at infinity and the second condition of Definition 8.4.1 is satisfied. It follows that the 2-Laplace transforms $E_\theta^0(x)$ of $U(\xi)$ are defined in all direction $\theta \in \Sigma$. The functions $E_\theta^0(x)$ are analytic continuation from each others since $U(\xi)$ admits no singular point in these directions and therefore, they define the (1,2)-Li-sum $E^0(x)$ of $\tilde{E}^0(x)$ on (I_1, I_2) . To summarize:

$$E_\theta^0(x) = \mathcal{L}_{2,\theta} \circ \mathcal{L}_{2,\theta} \circ \mathcal{B}_{2,\theta} \circ \mathcal{B}_{2,\theta}(\tilde{E}^0(x))$$

where the Borel series are replaced by their sum and analytic continuation when necessary. Notice that the function $E_\theta^0(x)$, although asymptotic to $\tilde{E}^0(x)$, is not 1/2-Gevrey asymptotic since, otherwise, the series $\tilde{E}^0(x)$ would be a 1/2-Gevrey series (cf. 2.3.10), which is not.

Look what happens to the series $\tilde{L}(x)$ in this procedure. The 2-Borel transform of $\tilde{L}(x)$ produces the convergent series $V(\xi) = \sum_{n \geq 0} (-1)^n \xi^{2n} = 1/(1 + \xi^2)$. It is then, 2-summable in all direction, and especially on \hat{I}_1 . It can be continued up to infinity with moderate growth but in the directions $\theta = \pm\pi/2$. It can then be applied a 2-Laplace transform in any direction of an unlimited open sector Σ containing $\hat{I}_2 \times]0, +\infty[$ to define the (1,2)-Li-sum $L(x)$ of $\tilde{L}(x)$ on (I_1, I_2) .

We conclude that the series $\tilde{M}(x)$ is (1,2)-Li-summable on (I_1, I_2) .

Compare (1,2)-sum and (1,2)-Li-sum. Denote by $M^{Li}(x)$ the (1,2)-sum of $\tilde{M}(x)$ on (I_1, I_2) . The (1,2)-Li-sum $E^0(x)$ of $\tilde{E}^0(x)$ can be continued all over I_1 by applying 2-Laplace transforms in directions θ from $\pi/4 - \varepsilon$ to $3\pi/4 + \varepsilon$ (indeed, the unique anti-Stokes directions for $\hat{U}(\xi)$ are $\theta = 0$ and $\theta = \pi$; therefore, the 2-sum $U(x)$ of $\hat{U}(\xi)$ exists. It can be continued with moderate growth at infinity on the unlimited sector $-\pi/4 < \arg(\xi) < 5\pi/4$). Taking 2-Laplace transforms in directions $\theta \in]-\pi/4, -\pi/4[$ allows to complete $E^0(x)$ into an element $\mathcal{E}^0 \in H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -1})$. Similarly, the section L over I_2 can be completed into an element $\mathcal{L} \in H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ and the sum $\mathcal{E}^0 + \mathcal{L}$

provides a k_1 -quasi-sum of $\tilde{M}(x)$ in the form of a section over S^1 with no jump on I_2 and flat jumps of exponential order 2 on I_1 and of order 1 out of I_1 , so that, Definition 8.3.1 is satisfied. By restriction to I_1 and I_2 , this k_1 -quasi-sum determines the $(1, 2)$ -sum (M_1, M_2) of $\tilde{M}(x)$ on (I_1, I_2) in the sense of Definition 8.3.1. It follows that $M_2(x) = M^{\text{Li}}(x)$ on I_2 . This fact is general as it is proved below (Balsler-Tougeron Theorem).

Let us finally observe that the previous procedure can as well be applied to the odd part of $\widetilde{\text{RS}}(x)$ after factoring x . This shows that the procedure applies to the Ramis-Sibuya series $\widetilde{\text{RS}}(x)$ itself giving rise to the same $(1, 2)$ -sum as before.

The main result is as follows.

Theorem 8.4.3 (Balsler-Tougeron: case of two levels)

(k_1, k_2) -Li-summability on (I_1, I_2) and (k_1, k_2) -summability on (I_1, I_2) are equivalent with “same” sum.

Precisely, if f^{Li} denotes the (k_1, k_2) -Li-sum of a series $\tilde{f}(x)$ and (f_1, f_2) its (k_1, k_2) -sum on (I_1, I_2) then, $f_2(x) = f^{\text{Li}}(x)$ on I_2 and thus,

$$f_2(x) = \mathcal{L}_{k_2, \theta'_2} \circ \mathcal{L}_{\kappa_1, \theta'_1} \circ \mathcal{B}_{\kappa_1} \circ \mathcal{B}_{k_2} \circ \tilde{f}(x)$$

when the formula makes sense and especially, for directions θ'_1 and θ'_2 close enough to the bisecting direction θ_2 of I_2 .

The latter formula explains the denomination “by Laplace iteration”.

Proof. — For simplicity of language assume that I_1 and I_2 are closed arcs.

▷ *Prove that (k_1, k_2) -summability implies (k_1, k_2) -Li-summability.*

We use the notations of Definition 8.4.1 and above. In particular, κ_1 is given by the formula $1/\kappa_1 = 1/k_1 - 1/k_2$. By hypothesis, there exists a k_1 -quasi-sum of $\tilde{f}(x)$ which induces the function $f_2(x)$ on I_2 and the 0-cochain $f_1(x)$ on I_1 (using the same notation for the 0-cochain and the element of $H^0(I_1; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ it defines); the coboundary of f_1 has values in $\mathcal{A}^{\leq -k_2}$ (no jump allowed on I_2 and exponentially flat jumps of order at most k_2 on $I_1 \setminus I_2$; cf. Def. 8.3.1). Denote again $\widehat{g}(\xi) = \mathcal{B}_{k_2}(\tilde{f})(\xi)$.

For simplicity, we assume that $f_1(x)$ has only one jump, the case of more jumps being treated similarly. Thus, assume that, in restriction to I_1 (i.e., to a neighborhood of I_1), the k_1 -quasi-sum reduces to two components: $f(x) = f_2(x)$ over an open arc I containing I_2 and $f^*(x)$ over an open arc I^* which we can assume to satisfy $I^* \cap I_2 = \emptyset$ jointly with $I \cup I^* \supset I_1$.

The proof of Theorem 6.3.11 (part (i) \implies (ii)) remains valid for $f(x)$ and $I \supset I_2$ although I is shorter than π/k_1 . Like in Theorem 6.3.11, denote

by \widehat{I} the (open since I is open) arc deduced from I by shortening it of $\pi/(2k_2)$ on both sides. It follows that there exists an unlimited sector Σ containing $\widehat{I} \times]0, +\infty[$ on which there is an analytic function $g(\xi)$ both κ_1 -Gevrey asymptotic to $\widehat{g}(\xi)$ at 0 and having exponential growth of order at most k_2 at infinity. As I is smaller than I_1 the sector Σ has opening smaller than $|\widehat{I}_1| = \pi/\kappa_1$ but contains the direction θ_2 . The question is to analytically continue $g(\xi)$ into a κ_1 -Gevrey asymptotic function on \widehat{I}_1 . To this end, let us use again the variables $Z = 1/x^{k_2}$, $\zeta = \xi^{k_2}$ and notations as in the proof of Theorem 6.3.11 choosing $\theta_2 = 0$ by means of a rotation. In particular, the series $\widetilde{f}(x)$ reads now $\widetilde{F}(Z) = \sum_{n \geq k_0} a_n/Z^{n/k_2}$. We are led to the following situation:

- : \triangleright a function $F(Z)$ satisfying the asymptotic Condition (42) at infinity on a sector
 $\mathfrak{D} = [-\omega_1, +\omega_2] \times [R_0, +\infty[$
 which contains the right half-plane $\Re(Z) > 0$ but the disc $|Z| < R_0$,
- : \triangleright a function $F^*(Z)$ satisfying the asymptotic Condition (42) at infinity on a sector
 $\mathfrak{D}^* = [\omega_1^*, \omega_2^*] \times [R_0, +\infty[$ where, say, $\pi/2 < \omega_1^* < \omega_2 < \omega_2^*$,
- : \triangleright the difference $F(Z) - F^*(Z)$ being exponentially flat of order 1 on the intersection
 $\mathfrak{D} \cap \mathfrak{D}^* = [\omega_1^*, \omega_2] \times [R_0, +\infty[$.

Recall Condition (42) for $F(Z)$ on \mathfrak{D} : there exist $A, C > 0$ such that

$$\left| F(Z) - \sum_{n=k_0}^{N-1} \frac{a_n}{Z^{n/k_2}} \right| \leq C N^{N/k_1} \frac{A^N}{|Z|^{N/k_2}} \quad \text{for all } N \text{ and all } Z \in \mathfrak{D}.$$

We also assume that $-\pi < -\omega_1$ and $\omega_1^* < \omega_2 < \omega_2^* < \pi$. Otherwise, we would proceed in several steps like in the proof of Theorem 6.3.11.

Defining $G(\zeta)$ by $G(\zeta) = 1/(2\pi i) \int_{\gamma} F(U) e^{\zeta U} dU$ where $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ (see Fig 4) is the boundary of \mathfrak{D} provides a function satisfying Condition (ii) of Theorem 6.3.11 on the sector $\Sigma = \{-\omega_2 + \pi/2 < \arg(\zeta) < +\omega_1 - \pi/2\}$ (condition translated in the variable $\zeta = \xi^{k_2}$). Observe that Σ contains the direction $\theta_2 = 0$. Extend now the domain of definition of $G(\zeta)$ by moving γ_1 towards γ_1^* in the direction ω_2^* . To this end, set

$$G^*(\zeta) = \frac{1}{2\pi i} \left(\int_{\gamma_1^*} F^*(U) e^{\zeta U} dU + \int_{\gamma_2 \cup \gamma_3} F(U) e^{\zeta U} dU \right).$$

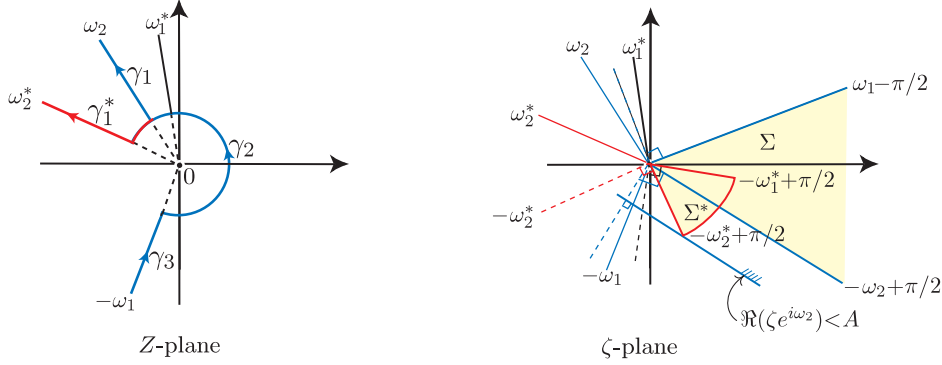


Figure 4

By Cauchy’s Theorem the difference $\Delta(\zeta) = G(\zeta) - G^*(\zeta)$ is given by

$$\begin{aligned} \Delta(\zeta) &= \frac{1}{2\pi i} \int_{\gamma_1} (F(U) - F^*(U)) e^{\zeta U} dU \\ &= \frac{e^{i\omega_2}}{2\pi i} \int_{R_0}^{+\infty} (F(Ve^{i\omega_2}) - F^*(Ve^{i\omega_2})) e^{\zeta V e^{i\omega_2}} dV. \end{aligned}$$

However, by hypothesis, there exists $A > 0$ such that $|F(U) - F^*(U)| \leq e^{-A|U|}$ (the coboundary in the variable x is exponentially flat of order k_2) so that

$$|(F(Ve^{i\omega_2}) - F^*(Ve^{i\omega_2})) e^{\zeta V e^{i\omega_2}}| \leq e^{(-A + \Re(\zeta e^{i\omega_2}))V}$$

and the function $\Delta(\zeta)$ is defined and holomorphic on the half-plane $\Re(\zeta e^{i\omega_2}) < A$. The function $G^*(\zeta) + \Delta(\zeta)$ provides the analytic continuation of $G(\zeta)$ to a (limited) sector Σ^* based on $\tilde{I}^* = [-\omega_2^* + \pi/2, -\omega_1^* + \pi/2]$ near 0. From now on, denote by $G(\zeta)$ this analytic continuation of the initial $G(\zeta)$ to $\Sigma^* \cup \Sigma$. Observe that the arguments above are unable to prove that the analytic continuation on Σ^* can be pushed up to infinity. This might however be possible, for instance, by Theorem 6.3.11, when the series $\tilde{f}(x)$ is not only (k_1, k_2) -summable on (I_1, I_2) but k_1 -summable on I_1 .

Recall Condition (41) for $G(\zeta)$ on Σ : there exist $A', C' > 0$ such that

$$\left| G(\zeta) - \sum_{n=k_0}^{N-1} \frac{a_n}{\Gamma(n/k_2)} \zeta^{n/k_2-1} \right| \leq C' N^{N/\kappa_1} A'^N |\zeta|^{N/k_2-1} \quad \text{for all } N \text{ and all } \zeta \in \Sigma.$$

Like the initial one the new function $G(\zeta)$ satisfy an asymptotic condition of the same type as Condition (41) at 0 on $\Sigma \cup \Sigma^*$ since this is the case for $G(\zeta)$ on Σ and for $G^*(\zeta)$ and $\Delta(\zeta)$ on Σ^* . We can thus conclude that $g(\xi) = G(\xi^{k_2})$

is κ_1 -asymptotic to $\tilde{g}(\xi)$ on \widehat{I}_1 . This achieves the proof of the fact that $\tilde{f}(x)$ is (k_1, k_2) -Li-summable on (I_1, I_2) .

▷ *Prove that (k_1, k_2) -Li-summability implies (k_1, k_2) -summability.*

Conversely, suppose $G(\zeta)$ is defined on $\Sigma \cup \Sigma^*$, has exponential growth at infinity on Σ and satisfies at 0 the asymptotic Condition (41) on $\Sigma \cup \Sigma^*$.

From the proof of Theorem 6.3.11 we know that Laplace transforms in directions belonging to Σ provide a function $F(Z)$ which satisfies Condition (42) on $[-\omega_1, \omega_2]$.

Let $b = |b|e^{i\beta} \in \Sigma \cap \Sigma^*$ with, say, $\beta = -\omega_2 + \pi/2$, and d_β be the half-line issuing from b in direction β . Consider the truncated Laplace transform $F^b(Z) = \int_0^b G(\zeta)e^{-Z\zeta}d\zeta$. We prove, like for P_1 in the proof of Theorem 6.3.11, that $F^b(Z)$ satisfies an estimate of the type of (42) on a half-plane $\Re(Ze^{i\beta}) > 0$ (with new constants). Since $G(\zeta)$ has exponential growth $|G(\zeta)| \leq C''e^{B|\zeta|}$ on d_β the difference $F(Z) - F^b(Z) = \int_{d_b} G(\zeta)e^{-Z\zeta}d\zeta$ satisfies

$$|F(Z) - F^b(Z)| \leq C'' \int_{|b|}^{+\infty} e^{(B - \Re(Ze^{i\beta}))\tau} d\tau$$

and has then exponential decay on the half-plane $\Re(Ze^{i\beta}) > B$ based on the arc $]\omega_2 - \pi, \omega_2[$ (i.e., bisected by $-\beta = \omega_2 - \pi/2$).

Consider now $b^* = |b|e^{i\beta^*}$ with $\beta^* = -\omega_2^* + \pi/2$ in Σ^* .

The function $F^{b^*}(Z) = \int_0^{b^*} G(\zeta)e^{-Z\zeta}d\zeta$ satisfies an estimate of type (51) on the half-plane $\Re(Ze^{i\beta^*}) > 0$ and the difference $F^b(Z) - F^{b^*}(Z) = \int_{\widehat{\beta^*\beta}} G(\zeta)e^{-Z\zeta}d\zeta$ is exponentially small of order one on the intersection of the two half-planes $\Re(Ze^{i\beta}) > 0$ and $\Re(Ze^{i\beta^*}) > 0$ (since $G(\zeta)$ is bounded on the arc $\widehat{\beta^*\beta}$). Turning back to the variable x this provides functions $f(x) = F(x^{k_2})$ and $f^{b^*}(x) = F^{b^*}(x^{k_2})$ that are k_1 -Gevrey asymptotic to $\tilde{f}(x)$ on $I \supset I_2$ and $I^* \supset I_1 \setminus I_2$ respectively, the difference $f(x) - f^{b^*}(x)$ being exponentially flat of order k_2 on $I \cap I^*$. In other words, the couple $(f^{b^*}(x) = F^{b^*}(x^{k_2}), f(x) = F(x^{k_2}))$ defines a (k_1, k_2) -sum of $\tilde{f}(x)$ on (I_1, I_2) .

▷ *Prove the formula.* We have proved that the (k_1, k_2) -Li-sum and the final (k_1, k_2) -sum are both equal to $f(x)$. By construction, $F(Z)$ is the Laplace transform of $G(\zeta)$ in directions belonging to Σ , that is, $f(x)$ is the k_2 -Laplace transform of $g(\xi)$ in directions θ'_2 close to the bisecting line θ_2 of I_2 . The κ_1 -sum $g(\xi)$ of $\widehat{g}(\xi)$ reads $g(\xi) = \mathcal{L}_{\kappa_1, \theta'_1} \circ \mathcal{B}_{\kappa_1}(\widehat{g})(\xi)$ for $\theta'_1 \in \widehat{I}_1$ and $\widehat{g}(\xi) = \mathcal{B}_{k_2}(\tilde{f})(\xi)$

by definition. Hence, the result

$$f(x) = \mathcal{L}_{k_2, \theta'_2} \circ \mathcal{L}_{\kappa_1, \theta'_1} \circ \mathcal{B}_{\kappa_1} \circ \mathcal{B}_{k_2} \circ \tilde{f}(x)$$

for all compatible choices of θ'_1, θ'_2 and x . This ends the proof. \square

Summability by Laplace iteration can be generalized by induction to the case of any multi-level $\underline{k} = (k_1, k_2, \dots, k_\nu)$ as follows.

Let $\underline{I} = (I_1, I_2, \dots, I_\nu)$ be a \underline{k} -wide multi-arc. Recall (cf. Def. 8.2.9) that this means that $0 < k_1 < k_2 < \dots < k_\nu$, the arcs I_1, I_2, \dots, I_ν are respectively k_1 -, k_2 -, \dots , k_ν -wide and satisfy $I_1 \supseteq I_2 \supseteq \dots \supseteq I_\nu$. Denote by $\widehat{I}_1, \widehat{I}_2, \dots, \widehat{I}_\nu$ the arcs deduced from I_1, I_2, \dots, I_ν by truncating an arc of length $\pi/(2k_\nu)$ on both sides of each arc and set $\widehat{\underline{I}} = (\widehat{I}_1, \widehat{I}_2, \dots, \widehat{I}_{\nu-1})$. Define $\widehat{\underline{k}} = (\widehat{k}_1, \widehat{k}_2, \dots, \widehat{k}_{\nu-1})$ by setting

$$\frac{1}{\widehat{k}_j} = \frac{1}{k_j} - \frac{1}{k_\nu} \quad \text{for all } j = 1, 2, \dots, \nu - 1.$$

Definition 8.4.4 (summability by Laplace iteration: general case)

A series $\tilde{f}(x)$ is said to be \underline{k} -summable by Laplace iteration on \underline{I} (in short, \underline{k} -Li-summable on \underline{I}) if its k_ν -Borel transform $\widehat{g}(\xi) = \mathcal{B}_{k_\nu}(\tilde{f})(\xi)$ satisfies the following two conditions:

▷ $\widehat{g}(\xi)$ is $\widehat{\underline{k}}$ -Li-summable on $\widehat{\underline{I}}$,

▷ its $\widehat{\underline{k}}$ -Li-sum $g(\xi)$ can be analytically continued to an unlimited open sector Σ containing $\widehat{I}_\nu \times]0, +\infty[$ with exponential growth of order k_ν at infinity.

The \underline{k} -Li-sum $f(x)$ of $\tilde{f}(x)$ on \underline{I} is defined as $f^{\text{Li}}(x) = \mathcal{L}_{k_\nu, \theta}(g)(x)$ for all direction $\theta \in \Sigma$ and corresponding x .

From the definition, the \underline{k} -Li-sum f^{Li} when it exists is unique.

Denote by $\kappa_1, \kappa_2, \dots, \kappa_\nu$ the numbers given by

$$\frac{1}{\kappa_j} = \frac{1}{k_j} - \frac{1}{k_{j+1}} \quad \text{for } j = 1, 2, \dots, \nu \quad \text{setting } \frac{1}{k_{\nu+1}} = 0$$

or equivalently, by

$$\left\{ \begin{array}{l} \frac{1}{k_\nu} = \frac{1}{\kappa_\nu} \\ \frac{1}{k_{\nu-1}} = \frac{1}{\kappa_\nu} + \frac{1}{\kappa_{\nu-1}} \\ \vdots \\ \frac{1}{k_1} = \frac{1}{\kappa_\nu} + \frac{1}{\kappa_{\nu-1}} + \cdots + \frac{1}{\kappa_1} \end{array} \right.$$

Theorem 8.4.3 can be generalized as follows.

Theorem 8.4.5 (Balsler-Tougeron: general case)

\underline{k} -Li-summability on \underline{I} and \underline{k} -summability on \underline{I} are equivalent with “same” sum.

Precisely, the \underline{k} -Li-sum of a series $\tilde{f}(x)$ with \underline{k} -sum (f_1, f_2, \dots, f_ν) on \underline{I} is equal to f_ν and consequently, f_ν reads

$$f_\nu(x) = \mathcal{L}_{\kappa_\nu, \theta'_\nu} \circ \cdots \circ \mathcal{L}_{\kappa_2, \theta'_2} \circ \mathcal{L}_{\kappa_1, \theta'_1} \circ \mathcal{B}_{\kappa_1} \circ \cdots \circ \mathcal{B}_{\kappa_{\nu-1}} \circ \mathcal{B}_{\kappa_\nu} \circ \tilde{f}(x)$$

when the formula makes sense and, especially, for directions $\theta'_1, \dots, \theta'_\nu$ close to the bisecting direction θ_ν of I_ν and corresponding x .

Proof. — The theorem can be proved by recurrence as follows. It is trivially true for $\nu = 1$ (and proved for $\nu = 2$ in Theorem 8.4.3). Suppose it is true for $\nu - 1$ and prove it for ν . The fact that $\tilde{f}(x)$ be \underline{k} -Li-summable on \underline{I} is now equivalent to the fact $\widehat{g}(\xi)$ be \widehat{k} -summable on \widehat{I} with \underline{k} -sum $(g_1, g_2, \dots, g_{\nu-1})$; and that moreover, $g_{\nu-1}$ be defined with exponential growth of order k_ν at infinity on Σ . The proof that this is equivalent to saying that $\tilde{f}(x)$ is \underline{k} -summable on \underline{I} with sum (f_1, f_2, \dots, f_ν) satisfying $f_\nu = \mathcal{L}_{k_\nu}(g_{\nu-1})$ is similar to the proof of Theorem 8.4.3 but the fact that the 1-cochain f_1 has to be replaced by $(f_1, f_2, \dots, f_{\nu-1})$ with jumps of order k_ν on $I_{\nu-1} \setminus I_\nu$, $k_{\nu-1}$ on $I_{\nu-2} \setminus I_{\nu-1}$, \dots , k_2 on $I_1 \setminus I_2$. We leave the details to the reader.

Suppose by recurrence that the sum $g_{\nu-1}(\xi)$ of $\widehat{g}(\xi)$ satisfies the formula of the theorem computed with values (k_1, k_2, \dots, k_ν) replaced by $(\widehat{k}_1, \widehat{k}_2, \dots, \widehat{k}_{\nu-1})$. Then, the associated values $(\kappa_1, \kappa_2, \dots, \kappa_{\nu-1})$ remain unchanged. Moreover, f_ν is given by $\mathcal{L}_{\kappa_\nu, \theta'_\nu}(g_{\nu-1})$ (observe that $k_\nu = \kappa_\nu$) and the formula follows. This ends the proof. \square

8.5. Fourth approach: Balser's decomposition into sums

Suppose again that we are given a multi-level $\underline{k} = (k_1, k_2, \dots, k_\nu)$ (cf. Def. 8.2.9) and a \underline{k} -multi-arc $\underline{I} = (I_1, I_2, \dots, I_\nu)$. We saw in Proposition 8.2.14 (ii) that a sum $\sum_{j=1}^\nu \tilde{f}_j(x)$ of k_j -summable series on I_j is a \underline{k} -summable series on \underline{I} . We address now the converse question:

Do such splittings characterize \underline{k} -summable series on \underline{I} ?

The answer is yes when $k_1 > 1/2$. Otherwise, one might have to introduce ramified series. The condition $k_1 > 1/2$ is weakened in Theorem 8.6.7.

8.5.1. Case when $k_1 > 1/2$. — Look first at the relations between the various splittings of a given series.

Proposition 8.5.1. — *Splittings are essentially unique.*

Precisely, suppose the series $\tilde{f}(x)$ admits two splittings

$$\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x) + \dots + \tilde{f}_\nu(x) = \tilde{f}'_1(x) + \tilde{f}'_2(x) + \dots + \tilde{f}'_\nu(x)$$

where, for $j = 1, 2, \dots, \nu$, the series $\tilde{f}_j(x)$ and $\tilde{f}'_j(x)$ are k_j -summable on I_j . Then, there exist series $\tilde{u}_j(x)$ such that, for $j = 1, 2, \dots, \nu$,

$$\tilde{f}'_j(x) = \tilde{u}_j(x) + \tilde{f}_j(x) - \tilde{u}_{j+1}(x)$$

where $\tilde{u}_1 = \tilde{u}_{\nu+1} = 0$ and, for $j = 2, \dots, \nu$, the series \tilde{u}_j is k_j -summable on I_{j-1} .

Moreover, the k_j -sums f_j of the \tilde{f}_j 's and f'_j of the \tilde{f}'_j 's satisfy

$$f_1(x) + f_2(x) + \dots + f_\nu(x) = f'_1(x) + f'_2(x) + \dots + f'_\nu(x) \quad \text{on } I_\nu$$

Notice that since $\tilde{u}_j(x)$ is k_j -summable not only on the k_j -wide arc I_j but on the k_{j-1} -wide arc I_{j-1} it is also k_{j-1} -summable on I_{j-1} .

Proof. — The series $\tilde{u}_\nu(x) = \tilde{f}'_\nu(x) - \tilde{f}_\nu(x)$ is k_ν -summable on I_ν and, in particular, is an s_ν -Gevrey series. Being equal to

$$(\tilde{f}_1(x) + \dots + \tilde{f}_{\nu-1}(x)) - (\tilde{f}'_1(x) + \dots + \tilde{f}'_{\nu-1}(x))$$

it is also $(k_1, \dots, k_{\nu-1})$ -summable on $(I_1, \dots, I_{\nu-1})$. From the Tauberian Theorem 8.7.5 we deduce that $\tilde{u}_\nu(x)$ is k_ν -summable on $I_{\nu-1}$ and, a fortiori, is $k_{\nu-1}$ -summable on $I_{\nu-1}$. Applying the same argument to the series $\tilde{f}(x) - \tilde{f}_\nu(x)$ and to its two splittings

$$\tilde{f}_1(x) + \dots + \tilde{f}_{\nu-1}(x) \quad \text{and} \quad \tilde{f}'_1(x) + \dots + \tilde{f}'_{\nu-2}(x) + (\tilde{f}'_{\nu-1}(x) + \tilde{u}_\nu(x))$$

proves the existence of $\tilde{u}_{\nu-1}(x)$ and we conclude to the existence of all \tilde{u}_j 's by decreasing recurrence. The equality of the sums follows directly. \square

We can now give a new definition of multisummability as follows.

Definition 8.5.2. — Assume $k_1 > 1/2$.

▷ A series $\tilde{f}(x)$ is said to be \underline{k} -split-summable on \underline{I} if, for $j = 1, 2, \dots, \nu$, there exist k_j -summable series $\tilde{f}_j(x)$ on I_j such that

$$\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x) + \dots + \tilde{f}_\nu(x).$$

▷ The \underline{k} -split-sum of $\tilde{f}(x)$ on \underline{I} is the function $f(x)$ uniquely defined on I_ν from any splitting of $\tilde{f}(x)$ by

$$f(x) = f_1(x) + f_2(x) + \dots + f_\nu(x),$$

where, for $j = 1, 2, \dots, \nu$, $f_j(x)$ denotes the k_j -sum of $\tilde{f}_j(x)$ on I_j .

Theorem 8.5.3 (Balsler [Bal92a]). — Assume $k_1 > 1/2$.

A series $\tilde{f}(x)$ is \underline{k} -split-summable on \underline{I} if and only if it is \underline{k} -summable on \underline{I} . Moreover, the \underline{k} -split-sum and the \underline{k} -sum agree.

Proof. — The “only if” part was considered in Remark 8.2.15 and above. Prove the converse assertion: if $\tilde{f}(x)$ is \underline{k} -summable on \underline{I} then it is \underline{k} -split-summable on \underline{I} .

Treat first the case when $\nu = 2$. Set $s_1 = 1/k_1$ and $s_2 = 1/k_2$ as usually. The series $\tilde{f}(x)$ being s_1 -Gevrey has a k_1 -quasi-sum $f_0(x) \in H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_1})$ (cf. Def. 6.2.3) and, by hypothesis (cf. Def. 8.3.1), there exists $f_1(x)$ in $H^0(I_1; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ such that $f_1 \bmod \mathcal{A}^{\leq -k_1} = f_0$ on I_1 and there exists $f_2(x)$ in $H^0(I_2; \mathcal{A})$ such that $f_2 \bmod \mathcal{A}^{\leq -k_2} = f_1$ on I_2 . In other words, f_0 can be represented by a 0-cochain φ_0 with values in $\mathcal{A}/\mathcal{A}^{\leq -k_1}$ and satisfying the following properties: its restriction $\varphi_1 = \varphi_0|_{I_1}$ to I_1 represents f_1 and has values in $\mathcal{A}/\mathcal{A}^{\leq -k_2}$; its restriction to I_2 is the asymptotic function $\varphi_0|_{I_2} = f_2$. From Lemma 8.2.3 applied to $f_1(x)$ on I_1 we are given $f'_1(x) \in H^0(I_1; \mathcal{A})$ and $f''_1(x) \in H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_2})$ such that

$$f_1 = f'_1 \bmod \mathcal{A}^{\leq -k_2} + f''_1 \quad \text{on } I_1.$$

There exists then a 0-cochain φ''_1 with values in $\mathcal{A}/\mathcal{A}^{\leq -k_2}$ representing f''_1 which satisfies $\varphi''_1 = \varphi_1 - f'_1$ in restriction to I_1 . From Corollary 6.2.2, $f''_1(x)$ can be identified to an s_2 -Gevrey series $\tilde{f}''_1(x)$ of which $f''_1(x)$ is a k_2 -quasi-sum. In restriction to I_2 the 0-cochain φ''_1 belongs to $H^0(I_2; \mathcal{A})$ since $\varphi''_1|_{I_2} = f_2 - f'_1|_{I_2}$. Therefore, according to Definition 6.2.4, the series $\tilde{f}''_1(x)$ is k_2 -summable on I_2 with k_2 -sum $f_2(x) - f'_1|_{I_2}(x)$. Consider now the 0-cochain $\varphi'_1 = \varphi_0 - \varphi''_1$

which belongs to $H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_1})$ and denote by $\tilde{f}_1(x)$ the s_1 -Gevrey series it defines (cf. Cor. 6.2.2). The 0-cochain φ'_1 has no jump on I_1 since

$$\varphi_{0|_{I_1}} - \varphi''_{1|_{I_1}} = \varphi_1 - (\varphi_1 - f'_1) = f'_1.$$

And this, again by Definition 6.2.4, means that $\tilde{f}_1(x)$ is k_1 -summable on I_1 . We have thus proved that $\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$ where \tilde{f}_1 is k_1 -summable on I_1 and $\tilde{f}_2 = \tilde{f}'_1$ is k_2 -summable on I_2 .

To prove the general case one proceeds by recurrence. It suffices to prove that when $\tilde{f}(x)$ is \underline{k} -summable on \underline{I} there exist a k_ν -summable series $\tilde{f}_\nu(x)$ on I_ν and a \underline{k}' -summable series $\tilde{g}(x)$ on \underline{I}' (where $\underline{k}' = (k_1, k_2, \dots, k_{\nu-1})$ and $\underline{I}' = (I_1, I_2, \dots, I_{\nu-1})$) such that $\tilde{f}(x) = \tilde{g}(x) + \tilde{f}_\nu(x)$. Indeed, let (f_1, f_2, \dots, f_ν) denote the \underline{k} -sum of $\tilde{f}(x)$ on \underline{I} . The k_1 -quasi-sum $f_0(x)$ is now represented by a 0-cochain φ_0 with values in $\mathcal{A}/\mathcal{A}^{\leq -k_1}$ with the following properties: for $j = 1, 2, \dots, \nu$ its restriction φ_j to I_j represents f_j on I_j ; for $j = 1, 2, \dots, \nu-1$ the restriction to I_j has values in $\mathcal{A}/\mathcal{A}^{-k_{j+1}}$ and for $j = \nu$ the restriction to I_ν is $\varphi_{0|_{I_\nu}} = f_\nu$. Apply Lemma 8.2.3 to $f_{\nu-1}$ on $I_{\nu-1}$ to get $f'(x) \in H^0(I_{\nu-1}; \mathcal{A})$ and $f''(x) \in H^0(S^1; \mathcal{A}/\mathcal{A}^{\leq -k_\nu})$ such that

$$f_{\nu-1} = f' \bmod \mathcal{A}^{\leq -k_\nu} + f'' \text{ on } I_{\nu-1}.$$

Like for f'_1 above, the section f'' determines a series $\tilde{f}_\nu(x)$ which is k_ν -summable on I_ν . There exists a 0-cochain φ'' with values in $\mathcal{A}/\mathcal{A}^{\leq -k_\nu}$ which represents f'' and satisfies the condition $\varphi'' = \varphi_{\nu-1} - f'$ on $I_{\nu-1}$. The 0-cochain $\varphi_0 - \varphi''$ shows that the series $\tilde{g}(x) = \tilde{f}(x) - \tilde{f}_\nu(x)$ is \underline{k}' -summable on \underline{I}' . Hence, the result. \square

Remark 8.5.4. — One must be aware of the fact that the splitting strongly depends on the choice of the multi-arc of summation (on the direction of summation if all arcs are bisected by the same direction). It would be interesting to know which series admit a global splitting, *i.e.*, the same splitting in almost all direction.

8.5.2. Case when $k_1 \leq 1/2$. — Choose $r \in \mathbb{N}$ such that $rk_1 > 1/2$. We know from Proposition 8.2.16 (i) that the series $\tilde{f}(x)$ is \underline{k} -summable on \underline{I} if and only if the series $\tilde{g}(x) = \tilde{f}(x^r)$ is $r\underline{k}$ -summable on \underline{I}/r . We can then apply Balser's Theorem 8.5.3 to $\tilde{g}(x)$ to write $\tilde{g}(x) = \sum_{j=1}^\nu \tilde{g}_j(x)$ where the series $\tilde{g}_j(x)$ are rk_j -summable on \underline{I}/r . This way, we obtain a splitting $\tilde{f}(x) = \sum_{j=1}^\nu \tilde{f}_j(x)$ of $\tilde{f}(x)$ by setting $\tilde{f}_j(x) = \tilde{g}_j(x^{1/r})$ for all j . However, the series $\tilde{f}_j(x)$ thus obtained are, in general, ramified series (in the

variable $x^{1/r}$) and the splitting does not fit the statement of Theorem 8.5.3. Actually, as shown by the example below in the case when $k_1 = 1/2$ there might exist no splitting in integer powers of x and we are driven to set the following definition.

Definition 8.5.5. — Suppose $k_1 \leq 1/2$.

A series $\tilde{f}(x)$ is said to be \underline{k} -split-summable on \underline{I} if, given $r \in \mathbb{N}$ such that $rk_1 > 1/2$, the series $\tilde{g}(x) = \tilde{f}(x^r)$ satisfies Definition 8.5.2.

With this definition and Proposition 8.2.16 (i) we can assert in all cases the equivalence of \underline{k} -summability and \underline{k} -split-summability on \underline{I} with “same” sum.

Uniqueness holds as follows:

Let r and r' be such that rk_1 and $r'k_1 > 1/2$. Set $\tilde{g}(x) = \tilde{f}(x^r)$ and $\tilde{g}'(x) = \tilde{f}(x^{r'})$ with splittings $\tilde{g}(x) = \tilde{g}_1(x) + \cdots + \tilde{g}_\nu(x)$ and $\tilde{g}'(x) = \tilde{g}'_1(x) + \cdots + \tilde{g}'_\nu(x)$ respectively. Denote by $R = \rho r = \rho' r'$ the l.c.m. of r and r' . Then, $\tilde{g}_1(x^\rho) + \cdots + \tilde{g}_\nu(x^\rho)$ and $\tilde{g}'_1(x^{\rho'}) + \cdots + \tilde{g}'_\nu(x^{\rho'})$ are two splittings of $\tilde{f}(x^R)$ into Rk_1 -, \dots , Rk_ν -summable series. Henceforth, they are essentially equal (cf. Prop. 8.5.1).

Show now that there might exist no splitting into integer power series.

To this end, consider the case of a multi-level $\underline{k} = (k_1, k_2)$ satisfying

$$1/\kappa_1 := 1/k_1 - 1/k_2 \geq 2.$$

Let (I_1, I_2) be a \underline{k} -multi-arc. Assume, for instance, that I_1 and I_2 are closed with same middle point θ_0 and, by means of a rotation, that $\theta_0 = 0$. For simplicity, assume that $1/\kappa_1 < 4$ and that $|I_2| = \pi/k_2$. Thus, with notations of Sections 6.3.3 and 8.4, the closed arc \widehat{I}_1 centered at 0 with length $|\widehat{I}_1| = |I_1| - \pi/k_2$ overlaps just once (since $2\pi \leq |\widehat{I}_1| < 4\pi$) and \widehat{I}_2 reduces to $\theta = 0$.

Consider a series $\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$ which is the sum of a k_1 -summable series $\tilde{f}_1(x)$ on I_1 and of a k_2 -summable series $\tilde{f}_2(x)$ on I_2 (series in integer powers of x). Denote by $\widehat{g} = \mathcal{B}_{k_2}(\tilde{f})$, $\widehat{g}_1 = \mathcal{B}_{k_2}(\tilde{f}_1)$ and $\widehat{g}_2 = \mathcal{B}_{k_2}(\tilde{f}_2)$ the series deduced from \tilde{f} , \tilde{f}_1 and \tilde{f}_2 by a k_2 -Borel transform. We know from Theorem 6.3.11 that $\widehat{g}_1(\xi)$ is κ_1 -summable on \widehat{I}_1 with κ_1 -sum $g_1(\xi)$. The theorem asserts also that $g_1(\xi)$ has an analytic continuation to an unlimited open sector $\Sigma = \widehat{J}_1 \times]0, +\infty[$ containing $\widehat{I}_1 \times]0, +\infty[$ with exponential growth of order k_2 at infinity. Since Σ is wider than 2π it has a self-intersection $\dot{\Sigma}$ that contains the negative real axis. Narrowing it if necessary we can assume that Σ overlaps just once like \widehat{I}_1 does. Denote by $\dot{g}_1(\xi) = g_1(\xi) - g_1(\xi e^{2\pi i})$

the difference of the two determinations of g_1 on $\dot{\Sigma}$. On the other hand, $\hat{g}_2(\xi)$ is convergent with sum $g_2(\xi)$. Hence, the difference of two determinations $\dot{g}_2(\xi) = g_2(\xi) - g_2(\xi e^{2\pi i})$ is identically 0 near 0 and can then be continued all over $\dot{\Sigma}$ by 0. Set $\dot{g}(\xi) = \dot{g}_1(\xi) + \dot{g}_2(\xi)$. We can thus state:

Lemma 8.5.6. — *The germ $\dot{g}(\xi)$ can be analytically continued all over $\dot{\Sigma}$.*

Proposition 8.5.7. — *With notations and conditions as before (and especially, the condition $1/\kappa_1 := 1/k_1 - 1/k_2 \geq 2$) there exists series that are (k_1, k_2) -summable on (I_1, I_2) but cannot be split into the sum of a k_1 -summable series on I_1 and a k_2 -summable series on I_2 if one restricts the splitting to series in integer powers of x .*

Proof. — To exhibit a counter-example to the splitting of (k_1, k_2) -summable series suppose that $\dot{g}(-1) \neq 0$ and consider the series $\hat{G}(\xi) = \hat{g}(\xi) \cdot \sum_{n \geq 0} (-1)^n \xi^n$. Set then $G(\xi) = g(\xi)/(1 + \xi)$ and $\dot{G}(\xi) = \dot{g}(\xi)/(1 + \xi)$ and denote the k_2 -Laplace transform of the series $\hat{G}(\xi)$ by $\tilde{F}(x) = \mathcal{L}_{k_2}(\hat{G})(x)$. The function $G(\xi)$ is κ_1 -asymptotic to $\hat{G}(\xi)$ on \hat{I}_1 (cf. Prop.2.3.12) and it can be analytically continued with exponential growth of order k_2 at infinity to an unlimited open sector σ containing $\hat{I}_2 \times]0, +\infty[$. Indeed, the function $1/(1 + \xi)$ is bounded at infinity and has a pole at -1. The function $g(\xi)$ is analytic with exponential growth of order k_2 at infinity on an unlimited open sector σ' . In case σ' does not contain the negative real axis one can take $\sigma = \sigma'$; otherwise, set $\sigma = \sigma' \cap \Re(\xi) > 0$ for instance. According to Definition 8.4.1 and Balser-Tougeron Theorem 8.4.3 this shows that the series $\tilde{F}(x)$ is (k_1, k_2) -summable on (I_1, I_2) . However, $\dot{G}(\xi) = \dot{g}(\xi)/(1 + \xi)$ has a pole at $\xi = -1$ which belongs to $\dot{\Sigma}$ and, thus, $\dot{G}(\xi)$ cannot be continued up to infinity over $\dot{\Sigma}$. From the lemma we conclude that the series $\tilde{F}(x)$, however (k_1, k_2) -summable on (I_1, I_2) , is not the sum of a k_1 -summable series on I_1 and of a k_2 -summable series on I_2 if one requires series in integer powers of x . \square

8.6. Fifth approach: Écalle's acceleration

Historically, this approach called *accelero-summation* was the first able to solve the problem of summation in a case of several levels. First introduced by J. Écalle in a very general setting applying to series solutions of non-linear equations and more general functional equations, it was adapted by J. Martinet and J.-P. Ramis to the case of solutions of linear differential equations in [MR91]. The method proceeds by recursion on increasing levels whereas

the iterated Laplace approach runs with decreasing levels. Each step is performed with the use of special integral operators called *accelerators* or *Écalle's accelerators* which involve the successive levels taken two-by-two.

In this section, for simplicity, we work in a given direction θ_0 , *i.e.*, we consider only multi-arcs $\underline{I} = (I_1, I_2, \dots, I_\nu)$ with common middle point θ_0 .

Begin by observing what happens on the example of the Ramis-Sibuya series $\widetilde{\text{RS}}(x)$ (Exa. 8.1.1).

Example 8.6.1 (accelero-summation of $\widetilde{\text{RS}}(x)$). — We saw in Example 8.1.1 that the series $\widetilde{\text{RS}}(x)$ is k -summable for no $k > 0$ in the directions $\theta \in [\pi/4, 3\pi/4] \bmod \pi$ and therefore, no k -Borel-Laplace process applies in these directions. In the case of $\widetilde{\text{RS}}(x)$ and more generally of a $(1, 2)$ -summable series the method consists, in some way, in applying simultaneously a 1- and a 2-Borel-Laplace process as shown below.

Fix a non anti-Stokes direction θ belonging to $] \pi/4, 3\pi/4[\bmod \pi$ and, when no confusion is possible, denote simply by \mathcal{B}_1 and \mathcal{B}_2 instead of $\mathcal{B}_{1,\theta}$ and $\mathcal{B}_{2,\theta}$ the 1- and the 2-Borel transforms in direction θ and by \mathcal{L}_1 and \mathcal{L}_2 instead of $\mathcal{L}_{1,\theta}$ and $\mathcal{L}_{2,\theta}$ the 1- and the 2-Laplace integrals in direction θ (*cf.* Def. 6.3.5). Contrary to the 2-Borel transform the (formal) 1-Borel transform applied to $\widetilde{\text{E}}(x)$ and $\widetilde{\text{L}}(x)$ provides convergent series. This invites us to begin with the 1-Borel-Laplace process followed by the 2-Borel-Laplace process. The 1-Borel transform of $\text{E}(x)$ can be continued to infinity in direction θ (and neighboring directions) with exponential growth of order one and can then be applied a Laplace operator \mathcal{L}_1 . On the contrary, the 1-Borel transform of $\text{L}(x)$ can only be continued with exponential growth of order two (*cf.* Exa. 6.3.14). Hence, the Laplace operator \mathcal{L}_1 does not apply to $\mathcal{B}_1(\widetilde{\text{RS}}(x))(\xi)$.

A solution to this problem consists in merging the next two arrows of the process as indicated in the diagram:

$$\widetilde{\text{RS}}(x) \xrightarrow{\mathcal{B}_1} \varphi(\xi) \xrightarrow{\mathcal{L}_1} \cdot \xrightarrow{\mathcal{B}_2} \psi(\zeta) \xrightarrow{\mathcal{L}_2} \text{RS}(x).$$

$\underbrace{\hspace{10em}}_{\mathbf{A}_{2,1}}$

Formally, we can write

$$\begin{aligned} \mathbf{A}_{2,1}(\varphi)(\zeta) &= \mathcal{B}_2(\mathcal{L}_1(\varphi))(\zeta) \\ &= \frac{1}{2\pi i} \int_{\gamma_{2\theta}} \left(e^{\zeta^2/t} \int_0^{e^{i\theta}\infty} \varphi(\xi) e^{-\xi/t^{1/2}} d\xi \right) \frac{dt}{t^2} \\ &= \frac{1}{2\pi i} \int_0^{e^{i\theta}\infty} \varphi(\xi) \int_{\gamma_{2\theta}} \exp\left(\frac{\zeta^2}{t} - \frac{\xi}{t^{1/2}}\right) \frac{dt}{t^2} d\xi \quad (\text{commuting the integrals}) \\ &= \frac{1}{2\pi i} \int_0^{e^{i\theta}\infty} \varphi(\xi) \int_{\mathcal{H}} \exp\left(u - \frac{\xi}{\zeta} u^{1/2}\right) \frac{du}{\zeta^2} d\xi \quad (\text{setting } u = \zeta^2/t) \end{aligned}$$

where \mathcal{H} denotes a Hankel contour around the negative real axis. Setting $u^{1/2} = iv$ in the integral kernel $\mathcal{C}_2(\tau) = \frac{1}{2\pi i} \int_{\mathcal{H}} e^{u-\tau u^{1/2}} du$ we recognize the derivative of the Fourier transform of the Gauss function e^{-v^2} and we obtain $\mathcal{C}_2(\tau) = \tau e^{-\tau^2/4}/(2\sqrt{\pi})$. This kernel

is (for $\Re\tau > 0$) exponentially small of order two and can then be applied to the 1-Borel transform $\varphi(\xi)$ of the Ramis-Sibuya series $\widetilde{\text{RS}}(x)$. The operator $\mathbf{A}_{2,1}$ defined by

$$\mathbf{A}_{2,1}(\varphi(\xi))(\zeta) = \frac{1}{\zeta^2} \int_0^{e^{i\theta}\infty} \varphi(\xi) \mathcal{C}_2(\xi/\zeta) d\xi$$

is called *(2, 1)-accelerator*.

Now, the function $\mathbf{A}_{2,1}(\varphi(\xi))(\zeta)$ satisfies the same equation as $\mathcal{B}_2(\widetilde{\text{RS}}(x))(\zeta)$ and can thus be applied a 2-Laplace transform (see Exa. refRSsum3). Finally, we obtain an asymptotic function on a quadrant bisected by θ . This function has the Ramis-Sibuya series $\widetilde{\text{RS}}(x)$ as Taylor series since, formally, the followed process is the identity.

The formal calculation made above to define the accelerator $\mathbf{A}_{2,1}$ can be made with $\mathbf{A}_{k_2,k_1} = \mathcal{B}_{k_2}(\mathcal{L}_{k_1}(\varphi))$ for any pair of levels $k_1 < k_2$. We obtain

$$\mathbf{A}_{k_2,k_1}(\varphi(\xi))(\zeta) = \frac{1}{\zeta^{k_2}} \int_{\xi=0}^{e^{i\theta}\infty} \varphi(\xi) \mathcal{C}_{k_2/k_1}((\xi/\zeta)^{k_1}) d(\xi^{k_1})$$

where the kernel \mathcal{C}_{k_2/k_1} is defined by

$$\mathcal{C}_\alpha(\tau) = \frac{1}{2\pi i} \int_{\mathcal{H}} e^{w-\tau w^{1/\alpha}} dw, \quad \alpha > 1.$$

When it is useful to make explicit the direction θ in which the integral is taken we denote $\mathbf{A}_{k_2,k_1}^\theta$.

Proposition 8.6.2. — *Given $\alpha > 1$ let β denote its conjugate number:*

$$1/\alpha + 1/\beta = 1.$$

The kernel $\mathcal{C}_\alpha(\tau)$ is flat of exponential order β at infinity on the sector $|\arg(\tau)| < \pi/(2\beta)$, i.e., for all $\delta > 0$ there exist constants $c_1, c_2 > 0$ such that

$$|\mathcal{C}_\alpha(\tau)| \leq c_1 \exp(-c_2|\tau|^\beta) \quad \text{on} \quad |\arg(\tau)| \leq \pi/(2\beta) - \delta.$$

Proof. — The proof was already given in the part “(i) implies (ii) point 3” of the proof of the pre-Tauberian Theorem 6.3.11. To use the same notations, perform the change of variable $U = \tau^\alpha w$ in the integral defining $\mathcal{C}_\alpha(\tau)$. Set $F(U) = \exp(-U^{1/\alpha})$, $\zeta = 1/\tau^\alpha$ and $G(\zeta) = \mathcal{C}_\alpha(\zeta^{1/\alpha})/\zeta$. We obtain $G(\zeta) = 1/(2\pi i) \int_{\mathcal{H}} F(U) e^{\zeta U} dU$. Then, set $k_1 = 1$, $k_2 = \alpha$ and $\kappa_1 = \beta$. Set $a_n = 0$ for all n in (42) (the exponential $F(U)$ is flat on $\arg(U) < \alpha\pi/2$). We conclude from Estimate (41) and Proposition 2.3.17. \square

Corollary 8.6.3. — *Denote, as before, $1/\kappa_1 = 1/k_1 - 1/k_2$.*

The accelerator \mathbf{A}_{k_2,k_1} applies to any function φ with exponential growth of order κ_1 at infinity in direction θ .

With this result the accelerator \mathbf{A}_{k_2, k_1} appears like similar to a κ_1 -Laplace operator and has similar properties. Let us state the following result generalizing Theorem 6.3.11 “(ii) \implies (i)” (cf. [Bal94, Thm 5.2.1]).

Lemma 8.6.4. — *Let \widehat{k} , k_1 , $k_2 > 0$ be given. Assume $k_1 < k_2$ and define κ and k by*

$$\frac{1}{\kappa} = \frac{1}{k_1} - \frac{1}{k_2} \quad \text{and} \quad \frac{1}{k} = \frac{1}{\kappa} + \frac{1}{\widehat{k}}.$$

Let $\widehat{\sigma} = \widehat{I} \times]0, +\infty[$ be an unlimited sector and denote by I the arc with same middle point θ_0 as \widehat{I} and length $|\widehat{I}| + \pi/\kappa$.

Suppose that $g(\xi)$ is analytic on $\widehat{\sigma}$, it belongs to $\mathcal{A}_{1/\widehat{k}}(\widehat{I})$ at 0 and it has exponential growth of order κ at infinity on $\widehat{\sigma}$.

Then, the function $f(\xi) = \mathbf{A}_{k_2, k_1}(g)(\xi)$ belongs to $\mathcal{A}_{1/k}(I)$.

Definition 8.6.5. — *Let $\underline{k} = (k_1, k_2, \dots, k_\nu)$ be a multi-level and θ_0 a direction. A series $\widetilde{f}(x)$ is said to be \underline{k} -accelero-summable (or, accelero-summable) in direction θ_0 if it can be applied the following sequence of operators in direction $\theta = \theta_0$ and neighboring ones resulting in the accelero-sum $f(x)$:*

$$\widetilde{f}(x) \xrightarrow{\mathcal{B}_{k_1}} \cdot \xrightarrow{\mathbf{A}_{k_2, k_1}} \cdot \xrightarrow{\mathbf{A}_{k_3, k_2}} \cdot \dots \cdot \xrightarrow{\mathbf{A}_{k_{\nu-1}, k_\nu}} \cdot \xrightarrow{\mathcal{L}_{k_\nu}} f(x).$$

By the expression “can be applied” we mean that the kernels of the integral operators have, at each step, the right growth rate at infinity for the integral to exist.

The term accelero-summation is commonly used for a larger class of operators associated with various kernels depending on the type of problem one wants to solve. Here, we refer always to the definition given above.

Theorem 8.6.6. — *\underline{k} -multisummability and \underline{k} -accelero-summability in a given direction θ_0 are equivalent with “same” sum.*

Precisely, if (f_1, f_2, \dots, f_ν) is the \underline{k} -multisum of a series $\widetilde{f}(x)$ in direction θ_0 then, $f = f_\nu$ is its \underline{k} -accelero-sum in direction θ_0 .

Proof. — We sketch the case of two levels $\underline{k} = (k_1, k_2)$ letting the reader perform the general case by iteration. Since the theorem holds true for polynomials we can assume that the given series have valuation greater than k_2 so that their k_2 -Borel series contain only positive powers of ξ .

▷ *multisummability implies accelero-summability.*

Without loss of generality we can assume that $k_1 > 1/2$. From Theorem 8.5.3 it is then sufficient to prove that k_1 - and k_2 -summable series in direction θ_0 are accelero-summable in the same direction. Suppose $\widetilde{f}(x)$ is k_1 -summable

in direction θ_0 with k_1 -sum $f_1(x)$. Then, $\mathcal{B}_{k_1}(\tilde{f})(\xi)$ is convergent at 0 and its sum $\varphi(\xi)$ can be analytically continued with exponential growth of order k_1 at infinity in direction θ_0 and neighboring ones. Since $\kappa_1 > k_1$ one can apply the accelerator \mathbf{A}_{k_2, k_1} to $\varphi(\xi)$ in direction θ_0 and neighboring ones and the resulting function can be analytically continued to infinity with moderate growth; it can then be applied a k_2 -Laplace transform to produce a (k_1, k_2) -sum $f(x)$. Actually in that case, $\mathbf{A}_{k_2, k_1}(\varphi) = \mathcal{B}_{k_2} \circ \mathcal{L}_{k_1}(\varphi)$ (the integrals commute). It follows that $f(x) = \mathcal{L}_{k_2} \circ \mathcal{B}_{k_2}(f_1)(x)$ and $f(x)$ is the restriction of f_1 to an open sector with opening larger than π/k_2 (but possibly smaller than π/k_1) centered at θ_0 .

Suppose $f(x)$ is k_2 -summable in direction θ_0 with k_2 -sum $f(x)$ and let κ_1 be defined by $1/\kappa_1 = 1/k_1 - 1/k_2$. The k_1 -Borel transform $\hat{g}_1(\xi) = \mathcal{B}_{k_1}(\tilde{f})(\xi)$ of $\tilde{f}(x)$ defines an entire function $g_1(\xi)$ with exponential growth of order κ_1 at infinity. Lemma 8.6.4 applied to $g_1(\xi)$ with $1/\hat{k} = 0$ shows that $\mathbf{A}_{k_2, k_1}(g_1)(\xi)$ is κ_1 -asymptotic to $\hat{g}_2(\xi) = \mathcal{B}_{k_2}(\tilde{f})(\xi)$ on a sector of opening larger than π/κ_1 bisected by θ_0 . Since $\tilde{f}(x)$ is k_2 -summable in direction θ_0 its k_2 -Borel series $\hat{g}_2(\xi)$ is convergent and $\mathbf{A}_{k_2, k_1}(g_1)(\xi)$ coincide with the sum $g_2(\xi)$ of series $\hat{g}_2(\xi)$. A k_2 -Laplace transform provides then the k_2 -sum $f(x)$ of $\tilde{f}(x)$ in direction θ_0 .

▷ *Accelerator-summability implies multisummability.*

Suppose $\tilde{f}(x)$ is (k_1, k_2) -accelero-summable in direction θ_0 . Thus, its k_1 -Borel transform $\hat{g}(\xi) = \sum_{p > k_2} c_p x^p$ converges on a disc $\hat{D}_\rho = \{|\xi| < \rho\}$. Its sum $g(\xi)$ can be analytically continued to an open sector $\hat{\sigma}$ neighboring the direction θ_0 with exponential growth of order κ_1 at infinity (recall $1/\kappa_1 = 1/k_1 - 1/k_2$). We choose $\hat{\sigma}$ so narrow about θ_0 that $\tilde{f}(x)$ is (k_1, k_2) -accelero-summable in all direction θ belonging to $\hat{\sigma}$. Without loss of generality, we assume that $1/\kappa_1 < 2$. Show now that under that condition the series \tilde{f} splits into a sum $\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$ where $\tilde{f}_1(x)$ and $\tilde{f}_2(x)$ are k_1 - and k_2 -summable in direction θ_0 respectively (cf. [Bal92a, Lem. 1 and 2]).

Show first that it suffices to consider the case when $\tilde{f}(x)$ is (k_1, k_2) -summable (i.e., (k_1, k_2) -summable in almost all direction). Let $0 < r < \rho$. The circle γ centered at 0 with radius r belongs to \hat{D}_ρ . Denote by γ_1 the arc of γ oriented positively outside the sector $\hat{\sigma}$ and by γ_2 the arc oriented positively inside $\hat{\sigma}$. From Cauchy's integral formula we know that, on the interior \hat{D}_r of γ , the function $g(\xi)$ satisfies

$$g(\xi) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\eta)}{\eta - \xi} d\eta$$

and $g(\xi) = g_1(\xi) + g_2(\xi)$ there if one sets

$$g_1(\xi) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{g(\eta)}{\eta - \xi} d\eta \quad \text{and} \quad g_2(\xi) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{g(\eta)}{\eta - \xi} d\eta.$$

The function $g_1(\xi)$ has an analytic continuation to $\widehat{D}_r \cup \widehat{\sigma}$ which is bounded at infinity. The k_1 -Laplace transform $\widehat{f}_1(x)$ of the Taylor series $\widehat{g}_1(\xi)$ of $g_1(\xi)$ is therefore a k_1 -summable series in direction θ_0 . Denote by $\text{cl}(\widehat{\sigma})$ the closure of $\widehat{\sigma}$ and set $\widehat{\sigma}' = \mathbb{C} \setminus \text{cl}(\widehat{\sigma})$. Thus, $\mathbb{C} = \widehat{\sigma} \cup \widehat{\sigma}' \cup d' \cup d''$ where d' and d'' are the two half-lines limiting $\widehat{\sigma}$ and $\widehat{\sigma}'$. Similarly to $g_1(\xi)$, the function $g_2(\xi)$ has an analytic continuation to $\widehat{D}_r \cup \widehat{\sigma}'$ which is bounded at infinity. Its k_1 -Laplace transform $\widehat{f}_2(x)$ is thus k_1 - hence (k_1, k_2) -summable in all directions of $\widehat{\sigma}'$. On another hand, $\widehat{f}_2(x) = \widehat{f}(x) - \widehat{f}_1(x)$, is (k_1, k_2) -summable in all directions of $\widehat{\sigma}$. We conclude that $\widehat{f}_2(x)$ being (k_1, k_2) -summable in all directions but, maybe, the singular directions d' and d'' , is (k_1, k_2) -summable. To continue the proof we can then assume that $\widetilde{f}(x)$ is (k_1, k_2) -summable (in almost all direction).

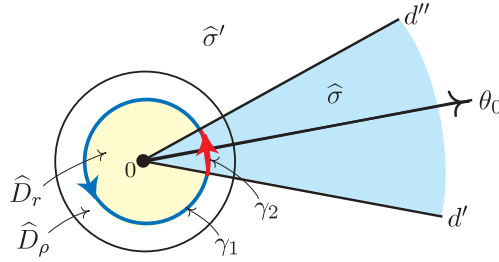


Figure 5

Prove now the followed splitting $\widetilde{f}(x) = \widetilde{f}_1(x) + \widetilde{f}_2(x)$ where $\widetilde{f}_1(x)$ and $\widetilde{f}_2(x)$ are respectively k_1 - and k_2 -summable in direction θ_0 . From Lem. 8.6.4 the function $h^\theta(\zeta) = \mathbf{A}_{k_2, k_1}^\theta(g)(\zeta)$ is κ_1 -asymptotic to a same series (precisely, to the k_2 -Borel transform of \widetilde{f}) on a sector of size π/κ_1 in all non-singular direction θ . These functions are analytic continuations from each others as long as θ does not pass a singular direction. With $\widetilde{f}(x)$ we can thus associate a 0-cochain $(\varphi_j(\zeta))_{j \in J}$ with $\varphi_j \in \overline{\mathcal{A}}_{1/\kappa_1}(U_j)$ all asymptotic to the same series $\widehat{f}(\zeta) = \mathcal{B}_{k_2}(\widetilde{f})(\zeta)$, the sectors U_j having opening $|U_j| > \pi/\kappa_1$ and making a good covering $\mathcal{U} = (U_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$ of a punctured neighborhood of 0 in \mathbb{C} (cf. Def. 3.2.9). We can choose the covering \mathcal{U} such that U_0 is bisected by θ_0 . Denoting by θ_j the direction bisecting U_j observe that, under such conditions, there might be several singular directions between θ_j and θ_{j+1} . Notice that such a covering is made possible due to the condition $1/\kappa_1 < 2$. Denote, as

previously, by $\dot{U}_j = U_j \cap U_{j+1}$ the nerve of \mathcal{U} . For all $j \in \mathbb{Z}/p\mathbb{Z}$ choose $a_j \in \dot{U}_j$ and apply the Cauchy-Heine Theorem (Thm. 2.5.2) to build a new 0-cochain with associated 1-cocycle $(\dot{\varphi}_j = \varphi_j - \varphi_{j+1})$. The construction is as follows. Decompose the 1-cocycle (φ_j) into the sum of the elementary 1-cocycles $\varphi'_j = \varphi_j$ on \dot{U}_j and 0 on \dot{U}_ℓ when $\ell \neq j$. Set $r' = \min(|a_j|)$ and $\dot{U}'_j = \dot{U}_j \cap \{|\zeta| < r'\}$. Denote by U'_j the sector with self intersection \dot{U}'_j . The Cauchy-Heine Theorem (Thm. 2.5.2) says that the function

$$\psi'_j(\zeta) = \frac{1}{2\pi i} \int_0^{a_j} \frac{\dot{\varphi}_j(t)}{t - \zeta} dt$$

can be analytically continued to U'_j with 1-cocycle $\varphi'_j(\zeta)$ and $\psi'_j(\zeta)$ is κ_1 -Gevrey asymptotic to the series $\sum c_m \zeta^m$ where $c_m = 1/(2\pi i) \int_0^{a_j} \dot{\varphi}_j(t)/t^{m+1} dt$. It has also an analytic continuation to \mathbb{C} deprived of the half-line $d_j =]0, \alpha_j[$, $\alpha_j = \arg(a_j)$ and it tends to 0 at infinity. Define the analytic function $\psi_j(\zeta)$ on U_j by setting

$$\psi_j(\zeta) = \sum_{\ell \in \mathbb{Z}/p\mathbb{Z}} \psi'_\ell(\zeta), \quad \zeta \in U_j$$

(choose the determinations of the ψ'_ℓ that are analytic on all of U_j). Suppose now that a_0 and a_{p-1} have been chosen so that the angle $|\alpha_0 - \alpha_{p-1}|$ is $> \pi/\kappa_1$ and bisected by θ_0 . This is possible since the opening of U_0 is larger than $\pi/(2\kappa_1)$ on both sides of θ_0 . Denote by V_0 the unlimited open sector $]\alpha_{p-1}, \alpha_0[\times]0, \infty[$ and by $\Psi_0(\zeta)$ the analytic continuation of $\psi_0(\zeta)$ to V_0 . The sector V_0 is κ_1 -wide; the function $\Psi_0(\zeta)$ has a κ_1 -asymptotic expansion $\widehat{\Psi}_0(\zeta)$ at 0 and an exponential growth of order less than k_2 at infinity on V_0 . Denote by $\widetilde{f}_1(x)$ the k_2 -Laplace transform of the series $\widehat{\Psi}_0(\zeta)$. It follows from Theorem 6.3.11 (ii) \implies (i) that the series $\widetilde{f}_1(x)$ is k_1 -summable in direction θ_0 .

On another hand, denote by $\widehat{\phi}_0(\zeta)$ the asymptotic series of φ_0 (it is actually the k_2 -Borel transform of $\widetilde{f}(x)$). By hypothesis, one can apply a \mathcal{L}_{k_2} -Laplace transform to φ_0 in direction θ_0 and neighboring ones. This means that φ_0 has an analytic continuation to an unlimited sector V'_0 containing the direction θ_0 with exponential growth of order k_2 . The 0-cochains $\varphi_j(\zeta)$ and $\psi_j(\zeta)$ induce the same 1-cocycle on $(\dot{U}'_j)_{j \in \mathbb{Z}/p\mathbb{Z}}$. It follows that $\varphi_j(\zeta) - \varphi_{j+1}(\zeta) = \psi_j(\zeta) - \psi_{j+1}(\zeta)$ for all j and the functions $\varphi_j(\zeta) - \psi_j(\zeta)$ glue together into an analytic function on the disc $D' = \{|\zeta| < r'\}$ which is the sum of the series $\widehat{\phi}_0(\zeta) - \widehat{\Psi}_0(\zeta)$. Denote by $\widetilde{f}_2(x)$ the k_2 -Laplace transform of that series. Therefore, the function

$\varphi_0(\zeta) - \psi_0(\zeta)$ can be continued into an analytic function on $V'_0 \cup D'$ with Taylor series $\mathcal{B}_{k_2}(\tilde{f}_2)(\zeta)$ and it has exponential growth of order k_2 at infinity. This means that the series $\tilde{f}_2(x)$ is k_2 -summable in direction θ_0 . Moreover, $\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$ and the result follows. \square

The second part of the proof of Theorem 8.6.6 provides the following improvement of Theorem 8.5.3:

Theorem 8.6.7 (Balsler [Bal93]). — *Let be given a multi-level $\underline{k} = (k_1, k_2)$ and a \underline{k} -multi-arc $\underline{I} = (I_1, I_2)$. Denote $1/\kappa_1 = 1/k_1 - 1/k_2$. Under the condition*

$$\kappa_1 > 1/2$$

then, \underline{k} -split-summability and \underline{k} -summability on \underline{I} are equivalent with same sum.

The property extends to multi-arcs $\underline{I} = (I_1, \dots, I_\nu)$ of length $\nu \geq 2$ under the conditions

$$\kappa_j > 1/2 \quad \text{where} \quad 1/\kappa_j = 1/k_j - 1/k_{j+1} \quad \text{for} \quad j = 1, \dots, \nu - 1.$$

Observe that the counter-example in Proposition 8.5.7 corresponds to $\nu = 2$ and $\kappa_1 = 1/2$.

8.7. Sixth approach: wild analytic continuation

8.7.1. \underline{k} -wild-summability. — Like Ramis-Sibuya definition of k -summability was translated in terms of analytic continuation in the infinitesimal neighborhood X^k of 0 endowed with the sheaf \mathcal{F}^k (cf. Sect. 6.4.1) Malgrange-Ramis definition of \underline{k} -multisummability can be translated in terms of analytic continuation in the infinitesimal neighborhood $X^{\underline{k}}$ of 0 endowed with the sheaf $\mathcal{F}^{\underline{k}}$ (cf. Sect. 4.5.3).

Definition 8.7.1 (\underline{k} -wild-summability). — *Let $\underline{k} = (k_1, \dots, k_\nu)$ be a multi-level and let $\underline{I} = (I_1, \dots, I_\nu)$ be a \underline{k} -multi-arc (cf. Def. 8.2.9). Set $\{k_{\nu+1}, 0\} = +\infty$ (cf. Nots. in Sect. 4.5.2).*

▷ A series $\tilde{f}(x) = \sum_{n \geq 0} a_n x^n$ is said to be \underline{k} -wild-summable on \underline{I} if it can be wild analytically continued in the infinitesimal neighborhood $(X^{\underline{k}}, \mathcal{F}^{\underline{k}})$ of 0 to a domain containing the closed disc $\overline{D}(0, \{k_1, 0\})$ and the sectors $I_j \times]0, \{k_{j+1}, 0\}]$ for all $j = 1, \dots, \nu$.

▷ Its sum is the germ of analytic function defined on I_ν by this wild analytic continuation. It is said to be \underline{k} -wild-summable in direction θ if all arcs I_j are bisected by the direction θ .

▷ The series is said to be \underline{k} -wild-summable if it is \underline{k} -wild-summable in almost all direction, i.e., all direction but finitely many called singular directions.

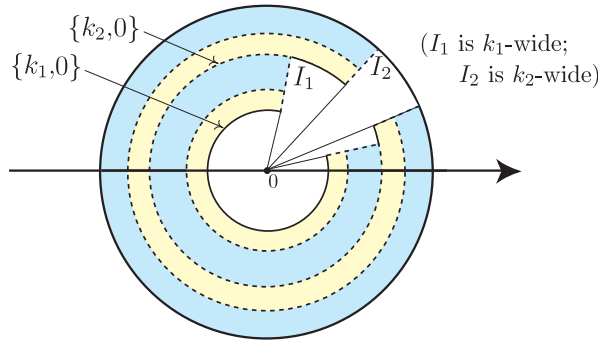


Figure 6. Domain for a (k_1, k_2) -sum in X^{k_1, k_2} (in white)

It follows from the Relative Watson's Lemma 8.2.1 and Watson's Theorem 6.1.3 that the continuation, hence the sum in the sense of wild-summation, when it exists, is unique.

Since Definition 8.7.1 exactly translates Malgrange-Ramis definition of multisummability (Def. 8.3.1) we can state:

Proposition 8.7.2. — \underline{k} -wild-summability is equivalent to \underline{k} -summability in any of the previous sense with same sum.

Definition 8.7.3. — Let \underline{I} be a \underline{k} -multi-arc.

▷ A sector built on \underline{I} like in Definition 8.7.1 (cf. Fig. 6) is called a \underline{k} -sector in $X^{\underline{k}}$.

▷ A k_j -arc I_j such that $\tilde{f}(x)$ can be wild analytically continued to the open sector $I_j \times]0, \{k_j, 0\}[$ but not to the closed sector $I_j \times]0, \{k_j, \infty\}]$ in $X^{\underline{k}}$ is said to be a singular arc of level k_j (for $\tilde{f}(x)$); otherwise it is said to be non singular.

▷ A direction θ bisecting one or several singular arcs is said to be a singular direction for $\tilde{f}(x)$; otherwise it is said to be non singular.

From the viewpoint of wild analytic continuation the following results are straightforward.

Proposition 8.7.4. — Let $\underline{k} = (k_1, k_2, \dots, k_\nu)$ be a multi-level.

▷ A series is \underline{k} -summable if and only if it admits finitely many singular arcs in $X^{\underline{k}}$.

▷ Let \underline{k}' be a multi-level containing all levels k_1, k_2, \dots, k_ν of \underline{k} . A series which is \underline{k} -summable in a direction is also \underline{k}' -summable in that direction. In other words, \underline{k} -summability is stronger than \underline{k}' -summability.

Proof. — ▷ If there is finitely many singular arcs then the series is \underline{k} -summable in all directions but the finitely many bisecting directions of the singular arcs. Conversely, suppose that the series has infinitely many singular arcs. If \underline{k} contains several levels then there is at least one level supporting infinitely many singular arcs and all bisecting directions of these arcs are singular directions for the series. Hence, the non-summability of the series.

▷ From the viewpoint of wild analytic continuation the domain one has to continue the series towards its \underline{k} -summability in direction θ contains the domain one has to continue it towards its \underline{k}' -summability in the same direction. \square

8.7.2. Application to Tauberian Theorems. — The Tauberian Theorems 6.3.12 and 6.3.13 are easily generalized to multisummable series (cf. [MR91, Prop. 8 p. 349]). Without loss of generality we assume that the smallest level k_1 is greater than $1/2$.

Theorem 8.7.5. — Let $\underline{k} = (k_1, \dots, k_\nu)$ be a multi-level, \underline{I} be a \underline{k} -multi-arc and $k_{\nu+1} = \infty$. Suppose k' satisfies $k_j \leq k' < k_{j+1}$ for some $j \in \{1, 2, \dots, \nu\}$.

Then, a series $\tilde{f}(x)$ which is both \underline{k} -summable on \underline{I} and $1/k'$ -Gevrey is $(k', k_{j+1}, \dots, k_\nu)$ -summable on $(I_j, I_{j+1}, \dots, I_\nu)$.

Proof. — Denote by \underline{k}' the multi-level \underline{k} augmented by k' in case $k' \neq k_j$. The proof is performed in the space $(X^{\underline{k}'}, \mathcal{F}^{\underline{k}'})$. The series $\tilde{f}(x)$ being $1/k'$ -Gevrey it can be continued as a section of $\mathcal{F}^{\underline{k}'}$ to the closed disc $D_{k'}$ with

radius $\{k', 0\}$. Being \underline{k} -summable on \underline{I} it can be continued to the \underline{k} -sector $\Delta_{\underline{k}, \underline{I}}$ built on \underline{I} . With the same arguments as in Section 6.4.2 one proves that the two continuations agree on $D_{k'} \cap \Delta_{\underline{k}, \underline{I}}$. Hence, $\tilde{f}(x)$ can be continued to the $(k', k_{j+1}, \dots, k_\nu)$ -sector $\Delta' = D_{k'} \cup \Delta_{\underline{k}, \underline{I}}$ and the result follows. \square

Given two multi-levels $\underline{k}' = (k'_1, k'_2, \dots, k'_{\nu'})$ and $\underline{k}'' = (k''_1, k''_2, \dots, k''_{\nu''})$ we define the multi-level $\underline{K} = (K_1, \dots, K_\nu)$ as being the shuffle of \underline{k}' and \underline{k}'' starting from $K_1 = \max(k'_1, k''_1)$.

Proposition 8.7.6. — *A series $\tilde{f}(x)$ which is both \underline{k}' - and \underline{k}'' -summable in a direction θ is \underline{K} -summable in direction θ .*

The proof is similar to the previous one.

Such a result is of poor interest since, in general, K has more levels than both \underline{k}' and \underline{k}'' and being \underline{K} -summable in direction θ is more complicated than being \underline{k}' - or \underline{k}'' -summable in direction θ . It would be more interesting to get \underline{K} -summable series for a \underline{K} equal to the intersection of \underline{k}' and \underline{k}'' . This is impossible when one considers summable series in a given direction. More interesting is the case of series that are both globally (*i.e.*, in almost all directions) \underline{k}' - and \underline{k}'' -summable since, then, one has the following generalization of the Tauberian Theorem 6.3.13.

Theorem 8.7.7. — *With notations as before let $\underline{\kappa} = \underline{k}' \cap \underline{k}''$ be the multi-level defined on the common values of \underline{k}' and \underline{k}'' : $\underline{\kappa} = (\kappa_1, \dots, \kappa_{\nu_0})$ satisfies*

$$\{\kappa_1, \dots, \kappa_{\nu_0}\} = \{k'_1, k'_2, \dots, k'_{\nu'}\} \cap \{k''_1, k''_2, \dots, k''_{\nu''}\}.$$

A series $\tilde{f}(x)$ which is both \underline{k}' - and \underline{k}'' -summable satisfies the following properties:

- (i) *if $\underline{\kappa} = \emptyset$ then $\tilde{f}(x)$ is convergent;*
- (ii) *if $\underline{\kappa} \neq \emptyset$ then $\tilde{f}(x)$ is $\underline{\kappa}$ -summable.*

Proof. — (i) Case when $\underline{\kappa}$ is empty. Suppose for instance that \underline{k}' and \underline{k}'' satisfy

$$k'_1 < \dots < k'_{j'_1-1} < k''_1 < \dots < k''_{j''_1-1} < k'_{j'_1} < \dots < k'_{j'_2-1} < k''_{j''_1} < \dots$$

It is sufficient to prove that the series has no singular arc in $(X^{\underline{k}' \cup \underline{k}''}, \mathcal{F}^{\underline{k}' \cup \underline{k}''})$.

Prove first that a \underline{k} -summable series ($\underline{k} = (k_1, k_2, \dots, k_\nu)$) with no singular arc of level k_1 in $(X^{\underline{k}}, \mathcal{F}^{\underline{k}})$ is (k_2, \dots, k_ν) -summable. Indeed, in that case, one can choose a covering of $X^{\underline{k}}$ by \underline{k} -sectors Δ^ℓ based on \underline{k} -arcs $(I_1^\ell, I_2^\ell, \dots, I_\nu^\ell)$ with the following properties: the sectors Δ^ℓ are sectors of \underline{k} -summation of the

series; the I_j^ℓ 's form a cyclic covering of S^1 and the consecutive intersections $I_1^\ell \cap I_1^{\ell+1}$ are made of arcs of length larger than π/k_1 . From the Ramis-Sibuya Corollary 6.2.2 and the Relative Watson Lemma 8.2.1 we conclude like in Section 6.4.2 that the corresponding sums glue together into a section of \mathcal{F}^k over the closed disc D_{k_2} with radius $\{k_2, 0\}$. The series is then $1/k_2$ -Gevrey and, by Theorem 8.7.5, it is (k_2, \dots, k_ν) -summable.

The series $\tilde{f}(x)$ being both \underline{k}' -summable and $1/k_1''$ -Gevrey we know from Theorem 8.7.5 that it is at worst $(k_1'', k_{j_1}', \dots, k_{\nu'}')$ -summable. As a \underline{k}' -summable series it has then no singular arc of level $< k_1''$ and then of level $< k_{j_1}'$. This proves that $\tilde{f}(x)$ is $(k_{j_1}', k_{j_1'+1}', \dots, k_{\nu'}')$ -summable. Exchanging the role of the k' 's and of the k'' 's we show the same way that the series is $(k_{j_1}'', k_{j_1'+1}'', \dots, k_{\nu}''')$ -summable; then $(k_{j_2}', k_{j_2'+1}', \dots, k_{\nu'}')$ -summable and so on... until no singular arc is left.

(ii) Suppose for instance that $k_1'' = k_{j_1}'$.

The previous reasoning remains valid on any arc I_1'' of summability. Instead of a continuation to the full infinitesimal neighborhood $(X^{\underline{k}' \cup \underline{k}''}, \mathcal{F}^{\underline{k}' \cup \underline{k}''})$ we obtain the continuation to any \underline{k} -multi-sector but the finitely many ones that are based on arcs of levels in $\underline{k}' \cap \underline{k}''$ that are singular for the \underline{k}' - and the \underline{k}'' -summation of $\tilde{f}(x)$. This ends the proof of the theorem. \square

With this result we see that any multisummable series is \underline{k} -summable for a unique \underline{k} of smallest length. This is no more true for directional summability. Recall, for instance, the case of the Leroy series which is both 1- and 2-summable in infinitely many directions (*cf.* Exa. 6.3.14).

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INDEX OF NOTATIONS

\Subset	proper inclusion of sectors	5
\mathcal{A}	sheaf over S^1 of germs of asymptotic functions	47
\mathcal{A}_s	subsheaf of \mathcal{A} of s -Gevrey germs	47
$\mathcal{A}^{<0}$	subsheaf of \mathcal{A} of flat germs, <i>i.e.</i> , asymptotic to 0	47
$\mathcal{A}^{\leq -k}$	subsheaf of $\mathcal{A}^{<0}$ of germs exponentially flat of order k	47
\mathcal{B}	formal Borel transform	125
\mathcal{B}_θ	Borel transform in direction θ	125
$\mathcal{B}_{k,\theta}$	k -Borel transform in direction θ	129
$\check{\mathcal{C}}$	Riemann surface of logarithm	5
$\mathbb{C}[[x]]$	differential algebra of formal power series with complex coefficients and derivation d/dx	
$\mathbb{C}\{x\}$	differential sub-algebra of $\mathbb{C}[[x]]$ restricted to convergent series	
$\mathbb{C}[[x]]_s$	differential sub-algebra of $\mathbb{C}[[x]]$ restricted to Gevrey series of order s , <i>i.e.</i> , of level $k = 1/s$	18
$\mathbb{C}\{x\}_{\{k,I\}}$	differential sub-algebra of $\mathbb{C}[[x]]$ of k -summable series on I	116
$\mathbb{C}\{x\}_{\{\underline{k},\underline{I}\}}$	differential sub-algebra of \underline{k} -multisummable series on \underline{I}	181
$\tilde{\mathbb{E}}(x)$	Euler series	8
$\mathbb{E}(x)$	Euler function	8
\mathcal{E}_0	homogeneous Euler operator	49
$\mathbb{E}i(x)$	exponential integral function	10
$\overline{\mathcal{F}}, \mathcal{F}$	presheaf and associated sheaf	42
${}_3F_0$	example of a hypergeometric series	11
$\tilde{g}(z)$	$\tilde{g}(z) = z^{-4}{}_3F_0(\{3, 4, 5\} 1/z)$	11
$\tilde{h}(z)$	example of a solution of a mild difference equation	13
$\tilde{\ell}(z)$	example of a solution of a wild difference equation	14
k, k_1, \dots	levels (positive numbers)	

$\underline{k}, \underline{I}$	multi-level and multi-arc	180
\mathcal{L}_θ	Laplace transform in direction θ	125
$\mathcal{L}_{k,\theta}$	k -Laplace transform in direction θ	129
s, s_1, \dots	orders, <i>i.e.</i> , inverses of levels $s = 1/k, s_1 = 1/k_1, \dots$	
$\mathcal{A}_{\alpha,\beta}(R)$	open sector $\{x; \alpha < \arg(x) < \beta \text{ and } 0 < x < R\}$ in \mathbb{C} or $\check{\mathbb{C}}$	5
$\mathcal{A}, \mathcal{A}_1, \dots$	open sectors in \mathbb{C}^* or $\check{\mathbb{C}}$ at 0	
\mathcal{A}	open sector in \mathbb{C}^* or $\check{\mathbb{C}}$ at infinity	
$\bar{\mathcal{A}}$	closure of the sector \mathcal{A} in \mathbb{C}^* or $\check{\mathbb{C}}$	
$\mathcal{S}_{\{k,I\}}$	k -summation operator on the arc I	115
$\mathcal{S}_{\{k,\theta\}}$	k -summation operator in direction θ	115
$\mathcal{S}_{\{\underline{k},\underline{I}\}}$	\underline{k} -summation operator on the multi-arc \underline{I}	181
T_Δ	Poincaré asymptotic expansion at 0 on Δ (usual Taylor map)	7
$T_{s,\Delta}$	s -Gevrey asymptotic expansion at 0 on Δ (Taylor map)	23
$x, z = 1/x$	coordinates on the Riemann sphere near 0 and ∞ respectively (Laplace plane)	125
ξ, ζ	coordinates after Borel transform (Borel plane)	125

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